Motivation

This section is motivation and will not be rigorous. We have a 'Dirac delta function' such that for all 'nice' functions f

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \mathrm{d}x = f(x_0).$$

Can we define $\delta'(x - x_0)$? Could try

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x &= \lim_{h \to 0} \int_{-\infty}^{\infty} \left[\frac{\delta(x - x_0 + h) - \delta(x - x_0)}{h} \right] f(x) \\ &= \lim_{h \to 0} \frac{1}{h} \left[f(x_0 - h) - f(x_0) \right] \\ &= -f'(x_0).\end{aligned}$$

i.e

$$\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x = -\int_{-\infty}^{\infty} \delta(x - x_0) f'(x) \mathrm{d}x$$

which looks like some form of integration-by-parts. We can make this rigorous using distribution theory.

Fourier transform of polynomials

If $f \in L^1(\mathbb{R})$ then

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) \mathrm{d}x.$$

How could we take the Fourier transform of something like $f(x) = x^n$? May recall identity

$$\delta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x$$

and then get

$$\begin{aligned}\hat{f}(\lambda) &= \int_{-\infty}^{\infty} x^n e^{-\lambda x} \mathrm{d}x \\ &= \left(i \frac{\partial}{\partial \lambda} \right)^n \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x \\ &= i^n 2\pi \delta^{(n)}(\lambda).\end{aligned}$$

Recall Parseval's theorem: for suitable f, g

$$\int_{-\infty}^{\infty} \hat{g}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} g(x) \hat{f}(x) \mathrm{d}x.$$

Could define the Fourier transform of $g(x) = x$ to be the function $\lambda \mapsto \hat{x}(\lambda)$ such that

$$\int_{-\infty}^{\infty} \hat{x}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} x \hat{f}(x) \mathrm{d}x$$

for all 'nice' functions f. We can make this rigorous using distributions.

Discontinuous solutions to PDEs

From linear acoustics, air pressure $p = p(x,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0. \tag{$*$}$$

Could introduce a 'nice' $f = f(x,t)$, say $f \in C_c^\infty(\mathbb{R}^2)$. Then $(*)$ implies

$$\int\int (p_{xx} - p_{tt}) f(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

So we can integrate by parts to interpret this as

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

We say that $p = p(x,t)$ is a *weak solution* to $(*)$ if

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. In each case, to extend a definition to a larger domain of applicability, we had to introduce a space of "nice" functions.

This is the theme of distribution theory: functions get replaced by linear maps on some auxilliary space of test functions V. A *distribution* is a linear map $u : V \to \mathbb{C}$, i.e we study the topological dual of V. Let $\langle \cdot, \cdot \rangle$ denote pairing between v and V^*, i.e for $u \in V^*$, $f, g \in V$, $\alpha, \beta \in \mathbb{C}$

$$\langle u, \alpha f + \beta g \rangle = \alpha \langle u, f \rangle + \beta \langle u, g \rangle.$$

The topological dual V^* consists of linear $u : V \to \mathbb{C}$ such that whenever $f_n \to f$ in V, we have $\langle u, f_n \rangle \to \langle u, f \rangle$ in $\mathbb{C}$. For example we could take $V = C^\infty(\mathbb{R})$ equipped with the topology of uniform convergence (i.e $f_n \to f$ in V if for all compact $K \subseteq \mathbb{R}$ and all $n \geq 0$, $\left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (f_n - f) \right| \to 0$) then $\delta_{x_0} : V \to bbC$ defined by $\langle \delta_{x_0}, f \rangle = f(x_0)$. Note that this is indeed continuous.

1 Distributions

1.1 Notation & Preliminaries

Throughout (unless otherwise specified) X, Y denote open subsets of $\mathbb{R}^n$, K a compact subset of $\mathbb{R}^n$. Integrals over $X, \mathbb{R}^n$ are written as $\int_X [\cdot]\mathrm{d}x$, $\int [\cdot]\mathrm{d}x$ respectively.

1.2 Distributions & Test Functions

Definition. The space $\mathcal{D}(X)$ consists of smooth functions $\varphi : X \to \mathbb{C}$ of compact support. We say a sequence $(\varphi_m)_{m\geq 0}$ in $\mathcal{D}(X)$ converges to 0 in $\mathcal{D}(X)$ if there exists $K \subseteq X$ compact such that $\mathrm{supp}(\varphi_m) \subseteq K$ and $\sup_K |\partial^\alpha \varphi_m| \to 0$ for all multi-indices α.

