Theorem (Lawler,Schramm,Werner). $\xi(1,1) = \frac{5}{4}$, $\xi(2,0) = \frac{2}{3}$.

1 Conformal maps

We consider a domain $U \subseteq \mathbb{C}$ (i.e an open and connected subset of the complex plane). We say U is *simply connected* if $\mathbb{C} \setminus U$ is connected.

We say $f : U \to \mathbb{C}$ is *holomorphic* if it is complex differentiable. If f is holomorphic and injective we say it is *univalent*. If $f : U \to V$ is holomorphic and bijective we say f is a *conformal map*.

Remark. If $f : U \to V$ is conformal then

$$f(w) = f(z) + f'(z)(w - z) + o(|w - z|)$$

and $f'(z) \neq 0$. Hence f locally looks like a translation combined with a scaling and rotation.

We will work in 2d throughout this course. This gives a richness to the conformal maps, as shown by the following theorem.

Theorem (Riemann mapping theorem). *If $U \subsetneq \mathbb{C}$ is a simply connected domain and $z \in U$ then there exists a unique conformal map $f : \mathbb{D} \to U$ with $f(0) = z$ and* $\arg f'(0) = 0$.

Where we have taken $\mathbb{D} = \{z : |z| < 1\}$ to be the open unit disc. We will also take $\mathbb{H} = \{z : \Im z > 0\}$ to be the open upper half-plane.

Examples.

- Let $f(z) = \frac{z-i}{z+i}$. Then $f : \mathbb{H} \to \mathbb{D}$ is a conformal map.
- $f : \mathbb{D} \to \mathbb{D}$ is conformal if and only if $f(w) = \lambda \frac{w-z}{\bar{z}w-1}$ for some $\lambda, z \in \mathbb{C}$ with $|\lambda| = 1$, $z \in \mathbb{D}$.
- $f : \mathbb{H} \to \mathbb{H}$ is conformal if and only if $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
- Given a simply connected domain D and disjioint subarcs $A, B \subseteq \partial D$, there is a unique conformal map from U to the rectangle such that A, B are mapped to parallel sides with length 1. The length L of the other sides is called the extremal length $\mathrm{EL}_D(A, B)$ and is unique.

Recall that if $f = u + iv$ (with u, v denoting the real/imaginary parts of f respectively) then f is holomorphic iff it satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It follows from this that if f is holomorphic,

$$\Delta u = \left(\frac{\partial}{\partial x}\right)^2 u + \left(\frac{\partial}{\partial y}\right)^2 u = \frac{\partial^2}{\partial x \partial y} v - \frac{\partial^2}{\partial x \partial y} v = 0$$

and similarly $\Delta v = 0$.

Conversely, if $u : U \to \mathbb{R}$ (for U a simply connected domain) is harmonic there exists $v : U \to \mathbb{R}$ such that $u + iv$ is holomorphic.

A consequence of this is that if u is harmonic on a bounded domain D and continuous on $\overline{D}$, for $z \in D$ and B a Brownian motion starting from z and $\tau := \inf\{t : B_t \notin D\}$, we have $u(z) = \mathbb{E}_z[u(B_\tau)]$ (see Part III Advanced Probability).

Conformal invariance of 2d Brownian motion

Let $f : D \to \tilde{D}$ be a conformal map and B be a Brownian motion starting at $z \in \mathbb{C}$. Define $\tau = \inf\{t : B_t \notin D\}$ and let $\sigma(t) = \inf\{s : \int_0^s |f'(B_r)|^2 \mathrm{d}r = t\}$. Then $f(B_{\sigma(t)})$ has the law of a Brownian motion starting from $f(z)$ until exiting $\tilde{D}$.

Proof. See Part III Stochastic Calculus. $\square$

We have seen that for u harmonic on D and continuous on $\overline{D}$ we have $u(z) = \mathbb{E}_z[u(B_{\tau_D})]$. We get the following corollary by taking a Brownian motion until it hits $\partial B(z, r)$.

Corollary (Mean value property). *For $B(z, r) \subseteq D$*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \mathrm{d}\theta.$$

Proposition (Strong maximum principle). Let u be harmonic in D, D a domain. If u attains a global maximum in D then u is constant.

Proof. Follows fom mean value property and compactness of paths connecting points. $\square$

Proposition (Maximum modulus principle). Let $f : D \to \mathbb{C}$ holomorphic, D a domain. Then if $|f|$ attains a global maximum in D, f is constant.

Proof. Let $K \subseteq D$ be compact. By considering $f + M$ for $M > 0$ large enough we may assume $|f| > 0$ on K. Thus $\log |f|$ is harmonic. So we can apply the strong maximum principle to see $\log |f|$ is constant on K, i.e f takes values on a circle. But this is impossible unless $f' = 0$ on K. $\square$

Proposition (Schwarz lemma). Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Furthermore if $|f(z)| = |z|$ for some $z \neq 0$ then $f(w) = we^{i\theta}$ for some $\theta \in \mathbb{R}$.

Proof. Define the holomorphic function $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{for } z \neq 0 \\ f'(0) & \text{for } z = 0 \end{cases}.$$

Then $|z| = 1$ on $\partial\mathbb{D}$, implying $|g| \leq 1$ on $\partial\mathbb{D}$. Thus $|g| \leq 1$ on $\mathbb{D}$ by the maximum modulus principle.

If $|g(z)| = 1$ for some $z \in \mathbb{D}$ then g is constant since this is a maximum. $\square$

Distortion theorems for conformal maps

Let $\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} \text{ univalent} : f(0) = 0, f'(0) = 1\}$.

Remark. We can write such f as $f(z) = z + a_2 z^2 + a_3 z^2 + \ldots$.

Goal: for $f \in \mathcal{S}$

- Koebe 1/4-theorem: $f(\mathbb{D}) \supseteq B(0, 1/4)$;
- Koebe distortion theorem: $\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$.

Corollary. *If $f : D \to \tilde{D}$ is conformal then*

$$\frac{\text{dist}(f(z), \partial\tilde{D})}{4\text{dist}(z, \partial D)} \leq |f'(z)| \leq \frac{4\text{dist}(f(z), \partial\tilde{D})}{\text{dist}(z, \partial D)}.$$

Corollary. *If f univalent in D, $B(z, R) \subseteq D$ then for $r < 1$ we have $|f'(u)| \leq c(r)|f'(v)|$ for all $u, v \in B(z, rR)$.*

Define

$$\Sigma = \{g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} : g \text{ univalent}, \ g(\infty) = \infty, \ g'(\infty) = 1\}.$$

Theorem (Area theorem)**.** *Let $g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$ be univalent with $g(z) \to \infty$ as $z \to \infty$ and $g'(z) \to 1$ as $z \to \infty$. Write $g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z} + \dots$ for g near ∞. Then*

$$\sum_{n \geq 1} n|b_n|^2 \leq 1$$

and moreover

$$\operatorname{area}(\mathbb{C} \setminus g(\mathbb{C} \setminus \overline{\mathbb{D}})) = \pi \left(1 - \sum_{n \geq 1} n|b_n|^2\right).$$

Proof. Let $r > 1$ and define $C_r = g(\partial D(0, r))$. Let E_r be the inner component of $\mathbb{C} \setminus C_r$. By Green's theorem

$$\begin{aligned}
\frac{1}{2i} \int_{C_r} \overline{w} \mathrm{d}w &= \frac{1}{2i} \int_{C_r} (x - iy)(\mathrm{d}x + i\mathrm{d}y) \\
&= \frac{1}{2i} \int_{C_r} ((x - iy)\mathrm{d}x + (ix + y)\mathrm{d}y) \\
&= \frac{1}{2i} \int_{E_r} 2i\mathrm{d}x\mathrm{y} && \text{(Green's thm)} \\
&= \operatorname{area}(E_r).
\end{aligned}$$

while we also have

$$\begin{aligned}
\frac{1}{2i} \int_{C_r} \overline{w} \mathrm{d}w &= \frac{1}{2i} \int_{\partial B(0,r)} \overline{g(z)} g'(z) \mathrm{d}z \\
&= \frac{1}{2} \int_0^{2\pi} \left(re^{-i\theta} + \sum_{n \geq 1} \overline{b_n} r^{-n} e^{in\theta}\right) \left(1 - \sum_{n \geq 1} b_n r^{-n-1} e^{i(n+1)\theta}\right) re^{i\theta} \mathrm{d}\theta \\
&= \pi \left(r^2 - \sum_{n \geq 1} n|b_n|^2 r^{-2n}\right).
\end{aligned}$$

Now take $r \downarrow 1$. $\square$

Theorem. *Let $f : \mathbb{D} \to \mathbb{C} \in \mathcal{S}$ write $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$. Then $|a_2| \leq 2$.*

Proof. We claim there exists $g \in \mathcal{S}$ with $g(z)^2 = f(z^2)$ (we call g the "square-root transform" of f). Note

$$f(z^2) = z^2 (\underbrace{1 + a_2 z^2 + a_3 z^4 + \dots}_{:= h(z)})$$

and since $h \neq 0$ (by $f(0) = 0$ and injectivity of f), we can define $g(z) = z\sqrt{h(z)}$. Also $g(0) = 0$ and $g'(0) = 1$. To show g is univalent, suppose $g(z_1) = g(z_2)$ for some $z_1, z_2 \in \mathbb{D}$. Then $f(z_1^2) = f(z_2^2)$ so $z_1^2 = z_2^2$, i.e $z_1 = \pm z_2$. But g is an odd function and only zero at $z = 0$ so we have $z_1 = z_2$.

To conlcude take $z \mapsto \frac{1}{g(1/z)} \in \Sigma$. This map is the same as

$$z \mapsto \frac{1}{\sqrt{f(1/z^2)}} = z - \frac{a_2}{2}\frac{1}{z} + \ldots$$

so by the area theorem, $|a_2/2| \leq 1$. $\square$

Theorem (Koebe 1/4-theorem)**.** *Let $f \in \mathcal{S}$. Then $f(\mathbb{D}) \supseteq B(0, 1/4)$.*

Proof. Let $w \notin f(\mathbb{D})$. Then

$$z \mapsto \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \ldots$$

is in $\mathcal{S}$ so by the above $\left|a_2 + \frac{1}{w}\right| \leq 2$. Since $|a_2| \leq 2$ we must have $|1/w| \leq 4$. $\square$

If we define

$$F(w) = \frac{f\left(\frac{w+z}{1+\overline{z}w}\right) - f(z)}{(1-|z|^2)f'(z)} = w + \frac{1}{2}\left((1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right) w^2 + \ldots$$

we see

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \leq 4.$$

Note

$$\begin{aligned} z\frac{f''(z)}{f'(z)} &= z\partial_z \log f'(z) = r\partial_r \log f'(z) \\ &= r\partial_r \log |f'(z)| + ir\partial_r \arg(f'(z)) \end{aligned}$$

and

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}$$

which implies

$$\frac{2r^2}{1-r^2} - \frac{4r}{1-r^2} \leq \Re\left(z\frac{f''(z)}{f'(z)}\right) \leq \frac{2r^2}{1-r^2} + \frac{4r}{1-r^2}.$$

Integrating from $r = 0$ to R,

$$\log\frac{1-R}{(1+R)^3} \leq \log|f'(Re^{i\theta})| \leq \log\frac{1+R}{(1-R)^3}.$$

So we get

Theorem (Kobe's distortion theorem)**.** *For $f \in \mathcal{S}$,*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Definition. $A \subseteq \mathbb{H}$ is a *compact* $\mathbb{H}$*-hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. We write $A \in \mathcal{Q}$ for such a set.

