Motivation

This section is motivation and will not be rigorous. We have a 'Dirac delta function' such that for all 'nice' functions f

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \mathrm{d}x = f(x_0).$$

Can we define $\delta'(x - x_0)$? Could try

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x &= \lim_{h \to 0} \int_{-\infty}^{\infty} \left[\frac{\delta(x - x_0 + h) - \delta(x - x_0)}{h} \right] f(x) \\
&= \lim_{h \to 0} \frac{1}{h} \left[f(x_0 - h) - f(x_0) \right] \\
&= -f'(x_0).
\end{aligned}$$

i.e

$$\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x = -\int_{-\infty}^{\infty} \delta(x - x_0) f'(x) \mathrm{d}x$$

which looks like some form of integration-by-parts. We can make this rigorous using distribution theory.

Fourier transform of polynomials

If $f \in L^1(\mathbb{R})$ then

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) \mathrm{d}x.$$

How could we take the Fourier transform of something like $f(x) = x^n$? May recall identity

$$\delta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x$$

and then get

$$\begin{aligned}
\hat{f}(\lambda) &= \int_{-\infty}^{\infty} x^n e^{-\lambda x} \mathrm{d}x \\
&= \left(i \frac{\partial}{\partial \lambda} \right)^n \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x \\
&= i^n 2\pi \delta^{(n)}(\lambda).
\end{aligned}$$

Recall Parseval's theorem: for suitable f, g

$$\int_{-\infty}^{\infty} \hat{g}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} g(x) \hat{f}(x) \mathrm{d}x.$$

Could define the Fourier transform of $g(x) = x$ to be the function $\lambda \mapsto \hat{x}(\lambda)$ such that

$$\int_{-\infty}^{\infty} \hat{x}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} x \hat{f}(x) \mathrm{d}x$$

for all 'nice' functions f. We can make this rigorous using distributions.

Discontinuous solutions to PDEs

From linear acoustics, air pressure $p = p(x,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0. \qquad (*)$$

Could introduce a 'nice' $f = f(x,t)$, say $f \in C_c^\infty(\mathbb{R}^2)$. Then $(*)$ implies

$$\int\int (p_{xx} - p_{tt}) f(x,t) \mathrm{d}x\mathrm{d}t = 0.$$

So we can integrate by parts to interpret this as

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x\mathrm{d}t = 0.$$

We say that $p = p(x,t)$ is a *weak solution* to $(*)$ if

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x\mathrm{d}t = 0$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. In each case, to extend a definition to a larger domain of applicability, we had to introduce a space of "nice" functions.

This is the theme of distribution theory: functions get replaced by linear maps on some auxilliary space of test functions V. A *distribution* is a linear map $u : V \to \mathbb{C}$, i.e we study the topological dual of V. Let $\langle \cdot, \cdot \rangle$ denote pairing between v and V^*, i.e for $u \in V^*$, $f, g \in V$, $\alpha, \beta \in \mathbb{C}$

$$\langle u, \alpha f + \beta g \rangle = \alpha \langle u, f \rangle + \beta \langle u, g \rangle.$$

The topological dual V^* consists of linear $u : V \to \mathbb{C}$ such that whenever $f_n \to f$ in V, we have $\langle u, f_n \rangle \to \langle u, f \rangle$ in $\mathbb{C}$. For example we could take $V = C^\infty(\mathbb{R})$ equipped with the topology of uniform convergence (i.e $f_n \to f$ in V if for all compact $K \subseteq \mathbb{R}$ and all $n \geq 0$, $\left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (f_n - f) \right| \to 0$) then $\delta_{x_0} : V \to bbC$ defined by $\langle \delta_{x_0}, f \rangle = f(x_0)$. Note that this is indeed continuous.

1 Distributions

1.1 Notation & Preliminaries

Throughout (unless otherwise specified) X, Y denote open subsets of $\mathbb{R}^n$, K a compact subset of $\mathbb{R}^n$. Integrals over $X, \mathbb{R}^n$ are written as $\int_X [\cdot] \mathrm{d}x$, $\int [\cdot] \mathrm{d}x$ respectively.

1.2 Distributions & Test Functions

Definition. The space $\mathcal{D}(X)$ consists of smooth functions $\varphi : X \to \mathbb{C}$ of compact support. We say a sequence $(\varphi_m)_{m \geq 0}$ in $\mathcal{D}(X)$ converges to 0 in $\mathcal{D}(X)$ if there exists $K \subseteq X$ compact such that $\operatorname{supp}(\varphi_m) \subseteq K$ and $\sup_K |\partial^\alpha \varphi_m| \to 0$ for all multi-indices α.

Functions in $\mathcal{D}(X)$ have nice properties. For example, if $\varphi \in \mathcal{D}(X)$ then $\varphi = 0$ before you reach the boundary of X. This means integration-by-parts is easy since

$$\int_X \varphi \partial^\alpha \psi \mathrm{d}x = (-1)^{|\alpha|} \int_X \psi \partial^\alpha \varphi \mathrm{d}x.$$

Since $\varphi \in \mathcal{D}(X)$ is smooth we have

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h)$$

where R_N is $o(|h|^N)$ uniformly in x.

Definition. A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is called a *distribution* if for all $K \subseteq X$ compact there exist $C, N \geq 0$ such that

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi| \tag{$*$}$$

for all $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$. The space of such linear maps is denoted by $\mathcal{D}'(X)$, i.e "distributions on X". If the same N can be used in $(*)$ for all compact $K \subseteq X$, say the least such N is the order of u, written $\operatorname{ord}(u)$.

For $x_0 \in X$ define $\delta_{x_0}(\varphi) = \varphi(x_0)$ for $\varphi \in \mathcal{D}(X)$. Then $\delta_{x_0} : \mathcal{D}(X) \to \mathbb{C}$ is linear and

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \sup |\varphi|$$

so we can take $C = 1, N = 0$ in $(*)$, so $\operatorname{ord}(\delta_{x_0}) = 0$.

For $\{f_\alpha\}$ in $C(X)$, define $T : \mathcal{D}(X) \to \mathbb{C}$ by

$$T(\varphi) = \sum_{|\alpha| \leq M} \int_X f_\alpha \partial^\alpha \varphi \mathrm{d}x.$$

Take $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$. Then

$$\begin{aligned} |T(\varphi)| &\leq \sum_{|\alpha|\leq M} \int_K |f_\alpha||\partial^\alpha \varphi| \mathrm{d}x \\ &\leq \left(\max_\alpha \int_K |f_\alpha| \mathrm{d}x\right) \sum_{|\alpha|\leq M} \sup |\partial^\alpha \varphi| \end{aligned}$$

so $(*)$ holds with $C = \max_\alpha \int_K |f_\alpha|\mathrm{d}x$, $N = M$. Hence $T \in \mathcal{D}'(X)$.

Note this estimate would hold if the $\{f_\alpha\}$ were only assumed locally integrable, written $f_\alpha \in L^1_{\text{loc}}(X)$.

Remark. For $f \in L^1_{\text{loc}}$ we have a corresponding distribution $T_f : \mathcal{D}(X) \to \mathbb{C}$ defined by $T_f(\varphi) = \int_X f\varphi \mathrm{d}x$. We often simply write $T_f = f$.

Lemma. *A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is a distribution if and only if $u(\varphi_m) \to 0$ for all sequences $\varphi_m \to 0$ in $\mathcal{D}(X)$.*

Proof. Suppose $u \in \mathcal{D}'(X)$ and $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then $\operatorname{supp}(\varphi_m) \subseteq K$ for some K independent of m and there exist $C, N \geq 0$

$$|\varphi_m(u)| \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha \varphi_m| \to 0$$

for all α.

Suppose not, i.e $u : \mathcal{D}(X) \to \mathbb{C}$ is linear and $u(\varphi_m) \to 0$ whenever $\varphi_m \to 0$ in $\mathcal{D}(X)$, but u is not a distribution. Then there is a compact set $K \subseteq X$ such that for all C, N, $(*)$ fails on some φ with support contained in K. So there must be some $\varphi_m \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi_m) \subseteq K$ and

$$|u(\varphi_m)| > m \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Now replace φ_m with $\varphi'_m = \frac{\varphi_m}{u(\varphi_m)}$. So we may assume $u(\varphi_m) = 1$ WLOG. Hence

$$1 > m \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Therefore $\sup_K |\partial^\alpha \varphi_m| < \frac{1}{m}$ for all $|\alpha| \leq m$. Hence $\varphi_m \to 0$ in $\mathcal{D}(X)$, giving a contradiction since $u(\varphi_m) \not\to 0$. □

1.3 Limits in $\mathcal{D}'(X)$

We often have some sequence (u_m) in $\mathcal{D}'(X)$. If there is some $u \in \mathcal{D}'(X)$ such that $\varphi(u_m) \to \varphi(u)$ for all φ we say $u_m \to u$ in $\mathcal{D}'(X)$.

Theorem (*Non-examinable*)**.** *If (u_m) is a sequence in $\mathcal{D}'(X)$ and $u(\varphi) = \lim_{m\to\infty} u(\varphi_m)$ exists for all $\varphi \in \mathcal{D}(X)$, then $u \in \mathcal{D}'(X)$.*

Proof. Not given. □

Take $u_m \in \mathcal{D}'(\mathbb{R})$ defined by $u_m(\varphi) = \int \sin(mx)\varphi(x)\mathrm{d}x$. By integration-by-parts we have

$$|\varphi(u_m)| = \left|\frac{1}{m}\int \cos(mx)\varphi'(x)\mathrm{d}x\right| \to 0.$$

i.e $\sin(mx) \to 0$ in $\mathcal{D}'(\mathbb{R})$.

1.4 Basic Operations

1.4.1 Differentiation & Multiplication by Smooth Functions

For $u \in C^\infty(X) \subseteq L^1_{\text{loc}}(X)$, $\partial^\alpha u \in D'(X)$ by

$$\begin{aligned}\langle \partial^\alpha u, \phi\rangle &= \int_X \phi \partial^\alpha u \mathrm{d}x \\ &= (-1)^{|\alpha|}\int_X u\partial^\alpha \phi \mathrm{d}x \\ &= (-1)^{|\alpha|}\langle u, \partial^\alpha \phi\rangle.\end{aligned}$$

This leads to

Definition. For $u \in D'(X)$, $f \in C^\infty(X)$ define

$$\langle \partial^\alpha (fu), \phi\rangle := (-1)^\alpha \langle u, f\partial^\alpha \phi\rangle$$

for $\phi \in \mathcal{D}(X)$ [note $\partial^\alpha(fu) \in \mathcal{D}'(X)$]. We call $\partial^\alpha u$ the *distributional derivatives* of u.

For δ_x we have

$$\begin{aligned}\langle \partial^\alpha \delta_x, \phi\rangle &= (-1)^{|\alpha|}\langle \delta_x, \partial^\alpha \phi\rangle \\ &= (-1)^{|\alpha|}\partial^\alpha \phi(x).\end{aligned}$$

Define the *Heaviside function*

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0\end{cases}.$$

Then $H \in L^1_{\text{loc}}(\mathbb{R})$ so

$$\langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_0^\infty \phi'(x)\mathrm{d}x = \phi(0) = \langle \delta_0, \phi\rangle.$$

Hence $H' = \delta_0$. Generally we say $u = v$ in $\mathcal{D}'(X)$ if $\langle u, \cdot\rangle = \langle v, \cdot\rangle$.

Lemma. *If $u \in \mathcal{D}'(\mathbb{R})$ and $u' = 0$ in $\mathcal{D}'(\mathbb{R})$ then u is constant.*

Proof. Fix $\theta \in \mathcal{D}(\mathbb{R})$ with $\langle 1, \theta\rangle = \int_{\mathbb{R}} \theta \mathrm{d}x = 1$. For $\phi \in \mathcal{D}(\mathbb{R})$ write

$$\phi = \underbrace{(\phi - \langle 1, \phi\rangle\theta)}_{:=\phi_A} + \underbrace{\langle 1, \phi\rangle\theta}_{:=\phi_B}.$$

Note that $\langle 1, \phi_A\rangle = \int_{\mathbb{R}} \phi_A \mathrm{d}x = 0$ so we have

$$\Phi_A(x) := \int_{-\infty}^x \phi_A(t)\mathrm{d}t$$

defines $\Phi_A \in \mathcal{D}(\mathbb{R})$ with $\Phi_A' = \phi_A$. So

$$\begin{aligned}\langle u, \phi\rangle &= \langle u, \phi_A\rangle + \langle u, \phi_B\rangle \\ &= \langle u, \Phi_A'\rangle + \langle 1, \phi\rangle\langle u, \theta\rangle \\ &= \underbrace{-\langle u', \phi_A\rangle + \langle 1, \phi\rangle}_{=0} + \underbrace{\langle 1, \phi\rangle\langle u, \theta\rangle}_{:=c \text{ constant}}\end{aligned}$$

so u is constant in $\mathcal{D}'(\mathbb{R})$. $\square$

1.4.2 Translation & Reflection

If $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ define *reflection* and *translation* by

$$\check{\phi}(x) = \phi(-x),\ (\tau_h\phi)(x) = \phi(x-h).$$

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ we define

$$\langle \check{u}, \phi\rangle = u, \check{\phi} \qquad \text{(reflection)}$$

and

$$\langle \tau_h u, \phi\rangle = \langle u, \tau_{-h}\phi\rangle \qquad \text{(translation)}$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

Lemma. *For $u \in \mathcal{D}'(\mathbb{R}^n)$ define*

$$v_h = \frac{\tau_{-h}u - u}{h}.$$

fIf $\frac{h}{|h|} \to m \in \mathbb{S}^{n-1}$ as $|h| \to 0$ then $v_h \to m \cdot \partial u$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. For $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\langle v_h, \phi\rangle = \langle u, \frac{\tau_h\phi - \phi}{h}\rangle.$$

By Taylor's theorem

$$(\tau_h\phi - \phi)(x) = \phi(x-h) - \phi(x) = -\sum_i h_i \frac{\partial\phi}{\partial x_i}(x) + R_1(x,h)$$

where $R_1 = o(|h|)$ in $\mathcal{D}(\mathbb{R}^n)$ [see Example Sheet 1] so by sequential continuity

$$\begin{aligned}\langle v_h, \phi\rangle &= -\sum_i \frac{h_i}{|h|}\langle u, \frac{\partial\phi}{\partial x_i}\rangle + o(1) \\ &= \langle \sum_i \frac{h_i}{|h|}\frac{\partial u}{\partial x_i}, \phi\rangle + o(1) \\ &\to \langle m \cdot \partial u, \phi\rangle \text{ as } |h| \to 0.\end{aligned}$$

$\square$

1.4.3 Convolution in $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$

For $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(\tau_x \check{\phi})(y) = \check{\phi}(y - x) = \phi(x - y).$$

If $u \in C^\infty(\mathbb{R}^n)$ define convolution with $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned} u * \phi(x) &= \int_{\mathbb{R}^n} u(x-y)\phi(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \phi(x-y)u(y)\mathrm{d}y \\ &= \langle u, \tau_x \check{\phi} \rangle. \end{aligned}$$

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ define

$$(u * \phi)(x) = \langle u, \tau_x \check{\phi} \rangle.$$

How regular is $u * \phi$?

