# Introduction

Quadratics (Babylonians):

$$X^2 + bX = c = (X + \frac{1}{2}b)^2 + c - \frac{b^2}{4}$$

$$= (X - x_1)(X - x_2) \implies x_1x_2 = c, x_1 + x_2 = -b$$

$$x_1 = \frac{1}{2}\left[(x_1 + x_2) + (x_1 - x_2)\right] = \frac{1}{2}\left[-b + \sqrt{b^2 - 4c}\right]$$

Cubics (Italy, 16th Century):

$$X^3 + aX^2 + bX + c = (X - x_1)(X - x_2)(X - x_3)$$

$$\implies x_1 + x_2 + x_3 = -a, x_1x_2 + x_1x_3 + x_2x_3 = b, x_1x_2x_3 = -c$$

WLOG $X \to X - a/3$ and $a = 0$

$$x_1 = \frac{1}{3}\left[(x_1 + x_2 + x_3) + \underbrace{(x_1 + \omega x_2 + \omega^2 x_3)}_{=u} + \underbrace{(x_1 + \omega^2 x_2 + \omega x_3)}_{=v}\right]$$

where $\omega = e^{2\pi i/3}$ so $\omega^2 + \omega + 1 = 0$. Cyclic permutation of $x_1, x_2, x_3$ gives $u \to \omega u \to \omega^2 u$ and $v \to \omega v \to \omega^2 v$ which implies $u^3$ and $v^3$ are invariant under cyclic permutations of the roots.

Also $u \leftrightarrow v$ under $x_2 \leftrightarrow x_3$. So $u^3 + v^3$, $u^3v^3$ are invariant under permutations of roots.

In fact,

$$u^3 + v^3 = 27x_1x_2x_3 = -27c$$

$$u^3v^3 = -27b^2$$

So $u^3, v^3$ are roots of $Y^2 + 27cY - 27b^2$. This gives a formula for $x_1$ (Cardano's formula).

Can follow a similar method for quartics - auxilliary cubic equation. Unfortunately it doesn't work for quintics - the reason being group theory.

# 1 Polynomials

In this course, all rings are commutative and non-zero. Let $R$ be a ring, then $R[X]$ denotes the ring of polynomials $\sum_{i=0}^{n} a_i X^i$, $a_i \in R$. A polynomial $f \in R[X]$ determines a function $R \to R$, $r \mapsto f(r)$.

The polynomial is not in general determined by this function, e.g let $R = \mathbb{Z}/p\mathbb{Z}$ ($p$ prime). Then for all $a \in R$, $a^p = a$ so the polynomials $X^p$ and $X$ represent the same function.

In the case when $R = K$ (a field), $K[X]$ is a Euclidean domain. The "division algorithm" says that if $f, g \in K[X]$, $g \neq 0$ then there exists unique $q, r \in K[X]$ such that $f = gq + r$ and $\deg r < \deg g$ (define $\deg(0) = -\infty$).

In particular, if $g = X - a$ is linear then $f = (X - a)q + f(a)$ ("remainder theorem"). So $K[X]$ is also a PID and a UFD - every polynomial is a product of irreducible polynomials, and there are GCD's, computable via Euclids algorithm in the usual way.

**Proposition 1.1.** *If $K$ is a field, $0 \neq f \in K[X]$, then $f$ has at most $\deg f$ roots in $K$.*

*Proof.* If $f$ has no roots then we are done. Otherwise, suppose $f(a) = 0$ for $a \in K$. Then

$$f = (X - a)g$$

for some $g \in K[X]$ and $\deg g = \deg f - 1$. If $b \in K$ is a root of $f$ then either $b = a$ or $g(b) = 0$ so the number of roots of $f$ is at most one more than the number of roots of $g$. Now done by induction. □

# 2 Symmetric polynomials

Let $R$ be a ring, consider $R[X_1, \ldots, X_n]$ for $n \geq 1$.

**Definition.** A polynomial $f \in R[X_1, \ldots, X_n]$ is *symmetric* if for every $\sigma \in S_n$, $f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = f$.

The set of symmetric polynomials is a subring of $R[X_1, \ldots, X_n]$.

**Example.** $X_1 + \ldots + X_n$, or more generally, $p_k = X_1^k + \ldots + X_n^k = \sum_{i=1}^n X_i^k$.

Alternative definition: if $f \in R[X_1, \ldots, X_n]$, define $f\sigma = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$. This is an action (on the right) of $S_n$ on $R[X_1, \ldots, X_n]$. A polynomial $f$ is symmetric if and only if it is fixed by this action.

**Definition.** *The elementary symmetric polynomials* are

$$s_r(X_1, \ldots, X_n) = \sum_{1 \leq i_1 < \ldots < i_r \leq n} X_{i_1} X_{i_2} \ldots X_{i_r}$$

**Example.** When $n = 3$ we have

$$\begin{aligned} s_1 &= X_1 + X_2 + X_3 \\ s_2 &= X_1X_2 + X_1X_3 + X_2X_3 \\ s_3 &= X_1X_2X_3 \end{aligned}$$

**Theorem 2.1.**

*(i) Every symmetric polynomial over $R$ can be expressed as a polynomial in $\{s_r : 1 \leq r \leq n\}$, with coefficients in $R$.*

*(ii) There are no non-trivial relations between $s_1, \ldots, s_n$.*

**Remark**:

(a) Consider the ring homomorphism

$$\theta : R[Y_1, \ldots, Y_n] \to R[X_1, \ldots, X_n],\ Y_r \mapsto s_r$$

then (i) says the image of $\theta$ is the set of symmetric polynomials. (ii) says that $\theta$ is injective.

(b) Equivalent definition of the $s_r$'s is

$$\prod_{i=1}^{n}(T + X_i) = T^n + s_1 T^{n-1} + \ldots + s_{n-1}T + s_n$$

If we need to specify the number of variables, write $s_{r,n}$ instead of $s_r$.

*Proof.* Terminology:

- A *monomial* is some $X_I = X_1^{i_1} \ldots X_n^{i_n}$ for $I \in \mathbb{N}^n = \{0, 1, 2, \ldots\}^n$. Its (total) degree is $\sum_\alpha i_\alpha$.
- A *term* is some $cX_I$, for $0 \neq c \in R$. So a polynomial is uniquely a sum of terms.
- *Total degree* of $f$ is the maximum degree over its terms

Lexicographical ordering on monomials $X_I$: write $X_I > X_J$ if either $i_1 > j_1$ or, for some $1 \leq r < n$, $i_1 = j_1, \ldots, i_r = j_r$ and $i_{r+1} > j_{r+1}$.

This is a total ordering: for each pair $I \neq J$, exactly one of $X_I > X_J$ or $X_J > X_I$ holds.

First we prove (ii):

Let $d$ be the total degree of some symmetric polynomial $f$, and let $X_I$ be the largest (in lexicographical order) monomial which occurs in $f$, with coefficient $c \in R$. As $f$ is symmetric, we must have $i_1 \geq i_2 \geq \ldots \geq i_n$ (otherwise we could exchange variables to get a larger monomial).

So

$$X_I = X_1^{i_1 - i_2} (X_1 X_2)^{i_2 - i_3} \ldots (X_1, \ldots X_n)^{i_n}$$

consider

$$g = s_1^{i_1 - i_2} s_2^{i_2 - i_3} \ldots s_{n-1}^{i_{n-1} - i_n} s_n^{i_n}$$

the leading monomial (i.e largest in lexicographical order) of $g$ is $X_I$, and $g$ is symmetric. So $f - cg$ is symmetric of total degree $\leq d$, and its leading monomial term is smaller (lexicographical) than $X_I$. As the set of monomials of degree at most $d$ is finite, this process terminates.

To prove (ii): induct on $n$. Suppose we have $G \in R[Y_1, \ldots, Y_n]$ with $G(s_{n,1}, \ldots, s_{n,n}) = 0$. We want to show $G = 0$. If $n = 1$, this is trivial ($s_{1,1} = X_1$). If $G = Y_n^k H$, with $Y_n \nmid H$, then $s_{n,n}^k H(s_{n,1}, \ldots, s_{n,n}) = 0$. As $s_{n,n} = X_1 \ldots X_n$, $s_{n,n}$ is not a zero divisor in $R[X_1, \ldots, X_n]$ so $H(s_{n,1}, \ldots, s_{n,n}) = 0$.

So we may assume $G$ is not divisible by $Y_n$. Replace $X_n$ by 0. Then

$$s_{n,r}(X_1, \ldots, X_{n-1}, 0) = \begin{cases} s_{n-1,r}(X_1, \ldots, X_{n-1}) & \text{if } r < n \\ 0 & \text{if } r = n \end{cases}$$

and so $G(s_{n-1,1}, \ldots, s_{n-1,n-1}, 0) = 0$. So by induction, $G(Y_1, \ldots, Y_{n-1}, 0) = 0$, i.e $Y_n \mid G$, a contradiction.

□

**Example.** $f = \sum_{i \neq j} X_i^2 X_j$ for $n \geq 3$. The leading term is $X_1^2 X_2 = X_1(X_1 X_2)$. Then compute

$$s_1 s_2 = \sum_i \sum_{j<k} X_i X_j X_k = \sum_{i \neq j} X_i^2 X_j + 3 \sum_{i<j<k} X_i X_j X_k$$

so $f = s_1 s_2 - 3 s_3$.

Computing say $\sum X_i^5$ by hand is tedious. But there are alternative formulae.

Recall $p_k = \sum_{i=1}^n X_i^k$ for $k \geq 1$.

**Theorem 2.2** (Newton's formulae)**.** *Let $n \geq 1$. Then for all $k \geq 1$*

$$p_k - s_1 p_{k-1} + \ldots + (-1)^{k-1} s_{k-1} p_1 + (-1)^k k s_k = 0$$

*by convention, $s_0 = 1$, and $s_r = 0$ if $r > n$.*

*Proof.* We may assume $R = \mathbb{Z}$ (or $\mathbb{R}$). Generating function

$$F(T) = \prod_{i=1}^n (1 - X_i T) = \sum_{r=0}^n (-1)^r s_r T^r$$

Take logarithmic derivative with respect to $T$:

$$\frac{F'(T)}{F(T)} = \sum_{i=1}^n \frac{-X_i}{1 - X_i T} = -\frac{1}{T} \sum_{i=1}^n \sum_{r=1}^\infty X_i^r T^r = -\frac{1}{T} \sum_{r=1}^\infty p_r T^r$$

So

$$-TF'(T) = s_1 T - 2 s_2 T^2 + \ldots + (-1)^{n-1} n s_n T^n$$

$$= F(T) \sum_{r=1}^\infty p_r T^r = (s_0 - s_1 T + \ldots + (-1)^n s_n T^n) \left(p_1 T + p_2 T^2 + \ldots\right)$$

comparing coefficients of $T^k$ gives the result. □

**Definition.** The *discriminant polynomial* is

$$D(X_1, \ldots, X_n) = \Delta(X_1, \ldots, X_n)^2$$

where $\Delta = \prod_{i<j}(X_i - X_j)$. (Recall from IA Groups that applying $\sigma \in S_n$ to $\Delta$ multiplies $\Delta$ by $\mathrm{sgn}(\sigma)$, so $D$ is symmetric.)

