Disclaimer

These notes have been written with the intention of being used by people who have already taken previous analysis courses in the Tripos (beyond IA Analysis). For this reason, some of the results which have been seen already in Tripos are omitted.

1 Basic Analysis

Definition. For $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, define

$$d(x, A) = \inf\{d(x, a) = a \in A\}.$$

Remark. If A is closed, $d(x, A) = 0$ if and only if $x \in A$. Indeed if $d(x, A) = 0$, there exists a sequence $(a_n)_{n\geq 1}$ such that $d(x, a_n) \to 0$ by the definition of infima. This is the same as $a_n \to x$, and since A is closed $x \in A$. The converse is obvious.

Proposition. The map $x \mapsto d(x, A)$ is continuous.

Proof. It is easy to show $|d(x, A) - d(y, A)| \leq d(x, y)$ and then the claim follows. □

Theorem. *If $E \subseteq \mathbb{R}^n$ is compact and $f : E \to f(E)$ is continuous, then E is compact.*

Proof. Take a sequence $(f(x_n))_{n\geq 1}$ in $f(E)$. Then $(x_n)_{n\geq 1}$ has a convergent subsequence $(x_{n_j})_{j\geq 1}$, and $f(x_{n_j}) \to f(x) \in f(E)$. □

Theorem (Fundamental Theorem of Algebra). *If $P(z)$ is a polynomial with coefficients in $\mathbb{C}$, then P has a root.*

Proof. Write $P(z) = \sum_{j=0}^{n} a_j z^j$ (wlog $a_n = 1$). Note

$$|P(z)| \geq |z|^n (|a_n| - |a_{n-1}||z|^{-1} - \ldots - |a_0||z|^{-n}).$$

Hence if $R = 2\left(2 + \sum_{j=0}^{n-1} |a_j|\right)$, whenever $|z| \geq R$ we have $|P(z)| > |z|^n/2$. Then $\overline{B}_R(0)$ is compact so $|P(z)|$ achieves a minimum on $B_R(0)$ at some $p_0 \in B_R(0)$. Since $|P(z)| \geq |z|^n/2 \geq R^n/2 > |a_0|$ for $|z| \geq R$, p_0 is a global minimum.

Now by considering $P(z + p_0)$, we can assume $p_0 = 0$. By considering $e^{i\theta}P(z)$ we can assume $a_0 \in \mathbb{R}$ and $a_0 > 0$. By considering $P(e^{i\theta}z)$ we can assume $P(z) = a_0 + a_m z^m + \sum_{m+1}^{n} a_j z^j$ for $a_0, a_m \in \mathbb{R}$, $a_0 > 0$ and $a_m < 0$. Then $\frac{P(\delta)-P(0)}{\delta^m} \to a_m$ as $\delta \to 0$.

So the real part of $\frac{P(\delta)-P(0)}{\delta^m}$ tends to a_m and the imaginary part tends to 0. Hence if δ is sufficiently small and positive $|P(\delta)| < |P(0)|$. □

Remark. Clearly this implies every complex polynomial of degree n has exactly n roots with multiplicity (division algorithm).

2 Laplace's Equation

Definition. *Laplace's equation* is $\nabla^2\phi = 0$.

Definition (Boundary)**.** Let (X, d) be a metric space, $E \subseteq X$.

1. The *closure* $\overline{E} = \mathrm{Cl}(E)$ is the collection of points $x \in E$ such that there exists a sequence $(x_n)_{n\geq 1}$ in E with $x_n \to x$.
2. The *interior* $\mathrm{int}(E)$ is the set of points $x \in E$ such that there exists $\delta > 0$ with $B(x, \delta) \subseteq E$.
3. The *boundary* ∂E is defined by $\partial E = \mathrm{Cl}(E) \setminus \mathrm{int}(E)$.

