Introduction

Course structure:

(i) Preliminary toolbox: inequalities

(ii) Normed vector spaces (NVS)

(iii) (Recalls on) finite-dimensional case

(iv) Hahn-Banach Theorems (how big is the dual?)

(v) Completeness: Baire's Theorem & consequences for NVS

(vi) Detailed study of the topology of $C(K)$

(vii) The Hilbert space

(viii) Projection & duality

(ix) Introduction to operators and spectral theory

1 Preliminary toolbox: Young's, Hölder's & Minkowski's inequalities for vectors & sequences

Proposition (Young's inequality for products)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\forall a, b \geq 0,\ ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The result is clear for $a = 0$ or $b = 0$. Assume $a, b > 0$ and note $L : (0, \infty) \to \mathbb{R}$, $t \mapsto \ln t$ is strictly concave: $L''(t) = -\frac{1}{t^2} < 0$.

Therefore for all $A, B > 0$, $\lambda \in (0, 1)$

$$\ln(\lambda A + (1 - \lambda)B) \geq \lambda \ln A + (1 - \lambda) \ln B$$

with equality iff $A = B$. Apply this to $A = a^p$, $B = b^q > 0$ and $\lambda = \frac{1}{p}$. This gives

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) = \ln(ab)$$

so applying exp to both sides gives the result and furthermore we have equality iff $a^p = b^q$. □

Proposition (Hölder's inequality for vectors & sequences)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

(i) for any $n \in \mathbb{N} \setminus \{0\} = \mathbb{N}^*$, $\forall x, y \in \mathbb{R}^n$

$$\sum_{k=1}^{n} |x_k y_k| \leq ||x||_p ||y||_q \tag{$*$}$$

with $||x||_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ and similarly for $||y||_q$.

(ii) define

$$\ell^p = \{x \in \mathbb{R}^{\mathbb{N}^*} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

then $\forall x \in \ell^p$, $y \in \ell^q$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq ||x||_{\ell^p} ||y||_{\ell^q}$$

where $||x||_{\ell^p} = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ and similar for $||y||_{\ell^q}$.

Proof. To show (i) implies (ii): take $n \to \infty$ in (i) so

$$\sum_{k=1}^{n} |x_k|^p \to ||x||_{\ell^p}^p$$

and similarly

$$\sum_{k=1}^{n} |y_k|^q \to ||y||_{\ell^q}^q$$

By (i)

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}$$

so

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k y_k| = \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k y_k|\right) &\leq \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q} \\ &= ||x||_{\ell^p} ||y||_{\ell^q} \end{aligned}$$

Proof of (i): if $||x||_{\ell^p}$ or $||y||_{\ell^q} = 0$, result is clear. Otherwise define $\tilde{x}$, $\tilde{y}$ sequences in ℓ^p and ℓ^q by

$$\tilde{x}_k = \frac{x_k}{||x||_{\ell^p}}, \ \tilde{y}_k = \frac{y_k}{||y||_{\ell^q}}$$

Then $||\tilde{x}||_{\ell^p} = 1$, $||\tilde{y}||_{\ell^q} = 1$. Then $(*)$ is equivalent to showing

$$\sum_{k=1}^{n} |\tilde{x}_k \tilde{y}_k| \leq 1 \tag{$**$}$$

Apply Young's inequality on each $k = 1, \dots, n$ so

$$|\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}|\tilde{x}_k|^p + \frac{1}{q}|\tilde{y}_k|^q$$

Summing over k:

$$\sum_{k=1}^{n} |\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}\left(\sum_{k=1}^{n} |\tilde{x}_k|^p\right) + \frac{1}{q}\left(\sum_{k=1}^{n} |\tilde{y}_k|^q\right) \leq \frac{1}{p} + \frac{1}{q} = 1$$

$\square$

Remark: Equality in $(*)$ is equivalent to equality in $(**)$ which is equivalent to equality in Young's for each k so $|\tilde{x}_k|^p = |\tilde{y}_k|^q$ for $k = 1, \dots, n$. Also, the $p = 1$, $q = \infty$ case is easy.

Proposition (Minkowski's inquality for vectors & sequences)**.** Let $p \in [1, \infty)$, then

(i) for all $x, y \in \mathbb{R}^n$

$$||x + y||_p \leq ||x||_p + ||y||_p$$

(ii) for all $x, y \in \ell^p$

$$||x + y||_{\ell^p} = ||x||_{\ell^p} + ||y||_{\ell^p}$$

Proof. To show (i) implies (ii): by taking $n \to \infty$ as before

$$\sum_{k=1}^{\infty} |x_k|^p \to ||x||_{\ell^p}^p$$

$$\sum_{k=1}^{\infty} |y_k|^p \to ||y||_{\ell^p}^p$$

$$\sum_{k=1}^{n} |x_k + y_k|^p \to ||x + y||_{\ell^p}^p$$

Proof of (i): if $p = 1$ this is just the usual triangle inequality on each coordinate. So let $p \in (1, \infty)$ and

$$\sum_{k=1}^{n} |x_k + y_k|^p = \sum_{k=1}^{n} |x_k + y_k| \cdot |x_k + y_k|^{p-1}$$

$$\leq \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}$$

$$\underbrace{\leq}_{\text{Hölder: } q=\frac{p}{p-1}} ||x||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + ||y||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\leq (||x||_p + ||y||_p)\, ||x+y||_p^{p-1}$$

so we have proved

$$||x+y||_p^p \leq (||x||_p + ||y||_p)\, ||x+y||_p^{p-1}$$

If $||x+y||_p = 0$, result is clear. Otherwise divide by $||x+y||_p^{p-1}$ to get

$$||x+y||_p \leq ||x||_p + ||y||_p$$

□

Remark: equality occurs iff there is equality in the triangle inequality and Hölder's.
Remarks:

1. Equality case: $p = 1$: $|x_k + y_k| \leq |x_k| + |y_k|$, i.e the usual triangle inquality
2. For $p = 2$ there's another proof: define $\mathcal{P} : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto ||x + \lambda y||^2$. Then $\mathcal{P}(\lambda) = a\lambda^2 + 2b\lambda + c$ and $\mathcal{P} \geq 0$. So

$$\langle x, y\rangle = b^2 \leq ac = ||x||^2||y||^2, \text{ Hölder's inequality}$$

2 Normed Vector Spaces (NVS)

Remark: this is not the most general structure for linear analysis - topological vector spaces (TVS).

Recall:

Definition. A *vector space* V over a field $\mathbb{F}$ is a set (of elements called *vectors*) with two operations:

$$A : V \times V \to V,\ (v, w) \mapsto v + w \text{ addition}$$

$$M : \mathbb{F} \times V \to V,\ (\lambda, v) \mapsto \lambda v \text{ scalar multiplication}$$

such that

- $(V, +)$ is an abelian group with identity 0.
- M is compatible with $(\mathbb{F}, 0)$ in the sense that $\lambda_1(\lambda_2 v) = \lambda_1\lambda_2 v$
- M distributes over $(V, +)$ and $(\mathbb{F}, +)$.

In this course $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$ unless stated otherwise.

Definition. Given a vector space V over $\mathbb{F}$:

- a *subspace* $W \subseteq V$ is a vector space over $\mathbb{F}$ included in V
- for a set $S \subseteq V$, a *linear combination of elements of* S is a finite sum of elements of S with coefficients in $\mathbb{F}$
- for a set $S \subseteq V$, the *span of* S, $\text{span}(S)$ is the smallest subspace of V containing S, and is also the set of linear combinations of S.

Definition. Given V a vector space over $\mathbb{F}$ and a set $S \subseteq V$:

- S is *linearly independent* if for all $m \in \mathbb{N}^*$ and for all $\alpha_1, \dots, \alpha_m \in \mathbb{F}$, for all $s_1, \dots, s_m \in S$, $\sum_{i=1}^m \alpha_i s_i = 0$ if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_m$.
- S is a *basis* of V if it is linearly independent and $\text{span}(S) = V$.
- If there exists a finite basis S of V, then V has finite dimension, otherwise it is infinite-dimensional.

Remark: later we'll prove with Zorn's lemma that any vector space has a basis.

Definition. A *normed vector space* (NVS) V over $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a function $N : V \to \mathbb{R}_+$, $v \mapsto ||v||$ (the *norm*), with

1. $||v|| \geq 0$ for all $v \in V$, with equality only at $v = 0$ (*positive definiteness*)
2. For all $\lambda \in \mathbb{F}$, $v \in V$ $||\lambda v|| = |\lambda|||v||$ (compatibility between N and M)

3. For all $v, w \in V$, $||v+w|| \leq ||v|| + ||w||$ (compatibility between N and A)

Example. $V = \mathbb{R}^n$, $v = (v_1, \ldots, v_n)$, $||v|| = (v_1^2 + \ldots + v_n^2)^{1/2}$ or

$$\begin{cases} ||v||_p = (|v_1|^p + \ldots + |v_n|^p)^{1/p} & \text{for } p \in [1, \infty) \\ ||v||_\infty = \sup_{i=1}^n |v_i| & \text{for } p = \infty \end{cases}$$

Definition. Given a set X, a *topology* τ on X is a collection of subsets of X ("open sets") such that

- $\emptyset \in \tau$, $X \in \tau$
- τ is stable under <u>any</u> union
- τ is stable under finite intersections

Definition.

- For (X, d) a metric space, the *induced topology* is the smallest topology that contains open balls in d
- For a NVS $(V, ||\cdot||)$, the induced topology is that associated with $d(v, w) = ||v - w||$

Natural question: $\mathbb{F}$ field, V vector space over $\mathbb{F}$. Norm on V, $\tau_{||\cdot||}$. Continuity of operations M and A?

