1 Motivation

This section is motivation and will not be rigorous. We have a 'Dirac delta function' such that for all 'nice' functions f

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \mathrm{d}x = f(x_0).$$

Can we define $\delta'(x - x_0)$? Could try

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x &= \lim_{h \to 0} \int_{-\infty}^{\infty} \left[\frac{\delta(x - x_0 + h) - \delta(x - x_0)}{h} \right] f(x) \\
&= \lim_{h \to 0} \frac{1}{h} \left[f(x_0 - h) - f(x_0) \right] \\
&= -f'(x_0).
\end{aligned}$$

i.e

$$\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x = - \inf_{-\infty}^{\infty} \delta(x - x_0) f'(x) \mathrm{d}x$$

which looks like some form of integration-by-parts. We can make this rigorous using distribution theory.

Fourier transform of polynomials

If $f \in L^1(\mathbb{R})$ then

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) \mathrm{d}x.$$

How could we take the Fourier transform of something like $f(x) = x^n$? May recall identity

$$\delta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x$$

and then get

$$\begin{aligned}
\hat{f}(\lambda) &= \int_{-\infty}^{\infty} x^n e^{-\lambda x} \mathrm{d}x \\
&= \left(i \frac{\partial}{\partial \lambda} \right)^n \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x \\
&= i^n 2\pi \delta^{(n)}(\lambda).
\end{aligned}$$

Recall Parseval's theorem: for suitable f, g

$$\int_{-\infty}^{\infty} \hat{g}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} g(x) \hat{f}(x) \mathrm{d}x.$$

Could define the Fourier transform of $g(x) = x$ to be the function $\lambda \mapsto \hat{x}(\lambda)$ such that

$$\int_{-\infty}^{\infty} \hat{x}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} x \hat{f}(x) \mathrm{d}x$$

for all 'nice' functions f. We can make this rigorous using distributions.

Discontinuous solutions to PDEs

From linear acoustics, air pressure $p = p(x,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0. \qquad (*)$$

Could introduce a 'nice' $f = f(x,t)$, say $f \in C_c^\infty(\mathbb{R}^2)$. Then $(*)$ implies

$$\int\int (p_{xx} - p_{tt}) f(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

So we can integrate by parts to interpret this as

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

We say that $p = p(x,t)$ is a *weak solution* to $(*)$ if

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. In each case, to extend a definition to a larger domain of applicability, we had to introduce a space of "nice" functions.

This is the theme of distribution theory: functions get replaced by linear maps on some auxilliary space of test functions V. A *distribution* is a linear map $u : V \to \mathbb{C}$, i.e we study the topological dual of V. Let $\langle \cdot, \cdot \rangle$ denote pairing between v and V^*, i.e for $u \in V^*$, $f, g \in V$, $\alpha, \beta \in \mathbb{C}$

$$\langle u, \alpha f + \beta g \rangle = \alpha \langle u, f \rangle + \beta \langle u, g \rangle.$$

The topological dual V^* consists of linear $u : V \to \mathbb{C}$ such that whenever $f_n \to f$ in V, we have $\langle u, f_n \rangle \to \langle u, f \rangle$ in $\mathbb{C}$. For example we could take $V = C^\infty(\mathbb{R})$ equipped with the topology of uniform convergence (i.e $f_n \to f$ in V if for all compact $K \subseteq \mathbb{R}$ and all $n \geq 0$, $\left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (f_n - f) \right| \to 0$) then $\delta_{x_0} : V \to bbC$ defined by $\langle \delta_{x_0}, f \rangle = f(x_0)$. Note that this is indeed continuous.

2 Distributions

2.1 Notation & Preliminaries

Throughout (unless otherwise specified) X, Y denote open subsets of $\mathbb{R}^n$, K a compact subset of $\mathbb{R}^n$. Integrals over $X, \mathbb{R}^n$ are written as $\int_X [\cdot]\mathrm{d}x$, $\int [\cdot]\mathrm{d}x$ respectively.

2.2 Distributions & Test Functions

Definition. The space $\mathcal{D}(X)$ consists of smooth functions $\varphi : X \to \mathbb{C}$ of compact support. We say a sequence $(\varphi_m)_{m\geq 0}$ in $\mathcal{D}(X)$ converges to 0 in $\mathcal{D}(X)$ if there exists $K \subseteq X$ compact such that $\operatorname{supp}(\varphi_m) \subseteq K$ and $\sup_K |\partial^\alpha \varphi_m| \to 0$ for all multi-indices α.

Functions in $\mathcal{D}(X)$ have nice properties. For example, if $\varphi \in \mathcal{D}(X)$ then $\varphi = 0$ before you reach the boundary of X. This means integration-by-parts is easy since

$$\int_X \varphi \partial^\alpha \psi \mathrm{d}x = (-1)^{|\alpha|} \int_X \psi \partial^\alpha \varphi \mathrm{d}x.$$

Since $\varphi \in \mathcal{D}(X)$ is smooth we have

$$\varphi(x+h) = \sum_{|\alpha|\leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h)$$

where R_N is $o(|h|^N)$ uniformly in x.

