1 Lebesgue Integration Theory

1.1 Review of measure theory

Definition. Given a set E, a *σ-algebra* on E is a collection $\mathcal{E}$ of subsets of E such that:

(i) $E \in \mathcal{E}$;

(ii) $A \in \mathcal{E} \Rightarrow A^c = E \setminus A \in \mathcal{A}$;

(iii) $A_n \in \mathcal{E}$, $n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$.

$(E, \mathcal{E})$ is called a *measurable space*, and any $A \in \mathcal{E}$ is called a *measurable set.*

Given a collection $\mathcal{A}$ of subsets of E, $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$.

Definition. A *measure* on $(E, \mathcal{E})$ is a function $\mu : \mathcal{E} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$;

(ii) $A_n \in \mathcal{E}$, $n \in \mathbb{N}$ disjoint $\Rightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

$(E, \mathcal{E}, \mu)$ is called a *measure space.*

Definition (Borel measure)**.** If (E, τ) is a topological space, then $\sigma(\tau)$ is called a *Borel algebra*, denoted $\mathcal{B}(E)$, and a measure on $(E, \mathcal{B}(E))$ is called a *Borel measure.*

Example. $E = \mathbb{R}^n$, μ the Lebesgue measure satisfying $\mu((a_1, b_1) \times \ldots \times (a_n, b_n)) = (b_1 - a_1) \ldots (b_n - a_n)$.

Notation: we write $\mu(\mathrm{d}x) = \mathrm{d}x$ and $\mu(A) = |A|$ when μ is the Lebesgue measure.

Definition (Measurable function)**.** Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Then $f : E \to F$ is *measurable* if $f^{-1}(A) \in \mathcal{E}$ whenever $A \in \mathcal{F}$. If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are Borel algebras, a measurable function is called a *Borel function.* Special case: $(F, \mathcal{F}) = ([0, \infty], \mathcal{B}([0, \infty]))$, then $f : E \to F$ is called a *non-negative measurable function.*

Fact. The class of measurable functions is closed under addition, multiplication and taking (pointwise) limits.

Definition. $f : E \to F$ ($F = [0, \infty]$ or $\mathbb{R}^n$ or $\mathbb{C}^n$) is a *simple function* if $f = \sum_{k=1}^{K} a_k \mathbb{1}_{A_k}$ for some $K \in \mathbb{N}$, $a_k \in F$, $A_k \in \mathcal{E}$. For a simple function, the integral is

$$\int f \mathrm{d}\mu = \int f(x) \mathrm{d}\mu(x) = \sum_{k=1}^{K} a_k \mu(A_k) \ (0 \cdot \infty := 0).$$

For a non-negative measurable f, we define

$$\int f \mathrm{d}\mu = \sup \left\{ \int g \mathrm{d}\mu : g \text{ simple }, 0 \leq g \leq f \right\}.$$

Definition. A measurable function $f : E \to \mathbb{R}$ is said to be *integrable* if $\int |f| \mathrm{d}\mu < \infty$. Write $f = f_+ - f_-$ with $f_\pm$ non-negative, measurable, $\int f_\pm \mathrm{d}\mu < \infty$, and then $\int f \mathrm{d}\mu = \int f_+ \mathrm{d}\mu - \int f_- \mathrm{d}\mu$. For $f : E \to \mathbb{R}^n$, this is applied in each component.

Theorem (Monotone convergence theorem)**.** *Let $(E, \mathcal{E}, \mu)$ be a measure space, and let (f_n) be a (pointwise) increasing sequence of non-negative functions on E converging to f. Then*

$$\lim_{n\to\infty} \int_E f_n \mathrm{d}\mu = \int_E f \mathrm{d}\mu.$$

Theorem (Dominated convergence theorem)**.** *Let (f_n) be a sequence of measurable functions on a measure space $(E, \mathcal{E}, \mu)$ such that:*

(i) $f_n \to f$ *pointwise almost everywhere;*

(ii) $|f_n| \leq g$ *almost everywhere for some integrable g.*

Then

$$\lim_{n\to\infty} \int_E f_n \mathrm{d}\mu = \int_E f \mathrm{d}\mu.$$

1.2 L^p spaces

Definition. Let $(E, \mathcal{E}, \mu)$ be a measure space. For $p \in [1, \infty)$ and $f : E \to \mathbb{R}$ define

$$||f||_{L^p} = \left(\int_E |f|^p \mathrm{d}\mu \right)^{1/p}$$

and

$$||f||_{L^\infty} = \mathrm{esssup}|f| = \inf\{K : |f| \leq K \text{ a.e}\}.$$

The space L^p, $p \in [1, \infty]$ is defined by

$$L^p = L^p(E, \mathcal{E}, \mu) = \{f : E \to \mathbb{R} \text{ measurable} : ||f||_{L^p} < \infty\}/\sim.$$

Where $f \sim g$ if $f = g$ a.e.

