**Intro**

The Jupiter Trojans are two groups of asteroids which flank Jupiter’s orbit of the Sun, at approximately in front and behind. The clusters of asteroids appear to be on an identical orbit around the Sun as Jupiter – orbiting roughly at the same radius as Jupiter, and with the same period. These orbits are only possible at those positions relative to Jupiter, due to the stable Lagrange points they inhabit. This provides the asteroids which make up the Jupiter Trojans with long-term stability which allows them to persist to this day. Currently, only asteroids which inhabit either one of the two flanking Lagrange points, having Tadpole orbits, are known to exist. This paper aims to investigate another possible type of quasi-stable orbit around both of the Lagrange points these Trojans could enter, or which has possibly existed in the past – known as a Horseshoe orbit.

The restricted three body problem can be used to determine the evolution of a system of three bodies – two massive, and one of negligible mass. The system considered in this paper takes initial parameters resembling the Sun and Jupiter, in circular orbits, in order to analyse the evolution of a third, massless, object representative of an asteroid.

Such a system results in a complex time and space varying gravitational field resulting from the superposition of the gravitational fields of the Sun and Jupiter as they orbit about a common barycentre – there is no full analytic solution to this problem, and thus numerical approximations must be used.

The following assumptions were applied to the system:

1. The Sun and Jupiter have constant fixed circular orbits around one another which are invariant over time – they orbit “on rails” and cannot be perturbed.
2. All initial positions and velocities are co-planar for both the massive objects, and the massless asteroids – the system can be represented as two-dimensional.

Jacobi’s integral can be used in this system to determine the regions in the Sun-Jupiter system that an asteroid with given initial position and velocity parameters can move within. is the asteroid velocity; is the specific potential energy – including both the gravitational and centrifugal potentials – and is a constant relating to the initial conditions of the asteroid.

The asteroid must always satisfy , which constrains it in regions of space, dependent on the initial conditions . This property can be used to construct equipotential zero-velocity curves which satisfy , and constrain the particle to move where .

Equilibrium points in the system exist where the net force (sum of gravitational and centrifugal forces) is zero in a rotating reference frame of angular velocity . In a non-rotating frame, this is equivalent to the gravitational force being equal to the centrifugal force required to rotate at angular velocity at a radius .

Points of zero net force in the rotating frame can be related to the potential via by taking a derivative.

Therefore, these equilibrium points can be identified where the potential reaches a stationary point in both dimensions. From a potential contour plot, five of these points can be identified (for ), known as Lagrange points. There are three unstable Lagrange points (L1, L2, and L3), which correlate to the three saddle stationary points along the Jupiter-Sun axis, and two stable Lagrange points (L4, and L5), which correlate to the two maxima either side of Jupiter. It can be shown that

The consequence of this, is that an asteroid placed at any of these points with an initial velocity perpendicular to the radius drawn from the asteroid to the Sun-Jupiter barycentre with magnitude – where is the orbital period of the system, will remain in that position relative to both Jupiter and the Sun forever. This results in the asteroid remaining stationary in the comoving frame, and the distance between the asteroid and both of the Sun and Jupiter remaining constant.

However, in reality, small perturbations will cause an asteroid to drift from the Lagrange point. For L1, L2, and L3 – the saddle points – a small movement from equilibrium will cause the asteroid to be ejected from the orbit. For the maxima L4 and L5, however; a perturbed asteroid will be able to follow a closed equipotential line orbiting the Lagrange point and remain stable. This causes the asteroid to move in the comoving frame (drawing a tadpole shaped orbit, similar to the equipotential lines surround L4/L5) and for the asteroid-Jupiter/Sun distance to oscillate over time. These are known as “Tadpole” orbits. A further perturbation can push the asteroid out towards the closed equipotential contours encapsulating both L4 and L5, shaped like a horseshoe. These are known as “Horseshoe” orbits.

In the Sun-Jupiter system, all known Trojan asteroids have a variation of the Tadpole orbit; and none being of the Horseshoe variety. This paper looks to investigate aspects of the Horseshoe orbit in the Sun-Jupiter system, including theoretical lifetime and regions where they could exist, to explore reasons for their absence and ways they could form in the future.