For $A \in \mathcal{Q}$, pick $g : \mathbb{H} \setminus A \to \mathbb{H}$ conformal (possible by Riemann mapping theorem) with $g(\infty) = \infty$.

Question: when does a holomorphic function extend analytically to the boundary?

Theorem (Schwarz reflection principle)**.** *Let $U \subseteq \mathbb{C}$ be a domain such that $U = \{\overline{z} : z \in U\}$. Let $U^+ = U \cap \mathbb{H}$. Let $f : U^+ \to \mathbb{C}$ be holomorphic with $\lim_{\Im z \downarrow 0} \Im f(z) = 0$. Then f extends to a holomorphic function on U with $f(\overline{z}) = \overline{f(z)}$ for all $z \in U$.*

Proof. On $U^- := U \cap \{z : \Im(z) < 0\}$ set $f(z) := \overline{f(\overline{z})}$. To extend f to $U \cap \mathbb{R}$, write $f = u + iv$ for u, v harmonic and note $\lim_{\Im z \downarrow 0} v(z) = 0$. So we have extended v via

$$v(z) = \begin{cases} -v(\overline{z}) & \Im z < 0 \\ 0 & \Im z = 0 \end{cases}.$$

Then v is still harmonic as it satisfies the mean value property.

For $z \in U \cap \mathbb{R}$ pick $\varepsilon > 0$ so that $B(z, \varepsilon) \subseteq U$. Let $\tilde{u}$ be the harmonic conjugate of v on $B(z, \varepsilon)$ (unique up to an additive constant). Then $f = u + iv = \tilde{u} + iv + \text{const}$ so f extends to $B(z, \varepsilon)$. Furthermore this matches with $f(z) = \overline{f(\overline{z})}$ on U^-. For different z these extensions match so by the identity principle we are done. $\square$

Now for $A \in \mathcal{Q}$, $g : \mathbb{H} \setminus A \to \mathbb{H}$ conformal with $g(\infty) = \infty$, we can Schwarz reflect. g has a simple pole at ∞ so

$$g(z) = b_{-1} z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots .$$

Also $g(z) = \overline{g(\overline{z})} = \overline{g(z)}$ for $z \in \mathbb{R}$ which implies $b_n \in \mathbb{R}$ for all $n \geq -1$. So we can scale and then translate g so that $b_{-1} = 1$ and $b_0 = 0$.

Definition. For $A \in \mathcal{Q}$, let $g_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ the conformal map with $g_A(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$.

Define the *half-plane capacity* $\operatorname{hcap}(A)$ to be equal to $b_1 \in \mathbb{R}$ as above.

For example we have $g_{[0,i]}(z) = \sqrt{z^2 + 1}$ and so $\operatorname{hcap}([0, i]) = \frac{1}{2}$ (we can see this by looking at what happens to $\mathbb{H} \setminus [0, i]$ under $z \mapsto z^2 \mapsto z^2 + 1 \mapsto \sqrt{z^2 + 1}$).

If A is instead a $\overline{\mathbb{D}} \cap \mathbb{H}$ with radius 1 centred at 0, we have $g_A(z) = z + \frac{1}{z}$ so $\operatorname{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = 1$.

It is straighforward to see $g_{rA}(z) = rg_A(z/r)$ for any $r > 0$ and so $\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A)$. Can also see that $\operatorname{hcap}(A + x) = \operatorname{hcap}(A)$ for any $x \in \mathbb{R}$.

For $A \subseteq \tilde{A}$ can also see that

$$g_{\tilde{A}} = g_{g_A(\tilde{A}\setminus A)} \circ g_A = z + \frac{\operatorname{hcap}(A)}{z} + \frac{\operatorname{hcap}(g_A(\tilde{A} \setminus A))}{z} + \dots$$

so $\operatorname{hcap}(\tilde{A}) = \operatorname{hcap}(A) + \operatorname{hcap}(g_A(\tilde{A} \setminus A))$. Thus $\operatorname{hcap}(A) \leq \operatorname{hcap}(\tilde{A})$ (after seeing later that hcap is non-negative). Also $\operatorname{hcap}(A) \leq \operatorname{hcap}(\operatorname{rad}(A) \cdot \overline{\mathbb{D}} \cap \mathbb{H}) \leq \operatorname{rad}(A)^2$ where $\operatorname{rad}(A) = \sup\{|z| : z \in A\}$.

Proposition. Let $A \in \mathcal{Q}$, B be a 2D Brownian motion and $\tau = \inf\{t : B_t \notin \mathbb{H} \setminus A\}$. Then

(i) For all $z \in \mathbb{H} \setminus A$, $\Im(z - g_A(z)) = \mathbb{E}_z[\Im(B_\tau)]$;

(ii) We have $\operatorname{hcap}(A) = \lim_{y\to\infty} y\mathbb{E}_{iy}[\Im(B_\tau)]$.

Remark. (ii) shows that $\operatorname{hcap}(A) \geq 0$.

Proof.
(i) Note $z \mapsto \Im(z - g_A(z))$ is harmonic and bounded. Hence

$$\Im(z - g_A(z)) = \mathbb{E}_z[\Im(B_\tau - g_A(B_\tau))] = \mathbb{E}_z[\Im(B_\tau)].$$

(ii) We have

$$\begin{aligned}
\operatorname{hcap}(A) = \lim_{z\to\infty} z(g_A(z) - z) &= \lim_{y\to\infty} iy(g_A(iy) - iy) \\
&= \lim_{y\to\infty} \Re(iy(g_A(iy) - iy)) && (\operatorname{hcap}(A) \in \mathbb{R}) \\
&= \lim_{y\to\infty} y\Im(iy - g_A(iy)) \\
&= \lim_{y\to\infty} y\mathbb{E}_{iy}[\Im(B_\tau)]. && \text{(by (i))}
\end{aligned}$$

□

The law of B_τ for $\tau = \inf\{t : B_t \notin D\}$ is often called the *harmonic measure for z relative to D*. For $z \in D$, $\omega(z, \cdot, D)$ is a probability measure on ∂D. For $A \in \mathcal{B}(\partial D)$, $\omega(\cdot, A, D)$ is harmonic (strong Markov property so satisfies mean value property).

Example.

- $\omega(0, \cdot, \mathbb{D})$ is the uniform distribution on $\partial\mathbb{D}$;
- $\omega(z, \cdot, \mathbb{D})$ may be computed using conformal invariance of Brownian motion (Example Sheet);
- $\omega(z, \cdot, \mathbb{H})$ may also be computed using conformal invariance (Example Sheet). If $z = x + iy$ it has density on $\mathbb{R}$ given by

$$u \mapsto \frac{1}{\pi}\frac{y}{(x-u)^2 + y^2}.$$

.

Proposition. There exists $c > 0$ such that for any $A \in \mathcal{Q}$ and $|z| \geq 2\operatorname{rad}(A)$ we have

$$\left| g_A(z) - z - \frac{\operatorname{hcap}(A)}{z} \right| \leq c\frac{\operatorname{rad}(A)\operatorname{hcap}(A)}{|z|^2}.$$

Proof. By scaling we may assume $\mathrm{rad}(A) \leq 1$. We have

$$\Im(z - g_A(z)) = \mathbb{E}_z[\Im(B_\tau)] = \int_0^\pi \mathbb{E}_{e^{i\theta}}[\Im(B_\tau)] p(z, e^{i\theta}) \mathrm{d}\theta$$

where $p(z, e^{i\theta})$ is the density of $w(z, \theta, \mathbb{H} \setminus \overline{\mathbb{D}})$. On the Example Sheet it will be shown that

$$p(z, e^{i\theta}) = \frac{2}{\pi} \frac{\Im(z)}{|z|^2} \sin(\theta)(1 + \mathcal{O}(|z|^{-1})) \text{ as } z \to \infty.$$

Hence

$$\begin{aligned}\Im(z - g_A(z)) &= \frac{2}{\pi} \frac{\Im(z)}{|z|^2} \int_0^\pi \mathbb{E}_{i\theta}[\Im(B_\tau)] \sin(\theta) \mathrm{d}\theta (1 + \mathcal{O}(|z|^{-1})) \\ &:= a \frac{\Im(z)}{|z|^2}(1 + \mathcal{O}(|z|^{-1}))\end{aligned}$$

and so $\Im(z - g_A(z) - \frac{a}{2}) = \mathcal{O}(a \frac{\Im z}{|z|^3})$. Define $h(z) := z - g_A(z) - \frac{a}{2}$. Then $\Im(h(z))$ is harmonic. Also $|\partial_x \Im(h(z))|, |\partial_y \Im(h(z))| \leq \tilde{c} \frac{a}{|z|^3}$. Then the Cauchy-Riemann equations imply similar inequalities for the real parts of $h(z)$ so $|h'(z)| \leq \tilde{c} \frac{a}{|z|^3}$. We have $h(\infty) = 0$ so $|h(re^{i\theta})| \leq \int_r^\infty |h'(se^{i\theta})| \mathrm{d}s \lesssim \frac{a}{r^2}$. $\square$

Loewner differential equation

Definition. Let $(A_t)_{t \geq 0}$ be a family of compact $\mathbb{H}$-hulls. We say $(A_t)_{t \geq 0}$

(i) *is strictly increasing* if $A_s \subsetneq A_t$ whenever $s < t$;

(ii) *satisfies the local growth property* if for all $T, \varepsilon > 0$ there exists $\delta > 0$ such that whenever $0 \leq s \leq t \leq s + \delta \leq T$ we have $\mathrm{diam}(g_s(A_t \setminus A_s)) \leq \varepsilon$.