Functions in $\mathcal{D}(X)$ have nice properties. For example, if $\varphi \in \mathcal{D}(X)$ then $\varphi = 0$ before you reach the boundary of X. This means integration-by-parts is easy since

$$\int_X \varphi \partial^\alpha \psi \mathrm{d}x = (-1)^{|\alpha|} \int_X \psi \partial^\alpha \varphi \mathrm{d}x.$$

Since $\varphi \in \mathcal{D}(X)$ is smooth we have

$$\varphi(x+h) = \sum_{|\alpha|\leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h)$$

where R_N is $o(|h|^N)$ uniformly in x.

Definition. A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is called a *distribution* if for all $K \subseteq X$ compact there exist $C, N \geq 0$ such that

$$|u(\varphi)| \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha \varphi| \qquad (*)$$

for all $\varphi \in \mathcal{D}(X)$ with $\mathrm{supp}(\varphi) \subseteq K$. The space of such linear maps is denoted by $\mathcal{D}'(X)$, i.e "distributions on X". If the same N can be used in $(*)$ for all compact $K \subseteq X$, say the least such N is the order of u, written $\mathrm{ord}(u)$.

For $x_0 \in X$ define $\delta_{x_0}(\varphi) = \varphi(x_0)$ for $\varphi \in \mathcal{D}(X)$. Then $\delta_{x_0} : \mathcal{D}(X) \to \mathbb{C}$ is linear and

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \sup |\varphi|$$

so we can take $C = 1, N = 0$ in $(*)$, so $\mathrm{ord}(\delta_{x_0}) = 0$.

For $\{f_\alpha\}$ in $C(X)$, define $T : \mathcal{D}(X) \to \mathbb{C}$ by

$$T(\varphi) = \sum_{|\alpha|\leq M} \int_X f_\alpha \partial^\alpha \varphi \mathrm{d}x.$$

Take $\varphi \in \mathcal{D}(X)$ with $\text{supp}(\varphi) \subseteq K$. Then

$$\begin{aligned} |T(\varphi)| &\leq \sum_{|\alpha|\leq M} \int_K |f_\alpha||\partial^\alpha \varphi|\mathrm{d}x \\ &\leq \left(\max_\alpha \int_K |f_\alpha|\mathrm{d}x\right) \sum_{|\alpha|\leq M} \sup |\partial^\alpha \varphi| \end{aligned}$$

so $(*)$ holds with $C = \max_\alpha \int_K |f_\alpha|\mathrm{d}x$, $N = M$. Hence $T \in \mathcal{D}'(X)$.

Note this estimate would hold if the $\{f_\alpha\}$ were only assumed locally integrable, written $f_\alpha \in L^1_{\text{loc}}(X)$.

Remark. For $f \in L^1_{\text{loc}}$ we have a corresponding distribution $T_f : \mathcal{D}(X) \to \mathbb{C}$ defined by $T_f(\varphi) = \int_X f\varphi \mathrm{d}x$. We often simply write $T_f = f$.

Lemma. *A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is a distribution if and only if $u(\varphi_m) \to 0$ for all sequences $\varphi_m \to 0$ in $\mathcal{D}(X)$.*

Proof. Suppose $u \in \mathcal{D}'(X)$ and $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then $\text{supp}(\varphi_m) \subseteq K$ for some K independent of m and there exist $C, N \geq 0$

$$|\varphi_m(u)| \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha \varphi_m| \to 0$$

for all α.

Suppose not, i.e $u : \mathcal{D}(X) \to \mathbb{C}$ is linear and $u(\varphi_m) \to 0$ whenever $\varphi_m \to 0$ in $\mathcal{D}(X)$, but u is not a distribution. Then there is a compact set $K \subseteq X$ such that for all C, N, $(*)$ fails on some φ with support contained in K. So there must be some $\varphi_m \in \mathcal{D}(X)$ with $\text{supp}(\varphi_m) \subseteq K$ and

$$|u(\varphi_m)| > m \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Now replace φ_m with $\varphi'_m = \frac{\varphi_m}{u(\varphi_m)}$. So we may assume $u(\varphi_m) = 1$ WLOG. Hence

$$1 > m \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Therefore $\sup_K |\partial^\alpha \varphi_m| < \frac{1}{m}$ for all $|\alpha| \leq m$. Hence $\varphi_m \to 0$ in $\mathcal{D}(X)$, giving a contradiction since $u(\varphi_m) \not\to 0$. □

1.3 Limits in $\mathcal{D}'(X)$

We often have some sequence (u_m) in $\mathcal{D}'(X)$. If there is some $u \in \mathcal{D}'(X)$ such that $\varphi(u_m) \to \varphi(u)$ for all φ we say $u_m \to u$ in $\mathcal{D}'(X)$.