**Method**

The dynamics of a restricted 3-body system were simulated over a period of time, , using steps of interval . A two body “on rails” circular orbit system, identical to the Sun-Jupiter system, was developed. Values for the masses, and , and the barycentre distances, and , were taken from literature, and subsequently the system period and angular velocity were calculated using Kepler’s Third Law. At a given time , the Sun and Jupiter are at cartesian positions and . The asteroid is at cartesian position with velocity . To calculate velocity and position at , the net gravitational force due to the Sun and Jupiter is calculated, and the RK4 method is used to accurately approximate the solution to the differential equations in 4 steps, in order to find and . The positions of the Sun and Jupiter are then moved along their orbit – from polar positions and to and through an angle change of . This process was then repeated over iterations to obtain the final state of the system at time , and a history of the positions each body had been at over the simulation period. From this list of positions, several key indicators of how the system evolved can be extracted.

Firstly, the raw positions can be used to track the orbits and orbit paths in the “stationary” frame – i.e. the absolute values of object position relative to the system barycentre can be seen.

Secondly, by calculating the aggregate of over time, and removing its value at each timestep, the raw positions can be transformed to show the positions relative to the rotation of the Sun-Jupiter system. This “comoving frame” results in the positions of Jupiter, the Sun, and all Lagrange points to be stationary over time, and shows how the position of the asteroid evolves relative to the system.

Thirdly, by finding the difference between the positions of the asteroid and Jupiter at each time step, one can see how the asteroid’s proximity to Jupiter evolves over time. This can be compared to the constant distance of L4 and L5 from Jupiter to see how the asteroid orbit compared to a stable Lagrange orbit over time.

Asteroids were injected into the system using an initial conditions generator. The number of asteroids to be injected, and the region of and in which the asteroid should be placed were taken as arguments, and the asteroids were randomly placed within the given parameters. Velocities were based off the position of each asteroid, being selected to initially give the asteroid the angular velocity to match Jupiter at a given radius.

Where the magnitude gives the asteroid the same angular speed of Jupiter, , andthe vector gives the direction of the asteroid velocity. This gives the asteroid an initially stable velocity to evolve with.

The Monte Carlo method was used to test regions in which horseshoe orbits could exist within the Sun-Jupiter system. The asteroid generator was used to place a number of asteroids, , with initial positions between and . Each asteroid was simulated for a period of time (, and had its evolution tested to determine if its trajectory was one of a Tadpole, Horseshoe, or unstable asteroid. The asteroids were analysed in the comoving frame – with Tadpoles remaining in the half of the system; Horseshoes reaching the , quadrant of the system; and unstable asteroids exceeding a range for , or reaching the , quadrant before a specified time. These critical parameters in and used to define the separation between Horseshoes and unstable orbits were calculated by comparing fringe cases and adjusting each value until each case was consistently displayed as the correct orbit type. This method results in a series of initial positions for classified asteroid orbits, which can be plotted to demonstrate the regions in which each type of orbit can exist.

A simulation was also used to test the long-term lifetime of Horseshoe orbits in general. A small deviation in initial conditions can lead to up to a tenfold reduction in lifetime, and thus a precise determination of initial conditions which led to the highest stability orbit was required. This was accomplished by running a parameter search looking to maximise the length of time an orbit remained within the previously defined horseshoe orbit zone. A sufficiently stable point was found by initially finding a somewhat stable position, and searching within a close proximity locus for points of greater stability, and repeating the process until a local maximum was found.

**Results**

**System Grav Pot**

**FIG X.** A plot of the Sun-Jupiter comoving gravitational potential. The top right plot shows the Sun-Jupiter system overlayed on a contour and heatmap plot of the comoving potential. The bottom plot shows the comoving potential along the slice of the system. The left plot shows the comoving potential along the slice of the system. The two local maxima seen in the slice correspond to the positions of L4 and L5. In the main plot, these are the points where the tadpole shaped contours would contract to a single point. The three local maxima seen in the slice correspond to the positions of L1, L2, and L3. In the main plot, these are the points where the tails of tadpole shaped contours would touch opposite Jupiter, and where the horns of the horseshoe shaped contours would touch in front and behind Jupiter.

Diagram

Description automatically generated

An investigation into the Sun-Jupiter system confirmed that, as expected, the mass ratio of Jupiter to the Sun, , is within the mass ratio for the existence for Lagrange points L4 and L5, . As a consequence, the zero-velocity contours indicate two maxima flanking Jupiter at the positions of L4 and L5. If space, contour plots of the L4 and L5. This agrees with the values suggested by the literature and mathematics and lends validity to the modelling methods of this investigation.