So $D(X_1, \ldots, X_n) = d(s_1, \ldots, s_n)$ for some polynomial $d$ ($\mathbb{Z}$-coefficients). For example, when $n = 2$, $D = (X_1 - X_2)^2 = s_1^2 - 4s_2$.

**Definition.** Let $f = T^n + \sum_{i=0}^{n-1} a_{n-i}T^i \in R[T]$. Its *discriminant* is $\mathrm{Disc}(f) = d(-a_1, a_2, -a_3, \ldots, (-1)^n a_n) \in R$.

Observe that if $f = \prod_{i=1}^{n}(T - x_i)$, $x_i \in R$, then $a_r = (-1)^r s_r(x_1, \ldots, x_n)$, so

$$\mathrm{Disc}(f) = \prod_{i<j}(x_i - x_j)^2 = D(x_1, \ldots, x_n)$$

If moreover $R = K$ is a field, then $\mathrm{Disc}(f) = 0$ iff $f$ has a repeated root (i.e $x_i = x_j$ for some $i \neq j$). E.g when $n = 2$, $\mathrm{Disc}(T^2 + bT + c) = b^2 - 4c$.

# 3 Fields

Recall:

**Definition.** A *field* is a ring $K$ (commutative with a 1) in which every non-zero element has a multiplicative inverse. The set of non-zero elements of $K$ is a group under multiplication, written $K^\times$ or $K^*$, called the *multiplicative group of* $K$.

**Definition.** The *characteristic of a field* $K$ is the least positive integer $p$ (if it exists) such that $p \cdot 1_K = 0_K$, or is said to be 0 if no such $p$ exists.

**Example.** $\mathbb{Q}$ has characteristic 0 and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ has characteristic $p$ ($p$ prime).

The characteristic $\text{char}(K)$ of $K$ is either 0 or a prime. Inside $K$, there is a smaller subfield, called the *prime subfield* of $K$. It is either isomorphic to $\mathbb{Q}$ (if characteristic is 0), or to $\mathbb{F}_p$ (if $\text{char}(K) = p$).

**Proposition 3.1.** *Let $\varphi : K \to L$ be a homomorphism of fields. Then $\varphi$ is an injection.*

*Proof.* $\varphi(1_K) = 1_L \neq 0$, so $\text{Ker}(\varphi) \subsetneq K$ is a proper ideal of $K$, so $\text{Ker}(\varphi) = (0)$ □

**Definition.** Let $K \subseteq L$ be fields (where the field operations on $K$ are the same as those on $L$). We say $K$ is a *subfield of* $L$, and $L$ is *an extension of* $K$, denoted $L/K$.

**Remarks**:

(i) The notation $L/K$ has nothing to do with the quotient (some write $L \mid K$)

(ii) It is useful to be more general - if $i : K \to L$ is a homomorphism of fields, then Proposition 3.1 says that $K$ is isomorphic to its image $i(K) \subseteq L$. In this situation, also say $L$ is an extension of $K$.

**Example.** Some extensions include

- $\mathbb{C}/\mathbb{R}$
- $\mathbb{R}/\mathbb{Q}$
- $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}/\mathbb{Q}$

**Definition.** $K \subseteq L$, $x \in L$. Define $K[x] = \{p(x) : p \in K[T]\}$ (a subring of $L$). Define $K(x) = \{\frac{p(x)}{q(x)} : p, q \in K[T], q(x) \neq 0\}$ (a subfield of $L$) "$K$ adjoin $x$". For $x_1, \ldots, x_n \in L$, define

$$K(x_1, \ldots, x_n) = \left\{ \frac{p(x_1, \ldots, x_n)}{q(x_1, \ldots, x_n)} : p, q \in K[T_1, \ldots, T_n], q(x_1, \ldots, x_n) \neq 0 \right\}$$

(Easy to check $K(x_1, \ldots, x_{n-1})(x_n) = K(x_1, \ldots, x_n)$). Likewise $K[x_1, \ldots, x_n]$ is defined analagously.

**Definition.** Suppose $L/K$ is a field extension. Then $L$ is naturally a vector space over its subfield $K$ (forget multiplication by elements of $L$). We can ask if it is a finite-dimensional vector space, if so we say that $L/K$ is a *finite extension* and write $[L : K] = \dim_K(L)$ for the dimension. The dimension is called the *degree of the extension $L$ over $K$*. If the dimension is infinite write $[L : K] = \infty$.

$\dim_K$ denotes the dimension as a $K$-vector space. Of course $L$ has dimension 1 over itself. As a $K$-vector space, $L \cong K^{[L:K]}$.

**Example.**

(i) $\mathbb{C}/\mathbb{R}$, $[\mathbb{C} : \mathbb{R}] = 2$

(ii) For any field $K$, $K(X)$ = field of rational functions in $X$ = field of fractions of polynomial ring $K[X] = \{\frac{p}{q} : p, q \in K[X], q \neq 0\}$. Then $[K(X) : K] = \infty$ since $1, X, X^2, \ldots$ are linearly independent.

(iii) $\mathbb{R}/\mathbb{Q}$, $[\mathbb{R} : \mathbb{Q}] = \infty$. This follows from countability - every finite dimensional vector space over $\mathbb{Q}$ is countable.

This course is largely about properties (and symmetries) of finite extensions of fields.

**Definition.** We say an extension $L/K$ is *quadratic* (*cubic*,...) if $[L : K] = 2(, 3, \ldots)$

**Proposition 3.2.** *Suppose $K$ is a finite field (necessarily of characteristic $p > 0$). Then $|K|$ is a power of $p$.*

*Proof.* Certainly $K/\mathbb{F}_p$ is finite, so $K \cong (\mathbb{F}_p)^n$ (as a vector space), where $n = [K : \mathbb{F}_p]$, so $|K| = p^n$. □

Later on we will see that every prime power $q = p^n$ admits a field $\mathbb{F}_q$ with $q$ elements.

Here is a simple but powerful fact:

**Theorem 3.3** ("Tower Law")**.** *Suppose $M/L$ and $L/K$ are field extensions. Then $M/K$ is a finite extension if and only if both $M/L$ and $L/K$ are finite. If so, then $[M : K] = [M : L][L : K]$.*

In fact, a slightly more general statement holds:

**Theorem 3.4.** *Let $L/K$ be an extension, $V$ an $L$-vector space. Then $\dim_K(V) = [L : K] \dim_L(V)$ (and obvious conclusions if any quantities are infinite).*

**Example.** If $V = \mathbb{C}^n$ then $V \cong \mathbb{R}^{2n}$.

*Proof.* Let $\dim_L(V) = d < \infty$. Then $V \cong L \oplus \ldots \oplus L = L^d$ as an $L$-vector space, so also as a $K$-vector space. If $[L : K] = n < \infty$, then $L \cong K^n$ as a $K$-vector space, so

$$V \cong \underbrace{K^n \oplus \ldots \oplus K^n}_{d \text{ times}} = K^{nd}$$

so $\dim_K(V) = [L : K] \dim_L(V)$. If $V$ is finite-dimensional over $K$, then a $K$-basis for $V$ certainly spans $V$ over $L$. So if $\dim_L(V) = \infty$ then $\dim_K(V) = \infty$. Likewise, if $[L : K] = \infty$ and $V \neq \{0\}$, then $V$ has an infinite linearly independent subset, so $\dim_K(V) = \infty$. □

Another important fact:

**Proposition 3.5.**

*(i) Let $K$ be a field, $G \subseteq K^\times$ a finite subgroup. Then $G$ is cyclic*

*(ii) If $K$ is finite, then $K^\times$ is cyclic*

*Proof.* We prove (i) ((ii) follows immediately): (recall from IB GRM) we can write

$$G \cong \frac{\mathbb{Z}}{m_1\mathbb{Z}} \oplus \ldots \oplus \frac{\mathbb{Z}}{m_k\mathbb{Z}}$$

where $1 < m_1 \mid m_2 \mid \ldots \mid m_k = m$. So for all $x \in G$, $x^m = 1$. As $K$ is a field, the polynomial $T^m - 1$ has at most $m$ roots. So $|G| < m$. Hence $k = 1$ and $G$ is cyclic.

□

**Remark**: Let $K = F = \mathbb{Z}/p\mathbb{Z}$. The above says there exists $a \in \{1, \ldots, p-1\}$ such that $\mathbb{Z}/pZ = \{0\} \cup \{a, a^2, \ldots, a^{p-1}\}$. $a$ is called a primitive root modulo $p$.

**Proposition 3.6.** *Let $R$ be a ring, $p$ a prime such that $p \cdot 1_R = 0_R$ (e.g $R$ a field of characteristic $p$). Then the map*

$$\varphi_p : R \to R,\ \varphi_p(x) = x^p$$

*is a homomorphism from $R$ to itself (called the Frobenius endomorphism of $R$).*

*Proof.* Have to show:

- $\varphi_p(1) = 1$
- $\varphi_p(xy) = \varphi_p(x)\varphi_p(y)$
- $\varphi_p(x+y) = \varphi_p(x) + \varphi_p(y)$

The first two are obvious. For the last one,

$$\begin{aligned}\varphi_p(x+y) &= x^p + \sum_{i=1}^{p-1} \underbrace{\binom{p}{i}}_{\equiv 0 \pmod p} x^i y^{p-i} + y^p \\ &= \varphi_p(x) + \varphi_p(y)\end{aligned}$$

□

**Example.** This gives another proof of Fermat's Little Theorem: $x^p \equiv x \pmod p$ (induction on $x$: $(x+1)^p = x^p + 1$).

# 4 Algebraic elements and extensions

**Definition.** Have $L/K$ an extension, $x \in L$. We say $x$ is *algebraic over $K$* if there exists a non-zero polynomial $f \in K[T]$ such that $f(x) = 0$. Otherwise we say $x$ is *transcendental over $K$*.

Suppose $f \in K[T]$; evaluation $f(x) \in L$. This gives a map $\mathrm{ev}_x : K[T] \to L$, $f \mapsto f(x)$. This is obviously a homomorphism of rings.