Theorem (*Non-examinable*). *If* (u_m) *is a sequence in* $\mathcal{D}'(X)$ *and* $u(\varphi) = \lim_{m\to\infty} u(\varphi_m)$ *exists for all* $\varphi \in \mathcal{D}(X)$*, then* $u \in \mathcal{D}'(X)$*.*

Proof. Not given. □

Take $u_m \in \mathcal{D}'(\mathbb{R})$ defined by $u_m(\varphi) = \int \sin(mx)\varphi(x)\mathrm{d}x$. By integration-by-parts we have

$$|\varphi(u_m)| = \left|\frac{1}{m}\int \cos(mx)\varphi'(x)\mathrm{d}x\right| \to 0.$$

i.e $\sin(mx) \to 0$ in $\mathcal{D}'(\mathbb{R})$.

1.4 Basic Operations

1.4.1 Differentiation & Multiplication by Smooth Functions

For $u \in C^\infty(X) \subseteq L^1_{\text{loc}}(X)$, $\partial^\alpha u \in D'(X)$ by

$$\begin{aligned}\langle \partial^\alpha u, \phi\rangle &= \int_X \phi \partial^\alpha u \mathrm{d}x \\ &= (-1)^{|\alpha|} \int_X u \partial^\alpha \phi \mathrm{d}x \\ &= (-1)^{|\alpha|} \langle u, \partial^\alpha \phi\rangle.\end{aligned}$$

This leads to

Definition. For $u \in D'(X)$, $f \in C^\infty(X)$ define

$$\langle \partial^\alpha (fu), \phi\rangle := (-1)^\alpha \langle u, f\partial^\alpha \phi\rangle$$

for $\phi \in \mathcal{D}(X)$ [note $\partial^\alpha (fu) \in \mathcal{D}'(X)$]. We call $\partial^\alpha u$ the *distributional derivatives* of u.

For δ_x we have

$$\begin{aligned}\langle \partial^\alpha \delta_x, \phi\rangle &= (-1)^{|\alpha|} \langle \delta_x, \partial^\alpha \phi\rangle \\ &= (-1)^{|\alpha|} \partial^\alpha \phi(x).\end{aligned}$$

Define the *Heaviside function*

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Then $H \in L^1_{\text{loc}}(\mathbb{R})$ so

$$\langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_0^\infty \phi'(x) \mathrm{d}x = \phi(0) = \langle \delta_0, \phi\rangle.$$

Hence $H' = \delta_0$. Generally we say $u = v$ in $\mathcal{D}'(X)$ if $\langle u, \cdot\rangle = \langle v, \cdot\rangle$.

Lemma. *If $u \in \mathcal{D}'(\mathbb{R})$ and $u' = 0$ in $\mathcal{D}'(\mathbb{R})$ then u is constant.*

Proof. Fix $\theta \in \mathcal{D}(\mathbb{R})$ with $\langle 1, \theta\rangle = \int_{\mathbb{R}} \theta \mathrm{d}x = 1$. For $\phi \in \mathcal{D}(\mathbb{R})$ write

$$\phi = \underbrace{(\phi - \langle 1, \phi\rangle \theta)}_{:=\phi_A} + \underbrace{\langle 1, \phi\rangle \theta}_{:=\phi_B}.$$

Note that $\langle 1, \phi_A\rangle = \int_{\mathbb{R}} \phi_A \mathrm{d}x = 0$ so we have

$$\Phi_A(x) := \int_{-\infty}^x \phi_A(t) \mathrm{d}t$$

defines $\Phi_A \in \mathcal{D}(\mathbb{R})$ with $\Phi_A' = \phi_A$. So

$$\begin{aligned}\langle u, \phi\rangle &= \langle u, \phi_A\rangle + \langle u, \phi_B\rangle \\ &= \langle u, \Phi_A'\rangle + \langle 1, \phi\rangle\langle u, \theta\rangle \\ &= \underbrace{-\langle u', \phi_A\rangle + \langle 1, \phi\rangle}_{=0} + \underbrace{\langle 1, \phi\rangle\langle u, \theta\rangle}_{:=c \text{ constant}}\end{aligned}$$

so u is constant in $\mathcal{D}'(\mathbb{R})$. □

1.4.2 Translation & Reflection

If $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ define *reflection* and *translation* by

$$\check{\phi}(x) = \phi(-x), \ (\tau_h\phi)(x) = \phi(x-h).$$

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ we define

$$\langle \check{u}, \phi\rangle = u, \check{\phi} \qquad \text{(reflection)}$$

and

$$\langle \tau_h u, \phi\rangle = \langle u, \tau_{-h}\phi\rangle \qquad \text{(translation)}$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

Lemma. *For $u \in \mathcal{D}'(\mathbb{R}^n)$ define*

$$v_h = \frac{\tau_{-h}u - u}{h}.$$

fIf $\frac{h}{|h|} \to m \in \mathbb{S}^{n-1}$ as $|h| \to 0$ then $v_h \to m \cdot \partial u$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. For $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\langle v_h, \phi\rangle = \langle u, \frac{\tau_h\phi - \phi}{h}\rangle.$$

