Introduction

Quadratics (Babylonians):

$$X^2 + bX = c = (X + \frac{1}{2}b)^2 + c - \frac{b^2}{4}$$

$$= (X - x_1)(X - x_2) \implies x_1 x_2 = c, x_1 + x_2 = -b$$

$$x_1 = \frac{1}{2}\left[(x_1 + x_2) + (x_1 - x_2)\right] = \frac{1}{2}\left[-b + \sqrt{b^2 - 4c}\right]$$

Cubics (Italy, 16th Century):

$$X^3 + aX^2 + bX + c = (X - x_1)(X - x_2)(X - x_3)$$

$$\implies x_1 + x_2 + x_3 = -a, x_1x_2 + x_1x_3 + x_2x_3 = b, x_1x_2x_3 = -c$$

WLOG $X \to X - a/3$ and $a = 0$

$$x_1 = \frac{1}{3}\left[(x_1 + x_2 + x_3) + \underbrace{(x_1 + \omega x_2 + \omega^2 x_3)}_{=u} + \underbrace{(x_1 + \omega^2 x_2 + \omega x_3)}_{=v}\right]$$

where $\omega = e^{2\pi i/3}$ so $\omega^2 + \omega + 1 = 0$. Cyclic permutation of x_1, x_2, x_3 gives $u \to \omega u \to \omega^2 u$ and $v \to \omega v \to \omega^2 v$ which implies u^3 and v^3 are invariant under cyclic permutations of the roots.

Also $u \leftrightarrow v$ under $x_2 \leftrightarrow x_3$. So $u^3 + v^3$, $u^3 v^3$ are invariant under permutations of roots.

In fact,

$$u^3 + v^3 = 27x_1x_2x_3 = -27c$$

$$u^3 v^3 = -27b^2$$

So u^3, v^3 are roots of $Y^2 + 27cY - 27b^2$. This gives a formula for x_1 (Cardano's formula).

Can follow a similar method for quartics - auxilliary cubic equation. Unfortunately it doesn't work for quintics - the reason being <u>group theory</u>.

1 Polynomials

In this course, all rings are commutative and non-zero. Let R be a ring, then $R[X]$ denotes the ring of polynomials $\sum_{i=0}^{n} a_i X^i$, $a_i \in R$. A polynomial $f \in R[X]$ determines a function $R \to R$, $r \mapsto f(r)$.

The polynomial is not in general determined by this function, e.g let $R = \mathbb{Z}/p\mathbb{Z}$ (p prime). Then for all $a \in R$, $a^p = a$ so the polynomials X^p and X represent the same function.

In the case when $R = K$ (a field), $K[X]$ is a Euclidean domain. The "division algorithm" says that if $f, g \in K[X]$, $g \neq 0$ then there exists unique $q, r \in K[X]$ such that $f = gq + r$ and $\deg r < \deg g$ (define $\deg(0) = -\infty$).

In particular, if $g = X - a$ is linear then $f = (X - a)q + f(a)$ ("remainder theorem"). So $K[X]$ is also a PID and a UFD - every polynomial is a product of irreducible polynomials, and there are GCD's, computable via Euclids algorithm in the usual way.

Proposition 1.1. *If K is a field, $0 \neq f \in K[X]$, then f has at most $\deg f$ roots in K.*

Proof. If f has no roots then we are done. Otherwise, suppose $f(a) = 0$ for $a \in K$. Then

$$f = (X - a)g$$

for some $g \in K[X]$ and $\deg g = \deg f - 1$. If $b \in K$ is a root of f then either $b = a$ or $g(b) = 0$ so the number of roots of f is at most one more than the number of roots of g. Now done by induction. □

2 Symmetric polynomials

Let R be a ring, consider $R[X_1, \ldots, X_n]$ for $n \geq 1$.

Definition. A polynomial $f \in R[X_1, \ldots, X_n]$ is *symmetric* if for every $\sigma \in S_n$, $f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = f$.

The set of symmetric polynomials is a subring of $R[X_1, \ldots, X_n]$.

Example. $X_1 + \ldots + X_n$, or more generally, $p_k = X_1^k + \ldots + X_n^k = \sum_{i=1}^n X_i^k$.

Alternative definition: if $f \in R[X_1, \ldots, X_n]$, define $f\sigma = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$. This is an action (on the right) of S_n on $R[X_1, \ldots, X_n]$. A polynomial f is symmetric if and only if it is fixed by this action.

Definition. *The elementary symmetric polynomials* are

$$s_r(X_1, \ldots, X_n) = \sum_{1 \leq i_1 < \ldots < i_r \leq n} X_{i_1} X_{i_2} \ldots X_{i_r}$$

Example. When $n = 3$ we have

$$\begin{aligned} s_1 &= X_1 + X_2 + X_3 \\ s_2 &= X_1X_2 + X_1X_3 + X_2X_3 \\ s_3 &= X_1X_2X_3 \end{aligned}$$

Theorem 2.1.

(i) *Every symmetric polynomial over* R *can be expressed as a polynomial in* $\{s_r : 1 \leq r \leq n\}$*, with coefficients in* R*.*

(ii) *There are no non-trivial relations between* $s_1, \ldots, s_n$*.*

Remark:

(a) Consider the ring homomorphism

$$\theta : R[Y_1, \dots, Y_n] \to R[X_1, \dots, X_n],\ Y_r \mapsto s_r$$

then (i) says the image of θ is the set of symmetric polynomials. (ii) says that θ is injective.

(b) Equivalent definition of the s_r's is

$$\prod_{i=1}^{n}(T + X_i) = T^n + s_1 T^{n-1} + \ldots + s_{n-1}T + s_n$$

If we need to specify the number of variables, write $s_{r,n}$ instead of s_r.

Proof. Terminology:

- A *monomial* is some $X_I = X_1^{i_1} \dots X_n^{i_n}$ for $I \in \mathbb{N}^n = \{0, 1, 2, \dots\}^n$. Its (total) degree is $\sum_\alpha i_\alpha$.
- A *term* is some cX_I, for $0 \neq c \in R$. So a polynomial is uniquely a sum of terms.
- *Total degree* of f is the maximum degree over its terms

Lexicographical ordering on monomials X_I: write $X_I > X_J$ if either $i_1 > j_1$ or, for some $1 \leq r < n$, $i_1 = j_1, \dots, i_r = j_r$ and $i_{r+1} > j_{r+1}$.

