

## Introduction

### Course structure:

- (i) Preliminary toolbox: inequalities
- (ii) Normed vector spaces (NVS)
- (iii) (Recalls on) finite-dimensional case
- (iv) Hahn-Banach Theorems (how big is the dual?)
- (v) Completeness: Baire's Theorem & consequences for NVS
- (vi) Detailed study of the topology of  $C(K)$
- (vii) The Hilbert space
- (viii) Projection & duality
- (ix) Introduction to operators and spectral theory

## 1 Preliminary toolbox: Young's, Hölder's & Minkowski's inequalities for vectors & sequences

**Proposition** (Young's inequality for products). Let  $p, q \in (1, \infty)$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$\forall a, b \geq 0, \quad ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

*Proof.* The result is clear for  $a = 0$  or  $b = 0$ . Assume  $a, b > 0$  and note  $L : (0, \infty) \rightarrow \mathbb{R}$ ,  $t \mapsto \ln t$  is strictly concave:  $L''(t) = -\frac{1}{t^2} < 0$ .

Therefore for all  $A, B > 0$ ,  $\lambda \in (0, 1)$

$$\ln(\lambda A + (1 - \lambda)B) \geq \lambda \ln A + (1 - \lambda) \ln B$$

with equality iff  $A = B$ . Apply this to  $A = a^p$ ,  $B = b^q > 0$  and  $\lambda = \frac{1}{p}$ . This gives

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) = \ln(ab)$$

so applying exp to both sides gives the result and furthermore we have equality iff  $a^p = b^q$ .  $\square$

**Proposition** (Hölder's inequality for vectors & sequences). Let  $p, q \in (1, \infty)$  be such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

(i) for any  $n \in \mathbb{N} \setminus \{0\} = \mathbb{N}^*$ ,  $\forall x, y \in \mathbb{R}^n$

$$\sum_{k=1}^n |x_k y_k| \leq \|x\|_p \|y\|_q \quad (*)$$

with  $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$  and similarly for  $\|y\|_q$ .

(ii) define

$$\ell^p = \{x \in \mathbb{R}^{\mathbb{N}^*} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

then  $\forall x \in \ell^p, y \in \ell^q$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_{\ell^p} \|y\|_{\ell^q}$$

where  $\|x\|_{\ell^p} = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$  and similar for  $\|y\|_{\ell^q}$ .

*Proof.* To show (i) implies (ii): take  $n \rightarrow \infty$  in (i) so

$$\sum_{k=1}^n |x_k|^p \rightarrow \|x\|_{\ell^p}^p$$

and similarly

$$\sum_{k=1}^n |y_k|^q \rightarrow \|y\|_{\ell^q}^q$$

By (i)

$$\sum_{k=1}^n |x_k y_k| \leq \left( \sum_{k=1}^n |x_k|^p \right)^{1/p} \left( \sum_{k=1}^n |y_k|^q \right)^{1/q}$$

so

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k y_k| &= \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |x_k y_k| \right) \leq \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n |x_k|^p \right)^{1/p} \left( \sum_{k=1}^n |y_k|^q \right)^{1/q} \\ &= \|x\|_{\ell^p} \|y\|_{\ell^q} \end{aligned}$$

*Proof of (i):* if  $\|x\|_{\ell^p}$  or  $\|y\|_{\ell^q} = 0$ , result is clear. Otherwise define  $\tilde{x}, \tilde{y}$  sequences in  $\ell^p$  and  $\ell^q$  by

$$\tilde{x}_k = \frac{x_k}{\|x\|_{\ell^p}}, \quad \tilde{y}_k = \frac{y_k}{\|y\|_{\ell^q}}$$

Then  $\|\tilde{x}\|_{\ell^p} = 1, \|\tilde{y}\|_{\ell^q} = 1$ . Then (\*) is equivalent to showing

$$\sum_{k=1}^n |\tilde{x}_k \tilde{y}_k| \leq 1 \quad (**)$$

Apply Young's inequality on each  $k = 1, \dots, n$  so

$$|\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p} |\tilde{x}_k|^p + \frac{1}{q} |\tilde{y}_k|^q$$

Summing over  $k$ :

$$\sum_{k=1}^n |\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p} \left( \sum_{k=1}^n |\tilde{x}_k|^p \right) + \frac{1}{q} \left( \sum_{k=1}^n |\tilde{y}_k|^q \right) \leq \frac{1}{p} + \frac{1}{q} = 1$$

□

**Remark:** Equality in (\*) is equivalent to equality in (\*\*) which is equivalent to equality in Young's for each  $k$  so  $|\tilde{x}_k|^p = |\tilde{y}_k|^q$  for  $k = 1, \dots, n$ . Also, the  $p = 1$ ,  $q = \infty$  case is easy.

