Theorem (Lawler,Schramm,Werner)**.** $\xi(1,1)=\frac{5}{4}$, $\xi(2,0)=\frac{2}{3}$.

1 Conformal maps

We consider a domain $U \subseteq \mathbb{C}$ (i.e an open and connected subset of the complex plane). We say U is *simply connected* if $\mathbb{C} \setminus U$ is connected.

We say $f : U \to \mathbb{C}$ is *holomorphic* if it is complex differentiable. If f is holomorphic and injective we say it is *univalent*. If $f : U \to V$ is holomorphic and bijective we say f is a *conformal map*.

Remark. If $f : U \to V$ is conformal then

$$f(w) = f(z) + f'(z)(w - z) + o(|w - z|)$$

and $f'(z) \neq 0$. Hence f locally looks like a translation combined with a scaling and rotation.

We will work in 2d throughout this course. This gives a richness to the conformal maps, as shown by the following theorem.

Theorem (Riemann mapping theorem)**.** *If* $U \subsetneq \mathbb{C}$ *is a simply connected domain and* $z \in U$ *then there exists a unique conformal map* $f : \mathbb{D} \to U$ *with* $f(0) = z$ *and* $\arg f'(0) = 0$.

Where we have taken $\mathbb{D} = \{z : |z| < 1\}$ to be the open unit disc. We will also take $\mathbb{H} = \{z : \Im z > 0\}$ to be the open upper half-plane.

Examples.

- Let $f(z) = \frac{z-i}{z+i}$. Then $f : \mathbb{H} \to \mathbb{D}$ is a conformal map.
- $f : \mathbb{D} \to \mathbb{D}$ is conformal if and only if $f(w) = \lambda \frac{w-z}{\bar{z}w-1}$ for some $\lambda, z \in \mathbb{C}$ with $|\lambda| = 1$, $z \in \mathbb{D}$.
- $f : \mathbb{H} \to \mathbb{H}$ is conformal if and only if $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
- Given a simply connected domain D and disjioint subarcs $A, B \subseteq \partial D$, there is a unique conformal map from U to the rectangle such that A, B are mapped to parallel sides with length 1. The length L of the other sides is called the extremal length $\mathrm{EL}_D(A, B)$ and is unique.

Recall that if $f = u + iv$ (with u, v denoting the real/imaginary parts of f respectively) then f is holomorphic iff it satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It follows from this that if f is holomorphic,

$$\Delta u = \left(\frac{\partial}{\partial x}\right)^2 u + \left(\frac{\partial}{\partial y}\right)^2 u = \frac{\partial^2}{\partial x \partial y} v - \frac{\partial^2}{\partial x \partial y} v = 0$$

and similarly $\Delta v = 0$.

Conversely, if $u : U \to \mathbb{R}$ (for U a simply connected domain) is harmonic there exists $v : U \to \mathbb{R}$ such that $u + iv$ is holomorphic.

A consequence of this is that if u is harmonic on a bounded domain D and continuous on $\overline{D}$, for $z \in D$ and B a Brownian motion starting from z and $\tau := \inf\{t : B_t \notin D\}$, we have $u(z) = \mathbb{E}_z[u(B_\tau)]$ (see Part III Advanced Probability).

Conformal invariance of 2d Brownian motion

Let $f : D \to \tilde{D}$ be a conformal map and B be a Brownian motion starting at $z \in \mathbb{C}$. Define $\tau = \inf\{t : B_t \notin D\}$ and let $\sigma(t) = \inf\{s : \int_0^s |f'(B_r)|^2 \mathrm{d}r = t\}$. Then $f(B_{\sigma(t)})$ has the law of a Brownian motion starting from $f(z)$ until exiting $\tilde{D}$.

Proof. See Part III Stochastic Calculus. □

We have seen that for u harmonic on D and continuous on $\overline{D}$ we have $u(z) = \mathbb{E}_z[u(B_{\tau_D})]$. We get the following corollary by taking a Brownian motion until it hits $\partial B(z, r)$.