$I = \mathrm{Ker}(\mathrm{ev}_x) \subseteq K[T]$ is an ideal (the set of polynomials which vanish at $x$). As $\mathrm{Im}(\mathrm{ev}_x)$ is a subring of $L$, it is an integral domain. So $I$ is a prime ideal. Two possibilities:

(i) $I = \{0\}$. Then the only $f$ with $f(x) = 0$ is $f = 0$. Hence $x$ is transcendental over $K$.

(ii) $I \neq \{0\}$. As $K[T]$ is a PID, there exists a unique monic irreducible $g \in K[T]$ such that $I = (g)$. So $f(x) = 0$ if and only if $f$ is a multiple of $g$. So $x$ is algebraic over $K$; we call $g$ the *minimal polynomial* of $x$ over $K$. written $m_{x,K}$. It is the unique monic irreducible polynomial such that $x$ is a root (and the monic polynomial of least degree with this property). [Depends on $K$ as well as $x$]

**Example.**

- $x \in K$, $m_{x,K} = T - x$
- $p$ prime, $d \geq 1$. Then $T^d - p \in \mathbb{Q}[T]$ is irreducible (Eisenstein's criterion) so it is the minimal polynomial of $\sqrt[d]{p} = x \in \mathbb{R}$ over $\mathbb{Q}$.
- $z = e^{2\pi i/p}$ ($p$ prime) is a root of $T^p - 1$ and of $\frac{T^p-1}{T-1} = g(T) = T^{p-1} + \ldots + T + 1 \in \mathbb{Q}[T]$. As

$$g(T+1) = \frac{(T+1)^p - 1}{T} = T^{p-1} + \binom{p}{1}T^{p-2} + \ldots + \binom{p}{2}T = \binom{p}{1}$$

  which is irreducible by Eisenstein, so $g$ is irreducible and $g$ is the minimal polynomial of $z$ over $\mathbb{Q}$.

**Definition.** The *degree of $x$ over $K$* ($x$ algebraic over $K$) is the degree of $m_{x,K}$, written $\deg_K(x)$ or $\deg(x/K)$.

Ring/field characterisation of algebraicity:

**Proposition 4.1.** *Let $L/K$ be a field extension, $x \in L$. The following are equivalent*

*(i)* *$x$ is algebraic over $K$*

*(ii)* $[K(x) : K] < \infty$

*(iii)* $\dim_K K[x] < \infty$

*(iv)* $K[x] = K(x)$

*(v)* *$K[x]$ is a field*

*If these hold, then* $\deg_K(x) = [K(x) : K]$.

**Note**: recall $K[x] = \{p(x)\}$, $K(x) = \left\{\frac{p(x)}{q(x)} | q(x) \neq 0, p, q \in K[T]\right\}$.

*Proof.* (ii) $\iff$ (iii), (iv) $\iff$ (v) are obvious.

Show (iii) $\Rightarrow$ (v),(iv) and (ii): let $0 \neq y = g(x) \in K[x]$. Consider $K[x] \to K[x]$, $z \mapsto yz$. It is a $K$-linear transformation, injective as $y \neq 0$, and since $\dim_K K[x] < \infty$, it is a bijection. So there exists $z$ such that $yz = 1$. So $K[x]$ is a field, equal to $K(x)$ and $[K(x) : K] < \infty$.

Show (v)$\Rightarrow$(i): wlog $x \neq 0$, then $x^{-1} = a_0 + a_1 x + \ldots + a_n x^n \in K[x]$. Then $a_n x^{n-1} + \ldots + a_0 x - 1 = 0$, so $x$ is algebraic over $K$.

Show (i)$\Rightarrow$(iii) and degree formula: The image of $\mathrm{ev}_x : K[T] \to L$ is $K[x] \subseteq L$. $x$ is algebraic over $K$ so the kernel of this map is $(m_{x,K})$, which is a maximal ideal ($m_{x,K}$ is irreducible). Applying the first isomorphism theorem gives

$\underbrace{K[T]/(m_{x,K})}_{\text{field}} \cong K[x]$. $m_{x,K}$ is monic of degree $d = \deg_K(x)$. So $K[T]/(m_{x,K})$ has basis $1, T, \ldots, T^{d-1}$. So $\dim_K K[x] = d < \infty$. Furthermore $\deg_K(x) = [K(x) : K] = d$. □

**Corollary 4.2.**

*(i)* *$x_1, \ldots, x_n$ are algebraic over $K$ if and only if $L = K(x_1, \ldots, x_n)$ is a finite extension over $K$. If so, every element of $L$ is algebraic in $K$*

*(ii)* *If $x, y$ are algebraic over $K$, then so are $x \pm y$, $xy$ and $1/x$ (if $x \neq 0$).*

*(iii)* *Let $L/K$ any extension. Then $\{x \in L : x$ algebraic over $K\}$ is a subfield of $L$*

*Proof.*

(i) If $x_n$ is algebraic over $K$, it's certainly algebraic over $K(x_1, \ldots, x_{n-1})$, so $[L : K(x_1, \ldots, x_{n-1})]$. So by induction on $n$ and the Tower Law, $[L : K] < \infty$. Convsersely, if $[L : K] < \infty$, then the subfield $K(y)$ is finite over $K$ for all $y \in L$, so $y$ is algebraic over $K$ by Proposition 4.1.

(ii) $x + y, xy, \frac{1}{x} \in K(x, y)$. So algebraic by (i).

(iii) Trivial from (ii).

□

**Example.** $z = e^{2\pi i/p}$, $p$ prime. $z$ has degree $p-1$. Let $x = 2\cos 2\pi/p = z+z^{-1} \in \mathbb{Q}(z)$. So $x$ is algebraic over $\mathbb{Q}$. Note $\mathbb{Q}(z) \supseteq \mathbb{Q}(x) \supseteq \mathbb{Q}$, $z^2 - xz + 1 = 0$. Hence the degree of $z$ over $\mathbb{Q}(x)$ is at most 2. We have $[\mathbb{Q}(z) : \mathbb{Q}] = p-1$ so $[\mathbb{Q}(z) : \mathbb{Q}(x)] = 2$ or 1. But $z \notin \mathbb{Q}(x) \subseteq \mathbb{R}$. So $[\mathbb{Q}(z) : \mathbb{Q}(x)] = 2$ and by the tower law $\deg_{\mathbb{Q}}(x) = \frac{p-1}{2}$.

We have

$$z^{\frac{p-1}{2}} + z^{\frac{p-3}{2}} + \ldots + z^{-\frac{p-1}{2}} = 0$$

$z + z^{-1} = x$. So can express this polynomial as a polynomial in $z + z^{-1} = x$ of degree $\frac{p-1}{2}$.

**Example.** Let $x = \sqrt{m} + \sqrt{n}$, $m, n \in \mathbb{Z}$ such that $m, n, mn$ are not squares. We have

$$(x - \sqrt{m})^2 = n = x^2 - 2\sqrt{m}x + m$$

So $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{m})] \leq 2$, since the above is a quadratic with coefficients in $\mathbb{Q}(\sqrt{m})$. In the exact same way we have $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{n})] \leq 2$. The quadratic also implies $\sqrt{m} \in \mathbb{Q}(x)$. So by the tower law either $[\mathbb{Q}(x) : \mathbb{Q}] = 4$ or $[\mathbb{Q}(x) : \mathbb{Q}] = 2$ and $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{n})$ (since $m, n$ not squares, $[\mathbb{Q}(\sqrt{m}) : \mathbb{Q}] = 2$).

$\mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{n})$ implies $\sqrt{m} = a + b\sqrt{n}$, $a, b \in \mathbb{Q}$. This implies $m = a^2 + b^2 n + 2ab\sqrt{n}$. $b = 0$ implies $m = a^2$ and $a = 0$ implies $mn = b^2 n^2$, a contradiction. So $\deg_{\mathbb{Q}}(x) = 4$.

**Definition.** An extension $L/K$ is *algebraic* if every $x \in L$ is algebraic over $K$.

**Proposition 4.3.**

*(i) Finite extensions are algebraic*

*(ii) $K(x)$ is algebraic over $K$ if and only if $x$ is algebraic over $K$*

*(iii) Let $M/L/K$ be a series of extensions. Then $M/K$ is algebraic if and only if both $M/L$ and $L/K$ are algebraic*

*Proof.*

(i) If $[L : K] < \infty$ then $\forall x \in L$, $[K(x) : K] < \infty$, so $x$ is algebraic over $K$.

(ii) ($\Rightarrow$) is by definition, ($\Leftarrow$) follows from (i).

(iii) Assume $M/K$ is algebraic. Then for all $x \in M$, $x$ is algebraic over $K$, so certainly $x$ is algebraic over $L$. So $M/L$ is algebraic. Since $L \subseteq M$, $L/K$ must be algebraic as $M/K$ is.

The other direction follows from the below Lemma.

□

**Lemma 4.4.** *Let $M/L/K$ be a series of extensions, where $L/K$ is algebraic. Let $x \in M$. Suppose $x$ is algebraic over $L$. Then $x$ is algebraic over $K$.*

*Proof.* There exists $f = T^n + a_{n-1}T^n + \ldots + a_0 \in L[T]$ with $f \neq 0$ and $f(x) = 0$. Let $L_0 = K(a_0, \ldots, a_{n-1})$, then as each $a_i \in L$ is algebraic over $K$, by Corollary 4.2, $[L_0 : K]$ is finite. As $f \in L_0[T]$, $x$ is algebraic over $L_0$. So $[L_0(x) : L_0] < \infty$, so $[L_0(x) : K] < \infty$ by the tower law, and so $[K(x) : K] < \infty$ and $x$ is algebraic over $K$. □

**Example.** Let $K = \mathbb{Q}$, $L = \{x \in \mathbb{C} : x \text{ is algebraic over } \mathbb{Q}\} = \overline{\mathbb{Q}}$. This is a field by Corollary 4.2. Obviously $L/\mathbb{Q}$ is algebraic, but the extension is not finite. Indeed, for all $n \geq 1$, $\sqrt[n]{2} \in L$ and $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$ (as $T^n - 2$ is irreducible over $\mathbb{Q}$). So as this holds for any $n$, $L$ can't be finite. We'll see other fields like $\overline{\mathbb{Q}}$ later on (algebraically closed fields).

# 5 Algebraic numbers in $\mathbb{R}$ and $\mathbb{C}$

Traditionally, $x \in \mathbb{C}$ is said to be *algebraic* if it's algebraic over $\mathbb{Q}$, and otherwise said to be *transcendental*. $\overline{\mathbb{Q}}$ is a subfield of $\mathbb{C}$. It is a proper subfield since $\mathbb{Q}[T]$ is countable, and each polynomial has countably (finitely) many roots, so there are countably many elements of $\overline{\mathbb{Q}}$.

However $\mathbb{C}$ is uncountable. So there are "lots" of transcendental numbers. This argument is non-constructive - it is harder to write a transcendental number explicitly, or to show some given number is transcendental.

