1 Lebesgue Integration Theory

1.1 Review of measure theory

Definition. Given a set E, a *σ-algebra* on E is a collection $\mathcal{E}$ of subsets of E such that:

(i) $E \in \mathcal{E}$;

(ii) $A \in \mathcal{E} \Rightarrow A^c = E \setminus A \in \mathcal{A}$;

(iii) $A_n \in \mathcal{E}$, $n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$.

$(E, \mathcal{E})$ is called a *measurable space*, and any $A \in \mathcal{E}$ is called a *measurable set*.

Given a collection $\mathcal{A}$ of subsets of E, $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$.

Definition. A *measure* on $(E, \mathcal{E})$ is a function $\mu : \mathcal{E} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$;

(ii) $A_n \in \mathcal{E}$, $n \in \mathbb{N}$ disjoint $\Rightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

$(E, \mathcal{E}, \mu)$ is called a *measure space*.

Definition (Borel measure)**.** If (E, τ) is a topological space, then $\sigma(\tau)$ is called a *Borel algebra*, denoted $\mathcal{B}(E)$, and a measure on $(E, \mathcal{B}(E))$ is called a *Borel measure*.

Example. $E = \mathbb{R}^n$, μ the Lebesgue measure satisfying $\mu((a_1, b_1) \times \ldots \times (a_n, b_n)) = (b_1 - a_1) \ldots (b_n - a_n)$.

Notation: we write $\mu(\mathrm{d}x) = \mathrm{d}x$ and $\mu(A) = |A|$ when μ is the Lebesgue measure.

Definition (Measurable function)**.** Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Then $f : E \to F$ is *measurable* if $f^{-1}(A) \in \mathcal{E}$ whenever $A \in \mathcal{F}$. If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are Borel algebras, a measurable function is called a *Borel function*. Special case: $(F, \mathcal{F}) = ([0, \infty], \mathcal{B}([0, \infty]))$, then $f : E \to F$ is called a *non-negative measurable function*.

Fact. The class of measurable functions is closed under addition, multiplication and taking (pointwise) limits.

Definition. $f : E \to F$ ($F = [0, \infty]$ or $\mathbb{R}^n$ or $\mathbb{C}^n$) is a *simple function* if $f = \sum_{k=1}^{K} a_k \mathbb{1}_{A_k}$ for some $K \in \mathbb{N}$, $a_k \in F$, $A_k \in \mathcal{E}$. For a simple function, the integral is

$$\int f \mathrm{d}\mu = \int f(x) \mathrm{d}\mu(x) = \sum_{k=1}^{K} a_k \mu(A_k) \ (0 \cdot \infty := 0).$$

For a non-negative measurable f, we define

$$\int f \mathrm{d}\mu = \sup \left\{ \int g \mathrm{d}\mu : g \text{ simple }, 0 \leq g \leq f \right\}.$$

Definition. A measurable function $f : E \to \mathbb{R}$ is said to be *integrable* if $\int |f| \mathrm{d}\mu < \infty$. Write $f = f_+ - f_-$ with $f_\pm$ non-negative, measurable, $\int f_\pm \mathrm{d}\mu < \infty$, and then $\int f \mathrm{d}\mu = \int f_+ \mathrm{d}\mu - \int f_- \mathrm{d}\mu$. For $f : E \to \mathbb{R}^n$, this is applied in each component.

Theorem (Monotone convergence theorem)**.** *Let $(E, \mathcal{E}, \mu)$ be a measure space, and let (f_n) be a (pointwise) increasing sequence of non-negative functions on E converging to f. Then*

$$\lim_{n\to\infty} \int_E f_n \mathrm{d}\mu = \int_E f \mathrm{d}\mu.$$

Theorem (Dominated convergence theorem)**.** *Let (f_n) be a sequence of measurable functions on a measure space $(E, \mathcal{E}, \mu)$ such that:*

(i) $f_n \to f$ pointwise almost everywhere;

(ii) $|f_n| \leq g$ almost everywhere for some integrable g.

Then

$$\lim_{n\to\infty} \int_E f_n \mathrm{d}\mu = \int_E f \mathrm{d}\mu.$$

1.2 L^p spaces

Definition. Let $(E, \mathcal{E}, \mu)$ be a measure space. For $p \in [1, \infty)$ and $f : E \to \mathbb{R}$ define

$$||f||_{L^p} = \left(\int_E |f|^p \mathrm{d}\mu \right)^{1/p}$$

and

$$||f||_{L^\infty} = \mathrm{esssup}|f| = \inf\{K : |f| \leq K \text{ a.e}\}.$$

The space L^p, $p \in [1, \infty]$ is defined by

$$L^p = L^p(E, \mathcal{E}, \mu) = \{f : E \to \mathbb{R} \text{ measurable} : ||f||_{L^p} < \infty\}/\sim .$$

Where $f \sim g$ if $f = g$ a.e.

Theorem (Riesz-Fisher theorem)**.** *L^p is a Banach space for all $p \in [1, \infty]$.*

Notation: when $E = \mathbb{R}^n$, μ the Lebesgue measure, write $L^p(E, \mu) = L^p(\mathbb{R}^n)$.

Fact. For $p \in [1, \infty)$, the simple functions f with $\mu(\{x : f(x) \neq 0\}) < \infty$ are dense in L^p. For $p = \infty$ we can drop the condition on the measure of the support.

Theorem. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$.*

Remark. This theorem is false for $p = \infty$.