Corollary (Mean value property)**.** *For $B(z, r) \subseteq D$*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \mathrm{d}\theta.$$

Proposition (Strong maximum principle)**.** Let u be harmonic in D, D a domain. If u attains a global maximum in D then u is constant.

Proof. Follows fom mean value property and compactness of paths connecting points. □

Proposition (Maximum modulus principle)**.** Let $f : D \to \mathbb{C}$ holomorphic, D a domain. Then if $|f|$ attains a global maximum in D, f is constant.

Proof. Let $K \subseteq D$ be compact. By considering $f + M$ for $M > 0$ large enough we may assume $|f| > 0$ on K. Thus $\log |f|$ is harmonic. So we can apply the strong maximum principle to see $\log |f|$ is constant on K, i.e f takes values on a circle. But this is impossible unless $f' = 0$ on K. □

Proposition (Schwarz lemma)**.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Furthermore if $|f(z)| = |z|$ for some $z \neq 0$ then $f(w) = we^{i\theta}$ for some $\theta \in \mathbb{R}$.

Proof. Define the holomorphic function $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{for } z \neq 0 \\ f'(0) & \text{for } z = 0 \end{cases}.$$

Then $|z| = 1$ on $\partial\mathbb{D}$, implying $|g| \leq 1$ on $\partial\mathbb{D}$. Thus $|g| \leq 1$ on $\mathbb{D}$ by the maximum modulus principle.

If $|g(z)| = 1$ for some $z \in \mathbb{D}$ then g is constant since this is a maximum. □

Distortion theorems for conformal maps

Let $\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} \text{ univalent} : f(0) = 0, f'(0) = 1\}$.

Remark. We can write such f as $f(z) = z + a_2 z^2 + a_3 z^2 + \ldots$.

Goal: for $f \in \mathcal{S}$

- Koebe 1/4-theorem: $f(\mathbb{D}) \supseteq B(0, 1/4)$;
- Koebe distortion theorem: $\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$.

Corollary. *If $f : D \to \tilde{D}$ is conformal then*

$$\frac{\operatorname{dist}(f(z), \partial\tilde{D})}{4\operatorname{dist}(z, \partial D)} \leq |f'(z)| \leq \frac{4\operatorname{dist}(f(z), \partial\tilde{D})}{\operatorname{dist}(z, \partial D)}.$$

Corollary. *If f univalent in D, $B(z, R) \subseteq D$ then for $r < 1$ we have $|f'(u)| \leq c(r)|f'(v)|$ for all $u, v \in B(z, rR)$.*

Define

$$\Sigma = \{g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} : g \text{ univalent},\ g(\infty) = \infty,\ g'(\infty) = 1\}.$$

Theorem (Area theorem)**.** *Let $g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$ be univalent with $g(z) \to \infty$ as $z \to \infty$ and $g'(z) \to 1$ as $z \to \infty$. Write $g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z} + \dots$ for g near ∞. Then*

$$\sum_{n \geq 1} n|b_n|^2 \leq 1$$

and moreover

$$\operatorname{area}(\mathbb{C} \setminus g(\mathbb{C} \setminus \overline{\mathbb{D}})) = \pi \left(1 - \sum_{n \geq 1} n|b_n|^2 \right).$$

Proof. Let $r > 1$ and define $C_r = g(\partial D(0, r))$. Let E_r be the inner component of $\mathbb{C} \setminus C_r$. By Green's theorem

$$\begin{aligned} \frac{1}{2i} \int_{C_r} \overline{w} \mathrm{d}w &= \frac{1}{2i} \int_{C_r} (x - iy)(\mathrm{d}x + i\mathrm{d}y) \\ &= \frac{1}{2i} \int_{C_r} ((x - iy)\mathrm{d}x + (ix + y)\mathrm{d}y) \\ &= \frac{1}{2i} \int_{E_r} 2i\mathrm{d}x\mathrm{y} && \text{(Green's thm)} \\ &= \operatorname{area}(E_r). \end{aligned}$$