Proposition. Let $(V, ||\cdot||)$ be a NVS over $\mathbb{F}$ ($\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$), then

(i) A, M are continuous for the following topologies: $\tau_{||\cdot||}$ on V, then product topology of it on $V \times V$, $\tau_{|\cdot|}$ over $\mathbb{F}$, then product topology of $\tau_{|\cdot|}$ and $\tau||\cdot||$ on $\mathbb{F} \times V$

(ii) Translations $T_{v_0} : V \to V$, $v \mapsto v + v_0$, $v_0 \in V$ and dilations $D_{\lambda_0} : V \to V$, $v \mapsto \lambda_0 v$, $\lambda_0 \in \mathbb{F}^*$ are homeomorphisms

Proof.

(i) Let us prove that $A : V \times V \to V$ is continuous: consider an open set $\emptyset \neq U \subseteq V$ and $(v_1, v_2) \in A^{-1}(U)$, i.e $v_1 + v_2 \in U$. Since U is open, there is $\varepsilon > 0$ such that $\underbrace{B_V(v_1 + v_2, \varepsilon)}_{\text{open ball}} \subseteq U$.

We have that $A(B(v_1, \varepsilon/2), B_V(v_2, \varepsilon/2)) \subseteq B_V(v_1 + v_2, \varepsilon)$ (triangle inequality). Note also that $B(v_1, \varepsilon/2) \times B(v_2, \varepsilon/2)$ is open (product topology), so $A^{-1}(U)$ is open and A is continuous.

Now we show $M : \mathbb{F} \times V \to V$ is continuous. Consider an open set $U \neq \emptyset$ in V, $(\lambda, v) \in M^{-1}(U)$. Since U is open, there exists $\varepsilon > 0$ such that $B_V(\lambda v, \varepsilon) \subseteq U$ (WLOG $\varepsilon < 1$). Then (check)

$$M\left(B_{\mathbb{F}}\left(\lambda, \frac{\varepsilon}{3\max(1, ||v||)}\right), B_V\left(v, \frac{\varepsilon}{3\max(1, |\lambda|)}\right)\right) \subseteq B_V(\lambda v, \varepsilon)$$

(ii) T_{v_0} and D_{λ_0} are linear, continuous with inverses T_{-v_0} and $D_{\lambda_0^{-1}}$ respectively, so are homeomorphisms.

$\square$

3 Characterisation of NVS

Idea: in order to better understand the topology of NVS's, we ask how special is a "normable" topology among topologies compatible with vector space operations?

Definition (TVS)**.** A *topological vector space* (TVS) over $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a topology τ such that

(i) A and M are continuous

(ii) every singleton $\{x_0\}$ is closed

Remark:

1. (i) says that T_{v_0} and D_{λ_0}, $\lambda_0 \neq 0$ are homeomorphisms
2. (ii) is called T_1 in the classification of seperation properties, and implies Hausforff for TVS

Definition. Given V a TVS

- $C \subseteq V$ is *convex* if $C = \{\lambda c_1 + (1-\lambda)c_2 : c_1, c_2 \in C, \lambda \in [0,1]\}$
- V is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0
- $B \subseteq V$ is *bounded* if for any U open around 0, there exists $t_0 > 0$ such that $\forall t > t_0$, $B \subseteq tU$
- V is *locally bounded* if there is $U \in \tau$ containing 0 and bounded

Example. Let $(V, ||\cdot||)$ be a NVS, then for all $r > 0$, $U = B(0,r)$ (open ball) is open, bounded and convex. Indeed

- Convexity follows from the triangle inequality
- Boundedness: any other $\tilde{U}$ open around 0 contains some open $\tilde{U}_0 = B(0, r_0) \in \tilde{U}$. Then for any $t > \frac{r}{r_0}$, $U \subseteq t\tilde{U}_0 \subseteq t\tilde{U}$.

Question: can we reverse-engineer the norm if we have these two properties?

Theorem (Kolmogorov 1934)**.** *Let (V, τ) be a TVS such that there is a bounded convex neighborhood of 0, say C. Then V is "normable" - there is a norm $||\cdot||$ on V that induces the topology τ.*

Proof. Step 1: there is $\tilde{C} \subseteq C$ which is a *balanced* convex bounded neighborhood of 0. "Balanced" means that for all $\lambda \in \mathbb{F}$ such that $|\lambda| \leq 1$, $\lambda\tilde{C} \subseteq \tilde{C}$.

$M : \mathbb{F} \times V \to V$ is continuous so $M^{-1}(C)$ is a neighourhood of $(0,0)$. So there exists $B_{\mathbb{F}}(0, \varepsilon) \times U$ with $\varepsilon > 0$ and U open around 0 such that $M(B_{\mathbb{F}}(0,\varepsilon), U) \subseteq C$.

Define $\tilde{C}$ to be the convex hull (i.e smallest convex set supserset) of $M(B_{\mathbb{F}}(0,\varepsilon), U)$.

Then $\tilde{C}$ is clearly convex, is a subset of C since C is convex and $M(B_{\mathbb{F}}(0,\varepsilon), U) \subseteq C$. $\tilde{C}$ is also bounded since $\tilde{C} \subseteq C$ and C is bounded (obvious that boundedness is inherited by inclusion). Finally $\tilde{C}$ is balanced since $\lambda B_{\mathbb{F}}(0,\varepsilon) \subseteq B_{\mathbb{F}}(0,\varepsilon)$ for $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$ and

$$\underbrace{\lambda M(B_{\mathbb{F}}(0,\varepsilon), U)}_{=M(\lambda B_{\mathbb{F}}(0,\varepsilon), U)} \subseteq M(B_{\mathbb{F}}(0,\varepsilon), U)$$

Notice $\lambda[\text{Convex Hull}(S)] = \text{Convex Hull}(\lambda S)$ (exercise). So deduce $\lambda\tilde{C} \subseteq \tilde{C}$.

Step 2: define the *Minkowski guage* (functional) of $\tilde{C}$

$$\mu_{\tilde{C}} : V \to \mathbb{R}_+, \ v \mapsto \inf\{t \geq 0 : v \in t\tilde{C}\}$$

$\mu_{\tilde{C}}$ is well-defined in $[0,\infty)$ since: any v satisfies $\frac{v}{t} \to 0$ as $t \to \infty$ by continuity of M. So $\frac{v}{t}$ must "enter" the neighborhood $\tilde{C}$ of 0 for t large enough.

Step 3: let us prove $v \mapsto \mu_{\tilde{C}}(v)$ is a norm:

- $\mu_{\tilde{C}}(v) \geq 0$ by construction
- if $\mu_{\tilde{C}} = 0$, then (assume $v \neq 0$ for contradiction) there exists U open around 0 with $v \notin U$ (since $V \setminus \{v\}$ is open). Since $\tilde{C}$ is bounded, there exists $t_1 > 0$ such that $\tilde{C} \subseteq t_1 U$. Since $\mu_{\tilde{C}}(v) = 0$, there exists $t_2 \in (0, t_1^{-1})$ such that $v \in t_2\tilde{C}$, then $v \in t_2\tilde{C} \subseteq t_1^{-1}\tilde{C} \subseteq U$, a contradiction.
- Want to show $\mu_{\tilde{C}}(\lambda v) = |\lambda|\mu_{\tilde{C}}(v)$ for $\lambda \in \mathbb{F}^{\times}$, $v \in V$. Use $\tilde{C}$ balanced: for all $t > 0$ such that $\lambda v \in t\tilde{C}$, we have

  $$\frac{\lambda}{|\lambda|} v \in \frac{t}{|\lambda|}\tilde{C} \implies v \in \frac{t}{|\lambda|}\tilde{C} \implies \mu_{\tilde{C}}(v) \leq \frac{1}{|\lambda|}\mu_{\tilde{C}}(\lambda v)$$

  The inequality in the other direction follows by reasoning with λ^{-1}. So $|\lambda|\mu_{\tilde{C}}(v) = \mu_{\tilde{C}}(\lambda v)$.
- Want to show $\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$ for all $v_1, v_2 \in V$. Indeed, given $t_1, t_2 > 0$ such that $v_1 \in t_1\tilde{C}$, $v_2 \in t_2\tilde{C}$, we have

  $$v_1 + v_2 \in t_1\tilde{C} + t_2\tilde{C} = (t_1+t_2)\left[\frac{t_1}{t_1+t_2}\tilde{C} + \frac{t_2}{t_1+t_2}\tilde{C}\right] \subseteq (t_1+t_2)\tilde{C} \text{ (convexity)}$$

  so $\mu_{\tilde{C}}(v_1 + v_2) \leq t_1 + t_2$. By taking infima over t_1, t_2:

  $$\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$$

Step 4: prove $\mu_{\tilde{C}}$ induces the topology τ.

- Want to prove

$$\underbrace{B(v_0, \varepsilon)}_{\text{open ball for } \mu_{\tilde{C}}} = \{v \in V : \mu_{\tilde{C}}(v - v_0) < \varepsilon\} \in \tau$$

  Take $v \in B(v_0, \varepsilon)$ then by the triangle inequality

$$B(v, \varepsilon - |v|) \subseteq B(v_0, \varepsilon)$$

  and $B(v, \varepsilon') \supseteq v + \frac{\varepsilon'}{2}\tilde{C}$ by definition of the ball for $\mu_{\tilde{C}}$. And (since translations, dilations continuous) $v + \frac{\varepsilon'}{2}\tilde{C}$ is a neighborhood of v.

  $B(v_0, \varepsilon)$ open (in τ) around its points, so is in τ.