If (i) and (ii) are satisfied then $t \mapsto \mathrm{hcap}(A_t)$ is continuous and increasing. In this case we say $(A_t)_{t \geq 0}$

(iii) is *parameterised by half-plane capacity* if $\mathrm{hcap}(A_t) = 2t$ for all t.

We let $\mathcal{A}$ be the set of all such families satisfying (i)-(iii). We let $\mathcal{A}_T$ be the set of all such families satisfying (i)-(iii) but on time interval $[0, T]$.

Theorem ("Chordal Loewner differential equation"). *Let $(A_t)_{t\geq 0} \in \mathcal{A}$, let $g_t := g_{A_t}$ be the mapping-out function. Then there exists $U : [0,\infty) \to \mathbb{R}$ continuous such that*

$$\partial_t g_t(z) = \frac{z}{g_t(z) - U_t},\ g_0(z) = z. \tag{$*$}$$

Proof. We have that $\bigcap_{s>0} \overline{g_t(A_s \setminus A_t)}$ is a single point by the local growth property. Let U_t be this point. The local growth property and the proposition from last time, U is continuous.

Define $\tilde{g} = g_{g_t(A_{t+\delta}\setminus A_t)-U_t}$. Then

$$\tilde{g}(z) = z + \frac{\operatorname{hcap}(g_t(A_{t+\delta} \setminus A_t) - U_t)}{2} + \mathcal{O}\left(\frac{\operatorname{hcap}(g_t(A_{t+\delta} \setminus A_t)) \operatorname{rad}(g_t(A_{t+\delta} \setminus A_t))}{|z|^2}\right).$$

Defining $g_{t,t+\delta} = g_{t+\delta}^{-1} \circ g_t$ we have

$$g_{t,t+\delta}(z) = z + \frac{2\delta}{z - U_t} + 2\delta \operatorname{diam}(g_t(A_{t+\delta} \setminus A_t))\mathcal{O}\left(\frac{1}{|z - U_t|^2}\right)$$

uniformly in $t \in [0, T]$. Hence

$$g_{t+\delta}(z) - g_t(z) = \frac{2\delta}{g_t(z) - U_t} + 2\delta \operatorname{diam}(g_t(A_{t+\delta} \setminus A_t))\mathcal{O}\left(\frac{1}{|g_t(z) - U_t|^2}\right).$$

Now dividing through by δ and noting $\operatorname{diam}(g_t(A_{t+\delta} \setminus A_t)) \to 0$ we get the result. $\square$

Conversely, given U continuous and real valued, then $(*)$ has a unique solution for $t < \tau_z := \sup\{s : |g_s(z) - U_s| > 0\}$.

We use the notation

$$\begin{aligned} A_t &:= \{z \in \mathbb{H} : \tau_z \leq t\} \\ H_t &:= \mathbb{H} \setminus A_t. \end{aligned}$$

Then $g_t : H_t \to \mathbb{H}$ is conformal and $(A_t) \in \mathcal{A}$ and $g_{A_t} = g_t$ (see Example Sheet). We call (U_t) the "driving function" or "Loewner transform" of (g_t) or (A_t).

Schramm-Loewner Evolution (SLE)

Suppose $(A_t) \in \mathcal{A}$ is random with driving function U such that

(i) (rA_{t/r^2}) has the same law as (A_t) (scale invariance);

(ii) Conditional on $\mathcal{F}_t = \sigma(U_s : s \leq t)$, the conditional law of $(g_t(A_{t+s} \setminus A_t) - U_t)_{s\geq 0}$ is the same as that of $(A_s)_{s\geq 0}$.

These are called the *conformal Markov properties*.

Theorem. *There exists $\kappa \geq 0$ such that $U_t = \sqrt{\kappa} B_t$ for some Brownian motion B.*

Proof. U is continuous and by (ii) of the conformal Markov properties, we have that $(U_{t+s} - U_t)_{s \geq 0}$ has the same law as $(U_s)_{s \geq 0}$ conditional on $\mathcal{F}_t$. Therefore U has independent and stationary increments so $U_t = at + \sqrt{\kappa} B_t$ for some a, κ.

(i) of the conformal Markov properties implies $U_t = \sqrt{\kappa} B_t$. $\square$

Definition. The random Loewner chain with $U_t = \sqrt{\kappa} B_t$ for a Brownian motion B is denoted SLE_κ.

Remarks.

- SLE_κ is generated by a curve, i.e there exists a continuous path γ in $\overline{\mathbb{H}}$ such that $H_t = \mathbb{H} \setminus A_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$.
- If $\kappa \leq 4$ then SLE_κ is a simple curve, i.e $\gamma(t) \in \mathbb{H}$ for $t > 0$ and $\gamma(t) \neq \gamma(s)$ for $s \neq t$
- If $\kappa \in (4, 8)$ then SLE_κ is self-intersecting and boundary-intersecting and disconnects points from ∞
- If $\kappa \geq 8$ then SLE_κ is space-filling.
- For all κ, $\gamma(t) \to \infty$ as $t \to \infty$.

Definition. If $D \subsetneq \mathbb{C}$ is a simply-connected domain, $x, y \in \partial D$ (suppose ∂D is a curve). Define SLE_κ in (D, x, y) as the pushforward SLE_κ in $(\mathbb{H}, 0, \infty)$ under a conformal transformation $\varphi : \mathbb{H} \to D$ with $\varphi(0) = x$, $\varphi(\infty) = y$ (well-defined due to scaling invariance in $\mathbb{H}$).

Definition. We say that a Loewner chain (g_t) (or equivalently (A_t)) is *generated by a curve* if there exists $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ continuous such that for all t, $H_t := \mathbb{H} \setminus A_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$.

Lemma. *Suppose $\gamma(t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy)$ exists for all t and is continuous then (g_t) is generated by γ.*

Remark. The converse is also true.

We will need some facts:

(i) Let A be a compact $\mathbb{H}$-hull. If α is a continuous path and $\alpha(s) \in \mathbb{H} \setminus A$ for $s > 0$, $\alpha(0) \in \partial A$. Then $\lim_{s \downarrow 0} g_A(\alpha(s)) \in \mathbb{R}$ exists. [See Q3 Example Sheet 1]

(ii) If $\alpha, \tilde{\alpha}$ are two paths in $\mathbb{H} \setminus A$ and $\lim_{s \downarrow 0} g_A(\alpha(s)) = \lim_{s \downarrow 0} g_A(\tilde{\alpha}(s))$ then $\alpha(0) = \tilde{\alpha}(0)$. [Q3 Example Sheet 1 again applied to g_A^{-1}]

Proof of Lemma. Clearly $\gamma(t) \notin H_t$ so $H_t \subseteq \mathbb{H} \setminus \text{fill}(\gamma([0, t]))$. Now we show $\partial A_t \cap \mathbb{H} \subseteq \gamma([0, t])$. Let $z \in \partial A_t \cap \mathbb{H}$. Since γ is continuous it's enough to show $z \in \overline{\gamma([0, t])}$. Pick $w_n \to z$, $w_n \in H_t$. Let α be the line segment from w_n towads z until it hits the first point $z_n \in \partial A_t$.

So now we show $z_n \in \gamma([0, t])$. Since $z_n \in A_t$ we have $s := \tau_{z_n} \leq t$. We claim $\lim_{r \downarrow 0} g_s(\alpha(r)) = U_s$. Once we have this, by fact (ii) above since $\lim_{y \downarrow 0} g_t^{-1}(U_s + y) = \gamma(s)$ we just have $\alpha(0) = \gamma(s)$.

Indeed if not, $\text{dist}(g_s(\alpha), U_s) > 0$. But since $z_n \in A_s \setminus A_{s-\delta}$ for all $\delta > 0$, combined with the local growth property, we have $\lim_{\delta \downarrow 0} g_{s-\delta}(z_n) = U_s$ and so $\text{dist}(g_{s-\delta}(\alpha), U_{s-\delta}) \to 0$ as $\delta \to 0$, giving a contradiction. □

As we will throughout the course, we assume (U_t) is continuous and real-valued. So we can solve the Loewner differential equation $\partial_t g_t(z) = \frac{z}{g_t(z) - U_t}$, $g_0(z) = z$, $t < \tau_z := \sup\{s : |g_s(z) - U_s| > 0\}$. Then since $U_t \in \mathbb{R}$ we have $\partial_t \overline{g_t(\overline{z})} = \frac{z}{\overline{g_t(\overline{z}) - U_t}}$, so $g_t(\overline{z}) = \overline{g_t(z)}$ and $\tau_z = \tau_{\overline{z}}$ by uniqueness. On $\{\overline{z} : z \in H_t\}$, this agrees with the Schwarz reflection of $g_t : H_t \to \mathbb{H}$.

Lemma. *For $z \in \mathbb{R}$, $\tau_z \leq t$ if and only if $z \in \overline{A_t \cap \mathbb{H}}$, i.e the domain $\{z \in \mathbb{C} : \tau_z > t\}$ agrees exactly with the reflection of H_t across $\mathbb{R}$.*

Proof. If $\tau_z > t$ then $\tau_w > t$ in a neighbourhood of z by continuity. Conversely, suppose $z \in \mathbb{R} \setminus \{U_0\}$, WLOG $z > U_0$. By the local growth property, $z \notin \overline{A_{t+\delta} \cap \mathbb{H}}$ for some $\delta > 0$.

Let $\varepsilon > 0$ be such that $B(z, \varepsilon) \cap \mathbb{H} \subseteq H_{t+\delta}$. The Schwarz reflection g_s^* of g_s is defined and univalent on $B(z, \tau)$ for all $s \leq t$. Hence $g_s^*(z) \neq U_s$, otherwise there would be some $w \in A_{s+\delta} \setminus A_s$ with $w \in B(z, \varepsilon) \cap \mathbb{H}$. Also $g_s(w_n) \to g_s^*(z)$ as $w_n \to z$, $w_n \in \mathbb{H}$.