Lemma. *For $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ write $\Phi_x(y) = \phi(x, y)$. If for each $x \in \mathbb{R}^n$ there exists a neighbourhood $N_x \subseteq \mathbb{R}^n$ of x and compact set $K \subseteq \mathbb{R}^n$ such that*

$$\operatorname{supp}(\phi|_{N_x \times \mathbb{R}^n}) \subseteq N_x \times K$$

then $\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi \rangle$ for $u \in \mathcal{D}'(\mathbb{R}^n)$.

Proof. By Taylor's theorem

$$\Phi_{x+h}(y) - \Phi_x(y) = \sum_i h_i \frac{\partial \phi}{\partial x_i}(x, y) + R_1(x, y, h).$$

For $|h|$ sufficiently small we have $x + h \in N_x$ so $\operatorname{supp}(R_1(x, \cdot, h)) \subseteq K$ and also

$$sup_y |\partial_y^\alpha R(x, y, h)| = o(|h|)$$

so $R_1(x, \cdot, h) = o(|h|)$ in $\mathcal{D}(\mathbb{R}^n)$. By sequential continuity

$$\langle u, \Phi_{x+h} \rangle - \langle u, \Phi_x \rangle = \sum_i h_i \langle u, \frac{\partial}{\partial x_i} \Phi_x \rangle + o(|h|)$$

and so $\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial}{\partial x_i} \Phi_x \rangle$ and the result follows by induction. $\square$

Corollary. *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ then $u * \phi \in C^\infty(\mathbb{R}^n)$ and*

$$\partial^\alpha(u * \phi) = u * \partial^\alpha \phi.$$

Proof. Have $(u * \phi)(x) = \langle u, \tau_x \check{\phi} \rangle$ so take $\Phi_x = \tau_x \check{\phi}$ in previous lemma. □

1.5 Density of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{D}'(\mathbb{R}^n)$

Can use previous result to prove an important theorem. First we need

Lemma. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$ then*

$$(u * \phi) * \psi = u * (\phi * \psi).$$

Proof. Fix $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
(u * \phi) * \psi(x) &= \int_{\mathbb{R}^n} (u * \phi)(x - y)\psi(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \langle u, \tau_{x-y}\check{\phi}\rangle \psi(y)\mathrm{d}y \\
&= \lim_{h \to 0} \sum_{m \in \mathbb{Z}^n} \langle u, \tau_{x-hm}\check{\phi}\psi(hm)\rangle h^n && \text{(Riemann sum)} \\
&= \lim_{h \to 0} \langle u, \sum_{m \in \mathbb{Z}_n} \tau_{x-hm}\check{\phi}\psi(hm)h^n\rangle && \text{(Finite sum)} \\
&= \langle u, \lim_{h \to 0} \sum_{m \in \mathbb{Z}_n} \tau_{x-hm}\check{\phi}\psi(hm)h^n\rangle \\
&= \langle u, \tau_x \phi \check{*} \psi\rangle \\
&= u * (\phi * \psi).
\end{aligned}$$

□

****Non-examinable****

We can justify the exchange of the limit and the $\langle u, \cdot \rangle$ by defining for $|h| \leq 1$ the family of functions $\{F_h\}$ by

$$F_h(z) = \sum_{m \in \mathbb{Z}^n} \phi(x - z - hm)\psi(hm)h^m.$$

It is straightforward to see that $\mathrm{supp}(F_h)$ lies in some fixed compact $K \subseteq \mathbb{R}^n$. Also each F_h is in $C^\infty(\mathbb{R}^n)$. Note that for each multi-index α we have

$$\sup_z |\partial^\alpha F_h(z)| \leq M_\alpha.$$

So for each α, $z \mapsto \partial^\alpha F_h(z)$ is uniformly bounded and equi-continuous. Equi-continuity follows from

$$\begin{aligned}|\partial^\alpha F_h(x) - \partial^\alpha F_h(y)| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \partial^\alpha F_h(tx + (1-t)y) \mathrm{d}t \right| \\ &= \left| \int_0^1 (x-y) \cdot \nabla \partial^\alpha F_h(tx + (1-t)y) \mathrm{d}t \right| \\ &\lesssim_\alpha |x-y|.\end{aligned}$$

Applying Arzela-Ascoli and a diagonal argument we get a sequence (h_k) such that $\sup_z |\partial^\alpha (F_{h_k} - \tau_x \phi \check{*} \psi)| \to 0$ for each α.

Theorem. *For $u \in \mathcal{D}'(\mathbb{R}^n)$ there exists (ϕ_k) in $\mathcal{D}(\mathbb{R}^n)$ such that $\phi_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$ (i.e $\langle u_k, \theta \rangle \to \langle u, \theta \rangle$ for all $\theta \in \mathcal{D}(\mathbb{R}^n)$).*

Proof. Fix $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi \mathrm{d}x = 1$ and set $\psi_k(x) = k^n \psi(kx)$. Fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi = 1$ on $[-1,1]$ and $\mathrm{supp}(\chi) \subseteq [-2,2]$. Set $\chi_k(x) = \chi(x/k)$. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and arbitrary $\theta \in \mathcal{D}(\mathbb{R}^n)$ consider $\langle \phi_k, \theta \rangle$ where $\phi_k = (u * \psi_k)\chi_k$. Then

$$\begin{aligned}\langle \phi_k, \theta \rangle &= \langle u * \psi_k, \chi_k \theta \rangle \\ &= (u * \psi_k) * (\check{\chi_k \theta})(0) \\ &= u * (\psi_j * (\check{\chi_k \theta}))(0) && \text{(previous lemma)}\end{aligned}$$

where we used the fact $\langle v, f \rangle = v * \check{f}(0)$. Note

$$\begin{aligned}\psi_k * (\check{\chi_j \theta})(x) &= \int k^n \psi(k(x-y)) \chi(-y/k) \theta(-y) \mathrm{d}y \\ &= \int \psi(y') \chi\left(\frac{y'}{k^2} - \frac{x}{k}\right) \theta\left(\frac{y'}{k} - x\right) \mathrm{d}y' && (y' = k(x-y)) \\ &= \theta(-x) + R_k(-x)\end{aligned}$$

where

$$R_k(x) = \int \psi(y) \left[\chi\left(\frac{y}{k^2} + \frac{x}{k}\right) \theta\left(\frac{y}{k} + x\right) - \theta(x) \right] \mathrm{d}y.$$

So

$$\begin{aligned}\langle \phi_k, \theta \rangle &= u * \check{\theta}(0) + u * \check{R}_k(0) \\ &= \langle u, \theta \rangle + \langle u, R_k \rangle.\end{aligned}$$

It is straightforward to show $R_k \to 0$ in $\mathcal{D}(\mathbb{R}^n)$ [exercise]. □

2 Distributions of Compact Support

Let $Y \subseteq X$ be open. We say $u \in \mathcal{D}'(X)$ vanishes on Y if $\langle u, \phi \rangle = 0$ for all $\phi \in \mathcal{D}(Y)$.

Definition. For $u \in \mathcal{D}'(X)$ define the support of u by

$$\operatorname{supp}(u) = X \setminus \left(\bigcup_{\substack{Y \subseteq X \text{ open} \\ u \text{ vanishes on } Y}} Y \right).$$

E.g for $\delta_x \in \mathcal{D}'(\mathbb{R}^n)$ we have $\operatorname{supp}(\delta_x) = \{x\}$.

****Non-examinable****

If $u \in \mathcal{D}'(X)$ vanishes on a collection $\{U_\lambda\}$ of open sets, then it vanishes on the union. Indeed suppose $\operatorname{supp}(\phi) \subseteq \bigcup_\lambda U_\lambda$. By compactness there is a finite collection $\{U_i\}_{i=1}^N$ such that $\operatorname{supp}(\phi) \subseteq \bigcup_{i=1}^N U_i$.

Take a partition of unity $\{\psi_i\}_{i=1}^N$ subordinate to $\{U_i\}_{i=1}^N$, i.e $\operatorname{supp}(\psi_i) \subseteq U_i$ and $\sum_{i=1}^N \psi_i = 1$. Then

$$\langle u, \phi \rangle = \sum_{i=1}^N \langle u, \psi_i \phi \rangle = 0.$$

A corollary of this is that $\operatorname{supp}(u)$ is the complement of the largest open set on which u vanishes.

2.1 More test functions & distributions

Definition. Define $\mathcal{E}(X)$ to be the space of smooth functions $\phi : X \to \mathbb{C}$. We say $\phi_m \to 0$ in $\mathcal{E}(X)$ if for each multi-index α we have $\partial^\alpha \phi_m \to 0$ locally uniformly, i.e $\sup_K |\partial^\alpha \phi| \to 0$ for all $K \subseteq X$ compact.

Definition. A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ if there exists $K \subseteq X$ compact and constants $C, N \geq 0$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi|$$

for all $\phi \in \mathcal{E}(X)$.

Lemma. *A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ if and only if $\langle u, \phi_m \rangle \to 0$ whenever $\phi_m \to 0$ in $\mathcal{E}(X)$.*

Proof. Almost identical to that of $\mathcal{D}'(X)$. □

Lemma. *If $u \in \mathcal{E}'(X)$ then $u|_{\mathcal{D}(X)}$ defines an element of $\mathcal{D}'(X)$ with compact support. Conversely if $u \in \mathcal{D}'(X)$ has compact support there exists a unique $\tilde{u} \in calE'(X)$ which extends u to $\mathcal{E}(X)$.*

Proof. Note that $\mathcal{D}(X) \subseteq \mathcal{E}(X)$ so if $u \in \mathcal{E}'(X)$ then $u|_{\mathcal{D}(X)}$ is well-defined. There exist compact $K \subseteq X$ and constants $C, N \geq 0$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi|$$

for all $\phi \in \mathcal{D}(X)$. Hence $u|_{\mathcal{D}(X)} \in \mathcal{D}'(X)$ and $\mathrm{supp}(u) \subseteq K$.

If $u \in \mathcal{D}'(X)$ has compact support, fix $\rho \in \mathcal{D}(X)$ such that $\rho = 1$ on a neighbourhood of $\mathrm{supp}(u)$. Define $\tilde{u} : \mathcal{E}(X) \to \mathbb{C}$ by $\langle \tilde{u}, \phi \rangle = \langle u, \rho\phi \rangle$ for each $\phi \in \mathcal{E}(X)$. Then $\mathrm{supp}(\rho\phi) \subseteq \mathrm{supp}(\phi)$. Since $u \in \mathcal{D}'(X)$ there exist constants $C, N \geq 0$ such that

$$\begin{aligned} \langle \tilde{u}, \phi \rangle| = |\langle u, \rho\phi \rangle| &\leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha (\rho\phi)| \\ &\leq C' \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi| \end{aligned}$$

so $\tilde{u} \in \mathcal{E}'(X)$. Suppose $\tilde{v} \in \mathcal{E}'(X)$ has $\tilde{v}|_{\mathcal{D}(X)} = u$ and $\mathrm{supp}(\tilde{v}) = \mathrm{supp}(u)$. With $\rho \in \mathcal{D}(X)$ as before

$$\begin{aligned} \langle \tilde{v}, \phi \rangle &= \langle \tilde{v}, \rho\phi \rangle + \langle \tilde{v}, (1-\rho)\phi \rangle \\ &= \langle \tilde{u}, \rho\phi \rangle + \langle \tilde{u}, (1-\rho)\phi \rangle \\ &= \langle \tilde{u}, \phi \rangle \end{aligned}$$

for all $\phi \in \mathcal{E}(X)$, i.e $\tilde{u} = \tilde{v}$. □

2.2 Convolution between $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$

For $\phi \in \mathcal{E}(\mathbb{R}^n)$, $u \in \mathcal{E}'(\mathbb{R}^n)$ define convolution as before by

$$u * \phi(x) = \langle u, \tau_x \check{\phi} \rangle.$$

We find $u*\phi \in \mathcal{E}(\mathbb{R}^n)$. Note that $u*\phi(x) = 0$ unless $(x-y) \in \operatorname{supp}(\phi)$ for some $y \in \operatorname{supp}(u)$, i.e $\operatorname{supp}(u*\phi) \subseteq \operatorname{supp}(\phi) + \operatorname{supp}(u)$. In particular if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have $u * \phi \in \mathcal{D}(\mathbb{R}^n)$.

Definition. Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$ where at least one of u, v has compact support. Then define

$$(u * v) * \phi := u * (v * \phi)$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then $u * v \in \mathcal{D}'(\mathbb{R}^n)$ [see Example Sheet 2].

Lemma. *For u, v as in the above definition. $u * v = v * u$.*

Proof. Recall by a previous lemma that if $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$ then $(u*\phi)*\psi = u*(\phi*\psi)$. The same holds if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\phi, \psi \in \mathcal{E}(\mathbb{R}^n)$ with at least one of $\operatorname{supp}(\phi), \operatorname{supp}(\psi)$ compact. We use this repeatedly as follows: for $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned}(u * v) * (\phi * \psi) &= u * [v * (\phi * \psi)] \\ &= u * [(v * \phi) * \psi] \\ &= u * [\psi * (v * \phi)] \\ &= (u * \psi) * (v * \phi).\end{aligned}$$

So using $\phi * \psi = \psi * \phi$ we have

$$\begin{aligned}(v * u) * (\phi * \psi) &= (v * \phi) * (u * \psi) \\ &= (u * \psi) * (v * \phi) \\ &= (u * v) * (\phi * \psi).\end{aligned}$$

So if $E = u * v - v * u$ we have $E * (\phi * \psi) = 0$ for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$. Thus $(E * \phi) * \psi = 0$ and $E * \phi = 0$, so $E = 0$ in $\mathcal{D}'(\mathbb{R}^n)$, i.e $u * v = v * u$. □

The above implies that for any $u \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$\delta_0 * u = u * \delta_0 = u$$

since for $\psi \in \mathcal{D}(\mathbb{R}^n)$

$$(u * \delta_0) * \psi = u * (\delta_0 * \psi) = u * \psi$$

and

$$\begin{aligned}(\delta_0 * \psi)(x) &= \langle \delta_0, \tau_x \check{\psi} \rangle \\ &= (\tau_x \check{\psi})(0) \\ &= \check{\psi}(-x) \\ &= \psi(x).\end{aligned}$$

3 Tempered Distributions & Fourier Analysis

3.1 More test functions & distributions

Definition. The *Schwartz space* written $\mathcal{S}(\mathbb{R}^n)$, consists of smooth $\phi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|\phi\|_{\alpha,\beta} := \sup |x^\alpha D^\beta \phi| < \infty$$

for all multi-indices α, β. We say $\phi_m \to 0$ in $\mathcal{S}$ if $\|\phi_m\|_{\alpha,\beta} \to 0$ for all α, β. Elements of the Schwartz space are sometimes called *rapidly decaying functions.*

Definition. A linear map $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$, the *space of tempered distributions*, if there exist $C, N \geq 0$ such that

$$|\langle u, \phi\rangle| \leq C \sum_{|\alpha|,|\beta} \|\phi\|_{\alpha,\beta}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Lemma. *A linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ iff $\langle u, \phi_m\rangle \to 0$ whenever $\phi_m \to 0$ in $\mathcal{S}$.*

Proof. Exercise. $\square$

Note that $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n)$ in the sense of continuous inclusions, i.e

$$\phi_m \xrightarrow{\mathcal{D}} 0 \implies \phi_m \xrightarrow{\mathcal{S}} 0 \implies \phi_m \xrightarrow{\mathcal{E}} 0.$$

Which gives the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$.