- Take $U \in \tau$, and (wlog) $0 \in U$. Let us prove $0 \in B(0, \varepsilon_0) \subseteq U$ for some $\varepsilon_0 > 0$. Indeed $\tilde{C}$ is bounded so there exists $\varepsilon_0 > 0$ such that $\tilde{C} \subseteq \varepsilon_0^{-1} U$ hence $U \supseteq \varepsilon_0 \tilde{C}$ and so $U \supseteq \varepsilon \tilde{C}$ $\forall \varepsilon < \varepsilon_0$ and thus $U \supseteq B(0, \varepsilon_0)$.

$\square$

Remarks:

1. $B(0, \varepsilon_0) \subseteq \bigcup_{0 \leq \varepsilon < \varepsilon_0} \varepsilon \tilde{C}$
2. T_1 implies Hausforff (T_2). Consider $v_0 \neq v_1$ in V: so $0 \neq v_1 - v_0$, T_1 implies there is U open around 0 with $v_1 - v_0 \notin U$. Then (since A, M continuous) $(v, w) \mapsto v - w$ is continuous and there exists $\tilde{U}$ open around 0 such that $\tilde{U} - \tilde{U} \subseteq U$. Then $v_0 + \tilde{U}$ and $v_1 + \tilde{U}$ are open disjoint neighborhoods of v_0 and v_1 respectively (disjoint since otherwise $v_1 - v_0 \in \tilde{U} - \tilde{U} \subseteq U$).

4 Some examples of NVS'

Definition. Let $(V, ||\cdot||)$ be an NVS (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). If (V, d), d distance induced by $||\cdot||$ is a complete metric space, then $(V, ||\cdot||)$ is called a *Banach space.*

Example. $\mathbb{R}^n$, $\mathbb{C}^n$, $n \geq 1$ are Banach spaces, for $||\cdot||_p$, $p \in [1, \infty)$.

Example. Given (X, τ) a general topological space, define

$$B_{\mathbb{F}}(X) = \{\text{functions } : X \to \mathbb{F} \text{ bounded}\}$$

$$C_{\mathbb{F}}(X) = \{\text{functions } : X \to \mathbb{F} \text{ continuous}\}$$

$$C_{\mathbb{F},b} = C_{\mathbb{F}}(X) \cap B_{\mathbb{F}}(X)$$

If $X = K$ is compact, $C_{\mathbb{F}}(X) = C_{\mathbb{F},b}(X)$. These are vector spaces over $\mathbb{F}$ with addition $(f+g)(x) = f(x) + g(x)$ and multiplication $(fg)(x) = f(x)g(x)$.

Norm on $C_{\mathbb{F},b}(X)$: the supremum norm, $||f||_\infty = \sup_{x \in X} |f(x)|$

Proposition. $(C_{\mathbb{F},b}, ||\cdot||_\infty)$ is a Banach space over $\mathbb{F}$.

Proof.

- $||f||_\infty$ is well defined in $\mathbb{R}^+$ since f is bounded.
- $||f||_\infty = 0$ means $f(x) = 0$ for all $x \in X$ and so $f = 0$.
- Homogeneity and triangle inequality: inherited from $|\cdot|$ in $\mathbb{F}$ (exercise).
- Completeness: let $(f_k)_{k \geq 1}$ be a Cauchy sequence under $||\cdot||_\infty$. For each $x \in X$ we have $|f_m(x) - f_n(x)| \leq ||f_m - f_n||_\infty \to 0$ as $n, m \to \infty$. So $(f_k(x))_{k \geq 1}$ is Cauchy in $\mathbb{F}$, so (since $\mathbb{F}$ is complete) there exists a limit $f(x) = \lim_{k \to \infty} f_k(x)$. This defines a function $f : X \to \mathbb{F}$.
- For all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\forall m, n \geq n_0$, $\forall x \in X$,

$$|f_m(x) - \underbrace{f_n(x)}_{\to f(x)}| \leq \varepsilon$$

so for all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\forall m \geq n_0$, $\forall x \in X$ we have

$$|f_m(x) - f(x)| \leq \varepsilon$$

so $||f_m - f||_\infty \leq \varepsilon$ and $f_m \to f$ uniformly, so $f \in C_{\mathbb{F},b}$ by properties of the uniform limit.

$\square$

Example. Given $U \subseteq \mathbb{R}^n$ open, bounded and non-empty; $m \in \mathbb{N}^*$, consider

$$\begin{aligned} C^m(\overline{U}) = \{f : U \to \mathbb{R} :& f \text{ is } m \text{ times differentiable on } U, \forall \alpha \in \mathbb{N}^n \\ & \text{s.t } |\alpha| = \alpha_1 + \ldots + \alpha_m \leq m \\ & , \partial^\alpha f \text{ is continuous and bounded on } U\} \end{aligned}$$

Then $(C^m(\overline{U}), || \cdot ||_{C^m})$ is a Banach space where

$$||f||_{C^m} = \sup_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} \underbrace{\sup_{x \in U} |\partial^\alpha f(x)|}_{||\partial^\alpha f||_\infty}$$

Exercise: check that this is complete and $\partial^\alpha f$, $\alpha \leq m-1$, extends continuously to $\tilde{U}$.

Example. $C_\mathbb{R}([0,1])$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$. This is a vector space over $\mathbb{R}$.

- $(C_\mathbb{R}([0,1]), || \cdot ||_\infty)$ is a Banach space (Example sheet)
- Could take another norm such that

$$||f||_p = \left(\int_0^1 |f(x)|^p \mathrm{d}x \right)^{1/p}, \ p \in [1, \infty)$$

Study of $(C_\mathbb{R}([0,1]), || \cdot ||_p)$:

- $|| \cdot ||_p$ is well defined: Riemann and Lebesgue integrable.
- If $||f||_p = 0$ and $f \neq 0$ then there exists $\varepsilon > 0$ and $x_0 \in [0,1]$ such that $|f(x_0)| \geq \varepsilon$, so by continuity there exist $a < b \in [0,1]$ such that $\inf_{x \in [a,b]} |f(x)| \geq \frac{\varepsilon}{2}$. Then $\int_0^1 |f(x)|^p \mathrm{d}x \geq \left(\frac{\varepsilon}{2}\right)^p (b-a) > 0$ which is impossible.
- Homogeneity is clear.
- Triangle inequality:

$$||f+g||_p^p = \int_0^1 |f+g|^p \mathrm{d}x = \int_0^1 |f+g||f+g|^{p-1} \mathrm{d}x$$

$$\leq \int_0^1 |f||f+g|^{p-1}\mathrm{d}x + \int_0^1 |g||f+g|^{p-1}\mathrm{d}x$$
$$\underbrace{\leq}_{\text{Hölder:}} ||f||_p||f+g||_p^{p-1} + ||g||_p||f+g||_p^{p-1}$$

If $||f+g||_p = 0$ then its clear. Otherwise this implies $||f+g||_p \leq ||f||_p + ||g||_p$.

- Completeness? Define

$$f_k(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{4k} \\ \left[x - \left(\frac{1}{2} - \frac{1}{4k}\right)\right] 4k & \frac{1}{2} - \frac{1}{4k} \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

then $(f_k)_{k\geq 1}$ is Cauchy for $||\cdot||_p$, and the limit is $1_{[1/2,1]}$ which is not continuous. So not complete.

Remark: what about the completion? In general, abstract completions are often not very useful; however in this case, it is: Lebesgue space $L^p([0,1])$, defined as equivalence classes for the "almost everywhere" equality.

Example. Take functions from $X = \mathbb{N} \to \mathbb{R}$ or $\mathbb{C}$, get $\ell^p_{\mathbb{F}}$ for $p \in [1, \infty]$, with norm $||(x_k)||_p = \left(\sum_{k\geq 1} |x_k|^p\right)^{1/p}$ for $p < \infty$ and $||(x_k)||_\infty = \sup_{k\geq 1} |x_k|$. Exercise: show this is indeed a norm and this is complete, hence Banach.

Remark: for $p \in (0, 1)$, ℓ^p is similarly defined.

Non-examinable example of TVS:

- Define for $U \subseteq \mathbb{R}^n$ open & non-empty, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $C_{\mathbb{F}}(U)$ the set of continuous functions $U \to \mathbb{F}$.
- TVS for the topology τ defined by the translations of the following basis of neighborhoods around 0: take $(K_n)_{n\geq 1}$ a sequence of increasing compact sets, $\bigcup_{n\geq 1} K_n = U$. Define

$$U_n = \left\{ f \in C_{\mathbb{F}}(U) : \sup_{K_n} |f| \leq \frac{1}{n} \right\}$$

- Exercise: show this indeed a TVS and τ does not depend on the choice of the (K_n).
- Proposition: $(C(U), \tau)$ is a locally convex, not locally bounded TVS (therefore not normable). Furthermore, it is metrizable with $d(f, g) = \sum_{k\geq 1} \frac{1}{2^n} \left(\frac{\sup_{K_n} |f-g|}{1+\sup_{K_n} |f-g|} \right)$. Also $(C(U), d)$ is complete (Frechet space).

Remarks:

1. Not locally bounded: suppose there exists B bounded neighborhood of 0, then there exists $n_0 \geq 1$ such that $U_{n_0} \subseteq B$. B is bounded so there exists $t > 0$ such that $B \subseteq tU_{n_0+1}$ so $U_{n_0} \subseteq tU_{n_0+1}$. But this is impossible since we can always construct $f \in U_{n_0}$ such that $\sup_{K_{n_0+1}} |tf| > 1/n$
2. Let $C_c(U)$ be the set of continuous functions with compact support. Then V is a neighborhood of 0 if and only if $V \cap C(K_n)$ is a neighborhood of 0 in $C(K_n)$. This is a non-countable topology.

5 Bounded linear maps & duality

Definition. Given (V, τ_V) and (W, τ_W) TVS', $T : V \to W$ linear is *bounded* if it maps bounded sets to bounded sets: for any $B_V \subseteq V$ bounded, then $T(B_V)$ is bounded in W.