**Reproducing Orbits**

Horseshoe and Tadpole asteroids were generated and simulated successfully using the restricted 3 body approximation.

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**FIG X.** Comparison of Horseshoe (red, top left) and Tadpole (blue, top right) orbits in the comoving frame, and their distance from Jupiter over time. The Tadpole encapsulates only L5 in a narrow comoving orbit, while the Horseshoe encapsulates both L5 and L4 (symmetrical to L5 about the Sun-Jupiter axis) in a much wider comoving orbit. The distance plot (bottom) shows how far each orbit strays from L5. L5 is at a constant distance of . Tadpoles have a lower amplitude (in this case, and a regular period oscillation about L5. Horseshoes are much more erratic, with a higher amplitude ( and a varying oscillation period. The evolution of these systems (including in the stationary frame) can be seen in these animations: <https://imgur.com/a/C83i8QA> (Horseshoe); <https://imgur.com/a/53z5HOe> (Tadpole).

Tadpoles remain close to Jupiter over long periods of time – orbiting around one of L4 or L5, as can be seen from the Tadpole comoving frame in **FIG. X**. In the stationary frame, this equates to the asteroid closely flanking Jupiter, as seen from the existing Trojans and slowly oscillating back and forth, remaining extremely close to a constant orbit radius of . Horseshoes, on the other hand, move from one side of Jupiter to the other – orbiting both L4 and L5, as seen in **FIG. X**. In the stationary frame, the asteroid is in a pursuit orbit with Jupiter. Initially, the asteroid starts in a higher orbit with respect to the Sun than Jupiter, and thus moves slower relative to the planet. Over time, Jupiter will catch up to the asteroid, and the asteroid will begin to enter Jupiter’s zone of influence and be pulled towards the planet. This acceleration acts to pull the asteroid down into a lower orbit with respect to the Sun, at which point it is moving quicker relative to Jupiter. Over time, the asteroid catches up with Jupiter, and is again pulled towards the planet. This time, the acceleration acts to pull the asteroid up into a higher orbit with respect to the Sun (similar to the original orbit), and the cycle repeats – leading to the orbit making a horseshoe shape in the comoving frame.

The Tadpole follows a tight, closed, path around the L5 point. The low amplitude, predictable oscillations suggest that this orbit is incredibly stable. Long-term simulations of order years of an asteroid in a tadpole orbit showed the asteroid to remain stable, even with a relatively large timestep which is prone to introducing aggregating inaccuracies. This finding is supported by current observations of Jupiter’s Trojans – which all follow a variety of the Tadpole orbit – as well as numerical integrations in literature which find them stable up to years.

The Horseshoe follows a much looser, fluctuating path around both L4 and L5. As seen from **FIG X.**, the orbit does not appear to be closed – the final position of the orbit deviates slightly from the initial conditions. This suggests that the orbit is not stable, and the asteroid’s closest approach to Jupiter changes over time. One explanation for this is that with every approach to Jupiter, the acceleration from the planet results in a larger change in velocity than the previous approach – i.e the asteroid either accelerates to a slightly higher velocity than before when caught by Jupiter, or decelerates to a slightly lower velocity than before when catching Jupiter. The result of this would be the “horns” of the orbit in the comoving frame creeping closer to Jupiter after each cycle, or the distance to Jupiter decreasing over time, until the asteroid eventually enters Jupiter’s sphere of gravitational influence and is ejected from the Horseshoe orbit. This suggests that the stability of the Horseshoe orbit in the Sun-Jupiter system is far lower than the Tadpole orbit, which is supported by the lack of Horseshoe Trojans today – suggesting that any that have existed in the past have been destabilised over time.

The approximations used in this investigation lend a combination of additional unphysical stability and instability to the simulated asteroids – namely a lack of external gravitational perturbations (such as from other planets), the simulation being restricted to two dimensions (equivalent to all asteroids having a perfect inclination of ), a lack of inter-asteroid dynamics (on one hand preventing increased stability through mutual orbits, but on the other preventing collisions and velocity fluctuations), and a lack of solar effects/perturbations (such as the Yarkovsky effect, or the effects of solar radiation over a long time period). Over a long timescale this may cause some significant deviation between the model and the physical system; however, we believe that the reproduction of a stable tadpole orbit, and a quasi-stable orbit shows that the model is sufficiently accurate for short to mid-length timescale simulations of the system.