This is a total ordering: for each pair $I \neq J$, exactly one of $X_I > X_J$ or $X_J > X_I$ holds.

First we prove (ii):

Let d be the total degree of some symmetric polynomial f, and let X_I be the largest (in lexicographical order) monomial which occurs in f, with coefficient $c \in R$. As f is symmetric, we must have $i_1 \geq i_2 \geq \ldots \geq i_n$ (otherwise we could exchange variables to get a larger monomial).

So

$$X_I = X_1^{i_1 - i_2} (X_1 X_2)^{i_2 - i_3} \dots (X_1, \dots X_n)^{i_n}$$

consider

$$g = s_1^{i_1 - i_2} s_2^{i_2 - i_3} \dots s_{n-1}^{i_{n-1} - i_n} s_n^{i_n}$$

the leading monomial (i.e largest in lexicographical order) of g is X_I, and g is symmetric. So $f - cg$ is symmetric of total degree $\leq d$, and its leading monomial term is smaller (lexicographical) than X_I. As the set of monomials of degree at most d is finite, this process terminates.

To prove (ii): induct on n. Suppose we have $G \in R[Y_1, \ldots, Y_n]$ with $G(s_{n,1}, \ldots, s_{n,n}) = 0$. We want to show $G = 0$. If $n = 1$, this is trivial ($s_{1,1} = X_1$). If $G = Y_n^k H$, with $Y_n \nmid H$, then $s_{n,n}^k H(s_{n,1}, \ldots, s_{n,n}) = 0$. As $s_{n,n} = X_1 \ldots X_n$, $s_{n,n}$ is not a zero divisor in $R[X_1, \ldots, X_n]$ so $H(s_{n,1}, \ldots, s_{n,n}) = 0$.

So we may assume G is not divisible by Y_n. Replace X_n by 0. Then

$$s_{n,r}(X_1, \ldots, X_{n-1}, 0) = \begin{cases} s_{n-1,r}(X_1, \ldots, X_{n-1}) & \text{if } r < n \\ 0 & \text{if } r = n \end{cases}$$

and so $G(s_{n-1,1}, \ldots, s_{n-1,n-1}, 0) = 0$. So by induction, $G(Y_1, \ldots, Y_{n-1}, 0) = 0$, i.e $Y_n \mid G$, a contradiction.

□

Example. $f = \sum_{i \neq j} X_i^2 X_j$ for $n \geq 3$. The leading term is $X_1^2 X_2 = X_1(X_1 X_2)$. Then compute

$$s_1 s_2 = \sum_i \sum_{j<k} X_i X_j X_k = \sum_{i \neq j} X_i^2 X_j + 3 \sum_{i<j<k} X_i X_j X_k$$

so $f = s_1 s_2 - 3 s_3$.

Computing say $\sum X_i^5$ by hand is tedious. But there are alternative formulae.

Recall $p_k = \sum_{i=1}^n X_i^k$ for $k \geq 1$.

Theorem 2.2 (Newton's formulae)**.** *Let $n \geq 1$. Then for all $k \geq 1$*

$$p_k - s_1 p_{k-1} + \ldots + (-1)^{k-1} s_{k-1} p_1 + (-1)^k k s_k = 0$$

by convention, $s_0 = 1$, and $s_r = 0$ if $r > n$.

Proof. We may assume $R = \mathbb{Z}$ (or $\mathbb{R}$). Generating function

$$F(T) = \prod_{i=1}^n (1 - X_i T) = \sum_{r=0}^n (-1)^r s_r T^r$$

Take logarithmic derivative with respect to T:

$$\frac{F'(T)}{F(T)} = \sum_{i=1}^n \frac{-X_i}{1 - X_i T} = -\frac{1}{T} \sum_{i=1}^n \sum_{r=1}^\infty X_i^r T^r = -\frac{1}{T} \sum_{r=1}^\infty p_r T^r$$

So

$$-TF'(T) = s_1 T - 2 s_2 T^2 + \ldots + (-1)^{n-1} n s_n T^n$$

$$= F(T) \sum_{r=1}^\infty p_r T^r = (s_0 - s_1 T + \ldots + (-1)^n s_n T^n) \left(p_1 T + p_2 T^2 + \ldots\right)$$

comparing coefficients of T^k gives the result. □

Definition. The *discriminant polynomial* is

$$D(X_1, \ldots, X_n) = \Delta(X_1, \ldots, X_n)^2$$

where $\Delta = \prod_{i<j}(X_i - X_j)$. (Recall from IA Groups that applying $\sigma \in S_n$ to Δ multiplies Δ by $\mathrm{sgn}(\sigma)$, so D is symmetric.)

So $D(X_1, \ldots, X_n) = d(s_1, \ldots, s_n)$ for some polynomial d ($\mathbb{Z}$-coefficients). For example, when $n = 2$, $D = (X_1 - X_2)^2 = s_1^2 - 4s_2$.

Definition. Let $f = T^n + \sum_{i=0}^{n-1} a_{n-i}T^i \in R[T]$. Its *discriminant* is $\mathrm{Disc}(f) = d(-a_1, a_2, -a_3, \ldots, (-1)^n a_n) \in R$.

Observe that if $f = \prod_{i=1}^n (T - x_i)$, $x_i \in R$, then $a_r = (-1)^r s_r(x_1, \ldots, x_n)$, so

$$\mathrm{Disc}(f) = \prod_{i<j}(x_i - x_j)^2 = D(x_1, \ldots, x_n)$$

If moreover $R = K$ is a field, then $\mathrm{Disc}(f) = 0$ iff f has a repeated root (i.e $x_i = x_j$ for some $i \neq j$). E.g when $n = 2$, $\mathrm{Disc}(T^2 + bT + c) = b^2 - 4c$.

3 Fields

Recall:

Definition. A *field* is a ring K (commutative with a 1) in which every non-zero element has a multiplicative inverse. The set of non-zero elements of K is a group under multiplication, written $K^\times$ or K^*, called the *multiplicative group of* K.

Definition. The *characteristic of a field* K is the least positive integer p (if it exists) such that $p \cdot 1_K = 0_K$, or is said to be 0 if no such p exists.

Example. $\mathbb{Q}$ has characteristic 0 and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ has characteristic p (p prime).