Theorem (Riesz-Fisher theorem)**.** *L^p is a Banach space for all $p \in [1, \infty]$.*

Notation: when $E = \mathbb{R}^n$, μ the Lebesgue measure, write $L^p(E, \mu) = L^p(\mathbb{R}^n)$.

Fact. For $p \in [1, \infty)$, the simple functions f with $\mu(\{x : f(x) \neq 0\}) < \infty$ are dense in L^p. For $p = \infty$ we can drop the condition on the measure of the support.

Definition. For $f, g : \mathbb{R}^n \to \mathbb{R}$, the *convolution* $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(y) g(x - y) \mathrm{d}y,$$

provided the integral exists. Note that $f * g = g * f$, convolution is associative, and $\mu(f * g) = \mu(f)\mu(g)$.

Theorem. $C_c^\infty(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n)$ *for* $p \in [1, \infty)$.

Before we prove the theorem, we will need some preliminary results.

Remark. This theorem is false for $p = \infty$.

Notation: a multiindex is $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Set $|\alpha| = \alpha_1 + \ldots + \alpha_n$; $\alpha! = \alpha_1! \ldots \alpha_n!$; $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ for $X \in \mathbb{R}^n$; $\nabla^\alpha f = D^\alpha f = \frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$.

Definition. We say $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ if $f \mathbb{1}_K \in L^p(\mathbb{R}^n)$ for any $K \subseteq \mathbb{R}^n$ compact.

Proposition. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $g \in C^k_c(\mathbb{R}^n)$, some $k \geq 0$. Then $f * g \in C^k(\mathbb{R}^n)$ and $\nabla^\alpha (f * g) = f * (\nabla^\alpha g)$ for all $|\alpha| \leq k$.

Proof. First we check for $k = 0$. Set $T_z f(x) = f(x - z)$, $z \in \mathbb{R}^n$. Then $T_z(f * g) = f * (T_z g)$. Also $T_z g(x) \to g(x)$ for all x as $z \to 0$ (continuity of g). Furthermore $|T_z g(x)| \leq ||g||_{L^\infty} \mathbb{1}_{B_R(0)}(x)$ if $|x| + 1 \leq R$, $|z| < 1$ (we can just take R large enough so it holds everywhere since g has compact support). Then $|f(y) T_z g(x - y)| \leq C |f(y)| \mathbb{1}_{B_R(0)}(x - y)$, for $C := ||g||_{L^\infty}$.

Since $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $|f(y)| \mathbb{1}_{B_R(0)}(x - y)$ is integrable in y, so by the dominated convergence theorem,

$$T_z(f * g) = (f * T_z g)(x) = \int_{\mathbb{R}^n} f(y) T_z g(x - y) \mathrm{d}y \xrightarrow{z \to 0} \int_{\mathbb{R}^n} f(y) g(x - y) \mathrm{d}y = (f * g)(x).$$

And so $f * g \in C^0$. Now let $k = 1$. Let $\nabla_i^h g(x) = \frac{g(x + he_i) - g(x)}{h}$, where e_i is the ith unit vector. Then $\nabla_i^h g(x) \to \nabla_i g(x)$ as $h \to 0$.

By the mean value theorem, there exists $t \in [-h, h]$ such that

$$\nabla_i^h g(x) = \nabla_i g(x + te_i) \Rightarrow |\nabla_i^h g(x)| \leq ||\nabla_i g||_{L^\infty} \mathbb{1}_{B_R(0)}(x).$$

Again by the dominated convergence theorem, $\nabla_i^h (f * g) = f * (\nabla_i^h g) \to f * \nabla_i g$. Thus $f * g \in C^1$. The case $k > 1$ is similar, with induction. □

Proposition (Minkowski's integral inequality)**.** Let $p \in [1, \infty)$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ Borel. Then

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x, y) \mathrm{d}x \right|^p \mathrm{d}y \right]^{1/p} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x, y)|^p \mathrm{d}y \right]^{1/p} \mathrm{d}x.$$

Proof. Example sheet 1. □

Proposition. Let $p \in [1, \infty)$, $g \in L^p(\mathbb{R}^n)$. Then

$$||T_z g - g||_{L^p} \to 0 \text{ as } |z| \to 0.$$

Remark. This is not true for $p = \infty$. Let $\theta(x) = \mathbb{1}_{x \geq 0}$. Then $||T_z \theta - \theta||_{L^\infty} = 1$ if $z \neq 0$.