Liouville showed that $\sum_{n \geq 1} \frac{1}{10^{n!}}$ is transcendental ("algebraic numbers can't be very well approximated by rationals").

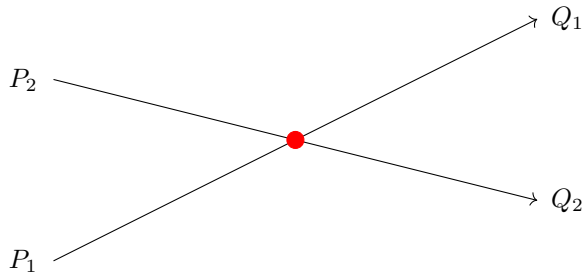
Hermite, Lindermann showed that $e$ and $\pi$ are transcendental.

In the 20th Century: Gelfond-Schneider Theorem: if $x, y$ are algebraic ($x \neq 1$), then $x^y$ is algebraic if and only if $y$ is rational. For example, this implies $\sqrt{2}^{\sqrt{3}}$ is transcendental. Also $e^{\pi} = (-1)^{-i/2}$ is transcendental.

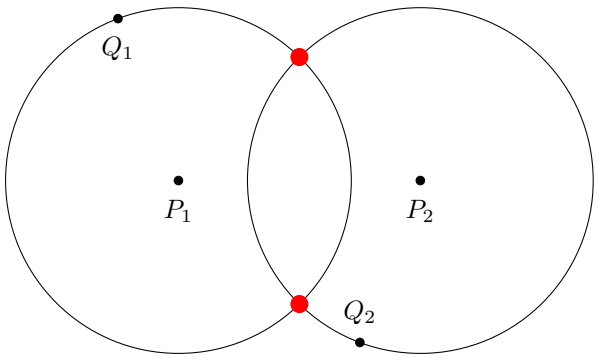
## Ruler & compass constructions

We have 3 basic geometric operations (in plane geometry).

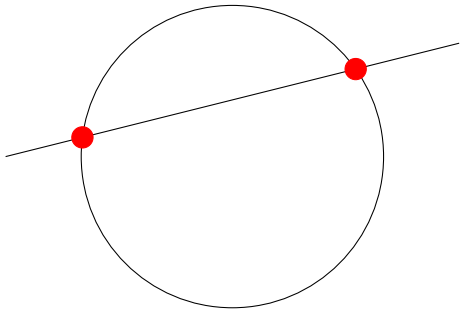
(A) Given $P_1, P_2, Q_1, Q_2 \in \mathbb{R}^2$ with $P_i \neq Q_i$, we can construct (with a ruler) the point of intersection of the lines $P_1Q_1$, $P_2Q_2$ (assuming they intersect properly).




(B) Given $P_1, P_2, Q_1, Q_2$ with $P_i \neq Q_i$, we can construct the intersection points of the circles with centres $P_i$ passing through $Q_i$.




(C) Can intersect lines with circles.



**Definition.** We say $(x, y) \in \mathbb{R}^2$ is *constructable from*

$$\{(x_1, y_1), \ldots, (x_n, y_n)\}$$

if it can be obtained by a finite sequence of constructions of type A,B,C, each involving only the starting points $\{(x_i, y_i) : 1 \leq i \leq n\}$ and any produced in a previous step.

**Definition.** We say $x \in \mathbb{R}$ is *constructable* if $(x, 0)$ is constructable from $\{(0, 0), (1, 0)\}$.

**Note**: every $x \in \mathbb{Q}$ is constructable, and so is $\sqrt{2}$.

**Definition.** Let $K \subseteq \mathbb{R}$ be a subfield. We say $K$ is *constructable* if there exists some $n \geq 0$ and some sequence of fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \mathbb{R}$ and $a_i \in F_i$ (for $1 \leq i \leq n$) such that

(i) $K \subseteq F_n$

(ii) $F_i = F_{i-1}(a_i)$

(iii) $a_i^2 \in F_{i-1}$

**Note**: (ii) and (iii) imply that $[F_i : F_{i-1}] \leq 2$. So by the tower law, $K/\mathbb{Q}$ is finite and $[K : \mathbb{Q}]$ is a power of 2.

**Theorem 5.1.** *If $x \in \mathbb{R}$ is constructable, then $K = \mathbb{Q}(x)$ is constructable.*

**Corollary 5.2.** *If $x \in \mathbb{R}$ is constructable, then $x$ is algebraic over $\mathbb{Q}$ and $\deg_{\mathbb{Q}}(x)$ is a power of 2 (follows from the above note and the theorem).*

*Proof of Theorem 5.1.* Induction on $k \geq 1$: we prove that if $(x, y) \in \mathbb{R}^2$ can be constructed with $k$ R&C (Ruler & Compass) constructions, then $\mathbb{Q}(x, y)$ is a constructable extension of $\mathbb{Q}$.

So assume we have

$$\mathbb{Q} = F_0 \subseteq \ldots \subseteq F_n$$

satisfying (ii),(iii) and such that the coordinates of all points obtained after $(k - 1)$ constructions lie in $F_n$.

Elementary analytic geometry tells us that in (A) the intersection point has coordinates which are rational functions of the coordinates of the points $\{P_i,Q_i\}$ with rational coefficients.

So if the $k$th construction is of type (A), then $x, y \in F_n$. For constructions (B) and (C), the coordinates of the two intersections can be written as $a \pm b\sqrt{e}$, $c \pm d\sqrt{e}$, where $a, e$ are rational functions of the coordinates of $\{P_i, Q_i\}$. So for the two newly constructed points $x, y \in F_n(\sqrt{e})$, which is a constructable extension of $\mathbb{Q}$. $\square$

**Remark**: it is not hard to show that the converse is true, i.e if $\mathbb{Q}(x)/\mathbb{Q}$ is constructable then $x$ is constructable.

**Examples of classical problems**:

1. "Squaring the circle" - construct a square whose area is that of a given circle, i.e have to construct $\sqrt{\pi}$. But since $\pi$ is transcendental, it (and therefore $\sqrt{\pi}$) is not constructable.

2. "Duplicating the cube" - Construct a cube with volume twice that of a given cube, i.e construct $\sqrt[3]{2}$. But $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ is not a power of two, so $\mathbb{Q}(\sqrt[3]{2})$ (and so $\sqrt[3]{2}$) is not constructable.

3. "Trisect the angle" - say we are trying to trisect $2\pi/3$, which is certainly constructable. So if we can trisect $2\pi/3$, we can construct the angle $2\pi/9$, i.e the real numbers $\cos(2\pi/9), \sin(2\pi/9)$ are constructable. By the formula
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
we note $\cos(2\pi/9)$ is a root of $8X^3 - 6X + 1$, and $2\cos(2\pi/9) - 2$ is a root of $X^3 + 6X^2 + 9X + 3$ which is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. So $\deg_{\mathbb{Q}}(\cos(2\pi/9)) = 3$ (not a power of two) so not constructable.

Later in the course we will see the following theorem

**Theorem** (Gauss)**.** *A regular $n$-gon is constructable if and only if $n$ is the product of a power of 2 and distinct primes of the form $2^{2^k} + 1$ ("Fermat primes").*

# 6 Splitting fields

**Problem**: we have a field $K$, $f \in K[T]$ - find an extension $L/K$ (preferably as small as possible) such that $f$ factors in $L[T]$ as a product of linear polynomials.

**Example.** Let $K = \mathbb{Q}$. By the Fundamental Theorem of Algebra, we can factor any monic $f \in \mathbb{Q}[T]$ as
$$f = \prod_{i=1}^{n} (T - x_i),\ x_i \in \mathbb{C}$$
(Later we will give another proof of the FTA.) So the "best" $L$ would be $\mathbb{Q}(x_1, \ldots, x_n)$, a finite extension of $\mathbb{Q}$.

**Example.** Let $K = \mathbb{F}_p$. Let $f$ be irreducible of degree $d > 1$. How to find $L$?

First step: find an extension in which $f$ has at least one root.

Key construction: suppose $f \in K[T]$ is (monic and) irreducible. Let $L_f = K[T]/(f)$. As $f$ is irreducible, $(f)$ is maximal and so $L_f$ is a field. By construction, if $x = T \pmod{(f)} \in L_f$ (the coset $T + (f)$), then $f(x) = 0$. Hence $L_f/K$

is a field extension in which $f$ has a root.

Questions:

- Is $L_f$ unique?
- What about the remaining roots?

**Theorem 6.1.** *Let $f \in K[T]$ be irreducible and monic. Let $L_f = K[T]/(f)$, $t \in L_f$ the residue class $T + (f)$. Then $L_f/K$ is a finite extension of fields, $[L_f : k] = \deg(f)$ and $f$ is the minimal polynomial of $t$ over $K$.*

*Proof.* See previous example. □

So we have an extension of $K$ in which $f$ has a root. To what extent is this unique?

Also recall that if $x$ is algebraic over $K$, then $K(x) \cong K[T]/(m_{x,K})$, where $m_{x,K}$ is the minimal polynomial of $x$ over $K$.

**Definition.** Suppose $K$ is a field, $L/K$ and $M/K$ extensions of $K$. A *$K$-homomorphism* from $L$ to $M$ is a field homomorphism $\sigma : L \to M$ such that $\sigma|_K = \mathrm{id}_K$. We also sometimes call this a *$K$-embedding*, since $\sigma$ is an injection.

**Theorem 6.2.** *Let $f \in K[T]$ be irreducible, $L/K$ be an arbitrary extension. Then*

*(i) If $x \in L$ is a root of $f$, then there exists a unique $K$-homomorphism $\sigma : L_f \to L$ sending $T + (f)$ to $x$.*

*(ii) Every $K$-homomorphism $L_f \to L$ arises as in (i). So there is a bijection between*

$$\{K\text{-homomorphisms } L_f \xrightarrow{\sigma} L\} \leftrightarrow \{\text{roots of } f \text{ in } L\}$$

*In particular, there are at most $\deg(f)$ such $\sigma$.*

*Proof.* Note

$$\begin{aligned} f(x) = 0 &\iff \mathrm{ev}_x(f) = 0 \\ &\iff \mathrm{Ker}(\mathrm{ev}_x) = (f) \\ &\iff \mathrm{ev}_x \text{ comes from a homomorphism } \sigma : K[T]/(f) \to L \\ &\qquad \text{which is the identity on } K \end{aligned}$$

where $\mathrm{ev}_x : K[T] \to L$ is the homomorphism $g \mapsto g(x)$. □

**Corollary 6.3.** *If $L = K(x)$ for $x$ algebraic over $K$, then there exists a unique isomorphism $\sigma : L_f \to K(x)$ such that $\sigma(t) = x$, with $f = m_{x,K}$.*

*Proof.* Take $L = K(x)$ in the above Theorem. □

**Definition.** Let $x, y$ be algebraic over $K$. We say $x, y$ are *$K$-conjugate* if they have the same minimal polynomial.