By Taylor's theorem

$$(\tau_h\phi - \phi)(x) = \phi(x-h) - \phi(x) = -\sum_i h_i \frac{\partial\phi}{\partial x_i}(x) + R_1(x,h)$$

where $R_1 = o(|h|)$ in $\mathcal{D}(\mathbb{R}^n)$ [see Example Sheet 1] so by sequential continuity

$$\begin{aligned}\langle v_h, \phi\rangle &= -\sum_i \frac{h_i}{|h|}\langle u, \frac{\partial\phi}{\partial x_i}\rangle + o(1) \\ &= \langle \sum_i \frac{h_i}{|h|}\frac{\partial u}{\partial x_i}, \phi\rangle + o(1) \\ &\to \langle m \cdot \partial u, \phi\rangle \text{ as } |h| \to 0.\end{aligned}$$

□

1.4.3 Convolution in $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^\ltimes)$

For $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(\tau_x \check{\phi})(y) = \check{\phi}(y-x) = \phi(x-y).$$

If $u \in C^\infty(\mathbb{R}^n)$ define convolution with $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned} u * \phi(x) &= \int_{\mathbb{R}^n} u(x-y)\phi(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \phi(x-y)u(y)\mathrm{d}y \\ &= \langle u, \tau_x \check{\phi} \rangle. \end{aligned}$$

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ define

$$(u * \phi)(x) = \langle u, \tau_x \check{\phi} \rangle.$$

How regular is $u * \phi$?

Lemma. *For $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ write $\Phi_x(y) = \phi(x, y)$. If for each $x \in \mathbb{R}^n$ there exists a neighbourhood $N_x \subseteq \mathbb{R}^n$ of x and compact set $K \subseteq \mathbb{R}^n$ such that*

$$\operatorname{supp}(\phi|_{N_x \times \mathbb{R}^n}) \subseteq N_x \times K$$

then $\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi \rangle$ for $u \in \mathcal{D}'(\mathbb{R}^n)$.

Proof. By Taylor's theorem

$$\Phi_{x+h}(y) - \Phi_x(y) = \sum_i h_i \frac{\partial \phi}{\partial x_i}(x, y) + R_1(x, y, h).$$

For $|h|$ sufficienly small we have $x + h \in N_x$ so $\operatorname{supp}(R_1(x, \cdot, h)) \subseteq K$ and also

$$sup_y |\partial_y^\alpha R(x, y, h)| = o(|h|)$$

so $R_1(x, \cdot, h) = o(|h|)$ in $\mathcal{D}(\mathbb{R}^n)$. By sequential continuity

$$\langle u, \Phi_{x+h} \rangle - \langle u, \Phi_x \rangle = \sum_i h_i \langle u, \frac{\partial}{\partial x_i} \Phi_x \rangle + o(|h|)$$

and so $\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial}{\partial x_i} \Phi_x \rangle$ and the result follows by induction. $\square$

Corollary. *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ then $u * \phi \in C^\infty(\mathbb{R}^n)$ and*

$$\partial^\alpha(u * \phi) = u * \partial^\alpha \phi.$$

Proof. Have $(u * \phi)(x) = \langle u, \tau_x \check{\phi} \rangle$ so take $\Phi_x = \tau_x \check{\phi}$ in previous lemma. □

1.5 Density of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{D}'(\mathbb{R}^n)$

Can use previous result to prove an important theorem. First we need

Lemma. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$ then*

$$(u * \phi) * \psi = u * (\phi * \psi).$$

Proof. Fix $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
(u * \phi) * \psi(x) &= \int_{\mathbb{R}^n} (u * \phi)(x - y)\psi(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \langle u, \tau_{x-y}\check{\phi} \rangle \psi(y)\mathrm{d}y \\
&= \lim_{h \to 0} \sum_{m \in \mathbb{Z}^n} \langle u, \tau_{x-hm}\check{\phi}\psi(hm) \rangle h^n && \text{(Riemann sum)} \\
&= \lim_{h \to 0} \langle u, \sum_{m \in \mathbb{Z}_n} \tau_{x-hm}\check{\phi}\psi(hm)h^n \rangle && \text{(Finite sum)} \\
&= \langle u, \lim_{h \to 0} \sum_{m \in \mathbb{Z}_n} \tau_{x-hm}\check{\phi}\psi(hm)h^n \rangle \\
&= \langle u, \tau_x \phi \check{*} \psi \rangle \\
&= u * (\phi * \psi).
\end{aligned}$$