Definition. A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is called a *distribution* if for all $K \subseteq X$ compact there exist $C, N \geq 0$ such that

$$|u(\varphi)| \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha \varphi| \tag{$*$}$$

for all $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$. The space of such linear maps is denoted by $\mathcal{D}'(X)$, i.e "distributions on X". If the same N can be used in $(*)$ for all compact $K \subseteq X$, say the least such N is the order of u, written $\operatorname{ord}(u)$.

For $x_0 \in X$ define $\delta_{x_0}(\varphi) = \varphi(x_0)$ for $\varphi \in \mathcal{D}(X)$. Then $\delta_{x_0} : \mathcal{D}(X) \to \mathbb{C}$ is linear and

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \sup |\varphi|$$

so we can take $C = 1, N = 0$ in $(*)$, so $\operatorname{ord}(\delta_{x_0}) = 0$.

For $\{f_\alpha\}$ in $C(X)$, define $T : \mathcal{D}(X) \to \mathbb{C}$ by

$$T(\varphi) = \sum_{|\alpha|\leq M} \int_X f_\alpha \partial^\alpha \varphi \mathrm{d}x.$$

Take $\varphi \in \mathcal{D}(X)$ with $\text{supp}(\varphi) \subseteq K$. Then

$$|T(\varphi)| \leq \sum_{|\alpha|\leq M} \int_K |f_\alpha||\partial^\alpha \varphi|\mathrm{d}x$$
$$\leq \left(\max_\alpha \int_K |f_\alpha|\mathrm{d}x\right) \sum_{|\alpha|\leq M} \sup |\partial^\alpha \varphi|$$

so $(*)$ holds with $C = \max_\alpha \int_K |f_\alpha|\mathrm{d}x$, $N = M$. Hence $T \in \mathcal{D}'(X)$.

Note this estimate would hold if the $\{f_\alpha\}$ were only assumed locally integrable, written $f_\alpha \in L^1_{\text{loc}}(X)$.

Remark. For $f \in L^1_{\text{loc}}$ we have a corresponding distribution $T_f : \mathcal{D}(X) \to \mathbb{C}$ defined by $T_f(\varphi) = \int_X f\varphi\mathrm{d}x$. We often simply write $T_f = f$.

Lemma. *A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is a distribution if and only if $u(\varphi_m) \to 0$ for all sequence $\varphi_m \to 0$ in $\mathcal{D}(X)$.*

Proof. Suppose $u \in \mathcal{D}'(X)$ and $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then $\text{supp}(\varphi_m) \subseteq K$ for some K independent of m and there exist $C, N \geq 0$

$$|\varphi_m(u)| \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha \varphi_m| \to 0$$

for all α.

Suppose not, i.e $u : \mathcal{D}(X) \to \mathbb{C}$ is linear and $u(\varphi_m) \to 0$ whenever $\varphi_m \to 0$ in $\mathcal{D}(X)$, but u is not a distribution. Then there is a compact set $K \subseteq X$ such that for all C, N, $(*)$ fails on some φ with support contained in K. So there must be some $\varphi_m \in \mathcal{D}(X)$ with $\text{supp}(\varphi_m) \subseteq K$ and

$$|u(\varphi_m)| > m \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Now replace φ_m with $\varphi'_m = \frac{\varphi_m}{u(\varphi_m)}$. So we may assume $u(\varphi_m) = 1$ WLOG. Hence

$$1 > m \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Therefore $\sup_K |\partial^\alpha \varphi_m| < \frac{1}{m}$ for all $|\alpha| \leq m$. Hence $\varphi_m \to 0$ in $\mathcal{D}(X)$, giving a contradiction since $u(\varphi_m) \not\to 0$. □

2.3 Limits in $\mathcal{D}'(X)$

We often have some sequence (u_m) in $\mathcal{D}'(X)$. If there is some $u \in \mathcal{D}'(X)$ such that $\varphi(u_m) \to \varphi(u)$ for all φ we say $u_m \to u$ in $\mathcal{D}'(X)$.

Theorem (*Non-examinable*)**.** *If* (u_m) *is a sequence in* $\mathcal{D}'(X)$ *and* $u(\varphi) = \lim_{m\to\infty} u(\varphi_m)$ *exists for all* $\varphi \in \mathcal{D}(X)$*, then* $u \in \mathcal{D}'(X)$*.*

Proof. Not given. □

Take $u_m \in \mathcal{D}'(\mathbb{R})$ defined by $u_m(\varphi) = \int \sin(mx)\varphi(x)\mathrm{d}x$. By integration-by-parts we have

$$|\varphi(u_m)| = \left|\frac{1}{m}\int \cos(mx)\varphi'(x)\mathrm{d}x\right| \to 0.$$

i.e $\sin(mx) \to 0$ in $\mathcal{D}'(\mathbb{R})$.