while we also have

$$\begin{aligned} \frac{1}{2i} \int_{C_r} \overline{w} \mathrm{d}w &= \frac{1}{2i} \int_{\partial B(0,r)} \overline{g(z)} g'(z) \mathrm{d}z \\ &= \frac{1}{2} \int_0^{2\pi} \left(re^{-i\theta} + \sum_{n \geq 1} \overline{b_n} r^{-n} e^{in\theta} \right) \left(1 - \sum_{n \geq 1} b_n r^{-n-1} e^{i(n+1)\theta} \right) re^{i\theta} \mathrm{d}\theta \\ &= \pi \left(r^2 - \sum_{n \geq 1} n|b_n|^2 r^{-2n} \right). \end{aligned}$$

Now take $r \downarrow 1$. $\square$

Theorem. *Let $f : \mathbb{D} \to \mathbb{C} \in \mathcal{S}$ write $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$. Then $|a_2| \leq 2$.*

Proof. We claim there exists $g \in \mathcal{S}$ with $g(z)^2 = f(z^2)$ (we call g the "square-root transform" of f). Note

$$f(z^2) = z^2 (\underbrace{1 + a_2 z^2 + a_3 z^4 + \dots}_{:= h(z)})$$

and since $h \neq 0$ (by $f(0) = 0$ and injectivity of f), we can define $g(z) = z\sqrt{h(z)}$. Also $g(0) = 0$ and $g'(0) = 1$. To show g is univalent, suppose $g(z_1) = g(z_2)$ for some $z_1, z_2 \in \mathbb{D}$. Then $f(z_1^2) = f(z_2^2)$ so $z_1^2 = z_2^2$, i.e $z_1 = \pm z_2$. But g is an odd function and only zero at $z = 0$ so we have $z_1 = z_2$.

To conlcude take $z \mapsto \frac{1}{g(1/z)} \in \Sigma$. This map is the same as

$$z \mapsto \frac{1}{\sqrt{f(1/z^2)}} = z - \frac{a_2}{2}\frac{1}{z} + \dots$$

so by the area theorem, $|a_2/2| \leq 1$. $\square$

Theorem (Koebe 1/4-theorem)**.** *Let $f \in \mathcal{S}$. Then $f(\mathbb{D}) \supseteq B(0, 1/4)$.*

Proof. Let $w \notin f(\mathbb{D})$. Then

$$z \mapsto \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \dots$$

is in $\mathcal{S}$ so by the above $\left|a_2 + \frac{1}{w}\right| \leq 2$. Since $|a_2| \leq 2$ we must have $|1/w| \leq 4$. $\square$

If we define

$$F(w) = \frac{f\left(\frac{w+z}{1+\overline{z}w}\right) - f(z)}{(1-|z|^2)f'(z)} = w + \frac{1}{2}\left((1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right) w^2 + \dots$$

we see

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \leq 4.$$

Note

$$\begin{aligned} z\frac{f''(z)}{f'(z)} = z\partial_z \log f'(z) &= r\partial_r \log f'(z) \\ &= r\partial_r \log |f'(z)| + ir\partial_r \arg(f'(z)) \end{aligned}$$

and

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}$$

which implies

$$\frac{2r^2}{1-r^2} - \frac{4r}{1-r^2} \leq \Re\left(z\frac{f''(z)}{f'(z)}\right) \leq \frac{2r^2}{1-r^2} + \frac{4r}{1-r^2}.$$

Integrating from $r = 0$ to R,

$$\log \frac{1-R}{(1+R)^3} \leq \log |f'(Re^{i\theta})| \leq \log \frac{1+R}{(1-R)^3}.$$

So we get

Theorem (Kobe's distortion theorem)**.** *For $f \in \mathcal{S}$,*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Definition. $A \subseteq \mathbb{H}$ is a *compact* $\mathbb{H}$*-hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. We write $A \in \mathcal{Q}$ for such a set.

