Introduction

Course structure:

(i) Preliminary toolbox: inequalities

(ii) Normed vector spaces (NVS)

(iii) (Recalls on) finite-dimensional case

(iv) Hahn-Banach Theorems (how big is the dual?)

(v) Completeness: Baire's Theorem & consequences for NVS

(vi) Detailed study of the topology of $C(K)$

(vii) The Hilbert space

(viii) Projection & duality

(ix) Introduction to operators and spectral theory

1 Preliminary toolbox: Young's, Hölder's & Minkowski's inequalities for vectors & sequences

Proposition (Young's inequality for products)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\forall a, b \geq 0,\ ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The result is clear for $a = 0$ or $b = 0$. Assume $a, b > 0$ and note $L : (0, \infty) \to \mathbb{R}$, $t \mapsto \ln t$ is strictly concave: $L''(t) = -\frac{1}{t^2} < 0$.

Therefore for all $A, B > 0$, $\lambda \in (0, 1)$

$$\ln(\lambda A + (1 - \lambda)B) \geq \lambda \ln A + (1 - \lambda) \ln B$$

with equality iff $A = B$. Apply this to $A = a^p$, $B = b^q > 0$ and $\lambda = \frac{1}{p}$. This gives

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) = \ln(ab)$$

so applying exp to both sides gives the result and furthermore we have equality iff $a^p = b^q$. ∎

Proposition (Hölder's inequality for vectors & sequences)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

(i) for any $n \in \mathbb{N} \setminus \{0\} = \mathbb{N}^*$, $\forall x, y \in \mathbb{R}^n$

$$\sum_{k=1}^{n} |x_k y_k| \leq ||x||_p ||y||_q \tag{$*$}$$

with $||x||_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ and similarly for $||y||_q$.

(ii) define

$$\ell^p = \{x \in \mathbb{R}^{\mathbb{N}^*} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

then $\forall x \in \ell^p$, $y \in \ell^q$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq ||x||_{\ell^p} ||y||_{\ell^q}$$

where $||x||_{\ell^p} = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ and similar for $||y||_{\ell^q}$.

Proof. To show (i) implies (ii): take $n \to \infty$ in (i) so

$$\sum_{k=1}^{n} |x_k|^p \to ||x||_{\ell^p}^p$$

and similarly

$$\sum_{k=1}^{n} |y_k|^q \to ||y||_{\ell^q}^q$$

By (i)

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}$$

so

$$\begin{aligned}\sum_{k=1}^{\infty} |x_k y_k| = \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k y_k|\right) &\leq \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q} \\ &= ||x||_{\ell^p} ||y||_{\ell^q}\end{aligned}$$

Proof of (i): if $||x||_{\ell^p}$ or $||y||_{\ell^q} = 0$, result is clear. Otherwise define $\tilde{x}$, $\tilde{y}$ sequences in ℓ^p and ℓ^q by

$$\tilde{x}_k = \frac{x_k}{||x||_{\ell^p}},\ \tilde{y}_k = \frac{y_k}{||y||_{\ell^q}}$$

Then $||\tilde{x}||_{\ell^p} = 1$, $||\tilde{y}||_{\ell^q} = 1$. Then $(*)$ is equivalent to showing

$$\sum_{k=1}^{n} |\tilde{x}_k \tilde{y}_k| \leq 1 \tag{$**$}$$

Aply Young's inequality on each $k = 1, \dots, n$ so

$$|\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}|\tilde{x}_k|^p + \frac{1}{q}|\tilde{y}_k|^q$$

Summing over k:

$$\sum_{k=1}^{n} |\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}\left(\sum_{k=1}^{n} |\tilde{x}_k|^p\right) + \frac{1}{q}\left(\sum_{k=1}^{n} |\tilde{y}_k|^q\right) \leq \frac{1}{p} + \frac{1}{q} = 1$$

$\square$

Remark: Equality in $(*)$ is equivalent to equality in $(**)$ which is equivalent to equality in Young's for each k so $|\tilde{x}_k|^p = |\tilde{y}_k|^q$ for $k = 1, \dots, n$. Also, the $p = 1$, $q = \infty$ case is easy.

