1 Basic Group Theory

1.1 Homomorphisms and Isomorphisms

- A function $\varphi\colon H \to G$ is a group homomorphism if for all $a, b \in H$ $\varphi(a \star_H b) = \varphi(a) \star_G \varphi(b)$
  - We call a homomorphism an isomorphism if it is injective.
  - We say that two groups G and K are isomorphic if there exists an isomorphism $\varphi\colon G \to K$
- $\text{Ker}(\varphi) \triangleleft H$ and $\text{Im}(\varphi) \leq G$ for any homomorphism $\varphi\colon H \to G$

1.2 Direct Products

- Direct Products give us a way to form a group which has two particular subgroups, G and H
- The direct product of two groups G and H is the set $G \times H$ with the operation of component-wise composition:

  $(g_1, h_1) \star (g_2, h_2) = (g_1 \star_G g_2, h_1 \star_H h_2)$

  Clearly then the groups $G \times \{e_H\}$ and $\{e_G\} \times H$ are subgroups of $G \times H$ and are also isomorphic to G and H respectively
- The direct product theorem states that for $H, K \leq G$ if the following properties hold, $H \times K \cong G$:
  1. $\forall g \in G\ \exists h \in H, k \in K$ such that $hk = g$
  2. $\forall h \in H, k \in K, hk = kh$
  3. $H \cap K = \{e\}$

  - The theorem can be shown to be true by the isomorphism $\varphi\colon H \times K \to G$ given by $(h, k) \mapsto hk$
  - Note that this is only one such isomorphism and the converse of the direct product theorem does not hold

2 Examples of Groups

2.1 Permutation groups

- Every $\sigma \in S_n$ is expressible in disjoint cycle notation
  - This can be shown by taking some $x \in \{1, 2, \ldots n\}$ and considering $\langle x \rangle$. Then pick some $y \in \{1, 2, \ldots n\}, y \notin \langle x \rangle$ and consider $\langle y \rangle$. Repeat until every element of $\{1, 2, \ldots n\}$ is in exactly one cycle.
  - It is easy to show that this is unique as it is a union of cyclic groups
- Every $\sigma \in S_n$ is expressible as a product of transpositions
  - This can be shown easily by writing σ in disjoint cycle notation and noting that $(a_1 a_2 a_3 \ldots a_k) = (a_1 a_2)(a_2 a_3) \ldots (a_{k-1} a_k)$
  - This product is not necessarily unique
- By considering $\#(\tau\sigma)$ for $\sigma \in S_n$ and a transposition τ it may be shown that when σ is written as a product of transpositions, the number of transpositions is either always even or always odd

2.2 Mobius Groups

- The Mobius Group $\mathcal{M}$ is the set of functions $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form:

  $f(z) = \dfrac{az + b}{cz + d}$

  And with $ad - bc \neq 0$ under the group operation $f \star g = f \circ g$ for $f, g \in \mathcal{M}$

3 Lagrange's Theorem

- The left cosets of H in G are the sets of the form $gH = \{gh : h \in H\}$ for $g \in G$

- Lagrange's Theorem says that for a subgroup $H \leq G$:

  $|G| = |H| \cdot |G : H|$

  Where $|G : H|$ is the number of distinct cosets of H in G

  - The theorem can be proven by showing that the cosets are disjoint and of the same size

4 Quotients of Groups

- Note that for $N \triangleleft G$, the cosets of N in G form a group under the operation $aN \star bN = abN$
  - The condition that N is normal in G is necessary to ensure that the operation is well-defined
  - This group is called the quotient group of G by N, written G/N
- The quotient group can be thought of as the group found by partitioning elements of G into equivalence classes with the equivalence relation $aRb \iff aN = bN$
- All of this leads towards the 1^{st} Isomorphism Theorem:

  Let $\varphi\colon G \to H$ be a homomorphism. Then $G/\text{Ker}\varphi \cong \text{Im}\varphi$

  - This can be shown by considering the isomorphism

    $\phi\colon G/\text{Ker}\varphi \to \text{Im}\varphi$ defined by $g \cdot \text{Ker}\varphi \mapsto \varphi(g)$
- Correspondence Theorem: Let $N \triangleleft G$, then the subgroups of G/N are in bijective correspondence with the subgroups of G containing N.

  Proven by considering some $H \leq G/N$ and the map

  $\pi\colon G \to G/N$
- 2nd Isomorphism Theorem: Let $H \leq G$, $N \triangleleft G$. Then

  $H \cap N \triangleleft H$ and $\dfrac{H}{H \cap N} \cong \dfrac{HN}{N}$

  Proven by considering the surjective homom $\varphi\colon H \to \frac{HN}{N}$ defined by $h \mapsto hN$ with $\text{Ker}\varphi = H \cap N$
- 3rd Isomorphism Theorem: Let $N \leq M \leq G$ such that $N \triangleleft G$, $M \triangleleft G$. Then

  $\dfrac{M}{N} \cong \dfrac{G}{N}$ and $\dfrac{\frac{G}{N}}{\frac{M}{N}} \cong \dfrac{G}{M}$

  Proven by considering the surjective homom $\varphi\colon G/N \to G/M$ given by $gN \mapsto gM$. This has kernel M/N so the result follows

5 Group Actions

- Let G be a group and X a set. An action of G on X is a function $\alpha\colon G \times X \to X$ defined by $(g, x) \mapsto \alpha_g(x)$ satisfying:
  1. $\alpha_g(x) \in X \ \forall g \in G, \forall x \in X$
  2. $\alpha_e(x) = x \ \forall x \in X$
  3. $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x) \ \forall g, h \in G, \forall x \in X$
- The orbit of some $x \in X$ is $\text{Orb}(x) = \{g(x)\colon g \in G\}$
- The stabiliser of x is $\text{Stab}(x) = \{g \in G\colon g(x) = x\} \leq G$
- Orbit-Stabiliser Theorem:

  Let the finite group G act on the set X. Then for any $x \in X$:

  $|G| = |\text{Orb}(x)| \cdot |\text{Stab}(x)|$