Taking limits in the Loewner differential equation for w_n implies $s \mapsto g_s^*(z)$ satisfies the Loewner differential equation on $[0, t]$ and $\tau_z > t$. $\square$

Bessel processes

Itô's formula says that if $X^1, \dots, X^d$ are semi-martingales and $F \in C^2$ then $F(X^1, \dots, X^d)$ is a semi-martingale and

$$\begin{aligned} F(X_t^1, \dots, X_t^d) = F(X_0^1, \dots, X_0^d) + \sum_{i=1}^d \int \partial_i F(X_t^1, \dots, X_t^d) \mathrm{d}X_t^i \\ + \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j F(X_t^1, \dots, X_t^d) \mathrm{d}\langle X^i, X^j \rangle_t. \end{aligned}$$

So

$$\mathrm{d}F(X_t^1, \dots, X_t^d) = \sum_{i=1}^d \partial_i F(X_t^1, \dots, X_t^d) \mathrm{d}X_t^i + \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j F(X_t^1, \dots, X_t^d) \mathrm{d}\langle X^i, X_j \rangle_t.$$

Let $(B^1, \dots, B^d)$ be a Brownian motion in $\mathbb{R}^d$ so $Z_t := \|B\|^2 = (B^1)^2 + \dots + (B^d)^2$. Then

$$\begin{aligned} \mathrm{d}\|B_t\|^2 &= \sum_{i=1}^d 2B_t^i \mathrm{d}B_t^i + d\mathrm{d}t \\ &= 2Z_t^{1/2} \mathrm{d}Y_t + d\mathrm{d}t \end{aligned}$$

where $\mathrm{d}Y_t = \sum_{i=1}^d \frac{1}{Z_t^{1/2}} B_t^i \mathrm{d}B_t^i$. In Part III Stochastic Calculus it is shown that Y has the law of a Brownian motion since

$$\mathrm{d}[Y]_t = \sum_{i=1}^d \frac{1}{Z_t} (B_t)^2 \mathrm{d}t = \mathrm{d}t.$$

By defining $X_t := Z_t^{1/2}$ we have

$$\begin{aligned} \mathrm{d}X_t &= \frac{1}{2} Z_t^{-1/2} \mathrm{d}Z_t + \frac{1}{2} \left(-\frac{1}{2} Z_t^{-3/2} \right) \mathrm{d}[Z]_t \\ &= \underbrace{\mathrm{d}Y_t}_{\text{B.m}} + \frac{d-1}{2} \frac{1}{X_t} \mathrm{d}t. \end{aligned}$$

This process is defined for all $d \in \mathbb{R}$ and is called a *Bessel process of dimension* d, denoted BES^d.

Proposition. Let $d \in \mathbb{R}$, $X \sim \mathrm{BES}^d$. If $d < 2$ then X_t hits 0 almost-surely. If $d \geq 2$ then X_t does not hit 0 almost-surely.

Idea: let $a < b$ and define $\tau_x = \inf\{t \geq 0 : X_t = x\}$. Then if $x \in (a, b)$ let $F(x) := \mathbb{P}_x(\tau_a < \tau_b)$. Then

$$F(x) = \mathbb{E}_x[\mathbb{P}(\tau_a < \tau_b | \mathcal{F}_t)] = \mathbb{E}_x[F(X_t)] \qquad \text{(MG property)}$$

which implies $F(X_t)$ should be a martingale. Suppose $F \in C^2$. Then Itô's formula gives

$$\begin{aligned} \mathrm{d}F(X_t) &= F'(X_t)\mathrm{d}X_t + \frac{1}{2}F''(X_t)\mathrm{d}[X]_t \\ &= \left(\frac{d-1}{2}\frac{F'(X_t)}{X_t} + \frac{1}{2}F''(X_t)\right)\mathrm{d}t + \text{ local martingale} \end{aligned}$$

and so if

$$\frac{d-1}{2}\frac{F'(u)}{u} + \frac{1}{2}F''(u) = 0 \text{ on } (a,b)$$

we will get a local martingale, giving

$$F'(u) = cu^{1-d} \implies F(u) = \begin{cases} c_1 u^{2-d} & d \neq 2 \\ c_1 \log u + c_2 & d = 2 \end{cases}.$$

Proof. Suppose first that $d \neq 2$. Define $f(u) = u^{2-d}$. Then by Itô's formula as above, $(f(X_t))$ is a local martingale. This gives $\mathbb{P}(\tau_u < \tau_b) = \frac{b^{2-d} - X_0^{2-d}}{b^{2-d} - a^{2-d}}$ by the Gamblers ruin formula for local martingales.

If $d < 2$ then sending $a \to 0$ and then $b \to \infty$ we have $\mathbb{P}(\tau_0 < \infty) = 1$.

If $d \geq 2$ then applying the same argument shows $\mathbb{P}(\tau_0 < \infty) = 0$. □

Proposition. If $\kappa \leq 4$ then $\gamma(t) \in \mathbb{H}$ for all t almost-surely. If $\kappa > 4$ then $\gamma(t) \in \mathbb{R}$ for some $t > 0$ almost-surely.

Proof. Note γ intersects $\mathbb{R}$ at $x > 0$ iff for all $y \in [0,x]$, $y \in A_t$. We have $y \in A_t$ iff $\tau_y \leq t$ where $\tau_y = \sup\{s : |g_s(y) - U_s| > 0\}$. Letting $g_s(y) - U_s := X_s(y)$ we have

$$\begin{aligned} &\mathrm{d}X_s(y) = \partial_t g_s(y)\mathrm{d}t - \mathrm{d}U_s \\ &\implies \mathrm{d}X_s(y) = \frac{2}{X_s(y)}\mathrm{d}s - \underbrace{\mathrm{d}U_s}_{\sqrt{\kappa}\mathrm{d}B_s} \qquad \text{(chordal eqn)} \\ &\iff \mathrm{d}\frac{X_s(y)}{\sqrt{\kappa}} = \frac{\frac{2}{\kappa}}{\frac{X_s(y)}{\sqrt{\kappa}}} - \mathrm{d}B_s \end{aligned}$$

a "Bessel process of dimension $1 + \frac{4}{\kappa}$". □

Proposition. If $\kappa \leq 4$ then SLE_κ corresponds to a simple curve almost-surely. If $\kappa > 4$ then SLE_κ has self-intersections almost-surely.

Proof. For $\kappa \leq 4$ note $s \mapsto g_t(\gamma(s+t)) - g_t(\gamma(t))$ is a SLE_κ curve and so a.s does not intersect $\mathbb{R}$. But intersections between $\gamma|_{[0,t]}$ and $\gamma|_{[t,\infty)}$ correspond to such curves hitting $\mathbb{R}$.

For $\kappa > 4$, by scale-invariance we have that for all $\varepsilon > 0$, $\tilde{\gamma}$ almost-surely intersects $(0,\varepsilon]$. So find $t > 0$ such that $g_t^{-1}[-\varepsilon,\varepsilon] \subseteq \mathbb{H}$, $\gamma(t+s) = g_t^{-1}(\tilde{\gamma}(s))$. □

Useful computations:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t} = 2\frac{(\Re g_t(z) - \sqrt{\kappa}B_t) - \Im g_t(z)}{|\underbrace{g_t(z) - \sqrt{\kappa}B_t}_{X_t(z)+iY_t(z)}|^2}$$

$$\mathrm{d}X_t = \frac{2X_t}{X_t^2 + Y_t^2}\mathrm{d}t - \sqrt{\kappa}\mathrm{d}B_t$$

$$\mathrm{d}Y_t = -\frac{2Y_t}{X_t^2 + Y_t^2}\mathrm{d}t$$

$$\partial_t g_t'(z) = -\frac{-2g_t'(z)}{(g_t(z) - \sqrt{\kappa}B_t)^2}$$

$$\partial_t \log|g_t'(z)| = -2\frac{X_t^2 - Y_t^2}{(X_t^2 + Y_t^2)^2}.$$

Theorem. *For $\kappa \geq 0$, $\kappa \neq 8$, SLE_κ is generated by a curve.*

Remark. Also true for $\kappa = 8$ but we will not prove this.

Proof. It suffices to show $\gamma(t) = \lim_{y\downarrow 0} g_t^{-1}(\sqrt{\kappa}B_t + iy)$ exists and is continuous in t. We will show that for $\kappa \neq 8$ there exists $\alpha = \alpha(\kappa) > 0$ such that

$$\sup_{t\in[0,T]} |(g_t^{-1})'(\sqrt{\kappa}B_t + iy)| \leq Cy^{-1+\alpha} \tag{$*$}$$

for some random almost-surely finite C. This will imply $t \mapsto g_t^{-1}(\sqrt{\kappa}B_t + iy)$ converges uniformly on $[0,T]$ as $y \downarrow 0$. By Koebe's distortion theorem it suffices to show $(*)$ for $y = 2^{-n}$, $n \in \mathbb{N}$.

We can restrict to the event $\{\|\sqrt{\kappa}B\|_{[0,T],\infty} \leq M\}$ i.e $A_T \subseteq B(0, M + 2\sqrt{T})$ for M sufficiently large.

Suppose that $|(g_t^{-1})'(\sqrt{\kappa}B_t + iy)| \geq u$ for some $t \in [0,T]$. Then by Koebe's 1/4 theorem we have $g_t^{-1}(B(\sqrt{\kappa}B_t + iy, y/2))$ contains a ball of radius $uy/8$. Then Koebe's distortion estimate says $|g_t'(w)| \simeq |g_t'(g_t^{-1}(\sqrt{\kappa}B_t + iy))| \leq 1/u$ for all $w \in B(g_t'(\sqrt{\kappa}B_t + iy), uy/16)$. So can pick $w \in \frac{uy}{16}\mathbb{Z}$ and $|w| \lesssim M + 2\sqrt{T}$ due to $A_T \subseteq B(0, M + 2\sqrt{T})$.

Note $g_t(w) \in B(\sqrt{\kappa}B_t + iy, y/2)$, i.e $\frac{X_t(w)}{Y_t(w)} \in [-1,1]$ and $Y_t(w) \in [y/2, 3y/2]$.

We try to compute $\mathbb{E}|g'_\sigma(w)|^\lambda$, $\sigma \in \inf\{s : Y_s(w) = y\}$. Let $\tilde{F}(a,b) = \mathbb{E}|g'_\sigma(a + ib)|^\lambda$ so

$$\begin{aligned}\tilde{F}(a,b) &= \mathbb{E}[|g'_t(a+ib)|^\lambda \mathbb{E}[|g'_\sigma(g_t(a+bi))||\mathcal{F}_t]] \\ &= \mathbb{E}[|g'_t(a+bi)|^\lambda \tilde{F}(X_t + iY_t)]\end{aligned}$$

which implies $|g'_t|^\lambda \tilde{F}(X_t, Y_t) := \tilde{M}_t$ is a martingale. Then Ito's formula gives $\mathrm{d}\tilde{M}_t$ as the sum of a local martingale and a term of the form $(\tilde{F}, \partial\tilde{F}, \ldots)\mathrm{d}t$. We won't be able to solve this exactly but we can find an approximation (see lemma following this proof).