Proposition. Given (V, τ_V), (W, τ_W) TVS' which are locally bounded (note this includes NVS'), and $T : V \to W$ is linear, then T is bounded if and only if T is continuous.

Proof.

Step 1: T bounded $\implies$ T continuous at 0. Let U_W be an open neighborhood of 0 in W, and U_V an open bounded neighborhood of 0 in V. Then $T(U_V)$ is bounded, so there exists $t > 0$ such that $T(U_V) \subseteq tU_W$. So $T^{-1}(U_W) \supseteq t^{-1}U_V$ and $t^{-1}U_V$ is open around 0 in V (using the fact dilations are continuous).

Step 2: T continuous at 0 $\implies$ T is continuous everywhere. Let $w \in W$, U_W open around w, $v \in V$ such that $T(v) = w$. Then $U_W - w$ is open around 0 in W (translation continuous), so by Step 1, $T^{-1}(U_W - w)$ is a neighborhood of 0 in V. So

$$\begin{aligned} T^{-1}(U_W) &= T^{-1}(\{w\}) + T^{-1}(U_W - w) \\ &= \bigcup_{v' \in T^{-1}(\{w\})} \big(v' + T^{-1}(U_W - w)\big) \\ &\supseteq \underbrace{v + T^{-1}(U_W - w)}_{\text{ngbd around } v} \end{aligned}$$

Step 3: T continuous $\implies$ T bounded. Let $B_V \subseteq V$ be bounded, and U_W an open neighborhood of 0 in W. Then $T^{-1}(U_W)$ is open around 0 in V. So (since B_V bounded) there exists $t > 0$ such that $B_V \subseteq tT^{-1}(U_W)$ and so $T(B_V) \subseteq tU_W$.

We have proved that $T(B_V)$ is covered by a dilation of any neighborhood of 0, so is bounded. $\square$

Definition. Given $(V, ||\cdot||_V)$, $(W, ||\cdot||_W)$ NVS' on $\mathbb{F}$, and $T : V \to W$ linear, T is bounded iff T is continuous iff there exists $t > 0$ such that $T(B_V(0,1)) \subseteq B_W(0,t)$. The infimum of such t's is denoted $|||T|||$.

Remark: can check that $|||T|||$ is equivalently defined as

$$|||T||| = \sup_{||v||_V \leq 1} ||Tv||_W = \sup_{||v||_V < 1} ||Tv||_W = \sup_{||v||_V = 1} ||Tv||_W \qquad (*)$$

Definition. Given $(V, ||\cdot||_V)$,$(W, ||\cdot||_W)$ NVS', denote

$$\mathcal{L}(V,W) = \{T : V \to W \text{ linear map}\}$$

$$\mathcal{B}(V,W) = \{T : V \to W \text{ linear bounded map}\}$$

Proposition. $(\mathcal{B}(V,W), |||\cdot|||)$ is an NVS.

Proof.

- $\mathcal{L}(V,W)$ is a vector space via $(\lambda_1 T_1 + \lambda_2 T_2)(v) = \lambda_1 T_1(v) + \lambda_2 T_2(v)$.
- $\mathcal{B}(V,W)$: dilation/(finite) sums of bounded sets are bounded. So T bounded implies λT is bounded and T_1, T_2 bounded implies $T_1 + T_2$ bounded.

- $|||T|||$ is well-defined in $\mathbb{R}_+$ for T bounded, $|||0||| = 0$ and if $|||T||| = 0$ then $T(B_V(0,1)) \subseteq B_W(0,t)$ for all $t > 0$ and so by continuity of dilation, $T(B_V(0,1)) = \{0\}$. By linearity, this implies $T = 0$.
- $|||\lambda T||| = |\lambda|\ |||T|||$ and $|||T_1 + T_2||| \leq |||T_1||| + |||T_2|||$ follows from $(*)$

$\square$

Proposition. Let $(V, ||\cdot||_V)$ be a NVS and $(W, ||\cdot||_W)$ a Banach space. Then $(\mathcal{B}(V,W), |||\cdot||||)$ is a Banach space.

Proof. We have proved that $(\mathcal{B}(V,W), |||\cdot|||)$ is an NVS above. So we prove completeness. Let $(T_k)_{k\geq 1}$ be a Cauchy sequence in $(\mathcal{B}(V,W), |||\cdot|||)$. Then

$$\sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \to 0 \text{ as } k_0 \to \infty \tag{$**$}$$

$$\forall v \in V, \sup_{k_1,k_2\geq k_0} ||T_{k_1}(v) - T_{k_2}(v)||_W \leq ||v||_V |||T_{k_1} - T_{k_2}||| \xrightarrow{k_0\to\infty} 0 \tag{$***$}$$

so $(T_k(v))_{k\geq 1}$ is a Cauchy sequence in W. Since W is complete, can let the associated limit be $T(v)$.

Then T is linear by pointwise limits:

$$\begin{aligned} T(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{k\to\infty} T_k(\lambda_1 v_1 + \lambda_2 v_2) &= \lim_{k\to\infty} [\lambda_1 T_k(v_1) + \lambda_2 T_k(v_2)] \\ &= \lambda_1 T(v_1) + \lambda_2 T(v_2) \end{aligned}$$

Use $(***)$, take $k_2 \to \infty$ so

$$\forall v \in V, \sup_{k_1\geq k_0} ||T_{k_1}(v) - T(v)||_W \leq ||v||_V \left(\sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \right) \to 0 \text{ as } k_0 \to \infty$$

Hence for $v \in V$ such that $||v|| \leq 1$ we have

$$\sup_{k_1\geq k_0} ||T_{k_1}(v) - T(v)||_W \leq \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \tag{$\dagger$}$$

Then (for $v \in V$ with $||v|| \leq 1$) by the triangle inequality

$$||T(v)||_W \leq ||\underbrace{T_{k_0}(v)}_{\text{bounded}}|| + \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}|||$$

$$\sup_{||v||\leq 1} ||T(v)||_W \leq |||T_{k_0}||| + \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}|||$$

So T is bounded. Now $(\dagger)$ implies

$$\sup_{k_1\geq k_0} |||T_{k_1} - T||| \leq \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \xrightarrow{k_0\to\infty} 0$$

So $T_{k_1} \xrightarrow{|||\cdot|||} T$. $\square$

Remark: can deduce from (†) that for all $v \in V$ with $||v|| \leq 1$,

$$||T_k(v)||_W - |||T_k - T||| \leq ||T(v)||_W \leq ||T_k(v)||_W + |||T_k - T|||$$

Then taking supremum over $||v|| \leq 1$

$$\left| \sup_{||v|| \leq 1} ||Tv||_W - \sup_{||v|| \leq 1} ||T_k(v)||_W \right| \leq |||T_k - T||| \xrightarrow{k \to \infty} 0$$

So $|||T_k||| \xrightarrow{k \to \infty} |||T|||$.

Definition. Let $(V, || \cdot ||_V)$ be a NVS over $\mathbb{F}$. Let

$$\mathcal{L}(V, \mathbb{F}) = \{\text{linear maps } V \to \mathbb{F}\}, \text{ the } \textit{algebraic dual}$$

$$\mathcal{B}(V, \mathbb{F}) = \{\text{bounded linear maps } V \to \mathbb{F}\} \text{ denoted } (V^*, || \cdot ||_{V^*})$$

Note that by the previous proposition $\mathcal{B}(V, \mathbb{F})$ is Banach (since $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is complete).

Definition. Let $(V, || \cdot ||_V)$, $(W, || \cdot ||_W)$ be NVS', $T \in \mathcal{B}(V, W)$. Then T^* (the *adjoint* of T) defined as $T^* : W^* \to V^*$, $\psi \mapsto \varphi = \psi \circ T$. i.e $T^*(\psi)(v) = \psi(T(v))$.

Proposition. T^* is well-defined $W^* \to V^*$, linear and bounded (for $|| \cdot ||_{W^*}$ and $|| \cdot ||_{V^*}$) with $|||T^*||| \leq |||T|||$.

Remark: soon, with the help of the Hahn-Banach Theorem, we'll prove that the duals are "big enough" so that $|||T^*||| = |||T|||$.

Proof.

- Well-defined: follows since linearity and boundedness are stable under composition, i.e if $T : V \to W$ is linear and bounded, $\psi : W \to \mathbb{F}$ is linear and bounded, so is $\psi \circ T : V \to \mathbb{F}$. So $\psi \circ T \in V^*$

- Linearity:

$$\begin{aligned} T^* \left(\lambda_1 \psi_1 + \lambda_2 \psi_2\right)(v) &= \left(\lambda_1 \psi_1 + \lambda_2 \psi_2\right)(Tv) \\ &= \lambda_1 \left[\psi_1(Tv)\right] + \lambda_2 \left[\psi_2(Tv)\right] \\ &= \lambda_1 T^*(\psi_1)(v) + \lambda_2 T^*(\psi_2)(v) \end{aligned}$$

- Boundedness:

$$|||T^*||| = \sup_{||\psi||_{W^*}} ||T^*(\psi)||_{V^*} = \sup_{||\psi||_{W^*} \leq 1} \sup_{||v||_V \leq 1} |T^*(\psi)(v)|$$

$$\leq \sup_{||\psi||_{W^*} \leq 1} \sup_{||v||_V \leq 1} |\psi(Tv)| \leq \sup_{||\psi||_{W^*} \leq 1} \sup_{||v||_V \leq 1} ||\psi||_{W^*} |||T||| \cdot ||v||_V \leq |||T|||$$

□

Definition. Let $(V, ||\cdot||_V)$ be an NVS. Since $(V^*, ||\cdot||_{V^*})$ is a NVS (Banach), we can define its dual, denoted $(V^{**}, ||\cdot||_{V^{**}})$ the *bidual* of V (again Banach).