**Proposition** (Minkowski's inequality for vectors & sequences). Let  $p \in [1, \infty)$ , then

(i) for all  $x, y \in \mathbb{R}^n$

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

(ii) for all  $x, y \in \ell^p$

$$\|x + y\|_{\ell^p} = \|x\|_{\ell^p} + \|y\|_{\ell^p}$$

*Proof.* To show (i) implies (ii): by taking  $n \rightarrow \infty$  as before

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k|^p &\rightarrow \|x\|_{\ell^p}^p \\ \sum_{k=1}^{\infty} |y_k|^p &\rightarrow \|y\|_{\ell^p}^p \\ \sum_{k=1}^n |x_k + y_k|^p &\rightarrow \|x + y\|_{\ell^p}^p \end{aligned}$$

Proof of (i): if  $p = 1$  this is just the usual triangle inequality on each coordinate. So let  $p \in (1, \infty)$  and

$$\begin{aligned} \sum_{k=1}^n |x_k + y_k|^p &= \sum_{k=1}^n |x_k + y_k| \cdot |x_k + y_k|^{p-1} \\ &\leq \sum_{k=1}^n |x_k| |x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k| |x_k + y_k|^{p-1} \\ &\leq \|x\|_p \left( \sum_{k=1}^n |x_k + y_k|^{(p-1) \frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \|y\|_p \left( \sum_{k=1}^n |x_k + y_k|^{(p-1) \frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\text{Hölder: } q = \frac{p}{p-1} \end{aligned}$$

$$\leq (||x||_p + ||y||_p) ||x + y||_p^{p-1}$$

so we have proved

$$||x + y||_p^p \leq (||x||_p + ||y||_p) ||x + y||_p^{p-1}$$

If  $||x + y||_p = 0$ , result is clear. Otherwise divide by  $||x + y||_p^{p-1}$  to get

$$||x + y||_p \leq ||x||_p + ||y||_p$$

□

**Remark:** equality occurs iff there is equality in the triangle inequality and Hölder's.

**Remarks:**

1. Equality case:  $p = 1$ :  $|x_k + y_k| \leq |x_k| + |y_k|$ , i.e the usual triangle inequality
2. For  $p = 2$  there's another proof: define  $\mathcal{P} : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\lambda \mapsto ||x + \lambda y||^2$ . Then  $\mathcal{P}(\lambda) = a\lambda^2 + 2b\lambda + c$  and  $\mathcal{P} \geq 0$ . So

$$\langle x, y \rangle = b^2 \leq ac = ||x||^2 ||y||^2, \text{ Hölder's inequality}$$

## 2 Normed Vector Spaces (NVS)

**Remark:** this is not the most general structure for linear analysis - topological vector spaces (TVS).

Recall:

**Definition.** A *vector space*  $V$  over a field  $\mathbb{F}$  is a set (of elements called *vectors*) with two operations:

$$A : V \times V \rightarrow V, (v, w) \mapsto v + w \text{ addition}$$

$$M : \mathbb{F} \times V \rightarrow V, (\lambda, v) \mapsto \lambda v \text{ scalar multiplication}$$

such that

- $(V, +)$  is an abelian group with identity 0.
- $M$  is compatible with  $(\mathbb{F}, 0)$  in the sense that  $\lambda_1(\lambda_2 v) = \lambda_1 \lambda_2 v$
- $M$  distributes over  $(V, +)$  and  $(\mathbb{F}, +)$ .

In this course  $\mathbb{F}$  will be  $\mathbb{R}$  or  $\mathbb{C}$  unless stated otherwise.

**Definition.** Given a vector space  $V$  over  $\mathbb{F}$ :

- a *subspace*  $W \subseteq V$  is a vector space over  $\mathbb{F}$  included in  $V$
- for a set  $S \subseteq V$ , a *linear combination of elements of  $S$*  is a finite sum of elements of  $S$  with coefficients in  $\mathbb{F}$
- for a set  $S \subseteq V$ , the *span of  $S$* ,  $\text{span}(S)$  is the smallest subspace of  $V$  containing  $S$ , and is also the set of linear combinations of  $S$ .

**Definition.** Given  $V$  a vector space over  $\mathbb{F}$  and a set  $S \subseteq V$ :

- $S$  is *linearly independent* if for all  $m \in \mathbb{N}^*$  and for all  $\alpha_1, \dots, \alpha_m \in \mathbb{F}$ , for all  $s_1, \dots, s_m \in S$ ,  $\sum_{i=1}^m \alpha_i s_i = 0$  if and only if  $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$ .
- $S$  is a *basis* of  $V$  if it is linearly independent and  $\text{span}(S) = V$ .
- If there exists a finite basis  $S$  of  $V$ , then  $V$  has finite dimension, otherwise it is infinite-dimensional.