Then by the last corollary, both $K(x)$ and $K(y)$ are isomorphic to $L_f$ (where $f$ is their common minimal polynomial).

**Corollary 6.4.** *$x, y$ are $K$-conjugate if and only if there exists a $K$-isomorphism $\sigma : K(x) \to K(y)$ with $\sigma(x) = y$.*

*Proof.* ($\Rightarrow$) follows by corollary 6.3.

($\Leftarrow$) follows since for all $g$ in $K[T]$ we have $\sigma(g(x)) = g(\sigma(x)) = g(y)$ so $x, y$ have the same minimal polynomial. $\square$

Moral: "the roots of an irreducible polynomial are algebraically indistinguishable".

It is useful (for inductive arguments) to have a generalisation of Theorem 6.2.

**Definition.** Let $L/K$, $L'/K'$ be field extensions. Let $\sigma : K \to K'$ be a homomorphism of fields. If $\tau : L \to L'$ is a homomorphism such that $\tau(x) = \sigma(x)$ whenever $x \in K$, we say $\tau$ is a *$\sigma$-homomorphism* from $L$ to $L'$. We also say $\tau$ *extends* $\sigma$ or that $\sigma$ is *the restriction of* $\tau$ to $K$. We write $\sigma = \tau|_K$.

From this definition we have the following variant of Theorem 6.2:

**Theorem 6.5.** *Let $f \in K[T]$ be irreducible, and $\sigma : K \to L$ be any homomorphism of fields. Let $\sigma f$ be the polynomial given by applying $\sigma$ to the coefficients of $f$. Then*

*(i) If $x \in L$ is a root of $\sigma f$, there exists a unique $\sigma$-homomorphism $\tau : L_f \to L$ such that $\tau(t) = \tau(T + (f)) = x$*

*(ii) Every $\sigma$-homomorphism $L_f \to L$ is of the form arising from (i), so we have a bijection*

$$\{\sigma\text{-homomorphisms } L_f \to L\} \leftrightarrow \{\text{roots of } \sigma f \text{ in } L\}$$

**Example.** $\sigma$ might not be the "obvious" homomorphism. Indeed take $K = \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$, and take $L = \mathbb{C}$. There is a homomorphism $\sigma : K \to L$ given by $x + y\sqrt{2} \mapsto x - y\sqrt{2}$. Now take $f = T^2 - (1 + \sqrt{2})$. The map $L_f \xrightarrow{\tau} \mathbb{C}$ must take $t = T + (f)$ to $\pm\sqrt{1 - \sqrt{2}} = \pm i\sqrt{\sqrt{2} - 1} \in \mathbb{C}$.

If instead we took $\sigma$ to be the inclusion $\tau$ takes $t$ to $\pm\sqrt{\sqrt{2} + 1}$.

What about all roots?

**Definition.** Let $f \in K[T]$ be a non-zero polynomial (not necessarily irreducible). An extension $L/K$ is a *splitting field* for $f$ over $K$ if

(i) $f$ splits into linear factors in $L[T]$.

(ii) $L = K(x_1, \ldots, x_n)$ where $\{x_1, \ldots, x_n\}$ are the roots of $f$ in $L$.

**Remark**: (ii) says that $f$ doesn't split into linear factors over any field $L'$ with $K \subseteq L' \subsetneq L$. Furthermore, any splitting field is necessarily finite since the $\{x_1, \ldots, x_n\}$ are algebraic.

**Theorem 6.6.** *Every non-zero polynomial in $K[T]$ has a splitting field.*

*Proof.* Induction on $\deg(f)$ (for all $K$). If $\deg(f) = 0$ or 1, then $K$ is a splitting field. So assume that for all fields $K'$ and all polynomials of degree less than $\deg(f)$, there is a splitting field.

Consider $g$, an irreducible factor of $f$. Consider $K' = L_g = K[T]/(g)$. Let $x_1 = T + (g)$. Then $g(x_1) = 0$, so $f(x_1) = 0$ and $f = (T - x_1)f_1$, for some $f_1 \in K'[T]$ and $\deg(f_1) < \deg(f)$. So by induction there is a splitting field $L$ for $f_1$ over $K'$. Let $x_2, \ldots, x_n \in L$ be the roots of $f_1$ in $L$. Then $f$ splits into linear factors in $L$, with roots $x_2, \ldots, x_n$, and $L = K'(x_2, \ldots, x_n) = K(x_1, \ldots, x_n)$. So $L$ is a splitting field for $f$ over $K$. □

**Theorem 6.7** ("Splitting fields are unique"). *Let $f \in K[T]$ be non-zero, let $L/K$ be a splitting field for $f$. Let $\sigma : K \to M$ be an extension such that $\sigma f \in M[T]$ splits [into linear factors] in $M[T]$. Then*

*(i) $\sigma$ can be extended to a homomorphism $\tau : L \to M$.*

*(ii) If $M$ is a splitting field for $\sigma f$ over $\sigma(K)$, then any $\tau$ as in (i) is an isomorphism. In particular, any two splitting fields for $f$ are $K$-isomorphic.*

**Remarks**:

- It is not obvious without this theorem that two splitting fields have the same degree, because of the choices we had in the construction.
- Typically there will be more than one $\tau$.

*Proof.*

(i) Induction on $n = [L : K]$. If $n = 1$ then $L = K$ and we are done.

Let $x \in L \setminus K$ be a root of an irreducible factor $g \in K[T]$ of $f$, with $\deg(g) > 1$. Let $y \in M$ be a root of $\sigma g \in M[T]$ (since $\sigma f$ splits in $M$ this exists). Theorem 6.4 implies there exists $\sigma_1 : K(x) \to M$ such that $\sigma_1(x) = y$ and $\sigma_1$ extends $\sigma$.

Now $[L : K(x)] < [L : K]$ and $L$ is certainly a splitting field for $f$ over $K(x)$ and $\sigma_1 f = \sigma f$ splits in $M$. So by induction we can extend $\sigma_1$ to a homomorphism $\tau : L \to M$.

(ii) Assume $M$ is a splitting field for $\sigma f$ over $\sigma(K)$. Let $\tau$ be as in (i) and $\{x_i\}$ the roots of $f$ in $L$. Then the roots of $\sigma f$ in $M$ are just $\{\tau(x_i)\}$. Since $M$ is a splitting field, $M = \sigma K(\tau(x_1), \ldots, \tau(x_n)) = \tau(L)$. So $\tau$ is an isomorphism. If $K \subseteq M$ and $\sigma$ is the inclusion, $\tau$ is a $K$-isomorphism from $L$ to $M$.

□

**Example.**

(i) $f = T^3 - 2 \in \mathbb{Q}[T]$. In $\mathbb{C}$, $f = (T - \sqrt[3]{2})(T - \omega\sqrt[3]{2})(T - \omega^2\sqrt[3]{2})$ where $\omega = \exp(2\pi i/3)$. So a splitting field for $f$ over $\mathbb{Q}$ is $L = \mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2})$. Then $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$, but $\omega \notin \mathbb{R}$, $\omega^2 + \omega + 1 = 0$, so $[L : \mathbb{Q}(\sqrt[3]{2})] = 2$ and $[L : \mathbb{Q}] = 6$.

(ii) $f = \frac{T^5-1}{T-1} = T^4 + T^3 + T^2 + T + 1 \in \mathbb{Q}[T]$. Let $z = \exp(2\pi i/5)$. Then $f = \prod_{1 \le a \le 4}(T - z^a)$. So $\mathbb{Q}(z)$ is already a splitting field over $\mathbb{Q}$ and $[\mathbb{Q}(z) : \mathbb{Q}] = 4$.

(iii) $f = T^3 - 2 \in \mathbb{F}_7[T]$. This is irreducible since 2 is not a cube modulo 7. Consider the field $L = \mathbb{F}_7[X]/(X^3 - 2) = \mathbb{F}_7(x)$. Then $x^3 = 2$. Now $2^3 = 1 = 4^3$ in $\mathbb{F}_7$. So $(2x)^3 = (4x)^3 = 2$ and so $f = (T - x)(T - 2x)(T - 4x) \in L[T]$

# 7 Normal extensions

Philosophy: pass from polynomials to fields generated by their roots.

Here we will see an "intrinsic" characterisation of splitting fields.

**Definition.** An extension $L/K$ is said to be *normal* if $L/K$ is algebraic and for every $x \in L$, $m_{x,K}$ splits into linear factors over $L$.

**Note**: this condition is equivalent to: for every $x \in L$, $L$ contains a splitting field for $m_{x,K}$. Or again, for every $f \in K[T]$ irreducible, if $f$ has a root in $L$, then it splits over $L$.

**Theorem 7.1** ("Splitting fields are normal")**.** *Let $L/K$ be a finite extension. Then $L$ is normal over $K$ if and only if $L$ is the splitting field for some $f \in K[T]$ (not necessarily irreducible).*

*Proof.* Suppose $L/K$ is normal, and write $L = K(x_1, \ldots, x_n)$. Then $m_{x_i,K}$ splits in $L$, and $L$ is generated by the roots of $f = \prod_i m_{x_i,K}$. So $L$ is a splitting field for $f$.

Conversely, if $L$ is the splitting field for $f \in K[T]$. Let $x \in L$, $m_{x,K} = g$ its minimal polynomial - we want to show $g$ splits in $f$. Let $M$ be a splitting field for $g$ over $L$, and $y \in M$ some root of $g$. We want to show $y \in L$. Since $L$ is a splitting field for $f$ over $K$, $L$ is a splitting field for $f$ over $K(x)$, and $L(y)$ is a splitting field for $f$ over $K(y)$.

Now there exists a $K$-isomorphism between $K(x)$ and $K(y)$ as $x, y$ are both roots of the same irreducible polynomial $g \in K[T]$. So $[L : K(x)] = [L(y) : K(y)]$ by uniqueness of splitting fields. Hence multiply both sides by $[K(x) : K] = [K(y) : K] = \deg(g)$, and use the tower law to see $[L : K] = [L(y) : K] = [L(y) : L][L : K]$. So $L(y) = L$, i.e $y \in L$. □

There is a "field-theoretic" version of a splitting field:

**Corollary 7.2** ("Normal closure")**.** *Let $L/K$ be a finite extension. Then there exists a finite extension $M/L$ such that*

*(i) $M/K$ is normal*

*(ii) If $L \subseteq M' \subseteq M$ and $M'/K$ is normal, then $M' = M$*

*Moreover, any two such extensions $M$ are $L$-isomorphic.*

*Proof.* Say $L = K(x_1, \ldots, x_k)$. Let $f = \prod_i m_{x_i,K}$. Let $M$ be a splitting field for $f$ over $L$. Then as the $x_i$'s are roots of $f$, $M$ is also a splitting field for $f$ over $K$. So $M/K$ is normal. Let $M'$ be as in (ii); then as $x_i \in M'$, $m_{x_i,K}$ splits in $M'$ (as $M'/K$ is normal). So $M' = M$.