□

****Non-examinable****

We can justify the exchange of the limit and the $\langle u, \cdot \rangle$ by defining for $|h| \leq 1$ the family of functions $\{F_h\}$ by

$$F_h(z) = \sum_{m \in \mathbb{Z}^n} \phi(x - z - hm)\psi(hm)h^m.$$

It is straightforward to see that $\mathrm{supp}(F_h)$ lies in some fixed compact $K \subseteq \mathbb{R}^n$. Also each F_h is in $C^\infty(\mathbb{R}^n)$. Note that for each multi-index α we have

$$\sup_z |\partial^\alpha F_h(z)| \leq M_\alpha.$$

So for each α, $z \mapsto \partial^\alpha F_h(z)$ is uniformly bounded and equi-continuous. Equi-continuity follows from

$$\begin{aligned}|\partial^\alpha F_h(x) - \partial^\alpha F_h(y)| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\partial^\alpha F_h(tx+(1-t)y)\mathrm{d}t\right| \\ &= \left|\int_0^1 (x-y)\cdot\nabla\partial^\alpha F_h(tx+(1-t)y)\mathrm{d}t\right| \\ &\lesssim_\alpha |x-y|.\end{aligned}$$

Applying Arzela-Ascoli and a diagonal argument we get a sequence (h_k) such that $\sup_z |\partial^\alpha(F_{h_k} - \tau_x\phi \check{*} \psi)| \to 0$ for each α.

Theorem. *For $u \in \mathcal{D}'(\mathbb{R}^n)$ there exists (ϕ_k) in $\mathcal{D}(\mathbb{R}^n)$ such that $\phi_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$ (i.e $\langle u_k, \theta\rangle \to \langle u, \theta\rangle$ for all $\theta \in \mathcal{D}(\mathbb{R}^n)$).*

Proof. Fix $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\psi \mathrm{d}x = 1$ and set $\psi_k(x) = k^n\psi(kx)$. Fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi = 1$ on $[-1,1]$ and $\mathrm{supp}(\chi) \subseteq [-2,2]$. Set $\chi_k(x) = \chi(x/k)$. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and arbitrary $\theta \in \mathcal{D}(\mathbb{R}^n)$ consider $\langle \phi_k, \theta\rangle$ where $\phi_k = (u * \psi_k)\chi_k$. Then

$$\begin{aligned}\langle \phi_k, \theta\rangle &= \langle u*\psi_k, \chi_k\theta\rangle \\ &= (u*\psi_k) * (\check{\chi_k\theta})(0) \\ &= u * (\psi_j * (\check{\chi_k\theta}))(0) && \text{(previous lemma)}\end{aligned}$$

where we used the fact $\langle v, f\rangle = v * \check{f}(0)$. Note

$$\begin{aligned}\psi_k * (\check{\chi_j\theta})(x) &= \int k^n\psi(k(x-y))\chi(-y/k)\theta(-y)\mathrm{d}y \\ &= \int \psi(y')\chi\left(\frac{y'}{k^2} - \frac{x}{k}\right)\theta\left(\frac{y'}{k} - x\right)\mathrm{d}y' && (y' = k(x-y)) \\ &= \theta(-x) + R_k(-x)\end{aligned}$$

where

$$R_k(x) = \int \psi(y)\left[\chi\left(\frac{y}{k^2}+\frac{x}{k}\right)\theta\left(\frac{y}{k}+x\right) - \theta(x)\right]\mathrm{d}y.$$

So

$$\begin{aligned}\langle \phi_k, \theta\rangle &= u * \check{\theta}(0) + u * \check{R}_k(0) \\ &= \langle u, \theta\rangle + \langle u, R_k\rangle.\end{aligned}$$

It is straightfoward to show $R_k \to 0$ in $\mathcal{D}(\mathbb{R}^n)$ [exercise]. □

2 Distributions of Compact Support

Let $Y \subseteq X$ be open. We say $u \in \mathcal{D}'(X)$ vanishes on Y if $\langle u, \phi\rangle = 0$ for all $\phi \in \mathcal{D}(Y)$.

Definition. For $u \in \mathcal{D}'(X)$ define the support of u by

$$\operatorname{supp}(u) = X \setminus \left(\bigcup_{\substack{Y \subseteq X \text{ open} \\ u \text{ vanishes on } Y}} Y \right).$$

E.g for $\delta_x \in \mathcal{D}'(\mathbb{R}^n)$ we have $\operatorname{supp}(\delta_x) = \{x\}$.

****Non-examinable****

If $u \in \mathcal{D}'(X)$ vanishes on a collection $\{U_\lambda\}$ of open sets, then it vanishes on the union. Indeed suppose $\operatorname{supp}(\phi) \subseteq \bigcup_\lambda U_\lambda$. By compactness there is a finite collection $\{U_i\}_{i=1}^N$ such that $\operatorname{supp}(\phi) \subseteq \bigcup_{i=1}^N U_i$.