Proposition. Define $\Phi : V \to V^{**}$, $v \mapsto \Phi(v)$ by

$$\forall \varphi \in V^*,\ \Phi(v)(\varphi) = \varphi(v)$$

Then Φ is well-defined, linear and bounded with $|||\Phi||| \leq 1$. Φ is called the *canonical bi-dual embedding.*

Remark: with the Hahn-Banach Theorem, we'll prove Φ is an isometry. In particular, $|||\Phi||| = 1$ and Φ is injective. However, Φ is not always surjective. In fact, V and V^{**} are not always isomorphic.

Proof.

- Well-defined: given $v \in V$, $\phi \in V^*$ is linear, and bounded since

$$\sup_{||\varphi||_{V^*} \leq 1} |\varphi(v)| \leq ||v||_V$$

- Linearity:

$$\begin{aligned}\Phi(\lambda_1 v_1 + \lambda_2 v_2)(\varphi) &= \varphi(\lambda_1 v_1 + \lambda_2 v_2) \\ &= \lambda_1 \varphi(v_1) + \lambda_2 \varphi(v_2) \\ &= \lambda_1 \Phi(v_1)(\varphi) + \lambda_2 \Phi(v_2)(\varphi)\end{aligned}$$

- Boundedness:

$$|||\Phi||| = \sup_{||v||_V \leq 1} ||\Phi(v)||_{V^{**}} = \sup_{||v||_V \leq 1} \sup_{||\varphi||_{V^*} \leq 1} |\underbrace{\Phi(v)(\varphi)}_{\varphi(v)}|$$

$$= \sup_{||v||_V \leq 1} \sup_{||\varphi_{V^*}|| \leq 1} \underbrace{|\varphi(v)|}_{\leq ||\varphi||_{V^*} ||v||_V} \leq 1$$

∎

Example. Let V, W be finite-dimensional NVS' with bases $(v_i)_{i=1}^m$ and $(w_j)_{j=1}^n$ respectively. Let $T : V \to W$ be linear (and thus bounded as finite dimensional). Take $(v_i^*)_{i=1}^m$ defined by $v_i^*(v_{i'}) = \delta_{ii'}$ and $(w_j^*)_{j=1}^n$ defined by $w_j^*(w_{j'}) = \delta_{jj'}$. Then V^*,W^* are finite-dimensional NVS' with bases (v_i^*) and (w_j^*) respectively. If T has a matrix $A = (a_{ij})_{i=1,j=1}^{i=m,j=n}$ in with respect to the bases (v_i) and (w_j), then

$$Tv_i = \sum_{j=1}^n a_{ij} w_j$$

and T^* has matrix $A^T = (a_{ji})_{j=1,i=1}^{j=n,i=m}$ with respect to the bases (w_j^*) and (v_i^*).

Example. Space of square summable spaces $\ell^2(\mathbb{F})$ (as usual $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is infinite dimensional. There are linear maps on this space that are

- Bounded, injective but not surjective: $T(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$ a "right shift" of the sequence
- Bounded, surjective but not injective: $T(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$ a "left shift" of the sequence
- Linear but not bounded: find a basis $(e_i)_{i \in I}$, extract $(e_n)_{n \geq 1}$ a countable subset. Then define $T : e_n \mapsto ne_n$, $e_i \mapsto 0$ for $i \notin \mathbb{N}$.

Duality: $(\ell^2)^* = \ell^2$ (Hilbert representation theorem)

Example. For ℓ^p, $p \in (1, \infty)$, $p \neq 2$, we have duals

$$\ell^p \to (\ell^p)^* = \ell^q \to (\ell^q)^* = \ell^p \text{ where } \frac{1}{p} + \frac{1}{q}$$

$$\ell^1 \to (\ell^1)^* = \ell^\infty \to (\ell^\infty)^* \neq \ell^1$$

Example. (Question 8 Example sheet 1) $\big(C^1([0,1]), ||\cdot||_{C^0}\big) \to \big(C^1([0,1]), ||\cdot||_{C^1}\big)$, $f \mapsto f$ is unbounded.

Zorn's Lemma

In a finite-dimensional NVS V, we have a "simple" dual V^*. In infinite-dimension, we have not even proved that if V is non-trivial (i.e not $\{0\}$) then V^* is non-trivial.

The Hahn-Banach Theorem will answer several questions:

- $V \neq \{0\} \implies V^* \neq \{0\}$
- V^* seperates points of V
- Φ (the bidual embedding) is isometric, $|||\Phi||| = 1$
- $|||T^*||| = |||T|||$

Idea of Hahn-Banach: extend linear bounded maps already defined on a subspace.

Strategy:

1. "Co-dimension 1" extension: any linear bounded map $V \to \mathbb{F}$ has an extension to $W \to \mathbb{F}$ where $V \subseteq W$ with codimension 1.
2. Transfinite induction: Zorn's Lemma (or equivalently the Axiom of Choice)

Remark: if $V = \bigcup_{n\geq 1} V_n$, V_n subspace, $V_n \subseteq V_{n+1}$, $\dim(V_n) = n$, could use step 1 above and standard (countable) induction. However, no Banach spaces are like this.

Definition. A set S is *partially ordered* (poset) if there is a binary relation "$\leq$" such that

- $\forall x, y \in S$, $x \leq y$ or not (partial order)
- $\forall x \in S$, $x \leq x$ (reflexive)
- $\forall x, y, z \in S$, if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive)
- $\forall x, y \in S$, if $x \leq y$ and $y \leq x$ then $x = y$ (non-ambiguous)

Definition. A poset S is *totally ordered* if $\forall x, y \in S$, if $x \not\leq y$ then $x \geq y$.

Definition. Given $S' \subseteq S$ (where $(S, \leq)$ is a poset), we say $l \in S$ is a *upper bound* of S' if $\forall x \in S'$, $x \leq l$. l is a *least upper bound* of S' if it is an upper bound and any other upper bound $l' \in S$ satisfies $l' \geq l$.

Definition. A subset S' of S ($(S, \leq)$ a poset) that is totally ordered is called a *chain*.

Definition. A poset $(S, \leq)$ has the *least upper bound property* if any non-empty chain has a least upper bound.

Definition. Given a poset $(S, \leq)$, $m \in S$ is said to be *maximal* if $\forall x \in S$, $x \geq m$ implies $x = m$.

Theorem (Zorn's Lemma). *Any non-empty poset $(S, \leq)$ with the least upper bound property has (at least one) maximal element.*

Remarks:

1. In fact Zorn's Lemma is true just with "upper bound" property on chains.
2. Zorn's Lemma is equivalent to the Axiom of Choice

5.1 Finite dimension

Definition. Let V be a NVS with two norms $||\cdot||_1$ and $||\cdot||_2$. Then these norms are said to be *equivalent*, denoted $||\cdot||_1 \sim ||\cdot||_2$ if there are two constants, $c, c' > 0$ such that

$$\forall v \in V,\ C||v||_1 \leq ||v||_2 \leq C'||v||_1$$

Remarks:

1. This defines equivalence classes on norms.
2. $||\cdot||_1 \sim ||\cdot||_2$ implies that their induced topologies are the same. The converse is also true: indeed $B_{||\cdot||_1}(0,1)$ is open around 0 for τ_2, so there exists $\varepsilon > 0$ such that $B_{||\cdot||_2}(0,\varepsilon) \subseteq B_{||\cdot||_1}(0,1)$, which implies that for all $v \in V \setminus \{0\}$

   $$\frac{\varepsilon v}{2||v||_2} \in B_{||\cdot||_2}(0,\varepsilon) \subseteq B_{||\cdot||_1}(0,1) \implies ||v||_1 \leq \frac{2}{\varepsilon}||v||_2$$

   and similarly for the opposite bound.
3. When 2 norms are equivalent, they generate te same notion of bounded linear maps, converging spaces & Cauchy sequences.

Proposition.

(i) All norms are equivalent in finite-dimension

(ii) Given $(V, ||\cdot||_V)$ a finite-dimensional NVS, $(W, ||\cdot||_W)$ a NVS, any linear map $T : V \to W$ is bounded

(iii) Given $(V, ||\cdot||_V)$ an NVS, if $\overline{B}_V(0,1)$ is compact, then V is finite dimensional.

Proof.

(i) Let us prove all norms are equivalent to $||\cdot||_\infty$, defined for a basis $(e_i)_{i=1}^n$ as $||v||_\infty = \sup_{1\leq i\leq n} |v_i|$ for $v = \sum v_i e_i$.

Let $||\cdot||$ be a norm on V

$$||v|| = \left|\left|\sum_{i=1}^n v_i e_i\right|\right| \leq \sum_{i=1}^n |v_i|||e_i|| \leq \underbrace{\left(\sum_{i=1}^n ||e_i||\right)}_{=C'} ||v||_\infty$$

Consider $\varphi : (V, ||\cdot||_\infty) \to \mathbb{R}_+$ defined by $v \mapsto ||v||$. Then φ is continuous:

$$|\varphi(v) - \varphi(w)| = |||v|| - ||w||| \leq ||v - w|| \leq C'||v - w||_\infty$$

Define $S_{||\cdot||_\infty}(0,1) = \{v \in V : ||v||_\infty = 1\}$. Then $\varphi : S_{||\cdot||_\infty}(0,1) \to \mathbb{R}_+$ continuous, so attains its minimum: there exists $v_0 \in S_{||\cdot||_\infty}(0,1)$ such that $\forall v \in S_{||\cdot||_\infty}(0,1)$, $\varphi(v) \geq \varphi(v_0)$.