It turns out that $\mathcal{S}$ is ideal for Fourier analysis.

3.2 Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$

Definition. For an integrable function $f \in L^1(\mathbb{R}^n)$ define the *Fourier transform* of f by

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} f(x)\mathrm{d}x,\ \lambda \in \mathbb{R}^n.$$

We use $\mathcal{F}$ to denote the linear map $f \mapsto \hat{f}$.

Note that $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ since for $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\phi|\mathrm{d}x &= \int_{\mathbb{R}^n} (1+|x|)^{-N}(1+|x|)^N|\phi|\mathrm{d}x \\ &\leq C \sum_{|\alpha|\leq N} \|\phi\|_{\alpha,0} \int_{\mathbb{R}^n} (1+|x|)^{-N}\mathrm{d}x \\ &< \infty\end{aligned}$$

for $N \geq n+1$.

Lemma. *If $f \in L^1(\mathbb{R}^n)$ then $\hat{f} \in C(\mathbb{R}^n)$.*

Proof. DCT. $\square$

Intuitively, the Fourier transform interchanges decay & smoothness.
Notation: we write D^α for $(-i)^\alpha \nabla^\alpha$.

Lemma. *For $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\begin{aligned}(\widehat{D^\alpha \phi})(\lambda) &= \lambda^\alpha \hat{\phi}(\lambda) \\ (\widehat{x^\beta \phi})(\lambda) &= (-D)^\beta \hat{\phi}(\lambda).\end{aligned}$$

Proof. Integration-by-parts gives

$$\begin{aligned}(\widehat{D^\alpha \phi})(\lambda) &= \int_{\mathbb{R}}^n e^{-i\lambda\cdot x} D^\alpha \phi \mathrm{d}x \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi D^\alpha [e^{-i\lambda\cdot x}]\mathrm{d}x \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} (-1)^{|\alpha|} \phi \lambda^\alpha e^{-i\lambda\cdot x}\mathrm{d}x \\ &= \lambda^\alpha \hat{\phi}(\lambda)\end{aligned}$$

and

$$\begin{aligned}(-D)^\beta \hat{\phi}(\lambda) &= (-D)^\beta \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} \phi(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} x^\beta e^{-i\lambda\cdot x}\phi(x)\mathrm{d}x \qquad \text{(DCT)} \\ &= (\widehat{x^\beta \phi})(\lambda).\end{aligned}$$

$\square$

Note that the above show that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

Theorem. *The Fourier transform is a continuous isomorphism on $\mathcal{S}(\mathbb{R}^n)$, i.e $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a homeomorphism.*

Proof. We know $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ so $C^\infty(\mathbb{R}^n)$. We also have

$$\begin{aligned}\left|\lambda^\alpha D^\beta \hat{\phi}(\lambda)\right| &= \left|\int_{\mathbb{R}^n} D^\alpha (x^\beta \phi) e^{-i\lambda\cdot x}\mathrm{d}x\right| \\ &\leq \int_{\mathbb{R}^n} |D^\alpha(x^\beta\phi)|\mathrm{d}x < \infty. \qquad (\dagger)\end{aligned}$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have $D^\alpha(x^\beta\phi) \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. Hence $\|\hat{\phi}\|_{\alpha,\beta} < \infty$ for all α, β, i.e $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$. Hence $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ to itself.

By suitably applying (†) to a sequence $\phi_m \to 0$ in $\mathcal{S}$, it's easy to see $\hat{\phi}_m \to 0$ in $\mathcal{S}$. We have

$$\int_{\mathbb{R}^n} e^{i\lambda\cdot x}\hat{\phi}(\lambda)\mathrm{d}\lambda = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} e^{i\lambda\cdot x}e^{-\varepsilon|\lambda|^2}\hat{\phi}(\lambda)\mathrm{d}\lambda.$$

Also

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{i\lambda\cdot x-\varepsilon|\lambda|^2}\hat{\phi}(\lambda)\mathrm{d}\lambda &= \int_{\mathbb{R}^n}\phi(y)\left[\int_{\mathbb{R}^n} e^{i\lambda\cdot(x-y)-\varepsilon|\lambda|^2}\mathrm{d}\lambda\right]\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\phi(y)\left[\prod_{j=1}^n\left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{-(x_j-y_j)^2/4\varepsilon}\right]\mathrm{d}y \qquad (*)\\
&= \int_{\mathbb{R}^n}\phi(y)\left(\frac{\pi}{\varepsilon}\right)^{n/2} e^{-|x-y|^2/4\varepsilon}\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\phi(x-2\sqrt{\varepsilon}y)\pi^{n/2}2^n e^{-|y'|^2}\mathrm{d}y' \qquad \left(y=\tfrac{x-y}{2\sqrt{\varepsilon}}\right)\\
&\xrightarrow{\varepsilon\downarrow 0}\phi(x)(2\pi)^n\left(\frac{1}{\sqrt{\pi}}\right)^n\int_{\mathbb{R}^n} e^{-|y|^2}\mathrm{d}y \\
&= (2\pi)^n\phi(x).
\end{aligned}$$

Thus $\phi(-x) = \mathcal{F}\left[\frac{\hat{\phi}}{(2\pi)^n}\right]$. So we get a homeomorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. $(*)$ follows from

$$\int_{\mathbb{R}^n} e^{i\lambda\cdot(x-y)-\varepsilon|\lambda|^2}\mathrm{d}\lambda = \prod_{j=1}^n\int_{\mathbb{R}} e^{i\lambda_j\cdot(x_j-y_j)}e^{-\varepsilon\lambda_j^2}\mathrm{d}\lambda_j$$

followed by

$$\begin{aligned}
\int_{\mathbb{R}} e^{i\lambda\sigma}e^{-\varepsilon\lambda^2}\mathrm{d}\lambda &= \int_{\mathbb{R}} e^{-\varepsilon\left(\lambda-\frac{i\sigma}{2\varepsilon}\right)^2-\frac{\sigma^2}{4\varepsilon}}\mathrm{d}\lambda \\
&= e^{-\frac{\sigma^2}{4\varepsilon}}\int_{\mathbb{R}} e^{-\varepsilon\left(\lambda-\frac{i\sigma}{2\varepsilon}\right)^2}\mathrm{d}\lambda.
\end{aligned}$$

□

3.3 Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$

Proposition. If $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} \phi(x)\hat{\psi}(x)\mathrm{d}x = \int_{\mathbb{R}^n} \hat{\phi}(x)\psi(x)\mathrm{d}x.$$

Proof. We have

$$\begin{aligned} \int_{\mathbb{R}^n} \phi(x)\hat{\psi}(x)\mathrm{d}x &= \int_{\mathbb{R}^n} \phi(x) \left[\int_{\mathbb{R}^n} e^{-i\lambda\cdot x}\psi(\lambda)\mathrm{d}\lambda\right] \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \psi(\lambda) \int_{\mathbb{R}^n} e^{-\lambda\cdot x}\phi(x)\mathrm{d}x\mathrm{d}\lambda \qquad \text{(Fubini)} \\ &= \int_{\mathbb{R}^n} \psi(\lambda)\hat{\phi}(\lambda)\mathrm{d}\lambda. \end{aligned}$$

□

If $u \in \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ then the previous lemma states

$$\langle \hat{u}, \phi\rangle = \langle u, \hat{\phi}\rangle \ \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Since $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, the RHS is well-defined for any $u \in \mathcal{S}'(\mathbb{R}^n)$.

Definition. For $u \in \mathcal{S}'(\mathbb{R}^n)$ define $\hat{u}$ by

$$\langle \hat{u}, \phi\rangle := \langle u, \hat{\phi}\rangle \ \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Example. Take $u = \delta_0$ so

$$\langle \hat{\delta_0}, \phi\rangle = \langle \delta_0, \hat{\phi}\rangle = \hat{\phi}(0) = \int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = \langle 1, \phi\rangle$$

hence $\hat{\delta_0} = 1$ in $\mathcal{S}'(\mathbb{R}^n)$. Also

$$\langle \hat{1}, \phi\rangle = \langle 1, \hat{\phi}\rangle = \int_{\mathbb{R}^n} \hat{\phi}(x)\mathrm{d}x = (2\pi)^n\phi(0) = \langle (2\pi)^n\delta_0, \phi\rangle$$

implying $\hat{1} = (2\pi)^n\delta_0$. Therefore

$$\text{“}\delta_0(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\lambda\cdot x}\mathrm{d}\lambda\text{”}.$$

It is straightforward to show

$$\begin{aligned} (D^{\hat{\alpha}}u) &= \lambda^\alpha\hat{u} \\ (x^{\hat{\beta}}u) &= (-D)^\beta\hat{u}. \end{aligned}$$

Theorem. *The Fourier transform defines a continuous bijection $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.*

Proof. Note

$$\check{u} = \frac{1}{(2\pi)^n}\hat{\hat{u}}.$$

Indeed

$$\begin{aligned}\langle \check{u}, \phi\rangle = \langle u, \check{\phi}\rangle &= \langle u, (2\pi)^{-n}\hat{\hat{\phi}}\rangle \qquad (*)\\ &= \langle (2\pi)^{-n}\hat{\hat{u}}, \phi\rangle\end{aligned}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, where $(*)$ follows by Fourier inversion. Hence $\mathcal{F}(\mathcal{S}'(\mathbb{R}^n)) \subseteq \mathcal{S}'(\mathbb{R}^n)$, note $\phi_m \xrightarrow{\mathcal{S}} 0$ iff $\hat{\phi}_m \xrightarrow{\mathcal{S}} 0$,so

$$\langle \hat{u}, \phi_m\rangle = \langle u, \hat{\phi}_m\rangle \to 0$$

whenever $\phi_m \xrightarrow{\mathcal{S}} 0$, i.e $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$. For continuity of $\mathcal{F}$, suppose $u_m \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$, i.e $u_m(\phi) \to 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. This happens if and only if $\langle u_m, \hat{\phi}\rangle \to 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ since $\mathcal{F}$ is a bijection from $\mathcal{S}(\mathbb{R}^n)$ to itself, so $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$. □

3.4 Sobolev Space

Definition. For $s \in \mathbb{R}$ define the *Sobolev Space* $H^s(\mathbb{R}^n)$ to be the $u \in \mathcal{S}'(\mathbb{R}^n)$ for which $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ can be identified with a measurable function $\lambda \mapsto \hat{u}(\lambda)$ that satisfies

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}^n} (1+|\lambda|^2)^s |\hat{u}(\lambda)|^2 \mathrm{d}\lambda < \infty.$$

We will use notation

$$\langle\lambda\rangle = (1+|\lambda|^2)^{1/2}$$

so $\lambda \sim |\lambda|$ as $|\lambda| \to \infty$. We see that $u \in H^s(\mathbb{R}^n)$ iff $\langle\lambda\rangle^s \hat{u} \in L^2(\mathbb{R}^n)$.

Lemma. *If $u \in H^s(\mathbb{R}^n)$ and $s > \frac{n}{2}$ then $u \in C(\mathbb{R}^n)$ (i.e u can be identified with a $C(\mathbb{R}^n)$ function).*

Proof. We establish that $\hat{u} \in L^1(\mathbb{R}^n)$. Indeed

$$\begin{aligned}\int_{\mathbb{R}^n} |\hat{u}(\lambda)|\mathrm{d}\lambda &= \left(\int_{\mathbb{R}^n} \langle\lambda\rangle^{-2s}\mathrm{d}\lambda\right)^{1/2} \left(\int_{\mathbb{R}^n} \langle\lambda\rangle^{2s}|\hat{u}(\lambda)|^2\mathrm{d}\lambda\right)^{1/2}\\ &= \left(\int_{S^{n-1}} \mathrm{d}\sigma \underbrace{\int_0^\infty (1+r^2)^{-s} r^{n-1}\mathrm{d}r}_{(\dagger)}\right)^{1/2} \|u\|_{H^s}\end{aligned}$$

where $\mathrm{d}\sigma$ is the surface element on the sphere S^{n-1}. Note $(\dagger)$ is $calO(r^{-2s+n-1})$ as $r \to \infty$ so the integral is finite if $s > n/2$. We cannot yet invoke the inverse

Fourier transform since we only proved that it works on $\mathcal{S}(\mathbb{R}^n)$. We have

$$\begin{aligned}\langle u, \hat{\phi}\rangle = \langle \hat{u}, \phi\rangle &= \int_{\mathbb{R}^n} \hat{u}(\lambda)\phi(\lambda)\mathrm{d}\lambda \\ &= \int_{\mathbb{R}^n} \hat{u}(\lambda)(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\lambda\cdot x}\hat{\phi}(x)\mathrm{d}x\mathrm{d}\lambda && \text{(inverse FT)} \\ &= \int_{\mathbb{R}^n} \hat{\phi}(x)(2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\lambda\cdot x}\hat{u}(\lambda)\mathrm{d}\lambda\mathrm{d}x && \text{(Fubini)} \\ &= \int_{\mathbb{R}^n} u(x)\hat{\phi}(x)\mathrm{d}x\end{aligned}$$

where

$$u(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\lambda\cdot x}\hat{u}(\lambda)\mathrm{d}\lambda.$$

Since $\hat{u} \in L^1(\mathbb{R}^n)$ the DCT implies $u(x) \in C(\mathbb{R}^n)$. □

Corollary. *If $u \in H^s(\mathbb{R}^n)$ for all $s > n/2$ then $u \in C^\infty(\mathbb{R}^n)$.*

Proof. Replace u with $D^\alpha u$ and show $\hat{(}D^\alpha u) = \lambda^\alpha \hat{u} \in L^1(\mathbb{R}^n)$ to conclude $D^\alpha u \in C(\mathbb{R}^n)$. □

When understanding regularity it suffices to confine attention to things of the form ϕu for $\phi \in \mathcal{D}(\mathbb{R}^n)$. Very rarely do we need to study u in isolation. Hence if $u \in \mathcal{D}'(X)$ for $X \supsetneq \mathbb{R}^n$ we can consider $\phi u \in \mathcal{D}'(X)$, $\phi \in \mathcal{D}(X)$ and make the extension $(\phi u)_{\text{ext}} \in \mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$.