Proposition (Minkowski's inquality for vectors & sequences)**.** Let $p \in [1, \infty)$, then

(i) for all $x, y \in \mathbb{R}^n$

$$||x + y||_p \leq ||x||_p + ||y||_p$$

(ii) for all $x, y \in \ell^p$

$$||x + y||_{\ell^p} = ||x||_{\ell^p} + ||y||_{\ell^p}$$

Proof. To show (i) implies (ii): by taking $n \to \infty$ as before

$$\sum_{k=1}^{\infty} |x_k|^p \to ||x||_{\ell^p}^p$$

$$\sum_{k=1}^{\infty} |y_k|^p \to ||y||_{\ell^p}^p$$

$$\sum_{k=1}^{n} |x_k + y_k|^p \to ||x + y||_{\ell^p}^p$$

Proof of (i): if $p = 1$ this is just the usual triangle inequality on each coordinate. So let $p \in (1, \infty)$ and

$$\sum_{k=1}^{n} |x_k + y_k|^p = \sum_{k=1}^{n} |x_k + y_k| \cdot |x_k + y_k|^{p-1}$$

$$\leq \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}$$

$$\underbrace{\leq}_{\text{Hölder: } q = \frac{p}{p-1}} ||x||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + ||y||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\leq (||x||_p + ||y||_p)\, ||x+y||_p^{p-1}$$

so we have proved

$$||x+y||_p^p \leq (||x||_p + ||y||_p)\, ||x+y||_p^{p-1}$$

If $||x+y||_p = 0$, result is clear. Otherwise divide by $||x+y||_p^{p-1}$ to get

$$||x+y||_p \leq ||x||_p + ||y||_p$$

□

Remark: equality occurs iff there is equality in the triangle inequality and Hölder's.
Remarks

1. Equality case: $p = 1$: $|x_k + y_k| \leq |x_k| + |y_k|$, i.e the usual triangle inquality
2. For $p = 2$ there's another proof: define $\mathcal{P} : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto ||x + \lambda y||^2$. Then $\mathcal{P}(\lambda) = a\lambda^2 + 2b\lambda + c$ and $\mathcal{P} \geq 0$. So

$$\langle x, y \rangle = b^2 \leq ac = ||x||^2 ||y||^2, \text{ Hölder's inequality}$$

2 Normed Vector Spaces (NVS)

Remark: this is not the most general structure for linear analysis - topological vector spaces (TVS).

Recall:

Definition. A *vector space* V over a field $\mathbb{F}$ is a set (of elements called *vectors*) with two operations:

$$A : V \times V \to V,\ (v, w) \mapsto v + w \text{ addition}$$

$$M : \mathbb{F} \times V \to V,\ (\lambda, v) \mapsto \lambda v \text{ scalar multiplication}$$

such that

- $(V, +)$ is an abelian group with identity 0.
- M is compatible with $(\mathbb{F}, 0)$ in the sense that $\lambda_1(\lambda_2 v) = \lambda_1\lambda_2 v$
- M distributes over $(V, +)$ and $(\mathbb{F}, +)$.

In this course $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$ unless stated otherwise.

Definition. Given a vector space V over $\mathbb{F}$:

- a *subspace* $W \subseteq V$ is a vector space over $\mathbb{F}$ included in V

- for a set $S \subseteq V$, a *linear combination of elements of* S is a finite sum of elements of S with coefficients in $\mathbb{F}$
- for a set $S \subseteq V$, the *span of* S, $\text{span}(S)$ is the smallest subspace of V containing S, and is also the set of linear combinations of S.

Definition. Given V a vector space over $\mathbb{F}$ and a set $S \subseteq V$:

- S is *linearly independent* if for all $m \in \mathbb{N}^*$ and for all $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$, for all $s_1, \ldots, s_m \in S$, $\sum_{i=1}^m \alpha_i s_i = 0$ if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_m$.
- S is a *basis* of V if it is linearly independent and $\text{span}(S) = V$.
- If there exists a finite basis S of V, then V has finite dimension, otherwise it is infinite-dimensional.