Take a partition of unity $\{\psi_i\}_{i=1}^N$ subordinate to $\{U_i\}_{i=1}^N$, i.e $\operatorname{supp}(\psi_i) \subseteq U_i$ and $\sum_{i=1}^N \psi_i = 1$. Then

$$\langle u, \phi \rangle = \sum_{i=1}^N \langle u, \psi_i \phi \rangle = 0.$$

A corollary of this is that $\operatorname{supp}(u)$ is the complement of the largest open set on which u vanishes.

2.1 More test functions & distributions

Definition. Define $\mathcal{E}(X)$ to be the space of smooth functions $\phi : X \to \mathbb{C}$. We say $\phi_m \to 0$ in $\mathcal{E}(X)$ if for each multi-index α we have $\partial^\alpha \phi_m \to 0$ locally uniformly, i.e $\sup_K |\partial^\alpha \phi| \to 0$ for all $K \subseteq X$ compact.

Definition. A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ if there exists $K \subseteq X$ compact and constants $C, N \geq 0$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi|$$

for all $\phi \in \mathcal{E}(X)$.

Lemma. *A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ if and only if $\langle u, \phi_m \rangle \to 0$ whenever $\phi_m \to 0$ in $\mathcal{E}(X)$.*

Proof. Almost identical to that of $\mathcal{D}'(X)$. □

Lemma. *If $u \in \mathcal{E}'(X)$ then $u|_{\mathcal{D}(X)}$ defines an element of $\mathcal{D}'(X)$ with compact support. Conversely if $u \in \mathcal{D}'(X)$ has compact support there exists a unique $\tilde{u} \in calE'(X)$ which extends u to $\mathcal{E}(X)$.*

Proof. Note that $\mathcal{D}(X) \subseteq \mathcal{E}(X)$ so if $u \in \mathcal{E}'(X)$ then $u|_{\mathcal{D}(X)}$ is well-defined. There exist compact $K \subseteq X$ and constants $C, N \geq 0$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi|$$

for all $\phi \in \mathcal{D}(X)$. Hence $u|_{\mathcal{D}(X)} \in \mathcal{D}'(X)$ and $\text{supp}(u) \subseteq K$.

If $u \in \mathcal{D}'(X)$ has compact support, fix $\rho \in \mathcal{D}(X)$ such that $\rho = 1$ on a neighbourhood of $\text{supp}(u)$. Define $\tilde{u} : \mathcal{E}(X) \to \mathbb{C}$ by $\langle \tilde{u}, \phi \rangle = \langle u, \rho\phi \rangle$ for each $\phi \in \mathcal{E}(X)$. Then $\text{supp}(\rho\phi) \subseteq \text{supp}(\phi)$. Since $u \in \mathcal{D}'(X)$ there exist constants $C, N \geq 0$ such that

$$\begin{aligned} \langle \tilde{u}, \phi \rangle| = |\langle u, \rho\phi \rangle| &\leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha (\rho\phi)| \\ &\leq C' \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi| \end{aligned}$$

so $\tilde{u} \in \mathcal{E}'(X)$. Suppose $\tilde{v} \in \mathcal{E}'(X)$ has $\tilde{v}|_{\mathcal{D}(X)} = u$ and $\text{supp}(\tilde{v}) = \text{supp}(u)$. With $\rho \in \mathcal{D}(X)$ as before

$$\begin{aligned} \langle \tilde{v}, \phi \rangle &= \langle \tilde{v}, \rho\phi \rangle + \langle \tilde{v}, (1-\rho)\phi \rangle \\ &= \langle \tilde{u}, \rho\phi \rangle + \langle \tilde{u}, (1-\rho)\phi \rangle \\ &= \langle \tilde{u}, \phi \rangle \end{aligned}$$

for all $\phi \in \mathcal{E}(X)$, i.e $\tilde{u} = \tilde{v}$. □

2.2 Convolution between $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$

For $\phi \in \mathcal{E}(\mathbb{R}^n)$, $u \in \mathcal{E}'(\mathbb{R}^n)$ define convolution as before by

$$u * \phi(x) = \langle u, \tau_x \check{\phi} \rangle.$$

We find $u * \phi \in \mathcal{E}(\mathbb{R}^n)$. Note that $u * \phi(x) = 0$ unless $(x - y) \in \operatorname{supp}(\phi)$ for some $y \in \operatorname{supp}(u)$, i.e $\operatorname{supp}(u * \phi) \subseteq \operatorname{supp}(\phi) + \operatorname{supp}(u)$. In particular if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have $u * \phi \in \mathcal{D}(\mathbb{R}^n)$.

Definition. Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$ where at least one of u, v has compact support. Then define

$$(u * v) * \phi := u * (v * \phi)$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then $u * v \in \mathcal{D}'(\mathbb{R}^n)$ [see Example Sheet 2].