**Remark:** later we'll prove with Zorn's lemma that any vector space has a basis.

**Definition.** A *normed vector space* (NVS)  $V$  over  $\mathbb{F}$  is a vector space over  $\mathbb{F}$  together with a function  $N : V \rightarrow \mathbb{R}_+$ ,  $v \mapsto \|v\|$  (the *norm*), with

1.  $\|v\| \geq 0$  for all  $v \in V$ , with equality only at  $v = 0$  (*positive definiteness*)
2. For all  $\lambda \in \mathbb{F}$ ,  $v \in V$   $\|\lambda v\| = |\lambda| \|v\|$  (compatibility between  $N$  and  $M$ )

3. For all  $v, w \in V$ ,  $\|v + w\| \leq \|v\| + \|w\|$  (compatibility between  $N$  and  $A$ )

**Example.**  $V = \mathbb{R}^n$ ,  $v = (v_1, \dots, v_n)$ ,  $\|v\| = (v_1^2 + \dots + v_n^2)^{1/2}$  or

$$\begin{cases} \|v\|_p = (|v_1|^p + \dots + |v_n|^p)^{1/p} & \text{for } p \in [1, \infty) \\ \|v\|_\infty = \sup_{i=1}^n |v_i| & \text{for } p = \infty \end{cases}$$

**Definition.** Given a set  $X$ , a *topology*  $\tau$  on  $X$  is a collection of subsets of  $X$  (“open sets”) such that

- $\emptyset \in \tau$ ,  $X \in \tau$
- $\tau$  is stable under any union
- $\tau$  is stable under finite intersections

**Definition.**

- For  $(X, d)$  a metric space, the *induced topology* is the smallest topology that contains open balls in  $d$
- For a NVS  $(V, \|\cdot\|)$ , the induced topology is that associated with  $d(v, w) = \|v - w\|$

**Natural question:**  $\mathbb{F}$  field,  $V$  vector space over  $\mathbb{F}$ . Norm on  $V$ ,  $\tau_{\|\cdot\|}$ . Continuity of operations  $M$  and  $A$ ?

**Proposition.** Let  $(V, \|\cdot\|)$  be a NVS over  $\mathbb{F}$  ( $\mathbb{F}$  either  $\mathbb{R}$  or  $\mathbb{C}$ ), then

- (i)  $A, M$  are continuous for the following topologies:  $\tau_{\|\cdot\|}$  on  $V$ , then product topology of it on  $V \times V$ ,  $\tau_{|\cdot|}$  over  $\mathbb{F}$ , then product topology of  $\tau_{|\cdot|}$  and  $\tau_{\|\cdot\|}$  on  $\mathbb{F} \times V$
- (ii) Translations  $T_{v_0} : V \rightarrow V$ ,  $v \mapsto v + v_0$ ,  $v_0 \in V$  and dilations  $D_{\lambda_0} : V \rightarrow V$ ,  $v \mapsto \lambda_0 v$ ,  $\lambda_0 \in \mathbb{F}^*$  are homeomorphisms

*Proof.*

- (i) Let us prove that  $A : V \times V \rightarrow V$  is continuous: consider an open set  $\emptyset \neq U \subseteq V$  and  $(v_1, v_2) \in A^{-1}(U)$ , i.e  $v_1 + v_2 \in U$ . Since  $U$  is open, there is  $\varepsilon > 0$  such that  $\underbrace{B_V(v_1 + v_2, \varepsilon)}_{\text{open ball}} \subseteq U$ .

We have that  $A(B(v_1, \varepsilon/2), B(v_2, \varepsilon/2)) \subseteq B_V(v_1 + v_2, \varepsilon)$  (triangle inequality). Note also that  $B(v_1, \varepsilon/2) \times B(v_2, \varepsilon/2)$  is open (product topology), so  $A^{-1}(U)$  is open and  $A$  is continuous.

Now we show  $M : \mathbb{F} \times V \rightarrow V$  is continuous. Consider an open set  $U \neq \emptyset$  in  $V$ ,  $(\lambda, v) \in M^{-1}(U)$ . Since  $U$  is open, there exists  $\varepsilon > 0$  such that  $B_V(\lambda v, \varepsilon) \subseteq U$  (WLOG  $\varepsilon < 1$ ). Then (check)

$$M\left(B_{\mathbb{F}}\left(\lambda, \frac{\varepsilon}{3 \max(1, \|v\|)}\right), B_V\left(v, \frac{\varepsilon}{3 \max(1, |\lambda|)}\right)\right) \subseteq B_V(\lambda v, \varepsilon)$$

- (ii)  $T_{v_0}$  and  $D_{\lambda_0}$  are linear, continuous with inverses  $T_{-v_0}$  and  $D_{\lambda_0^{-1}}$  respectively, so are homeomorphisms.

□

### 3 Characterisation of NVS

**Idea:** in order to better understand the topology of NVS's, we ask how special is a “normable” topology among topologies compatible with vector space operations?