Definition. We say $u \in \mathcal{D}'(X)$ belongs to the *local Sobolev space* $H^s_{\mathrm{loc}}(X)$ if $u\phi$ extends to an element of $H^s(\mathbb{R}^n)$ for each $\phi \in \mathcal{D}(X)$.

Note we interpret $\phi u \in \mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \phi u, \psi \rangle := \langle u, \phi\psi \rangle$$

which is well defined as $\mathrm{supp}(\phi\psi) \subseteq X$.

4 Applications of the Fourier Transform

4.1 Elliptic regularity

We're interested in problems of the form

$$P(D)u = f$$

where $u, f \in \mathcal{D}'(X)$ and P is a polynomial in n variables. For example if $P(\lambda) = \lambda_1^2 + \ldots + \lambda_n^2$ we have $P(D) = -\left(\frac{\partial}{\partial x_1}\right)^2 - \ldots - \left(\frac{\partial}{\partial x_n}\right)^2 = -\Delta$.

We are interested in the following question. If $f \in H^s_{\mathrm{loc}}(X)$, can we say that $u \in H^t_{\mathrm{loc}}(X)$ for some $t = t(s, P)$? We will answer this when P is elliptic.

Definition. An Nth order partial differential operator (P.D.O)

$$P(D) = \sum_{|\alpha| \leq N} C_\alpha D^\alpha$$

has *principal symbol* defined by

$$\sigma_P(\lambda) = \sum_{|\alpha| = N} c_\alpha \lambda^\alpha.$$

We say P is *elliptic* if $\sigma_P(\lambda) \neq 0$ on $\mathbb{R}^n \setminus \{0\}$.

Lemma. *If $P(D)$ is Nth order elliptic then for $|\lambda|$ sufficiently large, $|P(\lambda)| \gtrsim \langle \lambda \rangle^N$.*

Proof. By continuity and compactness, since $\sigma_p(\lambda)$ doesn't vanish on S^{n-1} we must have $\min_{|\lambda|=1} |\sigma_P(\lambda)| = C > 0$. Then for $\lambda \in \mathbb{R}^n \setminus \{0\}$

$$|\sigma_P(\lambda)| = |\lambda|^N \sum_{|\alpha|=N} C_\alpha (\lambda/|\lambda|)^\alpha \geq C|\lambda|^N.$$

By the triangle inequality

$$\begin{aligned} |P(\lambda)| &\geq |\sigma_P(\lambda)| - |P(\lambda) - \sigma_P(\lambda)| \\ &\geq \left[C - \frac{|P(\lambda) - \sigma_P(\lambda)|}{|\lambda|^N} \right] |\lambda|^N \\ &\geq \frac{C}{2} |\lambda|^N \end{aligned}$$

for $|\lambda|$ sufficiently large. Then $|P(\lambda)| \geq \frac{C}{2}|\lambda|^N \gtrsim \langle \lambda \rangle^N$. $\square$

Theorem. *If $P(D)$ is Nth order elliptic and $P(D)u \in H^s_{loc}(X)$, then $u \in H^{s+N}_{loc}(X)$.*

Now we will prove an easier version of this theorem, relevant if $u \in \mathcal{E}'(\mathbb{R}^n)$. We will use the fact that if $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\hat{u} \in \mathcal{E}(\mathbb{R}^n)$ and $|\hat{u}(\lambda)| \lesssim \langle\lambda\rangle^M$ for some $M \geq 0$.

When $u \in \mathcal{E}'(\mathbb{R}^n)$ we can use *parametrix* to prove a version of this theorem.

Definition. Say that $E \in \mathcal{D}'(\mathbb{R}^n)$ is a *parametrix* for $P(D)$ if there exists $\omega \in \mathcal{E}(\mathbb{R}^n)$ such that

$$P(D)E = \delta_0 + \omega.$$

Lemma. *Every (non-zero) elliptic $P(D)$ admits a parametrix $E \in \mathcal{E}(\mathbb{R}^n \setminus \{0\})$.*

Proof. Fix $R > 0$ so that $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$ for $|\lambda| > R$ and fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi = 1$ on $|\lambda| \leq R$ and $\chi = 0$ on $|\lambda| > R + 1$.

Define $E \in \mathcal{S}'(\mathbb{R}^n)$

$$\hat{E}(\lambda) = \frac{1 - \chi(\lambda)}{P(\lambda)}.$$

Then E is smooth and $|\hat{E}| \lesssim \langle\lambda\rangle^{-N}$ for $|\lambda| > R$, so $\hat{E} \in \mathcal{S}'(\mathbb{R}^n)$ and thus $E \in \mathcal{S}'(\mathbb{R}^n)$. By the inverse Fourier transform

$$P(D)E = \delta_0 + \omega$$

where

$$\hat{\omega} = -\chi \in \mathcal{D}(\mathbb{R}^n) \implies \omega \in \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n).$$

For $|\lambda| > R + 1$ have

$$\begin{aligned} |\widehat{(x^\beta E)}(\lambda)| &= |D^\beta \hat{E}(\lambda)| \\ &= \left| D^\beta \left(\frac{1}{P(\lambda)} \right) \right| \\ &\lesssim \langle\lambda\rangle^{-N-|\beta|} && \text{(induction)} \end{aligned}$$

so for every $s \in \mathbb{R}$ (in particular $s > n/2$) there is a β such that $x^\beta E \in H^s(\mathbb{R}^n)$. So for each α, $D^\alpha(x^\beta E)$ is continuous for $|\beta|$ sufficiently large [Sobolev lemma]. Hence E is smooth away from $x = 0$, i.e $E \in \mathcal{E}(\mathbb{R}^n \setminus \{0\})$. □

We now prove an easy version of

Theorem. *If $P(D)$ is Nth order elliptic and $P(D)u \in H^s_{loc}(X)$, then $u \in H^{s+N}_{loc}(X)$.*

Proof for special case. If $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\hat{u} \in \mathcal{E}(\mathbb{R}^n)$, using

$$P(\lambda)\hat{E}(\lambda) = 1 + \hat{\omega}$$

i.e $1 = P(\lambda)\hat{E} - \hat{\omega}$. Therefore $\hat{u} = [P(\lambda)\hat{u}]\hat{E} - \hat{\omega}\hat{u}$. Also $\langle\lambda\rangle^s P(\lambda)\hat{u} \in L^2$, and $\hat{E} \lesssim \langle\lambda\rangle^{-N}$. Furthermore $\hat{\omega} = o(\langle\lambda\rangle^{-k})$ for all k and $\hat{u} = \mathcal{O}(\langle\lambda\rangle^M)$ for some M. Hence

$$\langle\lambda\rangle^{s+N}\hat{u} = [\langle\lambda\rangle^s P(\lambda)\hat{u}]\hat{E}(\lambda)\langle\lambda\rangle^N - \hat{\omega}\hat{u}\langle\lambda\rangle^{s+N}$$

and we see that $\|\hat{u}\langle\lambda\rangle^{s+N}\|_{L^2} < \infty$, i.e $u \in H^{s+N}(\mathbb{R}^n)$. $\square$

Now we'll give a full proof

Proof. We use the following facts from Example Sheet 2:

- if $u \in \mathcal{E}'(\mathbb{R}^n)$ then there exists $t \in \mathbb{R}$ with $u \in H^t(\mathbb{R}^n)$;
- if $u \in H^s(\mathbb{R}^n)$ then $D^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$;
- if $s > t$ then $H^s(\mathbb{R}^n) \subseteq H^t(\mathbb{R}^n)$;
- if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n)$ then $\phi u \in H^s(\mathbb{R}^n)$.

Fix $\phi \in \mathcal{D}(X)$. Introduce test functions $\psi_0, \psi_1, \ldots, \psi_M$ such that $\psi_{i-1} = 1$ on $\operatorname{supp}(\psi_i)$ and $\operatorname{supp}(\phi) \subseteq \operatorname{supp}(\psi_M) \subseteq \ldots \operatorname{supp}(\psi_0)$.

Note $\psi_0 u \in \mathcal{E}'(\mathbb{R}^n)$ so $\psi_0 u \in H^t(\mathbb{R}^n)$ for some t. Then

$$\begin{aligned} P(D)[\psi_1 u] &= \psi_1 P(D)u + [P(D), \psi_1](u) \\ &= \psi_1 P(D)u + [P(D), \psi_1](\psi_0 u) \end{aligned}$$

since $\psi_0 = 1$ on $\operatorname{supp}(\psi_1)$. Because $P(D)u \in H^s(\mathbb{R}^n)$, we have $\psi_1 P(D)u \in H^s(\mathbb{R}^n)$, and also $[P(D), \psi_1](\psi_0 u) \in H^{t-N+1}(\mathbb{R}^n)$ since $\psi_0 u \in H^t(\mathbb{R}^n)$. Therefore

$$P(D)[\psi_1 u] \in H^{\tilde{A}_1}(\mathbb{R}^n)$$

where $\tilde{A}_1 = \min\{s, t - N + 1\}$, i.e

$$\int_{\mathbb{R}^n} \langle\lambda\rangle^{2\tilde{A}_1} |P(\lambda)[\psi_1 u]^n(\lambda)|^2 \mathrm{d}\lambda < \infty. \tag{†}$$

Since $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$, (†) implies

$$\int_{\mathbb{R}^n} \langle\lambda\rangle^{2(\tilde{A}_1+N)} |[\psi_1 u]^n(\lambda)|^2 \mathrm{d}\lambda < \infty$$

i.e $\psi_1 u \in H^{A_1}(\mathbb{R}^n)$ where $A_1 = \tilde{A}_1 + N = \min\{s + N, t + 1\}$. Similarly

$$\begin{aligned} P(D)[\psi_2 u] &= \psi_2 P(D)u + [P(D), \psi_2](u) \\ &= \psi_2 P(D)u + [P(D), \psi_2](\psi_1 u) \end{aligned}$$

and since $\psi_1 u \in H^{A_1}(\mathbb{R}^n)$, by the same argument we get $\psi_2 u \in H^{A_2}(\mathbb{R}^n)$ where $A_2 = \min\{s+N, A_1+1\} = \min\{s+N, \min\{s+N+1, t+2\}\} = \min\{s+N, t+2\}$. Proceeding inductively, $\psi_M u \in H^{A_M}(\mathbb{R}^n)$ where $A_m = \min\{s + N, t + M\} = s + N$ for M large enough. Since $\psi_M = 1$ on $\operatorname{supp}(\phi)$ we get $\phi u \in H^{s+N}(\mathbb{R}^n)$. As ϕ was arbitrary we see $u \in H^{s+N}_{\mathrm{loc}}(X)$. $\square$

4.2 Fundamental Solutions

To solve problems of the form $P(D)u = f$ we can use fundamental solutions.

Definition. We say $E \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution for $P(D)$ if $P(D)E = \delta_0$.

Lemma. *The fundamental solution for*

$$P(D) := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$

is given by $E = \frac{1}{\pi z}$.

Proof. We have $E \in L^1_{\text{loc}}(\mathbb{R}^2)$. For $\phi \in \mathcal{D}(\mathbb{R}^2)$ we have

$$\begin{aligned}
\langle \frac{\partial}{\partial \bar{z}} E, \phi \rangle &= -\langle E, \frac{\partial \phi}{\partial \bar{z}} \rangle \\
&= -\lim_{\varepsilon \downarrow 0} \int_{|z|>\varepsilon} frac1\pi z \frac{\partial \phi}{\partial \bar{z}} \mathrm{d}x && \text{(DCT)} \\
&= -\lim_{\varepsilon \downarrow 0} \int_{|z|>\varepsilon} \frac{\partial}{\partial \bar{z}} \left(\frac{\phi}{\pi z}\right) \mathrm{d}x \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{\phi}{z} \mathrm{d}z && \text{(Green's theorem)} \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\phi(\varepsilon\cos(\theta), \varepsilon\sin(\theta)) i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}} \mathrm{d}\theta \\
&= \frac{1}{2\pi} 2\pi \phi(0,0) \\
&= \langle \delta_0, \phi \rangle.
\end{aligned}$$

□

Lemma. *The fundamental solution for the heat operator*

$$P(D) = \frac{\partial}{\partial t} - \Delta_x$$

on $\mathbb{R}^n \times \mathbb{R}$ *is*

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{2t}\right) & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

Proof. Note that

$$P(D)E = 0$$

on $t \geq \varepsilon > 0$ (check). For $\phi \in \mathcal{D}(\mathbb{R}^{n+1})$ we have

$$\begin{aligned}
\langle\left(\frac{\partial}{\partial t} - \Delta_x\right) E, \phi\rangle &= -\langle E, \left(\frac{\partial}{\partial t} + \Delta_x\right)\phi\rangle \\
&= -\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty \int_\varepsilon^\infty \int_{\mathbb{R}^n} E(x,t)[\phi_t + \Delta_x\phi]\mathrm{d}x\mathrm{d}t && \text{(DCT)} \\
&= -\lim_{\varepsilon\downarrow 0}\left[\int_{\mathbb{R}^n} E(t,x)\phi(t,x)|_{t=\varepsilon}^\infty + \int_\varepsilon^\infty \int_{\mathbb{R}^n}\phi[\underbrace{E_t - \Delta_x E}_{=0}]\mathrm{d}x\mathrm{d}t\right] \\
&= \lim \varepsilon \downarrow 0 \int_{\mathbb{R}^n} (4\pi\varepsilon)^{-n/2} e^{-\frac{|x|^2}{2\varepsilon}}\phi(x,\varepsilon)\mathrm{d}x \\
&= \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n}(2\pi)^{-n/2}e^{-|y|^2}\phi(\sqrt{2\varepsilon}y,\varepsilon)\mathrm{d}y && (y = \sqrt{2\varepsilon}x) \\
&= \phi(0,0)\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|y|^2}\mathrm{d}y \\
&= \langle\delta_0, \phi\rangle.
\end{aligned}$$

□

We will try to construct a surface $\Sigma \subseteq \mathbb{C}^n$ such that $\Sigma \simeq \mathbb{R}^n$ (homotopic) and for which

$$\langle E, \phi \rangle = (2\pi)^{-n} \int_\Sigma \frac{\hat{\phi}(-\lambda)}{P(\lambda)} \mathrm{d}\lambda$$

defines an element of $\mathcal{D}'(\mathbb{R}^n)$. Note then

$$\begin{aligned} \langle P(D)E, \phi \rangle &= \langle E, P(-D)\phi \rangle \\ &= (2\pi)^{-n} \int_\Sigma \frac{P(\lambda)\hat{P}(-\lambda)}{P(\lambda)} \mathrm{d}\lambda \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi}(-\lambda) \mathrm{d}\lambda \qquad (\dagger) \\ &= \phi(0) \end{aligned}$$

where we hope Σ is nice enough that ($\dagger$) holds, using complex analysis and $\Sigma \simeq \mathbb{R}^n$. We will call Σ *Hörmander's Staircase.*

Lemma. *For $\lambda \in \mathbb{R}^n$ write $\lambda = (\lambda', \lambda_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For each $\lambda' \in \mathbb{R}^{n-1}$, if $\phi \in \mathcal{D}(\mathbb{R}^n)$ then*

$$\mathbb{C} \ni z \mapsto \hat{\phi}(\lambda', z)$$

is holomorphic and there exists $\delta > 0$ such that

$$|\hat{\phi}(\lambda', z)| \lesssim_m (1 + |z|)^{-m} e^{\delta |\Im(z)|}$$

for $m = 0, 1, 2, \ldots$, i.e we have fast decay at horizontal infinity so $\int_{\mathbb{R}+i\eta} \hat{\phi}(\lambda', z)\mathrm{d}z = \int_{\mathbb{R}} \hat{\phi}(\lambda', \lambda_n)\mathrm{d}\lambda_n$ for all $\eta \in \mathbb{R}$ by Cauchy's theorem.