Proof. Consider first $g = \mathbb{1}_R$, R a rectangle. Then the result is clear. Hence it also follows for any finite union of rectangles. If B is a Borel set, $|B| < \infty$, then for every $\varepsilon > 0$, there exists a finite union of rectangles R such that

$$||\mathbb{1}_B - \mathbb{1}_R||_{L^p} = |B \triangle R|^{1/p} < \varepsilon.$$

Hence

$$||T_z \mathbb{1}_B - \mathbb{1}_B||_{L^p} \leq \underbrace{||T_z \mathbb{1}_B - T_z \mathbb{1}_R||_{L^p}}_{=||\mathbb{1}_B - \mathbb{1}_R||_{L^p} < \varepsilon} + \underbrace{||T_z \mathbb{1}_R - \mathbb{1}_R||_{L^p}}_{<\varepsilon \text{ for } |z| \text{ small}} + \underbrace{||\mathbb{1}_R - \mathbb{1}_B||_{L^p}}_{<\varepsilon}.$$

Thus the result holds for $g = \mathbb{1}_B$, $B \in \mathcal{B}(\mathbb{R}^n)$. Thus the result holds for simple functions g. Finally, for any $g \in L^p$, there is a $\tilde{g}$ simple such that $||g - \tilde{g}||_{L^p} < \varepsilon$. Then

$$||T_z g - g||_{L^p} \leq \underbrace{||T_z g - T_z \tilde{g}||_{L^p}}_{=||g-\tilde{g}||_{L^p} < \varepsilon} + \underbrace{||T_z \tilde{g} - \tilde{g}||_{L^p}}_{<\varepsilon \text{ for } |z| \text{ small}} + \underbrace{||g - \tilde{g}||_{L^p}}_{<\varepsilon}.$$

$\square$

Theorem. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi \mathrm{d}x = 1$ and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then for any $g \in L^p$, $p \in [1, \infty)$, it follows that $\varphi_\varepsilon * g \in C^\infty(\mathbb{R}^n)$ and $\varphi_\varepsilon * g \to g$ in L^p.*

Proof. We have

$$\begin{aligned}\varphi_\varepsilon * g(x) - g(x)| &= \left| \int_{\mathbb{R}^n} [\varphi_\varepsilon(y) g(x-y) - g(x)] \, \mathrm{d}y \right| \\ &\underbrace{=}_{z:=y/\varepsilon} \left| \int_{\mathbb{R}^n} \varphi(z) \, [g(x-\varepsilon z) - g(x)] \, \mathrm{d}z \right| \\ &\leq \int_{\mathbb{R}^n} \varphi(z) \, |T_{\varepsilon z} g(x) - g(x)| \, \mathrm{d}z.\end{aligned}$$

Hence

$$\begin{aligned}\|\varphi_\varepsilon * g - g\|_{L^p} &= \left(\int_{\mathbb{R}^n} \underbrace{|\varphi_\varepsilon * g - g|^p}_{\int_{\mathbb{R}^n} \varphi(z)|T_{\varepsilon z} g - g| \mathrm{d}z} \, \mathrm{d}x \right)^{1/p} \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi(z)^p |T_{\varepsilon z} g(x) - g(x)|^p \mathrm{d}x \right)^{1/p} \mathrm{d}z \\ &= \int_{\mathbb{R}^n} \varphi(z) \underbrace{||T_{\varepsilon z} g - g||_{L^p}}_{\to 0 \text{ as } \varepsilon \to 0} \mathrm{d}z\end{aligned}$$

where the inequality comes from Minkowski's inequality. Also the final term tends to 0 as $\varepsilon \to 0$ by the DCT since $\varphi(z)||T_{\varepsilon z} g - g||_{L^p}|| \leq 2\varphi(z)||g||_{L^p}$ and φ is integrable. □

Definition. φ as above is called a (smooth) mollifier.

Corollary. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $p \in [1, \infty)$.*

Proof. The previous theorem implies $C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is dense in L^p. Since $||f - f\mathbb{1}_{B_R(0)}||_{L^p} \to 0$ as $R \to \infty$ by the DCT, for $f \in L^p$, applying the theorem with $g = f\mathbb{1}_{B_R(0)}$ it follows that $C_c^\infty(\mathbb{R}^n)$ is dense in L^p. □

1.3 Lebesgue Differentiation Theorem

Recall:

Theorem (Fundamental Theorem of Calculus)**.** *For $f : \mathbb{R} \to \mathbb{R}$ continuous, $F(x) := \int_0^x f(t)\mathrm{d}t$ is differentiable with $F'(x) = f(x)$.*

We actually have a stronger result:

Theorem (Lebesgue Differentiation Theorem)**.** *For $f : \mathbb{R}^n \to \mathbb{R}$ integrable,*

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \mathrm{d}y = 0 \text{ for almost all } x.$$

The x for which this holds are called Lebesgue points.