For $A \in \mathcal{Q}$, pick $g : \mathbb{H} \setminus A \to \mathbb{H}$ conformal (possible by Riemann mapping theorem) with $g(\infty) = \infty$.

Question: when does a holomorphic function extend analytically to the boundary?

Theorem (Schwarz reflection principle)**.** *Let $U \subseteq \mathbb{C}$ be a domain such that $U = \{\overline{z} : z \in U\}$. Let $U^+ = U \cap \mathbb{H}$. Let $f : U^+ \to \mathbb{C}$ be holomorphic with $\lim_{\Im z \downarrow 0} \Im f(z) = 0$. Then f extends to a holomorphic function on U with $f(\overline{z}) = \overline{f(z)}$ for all $z \in U$.*

Proof. On $U^- := U \cap \{z : \Im(z) < 0\}$ set $f(z) := \overline{f(\overline{z})}$. To extend f to $U \cap \mathbb{R}$, write $f = u + iv$ for u, v harmonic and note $\lim_{\Im z \downarrow 0} v(z) = 0$. So we have extended v via

$$v(z) = \begin{cases} -v(\overline{z}) & \Im z < 0 \\ 0 & \Im z = 0 \end{cases}.$$

Then v is still harmonic as it satisfies the mean value property.

For $z \in U \cap \mathbb{R}$ pick $\varepsilon > 0$ so that $B(z, \varepsilon) \subseteq U$. Let $\tilde{u}$ be the harmonic conjugate of v on $B(z, \varepsilon)$ (unique up to an additive constant). Then $f = u + iv = \tilde{u} + iv + \text{const}$ so f extends to $B(z, \varepsilon)$. Furthermore this matches with $f(z) = \overline{f(\overline{z})}$ on U^-. For different z these extensions match so by the identity principle we are done. $\square$

Now for $A \in \mathcal{Q}$, $g : \mathbb{H} \setminus A \to \mathbb{H}$ conformal with $g(\infty) = \infty$, we can Schwarz reflect. g has a simple pole at ∞ so

$$g(z) = b_{-1} z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots .$$

Also $g(z) = \overline{g(\overline{z})} = \overline{g(z)}$ for $z \in \mathbb{R}$ which implies $b_n \in \mathbb{R}$ for all $n \geq -1$. So we can scale and then translate g so that $b_{-1} = 1$ and $b_0 = 0$.

Definition. For $A \in \mathcal{Q}$, let $g_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ the conformal map with $g_A(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$.

Define the *half-plane capacity* $\text{hcap}(A)$ to be equal to $b_1 \in \mathbb{R}$ as above.

For example we have $g_{[0,i]}(z) = \sqrt{z^2 + 1}$ and so $\text{hcap}([0, i]) = \frac{1}{2}$ (we can see this by looking at what happens to $\mathbb{H} \setminus [0, i]$ under $z \mapsto z^2 \mapsto z^2 + 1 \mapsto \sqrt{z^2 + 1}$).

If A is instead a $\overline{\mathbb{D}} \cap \mathbb{H}$ with radius 1 centred at 0, we have $g_A(z) = z + \frac{1}{z}$ so $\text{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = 1$.

It is straighforward to see $g_{rA}(z) = rg_A(z/r)$ for any $r > 0$ and so $\text{hcap}(rA) = r^2 \,\text{hcap}(A)$. Can also see that $\text{hcap}(A + x) = \text{hcap}(A)$ for any $x \in \mathbb{R}$.

For $A \subseteq \tilde{A}$ can also see that

$$g_{\tilde{A}} = g_{g_A(\tilde{A}\setminus A)} \circ g_A = z + \frac{\text{hcap}(A)}{z} + \frac{\text{hcap}(g_A(\tilde{A} \setminus A))}{z} + \dots$$

so $\text{hcap}(\tilde{A}) = \text{hcap}(A) + \text{hcap}(g_A(\tilde{A} \setminus A))$. Thus $\text{hcap}(A) \leq \text{hcap}(\tilde{A})$ (after seeing later that hcap is non-negative). Also $\text{hcap}(A) \leq \text{hcap}(\text{rad}(A) \cdot \overline{\mathbb{D}} \cap \mathbb{H}) \leq \text{rad}(A)^2$ where $\text{rad}(A) = \sup\{|z| : z \in A\}$.