For the remainder of this section:

- We work in $\mathbb{R}^m$;
- Let $\Omega \neq \emptyset$ be a bounded open set;
- Let $\phi : \mathrm{Cl}(\Omega) \to \mathbb{R}$ be a continuous function which is twice differentiable on Ω;
- Let $f : \partial\Omega \to \mathbb{R}$ be a continuous function.

We want to show that the problem $\nabla^2\phi = 0$ on Ω, $\phi = f$ on $\partial\Omega$ has at most one solution.

Lemma. *If $\nabla^2\phi > 0$ on Ω, then ϕ achieves its maximum on $\partial\Omega$.*

Remark. The maximum must exist since $\mathrm{Cl}(\Omega)$ is closed and bounded.

Proof. Suppose ϕ attains its maximum at $x^* \in \Omega$. Then there exists $\delta > 0$ such that

$$(x_1^* - \delta, x^* + \delta) \times \ldots \times (x_n^* - \delta, x^* + \delta) \subseteq \Omega.$$

Define $f_j(t) = \phi(x_1^*, \ldots, x_{j-1}^*, x_j^* + t, x_{j+1}^*, \ldots, x_m^*)$. Then f_j has a maximum at 0 and f_j is twice differentiable with $f_j''(0) \leq 0$. Hence $\frac{\partial\phi(x^*)}{\partial x_j^2} \leq 0$ so $\nabla^2\phi(x^*) \leq 0$, a contradiction. □

Theorem. *If $\nabla^2\phi = 0$ on Ω, then the maximum is achieved at $\partial\Omega$.*

Proof. Set $\phi_n(x) = \phi(x) + \frac{1}{n}\sum_{j=1}^m x_j^2$. Then $\nabla^2\phi_n = \nabla^2\phi + \frac{2m}{n} = \frac{2m}{n}$. Then by the previous lemma, ϕ_n has a maximum at $x_n^* \in \partial\Omega$. The boundary is closed and bounded so compact, hence we can find x^* and a subsequence $(x_{n(j)}^*)_{j\geq 1}$ such that $x_{n(j)}^* \to x^*$. Then

$$\phi(x_{n(j)}^*) + \frac{1}{n(j)}\sum_{i=1}^m (x_k^*)^2 \geq \phi(x) + \frac{1}{n(j)}\sum_{i=1}^m x_k^2 \geq \phi(x)\ \forall x \in \mathrm{Cl}(\Omega).$$

So let $n(j) \to \infty$ and use continuity of ϕ to obtain $\phi(x^*) \geq \phi(x)$ for all $x \in \mathrm{Cl}(\Omega)$. $\square$

Remark. Now if $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$ on Ω and $\phi_1 = \phi_2 = f$ on $\partial\Omega$, then $\phi_1 - \phi_2 = 0$ on $\partial\Omega$ so $\phi_1 - \phi_2 \leq 0$ on $\mathrm{Cl}(\Omega)$ by the previous theorem. Similarly $\phi_2 - \phi_1 \leq 0$ on $\mathrm{Cl}(\omega)$ so $\phi_1 = \phi_2$.

What about existence?

Zaremba counterexample

Consider $\Omega = D \setminus \{0\} = \{x \in \mathbb{R}^2 : 0 < ||x|| < 1\}$. We'll show $\nabla^2\phi = 0$,

$$\phi(x) = \begin{cases} 0 & ||x|| = 1 \\ 1 & x = 0 \end{cases}$$

on $\partial\Omega$ has no solution.