Theorem. *For every non-zero $P(D)$ there exists a fundamental solution.*

Proof. By scaling and rotating coordinate axes can assume $P(\lambda)$ has the form

$$P(\lambda', \lambda_n) = \lambda_n^M + \sum_{m=0}^{M-1} a_m(\lambda')\lambda_n^M.$$

Let us fix $\mu' \in \mathbb{R}^{n-1}$. Then

$$P(\mu', \lambda_n) = \prod_{i=1}^{M} (\lambda_n - \tau_i(\mu'))$$

where $\{\tau_i(\mu')\}_i$ are the zeros of the polynomial $\lambda_n \to P(\mu', \lambda_n)$. We claim there exists a horizontal line $\Im\lambda_n = c(\mu')$ in the complex λ_n-plane, inside the strip $|\Im(\lambda_n)| \leq M + 1$ such that

$$|\Im(\lambda_n - \tau_i(\mu'))| > 1$$

for $i = 1, \ldots, M$. Indeed, $|\Im(\lambda_n)| \leq M + 1$ consists of $M + 1$ strips of width 2. So by the pigeonhole principle one of these strips contains no roots. So choose our horizontal line to bisect an empty strip. Hence

$$|P(\mu', \lambda_n)| > 1$$

on $\Im(\lambda_n) = c(\mu')$. Since the set of roots varies continuously in the coefficients of the polynomial, we deduce that the same statement holds for λ' in a sufficiently small open neighbourhood of μ', say $N(\mu')$. So we get

$$|P(\lambda', \lambda_n)| > 1 \text{ for } \Im(\lambda_n) = c(\mu'), \quad \lambda' \in N(\mu').$$

We can do this for every $\mu' \in \mathbb{R}^{n-1}$, to obtain an open cover $\{N(\mu')\}_{\mu' \in \mathbb{R}^{n-1}}$. By compactness we can extract a locally finite subcover $N_1 = N(\mu'_1), N_2 = N(\mu'_2), \ldots$ of $\mathbb{R}^{n-1}$. We have

$$|P(\lambda', \lambda_n)| > 1 \text{ on } \Im(\lambda_n) = c_i = c(\mu'_i), \quad \lambda' \in N_i.$$

Define open sets inductively by $\Delta_1 = N_1$ and $\Delta_i = N_i \setminus (\overline{N}_1 \cup \ldots \cup \overline{N}_{i-1})$. Now we have that $\{\Delta_i\}$ are open, disjoint and $\bigcup_i \overline{\Delta}_i = \mathbb{R}^{n-1}$ and

$$|P(\lambda', \lambda_n)| > 1 \text{ on } \Im(\lambda_n) = c_i, \quad \lambda' \in \Delta_i.$$

Now define

$$\langle E, \phi \rangle = (2\pi)^{-n} \sum_{i=1}^{\infty} \int_{\Delta_i} \int_{\Im(\lambda_n) = c_i} \frac{\hat{\phi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} \mathrm{d}\lambda_n \mathrm{d}\lambda'$$

then

$$\begin{aligned}
\langle P(D)E, \phi \rangle &= (2\pi)^{-n} \sum_{i=1}^{\infty} \int_{\Delta_i} \int_{\Im(\lambda_n) = c_i} \frac{P(\lambda', \lambda_n)\hat{\phi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} \mathrm{d}\lambda_n \mathrm{d}\lambda' \\
&= (2\pi)^{-n} \sum_{i=1}^{\infty} \int_{\Delta_i} \int_{\mathbb{R}} \hat{\phi}(-\lambda', -\lambda_n) \mathrm{d}\lambda_n \mathrm{d}\lambda \qquad \text{(Lemma+Cauchy)} \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \hat{\phi}(-\lambda', -\lambda_n) \mathrm{d}\lambda_n \mathrm{d}\lambda' \\
&= \phi(0) \\
&= \langle \delta_0, \phi \rangle.
\end{aligned}$$

Can show that E does indeed define a distribution [see Example Sheet 3] so $P(D)E = \delta_0$. $\square$

The existence of fundamental solutions is called the Malgrange-Ehrenpries theorem.

4.3 Structure Theorem for $\mathcal{E}'(X)$

We know that if $f \in C(X)$ then $\partial^\alpha f \in \mathcal{D}'(X)$ with

$$\langle \partial^\alpha f, \phi \rangle = (-1)^{|\alpha|} \int_X f \partial^\alpha \phi \mathrm{d}x$$

for all $\phi \in \mathcal{D}(X)$. Also note that

$$\delta_0 = (xH)'' \text{ in } \mathcal{D}'(\mathbb{R}).$$

Natural to ask: can all distributions be written in the form

$$u = \sum_\alpha \partial^\alpha f_\alpha \text{ in } \mathcal{D}'(X)$$

where $f_\alpha \in C(X)$? We will prove this in the case $\mathcal{E}'(X)$ but the result is true more generally.

Lemma. *If $u \in \mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$ then $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ can be identified with the smooth (analytic) function $\lambda \mapsto \hat{u}(\lambda) = \langle u, x \mapsto e^{-i\lambda\cdot x}\rangle$. Also there exists $M \geq 0$ such that $|\hat{u}(\lambda)| \lesssim \langle \lambda \rangle^M$.*

Proof. Fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi = 1$ on $|x| < 1$ and $\chi = 0$ on $|x| > 2$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ set $\phi_m(x) = \chi(x/m)\phi(x) \in \mathcal{D}(\mathbb{R}^n)$. We claim $\phi_m \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$. For arbitrary α, β,

$$\begin{aligned}\|\phi - \phi_m\|_{\alpha,\beta} &= \|x^\alpha D^\beta[\phi(x)(1-\chi(x/m))]\|_\infty \\ &= \left\| x^\alpha \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} D^\gamma \phi D^{\beta-\gamma}(1-\chi(x/m)) \right\|_\infty .\end{aligned}$$

All derivatives of $x \mapsto 1 - \chi(x/m)$ tend to 0 uniformly and

$$\begin{aligned}\|x^\alpha D^\gamma \phi(1-\chi(x/m))\|_\infty &\lesssim \sup_{|x|>m} |x^\alpha D^\gamma \phi| \\ &\lesssim \sup_{|x|>2m} \left| \frac{|x|}{2m} x^\alpha D^\gamma \phi \right| \\ &\lesssim \frac{\|\phi\|_{\alpha+1,\gamma}}{2m} \to 0.\end{aligned}$$

So by sequential continuity of $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$\begin{aligned}\langle \hat{u}, \phi \rangle &= \lim_{m\to\infty} \langle \hat{u}, \phi_m \rangle \\ &= \lim_{m\to\infty} \langle u, \hat{\phi}_m \rangle \\ &= \lim_{m\to\infty} \langle u, x \mapsto \int_{\mathbb{R}} e^{-i\lambda\cdot x} \phi_m(\lambda) \mathrm{d}\lambda \rangle.\end{aligned}$$

By a Riemann sum argument (note each ϕ_m has compact support), we have

$$\lim_{m\to\infty} \langle u, x \mapsto \int_{\mathbb{R}} e^{-i\lambda\cdot x}\phi_m(\lambda)\mathrm{d}\lambda\rangle = \lim_{m\to\infty} \langle u, x \mapsto e^{-i\lambda\cdot x}\rangle\phi_m(\lambda)\mathrm{d}\lambda.$$

Since the power series for $x \mapsto e^{-i\lambda\cdot x}$ converges locally uniformly, we were able to interchange $\langle\cdot,\cdot\rangle$ with the infinite sum, by sequential continuity. So $\hat{u}(\lambda) = \langle u, x \mapsto e^{-i\lambda\cdot x}\rangle$ is smooth and by the semi-norm estimate of $u \in \mathcal{E}'(\mathbb{R}^n)$, there exists $C, N \geq 0$ and compact $K \subseteq \mathbb{R}^n$ such that

$$\begin{aligned}|\hat{u}(\lambda)| &= |\langle u, x \mapsto e^{-i\lambda\cdot x}\rangle| \\ &\leq C \sum_{|\alpha|\leq N} \sup_K |\partial_x^\alpha(e^{-\lambda\cdot x})| \\ &\lesssim \langle\lambda\rangle^N\end{aligned}$$

for $\lambda \in \mathbb{R}^n$. Hence by the DCT

$$\lim_{m\to\infty} \int \langle u, x \mapsto \int_{\mathbb{R}} e^{-i\lambda\cdot x}\rangle\phi_m(\lambda)\mathrm{d}\lambda = \int \hat{u}(\lambda)\phi(\lambda)\mathrm{d}\lambda$$

i.e $\hat{u}$ can be identified with $\lambda \mapsto \hat{u}(\lambda)$. $\square$

Theorem. *For each $u \in \mathcal{E}'(X)$ there exists a finite collection $\{f_\alpha\}$, $f_\alpha \in C(X)$ and $\mathrm{supp}(f_\alpha) \subseteq X$ such that*

$$u = \sum_\alpha \partial^\alpha f_\alpha \text{ in } \mathcal{E}'(X).$$

Proof. Fix $\rho \in \mathcal{D}(X)$ such that $\rho = 1$ on $\mathrm{supp}(u)$. Then for $\phi \in \mathcal{E}(X)$ have

$$\langle u, \phi\rangle = \langle u, \rho\phi\rangle$$

and since u extends to an element of $\mathcal{E}'(\mathbb{R}^n)$ and $\rho\phi$ extends to $\rho\phi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ can write $(\rho\phi) = (\hat{\hat{\psi}})$. In fact

$$(2\pi)^n\check{\psi} = \rho\phi. \tag{$*$}$$

So we have

$$\langle u, \phi\rangle = \langle u, \hat{(\hat{\psi})}\rangle = \langle \hat{u}, \hat{\psi}\rangle.$$

Note that

$$\mathcal{F}\left([1-\Delta]^m\psi\right)(\lambda) = \langle\lambda\rangle^{2m}\hat{\psi}(\lambda)$$

where $\Delta = \sum_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^2$ is the Laplacian. Hence

$$\langle u, \psi\rangle = \langle\langle\lambda\rangle^{-2m}\hat{u}, \mathcal{F}\left([1-\Delta]^m\psi\right)\rangle.$$

By choosing m sufficiently large and defining $f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \langle \lambda \rangle^{-2m} \hat{u}(\lambda) \mathrm{d}\lambda$ we have that f is continuous by the DCT. Also

$$(2\pi)^n \check{f} = \mathcal{F}(\langle \lambda \rangle^{-2m} \hat{u})$$

and so

$$\begin{aligned}
\langle u, \phi \rangle &= \langle \mathcal{F}(\langle \lambda \rangle^{-2m} \hat{u}), [1 - \Delta]^m \psi \rangle \\
&= \langle (2\pi)^n \check{f}, [1 - \Delta]^m \psi \rangle \\
&= \langle f, [1 - \Delta]^m [(2\pi)^n \check{\psi}] \rangle \\
&= \langle f, [1 - \Delta]^m (\rho \phi) \rangle. && \text{(by } (*))
\end{aligned}$$

We can expand derivatives, so by Leibnitz

$$\langle u, \phi \rangle = \langle f, \sum_\alpha (-1)^{|\alpha|} \rho_\alpha \partial^\alpha \phi \rangle$$

where $\rho_\alpha \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(\rho_\alpha) \subseteq X$. So

$$\langle u, \phi \rangle = \langle \sum_\alpha \partial^\alpha (\rho_\alpha f), \phi \rangle$$

$$= \langle \sum_\alpha \partial^\alpha f_\alpha, \phi \rangle$$

where $f_\alpha = \rho_\alpha f \in C(X)$ and $\operatorname{supp}(f_\alpha) \subseteq X$. $\square$

Example. We know that $\delta_0 = (xH)''$. Also note that if $\phi \in \mathcal{D}(\mathbb{R})$ has $\phi(0) = 1$ then $\phi \delta_0 = \delta_0$. Hence for $f \in \mathcal{D}(\mathbb{R})$

$$\begin{aligned}
\langle \delta_0, f \rangle &= \langle \phi (xH)'', f \rangle \\
&= \langle xH, (f\phi)'' \rangle \\
&= \langle xH, f''\phi + 2\phi' f' + f\phi'' \rangle \\
&= \langle (\phi xH)'', f \rangle - 2\langle (\phi' xH)', f \rangle + \langle \phi'' xH, f \rangle
\end{aligned}$$

so $\delta_0 = (\phi xH)'' - 2(\phi' xH)' + \phi''(xH)$. Note that each of $\phi xH, \phi' xH, \phi'' xH$ have compact support in $\mathbb{R}$.