Proposition. Let $A \in \mathcal{Q}$, B be a 2D Brownian motion and $\tau = \inf\{t : B_t \notin \mathbb{H} \setminus A\}$. Then

(i) For all $z \in \mathbb{H} \setminus A$, $\Im(z - g_A(z)) = \mathbb{E}_z[\Im(B_\tau)]$;

(ii) We have $\operatorname{hcap}(A) = \lim_{y\to\infty} y\mathbb{E}_{iy}[\Im(B_\tau)]$.

Remark. (ii) shows that $\operatorname{hcap}(A) \geq 0$.

Proof.
(i) Note $z \mapsto \Im(z - g_A(z))$ is harmonic and bounded. Hence

$$\Im(z - g_A(z)) = \mathbb{E}_z[\Im(B_\tau - g_A(B_\tau))] = \mathbb{E}_z[\Im(B_\tau)].$$

(ii) We have

$$\begin{aligned}
\operatorname{hcap}(A) = \lim_{z\to\infty} z(g_A(z) - z) &= \lim_{y\to\infty} iy(g_A(iy) - iy) \\
&= \lim_{y\to\infty} \Re(iy(g_A(iy) - iy)) && (\operatorname{hcap}(A) \in \mathbb{R}) \\
&= \lim_{y\to\infty} y\Im(iy - g_A(iy)) \\
&= \lim_{y\to\infty} y\mathbb{E}_{iy}[\Im(B_\tau)]. && \text{(by (i))}
\end{aligned}$$

□

The law of B_τ for $\tau = \inf\{t : B_t \notin D\}$ is often called the *harmonic measure for z relative to D*. For $z \in D$, $\omega(z, \cdot, D)$ is a probability measure on ∂D. For $A \in \mathcal{B}(\partial D)$, $\omega(\cdot, A, D)$ is harmonic (strong markov property so satisfies mean value property).

Example.

- $\omega(0, \cdot, \mathbb{D})$ is the uniform distribution on $\partial\mathbb{D}$;
- $\omega(z, \cdot, \mathbb{D})$ may be computed using conformal invariance of Brownian motion (Example Sheet);
- $\omega(z, \cdot, \mathbb{H})$ may also be computed using conformal invariance (Example Sheet). If $z = x + iy$ it has density on $\mathbb{R}$ given by

$$u \mapsto \frac{1}{\pi} \frac{y}{(x-u)^2 + y^2}.$$

.

Proposition. There exists $c > 0$ such that for any $A \in \mathcal{Q}$ and $|z| \geq 2\operatorname{rad}(A)$ we have

$$\left| g_A(z) - z - \frac{\operatorname{hcap}(A)}{z} \right| \leq c \frac{\operatorname{rad}(A)\operatorname{hcap}(A)}{|z|^2}.$$

Proof. By scaling we may assume $\mathrm{rad}(A) \leq 1$. We have

$$\Im(z - g_A(z)) = \mathbb{E}_z[\Im(B_\tau)] = \int_0^\pi \mathbb{E}_{e^{i\theta}}[\Im(B_\tau)]p(z, e^{i\theta})\mathrm{d}\theta$$

where $p(z, e^{i\theta})$ is the density of $w(z, \theta, \mathbb{H} \setminus \overline{\mathbb{D}})$. On the Example Sheet it will be shown that

$$p(z, e^{i\theta}) = \frac{2}{\pi} \frac{\Im(z)}{|z|^2} \sin(\theta)(1 + \mathcal{O}(|z|^{-1})) \text{ as } z \to \infty.$$