**Horseshoe Zones**

The Monte Carlo method was used to search for the zones within the comoving frame horseshoe orbits could exist. As there are no exact mathematical definitions for any of the three types of orbits (horseshoe, tadpole, unstable), boundary conditions had to be approximated and then fine-tuned by manually classifying fringe cases, and altering parameters until they are correctly classified.

**FIG X.** A series of asteroids distributed near an unstable-horseshoe boundary in a section of the Sun-Jupiter system (top, blue indicating the polar radius, grey indicating the polar angle, where is taken as zero). The boundary exists between the green horseshoe position (ii, bottom right) and the nearest red unstable position (i, bottom left). The comoving evolution of the two fringe asteroids show the green position (ii) remaining in a clear horseshoe orbit over the simulation time, yrs, and the red position (i) spiralling out of the system after a short time in a horseshoe orbit. Based on these two cases, the boundary conditions for a horseshoe were defined as the asteroid remaining, in the comoving frame, within and reaching the quadrant after . A tadpole was defined as remaining within the half of the system. An unstable asteroid was defined as exceeding any of the horseshoe conditions.

Diagram

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A horseshoe orbit was defined as an orbit which creates at least one and a half, full, clearly stable horseshoes in the comoving frame – equivalent to surviving 3 closest approaches to Jupiter (excluding the initial closest approach – equivalent to visiting the opposite side of Jupiter twice, and not being ejected from the orbit). This definition was decided upon, as several fringe asteroids – including asteroid i – completed one full horseshoe orbit, however, did not return to a stable configuration, and were subsequently ejected at the next closest approach. We were not confident that anything less than this number of closest approaches could be called a horseshoe. It could be argued that simply finishing a single full horseshoe orbit classifies the asteroid as a horseshoe – in which case, the regions in which horseshoe orbits can exist would be wider than presented.

A picture containing diagram

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**FIG X.** The Monte Carlo simulation to show which initial positions resulted in which type of asteroid over a 340yr simulation. As expected, the general distribution of tadpoles (red) and horseshoes (green) roughly follow their orbit contours – closed around L5 and encapsulating both L4/L5, respectively. The simulation used asteroids with initial conditions of and , and simulated them for yrs. Of the 3231 asteroids, 2282 (71%) were unstable and escaped, 504 (16%) were Tadpoles, and 445 (14%) were Horseshoes. The half of the plot is identical, as the system is symmetrical. Generally, asteroids which stray away from the Jupiter orbital radius are unstable (black). A small region close to Jupiter ( also appears to be inherently unstable. Tadpoles (red) are mostly concentrated around the L5 stability point, , and also spread down towards along the radius . Horseshoes (green) tend to exist where the influence of L5 tapers off – in a small pocket closer to Jupiter, and generally populating the region of . They also exist in very narrow zones on the “outskirts” of the L5 influence zone, which is populated by Tadpoles.

The distribution of asteroid type across the system – as seen in **FIG X.** – appears to generally follow the equipotential contours in **FIG. X** as previously hypothesised. Tadpoles are generally confined to areas which are contained by a closed contour centred on L5 – such contours describe a zone of influence of L5, within which all orbits are bound about the point, and thus are tadpoles in the comoving frame. Horseshoes exist over a much wider region of the system – with two clear blocks either side of the L5 influence zone. Again, this loosely follows the shape of a potential contour encapsulating both L4 and L5 – approaching close to Jupiter, and connecting at the opposite side of the Sun, reflected by the abundance of Horseshoe orbits at and . Unstable asteroids tend to be on a near-horseshoe type orbit – following horseshoe contours which approach too close to Jupiter, leading to them to being destabilised and either completely ejected from the system, or falling into a closer orbit with the Sun.