To conclude, define

$$\tilde{\sigma} = \inf\{s : \frac{X_s}{Y_s} \in [-1,1],\ Y_s \in [y/2, 3y/2],\ |g'_s(w)| \le \tilde{c}/u\}$$

then if $\lambda \le 0$

$$\begin{aligned}\mathbb{P}(\tilde{\sigma} < \infty) &\lesssim u^\lambda y^{-\xi} \mathbb{E}[M_{\tilde{\sigma}} \mathbb{1}_{\tilde{\sigma}<\infty}] \\ &\le u^\lambda y^{-\xi} M_0 \\ &= u^\lambda y^{-\xi} (\Im w)^\xi \left(1 + \left(\frac{\Re(w)}{\Im(w)}\right)^2\right)^{r/2}.\end{aligned}$$

To summarise, we have

$$\begin{aligned}&\mathbb{P}\left(|(g_t^{-1})'(\sqrt{\kappa}B_t + iy)| \ge y^{-1+\alpha} \text{ for some } t \in [0,T],\ \|\sqrt{\kappa}B_t\|_{[0,T],\infty} \le M\right) \\ &\le \sum_{\substack{w \in y^\alpha \mathbb{Z}^2 \cap \mathbb{H} \\ |w| \lesssim M + 2\sqrt{T}}} \mathbb{P}(\tilde{\sigma}_w < \infty) \lesssim y^{(-1+\alpha)\lambda - \xi} \underbrace{\sum_w (\Im(w))^\xi \left(1 + \left(\frac{\Re(w)}{\Im(w)}\right)^2\right)^{r/2}}_{\lesssim y^{-2\alpha} \text{ for small } \xi, r}.\end{aligned}$$

Sum over all $y = 2^{-n}$, $n \in \mathbb{N}$ to get $\sum_n 2^{-n(-\lambda-\xi-\alpha(-\lambda+2))}$.

For $\kappa \ne 8$ we have $\min_r \left(2r - \frac{r\kappa}{4} + \frac{r^2\kappa}{4}\right) < 0$. So we can choose λ, ξ, r such that $(**)$ below holds, $\lambda \le 0$ and $\lambda + \xi < 0$. Hence we can take $\alpha > 0$ small so that $\sum_n 2^{-n(-\lambda-\xi-\alpha(-\lambda+2))} < \infty$ so we are done by Borel Cantelli. □

Lemma. *Let $\lambda, g, r \in \mathbb{R}$ be such that $\lambda + \xi \ge 2r - \frac{r\kappa}{4} + \frac{r^2\kappa}{4}$ and $\lambda - \xi \le -\frac{r\kappa}{4}$ (call these conditions $(**)$). Then*

$$M_t = |g'_t(w)|^\lambda Y_t(w)^\xi \left(1 + \frac{X_t^2}{Y_t^2}\right)^{r/2}$$

is a local supermartingale. If $(**)$ hold with equality then (M_t) is a local martingale.

Proof. By computation above. □

Conformality of SLE_6

Let $D \subseteq \mathbb{H}$ contain a neighbourhood of 0 and be simply connected. Let $\psi : D \to \mathbb{H}$ be a conformal map with $\psi(0) = 0$. Define $\tilde{A}_t = \psi(A_t)$ and let $\tilde{g}_t = g_{\tilde{A}_t}$. Define $\tilde{U}_t = \psi_t(U_t)$ where $\psi_y = \tilde{g}_t \circ \psi \circ g_t^{-1}$.

Remark. $(\tilde{A}_t)$ is not in general parameterised by half plane capacity anymore.

We have

$$\operatorname{hcap}(\tilde{g}_t(\tilde{A}_{t+s} \setminus \tilde{A}_t)) = (\psi_t'(U_t))^2 \operatorname{hcap}(g_t(A_{t+s} \setminus A_t)) + o(s) = (\psi_t'(U_t))^2 \cdot 2s + o(s).$$

Hence define

$$\tilde{a}(t) = \operatorname{hcap}(\tilde{A}_t) = \int_0^t 2\psi_s'(U_s)^2 \mathrm{d}s$$

so by the chain rule

$$\partial_t \tilde{g}_t(z) = \frac{\partial_t \tilde{a}(t)}{\tilde{g}_t(z) - \tilde{U}_t} = \frac{2\psi_t'(U_t)^2}{\tilde{g}_t(z) - U_t}$$

and Itô's formula applied to $(t, U_t) \mapsto \psi_t(U_t)$ gives

$$\mathrm{d}\underbrace{\psi_t(U_t)}_{\tilde{U}_t} = \partial_t \psi_t(U_t) + \psi_t'(U_t)\mathrm{d}U_t + \frac{1}{2}\psi_t''(U_t)\mathrm{d}[U]_t.$$

We can compute

$$\begin{aligned}
\partial_t \psi_t(z) &= \partial_t \tilde{g}_t(\psi(g_t^{-1}(z))) + \tilde{g}_t'(\psi(g_t^{-1}(z)))\psi'(g_t^{-1}(z))\partial_t g_t^{-1}(z) \\
&= 2\left(\frac{\psi_t'(U_t)^2}{\psi_t(z) - \psi_t(U_t)} - \frac{\psi_t'(z)}{z - U_t}\right) \\
&= -3\psi_t''(U_t) + \mathcal{O}(|z - U_t|)
\end{aligned}$$

which means

$$\begin{aligned}
\mathrm{d}\tilde{U}_t &= \partial_t \psi_t(U_t) + \psi_t'(U_t) \underbrace{\mathrm{d}U_t}_{\sqrt{\kappa}\mathrm{d}B_t} + \frac{1}{2}\psi_t''(U_t) \underbrace{\mathrm{d}[U]_t}_{\kappa \mathrm{d}t} \\
&= \frac{\kappa - 6}{2}\psi_t''(U_t)\mathrm{d}t + \sqrt{\kappa}\psi_t'(U_t)\mathrm{d}B_t
\end{aligned}$$

reparameterising by half-plane capacity, i.e setting $\sigma(s) = \inf\{t : \operatorname{hcap}(A_t) = 2\int_0^t \psi_t'(U_r)^2 \mathrm{d}r = 2s\}$ we have

$$\mathrm{d}\sigma(s) = \frac{\mathrm{d}s}{\psi_{\sigma(s)}'(U_{\sigma(s)})^2}$$

and therefore

$$\partial_s \tilde{g}_{\sigma(s)}(z) = \frac{2}{\tilde{g}_{\sigma(s)}(z) - \tilde{U}_{\sigma(s)}}, \qquad \tilde{g}_{\sigma(0)}(z) = z.$$

Then

$$\mathrm{d}\tilde{U}_{\sigma(s)} = \frac{\kappa - 6}{2} \frac{\psi''_{\sigma(s)}(U_{\sigma(s)})}{\psi'_{\sigma(s)}(U_{\sigma(s)})^2} \mathrm{d}s + \sqrt{\kappa} \mathrm{d}\tilde{B}_s$$

where $\tilde{B}_s = \int_0^{\sigma(s)} \psi'_r(U_r) \mathrm{d}B_r$ has the law of a Brownian motion [indeed $[\tilde{B}]_s = \int_0^{\sigma(s)} \psi'_r(U_r)^2 \mathrm{d}r = s$].

Theorem. *In the setup above, if $\kappa = 6$ the law of $\psi(\gamma)$ up to hitting $\pi(\partial D \cap H)$ is an* SLE_6.

Some further topics:

- variants of chordal SLE, such as the radial SLE;
- natural length;
- reversibility;
- duality;
- and more…

The Gaussian Free Field

Given a domain $D \subseteq \mathbb{C}$ the (zero boundary) Green's function on D is $G_D(x,y) = \log \frac{1}{|x-y|} \cdot \tilde{G}_x(y)$ where $\tilde{G}_x$ is a harmonic extension of $\partial D \ni y \mapsto \frac{1}{|x-y|}$.

Then we have $\Delta_y G_D(x,y) = -2\pi\delta_x(y)$ in a distributional sense.

Definition (Zero-boundary GFF on D)**.** A mean 0 Gaussian process $(\langle h, \rho \rangle)_{\rho \in C_c^\infty(D)}$ is called a (zero-boundary) *Gaussian free field* on D if it has covariance

$$\mathbb{E}[\langle h, \rho_1 \rangle \langle h, \rho_2 \rangle] = \int_{D^2} G(x,y)\rho_1(x)\rho_2(y)\mathrm{d}x\mathrm{d}y$$

where $\langle h, \rho_1 \rangle$ denotes $\int_{\mathbb{R}} h(x)\rho_1(x)\mathrm{d}x$, i.e the L^2-inner product.

Write

$$\langle f, g \rangle_\nabla = \frac{1}{2\pi} \nabla f(x) \cdot \nabla g(x) \mathrm{d}x$$

and let $H_0^1(D)$ be the Hilbert space completion of $C_c^\infty(D)$ with respect to $\langle \cdot, \cdot \rangle_\nabla$ (assuming $\operatorname{diam}(\partial D) > 0$).

Proposition. For $\varphi : D \to \tilde{D}$ conformallly invariant

- $\langle f, g \rangle_\nabla = (f \circ \varphi^{-1}, g \circ \varphi^{-1})_\nabla$ for $f, g \in C_c^\infty(D)$. Hence φ induces an isometry $H_0^1(D) \to H_0^1(\tilde{D})$;
- $G_D(x,y) = G_{\tilde{D}}(\varphi(x), \varphi(y))$.

Proof. The first statement is on the example sheet, the second follows from $\log|\varphi(x) - \varphi(y)| - \log|x-y| = \log\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|$ and the RHS is harmonic in D, so follows by uniqueness of harmonic extension. $\square$

Formally the GFF s a "standard normal in $H_0^1(D)$", i.e if $(f_n)_{n\geq 1}$ is an orthonormal basis of $H_0^1(D)$ and $(\alpha)_{n\geq 1}$ are iid $\mathcal{N}(0,1)$ then $\sum_{n\geq 1} \alpha_n f_n$ is a standard normal in $H_0^1(D)$. Indeed then $\langle h, f \rangle_\nabla = \sum_{n\geq 1} \alpha_n \langle f_n, f \rangle_\nabla$ will have the required mean and variance.

Definition. $(\langle h, f \rangle_\nabla)_{f \in H_0^1(D)}$ is defined as a mean 0 Gaussian process and

$$\langle h, f \rangle_\nabla = \sum_{n\geq 1} \alpha_n \langle f_n, f \rangle_\nabla$$

so $\mathbb{E}[\langle h, f \rangle_\nabla \langle h, g \rangle_\nabla] = \langle f, g \rangle_\nabla$.