Hence

$$\begin{aligned} \Im(z - g_A(z)) &= \frac{2}{\pi} \frac{\Im(z)}{|z|^2} \int_0^\pi \mathbb{E}_{i\theta}[\Im(B_\tau)] \sin(\theta)\mathrm{d}\theta(1 + \mathcal{O}(|z|^{-1})) \\ &:= a\frac{\Im(z)}{|z|^2}(1 + \mathcal{O}(|z|^{-1})) \end{aligned}$$

and so $\Im(z - g_A(z) - \frac{a}{2}) = \mathcal{O}(a\frac{\Im z}{|z|^3})$. Define $h(z) := z - g_A(z) - \frac{a}{2}$. Then $\Im(h(z))$ is harmonic. Also $|\partial_x \Im(h(z))|, |\partial_y \Im(h(z))| \leq \tilde{c}\frac{a}{|z|^3}$. Then the Cauchy-Riemann equations imply similar inequalities for the real parts of $h(z)$ so $|h'(z)| \leq \tilde{c}\frac{a}{|z|^3}$. We have $h(\infty) = 0$ so $|h(re^{i\theta})| \leq \int_r^\infty |h'(se^{i\theta})|\mathrm{d}s \lesssim \frac{a}{r^2}$. □

Loewner differential equation

Definition. Let $(A_t)_{t\geq 0}$ be a family of compact $\mathbb{H}$-hulls. We say $(A_t)_{t\geq 0}$

(i) *is strictly increasing* if $A_s \subsetneq A_t$ whenever $s < t$;

(ii) *satisfies the local growth property* if for all $T, \varepsilon > 0$ there exists $\delta > 0$ such that whenever $0 \leq s \leq t \leq s + \delta \leq T$ we have $\mathrm{diam}(g_s(A_t \setminus A_s)) \leq \varepsilon$.

If (i) and (ii) are satisfied then $t \mapsto \mathrm{hcap}(A_t)$ is continuous and increasing. In this case we say $(A_t)_{t\geq 0}$

(iii) is *parameterised by half-plane capacity* if $\mathrm{hcap}(A_t) = 2t$ for all t.

We let $\mathcal{A}$ be the set of all such families satisfying (i)-(iii). We let $\mathcal{A}_T$ be the set of all such families satisfying (i)-(iii) but on time interval $[0, T]$.

Theorem ("Chordal Loewner differential equation"). *Let $(A_t)_{t\geq 0} \in \mathcal{A}$, let $g_t := g_{A_t}$ be the mapping-out function. Then there exists $U : [0,\infty) \to \mathbb{R}$ continuous such that*

$$\partial_t g_t(z) = \frac{z}{g_t(z) - U_t}, \; g_0(z) = z. \tag{$*$}$$

Proof. We have that $\bigcap_{s>0} \overline{g_t(A_s \setminus A_t)}$ is a single point by the local growth property. Let U_t be this point. The local growth property and the proposition from last time, U is continuous.

Define $\tilde{g} = g_{g_t(A_{t+\delta}\setminus A_t)-U_t}$. Then

$$\tilde{g}(z) = z + \frac{\operatorname{hcap}(g_t(A_{t+\delta} \setminus A_t) - U_t)}{2} + \mathcal{O}\left(\frac{\operatorname{hcap}(g_t(A_{t+\delta} \setminus A_t)) \operatorname{rad}(g_t(A_{t+\delta} \setminus A_t))}{|z|^2}\right).$$

Defining $g_{t,t+\delta} = g_{t+\delta}^{-1} \circ g_t$ we have

$$g_{t,t+\delta}(z) = z + \frac{2\delta}{z - U_t} + 2\delta \operatorname{diam}(g_t(A_{t+\delta} \setminus A_t))\mathcal{O}\left(\frac{1}{|z - U_t|^2}\right)$$

uniformly in $t \in [0,T]$. Hence

$$g_{t+\delta}(z) - g_t(z) = \frac{2\delta}{g_t(z) - U_t} + 2\delta \operatorname{diam}(g_t(A_{t+\delta} \setminus A_t))\mathcal{O}\left(\frac{1}{|g_t(z) - U_t|^2}\right).$$

Now dividing through by δ and noting $\operatorname{diam}(g_t(A_{t+\delta} \setminus A_t)) \to 0$ we get the result. $\square$

Conversely, given U continuous and real valued, then $*$ has a unique solution for $t < \tau_z := \sup\{s : |g_s(z) - U_s| > 0\}$.