The characteristic $\mathrm{char}(K)$ of K is either 0 or a prime. Inside K, there is a smaller subfield, called the *prime subfield* of K. It is either isomorphic to $\mathbb{Q}$ (if characteristic is 0), or to $\mathbb{F}_p$ (if $\mathrm{char}(K) = p$).

Proposition 3.1. *Let $\varphi : K \to L$ be a homomorphism of fields. Then φ is an injection.*

Proof. $\varphi(1_K) = 1_L \neq 0$, so $\mathrm{Ker}(\varphi) \subsetneq K$ is a proper ideal of K, so $\mathrm{Ker}(\varphi) = (0)$ □

Definition. Let $K \subseteq L$ be fields (where the field operations on K are the same as those on L). We say K is a *subfield of* L, and L is *an extension of* K, denoted L/K.

Remarks:

(i) The notation L/K has nothing to do with the quotient (some write $L \mid K$)

(ii) It is useful to be more general - if $i : K \to L$ is a homomorphism of fields, then Proposition 3.1 says that K is isomorphic to its image $i(K) \subseteq L$. In this situation, also say L is an extension of K.

Example. Some extensions include

- $\mathbb{C}/\mathbb{R}$
- $\mathbb{R}/\mathbb{Q}$
- $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}/\mathbb{Q}$

Definition. $K \subseteq L$, $x \in L$. Define $K[x] = \{p(x) : p \in K[T]\}$ (a subring of L). Define $K(x) = \{\frac{p(x)}{q(x)} : p, q \in K[T], q(x) \neq 0\}$ (a subfield of L) "K adjoin x". For $x_1, \ldots, x_n \in L$, define

$$K(x_1, \ldots, x_n) = \left\{ \frac{p(x_1, \ldots, x_n)}{q(x_1, \ldots, x_n)} : p, q \in K[T_1, \ldots, T_n], q(x_1, \ldots, x_n) \neq 0 \right\}$$

(Easy to check $K(x_1, \ldots, x_{n-1})(x_n) = K(x_1, \ldots, x_n)$). Likewise $K[x_1, \ldots, x_n]$ is defined analagously.

Definition. Suppose L/K is a field extension. Then L is naturally a vector space over its subfield K (forget multiplication by elements of L). We can ask if it is a finite-dimensional vector space, if so we say that L/K is a *finite extension* and write $[L : K] = \dim_K(L)$ for the dimension. The dimension is called the *degree of the extension* L *over* K. If the dimension is infinite write $[L : K] = \infty$.

$\dim_K$ denotes the dimension as a K-vector space. Of course L has dimension 1 over itself. As a K-vector space, $L \cong K^{[L:K]}$.

Example.

(i) $\mathbb{C}/\mathbb{R}$, $[\mathbb{C} : \mathbb{R}] = 2$

(ii) For any field K, $K(X)$ = field of rational functions in X = field of fractions of polynomial ring $K[X] = \{\frac{p}{q} : p, q \in K[X], q \neq 0\}$. Then $[K(X) : K] = \infty$ since $1, X, X^2, \ldots$ are linearly independent.

(iii) $\mathbb{R}/\mathbb{Q}$, $[\mathbb{R} : \mathbb{Q}] = \infty$. This follows from countability - every finite dimensional vector space over $\mathbb{Q}$ is countable.

This course is largely about properties (and symmetries) of finite extensions of fields.

Definition. We say an extension L/K is *quadratic* (*cubic*,...) if $[L : K] = 2(, 3, \ldots)$

Proposition 3.2. *Suppose K is a finite field (necessarily of characteristic $p > 0$). Then $|K|$ is a power of p.*

Proof. Certainly $K/\mathbb{F}_p$ is finite, so $K \cong (\mathbb{F}_p)^n$ (as a vector space), where $n = [K : \mathbb{F}_p]$, so $|K| = p^n$. □

Later on we will see that every prime power $q = p^n$ admits a field $\mathbb{F}_q$ with q elements.

Here is a simple but powerful fact:

Theorem 3.3 ("Tower Law"). *Suppose M/L and L/K are field extensions. Then M/K is a finite extension if and only if both M/L and L/K are finite. If so, then $[M : K] = [M : L][L : K]$.*

In fact, a slightly more general statement holds:

Theorem 3.4. *Let L/K be an extension, V an L-vector space. Then $\dim_K(V) = [L : K] \dim_L(V)$ (and obvious conclusions if any quantities are infinite).*

Example. If $V = \mathbb{C}^n$ then $V \cong \mathbb{R}^{2n}$.

Proof. Let $\dim_L(V) = d < \infty$. Then $V \cong L \oplus \ldots \oplus L = L^d$ as an L-vector space, so also as a K-vector space. If $[L : K] = n < \infty$, then $L \cong K^n$ as a K-vector space, so

$$V \cong \underbrace{K^n \oplus \ldots \oplus K^n}_{d \text{ times}} = K^{nd}$$

so $\dim_K(V) = [L : K] \dim_L(V)$. If V is finite-dimensional over K, then a K-basis for V certainly spans V over L. So if $\dim_L(V) = \infty$ then $\dim_K(V) = \infty$. Likewise, if $[L : K] = \infty$ and $V \neq \{0\}$, then V has an infinite linearly independent subset, so $\dim_K(V) = \infty$. □

Another important fact:

Proposition 3.5.

(i) Let K be a field, $G \subseteq K^\times$ a finite subgroup. Then G is cyclic

(ii) If K is finite, then $K^\times$ is cyclic

Proof. We prove (i) ((ii) follows immediately): (recall from IB GRM) we can write

$$G \cong \frac{\mathbb{Z}}{m_1\mathbb{Z}} \oplus \ldots \oplus \frac{\mathbb{Z}}{m_k\mathbb{Z}}$$

where $1 < m_1 \mid m_2 \mid \ldots \mid m_k = m$. So for all $x \in G$, $x^m = 1$. As K is a field, the polynomial $T^m - 1$ has at most m roots. So $|G| < m$. Hence $k = 1$ and G is cyclic.

□

Remark: Let $K = F = \mathbb{Z}/p\mathbb{Z}$. The above says there exists $a \in \{1, \ldots, p-1\}$ such that $\mathbb{Z}/pZ = \{0\} \cup \{a, a^2, \ldots, a^{p-1}\}$. a is called a primitive root modulo p.