**Definition (TVS).** A *topological vector space* (TVS) over  $\mathbb{F}$  is a vector space over  $\mathbb{F}$  together with a topology  $\tau$  such that

- (i)  $A$  and  $M$  are continuous
- (ii) every singleton  $\{x_0\}$  is closed

**Remark:**

- 1. (i) says that  $T_{v_0}$  and  $D_{\lambda_0}$ ,  $\lambda_0 \neq 0$  are homeomorphisms
- 2. (ii) is called  $T_1$  in the classification of separation properties, and implies Hausdorff for TVS

**Definition.** Given  $V$  a TVS

- $C \subseteq V$  is *convex* if  $C = \{\lambda c_1 + (1 - \lambda)c_2 : c_1, c_2 \in C, \lambda \in [0, 1]\}$
- $V$  is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0
- $B \subseteq V$  is *bounded* if for any  $U$  open around 0, there exists  $t_0 > 0$  such that  $\forall t > t_0, B \subseteq tU$
- $V$  is *locally bounded* if there is  $U \in \tau$  containing 0 and bounded

**Example.** Let  $(V, \|\cdot\|)$  be a NVS, then for all  $r > 0$ ,  $U = B(0, r)$  (open ball) is open, bounded and convex. Indeed

- Convexity follows from the triangle inequality
- Boundedness: any other  $\tilde{U}$  open around 0 contains some open  $\tilde{U}_0 = B(0, r_0) \in \tilde{U}$ . Then for any  $t > \frac{r}{r_0}$ ,  $U \subseteq t\tilde{U}_0 \subseteq t\tilde{U}$ .

**Question:** can we reverse-engineer the norm if we have these two properties?

**Theorem** (Kolmogorov 1934). *Let  $(V, \tau)$  be a TVS such that there is a bounded convex neighborhood of 0, say  $C$ . Then  $V$  is “normable” - there is a norm  $\|\cdot\|$  on  $V$  that induces the topology  $\tau$ .*

*Proof.* Step 1: there is  $\tilde{C} \subseteq C$  which is a *balanced* convex bounded neighborhood of 0. “Balanced” means that for all  $\lambda \in \mathbb{F}$  such that  $|\lambda| \leq 1$ ,  $\lambda\tilde{C} \subseteq \tilde{C}$ .

$M : \mathbb{F} \times V \rightarrow V$  is continuous so  $M^{-1}(C)$  is a neighbourhood of  $(0, 0)$ . So there exists  $B_{\mathbb{F}}(0, \varepsilon) \times U$  with  $\varepsilon > 0$  and  $U$  open around 0 such that  $M(B_{\mathbb{F}}(0, \varepsilon), U) \subseteq C$ .



Define  $\tilde{C}$  to be the convex hull (i.e smallest convex set superset) of  $M(B_{\mathbb{F}}(0, \varepsilon), U)$ .

Then  $\tilde{C}$  is clearly convex, is a subset of  $C$  since  $C$  is convex and  $M(B_{\mathbb{F}}(0, \varepsilon), U) \subseteq C$ .  $\tilde{C}$  is also bounded since  $\tilde{C} \subseteq C$  and  $C$  is bounded (obvious that boundedness is inherited by inclusion). Finally  $\tilde{C}$  is balanced since  $\lambda B_{\mathbb{F}}(0, \varepsilon) \subseteq B_{\mathbb{F}}(0, \varepsilon)$  for  $\lambda \in \mathbb{F}$  with  $|\lambda| \leq 1$  and

$$\underbrace{\lambda M(B_{\mathbb{F}}(0, \varepsilon), U)}_{=M(\lambda B_{\mathbb{F}}(0, \varepsilon), U)} \subseteq M(B_{\mathbb{F}}(0, \varepsilon), U)$$

Notice  $\lambda[\text{Convex Hull}(S)] = \text{Convex Hull}(\lambda S)$  (exercise). So deduce  $\lambda\tilde{C} \subseteq \tilde{C}$ .

Step 2: define the *Minkowski gauge* (functional) of  $\tilde{C}$

$$\mu_{\tilde{C}} : V \rightarrow \mathbb{R}_+, v \mapsto \inf\{t \geq 0 : v \in t\tilde{C}\}$$

$\mu_{\tilde{C}}$  is well-defined in  $[0, \infty)$  since: any  $v$  satisfies  $\frac{v}{t} \rightarrow 0$  as  $t \rightarrow \infty$  by continuity of  $M$ . So  $\frac{v}{t}$  must “enter” the neighborhood  $\tilde{C}$  of 0 for  $t$  large enough.

Step 3: let us prove  $v \mapsto \mu_{\tilde{C}}(v)$  is a norm:

