Important theorems and their proofs

This is a list of some important theorems and proofs from IA Analysis. Please note that this list is not exhaustive and there may be a few inaccuracies.

Contents

List of Theorems

1 Series

Theorem 1.1: Bolzano Weirstrass

Every bounded sequence (x_n) has a convergent subsequence (x_{n_k}).

Proof 1.1

Let $[a_1, b_1] = [-K, K]$ and let $c = (b_1 - a_1)/2$. At least one of the intervals $[a_1, c_1]$ and $[c_1, b_1]$ must contain infinitely many terms of (x_n).

If $[a_1, c_1]$ does let $a_2 = a_1, b_2 = c_1$, otherwise let $a_2 = c_1, b_2 = b_1$. Proceed this way to consruct a sequence of nested intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \dots$$

Each of which contain infinitely many of the x_n. Furthermore

$$a_1 \leq a_2 \leq \dots a_n \leq b_n \leq \dots b_2 \leq b_1$$

The sequences formed by the a_n and the b_n are increasing and decreasing repectively, and bounded. Hence both converge. Also, $a_n - b_n \leq (a_1 - b_1)/2^n$ so $a_n - b_n \to 0$ as $n \to \infty$. Hence choosing $x_{n_k} \in [a_k, b_k]$ for each k gives a convergent subsequence. $\square$

Theorem 1.2: Cauchy Convergence

In $\mathbb{R}$ a sequence is convergent iff it is Cauchy convergent.

Proof 1.2

The triangle inequality quickly shows that convergent sequences are Cauchy:

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon$$

To show Cauchy sequences converge note the following:

1. Cauchy sequences are bounded (follows quickly since $|a_n| \leq |a_n - a_m| + |a_m|$)
2. Hence Bolzano-Weirstrass gives a convergent subsequence
3. Since the sequence is Cauchy, the convergent subsequence implies the whole sequence converges $\square$

Theorem 1.3: Root Test

Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then $\sum a_n$ converges if $a < 1$ and diverges if $a > 1$.

Proof 1.3

If $a < 1$:

Choose $a < r < 1$. Then by the definition of the limit, $\exists N$ such that $a_n^{1/n} < r$ for all $n \geq N$. Since $r < 1$, $\sum a_n$ then converges by comparison with the geometric series r^n.

If $a > 1$:

For $n \geq N$, we have $a_n > 1$. So a_n does not tend to zero, and $\sum a_n$ diverges. □

Theorem 1.4: Absolute Convergence Implies Convergence

If $\sum |a_n|$ converges, then so does $\sum a_n$.

Proof 1.4

Let $w_n = (|a_n| + a_n)/2$ and $v_n = (|a_n| - a_n)/2$.

Then clearly $w_n, v_n \geq 0$ so if $\sum |a_n| = \sum (w_n + v_n)$ converges, so do $\sum w_n$ and $\sum v_n$.

Hence $\sum (w_n - v_n) = \sum a_n$ does too. □

Theorem 1.5: Absolutely Convergent Series can be Rearranged

If $\sum a_n$ is absolutely convergent, every rearrangement $\sum a_{\sigma(n)}$ has the same sum.

Proof 1.5

Let a'_n be a rearrangement of a_n. Let $s_N = \sum_1^N a_n$, $t_N = \sum_1^N a'_n$.

Suppose further that $a_n \geq 0$. Then for every N, we can find q such that s_q contains every term of t_N. Hence $t_N \leq s_q \leq s$. As $N \to \infty$, $t_N \to t \Rightarrow t \leq s$. By a similar argument, $s \leq t$ as well, so $t = s$.

Now consider $w_n = (|a_n| - a)/2$ and $v_n = (|a_n| + a)/2$ and define w'_n and v'_n in the obvious way. Using the previous result for positive sums, it is simple to deduce that $\sum (w_n - v_n)$ and $\sum (w'_n - v'_n)$ converge to the same sum. □

2 Continuity

Theorem 2.1: Equivalence of Continuity Definitions

The following are equivalent:

1. $f : E \to \mathbb{C}$ is continuous at $a \in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) = f(a)$
2. $f : E \to \mathbb{C}$ is continuous at $a \in E$ if given $\varepsilon > 0$, $\exists \delta > 0$ such that $|z - a| < \delta \Rightarrow |f(z) - f(a)| < \varepsilon$

Proof 2.1

To show that (1)⇒(2):

Assume $f(z_n) \to f(a)$ whenever $z_n \to a$. Now suppose f did not satisfy the continuous definition given in (2). Then there exists some $\varepsilon > 0$ such that for all $\delta > 0$, there exists a z with $|z - a| < \delta$ and $|f(z) - f(a)| \geq \varepsilon$.

Not let $\delta = 1/n$. Then we have a z_n with $|z_n - a| < 1/n$ and $|f(z_n) - f(a)| \geq \varepsilon$. But then we have $z_n \to a$ while $f(z_n)$ does not tend to $f(a)$. Contradiction.

To show that (2)⇒(1):

Now let $z_n \to a$, then $\exists N$ such that $\forall n \geq N$ we have

$$|z_n - a| < \delta \Rightarrow |f(z_n) - f(a)| < \varepsilon$$

The implication follows from the definition given in (2). □

Theorem 2.2: Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) \neq f(b)$ and let $I \subseteq [a, b]$. Then $f(I)$ is an interval.