Proposition 3.6. *Let R be a ring, p a prime such that $p \cdot 1_R = 0_R$ (e.g R a field of characteristic p). Then the map*

$$\varphi_p : R \to R,\ \varphi_p(x) = x^p$$

is a homomorphism from R to itself (called the Frobenius endomorphism of R).

Proof. Have to show:

- $\varphi_p(1) = 1$
- $\varphi_p(xy) = \varphi_p(x)\varphi_p(y)$
- $\varphi_p(x + y) = \varphi_p(x) + \varphi_p(y)$

The first two are obvious. For the last one,

$$\begin{aligned}\varphi_p(x + y) &= x^p + \sum_{i=1}^{p-1} \underbrace{\binom{p}{i}}_{\equiv 0 \pmod p} x^i y^{p-i} + y^p \\ &= \varphi_p(x) + \varphi_p(y)\end{aligned}$$

□

Example. This gives another proof of Fermat's Little Theorem: $x^p \equiv x \pmod p$ (induction on x: $(x + 1)^p = x^p + 1$).

4 Algebraic elements and extensions

Definition. Have L/K an extension, $x \in L$. We say x is *algebraic over K* if there exists a non-zero polynomial $f \in K[T]$ such that $f(x) = 0$. Otherwise we say x is *transcendental over K*.

Suppose $f \in K[T]$; evaluation $f(x) \in L$. This gives a map $\mathrm{ev}_x : K[T] \to L$, $f \mapsto f(x)$. This is obviously a homomorphism of rings.

$I = \mathrm{Ker}(\mathrm{ev}_x) \subseteq K[T]$ is an ideal (the set of polynomials which vanish at x). As $\mathrm{Im}(\mathrm{ev}_x)$ is a subring of L, it is an integral domain. So I is a prime ideal. Two possibilities:

(i) $I = \{0\}$. Then the only f with $f(x) = 0$ is $f = 0$. Hence x is transcendental over K.

(ii) $I \neq \{0\}$. As $K[T]$ is a PID, there exists a unique monic irreducible $g \in K[T]$ such that $I = (g)$. So $f(x) = 0$ if and only if f is a multiple of g. So x is algebraic over K; we call g the *minimal polynomial* of x over K. written $m_{x,K}$. It is the unique monic irreducible polynomial such that x is a root (and the monic polynomial of least degree with this property). [Depends on K as well as x]

Example.

- $x \in K$, $m_{x,K} = T - x$
- p prime, $d \geq 1$. Then $T^d - p \in \mathbb{Q}[T]$ is irreducible (Eisenstein's criterion) so it is the minimal polynomial of $\sqrt[d]{p} = x \in \mathbb{R}$ over $\mathbb{Q}$.
- $z = e^{2\pi i/p}$ (p prime) is a root of $T^p - 1$ and of $\frac{T^p-1}{T-1} = g(T) = T^{p-1} + \ldots + T + 1 \in \mathbb{Q}[T]$. As

$$g(T+1) = \frac{(T+1)^p - 1}{T} = T^{p-1} + \binom{p}{1}T^{p-2} + \ldots + \binom{p}{2}T = \binom{p}{1}$$

  which is irreducible by Eisenstein, so g is irreducible and g is the minimal polynomial of z over $\mathbb{Q}$.

Definition. The *degree of* x *over* K (x algebraic over K) is the degree of $m_{x,K}$, written $\deg_K(x)$ or $\deg(x/K)$.

Ring/field characterisation of algebraicity:

Proposition 4.1. *Let L/K be a field extension, $x \in L$. The following are equivalent*

(i) *x is algebraic over K*

(ii) $[K(x) : K] < \infty$

(iii) $\dim_K K[x] < \infty$

(iv) $K[x] = K(x)$

(v) *$K[x]$ is a field*

If these hold, then $\deg_K(x) = [K(x) : K]$.

Note: recall $K[x] = \{p(x)\}$, $K(x) = \left\{\frac{p(x)}{q(x)} | q(x) \neq 0, p, q \in K[T]\right\}$.

Proof. (ii) $\iff$ (iii), (iv) $\iff$ (v) are obvious.

Show (iii) $\Rightarrow$ (v),(iv) and (ii): let $0 \neq y = g(x) \in K[x]$. Consider $K[x] \to K[x]$, $z \mapsto yz$. It is a K-linear transformation, injective as $y \neq 0$, and since $\dim_K K[x] < \infty$, it is a bijection. So there exists z such that $yz = 1$. So $K[x]$ is a field, equal to $K(x)$ and $[K(x) : K] < \infty$.

Show (v)$\Rightarrow$(i): wlog $x \neq 0$, then $x^{-1} = a_0 + a_1 x + \ldots + a_n x^n \in K[x]$. Then $a_n x^{n-1} + \ldots + a_0 x - 1 = 0$, so x is algebraic over K.

Show (i)$\Rightarrow$(iii) and degree formula: The image of $\mathrm{ev}_x : K[T] \to L$ is $K[x] \subseteq L$. x is algebraic over K so the kernel of this map is $(m_{x,K})$, which is a maximal ideal ($m_{x,K}$ is irreducible). Applying the first isomorphism theorem gives

$\underbrace{K[T]/(m_{x,K})}_{\text{field}} \cong K[x]$. $m_{x,K}$ is monic of degree $d = \deg_K(x)$. So $K[T]/(m_{x,K})$ has basis $1, T, \ldots, T^{d-1}$. So $\dim_K K[x] = d < \infty$. Furthermore $\deg_K(x) = [K(x) : K] = d$. □

Corollary 4.2.

(i) *$x_1, \ldots, x_n$ are algebraic over K if and only if $L = K(x_1, \ldots, x_n)$ is a finite extension over K. If so, every element of L is algebraic in K*

(ii) *If x, y are algebraic over K, then so are $x \pm y$, xy and $1/x$ (if $x \neq 0$).*

(iii) *Let L/K any extension. Then $\{x \in L : x \text{ algebraic over } K\}$ is a subfield of L*

Proof.

(i) If x_n is algebraic over K, it's certainly algebraic over $K(x_1, \ldots, x_{n-1})$, so $[L : K(x_1, \ldots, x_{n-1})]$. So by induction on n and the Tower Law, $[L : K] < \infty$. Convsersely, if $[L : K] < \infty$, then the subfield $K(y)$ is finite over K for all $y \in L$, so y is algebraic over K by Proposition 4.1.