Then $v_0 \neq 0$ since $||v_0||_\infty = 1$ and so $\varphi(v_0) = ||v_0|| = C > 0$. This implies

$$\left|\left|\frac{v}{||v||_\infty}\right|\right| \geq C,\ \forall v \in V \setminus \{0\} \implies \forall v \in V,\ ||v|| \geq C||v||_\infty$$

(ii) Completeness and the fact closed bounded sets are compact follows from (i) since true with $(\mathbb{F}^n, ||\cdot||_\infty)$.

$$||T(v)||_W = \left|\left|\sum_{i=1}^n v_i T(e_i)\right|\right|_W \leq \sum_{i=1}^n |v_i|||T(e_i)||_W$$

$$\leq ||v||_\infty \left(\sum_{i=1}^n ||T(e_i)||_W\right) \leq \frac{1}{C}||v||_V \left(\sum_{i=1}^n ||T(e_i)||_W\right)$$

so T is bounded

$\square$

Theorem (Riesz)**.** *If $(V, \|\cdot\|)$ is an NVS, $\overline{B}(0,1)$ compact then V finite dimensional.*

Proof. $\overline{B}(0,1) \subseteq \bigcup_{v\in\overline{B}(0,1)} B(v,1/2)$ open covering. Then compactness implies there exist $v_1,\ldots,v_n$ in $\overline{B}(0,1)$ such that $\overline{B}(0,1) \subseteq \bigcup_{i=1}^n B(v_i,1/2)$. Denote $W = \text{span}(v_1,\ldots,v_n)$ a subspace of V. Then $\overline{B}(0,1) \subseteq \bigcup_{i=1}^n (v_i + B(0,1/2))$.

$\overline{B}(0,1) \subseteq W + B^{\cdot}(0,1/2) \subseteq W + \overline{B}(0,1/2)$

Iterate on $\overline{B}(0,1/2) = \frac{1}{2}\overline{B}(0,1)$: $\overline{B}(0,1/2) \subseteq W + \overline{B}(0,1/4)$.

$$\overline{B}(0,1) \subseteq \bigcap_{k=1}^{K} \left(W + \overline{B}(0,2^{-k})\right), \ \forall K \geq 1$$

Then

$$\overline{B}(0,1) \subseteq \bigcap_{k\geq 1} \left(W + \overline{B}(0,2^{-k})\right) \subseteq \overline{W} = W$$

$\overline{B}(0,1) \subseteq W$ implies $V = W$. $\square$

Back to (Zorn's Lemma) and the Hahn-Banach Theorem

Construction of basis:

Proposition. Let $V \neq \{0\}$ be a vector space over $\mathbb{F}$ and $S \subseteq V$ subset which is linearly independent. Then there exists a subset $B \subseteq V$ linearly independent such that $S \subseteq B$ and $\text{span}(B) = V$ (i.e a basis).

Proof. Let $\mathcal{F} = \{$linearly independent subsets $S' \subseteq V$ such that $S \subseteq S'\}$. Then $\mathcal{S} \neq \emptyset$ since $S \in \mathcal{F}$.

$(\mathcal{F}, \subseteq)$ is a poset (easy check).

If $\Theta \subseteq \mathcal{F}$ is a chain (totally ordered for $\subseteq$) then it has a least upper bound: $\overline{S} = \bigcup_{S'\in\Theta} S'$.

Properties of $\overline{S}$:

- $\overline{S} \supseteq S'$, for all $S' \in \Theta$ so S' is an upper bound for Θ
- An upper bound for Θ will include each $S' \in \Theta$ so $\overline{S}$ is a least upper bound.
- $\overline{S} \supseteq S$ since $\overline{S} = \bigcup_{S'\in\Theta} S'$ and each $S' \supseteq S$.
- $\overline{S}$ is linearly independent: let $(v_1,\ldots,v_n) \in \overline{S}$ be distinct elements. Then for all $i = 1,\ldots,n$ there exists $S'_i \in \Theta$ such that $v_i \in S'_i$. Chain structure (total order) means there exists $i_0 \in \{1,\ldots,n\}$ such that $S'_j \subseteq S'_{i_0}$ for all $j = 1,\ldots,n$. So $\{v_1,\ldots,v_n\} \subseteq S'_{i_0}$ is linearly independent, and so $\overline{S}$ is.

Now Zorn's Lemma says that there exists a maximal element in $\mathcal{F}$: $B \supseteq S$, B linearly independent and maximal. Assume $\text{span}(B) \subsetneq V$, then we have $v_0 \in V \setminus \text{span}(B)$ and $B' = B \cup \{v_0\}$ is a strictly larger element of $\mathcal{F}$, a contradiction. Hence $V = \text{span}(B)$. $\square$

Note that the statement of the geometric form of Hahn-Banach below is ***non-examinable***

Theorem (Hahn-Banach "algebraic" form)**.**

(i) Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $p : V \to \mathbb{R}_+$ such that for all $v_1, v_2 \in V$, $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ and for all $\lambda \in \mathbb{F}$, $v \in V$ we have $p(\lambda v) = |\lambda| p(v)$.

Let $W \subseteq V$ be a subspace of V and $f : W \to \mathbb{F}$ linear with $|f(w)| \leq p(w)$ for all $w \in W$. Then there exists $\tilde{f} : V \to \mathbb{F}$ linear, with $\tilde{f}|_W = f$ and $|f(v)| \leq p(v)$ on all of V.

(ii) Let V be a vector space over $\mathbb{F} = \mathbb{R}$ and $p : V \to \mathbb{R}_+$ such that for all $v_1, v_2 \in V$, $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ and for all $\lambda > 0$, $v \in V$ we have $p(\lambda v) = \lambda p(v)$.

Let $W \subseteq V$ be a subspace of V and $f : W \to \mathbb{F}$ be linear with $f \leq p$ on W. Then there exists $\tilde{f} : V \to \mathbb{F}$ linear with $\tilde{f}|_W = f$, and $\tilde{f} \leq p$ on V.

Proof. Step 1: (i) in $\mathbb{R}$ implies (ii) in $\mathbb{C}$. Start from $f : W \to \mathbb{F} = \mathbb{C}$. Note that a vector space V over $\mathbb{C}$ can be seen as a vector space over $\mathbb{R}$. Indeed if $(e_i)_{i \in I}$ is a basis over $\mathbb{C}$, and $V_0 = \text{span}_{\mathbb{R}}((e_i)_{i \in I})$, $V = V_0 \oplus (iV_0)$ (same with W).

Define $g = \Re(f)$, this satisfies $|g| \leq p$. Then (i) on $\mathbb{R}$ implies there exists $\tilde{g} : V \to \mathbb{R}$ linear extending g such that $|\tilde{g}| \leq p$.

Define $\tilde{f}(v) := \tilde{g}(v) - i\tilde{g}(iv)$. Then $\tilde{f}(\lambda v) = \lambda \tilde{f}$ for all $\lambda \in \mathbb{R}$ (f linear). Also $\tilde{f}(iv) = i\tilde{f}(v)$. Hence f is linear over $\mathbb{C}$. This extends g to all of V.

Also for all $v \in V$, there exists $\theta \in [0, 2\pi)$ such that $|f(v)| = \Re(\tilde{f}(e^{i\theta} v)) = \tilde{g}(e^{i\theta} v) \leq p(e^{i\theta} v) = p(v)$.

Step 2: (ii) in $\mathbb{R}$ implies (i) in $\mathbb{R}$. If $W \subseteq V$ is a subspace, $p : V \to \mathbb{R}_+$ such that $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ for all $v_1, v_2 \in V$ and $p(\lambda v) = |\lambda| p(v)|$ for all $\lambda \in \mathbb{R}, v \in V$, and $f : W \to \mathbb{R}$ is linear such that $|f(v)| \leq p(v)$ for all $v \in W$ then (ii) can be applied to obtain $\tilde{f} : V \to \mathbb{R}$ linear extending f such that $\tilde{f}(v) \leq p(v)$ for all $v \in V$ (no modulus a priori in this conclusion).

We also deduce $\tilde{f}(-v) = p(-v) = p(v)$, so $|\tilde{f}(v)| \leq p(v)$.

Step 3: proof of (ii) in $\mathbb{R}$.

(a) Co-dimension 1 case: consider $V = W \oplus (\mathbb{R}v_0)$, $v_0 \neq 0$. We have $f : W \to \mathbb{R}$ linear, $f \leq p$ on W. To extend f it is enough to prescribe $\tilde{f}$ at v_0, then linearity does the rest: for $w \in W$, $\tilde{f}(w+av_0) = \tilde{f}(w)+a\tilde{f}(v_0) = f(w)+a\tilde{f}(v_0)$.