We use the notation

$$A_t := \{z \in \mathbb{H} : \tau_z \leq t\}$$
$$H_t := \mathbb{H} \setminus A_t.$$

Then $g_t : H_t \to \mathbb{H}$ is conformal and $(A_t) \in \mathcal{A}$ and $g_{A_t} = g_t$ (see Example Sheet). We call (U_t) the "driving function" or "Loewner transform" of (g_t) or (A_t).

Schramm-Loewner Evolution (SLE)

Suppose $(A_t) \in \mathcal{A}$ is random with driving function U such that

(i) (rA_{t/r^2}) has the same law as (A_t) (scale invariance);

(ii) Conditional on $\mathcal{F}_t = \sigma(U_s : s \leq t)$, the conditional law of $(g_t(A_{t+s} \setminus A_t) - U_t)_{s\geq 0}$ is the same as that of $(A_s)_{s\geq 0}$.

These are called the *conformal Markov properties.*

Theorem. *There exists $\kappa \geq 0$ such that $U_t = \sqrt{\kappa} B_t$ for some Brownian motion B.*

Proof. U is continuous and by (ii) of the conformal Markov properties, we have that $(U_{t+s} - U_t)_{s \geq 0}$ has the same law as $(U_s)_{s \geq 0}$ conditional on $\mathcal{F}_t$. Therefore U has independent and stationary increments so $U_t = at + \sqrt{\kappa} B_t$ for some a, κ.

(i) of the conformal Markov properties implies $U_t = \sqrt{\kappa} B_t$. $\square$

Definition. The random Loewner chain with $U_t = \sqrt{\kappa} B_t$ for a Brownian motion B is denoted SLE_κ.

Remarks.

- SLE_κ is generated by a curve, i.e there exists a continuous path γ in $\overline{\mathbb{H}}$ such that $H_t = \mathbb{H} \setminus A_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0,t])$.
- If $\kappa \leq 4$ then SLE_κ is a simple curve, i.e $\gamma(t) \in \mathbb{H}$ for $t > 0$ and $\gamma(t) \neq \gamma(s)$ for $s \neq t$
- If $\kappa \in (4, 8)$ then SLE_κ is self-intersecting and boundary-intersecting and disconnects points from ∞
- If $\kappa \geq 8$ then SLE_κ is space-filling.
- For all κ, $\gamma(t) \to \infty$ as $t \to \infty$.

Definition. If $D \subsetneq \mathbb{C}$ is a simply-connected domain, $x, y \in \partial D$ (suppose ∂D is a curve). Define SLE_κ in (D, x, y) as the pushformward SLE_κ in $(\mathbb{H}, 0, \infty)$ under a conformal transformation $\varphi : \mathbb{H} \to D$ with $\varphi(0) = x$, $\varphi(\infty) = y$ (well-defined due to scaling invariance in $\mathbb{H}$).

Definition. We say that a Loewner chain (g_t) (or equivalently (A_t)) is *generated by a curve* if there exists $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ continuous such that for all t, $H_t := \mathbb{H} \setminus A_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$.

Lemma. *Suppose $\gamma(t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + y)$ exists for all t and is continuous then (g_t) is generated by γ.*

Remark. The converse is also true.