Remark: later we'll prove with Zorn's lemma that any vector space has a basis.

Definition. A *normed vector space* (NVS) V over $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a function $N : V \to \mathbb{R}_+$, $v \mapsto ||v||$ (the *norm*), with

1. $||v|| \geq 0$ for all $v \in V$, with equality only at $v = 0$ (*positive definiteness*)
2. For all $\lambda \in \mathbb{F}$, $v \in V$ $||\lambda v|| = |\lambda| ||v||$ (compatibility between N and M)
3. For all $v, w \in V$, $||v + w|| \leq ||v|| + ||w||$ (compatibility between N and A)

Example. $V = \mathbb{R}^n$, $v = (v_1, \ldots, v_n)$, $||v|| = (v_1^2 + \ldots + v_n^2)^{1/2}$ or

$$\begin{cases} ||v||_p = (|v_1|^p + \ldots + |v_n|^p)^{1/p} & \text{for } p \in [1, \infty) \\ ||v||_\infty = \sup_{i=1}^n |v_i| & \text{for } p = \infty \end{cases}$$

Definition. Given a set X, a *topology* τ on X is a collection of subsets of X ("open sets") such that

- $\emptyset \in \tau$, $X \in \tau$
- τ is stable under any union
- τ is stable under finite intersections

Definition.

- For (X, d) a metric space, the *induced topology* is the smallest topology that contains open balls in d
- For a NVS $(V, ||\cdot||)$, the induced topology is that associated with $d(v, w) = ||v - w||$

Natural question: $\mathbb{F}$ field, V vector space over $\mathbb{F}$. Norm on V, $\tau_{||\cdot||}$. Continuity of operations M and A?

Proposition. Let $(V, ||\cdot||)$ be a NVS over $\mathbb{F}$ ($\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$), then

(i) A, M are continuous for the following topologies: $\tau_{||\cdot||}$ on V, then product topology of it on $V \times V$, $\tau_{|\cdot|}$ over $\mathbb{F}$, then product topology of $\tau_{|\cdot|}$ and $\tau|| \cdot ||$ on $\mathbb{F} \times V$

(ii) Translations $T_{v_0} : V \to V$, $v \mapsto v + v_0$, $v_0 \in V$ and dilations $D_{\lambda_0} : V \to V$, $v \mapsto \lambda_0 v$, $\lambda_0 \in \mathbb{F}^*$ are homeomorphisms

Proof. (i) Let us prove that $A : V \times V \to V$ is continuous: consider an open set $\emptyset \neq U \subseteq V$ and $(v_1, v_2) \in A^{-1}(U)$, i.e $v_1 + v_2 \in U$. Since U is open, there is $\varepsilon > 0$ such that $\underbrace{B_V(v_1 + v_2, \varepsilon)}_{\text{open ball}} \subseteq U$.

We have that $A(B(v_1, \varepsilon/2), B_V(v_2, \varepsilon/2)) \subseteq B_V(v_1+v_2, \varepsilon)$ (triangle inequality). Note also that $B(v_1, \varepsilon/2) \times B(v_2, \varepsilon/2)$ is open (product topology), so $A^{-1}(U)$ is open and A is continuous.

Now we show $M : \mathbb{F} \times V \to V$ is continuous. Consider an open set $U \neq \emptyset$ in V, $(\lambda, v) \in M^{-1}(U)$. Since U is open, there exists $\varepsilon > 0$ such that $B_V(\lambda v, \varepsilon) \subseteq U$ (WLOG $\varepsilon < 1$). Then

$$M\left(B_{\mathbb{F}}\left(\lambda, \frac{\varepsilon}{3\max(1, ||v||)}\right) B_V\left(v, \frac{\varepsilon}{3\max(1, |\lambda|)}\right)\right) \subseteq B_V(\lambda v, \varepsilon)$$

(ii) T_{v_0} and D_{λ_0} are linear, continuous with inverses T_{-v_0} and $D_{\lambda_0^{-1}}$ respectively, so are homeomorphisms.

□