Lemma. *For u, v as in the above definition. $u * v = v * u$.*

Proof. Recall by a previous lemma that if $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$ then $(u * \phi) * \psi = u * (\phi * \psi)$. The same holds if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\phi, \psi \in \mathcal{E}(\mathbb{R}^n)$ with at least one of $\operatorname{supp}(\phi), \operatorname{supp}(\psi)$ compact. We use this repeatedly as follows: for $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned}
(u * v) * (\phi * \psi) &= u * [v * (\phi * \psi)] \\
&= u * [(v * \phi) * \psi] \\
&= u * [\psi * (v * \phi)] \\
&= (u * \psi) * (v * \phi).
\end{aligned}$$

So using $\phi * \psi = \psi * \phi$ we have

$$\begin{aligned}
(v * u) * (\phi * \psi) &= (v * \phi) * (u * \psi) \\
&= (u * \psi) * (v * \phi) \\
&= (u * v) * (\phi * \psi).
\end{aligned}$$

So if $E = u * v - v * u$ we have $E * (\phi * \psi) = 0$ for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^n)$. Thus $(E * \phi) * \psi = 0$ and $E * \phi = 0$, so $E = 0$ in $\mathcal{D}'(\mathbb{R}^n)$, i.e $u * v = v * u$. $\square$

The above implies that for any $u \in \mathcal{D}'(\mathbb{R}^n)$ we have

$$\delta_0 * u = u * \delta_0 = u$$

since for $\psi \in \mathcal{D}(\mathbb{R}^n)$

$$(u * \delta_0) * \psi = u * (\delta_0 * \psi) = u * \psi$$

and

$$\begin{aligned}
(\delta_0 * \psi)(x) &= \langle \delta_0, \tau_x \check{\psi} \rangle \\
&= (\tau_x \check{\psi})(0) \\
&= \check{\psi}(-x) \\
&= \psi(x).
\end{aligned}$$

3 Tempered Distributions & Fourier Analysis

3.1 More test functions & distributions

Definition. The *Schwartz space* written $\mathcal{S}(\mathbb{R}^n)$, consists of smooth $\phi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|\phi\|_{\alpha,\beta} := \sup |x^\alpha D^\beta \phi| < \infty$$

for all multi-indices α, β. We say $\phi_m \to 0$ in $\mathcal{S}$ if $\|\phi_m\|_{\alpha,\beta} \to 0$ for all α, β. Elements of the Schwartz space are sometimes called *rapidly decaying functions.*

Definition. A linear map $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$, the *space of tempered distributions*, if there exist $C, N \geq 0$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{|\alpha|,|\beta} \|\phi\|_{\alpha,\beta}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Lemma. *A linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ iff $\langle u, \phi_m \rangle \to 0$ whenever $\phi_m \to 0$ in $\mathcal{S}$.*

Proof. Exercise. □

Note that $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n)$ in the sense of continuous inclusions, i.e

$$\phi_m \xrightarrow{\mathcal{D}} 0 \implies \phi_m \xrightarrow{\mathcal{S}} 0 \implies \phi_m \xrightarrow{\mathcal{E}} 0.$$

Which gives the continuous inclusions $\mathcal{E}'(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$.

It turns out that $\mathcal{S}$ is ideal for Fourier analysis.

3.2 Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$

Definition. For an integrable function $f \in L^1(\mathbb{R}^n)$ define the *Fourier transform* of f by

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} f(x) \mathrm{d}x, \ \lambda \in \mathbb{R}^n.$$

We use $\mathcal{F}$ to denote the linear map $f \mapsto \hat{f}$.

Note that $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ since for $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\phi| \mathrm{d}x &= \int_{\mathbb{R}^n} (1+|x|)^{-N} (1+|x|)^N |\phi| \mathrm{d}x \\ &\leq C \sum_{|\alpha| \leq N} \|\phi\|_{\alpha,0} \int_{\mathbb{R}^n} (1+|x|)^{-N} \mathrm{d}x \\ &< \infty\end{aligned}$$

for $N \geq n+1$.

Lemma. *If $f \in L^1(\mathbb{R}^n)$ then $\hat{f} \in C(\mathbb{R}^n)$.*

Proof. DCT. $\square$

Intuitively, the Fourier transform interchanges decay & smoothness.
Notation: we write D^α for $(-i)^\alpha \nabla^\alpha$.