For uniqueness: any $M$ satisfying (i) must contain a splitting field for $f$, and by the above, (ii) implies that $M$ is a splitting field for $f$. So uniqueness follows from uniqueness of splitting fields. □

# 8 Seperability

Over $\mathbb{C}$, we can tell if $f$ has multiple zeros by looking at its derivative. Over arbitrary fields, turns out the same is true if we replace the analytic notion of differentiation with an algebraic one.

**Definition.** The *(formal) derivative* of a polynomial $f = \sum_{0 \leq i \leq d} a_i T^i \in K[T]$ is $f' = \sum_{1 \leq i \leq d} i a_i T^{i-1}$.

It is easy to check that $(f+g)' = f' + g'$, $(fg)' = f'g + fg'$ and $(f^n)' = nf'f^{n-1}$.

**Example.** Let $K$ be a field of characteristic $p > 0$. Then if $f = T^p + a_0$, $f' = pT^{p-1} + 0 = 0$. So it is possible to have a non-constant polynomial with zero derivative.

**Proposition 8.1.** *Let $f \in K[T]$, $L/K$ an extension and $x \in L$ a root of $f$. Then $x$ is a simple root if and only if $f'(x) \neq 0$.*

*Proof.* Write $f = (T-x)g \in L[T]$. Then $f' = g + (T-x)g'$ so $f'(x) = g(x)$ and $g(x)$ is non-zero if and only if $(T-x) \nmid g$, i.e $x$ is a simple root of $f$. □

**Definition.** We say $f \in K[T]$ is *seperable* if it splits into distinct linear factors in a splitting field (i.e has $\deg(f)$ distinct roots).

**Corollary 8.2.** *$f$ is seperable if and only if* $\gcd(f, f') = 1$.

**Note**: we take $\gcd(f, g)$ to be the unique monic $h$ such that $(h) = (f, g)$. Then $h = af + bg$ for some $a, b$ which can be computed by Euclids algorithm. Observe that $\gcd(f, g)$ is the same in $K[T]$ or $L[T]$ for any $K \subseteq L$, since Euclids algorithm gives the same result.

*Proof of Corollary.* Replacing $K$ by a splitting field for $f$, we may assume $f$ has all its roots in $K$. Now $f$ is seperable if and only if $f, f'$ have no common root, which holds if and only if $\gcd(f, f') = 1$. □

**Example.** $\mathrm{char}(K) = p > 0$, $f = T^p - b$, $b \in K$. Then $f' = 0$ so $\gcd(f, f') = f \neq 1$. So $f$ is inseperable. Let $L$ be any extension of $K$ containing some $a \in L$ such that $a^p = b$. Then $f = (T-a)^p = T^p + (-a)^p = T^p - b$. So $f$ has only one root in a splitting field. In fact, if $b$ isn't a $p$th power in $K$, then $f$ is irreducible (Exercise).

**Theorem 8.3.**

*(i) Let $f \in K[T]$ be irreducible. Then $f$ is seperable if and only if $f' \neq 0$.*

*(ii) If $\mathrm{char}(K) = 0$ then every irreducible polynomial in $K[T]$ is seperable.*

*(iii) If $\mathrm{char}(K) = p > 0$ then an irreducible $f \in K[T]$ is inseperable if and only if $f = g(T^p)$ for some $g \in K[T]$.*

*Proof.*

(i) Assume wlog that $f$ is monic. Then as $f$ is irreducible, $\gcd(f, f') = f$ or 1. But $\deg(f) > \deg(f')$ so $\gcd(f, f') \neq f$ unless $f' = 0$, and converse is obvious.

(ii) Write $f = \sum_{0 \leq i \leq d} a_i T^i$, $f' = \sum_{1 \leq i \leq d} i a_i T^{i-1}$. So $f' = 0$ if and only if $ia_i = 0$ for all $1 \leq i \leq d$, so $a_i = 0$ for all $1 \leq i \leq d$ (since characteristic 0). Hence $f$ is constant, and not irreducible.

(iii) As above get $ia_i = 0$ for all $1 \leq i \leq d$, and $a_i = 0$ for all $i$ not divisible by $p$. Thus $f = g(T^p)$ where $g = \sum_i a_{pi} T^i$.

□

Now we go from polynomials to fields:

**Definition.** Let $L/K$ be an extension. Say $x \in L$ is *seperable over $K$* if $x$ is algebraic over $K$ and $m_{x,K}$ is seperable. Say $L/K$ is *seperable over $K$* if $x$ is seperable over $K$ for all $x \in L$.

**Theorem 8.4.** *Let $x$ be algebraic over $K$, and $L/K$ any extension in which $m_{x,K}$ splits. Then $x$ is seperable over $K$ if and only if there are exactly $\deg_K(x)$ $K$-homomorphisms from $K(x) \to L$.*

*Proof.* Recall (from 6.2) that the number of such homomorphisms is the number of roots of $m_{x,K}$ in $L$. This is equal to $\deg_K(x)$ if and only if $m_{x,K}$ splits. □

Notation: write $\mathrm{Hom}_K(L, M) = \{K\text{-homomorphisms } L \to M\}$ (not to be confused with linear maps $L \to M$).

**Theorem 8.5** ("Counting embeddings"). *Let $L = K(x_1, \ldots, x_k)$ be a finite extension of $K$, and $M/K$ any extension. Then $|\mathrm{Hom}_K(L, M)| \leq [L : K]$ with equality if and only if*

*(i) For all $i$, $m_{x_i,K}$ splits into linear factors over $M$*

*(ii) All the $x_i$ are seperable over $K$*

**Remarks**:

1. (i) and (ii) are the same as saying $m_{x_i,K}$ splits into distinct linear factors in $M$
2. Obvious variant: take any homomorphism $\sigma : K \to M$ and the condition becomes that the number of $\sigma$-homomorphisms is bounded by $[L : K]$ with equality if and only if for all $i$, $\sigma m_{x_i,K}$ splits over $M$

*Proof.* Induction on $k$. If $k = 0$ we're done. For $k \geq 1$ take $K_1 = K(x_1)$, $\deg_{K_1}(x_1) = d = [K_1 : K]$. Then $|\mathrm{Hom}_K(K_1, M)| = e = |\{\text{roots of } m_{x_1,K} \text{ in } M\}| \leq d$. Let $\sigma : K_1 \to M$ be a $K$-homomorphism. Apply induction to $L/K_1$. So there exist at most $[L : K_1]$ extensions of $\sigma$ to a homomorphism $L \to M$. So $|\mathrm{Hom}_K(L, M)| \leq e[L : K_1] \leq d[L : K_1] = [L : K]$.

If equality holds, then $e = d$, i.e $M_{x_1,K}$ has $d$ distinct roots in $M$. But we could have taken any other $x_i$ instead of $x_1$ in the above, to get (i) and (ii).

Conversely, assume (i) and (ii) hold. Then by the previous theorem $|\mathrm{Hom}_K(K_1, M)| = d$ and (i), (ii) still hold over $K_1$. So by induction on $k$, each $\sigma : K_1 \to M$ has $[L : K_1]$ extensions to $L \to M$, so $|\mathrm{Hom}_K(L, M)| = [L : K]$. □

**Theorem 8.6** ("Seperably generated implies seperable"). *Let $L = K(x_1, \ldots, x_k)$ be a finite extension of $K$. Then $L/K$ is seperable over $K$ if and only if $x_i$ is seperable over $K$ for all $i$.*

*Proof.* If $L/K$ is seperable, all the $x_i$ are seperable by definition. So assume all the $x_i$ are seperable over $K$, and let $M$ be a normal closure (splitting field of $\prod_i m_{x_i,K}$ over $L$). Then in the previous theorem, both (i) and (ii) are satisfied so $|\mathrm{Hom}_K(L, M)| = [L : K]$. But if $x \in L$, then $L = (x, x_1, \ldots, x_k)$ as well. So by the previous theorem again, $x$ is seperable. □

**Corollary 8.7.** *Let $x, y \in L$, $L/K$ an extension of $K$. If $x, y$ are seperable over $K$, so are $x + y, xy$ and $1/x$ (if $x \neq 0$).*

*Proof.* Apply previous theorem to $K(x, y)$. So $\{x \in L : x \text{ seperable over } K\}$ forms a subfield of $L$. □

**Theorem 8.8** ("Primitive element theorem for seperable extensions")**.** *Let $K$ be an infinite field, and $L = K(x_1, \ldots, x_k)$ a finite extension where $x_1, \ldots, x_k$ are seperable. Then there exists $x \in L$ such that $L = K(x)$ (by the previous, $x$ is also seperable over $K$).*

*Proof.* It is enough to consider the case $k = 2$, $L = K(x, y)$ with $x, y$ seperable over $K$. Let $n = [L : K]$ and let $M$ be a normal closure for $L/K$. Then there exist $n$ distinct $K$-homomorphisms $\sigma_i : L \to M$. Let $a \in K$ and consider $z = x + ay$. We will choose $a$ such that $L = K(z)$.

As $L = K(x, y)$, $\sigma_i(x) = \sigma_j(x)$ and $\sigma_i(y)$ and $\sigma_j(y)$ occurs iff $\sigma_i = \sigma_j$, i.e $i = j$. Consider $\sigma_i(z) = \sigma_i(x) + a\sigma_i(y)$. If $\sigma_i(x) = \sigma_j(x)$ then $[\sigma_i(x) - \sigma_j(x)] - a[\sigma_i(y) - \sigma_j(y)] = 0$ and if $i \neq j$, at least one of these brackets is non-zero, so there exists at most one $a \in K$ for which it holds. So there is at most one $a$ for which $\sigma_i(z) = \sigma_j(z)$. Since $K$ is infinite, there exists $a$ such that $\sigma_i(z)$ is distinct for all $1 \leq i \leq n$. But then $\deg_K(z) = n$, so $L = K(z)$. □

For finite fields, the result is much easier:

**Theorem 8.9.** *If $L/K$ is an extension of finite fields, then $L = K(x)$ for some $x \in K$.*

*Proof.* The multiplicative group $L^\times$ is cyclic. Let $x$ be a generator of this group. Then $L = K(x)$. □

# 9 Galois Theory

Automorphisms of fields: $\sigma : L \to L$ is an *automorphism* of the field $L$ if it is a bijective homomorphism.