- $\mu_{\tilde{C}}(v) \geq 0$  by construction
- if  $\mu_{\tilde{C}} = 0$ , then (assume  $v \neq 0$  for contradiction) there exists  $U$  open around 0 with  $v \notin U$  (since  $V \setminus \{v\}$  is open). Since  $\tilde{C}$  is bounded, there exists  $t_1 > 0$  such that  $\tilde{C} \subseteq t_1 U$ . Since  $\mu_{\tilde{C}}(v) = 0$ , there exists  $t_2 \in (0, t_1^{-1})$  such that  $v \in t_2 \tilde{C}$ , then  $v \in t_2 \tilde{C} \subseteq t_1^{-1} \tilde{C} \subseteq U$ , a contradiction.
- Want to show  $\mu_{\tilde{C}}(\lambda v) = |\lambda| \mu_{\tilde{C}}(v)$  for  $\lambda \in \mathbb{F}^\times$ ,  $v \in V$ . Use  $\tilde{C}$  balanced: for all  $t > 0$  such that  $\lambda v \in t\tilde{C}$ , we have

$$\frac{\lambda}{|\lambda|} v \in \frac{t}{|\lambda|} \tilde{C} \implies v \in \frac{t}{|\lambda|} \tilde{C} \implies \mu_{\tilde{C}}(v) \leq \frac{1}{|\lambda|} \mu_{\tilde{C}}(\lambda v)$$

The inequality in the other direction follows by reasoning with  $\lambda^{-1}$ . So  $|\lambda| \mu_{\tilde{C}}(v) = \mu_{\tilde{C}}(\lambda v)$ .

- Want to show  $\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$  for all  $v_1, v_2 \in V$ . Indeed, given  $t_1, t_2 > 0$  such that  $v_1 \in t_1 \tilde{C}$ ,  $v_2 \in t_2 \tilde{C}$ , we have

$$v_1 + v_2 \in t_1 \tilde{C} + t_2 \tilde{C} = (t_1 + t_2) \left[ \frac{t_1}{t_1 + t_2} \tilde{C} + \frac{t_2}{t_1 + t_2} \tilde{C} \right] \subseteq (t_1 + t_2) \tilde{C} \text{ (convexity)}$$

so  $\mu_{\tilde{C}}(v_1 + v_2) \leq t_1 + t_2$ . By taking infima over  $t_1, t_2$ :

$$\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$$

Step 4: prove  $\mu_{\tilde{C}}$  induces the topology  $\tau$ .

- Want to prove

$$\underbrace{B(v_0, \varepsilon)}_{\text{open ball for } \mu_{\tilde{C}}} = \{v \in V : \mu_{\tilde{C}}(v - v_0) < \varepsilon\} \in \tau$$

Take  $v \in B(v_0, \varepsilon)$  then by the triangle inequality

$$B(v, \varepsilon - |v|) \subseteq B(v_0, \varepsilon)$$

and  $B(v, \varepsilon') \supseteq v + \frac{\varepsilon'}{2}\tilde{C}$  by definition of the ball for  $\mu_{\tilde{C}}$ . And (since translations, dilations continuous)  $v + \frac{\varepsilon'}{2}\tilde{C}$  is a neighborhood of  $v$ .

$B(v_0, \varepsilon)$  open (in  $\tau$ ) around its points, so is in  $\tau$ .

- Take  $U \in \tau$ , and (wlog)  $0 \in U$ . Let us prove  $0 \in B(0, \varepsilon_0) \subseteq U$  for some  $\varepsilon_0 > 0$ . Indeed  $\tilde{C}$  is bounded so there exists  $\varepsilon_0 > 0$  such that  $\tilde{C} \subseteq \varepsilon_0^{-1}U$  hence  $U \supseteq \varepsilon_0\tilde{C}$  and so  $U \supseteq \varepsilon\tilde{C} \forall \varepsilon < \varepsilon_0$  and thus  $U \supseteq B(0, \varepsilon_0)$ .

□

**Remarks:**

1.  $B(0, \varepsilon_0) \subseteq \bigcup_{0 \leq \varepsilon < \varepsilon_0} \varepsilon \tilde{C}$
2.  $T_1$  implies Hausdorff ( $T_2$ ). Consider  $v_0 \neq v_1$  in  $V$ : so  $0 \neq v_1 - v_0$ ,  $T_1$  implies there is  $U$  open around 0 with  $v_1 - v_0 \notin U$ . Then (since  $A, M$  continuous)  $(v, w) \mapsto v - w$  is continuous and there exists  $\tilde{U}$  open around 0 such that  $\tilde{U} - \tilde{U} \subseteq U$ . Then  $v_0 + \tilde{U}$  and  $v_1 + \tilde{U}$  are open disjoint neighborhoods of  $v_0$  and  $v_1$  respectively (disjoint since otherwise  $v_1 - v_0 \in \tilde{U} - \tilde{U} \subseteq U$ ).

## 4 Some examples of NVS'

**Definition.** Let  $(V, \|\cdot\|)$  be an NVS (over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ). If  $(V, d)$ ,  $d$  distance induced by  $\|\cdot\|$  is a complete metric space, then  $(V, \|\cdot\|)$  is called a *Banach space*.