We will need some facts:

(i) Let A be a compact $\mathbb{H}$-hull. If α is a continuous path and $\alpha(s) \in \mathbb{H} \setminus A$ for $s > 0$, $\alpha(0) \in \partial A$. Then $\lim_{s \downarrow 0} g_A(\alpha(s)) \in \mathbb{R}$ exists. [See Q3 Example Sheet 1]

(ii) If $\alpha, \tilde{\alpha}$ are two paths in $\mathbb{H} \setminus A$ and $\lim_{s \downarrow 0} g_A(\alpha(s)) = \lim_{s \downarrow 0} g_A(\tilde{\alpha}(s))$ then $\alpha(0) = \tilde{\alpha}(0)$. [Q3 Example Sheet 1 again applied to g_A^{-1}]

Proof of Lemma. Clearly $\gamma(t) \notin H_t$ so $H_t \subseteq \mathbb{H} \setminus \text{fill}(\gamma([0, t]))$. Now we show $\partial A_t \cap \mathbb{H} \subseteq \overline{\gamma([0, t])}$. Let $z \in \partial A_t \cap \mathbb{H}$. Since γ is continuous it's enough to show $z \in \overline{\gamma([0, t])}$. Pick $w_n \to z$, $w_n \in H_t$. Let α be the line segment from w_n towads z until it hits the first point $z_n \in \partial A_t$.

So now we show $z_n \in \gamma([0, t])$. Since $z_n \in A_t$ we have $s := \tau_{z_n} \leq t$. We claim $\lim_{r \downarrow 0} g_s(\alpha(r)) = U_s$. Once we have this, by fact (ii) above since $\lim_{y \downarrow 0} g_t^{-1}(U_s + y) = \gamma(s)$ we just have $\alpha(0) = \gamma(s)$.

Indeed if not, $\text{dist}(g_s(\alpha), U_s) > 0$. But since $z_n \in A_s \setminus A_{s-\delta}$ for all $\delta > 0$, combined with the local growth property, we have $\lim_{\delta \downarrow 0} g_{s-\delta}(z_n) = U_s$ and so $\text{dist}(g_{s-\delta}(\alpha), U_{s-\delta}) \to 0$ as $\delta \to 0$, giving a contradiction. □

As we will throughout the course, we assume (U_t) is continuous and real-valued. So we can solve the Loewner differential equation $\partial_t g_t(z) = \frac{z}{g_t(z) - U_t}$, $g_0(z) = z$, $t < \tau_z := \sup\{s : |g_s(z) - U_s| > 0\}$. Then since $U_t \in \mathbb{R}$ we have $\partial_t \overline{g_t(\overline{z})} = \frac{z}{\overline{g_t(\overline{z})} - U_t}$, so $g_t(\overline{z}) = \overline{g_t(z)}$ and $\tau_z = \tau_{\overline{z}}$ by uniqueness. On $\{\overline{z} : z \in H_t\}$, this agrees with the Schwarz reflection of $g_t : H_t \to \mathbb{H}$.

Lemma. *For $z \in \mathbb{R}$, $\tau_z \leq t$ if and only if $z \in \overline{A_t \cap \mathbb{H}}$, i.e the domain $\{z \in \mathbb{C} : \tau_z > t\}$ agrees exactly with the reflection of H_t across $\mathbb{R}$.*

Proof. If $\tau_z > t$ then $\tau_w > t$ in a neighbourhood of z by continuity. Conversely, suppose $z \in \mathbb{R} \setminus \{U_0\}$, WLOG $z > U_0$. By the local growth property, $z \notin \overline{A_{t+\delta} \cap \mathbb{H}}$ for some $\delta > 0$.

Let $\varepsilon > 0$ be such that $B(z, \varepsilon) \cap \mathbb{H} \subseteq H_{t+\delta}$. The Schwarz reflection g_s^* of g_s is defined and univalent on $B(z, \tau)$ for all $s \leq t$. Hence $g_s^*(z) \neq U_s$, otherwise there would be some $w \in A_{s+\delta} \setminus A_s$ with $w \in B(z, \varepsilon) \cap \mathbb{H}$. Also $g_s(w_n) \to g_s^*(z)$ as $w_n \to z$, $w_n \in \mathbb{H}$.

Taking limits in the Loewner differential equation for w_n implies $s \mapsto g_s^*(z)$ satisfies the Loewner differential equation on $[0, t]$ and $\tau_z > t$. $\square$