If $(\tilde{f}_n)_{n\geq 1}$ is another orthonormal basis we can project $\tilde{\alpha}_n = \langle h, \tilde{f}_n \rangle_\nabla$ so $\mathbb{E}[\tilde{\alpha}_n, \tilde{\alpha}_m] = \langle \tilde{f}_n, \tilde{f}_m \rangle = \delta_{n,m}$ and $h = \sum_{n\geq 1} \tilde{\alpha}_n \tilde{f}_n$.

For $U \subseteq D$ we have a continuous embedding $C_c^\infty(U) \subseteq C_c^\infty(D)$ and so a continuous embedding $H_0^1(U) \subseteq H_0^1(D)$. We claim that $H_0^1(D) = H_0^1(U) \oplus H_{\text{harm}}(U)$, where $H_{\text{harm}}(U)$ denotes the elements f of $H_0^1(D)$ that are harmonic in the weak sense $\langle f, \Delta\rho\rangle = 0$ for all $\rho \in C_c^\infty(U)$.

Now we prove the claim. If $f \in H_0^1(U)$, $g \in H_{\text{harm}}(U)$ then

$$\langle f, g\rangle_\nabla = \frac{1}{2\pi}\int_U \nabla f \cdot \nabla g \mathrm{d}x = -\frac{1}{2\pi}\int_U f\Delta g \mathrm{d}x = 0.$$

Hence $H_{\text{harm}}(U)$ is a subset of the orthogonal complement of $H_0^1(U)$. To show that it is in fact the whole of $H_0^1(U)^\perp$, suppose $g \in H_0^1(U)^\perp$. Then for all $\rho \in C_c^\infty(U)$ we have

$$0 = \langle \rho, g\rangle_\nabla = -\frac{1}{2\pi}\int_U (\Delta\rho) g \mathrm{d}x$$

i.e $g \in H_{\text{harm}}(U)$.

Proposition. f being harmonic in the weak sense is equivalent to saying f can be represented by a function with $\Delta f = 0$ in U.

Proof. Example Sheet. □

Proposition. Let h be a GFF in D, $U \subseteq D$. Then there exists $h_{D\setminus U}, h_U^{D\setminus U}$ independent with $h = h_{D\setminus U} + h_U^{D\setminus U}$ such that $h_U^{D\setminus U}$ is a GFF in U and $h_{D\setminus U}$ is a GFF in $D \setminus U$ and harmonic in U.

This is called the *domain Markov property.*

Proof. Let (f_n^1) be an orthonormal basis of $H_0^1(U)$ and let (f_n^2) be an orthonormal basis of $H_{\text{harm}}(U)$. Then

$$h = \underbrace{\sum_{n\geq 1} \alpha_n^1 f_n^1}_{h_U^{D\setminus U}} + \underbrace{\sum_{n\geq 1} \alpha_n^2 f_n^2}_{h_{D\setminus U}}.$$

□

Note that we have conformal invariance: if $\varphi : D \to \tilde{D}$ is conformal then $\langle h\circ\varphi^{-1}, f\rangle_\nabla = \langle h, f\circ\varphi\rangle_\nabla$ and $\langle h\circ\varphi^{-1}, \rho\rangle = \langle h, (\rho\circ\varphi)|\varphi'|^2\rangle$ so $h\circ\varphi^{-1}$ is a GFF in $\tilde{D}$.

Local sets of the Gaussian Free Field

For $A_1 \subseteq A_2 \subseteq U$ take $h = h_{A_1} + h^{A_1}_{D\setminus A_1}$. Since $D \setminus A_1 \supseteq D \setminus A_2$ we can then take

$$h^{A_1}_{D\setminus A_1} = (h^{A_1}_{D\setminus A_1})_{A_2} + (h^{A_1}_{D\setminus A_1})^{A_2}_{D\setminus A_2}.$$

It turns out that $A_2 \supseteq A_1$ can depend randomly on h_{A_1}, as well as additional randomness which is independent of $h^{A_1}_{D\setminus A_1}$. Therefore we have a GFF (h, A_2) (with A_2 random) where

$$h = h_{A_2} + h^{A_2}_{D\setminus A_2}$$

and h_{A_2} is harmonic in $D\setminus A_2$, $h^{A_2}_{D\setminus A_2}$ a GFF in $D\setminus A_2$ (conditional on (h_{A_2}, A_2)).

Consider the space of relatively closed $A \subseteq D$. We can identify A with $\overline{A}$, so the Hausdorff metric makes this a Polish space.

Definition. Let h be a GFF in D, $A \subseteq D$ a random relatively closed subset. We say (h, A) is *local* if for all open $U \subseteq D$ the conditional probability $\mathbb{P}(A \cup U = \emptyset|h)$ is almost-surely measurable with respect to $h_{D\setminus U}$, i.e $\mathbb{P}(A \cup U = \emptyset|h) = \mathbb{P}(A \cup U = \emptyset|h_{D\setminus U})$ almost-surely.

Let $\mathcal{D}_n := \{([j2^{-n}, (j+1)2^{-n}] \times [k2^{-n}, (k+1)2^{-n}]) \cap D : j, k \in \mathbb{Z}\}$.

- Suppose A is almost-surely a union of squares in $\mathcal{D}_n$ and suppose B is a deterministic union of squares, so by the domain Markov property, $h = h_B + h^B_{D\setminus B}$. If (h, A) is local then the conditional law of $h^B_{D\setminus B}$ given h_B and $\{A \subseteq B\}$ is still a GFF in $D\setminus B$. For $B' \subseteq B$, $\sigma(h_{B'}) \subseteq \sigma(h_B)$. Hence conditional on $\{A = B\}$ we have $h = h_B + h^B_{D\setminus B}$, so $h = h_A + h^A_{D\setminus A}$ where h_A is harmonic in $D \setminus A$ and the conditional law of $h^A_{D\setminus A}$ given (A, h_A) is a GFF in $D \setminus A$.
- Conversely suppose we have $h = h_A + h^A_{D\setminus A}$ with these properties. We claim (h, A) is local. Let $U \subseteq D$ be open, $h = h_{D\setminus U} + h^{D\setminus U}_U$. Then conditional on $\{A\cap U = \emptyset\}$ and h_A, $h^A_{D\setminus A} = h^A_{D\setminus A\setminus U} + (h^A)^{D\setminus U}_U$. Projecting onto $H_{\text{harm}}(U), H^1_0(U)$ respectively implies $h_{D\setminus U} = h_A + h^A_{D\setminus A\setminus U}$ and $h^{D\setminus U}_U = (h^A)^{D\setminus U}_U$ almost-surely. Hence the conditional law of $h^{D\setminus U}_U = (h^A)^{D\setminus U}_U$ given $h_{D\setminus U} = h_A + h^A_{D\setminus A\setminus U}$ and $\{A \cap U = \emptyset\}$ is a GFF in U, i.e the same as the conditional law given just $h_{D\setminus U}$. Therefore conditionally on $h_{D\setminus U}$, the event $\{A \cap U = \emptyset\}$ is independent of $h^{D\setminus U}_U$, hence $\mathbb{P}(A\cap U = \emptyset|h) = \mathbb{P}(A \cap U = \emptyset|h_{D\setminus U})$.

Proposition. (h, A) is local if and only if there exist $h_A, h^A_{D\setminus A}$ where h_A is harmonic in $D \setminus A$, the conditional law of $h^A_{D\setminus A}$ given (A, h_A) is a GFF in $D \setminus A$, and $h = h_A + h^A_{D\setminus A}$ almost-surely.

Proof. We have shown the backward direction.

For the forward direction, let $A_n := \bigcup\{Q \in \mathcal{D}_n : A \cap Q \neq \emptyset\}$. We claim (h, A_n) is local. Let $U \subseteq D$. Then $A_n \cap U \neq \emptyset$ iff $A \cap U_n \neq \emptyset$ where $U_n = \bigcup\{Q \in \mathcal{D}_n : U \cap Q \neq \emptyset\}$, which happens iff $\bigcap_{\delta > 0}\{A \cap U_n^\delta\} \neq \emptyset$, where U_n^δ is a δ-neighbourhood of U_n. By locality of (h, A), $\mathbb{P}(A \cap U_n^\delta \neq \emptyset | h)$ is measurable with respect to $\sigma(h_{D \setminus U_n^\delta}) \subseteq \sigma(h_{D\setminus U})$, so taking $\delta \downarrow 0$, $\mathbb{P}(A \cap U_n \neq \emptyset | h)$ is $\sigma(h_{D\setminus U})$-measurable.

By the proposition for the dyadic square case, there exists a decomposition $h_{A_n}, h^{A_n}_{D\setminus A_n}$ for (h, A_n). Let $\mathcal{G}_n = \sigma(A, h_{A_n}) = \sigma(h_{A_n}, A_n, A_{n+1}, \ldots)$ which is decreasing in n. Then $8\mathbb{E}[\langle h, \rho\rangle | \mathcal{G}_n] = \langle h_{A_n}, \rho\rangle$ for all $\rho \in C_c^\infty$. Backward martingale convergence implies $\langle h_{A_n}, \rho\rangle$ converges to some C_ρ. Consider a countable dense set of such ρ, so we can construct h_A such that it is harmonic in $D \setminus A$ and $(h_A, \rho) = C_\rho$ for all ρ.

We have

$$\begin{aligned}\mathbb{E}[\exp(i\theta\langle h, \rho\rangle)|\mathcal{G}_n] &= \exp(i\theta\langle h_A, \rho\rangle)\exp\left(-\frac{\theta^2}{2}\iint G_{D\setminus A_n}(x,y)\rho(x)\rho(y)\right) \\ &\xrightarrow{n\to\infty} \exp(i\theta\langle h_A, \rho\rangle)\exp\left(-\frac{\theta^2}{2}\iint G_{D\setminus A}(x,y)\rho(x)\rho(y)\right)\end{aligned}$$

and so

$$\mathbb{E}[\exp(i\theta\langle h - h_A, \rho\rangle)|h_A, A] = \exp\left(-\frac{\theta^2}{2}\iint G_{D\setminus A}(x,y)\rho(x)\rho(y)\right).$$

$\square$

Level lines of the Gaussian Free Field

Definition. Let $\mathfrak{h}$ be a harmonic function, h a zero-boundary GFF. Call $\mathfrak{h}+h$ a GFF with boundary values $\mathfrak{h}$.

Let $\mathfrak{h}_0(z) = \lambda - \frac{2\lambda}{\pi}\arg(z)$ and $\mathfrak{h}_t = \mathfrak{h}_0(g_t(z) - U_t)$.