(ii) $x + y, xy, \frac{1}{x} \in K(x, y)$. So algebraic by (i).

(iii) Trivial from (ii).

□

Example. $z = e^{2\pi i/p}$, p prime. z has degree $p-1$. Let $x = 2\cos 2\pi/p = z + z^{-1} \in \mathbb{Q}(z)$. So x is algebraic over $\mathbb{Q}$. Note $\mathbb{Q}(z) \supseteq \mathbb{Q}(x) \supseteq \mathbb{Q}$, $z^2 - xz + 1 = 0$. Hence the degree of z over $\mathbb{Q}(x)$ is at most 2. We have $[\mathbb{Q}(z) : \mathbb{Q}] = p-1$ so $[\mathbb{Q}(z) : \mathbb{Q}(x)] = 2$ or 1. But $z \notin \mathbb{Q}(x) \subseteq \mathbb{R}$. So $[\mathbb{Q}(z) : \mathbb{Q}(x)] = 2$ and by the tower law $\deg_{\mathbb{Q}}(x) = \frac{p-1}{2}$.

We have

$$z^{\frac{p-1}{2}} + z^{\frac{p-3}{2}} + \ldots + z^{-\frac{p-1}{2}} = 0$$

$z + z^{-1} = x$. So can express this polynomial as a polynomial in $z + z^{-1} = x$ of degree $\frac{p-1}{2}$.

Example. Let $x = \sqrt{m} + \sqrt{n}$, $m, n \in \mathbb{Z}$ such that m, n, mn are not squares. We have

$$(x - \sqrt{m})^2 = n = x^2 - 2\sqrt{m}x + m$$

So $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{m})] \leq 2$, since the above is a quadratic with coefficients in $\mathbb{Q}(\sqrt{m})$. In the exact same way we have $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{n})] \leq 2$. The quadratic also implies $\sqrt{m} \in \mathbb{Q}(x)$. So by the tower law either $[\mathbb{Q}(x) : \mathbb{Q}] = 4$ or $[\mathbb{Q}(x) : \mathbb{Q}] = 2$ and $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{n})$ (since m, n not squares, $[\mathbb{Q}(\sqrt{m}) : \mathbb{Q}] = 2$).

$\mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{n})$ implies $\sqrt{m} = a + b\sqrt{n}$, $a, b \in \mathbb{Q}$. This implies $m = a^2 + b^2 n + 2ab\sqrt{n}$. $b = 0$ implies $m = a^2$ and $a = 0$ implies $mn = b^2 n^2$, a contradiction. So $\deg_{\mathbb{Q}}(x) = 4$.

Definition. An extension L/K is *algebraic* if every $x \in L$ is algebraic over K.

Proposition 4.3.

(i) Finite extensions are algebraic

(ii) $K(x)$ is algebraic over K if and only if x is algebraic over K

(iii) Let $M/L/K$ be a series of extensions. Then M/K is algebraic if and only if both M/L and L/K are algebraic

Proof.

(i) If $[L : K] < \infty$ then $\forall x \in L$, $[K(x) : K] < \infty$, so x is algebraic over K.

(ii) $(\Rightarrow)$ is by definition, $(\Leftarrow)$ follows from (i).

(iii) Assume M/K is algebraic. Then for all $x \in M$, x is algebraic over K, so certainly x is algebraic over L. So M/L is algebraic. Since $L \subseteq M$, L/K must be algebraic as M/K is.

The other direction follows from the below Lemma.

□

Lemma 4.4. *Let $M/L/K$ be a series of extensions, where L/K is algebraic. Let $x \in M$. Suppose x is algebraic over L. Then x is algebraic over K.*

Proof. There exists $f = T^n + a_{n-1}T^n + \ldots + a_0 \in L[T]$ with $f \neq 0$ and $f(x) = 0$. Let $L_0 = K(a_0, \ldots, a_{n-1})$, then as each $a_i \in L$ is algebraic over K, by Corollary 4.2, $[L_0 : K]$ is finite. As $f \in L_0[T]$, x is algebraic over L_0. So $[L_0(x) : L_0] < \infty$, so $[L_0(x) : K] < \infty$ by the tower law, and so $[K(x) : K] < \infty$ and x is algebraic over K. □

Example. Let $K = \mathbb{Q}$, $L = \{x \in \mathbb{C} : x \text{ is algebraic over } \mathbb{Q}\} = \overline{\mathbb{Q}}$. This is a field by Corollary 4.2. Obviously $L/\mathbb{Q}$ is algebraic, but the extension is <u>not</u> finite. Indeed, for all $n \geq 1$, $\sqrt[n]{2} \in L$ and $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$ (as $T^n - 2$ is irreducible over $\mathbb{Q}$). So as this holds for any n, L can't be finite. We'll see other fields like $\overline{\mathbb{Q}}$ later on (algebraically closed fields).

5 Algebraic numbers in $\mathbb{R}$ and $\mathbb{C}$

Traditionally, $x \in \mathbb{C}$ is said to be *algebraic* if it's algebraic over $\mathbb{Q}$, and otherwise said to be *transcendental*. $\overline{\mathbb{Q}}$ is a subfield of $\mathbb{C}$. It is a proper subfield since $\mathbb{Q}[T]$ is countable, and each polynomial has countably (finitely) many roots, so there are countably many elements of $\overline{\mathbb{Q}}$.

However $\mathbb{C}$ is uncountable. So there are "lots" of transcendental numbers. This argument is non-constructive - it is harder to write a transcendental number explicitly, or to show some given number is transcendental.

Liouville showed that $\sum_{n\geq 1} \frac{1}{10^{n!}}$ is transcendental ("algebraic numbers can't be very well approximated by rationals").

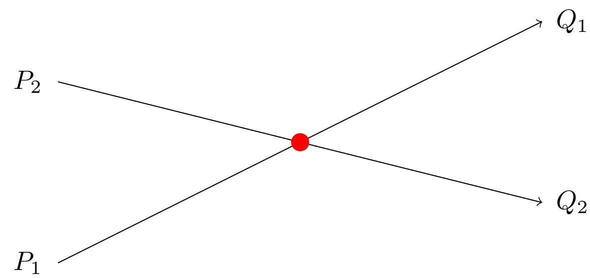
Hermite, Lindermann showed that e and π are transcendental.