4.4 Paley-Wiener Schwartz Theorem

Have seen that if $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\hat{u}$ can be identified with

$$\lambda \mapsto \hat{u}(\lambda) = \langle u, x \mapsto e^{-i\lambda \cdot x} \rangle.$$

Taking a complex analytic extension to $z \in \mathbb{C}^n$, call this $\hat{u}(z) = \langle u, z \mapsto e^{-iz\cdot x} \rangle$ we obtain the *Fourier-Laplace transform* of $u \in \mathcal{E}'(\mathbb{R}^n)$. We know there exists $C, N \geq 0$, $K \subseteq \mathbb{R}^n$ compact such that

$$\begin{aligned} |\hat{u}(z)| &= |\langle u, x \mapsto e^{iz\cdot x} \rangle| \\ &\leq C \sum_{|\alpha| \leq N} \sup_K |\partial_x^\alpha e^{-iz\cdot x}|. \end{aligned}$$

Also, $z \mapsto \hat{u}(z)$ is entire [power series of $x \mapsto e^{-iz\cdot x}$ converges locally uniformly, so can apply u termwise (sequential continuity of u) to get power series for $\hat{u}(z)$].

Lemma. *If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{supp}(u) \subseteq \overline{B}_\delta = \{x \in \mathbb{R}^n : |x| \leq \delta\}$ then there exist $C, N \geq 0$ such that*

$$|\hat{u}(z)| \leq C(1 + |z|)^N e^{\delta |\Im(z)|}.$$

Proof. Fix $\psi \in C^\infty(\mathbb{R})$ such that $\psi(\tau) = 1$ on $\tau \geq -\frac{1}{2}$ and $\psi(\tau) = 0$ on $\tau \leq -1$. For $\varepsilon > 0$, define

$$\phi_\varepsilon(x) = \psi(\varepsilon(\delta - |x|))$$

for $x \in \mathbb{R}^n$. Then $\phi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and

$$\phi_\varepsilon = \begin{cases} 1 & \text{on } |x| \leq \delta + \frac{1}{2\varepsilon} \\ 0 & \text{on } |x| \geq \delta + \frac{1}{\varepsilon} \end{cases}.$$

Note that $\phi_\varepsilon = 1$ on $\operatorname{supp}(u)$. Since $u \in \mathcal{E}'(\mathbb{R}^n)$ there exist $C, N \geq 0$ such that

$$\begin{aligned} |\hat{u}(z)| &= |\langle u, x \mapsto \phi_\varepsilon(x) e^{-iz\cdot x} \rangle \\ &\leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha [\phi_\varepsilon e^{-ix\cdot z}]|. \end{aligned}$$

Note $|\partial^\beta \phi_\varepsilon| \lesssim_\beta \varepsilon^{|\beta|}$ and $|\partial^\gamma e^{-iz\cdot x}| \lesssim |z|^{|\gamma|} e^{(\varepsilon + \frac{1}{2\delta})|\Im(z)|}$ on $\operatorname{supp}(\phi_\varepsilon)$. Hence

$$|\hat{u}(z)| \lesssim \sum_{|\beta| + |\gamma| \leq N} \varepsilon^{|\beta|} |z|^{|\gamma|} e^{(\varepsilon + \frac{1}{2\delta})|\Im(z)|}$$

so take $\varepsilon = |z|$ to get the result. $\square$

The Paley-Wiener-Schwartz theorem is about the converse: if $z \mapsto U(z)$ is entire and

$$|U(z)| \lesssim (1 + |z|)^N e^{\delta |\Im(z)|}$$

is it the case that $U = \hat{u}$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$ with $\operatorname{supp}(u) \subseteq \overline{B}_\delta$.

Theorem (Paley-Wiener-Schwartz)**.**

(A) If $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}(\phi) \subseteq \overline{B}_\delta$ then $z \mapsto \hat{\phi}(z)$ is entire and

$$|\hat{\phi}(z)| \lesssim_N (1+|z|)^{-N} e^{\delta|\Im(z)|},\ z \in \mathbb{C},\ N = 0,1,2,\dots. \tag{$\dagger$}$$

Conversely, if $z \mapsto \Phi(z)$ is entire and satisfies (†) then $\Phi = \hat{\phi}$ for some $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{supp}(\phi) \subseteq \overline{B}_\delta$.

(B) If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{supp}(u) \subseteq \overline{B}_\delta$, then $z \mapsto \hat{u}(z)$ is entire and there exists $N \geq 0$ such that

$$|\hat{u}(z)| \lesssim (1+|z|)^N e^{\delta|\Im(z)|},\ z \in \mathbb{C}. \tag{$\ddagger$}$$

Conversely, if $z \mapsto U(z)$ is entire and satisfies (‡) then $U = \hat{u}$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$ with $\operatorname{supp}(u) \subseteq \overline{B}_\delta$.

Proof.

(A) It is clear that

$$z \mapsto \hat{u}(z) = \int_{\mathbb{R}^n} e^{iz\cdot x}\phi(x)\mathrm{d}x$$

is entire (Morera+Fubini). For the estimate (†) note that for α a multi-index

$$\begin{aligned} |z^\alpha \hat{\phi}(z)| &= \left| \int_{\mathbb{R}^n} z^\alpha e^{-iz\cdot x}\phi(x)\mathrm{d}x \right| \\ &= \left| \int_{\mathbb{R}^n} e^{-iz\cdot x} D^\alpha \phi(x)\mathrm{d}x \right| \\ &\lesssim_\alpha e^{\delta|\Im(z)|} \end{aligned}$$

since $|e^{-z\cdot x}| = |e^{\Im(z)\cdot x} \leq e^{\delta|Im(z)|}$ on $\operatorname{supp}(\phi)$. The estimate (†) then follows. For the converse, given $z \mapsto \Phi(z)$ entire and obeying (†) define

$$\phi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\lambda\cdot x}\Phi(\lambda)\mathrm{d}\lambda.$$

Then by the DCT and (†) we have that $\phi \in C^\infty(\mathbb{R}^n)$. By Cauchy's theorem, entirety of $z \mapsto \Phi(z)$ and estimate (†), we have for arbitrary $\eta \in \mathbb{R}^n$ that

$$|\phi(x)| = (2\pi)^{-n} \left| \int_{\mathbb{R}^n} e^{i(\lambda+i\eta)\cdot x}\Phi(\lambda+i\eta)\mathrm{d}\lambda \right|$$

[justified by rapid horizontal decay of Φ]. So by (†)

$$\begin{aligned} |\phi(x)| &\lesssim_N \int_{\mathbb{R}^n} e^{-\eta\cdot x}(1+|\lambda+i\eta|)^{-N} e^{\delta|\eta|}\mathrm{d}\lambda \\ &\lesssim e^{\delta|\eta|-\eta\cdot x}. \end{aligned}$$

Take $\eta = \frac{x}{|x|}t$, $t > 0$. Then

$$e^{\delta|\eta| - \eta\cdot x} = e^{-t(|x|-\delta)}.$$

If $|x| > \delta$, take $t \to \infty$ to get $\phi = 0$. Hence $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}(\phi) \subseteq \overline{B}_\delta$. Taking Fourier transform shows $\Phi = \hat{\phi}$.

(B) We already established the forward direction. For the converse, let $z \mapsto U(z)$ satisfy (‡). Then $U|_{\mathbb{R}^n} \in \mathcal{S}'(\mathbb{R}^n)$ since $|U(\lambda)| \lesssim \langle\lambda\rangle^N$. Since $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is an isomorphism, there exists $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\hat{u} = U$.
Fix $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi \mathrm{d}x = 1$ and $\operatorname{supp}(\phi) \subseteq B_1$. Set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$. Then $\phi_\varepsilon \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp}(\phi_\varepsilon) \subseteq B_\varepsilon$. Hence $\hat{\phi_\varepsilon} \to 1$ in $\mathcal{S}'(\mathbb{R}^n)$.

Define

$$\hat{u}_\varepsilon(z) = \hat{\phi_\varepsilon}(z)U(z).$$

By (†) (for $\hat{\phi_\varepsilon}$) and (‡) (for U) we have

$$|\hat{u}_\varepsilon(z)| \lesssim_N (1+|z|)^{-N}e^{(\varepsilon+\delta)|\Im(z)|}, N = 0, 1, 2, \ldots.$$

Hence $u_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp}(u_\varepsilon) \subseteq \overline{B}_{\delta+\varepsilon}$. As $\varepsilon \downarrow 0$, $\hat{u}_\varepsilon \to \hat{u}$ in $\mathcal{S}'(\mathbb{R}^n)$.

$\square$

5 Oscillatory integrals

In this section we would like to make sense of

$$\int_{\mathbb{R}} e^{i\lambda x} \mathrm{d}\lambda$$

and more generally, objects of the form

$$\int_{\mathbb{R}^k} e^{i\Phi(x,\theta)} a(x,\theta) \mathrm{d}\theta$$

where $x \in X$. Call real valued $\Phi \in C^\infty(X \times \mathbb{R}^k \setminus \{0\})$ the *phase function* and a will belong to a class of functions called *symbols*. Note that the integral will not be well-defined in a classical sense as we will allow symbols which get large as $|\theta| \to \infty$.

Lemma (Riemann-Lebesgue)**.** *If $f \in L^1(\mathbb{R})$ then $|\hat{f}(\lambda)| \to 0$ as $|\lambda| \to \infty$.*

Proof. Assume $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. Then

$$\begin{aligned}
\hat{f}(\lambda) &= \frac{1}{2} \int_{\mathbb{R}} [e^{-i\lambda x} f(x) + e^{-i\lambda x} f(x)] \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}} [e^{-i\lambda x} f(x) + e^{-i\pi} e^{-i\lambda x} f(x + \pi/\lambda)] \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}} e^{-i\lambda x} [f(x) - f(x + \pi/\lambda)] \mathrm{d}x.
\end{aligned}$$

Since $f \in L^1(\mathbb{R})$, given $\varepsilon > 0$ there exists R such that

$$\frac{1}{2} \int_{|x|>R} |f(x) - f(x + \pi/\lambda)| \mathrm{d}x < \frac{\varepsilon}{4}.$$

Since $f \in C(\mathbb{R})$ we can take $|\lambda|$ sufficiently large so that

$$\left| \int_{|x|<R} e^{-i\lambda x} [f(x) - f(x + \pi/\lambda)] \mathrm{d}x \right| < \frac{\varepsilon}{4}$$

i.e $|\hat{f}(\lambda)| < \varepsilon/2$ for all $|\lambda|$ sufficiently large. Note $L^1(\mathbb{R}) \cap C(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ so given $g \in L^1(\mathbb{R})$ we can fix $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ such that $\|f - g\|_{L^1} < \varepsilon/2$ so

$$\begin{aligned}
|\hat{g}(\lambda)| &= |\hat{g}(\lambda) - \hat{f}(\lambda) + \hat{f}(\lambda)| \\
&= \|g - f\|_{L^1} + |\hat{f}(\lambda)| \\
&< \varepsilon
\end{aligned}$$

for $|\lambda|$ sufficiently large. □

The above result intuitively says that more oscillation in an integral implies faster decay. More generally, if $\phi \in \mathcal{D}(\mathbb{R})$ and $\Phi \in C^\infty(\mathbb{R})$ we expect

$$\int_{\mathbb{R}} \phi(x) e^{i\lambda\Phi(x)} \mathrm{d}x$$

to decay as $|\lambda| \to \infty$. For example if $\Phi' \neq 0$ then the operator $L = \frac{1}{i\lambda\Phi'(\theta)}\frac{\mathrm{d}}{\mathrm{d}\theta}$ is well-defined since $|\Phi'(\theta)| \gtrsim 1$ on $\operatorname{supp}(\phi)$. Note that $Le^{i\lambda\Phi} = e^{i\lambda\Phi}$. So

$$\begin{aligned}
\int_{\mathbb{R}} \phi(\theta) e^{i\lambda\Phi(\theta)} \mathrm{d}\theta &= \int_{\mathbb{R}} \phi(\theta) L e^{i\lambda\Phi} \mathrm{d}\theta \\
&= \int_{\mathbb{R}} L^t[\phi] e^{i\lambda\Phi} \mathrm{d}\theta
\end{aligned}$$

where $L^t = -\frac{1}{i\lambda}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{1}{\Phi'}\cdot\right]$ is the formal adjoint of L. We can do this as many times as we want so

$$\begin{aligned}
\left|\int_{\mathbb{R}} e^{i\lambda\Phi}\phi \mathrm{d}\theta\right| &= \left|\int_{\mathbb{R}} (L^t)^N(p) e^{i\lambda\Phi} \mathrm{d}\theta\right| \\
&\lesssim_N \langle\lambda\rangle^{-N}.
\end{aligned}$$

We expect to get a dominant contribution from points at which $\Phi' = 0$ (stationary points).