**Example.**  $\mathbb{R}^n, \mathbb{C}^n, n \geq 1$  are Banach spaces, for  $\|\cdot\|_p, p \in [1, \infty)$ .

**Example.** Given  $(X, \tau)$  a general topological space, define

$$B_{\mathbb{F}}(X) = \{\text{functions } : X \rightarrow \mathbb{F} \text{ bounded}\}$$

$$C_{\mathbb{F}}(X) = \{\text{functions } : X \rightarrow \mathbb{F} \text{ continuous}\}$$

$$C_{\mathbb{F},b}(X) = C_{\mathbb{F}}(X) \cap B_{\mathbb{F}}(X)$$

If  $X = K$  is compact,  $C_{\mathbb{F}}(X) = C_{\mathbb{F},b}(X)$ . These are vector spaces over  $\mathbb{F}$  with addition  $(f + g)(x) = f(x) + g(x)$  and multiplication  $(fg)(x) = f(x)g(x)$ .

Norm on  $C_{\mathbb{F},b}(X)$ : the supremum norm,  $\|f\|_{\infty} = \sup_{x \in X} |f(x)|$

**Proposition.**  $(C_{\mathbb{F},b}, \|\cdot\|_{\infty})$  is a Banach space over  $\mathbb{F}$ .

*Proof.*

- $\|f\|_{\infty}$  is well defined in  $\mathbb{R}^+$  since  $f$  is bounded.
- $\|f\|_{\infty} = 0$  means  $f(x) = 0$  for all  $x \in X$  and so  $f = 0$ .
- Homogeneity and triangle inequality: inherited from  $|\cdot|$  in  $\mathbb{F}$  (exercise).
- Completeness: let  $(f_k)_{k \geq 1}$  be a Cauchy sequence under  $\|\cdot\|_{\infty}$ . For each  $x \in X$  we have  $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_{\infty} \rightarrow 0$  as  $n, m \rightarrow \infty$ . So  $(f_k(x))_{k \geq 1}$  is Cauchy in  $\mathbb{F}$ , so (since  $\mathbb{F}$  is complete) there exists a limit  $f(x) = \lim_{k \rightarrow \infty} f_k(x)$ . This defines a function  $f : X \rightarrow \mathbb{F}$ .
- For all  $\varepsilon > 0$ , there exists  $n_0 \geq 1$  such that  $\forall m, n \geq n_0, \forall x \in X$ ,

$$|f_m(x) - \underbrace{f_n(x)}_{\rightarrow f(x)}| \leq \varepsilon$$

so for all  $\varepsilon > 0$ , there exists  $n_0 \geq 1$  such that  $\forall m \geq n_0, \forall x \in X$  we have

$$|f_m(x) - f(x)| \leq \varepsilon$$

so  $\|f_m - f\|_\infty \leq \varepsilon$  and  $f_m \rightarrow f$  uniformly, so  $f \in C_{\mathbb{R},b}$  by properties of the uniform limit.

□

**Example.** Given  $U \subseteq \mathbb{R}^n$  open, bounded and non-empty;  $m \in \mathbb{N}^*$ , consider

$$\begin{aligned} C^m(\overline{U}) = \{f : U \rightarrow \mathbb{R} : f \text{ is } m \text{ times differentiable on } U, \forall \alpha \in \mathbb{N}^n \\ \text{s.t. } |\alpha| = \alpha_1 + \dots + \alpha_m \leq m \\ , \partial^\alpha f \text{ is continuous and bounded on } U\} \end{aligned}$$

Then  $(C^m(\overline{U}), \|\cdot\|_{C^m})$  is a Banach space where

$$\|f\|_{C^m} = \sup_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} \underbrace{\sup_{x \in U} |\partial^\alpha f(x)|}_{\|\partial^\alpha f\|_\infty}$$

Exercise: check that this is complete and  $\partial^\alpha f, \alpha \leq m-1$ , extends continuously to  $\tilde{U}$ .

**Example.**  $C_{\mathbb{R}}([0,1])$ , the set of continuous functions from  $[0,1]$  to  $\mathbb{R}$ . This is a vector space over  $\mathbb{R}$ .

- $(C_{\mathbb{R}}([0,1]), \|\cdot\|_\infty)$  is a Banach space (Example sheet)
- Could take another norm such that

$$\|f\|_p = \left( \int_0^1 |f(x)|^p dx \right)^{1/p}, \quad p \in [1, \infty)$$

Study of  $(C_{\mathbb{R}}([0,1]), \|\cdot\|_p)$ :

- $\|\cdot\|_p$  is well defined: Riemann and Lebesgue integrable.
- If  $\|f\|_p = 0$  and  $f \neq 0$  then there exists  $\varepsilon > 0$  and  $x_0 \in [0,1]$  such that  $|f(x_0)| \geq \varepsilon$ , so by continuity there exist  $a < b \in [0,1]$  such that  $\inf_{x \in [a,b]} |f(x)| \geq \frac{\varepsilon}{2}$ . Then  $\int_0^1 |f(x)|^p dx \geq \left(\frac{\varepsilon}{2}\right)^p (b-a) > 0$  which is impossible.
- Homogeneity is clear.
- Triangle inequality:

$$\|f + g\|_p^p = \int_0^1 |f + g|^p dx = \int_0^1 |f + g| |f + g|^{p-1} dx$$

$$\begin{aligned} &\leq \int_0^1 |f| |f+g|^{p-1} dx + \int_0^1 |g| |f+g|^{p-1} dx \\ &\underbrace{\leq}_{\text{Hölder:}} \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1} \end{aligned}$$

If  $\|f+g\|_p = 0$  then it's clear. Otherwise this implies  $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ .

- Completeness? Define

$$f_k(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{4k} \\ \left[x - \left(\frac{1}{2} - \frac{1}{4k}\right)\right] 4k & \frac{1}{2} - \frac{1}{4k} \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

then  $(f_k)_{k \geq 1}$  is Cauchy for  $\|\cdot\|_p$ , and the limit is  $1_{[1/2, 1]}$  which is not continuous. So not complete.

**Remark:** what about the completion? In general, abstract completions are often not very useful; however in this case, it is: Lebesgue space  $L^p([0, 1])$ , defined as equivalence classes for the “almost everywhere” equality.

**Example.** Take functions from  $X = \mathbb{N} \rightarrow \mathbb{R}$  or  $\mathbb{C}$ , get  $\ell_{\mathbb{F}}^p$  for  $p \in [1, \infty]$ , with norm  $\|(x_k)\|_p = \left(\sum_{k \geq 1} |x_k|^p\right)^{1/p}$  for  $p < \infty$  and  $\|(x_k)\|_{\infty} = \sup_{k \geq 1} |x_k|$ . Exercise: show this is indeed a norm and this is complete, hence Banach.

**Remark:** for  $p \in (0, 1)$ ,  $\ell^p$  is similarly defined.

**\*Non-examinable example of TVS\*:**

- Define for  $U \subseteq \mathbb{R}^n$  open & non-empty,  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ,  $C_{\mathbb{F}}(U)$  the set of continuous functions  $U \rightarrow \mathbb{F}$ .
- TVS for the topology  $\tau$  defined by the translations of the following basis of neighborhoods around 0: take  $(K_n)_{n \geq 1}$  a sequence of increasing compact sets,  $\bigcup_{n \geq 1} K_n = U$ . Define

$$U_n = \left\{ f \in C_{\mathbb{F}}(U) : \sup_{K_n} |f| \leq \frac{1}{n} \right\}$$

- Exercise: show this indeed a TVS and  $\tau$  does not depend on the choice of the  $(K_n)$ .
- Proposition:  $(C(U), \tau)$  is a locally convex, not locally bounded TVS (therefore not normable). Furthermore, it is metrizable with  $d(f, g) = \sum_{k \geq 1} \frac{1}{2^k} \left( \frac{\sup_{K_n} |f - g|}{1 + \sup_{K_n} |f - g|} \right)$ . Also  $(C(U), d)$  is complete (Frechet space).

**Remarks:**

1. Not locally bounded: suppose there exists  $B$  bounded neighborhood of 0, then there exists  $n_0 \geq 1$  such that  $U_{n_0} \subseteq B$ .  $B$  is bounded so there exists  $t > 0$  such that  $B \subseteq tU_{n_0+1}$  so  $U_{n_0} \subseteq tU_{n_0+1}$ . But this is impossible since we can always construct  $f \in U_{n_0}$  such that  $\sup_{K_{n_0+1}} |tf| > 1/n$
2. Let  $C_c(U)$  be the set of continuous functions with compact support. Then  $V$  is a neighborhood of 0 if and only if  $V \cap C(K_n)$  is a neighborhood of 0 in  $C(K_n)$ . This is a non-countable topology.

## 5 Bounded linear maps & duality

**Definition.** Given  $(V, \tau_V)$  and  $(W, \tau_W)$  TVS',  $T : V \rightarrow W$  linear is *bounded* if it maps bounded sets to bounded sets: for any  $B_V \subseteq V$  bounded, then  $T(B_V)$  is bounded in  $W$ .

**Proposition.** Given  $(V, \tau_V)$ ,  $(W, \tau_W)$  TVS' which are locally bounded (note this includes NVS'), and  $T : V \rightarrow W$  is linear, then  $T$  is bounded if and only if  $T$  is continuous.

*Proof.*

Step 1:  $T$  bounded  $\implies T$  continuous at 0. Let  $U_W$  be an open neighborhood of 0 in  $W$ , and  $U_V$  an open bounded neighborhood of 0 in  $V$ . Then  $T(U_V)$  is bounded, so there exists  $t > 0$  such that  $T(U_V) \subseteq tU_W$ . So  $T^{-1}(U_W) \supseteq t^{-1}U_V$  and  $t^{-1}U_V$  is open around 0 in  $V$  (using the fact dilations are continuous).