Indeed, let R_θ be a rotation through θ about 0. If θ is a solution, then $x \mapsto \phi(R_\theta x)$ is a also a solution. So by uniqueness $\phi = \phi \circ R_\theta$, and we can write $\phi(x) = f(||x||)$ for some f. Now either $\nabla^2\phi$ is aready known in radial terms, or

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \frac{\partial}{\partial x} f(\sqrt{x^2+y^2}) = \frac{x}{\sqrt{x^2+y^2}} f'(\sqrt{x^2+y^2}) \\
\frac{\partial^2\phi}{\partial x^2} &= \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}}\right) f'(\sqrt{x^2+y^2}) + \frac{x^2}{x^2+y^2} f''(\sqrt{x^2+y^2}) \\
\frac{\partial^2\phi}{\partial y^2} &= \left(\frac{1}{r} - \frac{y^2}{r^3}\right) f'(r) + \frac{y^2}{r^2} f''(r) \\
&= \nabla^2\phi = \frac{1}{r} f'(r) + f''(r) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r f'(r))
\end{aligned}$$

Now we solve $0 = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rf'(r))$. This gives $A\log r + B = f(r)$. Since ϕ is continuous on $\mathrm{Cl}(E)$, $A = 0$ (otherwise singularity at 0). So ϕ is constant, a contradiction.

3 Brouwer's Fixed Point Theorem

In one dimension: if $f : [0,1] \to [0,1]$ is continuous, there exists $x \in [0,1]$ such that $f(x) = x$ (a fixed point). Indeed consider $g(x) = f(x) - x$ so $g(0) \geq 0$, $g(1) \leq 0$, and use the intermediate value theorem.

Theorem (Brouwer's Fixed Point Theorem)**.** *If $f : \overline{D} \to \overline{D}$ is continuous ($\overline{D} = \overline{D}(0,1) \subseteq \mathbb{R}^2$) then there exists x^* such that $f(x^*) = x^*$.*

Remark. This generalises to $\mathbb{R}^n$, with the proof essentially unchanged apart from one point ("Sperner's Lemma') where we have done a little simplification from the general case.

Remark. If E is homeomorphic to $\overline{D}$ then the fixed point result also holds for E. Indeed if $T : E \to \overline{D}$ is a homeomorphism, $f : E \to E$ is continuous, then $T \circ f \circ T^{-1} : \overline{D} \to \overline{D}$ is continuous. Hence there exists $x^* \in \overline{D}$ such that $T(f(T^{-1}(x^*))) = x^*$, i.e $T^{-1}(x^*)$ is a fixed point of f.

The proof of the fixed point theorem is a long line of lemmas, most of which assert that two statements are equivalent.

The first thing we'll do is show that the Fixed Point Theorem is equivalent to the "No Retraction Theorem".

Theorem (No Retraction Theorem)**.** *There does not exist a continuous map* $g : \overline{D} \to \partial D$ *such that* $g(x) = x$ *for* $x \in \partial D$ *(called a retraction).*

Proof of equivalence. Suppose the Fixed Point Theorem is true, but there exists a retraction $g : \overline{D} \to \partial D$. Let R be a rotation through π about 0. Then $R \circ g : \overline{D} \to \overline{D}$ has no fixed point, which is a contradiction.

Now suppose the No Retraction Theorem holds, but $f : \overline{D} \to \overline{D}$ is continuous with no fixed points. Let $F : \overline{D}^2 \setminus \{(x, x) : x \in \overline{D}\} \to \overline{D} \to \partial D$ be defined so that $F(x, y)$ is the point of intersection between the (directed) line from y to x and int∂D, we see F is continuous. Now consider $g(x) = F(x, f(x))$. Since f is continuous, g is continuous, but $g(\overline{D}) \subseteq \partial D$ and $g(x) = x$ for $x \in \partial D$, a contradiction. □

Now we proceed through minor lemmas with unofficial titles.

Lemma (Three Arcs Theorem)**.** *If we consider* 3 *equal arcs of* ∂D *in polar coordinates, say* $A = \{(1, \theta) : 0 \le \theta \le 2\pi/3\}$, $B = \{(1, \theta) : 2\pi/3 \le \theta \le 4\pi/3\}$, $C = \{(1, \theta) : 4\pi/3 \le \theta \le 2\pi\}$, *then there does not exist* $f : \overline{D} \to \partial D$ *such that* $f(A) \subseteq A$, $f(B) \subseteq B$, $f(C) \subseteq C$.