In the 20th Century: <u>Gelfond-Schneider Theorem</u>: if x, y are algebraic ($x \neq 1$), then x^y is algebraic if and only if y is rational. For example, this implies $\sqrt{2}^{\sqrt{3}}$ is transcendental. Also $e^\pi = (-1)^{-i/2}$ is transcendental.

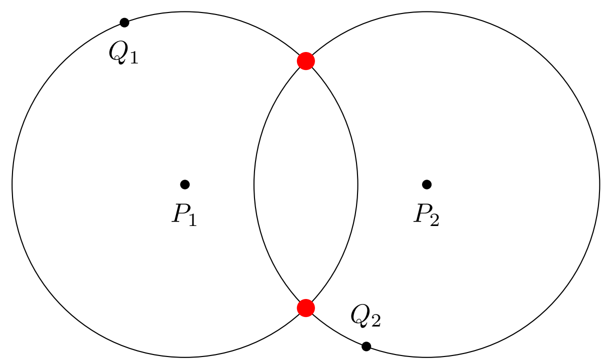
Ruler & compass constructions

We have 3 basic geometric operations (in plane geometry).

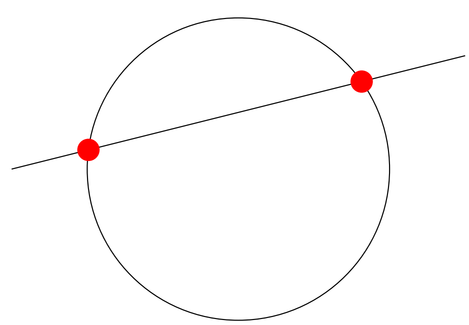
(A) Given $P_1, P_2, Q_1, Q_2 \in \mathbb{R}^2$ with $P_i \neq Q_i$, we can construct (with a ruler) the point of intersection of the lines P_1Q_1, P_2Q_2 (assuming they intersect properly).



(B) Given P_1, P_2, Q_1, Q_2 with $P_i \neq Q_i$, we can construct the intersection points of the circles with centres P_i passing through Q_i.



(C) Can intersect lines with circles.



Definition. We say $(x, y) \in \mathbb{R}^2$ is *constructable from*

$$\{(x_1, y_1), \dots, (x_n, y_n)\}$$

if it can be obtained by a finite sequence of constructions of type A,B,C, each involving only the starting points $\{(x_i, y_i) : 1 \leq i \leq n\}$ and any produced in a previous step.

Definition. We say $x \in \mathbb{R}$ is *constructable* if $(x, 0)$ is constructable from $\{(0, 0), (1, 0)\}$.

Note: every $x \in \mathbb{Q}$ is constructable, and so is $\sqrt{2}$.

Definition. Let $K \subseteq \mathbb{R}$ be a subfield. We say K is *constructable* if there exists some $n \geq 0$ and some sequence of fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \mathbb{R}$ and $a_i \in F_i$ (for $1 \leq i \leq n$) such that

(i) $K \subseteq F_n$

(ii) $F_i = F_{i-1}(a_i)$

(iii) $a_i^2 \in F_{i-1}$

Note: (ii) and (iii) imply that $[F_i : F_{i-1}] \leq 2$. So by the tower law, $K/\mathbb{Q}$ is finite and $[K : \mathbb{Q}]$ is a power of 2.

Theorem 5.1. *If $x \in \mathbb{R}$ is constructable, then $K = \mathbb{Q}(x)$ is constructable.*

Corollary 5.2. *If $x \in \mathbb{R}$ is constructable, then x is algebraic over $\mathbb{Q}$ and $\deg_{\mathbb{Q}}(x)$ is a power of 2 (follows from the above note and the theorem).*

Proof of Theorem 5.1. Induction on $k \geq 1$: we prove that if $(x, y) \in \mathbb{R}^2$ can be constructed with k R&C (Ruler & Compass) constructions, then $\mathbb{Q}(x, y)$ is a constructable extension of $\mathbb{Q}$.

So assume we have

$$\mathbb{Q} = F_0 \subseteq \ldots \subseteq F_n$$

satisfying (ii),(iii) and such that the coordinates of all points obtained after $(k - 1)$ constructions lie in F_n.

Elementary analytic geometry tells us that in (A) the intersection point has coordinates which are rational functions of the coordinates of the points $\{P_i, Q_i\}$ with rational coefficients.

So if the kth construction is of type (A), then $x, y \in F_n$. For constructions (B) and (C), the coordinates of the two intersections can be written as $a \pm b\sqrt{e}$, $c \pm d\sqrt{e}$, where a, e are rational functions of the coordinates of $\{P_i, Q_i\}$. So for the two newly constructed points $x, y \in F_n(\sqrt{e})$, which is a constructable extension of $\mathbb{Q}$. $\square$

Remark: it is not hard to show that the converse is true, i.e if $\mathbb{Q}(x)/\mathbb{Q}$ is constructable then x is constructable.

Examples of classical problems:

1. "Squaring the circle" - construct a square whose area is that of a given circle, i.e have to construct $\sqrt{\pi}$. But since π is transcendental, it (and therefore $\sqrt{\pi}$) is not constructable.

2. "Duplicating the cube" - Construct a cube with volume twice that of a given cube, i.e construct $\sqrt[3]{2}$. But $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ is not a power of two, so $\mathbb{Q}(\sqrt[3]{2})$ (and so $\sqrt[3]{2}$) is not constructable.

3. "Trisect the angle" - say we are trying to trisect $2\pi/3$, which is certainly constructable. So if we can trisect $2\pi/3$, we can construct the angle $2\pi/9$, i.e the real numbers $\cos(2\pi/9), \sin(2\pi/9)$ are constructable. By the formula
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
we note $\cos(2\pi/9)$ is a root of $8X^3 - 6X + 1$, and $2\cos(2\pi/9) - 2$ is a root of $X^3 + 6X^2 + 9X + 3$ which is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. So $\deg_{\mathbb{Q}}(\cos(2\pi/9)) = 3$ (not a power of two) so not constructable.

Later in the course we will see the following theorem

Theorem (Gauss). *A regular n-gon is constructable if and only if n is the product of a power of 2 and distinct primes of the form $2^{2^k} + 1$ ("Fermat primes").*

6 Splitting fields

Problem: we have a field K, $f \in K[T]$ - find an extension L/K (preferably as small as possible) such that f factors in $L[T]$ as a product of linear polynomials.