The set of automorphisms of $L$ forms a group under composition of functions and is denoted $\mathrm{Aut}(L)$ (the "automorphism group of $L$").

If $S \subseteq \mathrm{Aut}(L)$, is a subset, let $L^S = \{x \in L : \forall \sigma \in S,\ \sigma(x) = x\}$. This is a subfield of $L$ (since each $\sigma$ is a homomorphism) and is called the *fixed field* of $S$.

E.g $L = \mathbb{C}$, $\sigma =$ complex conjugation. Then $L^{\{\sigma\}} = \mathbb{R}$. Let $L/K$ be an extension. Define $\mathrm{Aut}(L/K) = \{K\text{-automorphisms of } L\} = \{\sigma \in \mathrm{Aut}(L) : \sigma(x) = x\ \forall x \in K\}$ (a subgroupf of $\mathrm{Aut}(L)$). Then $\sigma \in \mathrm{Aut}(L/K)$ if and only if $K \subseteq L^{\{\sigma\}}$.

**Theorem 9.1.** *Let $L/K$ be finite. Then $|Aut(L, K)| \leq [L : K]$.*

*Proof.* Take $M = L$ in Theorem 8.5. Then $\mathrm{Hom}_K(L, M) = \mathrm{Aut}(L/K)$. □

**Fact**: If $K = \mathbb{Q}$ or $\mathbb{F}_p$ then $\mathrm{Aut}(K) = \{1\}$ ($\sigma(1_K) = 1_K$ implies $\sigma(m1_K) = m\sigma(1_K)$ for all $m \in \mathbb{Z}$). So for any $L$, $\mathrm{Aut}(L) = \mathrm{Aut}(L/K)$ where $K$ is the prime subfield (copy of $\mathbb{Q}$ or $\mathbb{F}_p$).

There is a notion of when $L/K$ has "many" symmetries.

**Definition.** An extension $L/K$ is said to be *Galois* if it is algebraic and $L^{\mathrm{Aut}(L/K)} = K$, i.e automorphisms detect when an element of $L$ is in $K$.

**Examples**:

1. $\mathbb{C}/\mathbb{R}$ is Galois (e.g complex conjugation fixes only elements of $\mathbb{R}$). Likewise $\mathbb{Q}(i)/\mathbb{Q}$ is Galois.
2. $K/\mathbb{F}_p$ a finite extension. Then $K$ is a finite field. The Frobenius automorphism $\varphi_p : K \to K$, $x \mapsto x^p$ has $K^{\{\varphi_p\}} = \{x \in K : x \text{ root } of T^p - T\}$. $T^p - T$ has at most $p$ roots and everything in $\mathbb{F}_p$ is a root so $K^{\{\varphi_p\}} = \mathbb{F}_p$, i.e $K/\mathbb{F}_p$ is Galois.

**Definition.** If $L/K$ is Galois, write $\mathrm{Gal}(L/K) = \mathrm{Aut}(L/K)$, the Galois group of $L/K$.

**Theorem 9.2** (Classification of finite Galois extensions)**.** *Let $L/K$ be a finite extension, $G = Aut(L/K)$. The following are equivalent*

*(i) $L/K$ is Galois (i.e $L^G = K$)*

*(ii) $L/K$ is normal and seperable*

*(iii) $L$ is the splitting field of a seperable polynomial*

*(iv) $|Aut(L/K)| = [L : K]$.*

*If so then the minimal polynomial of $x \in L$ is $m_{x,K} = \prod_{i=1}^{r}(T - x_i)$, where $\{x_1, \ldots, x_r\} = \{\sigma(x) : \sigma \in G\}$ is the orbit of $G$ on $x$ (the $x_i$ are distinct)*

*Proof.* First we show (i)⇒(ii) and the last part. Let $x \in L$, $\{x_1, \ldots, x_r\}$ be the orbit of $G$ on $x$, $f = \prod(T - x_i)$. Then $f(x) = 0$. As $G$ permutes $\{x_i\}$, $f \in L^G[T] = K[T]$, so $m_{x,K} \mid f$. Also since $m_{x,K}(\sigma(x)) = \sigma(m_{x,K}(x)) = 0$, every $x_i$ is a root of $m_{x,K}$. So $f = m_{x,K}$ and $x$ is seperable over $K$, and $m_{x,K}$ splits in $L$, so $L/K$ is normal and seperable.

Now we show (ii)⇒(iii). By Theorem 7.1, $L$ is a splitting field for some $f \in K[T]$. Write $f = \prod q_i^{e_i}$, where $q_i$ are irreducible and $e_i \geq 1$. Since $L/K$ is seperable, $q_i$ are seperable, so $g = \prod q_i$ is seperable amd $L$ is also a splitting field for $g$.

Now we show (iii)⇒(iv). Write $L = K(x_1, \ldots, x_k)$, the splitting field of some seperable $f$ with roots $x_i$. Take $M = L$ and apply Theorem 8.5 as since $m_{x_i,K} \mid f$, the conditions for equality hold. Hence $|\mathrm{Hom}_K(L, M)| = [L : K]$

Finally we show (iv)⇒(i). Suppose $|G| = [L : K]$. Then $G \subseteq \mathrm{Aut}(L/L^G) \subseteq \mathrm{Aut}(L/K)$ so in fact $G = \mathrm{Aut}(L/L^G)$, and $[L : K] = |G| \leq [L : L^G]$. As $L^G \supseteq K$, this implies that $L^G = K$ by the tower law. □

**Corollary 9.3.** *Let $L/K$ be a finite Galois extension. Then $L = K(x)$ for some $x$ seperable over $K$ of degree $[L : K]$.*

*Proof.* By (ii) in the previous theorem, $L/K$ is seperable. So by the Primitive Element Theorem, $L = K(x)$ and the result follows. □

**Theorem 9.4** ("The Galois correspondence")**.** *Let $L/K$ be a finite Galois extension, $G = \mathrm{Gal}(L/K)$.*

*(a) Let $F \subseteq L$ be a subfield with $F \supseteq K$. Then $L/F$ is a Galois extension, $\mathrm{Gal}(L/F) \subseteq \mathrm{Gal}(L/K)$. The map $F \mapsto \mathrm{Gal}(L/F)$ is a bijection between $\{F \text{ field} : K \subseteq F \subseteq L\}$ and $\{\text{subgroups } H \text{ of } G\}$ whose inverse is the map taking $H$ to the fixed field $L^H$. This bijection is inclusion-reversing and if $F = L^H$, $[F : K] = (G : H)$ (where $(G : H)$ denotes the index of the subgroup).*

*(b) Let $\sigma \in G$, $H \subseteq G$ a subgroup, $F = L^H$. Then $\sigma H \sigma^{-1}$ corresponds to $\sigma F$.*

*(c) The following are equivalent (for a subgroup $H \subseteq G$)*

*(i) $L^H/K$ is Galois*

*(ii) $L^H/K$ is normal*

*(iii) For all $\sigma \in G$, $\sigma(L^H) = L^H$*

*(iv) $H$ is a normal subgroup of $G$*

*If so, $\mathrm{Gal}(L^H/K) \cong G/H$.*

*Proof.*

(a) Let $x \in L$. Then $m_{x,F}$ divides $m_{x,K}$ in $F[T]$. As $m_{x,K}$ splits into distinct linear factors in $L$, so does $m_{x,F}$. Hence $L/F$ is normal and seperable, hence is Galois. By definition $\mathrm{Gal}(L/F) \subseteq G$.

To check we have a bijection, with claimed inverse, note $F \mapsto H = \mathrm{Gal}(L/F) \mapsto L^H$. But $L^{\mathrm{Gal}(L/F)} = F$ as $L/F$ is Galois, i.e $L^H = F$. Also $H \mapsto L^H \mapsto \mathrm{Gal}(L/L^H)$. It is enough to show $[L : L^H] \leq |H|$ since certainly $H \subseteq \mathrm{Gal}(L/L^H)$ and $|\mathrm{Gal}(L/L^H)| \leq [L : L^H]$. By Corollary 9.3, $L = L^H(x)$ for some $x$, and $f = \prod_{\sigma \in H}(T - \sigma(x)) \in L^H[T]$, with $x$ a root. So $[L : L^H] = \deg_{L^H}(x) \leq \deg(f) = |H|$. So we have a bijection.

If $F \subseteq F'$, then $\mathrm{Gal}(L/F') \subseteq \mathrm{Gal}(L/F)$, so the bijection is inclusion-reversing. Finally if $F = L^H$ then

$$[F : K] = \frac{[L : K]}{[L : F]} = \frac{|\mathrm{Gal}(L/K)|}{|\mathrm{Gal}(L/F)|} = \frac{|G|}{|H|} = (G : H)$$

(b) Under (a), $\sigma H\sigma^{-1}$ corresponds to $L^{\sigma H\sigma^{-1}} = \{x \in L : \sigma\tau\sigma^{-1} = x,\ \forall \tau \in H\}$ and $\sigma\tau\sigma^{-1} = x$ if and only if $\tau\sigma^{-1}(x) = \sigma^{-1}(x)$, i.e $\tau(y) = y$ where $x = \sigma(y)$. So $x \in L^{\sigma H\sigma^{-1}}$ if and only if $x = \sigma(y)$ for $y \in L^H$, i.e $L^{\sigma H\sigma^{-1}} = \sigma F$.

(c) $L/K$ is seperable, so $L^H/K$ is seperable, so (i) is equivalent to (ii). Let $F = L^H$. Let $F = L^H$, $x \in F$. Then {roots of $m_{x,K}$} is the orbit of $x$ under $G$. So $m_{x,K}$ splits in $F$ if and only if $\forall \sigma \in G$, $\sigma(x) \in F$. As this must hold for all $x \in F$, $F$ is normal if and only if $\sigma F \subseteq F$. As $[\sigma F : K] = [F : K]$ ($K$-isomorphic extensions), this means $\sigma F = F$. By (b), this is equivalent to: $\forall \sigma \in G$, $\sigma H\sigma^{-1} = H$, i.e $H$ is a normal subgroup of $G$.

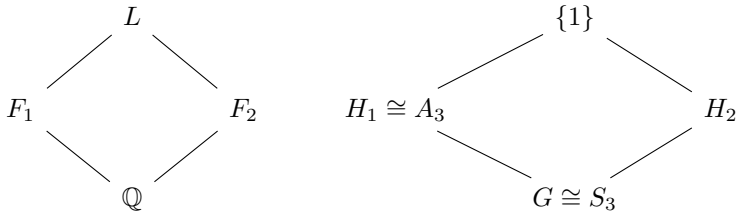
Last part: since $\forall \sigma \in G$, $\sigma F = F$, we have a homomorphism $G \to \mathrm{Gal}(F/K)$ given by restricting $\sigma \in G$ to $F$. This homomorphism has kernel $H$ (since $F = L^H$). So $G/H \to \mathrm{Gal}(F/K)$ is an isomorphism.