Lemma (Stationary phase)**.** *Let $\Phi \in C^\infty(\mathbb{R})$ such that $\Phi' \neq 0$ on $\mathbb{R} \setminus \{0\}$ and $\Phi(0) = \Phi'(0) = 0$, $\Phi''(0) \neq 0$. Then for $\phi \in \mathcal{D}(\mathbb{R})$*

$$\left|\int_{\mathbb{R}} e^{i\lambda\Phi(\theta)}\phi(\theta)\mathrm{d}\theta\right| \lesssim \frac{1}{|\lambda|^{1/2}} \text{ as } |\lambda| \to \infty.$$

Proof. Fix $\rho \in \mathcal{D}(\mathbb{R})$ such that $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Write

$$\begin{aligned}
\int_{\mathbb{R}} e^{i\lambda\Phi}\rho(\theta)\mathrm{d}\theta &= \underbrace{\int_{\mathbb{R}} e^{i\lambda\Phi}\rho(\theta/\delta)\phi(\theta)\mathrm{d}\theta}_{I_1} \\
&\quad + \underbrace{\int_{\mathbb{R}} e^{i\lambda\Phi}(1-\rho(\theta/\delta))\phi(\theta)\mathrm{d}\theta}_{I_2}
\end{aligned}$$

and since $\rho(\theta/\delta) = 0$ on $|\theta| > 2\delta$ we get the simple estimate

$$|I_1| \lesssim \delta.$$

Note $(1-\rho(\theta/\delta)) = 0$ on $|\theta| < \delta$ so we're essentially integrating over $|\theta| > \delta$, so

$$L = \frac{1}{i\lambda\Phi'}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

is well-defined and $Le^{i\lambda\Phi} = e^{i\lambda\Phi}$. So

$$I_2(\lambda) = \int_{\mathbb{R}} e^{i\lambda\Phi}(L^t)^2[(1-\rho(\theta/\delta))\phi(\theta)]\mathrm{d}\theta$$

where $L^t = -\frac{1}{i\lambda}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{1}{\Phi'}\cdot\right]$. Note that if

$$P := \frac{\mathrm{d}}{\mathrm{d}\theta}\left[a\cdot\right]$$

we have

$$\begin{aligned} P^2 &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[a\frac{\mathrm{d}}{\mathrm{d}\theta}\left(a\cdot\right)\right] \\ &= a^2\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + 3aa'\frac{\mathrm{d}}{\mathrm{d}\theta} + (aa'). \end{aligned}$$

Also

$$(L^t)^2 = -\frac{1}{\lambda^2}\left[\frac{1}{(\Phi')^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} - 3\frac{\Phi''}{(\Phi')^3}\frac{\mathrm{d}}{\mathrm{d}\theta} - \left(\frac{\Phi''}{(\Phi')^3}\right)'\right] - \frac{(\Phi'')^2}{(\Phi')^4}.$$

Note that

$$\begin{aligned} \Phi'(\theta) - \Phi'(0) &= \int_0^\theta \Phi''(t)\mathrm{d}t \\ &= \theta\int_0^1 \Phi''(t\theta)\mathrm{d}t \end{aligned}$$

i.e

$$\frac{\mathrm{d}\Phi'(\theta)}{\mathrm{d}\theta} = \int_0^1 \Phi'(t\theta)\mathrm{d}t.$$

The LHS of this is non-zero at $\theta \neq 0$ and converges to $\Phi''(0) \neq 0$ as $\theta \to 0$, i.e $|\Phi'(\theta)| \gtrsim |\theta|$ on $\mathrm{supp}(\phi)$. So

$$(L^t)^2[\phi(\theta)[1-\rho(\theta/\delta)]] = \mathcal{O}\left(\frac{1}{\lambda^2\theta^2\delta^2}\right) + \mathcal{O}\left(\frac{1}{\lambda^2\theta^2\delta}\right) + \mathcal{O}\left(\frac{1}{\lambda^2\theta^4}\right)$$

and integrating over $|\theta| > \delta$ we see

$$|I_2(\lambda)| = \mathcal{O}\left(\frac{1}{\lambda^2\delta^3}\right) + \mathcal{O}\left(\frac{1}{\lambda^2\delta^3}\right) + \mathcal{O}\left(\frac{1}{\lambda^2\delta^3}\right).$$

Matching with $I_1(\lambda) = \mathcal{O}(\delta)$ we take

$$\delta = \frac{1}{\lambda^2\delta^3} \implies \delta = \frac{1}{|\lambda|^{1/2}}.$$

□

This estmiate is sharp, e.g

$$\int_{\mathbb{R}} e^{i\lambda\theta^2}\phi(\theta)\mathrm{d}\theta = \frac{1}{\sqrt{\lambda}}\int_{\mathbb{R}} e^{i\eta^2}\phi(\eta/\sqrt{\lambda})\mathrm{d}\eta \qquad (\theta = \eta/\sqrt{\lambda})$$
$$\sim \frac{\text{constant}}{\sqrt{\lambda}} + \text{ lower order terms}$$

as $|\lambda| \to \infty$. Using this we expect

$$u(x) = \int_{\mathbb{R}} e^{i\Phi(x,\theta)}a(x,\theta)\mathrm{d}\theta$$

to be "badly behaved" at $x_0 \in X$ for which

$$\nabla_\theta \Phi(x_0, \theta) = 0 \text{ for some } \theta \in \mathbb{R}^k.$$

We will show

$$\operatorname{sing\,supp}(u) \subseteq \{x \in X : \nabla_\theta \Phi(x,\theta) = 0 \text{ for some } \theta \in \mathbb{R}^k \setminus \{0\}\}.$$

Definition. Let $X \subseteq \mathbb{R}^n$ be open. A smooth function $a : X \times \mathbb{R}^k \to \mathbb{C}$ is called a *symbol* of order $N \in \mathbb{R}$ if for each $K \subseteq X$ compact

$$|D_x^\alpha D_\theta^\beta a(x,\theta)| \lesssim_{K,\alpha,\beta} \langle\theta\rangle^{N-|\beta|}$$

for $(x,\theta) \in K \times \mathbb{R}^k$. Call the space of all such symbols $\mathrm{Sym}(X, \mathbb{R}^k; N)$.

For example if $\{\varphi_\alpha\}$ is in $C^\infty(X)$ then

$$a(x,\theta) = \sum_{|\alpha| \leq N} \varphi_\alpha(x)\theta^\alpha$$

belongs to $\mathrm{Sym}(X, \mathbb{R}^k; N)$. We only care about the behaviour of symbols for large $|\theta|$ since for any compact $L \subseteq \mathbb{R}^k$, if $a \in C^\infty(X \times \mathbb{R}^k)$ then

$$(x,\theta) \mapsto \frac{D_x^\alpha D^\beta(x,\theta)}{\langle\theta\rangle^{N-|\beta|}}$$

will always be bounded on $K \times L$ for $K \subseteq X$ compact.

Lemma.

- *If $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then $D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbb{R}^k; N - |\beta|)$;*
- *If $a_i \in \mathrm{Sym}(X, \mathbb{R}^k; N_i)$ then $a_1 a_2 \in \mathrm{Sym}(X, \mathbb{R}^k, N_1 + N_2)$;*

Proof. Obviously $D_x^\alpha D_\theta^\beta a$ is smooth on $X \times \mathbb{R}^k$. For $K \subseteq X$ compact

$$\begin{aligned} |D_x^{\alpha'} D_\theta^{\beta'}[D_x^\alpha D_\theta^\beta a]| &= |D_x^{\alpha+\alpha'} D_\theta^{\beta+\beta'} a| \\ &\lesssim_{K,\alpha,\beta} \langle\theta\rangle^{N-|\beta+\beta'|} \\ &= \langle\theta\rangle^{N-|\beta|-|\beta'|} \end{aligned}$$

and so $D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbb{R}^k; N - |\beta|)$. Again for $K \subseteq X$ compact

$$\begin{aligned} |D_x^\alpha D_\theta^\beta (a_1 a_2)| &\leq \left| \sum_{\alpha' \leq \alpha} \sum_{\beta' \leq \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} (D_x^{\alpha'} D_\theta^{\beta'} a_1)(D_x^{\alpha-\alpha'} D_\theta^{\beta-\beta'} a_2) \right| \\ &\lesssim_{\alpha,\beta} \sum_{\alpha' \leq \alpha} \sum_{\beta' \leq \beta} |D_x^{\alpha'} D_\theta^{\beta'} a_1||D_x^{\alpha-\alpha'} D_\theta^{\beta-\beta'} a_2| \\ &\lesssim_{K,\alpha,\beta} \sum_{\alpha' \leq \alpha} \sum_{\beta' \leq \beta} \langle\theta\rangle^{N_1-|\beta'|} \langle\theta\rangle^{N_2-|\beta|+|\beta'|} \\ &\lesssim_{\alpha,\beta} \langle\theta\rangle^{N_1+N_2-|\beta|} \end{aligned}$$

and hence $a_1 a_2 \in \mathrm{Sym}(X, \mathbb{R}^k; N_1 + N_2)$. □

Lemma. *If $a \in C^\infty(X \times \mathbb{R}^k)$ and a is positively homogenous of degree N (in θ) for $|\theta|$ sufficiently large, then $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$.*

Remark. Technically we mean that for $|\theta|$ sufficiently large, a agrees with a function which is positively homogenous everywhere.

Proof. For $|\theta|$ sufficiently large $a(x, t\theta) = t^N a(x, \theta)$ for $t > 0$. So for $|\theta|$ large

$$\begin{aligned} t^N D_x^\alpha D_\theta^\beta [a(x,\theta)] &= D_x^\alpha D_\theta^\beta [a(x, t\theta)] \\ &= t^{|\beta|} (D_x^\alpha D_\theta^\beta a)(x, t\theta) \end{aligned}$$

i.e $D_x^\alpha D_\theta^\beta a$ is also positively homogenous of order $N - |\beta|$ for $|\theta|$ large. For $K \subseteq X$ compact set $\omega = \theta/|\theta| \in S^{k-1}$ so

$$\begin{aligned} |D_x^\alpha D_\theta^\beta a(x,\theta)| &= |D_x^\alpha D_\theta^\beta a(x, |\theta|\omega)| \\ &= |\theta|^{N-|\beta|} |D_x^\alpha D_\theta^\beta a(x,\omega)| \\ &\lesssim_{K,\alpha,\beta} \langle\theta\rangle^{N-|\beta|} \end{aligned}$$

by compactness of S^{k-1}. $\square$

Definition. $\Phi : X \times \mathbb{R}^k \to \mathbb{R}$ is called a *phase function* if

(i) Φ is continuous on $X \times \mathbb{R}^k$ and positively homogenous of degree 1 in θ;

(ii) Φ is smooth $X \times (\mathbb{R}^k \setminus \{0\})$;

(iii) $\mathrm{d}\Phi = \nabla_\theta \Phi \cdot \mathrm{d}\theta + \nabla_x \Phi \cdot \mathrm{d}x \neq 0$ on $X \times (\mathbb{R}^k \setminus \{0\})$, i.e

$$\begin{pmatrix} \frac{\partial \Phi}{\partial \theta_1} \\ \vdots \\ \frac{\partial \Phi}{\partial \theta_k} \\ \frac{\partial \Phi}{\partial x_1} \\ \vdots \\ \frac{\partial \Phi}{\partial x_n} \end{pmatrix} \neq 0$$

on $X \times (\mathbb{R}^k \setminus \{0\})$.

We want to make sense of

$$D^\alpha \delta_0(x) = (2\pi)^{-n} \int_{\mathbb{R}^k} \theta^\alpha e^{ix\cdot\theta} \mathrm{d}\theta$$

i.e $\Phi(x,\theta) = x \cdot \theta$, $a(x,\theta) = (2\pi)^{-n}\theta^\alpha \in \mathrm{Sym}(\mathbb{R}^n, \mathbb{R}^n; |\alpha|)$.

More generally we want to make sense of

$$\int_{\mathbb{R}^k} e^{i\Phi(x,\theta)} a(x,\theta) \mathrm{d}\theta$$

for Φ a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$.

We could define a linear form $I_\Phi(a) : \mathcal{D}(X) \to \mathbb{C}$ by

$$\langle I_\Phi(a), \phi\rangle = \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{i\Phi(x,\theta)}a(x,\theta)\phi(x)\mathrm{d}x\mathrm{d}\theta$$

but this is difficult to work with due to lack of $\mathrm{d}\theta \otimes \mathrm{d}x$ integrability. Instead, fix $\chi \in \mathcal{D}(\mathbb{R}^k)$ such that $\chi = 1$ on $|\theta| < 1$ and set

$$I_\Phi^\varepsilon(a) = \int_{\mathbb{R}^k} e^{i\Phi(x,\theta)}a(x,\theta)\chi(\varepsilon\theta)\mathrm{d}\theta.$$

Then define

$$I_\Phi(a) = \lim_{\varepsilon\downarrow 0} I_\Phi^\varepsilon(a) \text{ in } \mathcal{D}'(X).$$

Lemma. *If L has form*

$$L = \sum_{j=1}^k a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^n b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

for $a_j \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$, $b_j, c \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$, then L^t has the same form.

Proof. We have

$$\begin{aligned} L^t &= -\sum_{j=1}^k \frac{\partial}{\partial\theta_j}(a_j\cdot) - \sum_{j=1}^n \frac{\partial}{\partial x_j}(b_j\cdot) + c \\ &= \sum_{j=1}^k \tilde{a}_j\frac{\partial}{\partial\theta_j} + \sum_{j=1}^n \tilde{b}_j\frac{\partial}{\partial x_j} + \tilde{c} \end{aligned}$$

where $\tilde{a}_j = -a_j \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$, $\tilde{b}_j = -b_j \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$ and

$$\tilde{c} = -\sum_{j=1}^k \frac{\partial a_j}{\partial\theta_j} - \sum_{j=1}^n \frac{\partial b_j}{\partial x_j} + c \in \mathrm{Sym}(X, \mathbb{R}^k; -1).$$

□

If we could find such an L for which

$$Le^{i\Phi} = e^{i\Phi}$$

then

$$\begin{aligned} \langle \Phi_I^\varepsilon(a), \phi\rangle &= \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} (L^N e^{i\Phi})a(x,\theta)\chi(\varepsilon\theta)\phi(x)\mathrm{d}x\mathrm{d}\theta \\ &= \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{i\Phi}(L^t)^N[a(x,\theta)\chi(\varepsilon\theta)\phi(x)]\mathrm{d}x\mathrm{d}\theta \end{aligned}$$

the form of L (and hence L^t) should then lower the order of $a(x,\theta)\chi(\varepsilon\theta)\phi(x)$ by 1 upon each application.

Lemma. *There exists a differential operator L of the form*

$$L = \sum_{j=1}^{k} a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n} b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

for $a_j \in \mathrm{Sym}(X,\mathbb{R}^k;0)$, $b_j, c \in \mathrm{Sym}(X,\mathbb{R}^k;-1)$ such that if $L^t e^{i\Phi} = e^{i\Phi}$, for Φ a phase function.

Proof. Clearly

$$\frac{\partial}{\partial\theta_j}e^{i\Phi} = i\frac{\partial\Phi}{\partial\theta_j}e^{i\Phi}, \qquad \frac{\partial}{\partial x_j}e^{i\Phi} = i\frac{\partial\Phi}{\partial x_j}e^{i\Phi}.$$

So

$$\left(-\sum_{j=1}^{k} i|\theta|^2\frac{\partial\Phi}{\partial\theta_j}\frac{\partial}{\partial\theta_j} - \sum_{j=1}^{n} i\frac{\partial\Phi}{\partial x_j}\frac{\partial}{\partial x_j}\right)e^{i\Phi} = (|\theta|^2|\nabla_\theta\Phi|^2 + |\nabla_x\Phi|^2)e^{i\Phi}.$$

Since $\Phi(x,t\theta) = t\Phi(x,\theta)$ for $t>0$,

$$t\frac{\partial}{\partial x_j}\Phi(x,\theta) = \frac{\partial}{\partial x_j}\Phi(x,t\theta) = \frac{\partial\Phi}{\partial x_j}(x,t\theta)$$

so $\frac{\partial\Phi}{\partial x_j}$ is positively homogenous of degree 1. Similarly

$$t\frac{\partial}{\partial\theta_j}\Phi(x,\theta) = \frac{\partial}{\partial\theta_j}\Phi(x,t\theta) = t\frac{\partial\Phi}{\partial\theta_j}(x,t\theta)$$

so $\frac{\partial\Phi}{\partial\theta_j}$ is positively homogenous of degree 0. Define

$$P = \sum_{j=1}^{k}\tilde{a}_j\frac{\partial}{\partial\theta_j} + \sum_{j=1}^{n}\tilde{b}_j\frac{\partial}{\partial x_j}$$

where

$$\tilde{a}_j = \frac{-i|\theta|^2\frac{\partial\Phi}{\partial\theta_j}}{|\theta|^2|\nabla_\theta\Phi|^2 + |\nabla_x\Phi|^2}$$
$$\tilde{b}_j = \frac{-i\frac{\partial\Phi}{\partial x_j}}{|\theta|^2|\nabla_\theta\Phi|^2 + |\nabla_x\Phi|^2}.$$

Then we see that $\tilde{a}_j$ is positively homogenous of degree 0 and $\tilde{b}_j$ is positively homogenous of degree -1. Note that the denominators may vanish at $\theta = 0$, so we fix $\rho \in \mathcal{D}(\mathbb{R}^k)$ with $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Define

$$L^t = (1-\rho)P + \rho.$$

Then $L^t e^{i\Phi} = (1-\rho)e^{i\Phi} + \rho e^{i\Phi} = e^{i\Phi}$. By the previous two lemmas we then have that L is of the required form. □

Note that $L : \mathrm{Sym}(X, \mathbb{R}^k; N) \to \mathrm{Sym}(X, \mathbb{R}^k; -1)$. Also, more generally for $\varphi \in \mathcal{D}(X)$ we have

$$L^[a(x,\theta)\varphi(x)] = \sum_{|\alpha| \le M} a_\alpha(x,\theta)\partial^\alpha \varphi$$

where $a_\alpha \in \mathrm{Sym}(X, \mathbb{R}^k; N - M)$ (by induction on M).