Example. Let $K = \mathbb{Q}$. By the Fundamental Theorem of Algebra, we can factor any monic $f \in \mathbb{Q}[T]$ as
$$f = \prod_{i=1}^{n}(T - x_i),\ x_i \in \mathbb{C}$$
(Later we will give another proof of the FTA.) So the "best" L would be $\mathbb{Q}(x_1, \ldots, x_n)$, a finite extension of $\mathbb{Q}$.

Example. Let $K = \mathbb{F}_p$. Let f be irreducible of degree $d > 1$. How to find L?

First step: find an extension in which f has at least one root.

Key construction: suppose $f \in K[T]$ is (monic and) irreducible. Let $L_f = K[T]/(f)$. As f is irreducible, (f) is maximal and so L_f is a field. By construction, if $x = T \pmod{(f)} \in L_f$ (the coset $T + (f)$), then $f(x) = 0$. Hence L_f/K

is a field extension in which f has a root.

Questions:

- Is L_f unique?
- What about the remaining roots?

Theorem 6.1. *Let $f \in K[T]$ be irreducible and monic. Let $L_f = K[T]/(f)$, $t \in L_f$ the residue class $T + (f)$. Then L_f/K is a finite extension of fields, $[L_f : k] = \deg(f)$ and f is the minimal polynomial of t over K.*

Proof. See previous example. □

So we have an extension of K in which f has a root. To what extent is this unique?

Also recall that if x is algebraic over K, then $K(x) \cong K[T]/(m_{x,K})$, where $m_{x,K}$ is the minimal polynomial of x over K.

Definition. Suppose K is a field, L/K and M/K extensions of K. A *K-homomorphism* from L to M is a field homomorphism $\sigma : L \to M$ such that $\sigma|_K = \mathrm{id}_K$. We also sometimes call this a *K-embedding*, since σ is an injection.

Theorem 6.2. *Let $f \in K[T]$ be irreducible, L/K be an arbitrary extension. Then*

(i) If $x \in L$ is a root of f, then there exists a unique K-homomorphism $\sigma : L_f \to L$ sending $T + (f)$ to x.

(ii) Every K-homomorphism $L_f \to L$ arises as in (i). So there is a bijection between

$$\{K\text{-homomorphisms } L_f \xrightarrow{\sigma} L\} \leftrightarrow \{\text{roots of } f \text{ in } L\}$$

In particular, there are at most $\deg(f)$ such σ.

Proof. Note

$$\begin{aligned} f(x) = 0 &\iff \mathrm{ev}_x(f) = 0 \\ &\iff \mathrm{Ker}(\mathrm{ev}_x) = (f) \\ &\iff \mathrm{ev}_x \text{ comes from a homomorphism } \sigma : K[T]/(f) \to L \\ &\qquad \text{which is the identity on } K \end{aligned}$$

where $\mathrm{ev}_x : K[T] \to L$ is the homomorphism $g \mapsto g(x)$. □

Corollary 6.3. *If $L = K(x)$ for x algebraic over K, then there exists a unique isomorphism $\sigma : L_f \to K(x)$ such that $\sigma(t) = x$, with $f = m_{x,K}$.*

Proof. Take $L = K(x)$ in the above Theorem. □

Definition. Let x, y be algebraic over K. We say x, y are *K-conjugate* if they have the same minimal polynomial.

Then by the last corollary, both $K(x)$ and $K(y)$ are isomorphic to L_f (where f is their common minimal polynomial).

Corollary 6.4. *x, y are K-conjugate if and only if there exists a K-isomorphism $\sigma : K(x) \to K(y)$ with $\sigma(x) = y$.*

Proof. ($\Rightarrow$) follows by corollary 6.3.

($\Leftarrow$) follows since for all g in $K[T]$ we have $\sigma(g(x)) = g(\sigma(x)) = g(y)$ so x, y have the same minimal polynomial. □

Moral: "the roots of an irreducible polynomial are algebraically indistinguishable".

It is useful (for inductive arguments) to have a generalisation of Theorem 6.2.

Definition. Let L/K, L'/K' be field extensions. Let $\sigma : K \to K'$ be a homomorphism of fields. If $\tau : L \to L'$ is a homomorphism such that $\tau(x) = \sigma(x)$ whenever $x \in K$, we say τ is a *σ-homomorphism* from L to L'. We also say τ *extends* σ or that σ is *the restriction of* τ to K. We write $\sigma = \tau|_K$.

From this definition we have the following variant of Theorem 6.2:

Theorem 6.5. *Let $f \in K[T]$ be irreducible, and $\sigma : K \to L$ be any homomorphism of fields. Let σf be the polynomial given by applying σ to the coefficients of f. Then*

(i) If $x \in L$ is a root of σf, there exists a unique σ-homomorphism $\tau : L_f \to L$ such that $\tau(t) = \tau(T + (f)) = x$

(ii) Every σ-homomorphism $L_f \to L$ is of the form arising from (i), so we have a bijection

$$\{\sigma\text{-homomorphisms } L_f \to L\} \leftrightarrow \{\text{roots of } \sigma f \text{ in } L\}$$

Example. σ might not be the "obvious" homomorphism. Indeed take $K = \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$, and take $L = \mathbb{C}$. There is a homomorphism $\sigma : K \to L$ given by $x + y\sqrt{2} \mapsto x - y\sqrt{2}$. Now take $f = T^2 - (1 + \sqrt{2})$. The map $L_f \xrightarrow{\tau} \mathbb{C}$ must take $t = T + (f)$ to $\pm\sqrt{1 - \sqrt{2}} = \pm i\sqrt{\sqrt{2} - 1} \in \mathbb{C}$.

If instead we took σ to be the inclusion τ takes t to $\pm\sqrt{\sqrt{2} + 1}$.

What about all roots?

Definition. Let $f \in K[T]$ be a non-zero polynomial (not necessarily irreducible). An extension L/K is a *splitting field* for f over K if

(i) f splits into linear factors in $L[T]$.

(ii) $L = K(x_1, \dots, x_n)$ where $\{x_1, \dots, x_n\}$ are the roots of f in L.