There are some notable discrepancies between the contour lines and the asteroid distribution, however. Tadpoles look to extend far past the zone of influence of L5 along , with a few even approaching the region. The width of the Horseshoe region is also incredibly thin, even compared to the Tadpole region. The L5 zone of influence is approximately as wide as would be suggested by **FIG X.**, however, the Horseshoe region past this is significantly smaller than would be expected before reaching the unstable asteroid region – leaving just a thin strip for Horseshoes to occupy. This mismatch between the potential contours and the asteroid positions suggests that the boundary conditions of the Monte Carlo method were too strict. The width agreement with Tadpoles – unaffected by the boundary conditions – and the disagreement in Horseshoe width – directly affected by the boundary conditions – shows that the requirement for three stable closest approaches to be too conservative. Reducing this to one or two closest approaches would increase the width without necessarily compromising the accuracy of the simulation – such a change would allow increasingly unstable horseshoe orbits to be classed as horseshoes, however it is difficult to judge where the line should be drawn. Tadpoles which extend past the L5 zone of influence, along to also suggest another error in the simulation. We believe that these “Tadpoles” are actually initially Horseshoes on the boundary of the L5 zone which get trapped near L3 (, ). It is possible for these asteroids to be slow moving compared to other, more apparent, Horseshoes due to their proximity to the L5 zone. Extreme cases may not have crossed to the half of the system at all, while some may have approached along the line but were ensnared by L3 and never crossed over to the other side of the system. While the latter is incredibly unlikely due to the unstable nature of L3 further amplified by the simulation timestep, the former suggests that the length of the simulation was insufficient.

Adjusting the definition of a Horseshoe does not change the fact that they have an inherit instability in the Sun-Jupiter system. Other systems are known to have stable horseshoe orbits – most notably the Saturn-Janus-Epimetheus system. There are two factors which differentiate this system from the Sun-Jupiter system. Firstly, Epimetheus is approximately a tenth of the mass of Janus, which affects orbits significantly compared to the asteroid having no mass compared to Jupiter. Secondly, the mass ratio of the two main bodies in the system, Janus and Saturn – , is significantly lower than that of Jupiter and the Sun – . Previous investigations into Horseshoe orbits found that the critical mass ratio for their stability is approximately . The Sun-Jupiter system lies a small margin beyond this stability requirement, whereas Saturn-Janus lies well within it. This proximity to the critical ratio could explain why Sun-Jupiter Horseshoe orbits can exist in a quasi-stable state, but never settle into a fully stable Horseshoe orbit over a long time period. Such a dependency on the mass ratio is logical, as Horseshoe orbits are a consequence of the stable Lagrange points L4 and L5, which require a mass ratio of . For , the L5 zone of influence takes up a large portion of the stability region along the orbiter radius. As such, the potential contours which correspond to Horseshoe orbits are too narrow for an asteroid to oscillate (the “wobble” in the comoving frame in **FIG. X**), and thus are not stable. For , the stability zones sufficiently increase in width for Horseshoe orbits to be fully contained in their related contours, and thus will be stable over long periods of time.

**Lifetime**

Being inherently unstable due to the system mass ratio, there exists a maximum possibly lifetime for Horseshoe orbits in the Sun-Jupiter system. An estimate for the upper limit of the lifetime of a Horseshoe orbit, , within a system of orbital period and mass ratio has been found to be

For the Sun-Jupiter system, this results in an approximate value of .

Chart

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**FIG X.** The velocity and distance evolution of a Horseshoe with initial conditions , . This position approximately coincides with the top of the L5 influence zone. The simulation had a timestep of 5000s, and ran for 50000yrs. This is As expected, the two plots are closely correlated – a close approach to Jupiter results in the asteroid having a higher velocity. The comoving oscillatory period varies throughout the period of simulation, shown by a reduction in frequency at yrs and an increase at . The increase in frequency appears to correlate to an increase in speed/decrease in closest approach, and vice versa. In general, the asteroid remains at distance from Jupiter and only briefly crosses this threshold during the high frequency, high velocity phase at . The asteroid then proceeds to return to the modal closet approach of , with a similar decrease in velocity.

The approximations used in this investigation lend a combination of additional unphysical stability and instability to the simulated asteroids – namely a lack of external gravitational perturbations (such as from other planets[3]), the simulation being restricted to two dimensions (asteroids having an inclination of zero results in higher stability[3]) Jupiter having a zero eccentricity orbit (increasing the stability of its co-orbitals[4]), a lack of inter-asteroid dynamics (on one hand preventing increased stability through mutual orbits, but on the other preventing collisions and velocity fluctuations), and a lack of solar effects/perturbations (such as the Yarkovsky effect[5], or the effects of solar radiation over a long time period).

Over a long timescale this may cause some significant deviation between the model and the physical system; however, we believe that the reproduction of a stable tadpole orbit, and a quasi-stable orbit shows that the model is sufficiently accurate for short to mid-length timescale simulations of the system.