Theorem. *If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then*

$$I_\Phi(a) = \lim_{\varepsilon \downarrow 0} I_\Phi^\varepsilon(a) \in \mathcal{D}'(X)$$

and $\mathrm{ord}(I_\Phi(a)) \le N + k + 1$.

Proof. For each $\varepsilon > 0$,

$$I_\Phi^\varepsilon(a) = \int_{\mathbb{R}^k} e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta) \mathrm{d}\theta$$

where $\chi \in \mathcal{D}(\mathbb{R}^k)$ has $\chi = 1$ on $|\theta| < 1$ and $\chi = 0$ on $|\theta| > 2$. So for $\varphi \in \mathcal{D}(X)$ and taking L as above

$$\begin{aligned}
\langle I_\Phi(a)^\varepsilon, \varphi \rangle &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} e^{i\Phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\varphi(x) \mathrm{d}x \mathrm{d}\theta \\
&= \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} [(L^t)^M e^{i\Phi(x,\theta)}] a(x,\theta)\chi(\varepsilon\theta)\varphi(x) \mathrm{d}x \mathrm{d}\theta \\
&= \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} e^{i\Phi} L^M [a(x,\theta)\chi(\varepsilon\theta)\varphi(x)] \mathrm{d}x \mathrm{d}\theta.
\end{aligned}$$

Note that since $\chi \in \mathcal{D}(\mathbb{R}^k)$,

$$\begin{aligned}
\left| \left(\frac{\partial}{\partial\theta} \right)^\alpha \chi(\varepsilon\theta) \right| &= \varepsilon^{|\alpha|} |(\partial^\alpha \chi)(\varepsilon\theta)| \\
&\lesssim \varepsilon^{|\alpha|} \langle \varepsilon\theta \rangle^{-|\alpha|} \\
&= c_\alpha \frac{\varepsilon^{|\alpha|}}{[1 + \varepsilon^2 |\theta|^2]^{|\alpha|/2}} \\
&= c_\alpha \frac{1}{\left[\frac{1}{\varepsilon^2} + |\theta|^2\right]^{|\alpha|/2}}
\end{aligned}$$

so for $0 < \varepsilon \le 1$

$$\left| \left(\frac{\partial}{\partial\theta} \right)^\alpha \chi(\varepsilon\theta) \right| \lesssim_\alpha \langle \theta \rangle^{-|\alpha|}$$

i.e $\chi(\varepsilon\theta) \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$ uniformly in ε. So $a(x,\theta)\chi(\varepsilon\theta) \in \mathrm{Sym}(X, \mathbb{R}^k, N)$ so

$$L^M [a(x,\theta)\chi(\varepsilon\theta)\varphi(x)] = \sum_{|\alpha| \le M} a_\alpha(x,\theta;\varepsilon)\partial^\alpha \varphi$$

where $a_\alpha \in \mathrm{Sym}(X, \mathbb{R}^k; N - M)$. Also $a_\alpha(x,\theta) := a(x,\theta;0) \in \mathrm{Sym}(X, \mathbb{R}^k, N - M)$. Choosing $M = N + k + 1$ (i.e with $N - M < -k$) by the DCT we have

$$\begin{aligned}\langle I_\phi(a), \varphi\rangle &= \lim_{\varepsilon\downarrow 0}\langle I_\Phi^\varepsilon(a), \varphi\rangle \\ &= \sum_{|\alpha|\le N+k+1}\int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{i\Phi(x,\theta)} a_\alpha(x,\theta)\partial^\alpha\varphi \mathrm{d}x\mathrm{d}\theta.\end{aligned}$$

If $\mathrm{supp}(\varphi) \subseteq K$ we have

$$\begin{aligned}|\langle I_\phi(a), \varphi\rangle| &\le \sum_{|\alpha|\le N+k+1}\int_{\mathbb{R}^k}\int_K |a_\alpha(x,\theta)||\partial^\alpha|\mathrm{d}x\mathrm{d}\theta \\ &\lesssim_K \sum_{|\alpha|\le N+k+1}\sup_K|\partial^\alpha\varphi|\end{aligned}$$

so $I_\Phi(a) \in \mathcal{D}'(X)$ and $\mathrm{ord}(I_\Phi(a)) \le N + k + 1$. □

Given $I_\Phi(a) \in \mathcal{D}'(X)$ we can show that $\frac{\partial}{\partial x_i} I_\Phi(a)$ coincides with the oscillatory integral

$$\int_{\mathbb{R}^k} e^{i\Phi(x,\theta)}\left[i\frac{\partial Phi}{\partial x_j}a(x,\theta) + \frac{\partial a}{\partial x_j}(x,\theta)\right]\mathrm{d}\theta.$$

Remark. Since $i\frac{\partial Phi}{\partial x_j}a(x,\theta) + \frac{\partial a}{\partial x_j}(x,\theta)$ may fail to be smooth at $\theta = 0$, write

$$\int_{\mathbb{R}^k} e^{i\Phi(x,\theta)}\rho(\theta)a(x,\theta)\mathrm{d}\theta + \int_{\mathbb{R}^k} e^{i\Phi(x,\theta)}(1-\rho(\theta))a(x,\theta)\mathrm{d}\theta$$

for $\rho \in \mathcal{D}(\mathbb{R}^k)$, $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Because of this technicality we often assume that the symbols have support away from a neighbourhood of the origin.

Consider

$$I_\phi(a) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\theta} \mathrm{d}\theta$$

for $x \in \mathbb{R}^n$. Then for $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\langle I_\Phi(a), \phi\rangle = \lim_{\varepsilon \downarrow 0} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix\cdot\theta} \chi(\varepsilon\theta)\phi(x) \mathrm{d}x \mathrm{d}\theta$$

so taking $(x, \theta) \mapsto (\varepsilon x, \theta/\varepsilon)$ this becomes

$$\begin{aligned}
&(2\pi)^{-n} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\cdot\theta}\chi(\theta)\phi(\varepsilon x)\mathrm{d}x\mathrm{d}\theta \\
&\qquad = (2\pi)^{-n} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} (2\pi)^{-n} \hat{\chi}(-x)\phi(\varepsilon x)\mathrm{d}x \\
&\qquad = \phi(0)(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\chi}(-x)\mathrm{d}x \\
&\qquad = \phi(0)\chi(0) \\
&\qquad = \phi(0)
\end{aligned}$$

i.e (abusing notation)

$$\delta(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\theta} \mathrm{d}\theta$$

which gives

$$D^\alpha \delta_0(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \theta^\alpha e^{ix\cdot\theta} \mathrm{d}\theta.$$

It is natural to ask when $I_\Phi(a) \in \mathcal{D}'(X)$ can be identified with a smooth function.

Definition. Let $Y \subseteq X$ be open. We say $u \in \mathcal{D}'(X)$ is *smooth on* Y if there exists $f \in C^\infty(Y)$ such that $\langle u, \phi\rangle = \int_Y f(x)\phi(x)\mathrm{d}x$ for all $\phi \in \mathcal{D}(Y)$. Define the *singular support* of $u \in \mathcal{D}'(X)$ by

$$\operatorname{sing\,supp}(u) = X \setminus \bigcup_{\substack{Y \subseteq X \\ \text{open}}} \{Y : u \text{ is smooth on } Y\}$$

i.e the complement of the largest open set on which u is smooth.

Example. We have

$$\operatorname{sing\,supp}(\delta_0) = \{0\}.$$

When looking at sing supp of $I_\Phi(a)$, the following lemma allows us to assume that $a(x, \theta) = 0$ on $|\theta| < 1$ WLOG.

Lemma. *If Φ is a phase function and a a symbol then the function $x \mapsto \int_{\mathbb{R}^k} e^{i\Phi(x,\theta)}\rho(\theta)a(x,\theta)\mathrm{d}\theta$ is smooth for any $\rho \in \mathcal{D}(\mathbb{R}^k)$.*

Proof. Easy. ☐

Fix $\rho \in \mathcal{D}(\mathbb{R}^k)$, $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Then we can write

$$I_\Phi(a) = \underbrace{I_\Phi(\rho a)}_{\text{smooth function}} + I_\Phi(\underbrace{(1-\rho)a}_{:=\tilde{a}}).$$

Note $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ so $\tilde{a} \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ and $\operatorname{sing\,supp}(I_\Phi(a)) = \operatorname{sing\,supp}(I_\Phi(a))$. Clearly $\tilde{a}(x,\theta) = 0$ on $|\theta| < 1$.
It is natural to expect $\int_{\mathbb{R}^k} e^{i\Phi(x,\theta)}a(x,\theta)\mathrm{d}\theta$ to behave badly at the $x \in X$ for which $\nabla_\theta\Phi(x,\theta) = 0$ for some $\theta \in \mathbb{R}^k$. In fact the singular support is contained in the set of such x.

Theorem. *We have*

$$\operatorname{sing\,supp}(I_\Phi(a)) \subseteq \{x \in X : \nabla_\theta\Phi(x,\theta) = 0 \textit{ for some } \theta \in \mathbb{R}^k \setminus \{0\}\}.$$

Proof. Fix $x_0 \in X$ for which

$$\nabla_\theta\Phi(x_0,\theta) \neq 0 \qquad \forall\theta \in \mathbb{R}^k \setminus \{0\}.$$

Note that $\theta \mapsto |\nabla_\theta\Phi(x_0,\theta)|$ is homogenous of degree 0, so is completely determined by values it takes on S^{k-1}. By continuity and compactness there exists $c > 0$ such that

$$|\nabla_\theta\Phi(x_0,\theta)| \geq c \text{ on } \mathbb{R}^k \setminus \{0\}.$$

By continuity, there exists a small open neighbourhood Y of x_0 such that

$$|\nabla_\theta\Phi(x,\theta)| \geq c/2 \text{ on } Y \times (\mathbb{R}^k \setminus \{0\}).$$

Consider

$$\mathcal{D}(Y) \ni \phi \mapsto \langle I_\Phi(a), \phi\rangle.$$

The differential operator

$$L^t = \sum_{j=1}^k \frac{-i\frac{\partial\Phi}{\partial\theta_j}}{|\nabla_\theta\Phi|^2}\frac{\partial}{\partial\theta_j}$$

is well defined on $Y \times (\mathbb{R}^k \setminus \{0\})$ and $L^t e^{i\Phi} = e^{i\Phi}$. Since we can assume $a(x,\theta) = 0$ on $|\theta| < 1$ WLOG, it follows that L^t is well defined on

$$(Y \times \mathbb{R}^k) \cap \operatorname{supp}[a(x,\theta)].$$

By the same arguments as the proof to the first theorem in this section, we can show $L : \mathrm{Sym}(X, \mathbb{R}^k; N) \to \mathrm{Sym}(X, \mathbb{R}^k; N-1)$ so for $\phi \in \mathcal{D}(Y)$ we have

$$\langle I_\Phi(a), \phi\rangle = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^k} \int_X e^{i\Phi} L^M [a(x,\theta)\chi(\varepsilon\theta)]\phi(x) \mathrm{d}x \mathrm{d}\theta$$

noting that $L[a(x,\theta)\chi(\varepsilon\theta)\phi(x)] = L[a(x,\theta)\chi(\varepsilon\theta)]\phi(x)$. Since $a(x,\theta)\chi(\varepsilon\theta) \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ (uniformly in ε) we can choose M large enough that the DCT allows us to take the limit. Then we see

$$\langle I_\Phi(a), \phi\rangle = \int_X \underbrace{\int_{\mathbb{R}^k} e^{i\Phi} L^M a(x,\theta) \mathrm{d}\theta}_{(\dagger)} \, \phi(x) \mathrm{d}x.$$

Since again we can take M as large as we want, we can differentiate under the integral sign to deduce $I_\Phi(a)$ is smooth on Y (i.e ($\dagger$) is smooth on Y). Since $x_0 \in Y$ we get the result. $\square$

Since we had

$$\delta_0(x) = (2\pi)^{-n} \int_{\mathbb{R}^k} e^{ix\cdot\theta} \mathrm{d}\theta$$

we deduce

$$\begin{aligned} \operatorname{sing\,supp}(\delta_0) &\subseteq \{x \in \mathbb{R}^n : \nabla_\theta(x\cdot\theta) = 0 \text{ for some } \theta \in \mathbb{R}^k \setminus \{0\}\} \\ &= \{0\}. \end{aligned}$$

Suppose we wanted to solve (for $c \in \mathbb{R}^n$ constant)

$$\frac{\partial u}{\partial t} + c \cdot \nabla u = 0, \qquad \lim_{t\downarrow 0} u(x', t) = \delta_0(x')$$

i.e $u(\cdot, t) \in \mathcal{D}'(\mathbb{R}^n)$ for each t and

$$\lim_{t\downarrow 0} u(\cdot, t) = \delta_0(x').$$

Writing $x = (x', t)$. We guess by Fourier transform

$$u(x', t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\theta\cdot(x' - ct)} \mathrm{d}\theta$$

and indeed differentiating under the integral we find

$$\frac{\partial u}{\partial t} + c \cdot \nabla u = 0$$

and

$$\begin{aligned} \lim_{t\downarrow 0} u(x', t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\theta\cdot x'} \mathrm{d}\theta \\ &= \delta_0(x) \text{ in } \mathcal{D}'(\mathbb{R}^n). \end{aligned}$$