Remark: (ii) says that f doesn't split into linear factors over any field L' with $K \subseteq L' \subsetneq L$. Furthermore, any splitting field is necessarily finite since the $\{x_1, \dots, x_n\}$ are algebraic.

Theorem 6.6. *Every non-zero polynomial in $K[T]$ has a splitting field.*

Proof. Induction on $\deg(f)$ (for all K). If $\deg(f) = 0$ or 1, then K is a splitting field. So assume that for all fields K' and all polynomials of degree less than $\deg(f)$, there is a splitting field.

Consider g, an irreducible factor of f. Consider $K' = L_g = K[T]/(g)$. Let $x_1 = T + (g)$. Then $g(x_1) = 0$, so $f(x_1) = 0$ and $f = (T - x_1)f_1$, for some $f_1 \in K'[T]$ and $\deg(f_1) < \deg(f)$. So by induction there is a splitting field L for f_1 over K'. Let $x_2, \ldots, x_n \in L$ be the roots of f_1 in L. Then f splits into linear factors in L, with roots $x_2, \ldots, x_n$, and $L = K'(x_2, \ldots, x_n) = K(x_1, \ldots, x_n)$. So L is a splitting field for f over K. $\square$

Theorem 6.7 ("Splitting fields are unique"). *Let $f \in K[T]$ be non-zero, let L/K be a splitting field for f. Let $\sigma : K \to M$ be an extension such that $\sigma f \in M[T]$ splits [into linear factors] in $M[T]$. Then*

(i) σ can be extended to a homomorphism $\tau : L \to M$.

(ii) If M is a splitting field for σf over $\sigma(K)$, then any τ as in (i) is an isomorphism. In particular, any two splitting fields for f are K-isomorphic.

Remarks:

- It is not obvious without this theorem that two splitting fields have the same degree, because of the choices we had in the construction.
- Typically there will be more than one τ.

Proof.

(i) Induction on $n = [L : K]$. If $n = 1$ then $L = K$ and we are done.

Let $x \in L \setminus K$ be a root of an irreducible factor $g \in K[T]$ of f, with $\deg(g) > 1$. Let $y \in M$ be a root of $\sigma g \in M[T]$ (since σf splits in M this exists). Theorem 6.4 implies there exists $\sigma_1 : K(x) \to M$ such that $\sigma_1(x) = y$ and σ_1 extends σ.

Now $[L : K(x)] < [L : K]$ and L is certainly a splitting field for f over $K(x)$ and $\sigma_1 f = \sigma f$ splits in M. So by induction we can extend σ_1 to a homomorphism $\tau : L \to M$.

(ii) Assume M is a splitting field for σf over $\sigma(K)$. Let τ be as in (i) and $\{x_i\}$ the roots of f in L. Then the roots of σf in M are just $\{\tau(x_i)\}$. Since M is a splitting field, $M = \sigma K(\tau(x_1), \ldots, \tau(x_n)) = \tau(L)$. So τ is an isomorphism. If $K \subseteq M$ and σ is the inclusion, τ is a K-isomorphism from L to M.

□

Example.

(i) $f = T^3 - 2 \in \mathbb{Q}[T]$. In $\mathbb{C}$, $f = (T - \sqrt[3]{2})(T - \omega\sqrt[3]{2})(T - \omega^2\sqrt[3]{2})$ where $\omega = \exp(2\pi i/3)$. So a splitting field for f over $\mathbb{Q}$ is $L = \mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2})$. Then $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$, but $\omega \notin \mathbb{R}$, $\omega^2 + \omega + 1 = 0$, so $[L : \mathbb{Q}(\sqrt[3]{2})] = 2$ and $[L : \mathbb{Q}] = 6$.

(ii) $f = \frac{T^5-1}{T-1} = T^4 + T^3 + T^2 + T + 1 \in \mathbb{Q}[T]$. Let $z = \exp(2\pi i/5)$. Then $f = \prod_{1 \le a \le 4}(T - z^a)$. So $\mathbb{Q}(z)$ is already a splitting field over $\mathbb{Q}$ and $[\mathbb{Q}(z) : \mathbb{Q}] = 4$.

(iii) $f = T^3 - 2 \in \mathbb{F}_7[T]$. This is irreducible since 2 is not a cube modulo 7. Consider the field $L = \mathbb{F}_7[X]/(X^3 - 2) = \mathbb{F}_7(x)$. Then $x^3 = 2$. Now $2^3 = 1 = 4^3$ in $\mathbb{F}_7$. So $(2x)^3 = (4x)^3 = 2$ and so $f = (T - x)(T - 2x)(T - 4x) \in L[T]$

7 Normal extensions

Philosophy: pass from polynomials to fields generated by their roots.

Here we will see an "intrinsic" characterisation of splitting fields.

Definition. An extension L/K is said to be *normal* if L/K is algebraic and for every $x \in L$, $m_{x,K}$ splits into linear factors over L.

Note: this condition is equivalent to: for every $x \in L$, L contains a splitting field for $m_{x,K}$. Or again, for every $f \in K[T]$ irreducible, if f has a root in L, then it splits over L.

Theorem 7.1 ("Splitting fields are normal")**.** *Let L/K be a finite extension. Then L is normal over K if and only if L is the splitting field for some $f \in K[T]$ (not necessarily irreducible).*

Proof. Suppoe L/K is normal, and write $L = K(x_1, \dots, x_n)$. Then $m_{x_i,K}$ splits in L, and L is generated by the roots of $f = \prod_i m_{x_i,K}$. So L is a splitting field for f.

Conversely, if L is the splitting field for $f \in K[T]$. Let $x \in L$, $m_{x,K} = g$ its minimal polynomial - we want to show g splits in f. Let M be a splitting field for g over L, and $y \in M$ some root of g. We want to show $y \in L$. Since L is a splitting field for f over K, L is a splitting field for f over $K(x)$, and $L(y)$ is a splitting field for f over $K(y)$.

Now there exists a K-isomorphism between $K(x)$ and $K(y)$ as x, y are both roots of the same irreducible polynomial $g \in K[T]$. So $[L : K(x)] = [L(y) : K(y)]$ by uniqueness of splitting fields. Hence multiply both sides by $[K(x) : K] = [K(y) : K] = \deg(g)$, and use the tower law to see $[L : K] = [L(y) : K] = [L(y) : L][L : K]$. So $L(y) = L$, i.e $y \in L$. $\square$