Given a "level line" $\mathfrak{h}_0 + h$ as above

$$\mathbb{E}[(\underbrace{h_0 + h}_{\mathfrak{h}_t + \text{indep GFF}}, \underbrace{\rho_z}_{\text{radially symmetric about } z})|\mathcal{F}_t] = \mathfrak{h}_t(z)$$

so $\mathfrak{h}_t(z)$ is a martingale. Therefore $\arg(g_t(z) - U_t)$ is a martingale. Let $g_t(z) - U_t = X_i + iY_t$ so $\frac{X_t}{Y_t}$ is a semimartingale, implying X_t a semimartingale and therefore U_t a semimartingale.

Itô's formula gives

$$\mathrm{d}\mathfrak{h}_t(z) = \frac{2\lambda}{\pi}\Im(\mathrm{d}\log(g_t(z) - U_t))$$

and we have

$$\mathrm{d}\log(g_t(z) - U_t) = -\frac{1}{g_t(z)U(t)}\mathrm{d}U_t + \frac{1}{(g_t(z) - U_t)^2}(2\mathrm{d}t - \frac{1}{2}\mathrm{d}[U]_t)$$

so if this bounded variation term vanishes for all z we must have $2\mathrm{d}t = \frac{1}{2}\mathrm{d}[U]_t$, and so $\mathrm{d}U_t = \sqrt{4}\mathrm{d}B_t$ by Lévy's characterisation.

Proposition (Martingale characterisation of SLE_4)**.** Suppose (g_t) is a random Loewner chain with continuous driving function U. Then it is an SLE_4 iff for all $z \in \mathbb{H}$ the process $\arg(g_t(z) - U_t)$ is a local martingale for $t < \tau_z$.

Theorem. *Let $\lambda = \frac{\pi}{2}$, γ an SLE_4 in $(\mathbb{H}, 0, \infty)$, (g_t) the corresponding Loewner chain with $U_t = \sqrt{4}B_t$. Let $\tilde{h}$ be a GFF in $\mathbb{H}$ independent of γ. For a stopping time τ let $\mathfrak{h}_\tau(z) = \mathfrak{h}_0(g_\tau(z) - U_\tau)\mathbb{1}_{H_\tau(z)}$ for $\mathfrak{h}_0(z) = \lambda - \frac{2\lambda}{\pi}\arg(z)$.*

Then $\mathfrak{h}_\tau + \tilde{h} \circ g_\tau$ has the same law as $\mathfrak{h}_0 + \tilde{h}$. The same is true for $\tau = \infty$. Hence h_∞ is takes constant values $-\lambda, \lambda$ in the two components to the left/right respectively of γ. $\tilde{h} \circ g_\infty = \tilde{\tilde{h}}$ consists of an independent GFF in each component.

We wll use the fact that if for all $z \in \mathbb{H}$ we have $\mathrm{dist}(z, \gamma) > 0$ almost-surely, then $\mathrm{area}(\gamma) = 0$ almost-surely.

Can check that

$$G_{\mathbb{H}}(z, w) = \log\frac{1}{|z - w|} - \log\frac{1}{|z - \bar{w}}$$
$$G_{H_t}(z, w) = G_{\mathbb{H}}(g_t(z), g_t(w)).$$

Lemma. *We have*

$$\partial_t G_{H_t}(z,w) = -\Im\frac{2}{g_t(z)-U_t}\Im\frac{2}{g_t(w)-U_t}.$$

Proof of theorem. We have

$$\mathrm{d}\mathfrak{h}_t(z) = \underbrace{\frac{2\lambda}{\pi}}_{=1}\Im\frac{2}{g_t(z)-U_t}\mathrm{d}B_t.$$

Let $\rho \in C_c^\infty(\mathbb{H})$. Since $\mathfrak{h}_t$ is bounded, $(\mathfrak{h}_t,\rho)$ is a martingale.

Intuitively,

$$\mathrm{d}(\mathfrak{h}_t,\rho) = \int \rho(z)\Im\frac{2}{g_t(z)-U_t}\mathrm{d}z\mathrm{d}B_t$$

and so

$$d[(h,\rho)]_t = \left(\iint \rho(z)\rho(w)\underbrace{\Im\frac{2}{g_t(z)-U_t}\Im\frac{2}{g_t(w)-U_t}}_{-G_{H_t}(z,w)}\mathrm{d}z\mathrm{d}w\right)\mathrm{d}t$$

but the exchange of integrals would need justification. We have

$$\begin{aligned}\mathrm{d}[\mathfrak{h}_0(z),\mathfrak{h}_0(w)]_t &= \Im\frac{2}{g_t(z)-U_t}\Im\frac{2}{g_t(w)-U_t}\mathrm{d}t\\ &= -G_{H_t}(z,w)\mathrm{d}t.\end{aligned}$$

Hence $\mathfrak{h}_t(z)\mathfrak{h}_t(w)+G_{H_t}(z,w)$ is a bounded martingale, so

$$(\mathfrak{h}_t,\rho)^2 + \iint \rho(z)\rho(w)G_{H_t}(z,w)\mathrm{d}z\mathrm{d}w$$

is a bounded martingale.

Now note

$$\begin{aligned}&\mathbb{E}\left[\exp\left(i\theta(\mathfrak{h}_\tau+\tilde{h}\circ g_\tau,\rho)\right)\right]\\ &= \mathbb{E}\left[\exp(i\theta(\mathfrak{h}_\tau))\underbrace{\mathbb{E}\left[\exp(i\theta(\tilde{h}\circ g_\tau))|\mathcal{F}_\tau\right]}_{\exp\left(-\frac{\theta^2}{2}\iint\rho(z)\rho(w)G_{H_t}(z,w)\mathrm{d}z\mathrm{d}w\right)}\right]\\ &= \mathbb{E}\left[\underbrace{\exp(i\theta(h_\tau,\rho))+\frac{\theta^2}{2}[(\mathfrak{h}_0,\rho)]_\tau}_{\exp(i\theta(\mathfrak{h}_0,\rho))}-\frac{\theta^2}{2}\iint\rho(z)\rho(w)G_{H_t}(z,w)\mathrm{d}z\mathrm{d}w\right]\\ &= \mathbb{E}[\exp(i\theta(\mathfrak{h}_0+\tilde{h},\rho))].\end{aligned}$$

□

We have $\mathfrak{h}_\infty + \tilde{\tilde{h}} = \mathfrak{h}_0 + h$ where $\tilde{\tilde{h}}, h$ are GFF's on each component of $\mathbb{H} \setminus \gamma$ and $\mathbb{H}$ respectively.

Proposition. In this setup, (h, γ) is local. For any stopping time τ for γ, $(h, \gamma[0,\tau])$ is local with $h_{\gamma[0,\tau]} = \mathfrak{h}_\tau - \mathfrak{h}_0$.

Proof. The first claim follows from the construction.

For the second claim, after mapping under g_τ we have the setup for the theorem, so we may conclude by characterisation of local sets. □

We already used the following fact:

Proposition. For $z \in \mathbb{H}$, $\mathrm{dist}(z, \gamma) > 0$ almost-surely if $\kappa < 8$.

Proof sketch for $\kappa \le 4$. Define $\Upsilon_t := \frac{Y_t(z)}{|g_t'(z)|} = \frac{1}{2}\mathrm{CR}(z, H_t)$. We have

$$\mathrm{d}\log\Upsilon_t = \frac{-4Y_t^2}{(X_t^2 + Y_t^2)}\mathrm{d}t.$$

Let $\theta_t := \arg(g_t(z) - U_t)$, $r := r_t = -\frac{\kappa}{4}\log\Upsilon_t$ giving

$$\mathrm{d}\theta_r = \left(1 - \frac{4}{\kappa}\right)\cot(\theta_r)\mathrm{d}r + \mathrm{d}\tilde{B}_r$$

which is approximately a Bessel process, so we see θ_r hits 0 or π in finite "r-time" iff $\kappa < 8$. This is only possible with $t(r) \to \infty$, hence $\mathrm{dist}(z, \gamma) \simeq \Upsilon_\infty = e^{-\frac{4}{\kappa}r} > 0$. □

We call the coupling (h, γ) the *level line coupling* and say γ is *the level line* of $\mathfrak{h}_0 + h$. Recall that by the martingale characterisation, this coupling with properties above is unique in law.

Theorem. *There is a measurable function of the GFF such that in the level line coupling, γ agrees with that function of h.*

Idea: we write $\mathrm{Law}(h, \gamma) = \mathrm{Law}(h) \otimes \mathrm{Law}(\gamma|h)$. Sample $\gamma, \tilde{\gamma}$ conditionally independently given h, i.e $(h, \gamma, \tilde{h}) \sim \mathrm{Law}(h) \otimes \mathrm{Law}(\gamma|h) \otimes \mathrm{Law}(\gamma|h)$. We want to show $\gamma = \tilde{\gamma}$ almost-surely, as this will imply γ is a function of h.

Note

$$\begin{aligned}\mathfrak{h}_t(z) &= \mathbb{E}[\mathfrak{h}_\infty(z)|\gamma[0,t]] \\ &= -\lambda\mathbb{P}(\gamma \text{ passes to the right of } z|\gamma[0,t]) \\ &\quad + \lambda\mathbb{P}(\gamma \text{ passes to the left of } z|\gamma[0,t]).\end{aligned}$$

Hence we would expect

$$\begin{aligned}\mathfrak{h}_\infty(z) &= \mathbb{E}[\mathfrak{h}_{\gamma\cup\tilde{\gamma}}(z)|\gamma] \\ &= \begin{cases} -\lambda & \text{if } z \text{ to the left of } \gamma \\ \lambda & \text{if } z \text{ to the right of } \gamma \end{cases}\end{aligned}$$

where $\mathfrak{h}_{\gamma\cup\tilde{\gamma}}$ is the harmonic extension of the GFF on $\gamma \cup \tilde{\gamma}$ to $\mathbb{H}$. This implies $\tilde{\gamma} = \gamma$ almost-surely. This is all quite heuristic and we cannot quite do this.

Lemma. *Suppose (h, A) and $(h, \tilde{A})$ are local. Sample $(h, A, \tilde{A})$ so that $A, \tilde{A}$ are conditionally independent given h. Then $(h, A \cup \tilde{A})$ is local. Furthermore the conditional law of $h_{D\setminus(A\cup\tilde{A})}$ given $(A, \tilde{A}, h_{A\cup\tilde{A}})$ is a GFF in $D \setminus (A \cup \tilde{A})$.*

Proof. For $U \subseteq D$ open we want to show that the conditional law of $h_U^{D\setminus U}$ given $(\mathbb{1}_{A\cap U=\emptyset}, A\mathbb{1}_{A\cap U=\emptyset}, \mathbb{1}_{\tilde{A}\cap U=\emptyset}, \tilde{A}\mathbb{1}_{\tilde{A}\cap U=\emptyset}, h_{D\setminus U})$ is a GFF in U.