The value of $\tilde{f}(v_0)$ must satisfy:

$$\tilde{f}(w + av_0) \leq p(w + av_0),\ a > 0 \text{ and for } a < 0$$

This gives

$$\underbrace{-p\left(-\frac{w}{a} - v_0\right) + f\left(-\frac{w}{a}\right)}_{A(w')} \underbrace{\leq}_{a<0} \tilde{f}(v_0) \underbrace{\leq}_{a>0} \underbrace{p\left(\frac{w}{a} + v_0\right) - f\left(\frac{w}{a}\right)}_{B(w'')}$$

where $w' = -\frac{w}{a}$ and $w'' = \frac{w}{a}$. Then for all $w', w'' \in W$, $\tilde{f}(v_0) \in [A(w'), B(w'')]$. Set $\beta = \tilde{f}(v_0)$. Then a consistent value of β exists if and only if

$$\sup_{w' \in W} A(w') \leq \inf_{w'' \in W} B(w'')$$

This is indeed satisfied since

$$f(w') + f(w'') = f(w' + w'') \leq p(w' + w'') \leq p(w' - v_0) + p(w'' + v_0)$$

(b) Transfinite induction: define

$$\mathcal{S} = \{(\tilde{f}, \tilde{W}) : \tilde{f} : \tilde{W} \to \mathbb{R} \text{ linear }, \tilde{f} \leq p \text{ and } \tilde{W} \supseteq W,\ \tilde{f}|_W = f\}$$

Now $\mathcal{S}$ is a poset under $(f_1, W_1) \subseteq (f_2, W_2)$ if $W_1 \subseteq W_2$ and $f_2|_{W_1} = f_1$. Also $\mathcal{S}$ has the least upper bound property: indeed consider $\Theta \subseteq \mathcal{S}$ a chain (totally ordered subset). Then for $(\bar{f}, \bar{W})$ defined by

$$\bar{W} = \bigcup_{W' : (f', W') \in \Theta} W'$$

and $\bar{f}(v) = f'(v)$ for all $v \in \bar{W}$, for $(f', W') \in \Theta$ such that $v \in W'$. Also $\bar{f}$ is well defined since Θ is totally ordered: so if $v \in W_1' \cap W_2'$ then wlog $W_1' \subseteq W_2'$, $f_2'|_{W_1'} = f_1'$ so $\bar{f}(v) = f_2'(v) = f_2'(v)$.

$\bar{f}$ is linear as Θ is totally ordered: $\bar{f}(\lambda v) = f'(\lambda v) = \lambda f'(v) = \lambda \bar{f}(v)$ for $(f', W') \in \Theta$ with $v \in W'$. Also

$$\bar{f}(v_1 + v_2) = f_2'(v_1 + v_2) = f_2'(v_1) + f_2'(v_2) = \bar{f}(v_1) + \bar{f}(v_2)$$

Finally $\bar{f} \leq p$ since for all $v \in \bar{W}$, $v \in W'$, $(f', W') \in \Theta$, $\bar{f}(v) = f'(v) \leq p(v)$.

So by Zorn's Lemma, there is a maximal element $(\tilde{f}, \tilde{W})$ in $\mathcal{S}$. If $\tilde{W} \subsetneq V$, then there exists $v_0 \in V \setminus \tilde{W}$ and the previous step applied to $\tilde{W} \subseteq \tilde{W} \oplus \mathbb{R}v_0$ and $\tilde{f} : \tilde{W} \to \mathbb{R}$ linear with $\tilde{f} \leq p$, gives the existence of a

$$\tilde{f}' : \underbrace{\tilde{W} \oplus \mathbb{R}v_0}_{\tilde{W}'} \to \mathbb{R}$$

linear with $\tilde{f}'|_{\tilde{W}} = \tilde{f}$. But then $(\tilde{f}', \tilde{W}')$ is strictly larger than $(\tilde{f}, \tilde{W})$, a contradiction.

$\square$

Theorem (Geometric form of Hahn-Banach)**.**

(i) *Let $(V, ||\cdot||)$ be an NVS over $\mathbb{R}$, $A \subseteq V$ open, convex and non-empty; $B \subseteq V$ convex and non-empty; $A \cap B = \emptyset$. Then there is a closed hyperplane weakly seperating A and B: there exists $f \in V^* \setminus \{0\}$, $\alpha \in \mathbb{R}$ such that $\sup_A f \leq \alpha \leq \inf_B f$ (the hyperplane is $f^{-1}(\{\alpha\})$)*

(ii) *Let $(V, ||\cdot||)$ be an NVS over $\mathbb{R}$, $A \subseteq V$ closed, conevex and non-empty; $B \subseteq V$ compact, convex and non-empty; $A \cap B = \emptyset$. Then there is a closed hyperplane strictly seperating A and B: there exists $f \in V^* \setminus \{0\}$, $\alpha_1 < \alpha_2 \in \mathbb{R}$ such that $\sup_A f \leq \alpha_1 < \alpha_2 \leq \inf_B f$.*

Proof.

(i) Let $C_0 = A - B = \{a - b : a \in A, b \in B\}$. Then $C_0 \neq \emptyset$ since A and B are non-empty, convex as

$$\lambda(a-b) + (1-\lambda)(a'-b') = (\underbrace{\lambda a + (1-\lambda a')}_{\in A}) - (\underbrace{\lambda b + (1-\lambda)b'}_{\in B})$$

Also C_0 is open since $C_0 = \bigcup_{b \in B} (\underbrace{A - b}_{\text{open}})$.

$0 \neq C_0$ since $A \cap B = \emptyset$. Let $v_0 \in C_0$, define $C = C_0 - v_0$. Then C is open, convex, non-empty and includes 0. Define $p = \mu_C$ (Minkowski gauge):

$$\forall v \in V,\ p(v) = \inf\{t \geq 0 : v \in tC\}$$

p satisfies (see proof of Kolmogorov)

- p is well-defined
- $p(\lambda v) = \lambda p(v),\ \forall \lambda > 0$
- $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ (using C convex)
- $p(-v)$ is not necessarily equal to $p(v)$ (C is not necessarily balanced)

Let $f : \mathbb{R}v_0 \to \mathbb{R}$ be linear defined by $f(-v_0) = 1$. Since $-v_0 \notin C$ ($0 \notin C_0$) we have $p(-v_0) \geq 1$, so $f \leq p$ ($\tilde{f}(-v_0) \leq p(-v_0)$) so $\tilde{f}(-\lambda v_0) \leq p(-\lambda v_0)$ for all $\lambda > 0$, and for $\lambda < 0$ $\tilde{f}(-\lambda v_0) \leq 0$.

The Hahn-Banach theorem (algebraic version) gives $\tilde{f} : V \to \mathbb{R}$ linear such that $\tilde{f}|_{\mathbb{R}v_0} = f$, $\tilde{f}(-v_0) = 1$. So $\tilde{f} \neq 0$, and since $p < 1$ in C, $\tilde{f}|_C < 1$, so since C is open around 0: there exists $B(0, \varepsilon) \subseteq C$ such that

$$\sup_{v \in B(0,\varepsilon)} \tilde{f}(v) \leq 1 \implies \sup_{v \in B(0,\varepsilon)} |\tilde{f}| \leq 1 \implies \tilde{f} \in V^*,\ ||\tilde{f}||_{V^*} \leq \varepsilon^{-1}$$

And

$$\tilde{f}|_C < 1 \implies \tilde{f}|_{C_0} < 0 \implies \sup_A \tilde{f} \leq \inf_B \tilde{f}$$

So there is $\alpha \in \mathbb{R}$ such that $\sup_A \tilde{f} \leq \alpha \leq \inf_B \tilde{f}$

(ii) $C_0 = B - A$ non-empty, convex, doesn't include 0, is closed: given $(a_n - b_n)_{n\geq 1}$ a sequence in C_0 with $(a_n - b_n) \to e$, we have (since B is compact), there exists a subsequence $(a_{n'} - b_{n'})_{n'\geq 1}$ such that $b_{n'}$ converges to $b \in B$, so $a_{n'}$ converges to $a \in A$ as A is closed. So $l = a - b \in C_0$.

So there exists an open ball $B(0, \varepsilon)$ such that $B(0, \varepsilon) \cap C_0 = \emptyset$. Apply (i) to $\tilde{A} = B(0, \varepsilon)$ (open, convex, non-empty) and $\tilde{B} = C_0$ (convex, non-empty). Then there exists $f : V \to \mathbb{R}$ bounded and linear, $f \neq 0$ such that

$$\sup_{B(0,\varepsilon)} f \leq \alpha \leq \inf_{C_0} f = \inf_B f - \sup_A f$$

Where $\alpha = \varepsilon ||f||_{V^*} = \sup_{v \in B(0,\varepsilon)} |f(v)| > 0$.

$\square$

Consequences of Hahn-Banach

Proposition.

(i) Given $(V, ||\cdot||)$ an NVS, W a subspace, $f \in W^*$ (linear and continuous on W), there exists $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$, and $||\tilde{f}||_{V^*} = ||f||_{W^*}$.

(ii) If $(V, ||\cdot||)$, is an NVS with $V \neq \{0\}$, then $V^* \neq \{0\}$.

(iii) Given $(V, ||\cdot||)$ an NVS with $V \neq \{0\}$, and $v, w \in V$ with $v \neq w$ then there exists $f \in V^*$ such that $f(v) \neq f(w)$.

Proof.

(i) Apply HB (algebraic form) with $p : V \to \mathbb{R}+$, $v \mapsto ||f||_{W^*}||v||$. This satisfies the assumptions trivially and $|f| \leq p$ on W, so there exists $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$ and $|\tilde{f}(v)| \leq p(v) \leq ||f||_{W^*}||v||$ for all $v \in V$. This implies $||f||_{V^*} \leq ||f||_{W^*}$ and we clearly have equality.

(ii) Consider $v_0 \in V \setminus \{0\}$. Then define ("support functional" for v_0) $f : W = \mathbb{F}v_0 \to \mathbb{F}$ the linear map such that $f(v_0) = ||v_0||$. Then (i) implies the existence of $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$ and $||f||_{W^*} = ||f||_{V^*} = 1$. Hence $\tilde{f} \neq 0$ and $V^* \neq \{0\}$.

(iii) Given $v \neq w$ in V, apply (ii) to $v_0 = v - w$. Then there is $\tilde{f} \in V^*$ such that $\tilde{f}(v_0) = \tilde{f}(v) - \tilde{f}(w) = ||v_0|| \neq 0$.

□

Proposition. Given $(V, ||\cdot||)$ an NVS, $\Phi : V \to V^{**}$ defined by $v \mapsto \Phi(v)$ where $\Phi(v)(f) = f(v)$ for any $f \in V^*$. This is an isometry (in particular $|||\Phi||| = 1$).