Lemma. *For $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\begin{aligned}(\hat{D^\alpha \phi})(\lambda) &= \lambda^\alpha \hat{\phi}(\lambda) \\ (\hat{x^\beta \phi})(\lambda) &= (-D)^\beta \hat{\phi}(\lambda).\end{aligned}$$

Proof. Integration-by-parts gives

$$\begin{aligned}(\hat{D^\alpha \phi})(\lambda) &= \int_{\mathbb{R}}^n e^{-i\lambda\cdot x} D^\alpha \phi \mathrm{d}x \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \phi D^\alpha [e^{-i\lambda\cdot x}] \mathrm{d}x \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} (-1)^{|\alpha|} \phi \lambda^\alpha e^{-i\lambda\cdot x} \mathrm{d}x \\ &= \lambda^\alpha \hat{\phi}(\lambda)\end{aligned}$$

and

$$\begin{aligned}(-D)^\beta \hat{\phi}(\lambda) &= (-D)^\beta \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} \phi(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^n} x^\beta e^{-i\lambda\cdot x} \phi(x) \mathrm{d}x \qquad \text{(DCT)} \\ &= (\hat{x^\beta \phi})(\lambda).\end{aligned}$$

$\square$

Note that the above show that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

Theorem. *The Fourier transform is a continuous isomorphism on $\mathcal{S}(\mathbb{R}^n)$, i.e $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a homeomorphism.*

Proof. We know $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ so $C^\infty(\mathbb{R}^n)$. We also have

$$\begin{aligned}\left|\lambda^\alpha D^\beta \hat{\phi}(\lambda)\right| &= \left|\int_{\mathbb{R}^n} D^\alpha (x^\beta \phi) e^{-i\lambda\cdot x} \mathrm{d}x\right| \\ &\leq \int_{\mathbb{R}^n} |D^\alpha (x^\beta \phi)| \mathrm{d}x < \infty. \qquad (\dagger)\end{aligned}$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have $D^\alpha(x^\beta \phi) \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. Hence $\|\hat{\phi}\|_{\alpha,\beta} < \infty$ for all α, β, i.e $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$. Hence $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ to itself.

By suitably applying (†) to a sequence $\phi_m \to 0$ in $\mathcal{S}$, it's easy to see $\hat{\phi}_m \to 0$ in $\mathcal{S}$. We have

$$\int_{\mathbb{R}^n} e^{i\lambda\cdot x}\hat{\phi}(\lambda)\mathrm{d}\lambda = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n} e^{i\lambda\cdot x}e^{-\varepsilon|\lambda|^2}\hat{\phi}(\lambda)\mathrm{d}\lambda.$$

Also

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{i\lambda\cdot x-\varepsilon|\lambda|^2}\hat{\phi}(\lambda)\mathrm{d}\lambda &= \int_{\mathbb{R}^n}\phi(y)\left[\int_{\mathbb{R}^n} e^{i\lambda\cdot(x-y)-\varepsilon|\lambda|^2}\mathrm{d}\lambda\right]\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\phi(y)\left[\prod_{j=1}^n\left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{-(x_j-y_j)^2/4\varepsilon}\right]\mathrm{d}y \qquad (*) \\
&= \int_{\mathbb{R}^n}\phi(y)\left(\frac{\pi}{\varepsilon}\right)^{n/2} e^{-|x-y|^2/4\varepsilon}\mathrm{d}y \\
&= \int_{\mathbb{R}^n}\phi(x-2\sqrt{\varepsilon}y)\pi^{n/2}2^n e^{-|y'|^2}\mathrm{d}y' \qquad \left(y=\tfrac{x-y}{2\sqrt{\varepsilon}}\right) \\
&\xrightarrow{\varepsilon\downarrow 0} \phi(x)(2\pi)^n\left(\frac{1}{\sqrt{\pi}}\right)^n\int_{\mathbb{R}^n} e^{-|y|^2}\mathrm{d}y \\
&= (2\pi)^n\phi(x).
\end{aligned}$$

Thus $\phi(-x) = \mathcal{F}\left[\frac{\hat{\phi}}{(2\pi)^n}\right]$. So we get a homeomorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. $(*)$ follows from

$$\int_{\mathbb{R}^n} e^{i\lambda\cdot(x-y)-\varepsilon|\lambda|^2}\mathrm{d}\lambda = \prod_{j=1}^n\int_{\mathbb{R}} e^{i\lambda_j\cdot(x_j-y_j)}e^{-\varepsilon\lambda_j^2}\mathrm{d}\lambda_j.$$

Followed by

$$\begin{aligned}
\int_{\mathbb{R}} e^{i\lambda\sigma}e^{-\varepsilon\lambda^2}\mathrm{d}\lambda &= \int_{\mathbb{R}} e^{-\varepsilon\left(\lambda-\frac{i\sigma}{2\varepsilon}\right)^2-\frac{\sigma^2}{4\varepsilon}}\mathrm{d}\lambda \\
&= e^{-\frac{\sigma^2}{4\varepsilon}}\int_{\mathbb{R}} e^{-\varepsilon\left(\lambda-\frac{i\sigma}{2\varepsilon}\right)^2}\mathrm{d}\lambda.
\end{aligned}$$

□