Write $S = (\mathbb{1}_{A\cap U=\emptyset}, A\mathbb{1}_{A\cap U=\emptyset})$, $\tilde{S} = (\mathbb{1}_{\tilde{A}\cap U=\emptyset}, \tilde{A}\mathbb{1}_{\tilde{A}\cap U=\emptyset})$. We have

$$
\begin{aligned}
&\mathbb{E}[\exp(i\theta(h_U^{D\setminus U}, \rho))g(S)\tilde{g}(\tilde{S})|h_{D\setminus U}] \\
&= \mathbb{E}[\exp(i\theta(h_U^{D\setminus U}, \rho))\mathbb{E}[g(S)\tilde{g}(\tilde{S})|h_{D\setminus U}, h_U^{D\setminus U}]|h_{D\setminus U}] \\
&= \mathbb{E}[\exp(i\theta(h_U^{D\setminus U}, \rho))\mathbb{E}[g(S)|h_{D\setminus U}, h_U^{D\setminus U}]\mathbb{E}[\tilde{g}(\tilde{S})|h_{D\setminus U}, h_U^{D\setminus U}]|h_{D\setminus U}] \\
&= \mathbb{E}[\exp(i\theta(h_U^{D\setminus U}, \rho))\mathbb{E}[g(S)|h_{D\setminus U}]\mathbb{E}[\tilde{g}(\tilde{S})|h_{D\setminus U}]|h_{D\setminus U}] \qquad \text{(local)} \\
&= \mathbb{E}[\exp(i\theta(h_U^{D\setminus U}))]\mathbb{E}[g(S)|h_{D\setminus U}]\mathbb{E}[\tilde{g}(\tilde{S})|h_{D\setminus U}] \\
&= \mathbb{E}[\exp(i\theta(h_U^{D\setminus U}))]\mathbb{E}[g(S)\tilde{g}(\tilde{S})|h].
\end{aligned}
$$

Now following the proof of the "strong Markov property", the conditional law of

$$h_{D\setminus(A\cup\tilde{A})_n}^{(A\cup\tilde{A})_n}$$

given $(A, \tilde{A}, h_{(A\cup\tilde{A})_n})$ is a GFF for all n (where $(A \cup \tilde{A})_n$ is the dyadic discretisation from before). Taking $n \to \infty$, $h_{(A\cup\tilde{A})_n} \to h_{A\cup\tilde{A}}$ and

$$
\begin{aligned}
&\mathbb{E}[\exp(i\theta(h - h_{A\cup\tilde{A}}, \rho))|A \cup \tilde{A}, h_{A\cup\tilde{A}}] \\
&= \mathbb{E}[\exp(i\theta(h - h_{A\cup\tilde{A}}, \rho))|A, \tilde{A}, h_{A\cup\tilde{A}}] \\
&= \exp\left(-\frac{\theta^2}{2}\int G_{D\setminus A}(x, y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y\right).
\end{aligned}
$$

$\square$

Lemma. *Conditional on $\tilde{A}$, $(h^{\tilde{A}}_{D\setminus\tilde{A}}, A \setminus \tilde{A})$ is local and $(h^{\tilde{A}}_{D\setminus\tilde{A}}) = h_{A\cup\tilde{A}} - h_{\tilde{A}}$ almost-surely.*

Proof. We have

$$h = h_{\tilde{A}} + h^{\tilde{A}}_{D\setminus\tilde{A}} = h_{A\cup\tilde{A}} + h^{A\cup\tilde{A}}_{D\setminus(A\cup\tilde{A})}$$

implying

$$h^{\tilde{A}}_{D\setminus\tilde{A}} = h_{A\cup\tilde{A}} + h^{A\cup\tilde{A}}_{D\setminus(A\cup\tilde{A})} - h_{\tilde{A}}$$

and from the previous lemma, $h^{A\cup\tilde{A}}_{D\setminus(A\cup\tilde{A})}$ is a GFF on $D\setminus(A\cup\tilde{A})$ given $(\tilde{A}, A, h_{A\cup\tilde{A}})$. □

Lemma. *For $(h, A, \tilde{A})$ as above then the following holds almost-surely. Let $x \in \partial(D\setminus(A\cup\tilde{A}))$ and suppose x lies on a connected component of $\partial(D\setminus(A\cup\tilde{A}))$ that consists of more than a single point. Suppose $\operatorname{dist}(x, A \setminus \tilde{A}) > 0$ and $x_n \in D \setminus (A \cup \tilde{A})$, $x_n \to x$. Then $h_{A\cup\tilde{A}}(x_n) - h_{\tilde{A}}(x_n) \to 0$.*

Proof. By the previous lemma it suffices to consider the case $\tilde{A} = \emptyset$. Consider a small ball $B(u, \varepsilon) \cap D$. It suffices to prove the claim conditional on the event $\{A \cap B(u, \varepsilon) = \emptyset\}$ (can pick a countable number of such balls).

Since A is local we can further write $h_A = h_{D\setminus B(u,\varepsilon)} - h^A_{D\setminus B(u,\varepsilon)}$ (on the event above). Hence it suffices to prove the case that A is deterministic. So it remains to show h_A extends continuously to 0 on $\partial D \setminus \overline{A}$. We have

$$h_A(z) = (h, \omega(z, \cdot, D \setminus A))$$

(see Example Sheet 3). Then

$$\begin{aligned}
&\mathbb{E}[(h_A(z) - h_A(z'))^2] \\
&= \int\int G_D(x, y)(\omega(z, \mathrm{d}x, D \setminus A) - \omega(z', \mathrm{d}x, D \setminus A))(\omega(z, \mathrm{d}y, D \setminus A) - \omega(z', \mathrm{d}y, D \setminus A)) \\
&= \int\int (\underbrace{G_D(x, z) - G_D(x, z')}_{\lesssim |z-z'| \text{ away from } \partial A})(\omega(z, \mathrm{d}x, D \setminus A) - \omega(z', \mathrm{d}x, D \setminus A)) \\
&\lesssim |z - z'|.
\end{aligned}$$

Note that this is also true for $z' \in \partial D \setminus \overline{A}$ when we set $h_A(z') = 0$. Since h_A is Gaussian, can raise to any power so we conclude by the Kolmogorov consistency theorem. □

Let (h, γ) be a level line coupling. Consider stopping times $\tau, \tilde{\tau}$ for γ and $\tilde{\gamma}$ respectively. Then if $\gamma[0, \tau] \cap \tilde{\gamma}[0, \tilde{\tau}] = \emptyset$, the values of $h_{\gamma[0,\tau]\cup\tilde{\gamma}[0,\tilde{\tau}]}$ are determined by the previous lemma.

By previous lemmas, conditional on $\tilde{\gamma}[0,\tilde{\tau}]$, (h,γ) is a level line coupling in $\mathbb{D}\setminus\tilde{\gamma}[0,\tilde{\tau}]$ until γ hits $\tilde{\gamma}[0,\tilde{\tau}]$. But by the martingale characterisation, the conditional law of γ given $\tilde{\gamma}[0,\tilde{\tau}]$ is SLE_4 in $(\mathbb{D}\setminus\tilde{\gamma}[0,\tilde{\tau}],-i,\tilde{\gamma}(\tilde{\tau}))$.

By transience of SLE_4 (proof of this later), γ must hit $\tilde{\gamma}(\tilde{\tau})$. This is true for any choice of $\tilde{\tau}$, so in particular we can pick a countable dense set of times to force $\gamma=\tilde{\gamma}$ almost-surely (modulo time-reversal). We have shown:

Theorem. *In the level line coupling, γ almost-surely agrees with a measurable function of h.*

as well as:

Theorem. *The time-reversal of SLE_4 in (D,x,y) has the law of SLE_4 in (D,y,x) (modulo parameterisation).*

Transience of the SLE_κ

Proposition. Let $\gamma \sim \mathrm{SLE}_\kappa$ in $(\mathbb{H}, 0, \infty)$. Then $\lim_{t\to\infty} \gamma(t) = \infty$ almost-surely.

Proof sketch for $\kappa \leq 4$. By scaling, it suffices to show there is a random $r > 0$ such that $|\gamma(t)| > r$ for all $t \geq 1$.

By the domain Markov property it suffices to show that for $x > 0$ (WLOG $x = 1$ by scaling) $\mathrm{dist}(x, \gamma) > 0$ almost-surely. Consider the local martingale

$$M_t = |g'(1+2r)|^{\frac{8}{\kappa}-1}(g_t(1+2r) - U_t)^{1-\frac{8}{\kappa}}$$

and let $\tau = \inf\{t : \gamma(t) \in \partial B(1, r)\}$. We have that

$$g_\tau(1+2r) - g_\tau(1) \approx g_\tau(1+2r) - U_\tau$$

by e.g considering extremal length. The Koebe 1/4-theorem shows

$$\begin{aligned}\underbrace{(g_\tau(1+2r) - U_\tau)|(g_\tau^{-1})'(g_\tau(1+2r))|}_{M_\tau^{-\frac{1}{\frac{8}{\kappa}-1}}} &\lesssim \mathrm{dist}(1+2r, \gamma[0,\tau]) \\ &\lesssim r.\end{aligned}$$

Hence

$$\mathbb{P}(M \text{ hits } r^{-(\frac{8}{\kappa}-1)} \text{ before } 0) \sim r^{\frac{8}{\kappa}-1} \xrightarrow{r\to 0} 0.$$

□

$\mathrm{SLE}_\kappa(\rho)$

We define $\mathrm{SLE}_\kappa(\rho)$ by taking driving function U satisfying

$$\mathrm{d}U_t = \sqrt{\kappa} B_t + \sum_j \Re \frac{\rho_j}{U_t - g_t(z_j)} \mathrm{d}t$$

Example. Given a SLE_κ in $(\mathbb{H}, 0, \infty)$, under the map $\psi(z) = \frac{z}{z+1}$ we get a $\mathrm{SLE}_\kappa(\kappa - 6)$ in $(\mathbb{H}, 0, 1)$.

Duality

Take an $\mathrm{SLE}_{16/\kappa}$ with $\kappa \in (0, 4)$ in $(\mathbb{D}, i, -i)$ and consider the outermost left/right curves from i to $-i$. These are in fact $\mathrm{SLE}_\kappa(\tilde{\rho}), \mathrm{SLE}_\kappa(\rho)$ for some $\tilde{\rho}, \rho$.