Proof. We have already proven that $||\Phi(v)||_{V^{**}} \leq ||v||_V$ for all $v \in V$. Let us prove this is an equality. Consider $v \in V \setminus \{0\}$, let f_v be a support functional for v, $f_v \in V^*$, $f_v(v) = ||v||_V$, $||f_v||_{V^*} = 1$ (constructed in the proof of (ii) in the previous proposition). Now $\Phi(v)(f_v) = f_v(v) = ||v||_V$. Hence

$$\sup_{\substack{f \in V^* \\ ||f||_{V^*} \leq 1}} |\Phi(V)(f)| \geq ||v||_V \implies ||\Phi(v)||_{V^{**}} \geq ||v||_V$$

□

Proposition. Let V, W be NVS', $T : V \to W$ linear and bounded. Then $T^* : W^* \to V^*$ (the adjoint) satisfies $|||T^*||| = |||T|||$.

Proof. We already proved $|||T^*||| \leq |||T|||$. So we show the reverse inequality. Consider $v \in V$ such that $||v|| = 1$ and $w = Tv \neq 0$. Let $g_w \in W^*$ be a support functional for $w \in W$. Then $T^*(g_w)(v) = g_w(Tv) = g_w(w) = ||w||_W$. So

$$||T^*(g_w)||_{V^*} = \sup_{\substack{v' \in V \\ ||v'||=1}} |T^*(g_w)(v')| \geq ||w||_W$$

so

$$|||T^*||| = \sup_{\substack{g \in W^* \\ ||g||_{W^*}=1}} ||T^*(g)||_{V^*} \geq ||T^*(g_w)|| \geq ||w||_W$$

so

$$|||T^*||| \geq ||w||_W = ||Tv|_W$$

So take the supremum over $v \in V, ||v|| = 1$ to get

$$|||T^*||| \sup_{\substack{v \in V \\ ||v||=1}} ||Tv||_W = |||T|||$$

□

6 The Baire Category Theorem

Hahn Banach: uses sublinearity of gauges/norms (convexity of associated unit ball) to study the dual space and build linear forms.

Baire: use completeness to prove that complete NVS' are necessarily "big" - used for existence of objects and local-to-global estimates.

The following theorem was proved by Osgood (1897) in $\mathbb{R}$ and by Baire (1899) in general.

Definition. Let (X, τ) be a topological space.

(i) A subset $B \subseteq X$ is *rare* (or *nowhere dense*) if $\overline{B}$ has empty interior, i.e for all $U \in \tau$, $B \cap U$ is not dense in U.

(ii) A subset $B \subseteq X$ is *meagre* (first category) in X if it can be written as a countable union of rare sets. Otherwise B is *non-meagre* (second category) in X.

(iii) (X, τ) is *meagre/non-meagre* (first/second category) if it is as a subset of itself.

Proposition. Given (X, τ) a topological space, the following are equivalent

(i) X is non-meagre

(ii) For all $(C_n)_{n \geq 1}$ a countable collection of closed sets covering X, at least one C_n has non-empty interior

(iii) For all $(O_n)_{n \geq 1}$ a countable collection of open sets which are all dense in X, $\bigcap_{n \geq 1} O_n \neq \emptyset$

Proof. (ii) implies (i): if $X = \bigcup_n A_n$, with A_n rare, then $C_n := \bar{A}_n$ are closed with empty interior, and $X = \bigcup_n C_n$.

(i) implies (ii): if $X = \bigcup_n C_n$, C_n closed with empty interior, then $A_n := C_n$ are rare.

(ii) implies (iii): given $(O_n)_{n\geq 1}$ open dense sets, $C_n = O_n^c$ are closed with empty interior: otherwise there exists $U \in \tau$, $U \subseteq C_n$ such that $U \cap O_n = \emptyset$ (contadicting density). Also $\bigcap_n O_n \neq \emptyset \iff \bigcup_n C_n \supsetneq X$.

(iii) implies (ii): Given $(C_n)_{n\geq 1}$ closed sets with $U_{n\geq 1} C_n = X$, if all C_n have empty interiors, then $O_n := C_n^c$ contradicts (iii) so at least one C_n has non empty interior ☐

Theorem (Baire's Theorem)**.** *Let (X, d) be a complete metric space. Then X is non-meagre. In fact it is a Baire space, a space in which countable intersections of dense open sets are dense.*

Proof. It is enough to prove that (X, d) is a Baire space. Consider $(O_n)_{n\geq 1}$ a sequence of open dense sets, and U an arbitrary open set. We will show $U \cap (\bigcap_n O_n) \neq \emptyset$.

Induction: since O_1 is dense, $O_1 \cap U$ is non-empty and open. Pick $x_1 \in O_1 \cap U$, with $B(x_1, r_1) \subseteq O_1 \cap U$ for some $r_1 > 0$. Then $O_2 \cap B(x_1, r_1/2) \neq \emptyset$ (density of O_2) and open. So there exists $x_2 \in O_2$ and $r_2 > 0$ such that $B(x_2, r_2) \subseteq O_2 \cap B(x_1, r_2/2)$.

General step: there exists $B(x_{k+1}, r_{k+1}) \subseteq O_{k+1} \cap B(x_k, r_k/2)$ for $x_{k+1} \in X$, $r_{k+1} > 0$. This builds a sequence $(x_k)_{k\geq 1}$ in X which is Cauchy: for all $k \geq k_0 \geq 1$, $x_k \in B(x_{k_0}, r_{k_0}/2)$ and inclusion of balls implies $r_{k+1} \leq r_k/2$, for $k \geq 1$. So $r_k \leq 2^{-k+1} r_1 \to 0$, so it is indeed Cauchy. Hence $x_k \to e$ for some $e \in X$ and $e \in \overline{B}(x_{k_0}, r_{k_0}/2)$ for all $k_0 \geq 1$. So $e \in O_{k+1} \cap B(x_k, r_k/2)$ for all k, and so $e \in (\bigcap_n O_n) \cap U$ (contained in U since $B(x_1, r_1)$ is). ☐

Theorem (Baire)**.** *If (X, τ) is a compact and Hausdorff space, then X is:*

- *(i) Normal: for all C_1, C_2 disjoint non-empty closed sets, there exist $U_1, U_2 \in \tau$ disjoint such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$.*
- *(ii) X is a Baire space.*

Proof.

(i) Let C_1, C_2 be as in the statement. For all $x \in C_1, y \in C_2$ there exist $U^1_{x,y}, U^2_{x,y} \in \tau$ such that $x \in U^1_{x,y}$, $y \in U^2_{x,y}$ and $U^1_{x,y} \cap U^2_{x,y} = \emptyset$. Fix $y \in C_2$, so $C_1 \subseteq \bigcup_{x\in C_1} U^1_{x,y}$ (since $x \in U^1_{x,y}$).
Since C_1 is a closed subset of a compact space X, it is compact. So extract a finite covering: take $x_1, \ldots, x_m \in C_1$ such that $C_1 \subseteq \bigcup_{i=1}^m U^1_{x_i,y}$. Denote

$V_y^1 = \bigcup_{i=1}^m U^1_{x_i,y}$ and $V_y^2 = \bigcap_{i=1}^m U^2_{x_i,y}$. Observe that V_y^1, V_y^2 are open and disjoint. Then C_2 is compact (closed in compact space), $C_2 \subseteq \bigcup_{y \in C_2} V_y^2$ (since $y \in V_y^2$). So can extract a finite covering: take $y_1, \ldots, y_n \in C_2$ such that $C_2 \subseteq \bigcup_{j=1}^m V^2_{y_j}$.

Finally denote $U^1 = \bigcap_{j=1}^n V^1_{y_j}$ and $U^2 = \bigcup_{j=1}^n V^2_{y_j}$. Then U^1, U^2 are open, disjoint and $C_1 \subseteq U_1$, $C_2 \subseteq U_2$.

(ii) Consider $(O_n)_{n \geq 1}$ open dense sets, and $U \in \tau$. We want to show $(\bigcap_n O_n) \cap U \neq \emptyset$.

Induction:

- Since O_1 is dense, there exists $x_1 \in O_1 \cap U$ ($O_1 \cap U$ non-empty and open). We want to show there exists U_1 open around x_1 such that $\overline{U}_1 \subseteq O_1 \cap U$.
- $\{x_1\}$ is disjoint from $(O_1 \cap U)^c$, and both sets closed. So there exist $U_1, U_1' \in \tau$ such that $x_1 \in U_1$, $(O_1 \cap U_1)^c \subseteq U_1'$ and $U_1 \cap U_1' = \emptyset$. Then $\overline{U}_1 \subseteq (U_1')^c \subseteq O_1 \cap U$
- Continuing the induction: $x_k \in U_k \subseteq \overline{U}_k \subseteq O_k \cap U_{k-1}$. Then $\bigcap_k \overline{U}_k$ is non empty (X compact) so $\bigcap_k \overline{U}_k \subseteq U \cap (\bigcap_n O_n)$

□

Applications:

- Existence of irrationals in $\mathbb{R}$: $(\mathbb{R}, |\cdot|)$ is a complete metrix space, so a Baire space. Then for all $x \in \mathbb{R}$, $\{x\}$ is closed with empty interior. So if $\mathbb{Q} = \{q_n : n \geq 1\}$, then $\mathbb{R} = \bigcup_n \{q_n\}$ would contradict (ii) in the above proposition (before the last two theorems). In fact a similar argument proves a stronger result: if (X, d) is a metric space with no isolated points, then X is uncountable.

- There exists $f \in C([0,1])$ that is nowhere differentiable. To show this, we instead prove

$$\mathcal{D} = \{f \in C([0,1]) : f \text{ differentiable at some } x \in [0,1]\}$$

is meagre in the Baire space $(C([0,1]), ||\cdot||_\infty)$.