□

**Example.** $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt[3]{2}, \omega) \subseteq \mathbb{C}$ where $\omega = \exp(2\pi i/3)$. Then $L$ is a splitting field for $T^3 - 2$ and $[L : \mathbb{Q}] = 6$. So $L/K$ is the splitting field of a seperable polynomial, hence is Galois, and if $G = \mathrm{Gal}(L/K)$ then $|G| = 6$. Obvious subfields of $L$: $F_1 = \mathbb{Q}(\omega)$, $F_2 = \mathbb{Q}(\sqrt[3]{2})$. Then $[F_1 : \mathbb{Q}] = 2$ and $[F_2 : \mathbb{Q}] = 3$.

$G$ must be isomorphic to either cyclic groups of order 6, or $S_3$. $F_2/\mathbb{Q}$ isn't normal, as $\omega\sqrt[3]{2} \notin F_2$. So $H_2 = \mathrm{Gal}(L/F_2)$ isn't a normal subgroup of $G$. So $G$ is non-abelian and $G \cong S_3$, and $H_2 \cong \{(12), e\}$, $H_1$ must be $\cong A_3$. The other subgroups are $\{(13), e\}$ and $\{(23), e\}$ which are the conjugates of $H_2$. So the corresponding subfields are $\{\sigma F_2 : \sigma \in G\}$, which are $\mathbb{Q}(\omega\sqrt[3]{2})$, $\mathbb{Q}(\omega^2\sqrt[3]{2})$ (conjugates of $\sigma(\sqrt[3]{2})$ are the roots of the minimal polynomial). So this describes all $F$ with $\mathbb{Q} \subseteq F \subseteq L$.

In fact, we could have seen at once that $G \cong S_3$: $f \in K[T]$ seperable polynomial, $x_1, \ldots, x_n$ roots in splitting field $L$. $G = \mathrm{Gal}(L/K)$ permutes $\{x_i\}$ as $f(\sigma x_i) = \sigma f(x_i) = 0$ and if $\sigma(x_1) = x_i$ for all $i$, then since $L = K(x_1, \ldots, x_n)$, $\sigma = \mathrm{id}$. This gives a homomorphism $G \to S_n$ which is injective (where $n = \deg f$).




**Definition.** The subgroup $\mathrm{Gal}(f/K) \subseteq S_n$ given by the image of $G$ is the Galois group of $f$ over $K$. Note that $[L : K] = |\mathrm{Gal}(L/K)| = |\mathrm{Gal}(f/K)|$ so divides $n!$.

There exist several methods for determining $\mathrm{Gal}(f/K)$.

**Proposition 9.5.** *A polynomial $f$ is irreducible if and only if $\mathrm{Gal}(f/K)$ is transitive (recall that a subgroup $G \subseteq S_n$ is transitive if $\forall i, j \in \{1, \ldots, n\}$, there exists $\sigma \in G$ with $\sigma(i) = j$, i.e there is only one orbit).*

*Proof.* Let $x$ be a root of $f$ in a splitting field $L$. Then its orbit under $G = \mathrm{Gal}(f/K)$ is the set of roots of $m_{x,K}$ (by 9.2). As $m_{x,K} \mid f$, have $m_{x,K} = f$ if and only if $f$ is irreducible. And $m_{x,K} = f$ if and only if every root of $f$ is in the orbit of $x$, i.e iff $G$ acts transitively on the roots of $f$. □

**Remark**: if $G \subseteq S_n$ is transitive, then by the orbit-stabiliser theorem, $n \mid |G|$.

Recall (from section 2) the discriminant: if $f \in K[T]$ is monic, $f = \prod_{1 \le i \le n}(T - x_i)$ in $L$ (splitting field) then $\mathrm{Disc}(f) = \Delta^2 \in K$ where $\Delta = \prod_{1 \le i \le j \le n}(x_i - x_j)$. $\mathrm{Disc}(f) \neq 0$ if and only if $f$ is seperable.

**Proposition 9.6.** *Assume $\mathrm{char}(K) \neq 2$. The fixed field of $G \cap A_n$ is $K(\Delta)$. In particular, $\mathrm{Gal}(f/K) \subseteq A_n$ if and only if $\mathrm{Disc}(f)$ is a square in $K$.*

*Proof.* If $\pi \in S_n$, the sign of $\pi$ is an element of $\{\pm 1\}$, and

$$\prod_{1 \leq i \leq j \leq n} (T_{\pi(i)} - T_{\pi(j)}) = \operatorname{sgn}(\pi) \prod_{1 \leq i \leq j \leq n} (T_i - T_j)$$

So if $\sigma \in G$, then $\sigma(\Delta) = \operatorname{sgn}(\sigma)\Delta$. Since $\operatorname{char}(K) \neq 2$, $-1 \neq 1$ so as $\Delta \neq 0$, this implies $\Delta \in K$ if and ony if $G \subseteq A_n$ and $\Delta$ lies in the fixed field $F$ of $G \cap A_n$. As

$$[F : K] = (G : G \cap A_n) = \begin{cases} 1 & \text{if } G \subseteq A_n \\ 2 & \text{otherwise} \end{cases}$$

we have $F = K(\Delta)$. □

**Example.** Let $f = T^3 + aT + b$, say $f = \prod_{i=1}^3 (T - x_i)$, $x_3 = -x_1 - x_2$ and

$$\begin{aligned} a &= x_1x_2 - (x_1 + x_2)^2 \\ b &= x_1x_2(x_1 + x_2) \end{aligned}$$

So (plugging in) $\operatorname{Disc}(f) = -4a^3 - 27b^2$. So $\operatorname{Gal}(f/K) \subseteq A_3$ if and only if $-4a^3 - 27b^2$ is a square in $K$. Suppose $a = -21$, $b = -7$. Then $f \in \mathbb{Q}[T]$ is irreducible. We have $\operatorname{Disc}(f) = 4 \cdot 21^3 - 27 \cdot 7^2 = (27 \cdot 7)^2$. So $\operatorname{Gal}(f/\mathbb{Q}) \subseteq A_3$. As $f$ is irreducible, the Galois group is transitive, so $\operatorname{Gal}(f/\mathbb{Q}) = A_3$. Thus this method computes the Galois group of any cubic polynomial (when $\operatorname{char}(K) \neq 2, 3$).

# 10 Finite fields

Let $p$ be prime, and write $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We aim to describe all finite fields of characteristic $p$ (i.e all finite extensions $F$ of $\mathbb{F}_p$), and their Galois theory. Recall:

- $|F| = p^n$, where $n = [F : \mathbb{F}_p]$.
- $F^\times$ is cyclic of order $p^n - 1$.
- $\varphi_p : F \to F$, $x \mapsto x^p$ is an automorphism of $F$.

**Theorem 10.1.** *Let $n \geq 1$. Then there exists a field with $q = p^n$ elements. Any such field is a splitting field of the polynomial $T^q - T$ over $\mathbb{F}_p$. In particular, any two finite fields of the same order are isomorphic.*

*Proof.* Let $F$ be a field with $q = p^n$ elements. Then if $x \in F^\times$, $x^{q-1} = 1$. So for all $x \in F$, $x^q = x$. So $f = T^q - T = \prod_{x \in F}(T - x)$ splits into linear factors in $F$, and not in any proper subfield of $F$. So $F$ is a splitting field for $f$ over $\mathbb{F}_p$. So by uniqueness of splitting fields, $F$ is unique up to isomorphism.

To show the existence of such an $F$, given $n$, let $L/\mathbb{F}_p$ be a splitting field of $f = T^q - T$ where $q = p^n$. Let $F \subseteq L$ be the fixed field of $\varphi_p^n : x \mapsto x^q$. So $F$ is the set of roots of $f$ in $L$. So $|F| = q$ (and $F = L$). □

**Notation**: write $\mathbb{F}_q$ for any finite field with $q$ elements (by the above theorem, any two such fields are isomorphic, although there is no canonical isomorphism).

**Theorem 10.2.** *$\mathbb{F}_{p^n}/\mathbb{F}_p$ is Galois with Galois group cyclic of order $n$, generated by $\varphi_p$.*

*Proof.* $T^{p^n} - T = \prod_{x \in \mathbb{F}_{p^n}} (T - x)$ is seperable, so $\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_p$ (as the splitting field of a seperable polynomial). Let $G \subseteq \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the subgroup generated by $\varphi_p$. Then $\mathbb{F}_{p^n}^G = \{x : x^p = x\} = \mathbb{F}_p$. So by the Galois correspondence $G = \mathrm{Gal}(\mathbb{F}_{p^n}, \mathbb{F}_p)$. □

**Theorem 10.3.** *$\mathbb{F}_{p^n}$ has a unique subfield of order $p^m$ for each $m \mid n$, and no others. If $m \mid n$ then $\mathbb{F}_{p^m} \subseteq \mathbb{F}_{p^n}$ is the fixed field of $\varphi_p^m$.*

*Proof.* $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$. The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are the $m\mathbb{Z}/n\mathbb{Z}$ for $m \mid n$, $m \geq 1$. So by Galois correspondence, the subfields of $\mathbb{F}_{p^n}$ are the fixed fields of these subgroups, i.e of the subgroups $\langle \varphi_p^n \rangle$, which have degree equal to the indices $(\mathbb{Z}/n\mathbb{Z} : m\mathbb{Z}/n\mathbb{Z}) = m$. □

**Remark**: if $m \mid n$, then $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^m}) = \langle \varphi_p^m \rangle$.

**Theorem 10.4.** *Let $f \in \mathbb{F}_p[T]$ be seperable of degree $n \geq 1$, whose irreducible factors have degrees $n_1, \ldots, n_r$, $\sum n_i = n$. Then $\mathrm{Gal}(f/\mathbb{F}_p) \subseteq S_n$ is cyclic, generated by an element of cycle type $(n_1, \ldots, n_r)$. In particular, $|\mathrm{Gal}(f/\mathbb{F}_p)|$ is equal to the lowest common multiple of $\{n_i\}$.*

*Proof.* Let $L$ be a splitting field for $f$ over $\mathbb{F}_p$, with roots $x_1, \ldots, x_n \in L$. Then $\mathrm{Gal}(L/F)$ is cyclic, generated by $\varphi_p$. As the irreducible factors of $f$ are the minimal polynomials of the $x_i$'s, and the set of roots of the minimal polynomial of $x_i$ is the orbit of $\varphi_p$ on $x_i$, the cycle type of $\varphi_p$ is $(n_1, \ldots, n_r)$. The order of any such permutation is then $\mathrm{lcm}(n_1, \ldots, n_r)$. □