Step 2:  $T$  continuous at 0  $\implies T$  is continuous everywhere. Let  $w \in W$ ,  $U_W$  open around  $w$ ,  $v \in V$  such that  $T(v) = w$ . Then  $U_W - w$  is open around 0 in  $W$  (translation continuous), so by Step 1,  $T^{-1}(U_W - w)$  is a neighborhood of 0 in  $V$ . So

$$\begin{aligned} T^{-1}(U_W) &= T^{-1}(\{w\}) + T^{-1}(U_W) \\ &= \bigcup_{v' \in T^{-1}(\{w\})} (v' + T^{-1}(U_W)) \\ &\supseteq \underbrace{v + T^{-1}(U_W)}_{\text{ngbd around } v} \end{aligned}$$

Step 3:  $T$  continuous  $\implies T$  bounded. Let  $B_V \subseteq V$  be bounded, and  $U_W$  an open neighborhood of 0 in  $W$ . Then  $T^{-1}(U_W)$  is open around 0 in  $V$ . So (since  $B_V$  bounded) there exists  $t > 0$  such that  $B_V \subseteq tT^{-1}(U_W)$  and so  $T(B_V) \subseteq tU_W$ .

We have proved that  $T(B_V)$  is covered by a dilation of any neighborhood of 0, so is bounded.  $\square$

**Definition.** Given  $(V, \|\cdot\|_V)$ ,  $(W, \|\cdot\|_W)$  NVS' on  $\mathbb{F}$ , and  $T : V \rightarrow W$  linear,  $T$  is bounded iff  $T$  is continuous iff there exists  $t > 0$  such that  $T(B_V(0, 1)) \subseteq B_W(0, t)$ . The infimum of such  $t$ 's is denoted  $\|T\|$ .

**Remark:** can check that  $\|T\|$  is equivalently defined as

$$\|T\| = \sup_{\|v\|_V \leq 1} \|T_v\|_W = \sup_{\|v\|_V < 1} \|T_v\|_W = \sup_{\|v\|_V = 1} \|T_v\|_W \quad (*)$$

**Definition.** Given  $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$  NVS', denote

$$\mathcal{L}(V, W) = \{T : V \rightarrow W \text{ linear map}\}$$

$$\mathcal{B}(V, W) = \{T : V \rightarrow W \text{ linear bounded map}\}$$

**Proposition.**  $(\mathcal{B}(V, W), \|\cdot\|)$  is an NVS.

*Proof.*

- $\mathcal{L}(V, W)$  is a vector space via  $(\lambda_1 T_1 + \lambda_2 T_2)(v) = \lambda_1 T_1(v) + \lambda_2 T_2(v)$ .
- $\mathcal{B}(V, W)$ : dilation/(finite) sums of bounded sets are bounded. So  $T$  bounded implies  $\lambda T$  is bounded and  $T_1, T_2$  bounded implies  $T_1 + T_2$  bounded.

- $|||T|||$  is well-defined in  $\mathbb{R}_+$  for  $T$  bounded,  $|||0||| = 0$  and if  $|||T||| = 0$  then  $T(B_V(0, 1)) \subseteq B_W(0, t)$  for all  $t > 0$  and so by continuity of dilation,  $T(B_V(0, 1)) = \{0\}$ . By linearity, this implies  $T = 0$ .
- $|||\lambda T||| = |\lambda| |||T|||$  and  $|||T_1 + T_2||| \leq |||T_1||| + |||T_2|||$  follows from (\*)

□

**Proposition.** Let  $(V, \|\cdot\|_V)$  be a NVS and  $(W, \|\cdot\|_W)$  a Banach space. Then  $(\mathcal{B}(V, W), |||\cdot|||)$  is a Banach space.

*Proof.* We have proved that  $(\mathcal{B}(V, W), |||\cdot|||)$  is an NVS above. So we prove completeness. Let  $(T_k)_{k \geq 1}$  be a Cauchy sequence in  $(\mathcal{B}(V, W), |||\cdot|||)$ . Then

$$\sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}||| \rightarrow 0 \text{ as } k_0 \rightarrow \infty \quad (**)$$

$$\forall v \in V, \sup_{k_1, k_2 \geq k_0} \|T_{k_1}(v) - T_{k_2}(v)\|_W \leq \|v\|_V |||T_{k_1} - T_{k_2}||| \xrightarrow{k_0 \rightarrow \infty} 0 \quad (***)$$

so  $(T_k(v))_{k \geq 1}$  is a Cauchy sequence in  $W$ . Since  $W$  is complete, can let the associated limit be  $T(v)$ .

Then  $T$  is linear by pointwise limits:

$$\begin{aligned} T(\lambda_1 v_1 + \lambda_2 v_2) &= \lim_{k \rightarrow \infty} T_k(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{k \rightarrow \infty} [\lambda_1 T_k(v_1) + \lambda_2 T_k(v_2)] \\ &= \lambda_1 T(v_1) + \lambda_2 T(v_2) \end{aligned}$$

Use (\*\*), take  $k_2 \rightarrow \infty$  so

$$\forall v \in V, \sup_{k_1 \geq k_0} \|T_{k_1}(v) - T(v)\|_W \leq \|v\|_V \left( \sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}||| \right) \rightarrow 0 \text{ as } k_0 \rightarrow \infty$$

□