Corollary. *If $g \in L^1(\mathbb{R})$ and $G(x) = \int_{-\infty}^x g(t)\mathrm{d}t$, then G is differentiable for almost every x with $G'(x) = g(x)$.*

Corollary. *If φ is a smooth mollifier and $g \in L^p(\mathbb{R}^n)$, then $\varphi_\varepsilon * g \xrightarrow{\varepsilon \to 0} g$ almost everywhere.*

Definition. For $f : \mathbb{R}^n \to \mathbb{R}$ integrable, the *Hardy-Littlewood Maximal Function* $\mathrm{Mf} : \mathbb{R}^n \to [0, \infty]$ is defined by

$$\mathrm{Mf}(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\mathrm{d}y.$$

Remark. We sometimes write $⨍_{B_r(x)} |f(y)|\mathrm{d}y$ for $\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\mathrm{d}y$.

Lemma (Wiener's covering lemma)**.** *If K is compact and $K \subseteq \bigcup_{i=1}^N B_i$ for open balls $(B_i)_{i=1}^N$, there exists a subcollection $(B_{i_k})_k$ of disjoint balls such that*

$$\left|\bigcup_{i=1}^N B_i\right| \leq 3^n \sum_k |B_{i_k}|.$$

Proof. Example sheet. ☐

Proposition. Take $f \in L^1(\mathbb{R}^n)$. Then Mf is a Borel function, finite almost everywhere, and

$$|\underbrace{\{\mathrm{Mf} > \lambda\}}_{:= A_\lambda}| \leq \frac{3^n}{\lambda} ||f||_{L^1}.$$

Proof. For each $x \in A_\lambda$, there exists $r_x > 0$ such that

$$\frac{1}{B_{r_x}(x)} \int_{B_{r_x}(x)} |f(y)|\mathrm{d}y > \lambda.$$

We claim that A_λ is open. Then we will have shown Mf is Borel as the $A_\lambda = (\mathrm{Mf})^{-1}((\lambda, \infty])$ are open, and the sets $(\lambda, \infty]$ generate the Borel σ-algebra.

We'll actually show A_λ^c is closed. Suppose $(x_k)_{k\geq 1}$ is a sequence in A_λ^c with $x_k \to x$. Suppose $x \in A_\lambda$. By the Dominated Convergence Theorem,

$$\frac{1}{B_{r_x}(x_k)} \int_{B_{r_x}(x_k)} |f(y)|\mathrm{d}y \to \frac{1}{B_{r_x}(x)} \int_{B_{r_x}(x)} |f(y)|\mathrm{d}y.$$

Since $x_k \notin A_\lambda$, the LHS is $\leq \lambda$ for all k, but the RHS is $> \lambda$ which is impossible. Hence $x \in A_\lambda^c$ and A_λ^c is closed.

To prove the inequality, let $K \subseteq A_\lambda$ be compact. Since $\{B_{r_x}(x)\}_{x \in A_\lambda}$ is an open cover of K, there exists a finite subcover $K \subseteq \bigcup_{i=1}^N B_i$, where $B_i = B_{r_x}(x)$ for some $x \in A_\lambda$. Now take a subcollection $(B_{i_k})_k$ of disjoint balls as in Wiener's covering lemma.

Since $\frac{1}{|B_i|}\int_{B_i}|f(y)|\mathrm{d}y > \lambda$, it follows that $|B_i| < \frac{1}{\lambda}\int_{B_i}|f(y)|\mathrm{d}y$. Hence

$$|K| \leq 3^n \sum_k |B_{i_k}| < \frac{3^n}{\lambda}\sum_k \int_{B_{i_k}} |f(y)|\mathrm{d}y \leq \frac{3^n}{\lambda}\int_{\mathbb{R}^n}|f(y)|\mathrm{d}y.$$

Since this holds for any $K \subseteq A_\lambda$ compact, by regularity of the Lebesgue measure, it also holds for A_λ. In particular, $|\{\mathrm{Mf} = \infty\}| \leq |\{\mathrm{Mf} > \lambda\}| \xrightarrow{\lambda\to\infty} 0$, i,e $\mathrm{Mf} < \infty$ almost everywhere.

$\square$