Proof 2.2

Wlog suppose $f(a) < f(b)$. Then we need to show f takes all values $f(a) < \eta < f(b)$. Fix some η and consider the set

$$S = \{x \in [a, b] : f(x) < \eta\}$$

$a \in S$ so the set is non-empty. Clearly S is bounded above by b. Hence $c = \sup(S)$ exists. Since f is continuous, for a given ε we can choose δ as above so that

$$f(x) - \varepsilon < f(c) < f(x) + \varepsilon$$

For all $x \in (c-\delta, c+\delta)$. By the definition of the supremum, there exists $a \in (c-\delta, c]$ with $a \in S$ such that

$$f(c) < f(a) + \varepsilon \leq \eta + \varepsilon$$

Now we can choose $b \in (c, c+\delta)$ and since b cannot be in S

$$\eta - \varepsilon < f(b) - \varepsilon < f(c)$$

Finally, combining the inequalities we have

$$\eta - \varepsilon < f(c) < \eta + \varepsilon$$

Since ε was arbitrary, we conclude that $f(c) = \eta$. $\square$

Theorem 2.3: Continuous Functions are Bounded

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists K such that $|f(x)| \leq K$ for all $x \in [a, b]$.

Proof 2.3

Suppose not. Then we can make a sequence x_n such that $|f(x_n)| > n$. x_n is bounded (by b), so by Bolzano-Weirstrass, x_n has a convergent subsequence $x_{n_j} \to x$. By continuiuty of f, $f(x_{n_j}) \to f(x)$ but $|f(x_{n_j})| \to \infty$. Contradiction $\square$

Theorem 2.4: Continuous Functions Achieve their Bounds

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

For all $x \in [a, b]$

Proof 2.4

Let $A = f([a, b])$. A is non-empty and bounded, so $M = \sup(A)$ exists. Furthermore, for all $n \in \mathbb{N}$ there is an x_n such that

$$M - 1/n < f(x_n) \leq M$$

By properties of the supremum. x_n is a bounded sequence, so $x_{n_j} \to x$ for some subsequence. Also $f(x_{n_j}) \to M$ by the above so $f(x_{n_j}) \to f(x) = M$ by continuity. Similar proof for minimum. $\square$

3 Differentiability

Theorem 3.1: Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then if $f(a) = f(b)$, there exists $c \in (a, b)$ with $f'(c) = 0$.

Proof 3.1

Let $M = \max_{[a,b]} f(x)$, $m = \min_{[a,b]} f(x)$. These exist since continuous functions achieve their bounds.

Let $k = f(a)$. If $M = m = k$ then f is contant and $f'(c) = 0$ for any $c \in (a, b)$.

Then $M > k$ or $m < k$. Suppose $M > k$. Then there exists $c \in (a, b)$ such that $f(c) = M$. Note that if $f'(c) > 0$

$$f(c + h) - f(c) = h(f'(c) + \varepsilon(h)) > 0$$

For $h > 0$ sufficiently small. Hence $f(c+h)$ is larger which contradicts that M is the maximum. Similarly if $f'(c) < 0$, $f(c - h) > f(c)$ for h sufficiently small. Therefore $f'(c) = 0$. □

Theorem 3.2: Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Proof 3.2

Let $\phi(x) = f(x) - kx$. Choose $k = (f(b) - f(a))/(b - a)$ so that $\phi(a) = \phi(b)$. Then by Rolle's Theorem, there exists $c \in (a, b)$ with

$$\phi'(c) = 0 \Rightarrow f'(c) = k = (f(b) - f(a))/(b - a)$$

□

Theorem 3.3: Inverse Function Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) > 0$ for all $x \in (a, b)$. Let $c = f(a)$, $d = f(b)$. Then the function $f : [a, b] \to [c, d]$ is bijective and f^{-1} is differentiable on (c, d) with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof 3.3

If $y > x$, then $f(y) - f(x) = f'(c)(y-x)$ for some $c \in (x,y)$ by the mean value theorem. Since $f'(c) > 0$, $f(y) - f(x) > 0$ and so f is increasing. Hence $f : [a,b] \to [c,d]$ is a bijection.

Let $f^{-1} = g$, let x and $k \neq 0$ be given, let $y = f(x)$, and let h be given by:

$$y + k = f(x+h) \iff g(y+k) = x + h$$

Then

$$\frac{g(y+k) - g(y)}{k} = \frac{h}{f(x+h) - f(x)}$$

Let $k \to 0$, then $h \to 0$ since g is continuous. Hence

$$g'(y) = \lim_{h\to 0} \frac{h}{f(x+h) - f(x)} = \frac{1}{f'(x)}$$

As required. $\square$

Theorem 3.4: Cauchy's Mean Value Theorem

Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $t \in (a,b)$ such that

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a))$$

Proof 3.4

Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$$

ϕ is continuous on $[a,b]$ and differentiable on (a,b). Also $\phi(a) = \phi(b)$. Apply Rolle's Theorem and the result follows. $\square$

Theorem 3.5: Taylor's Theorem with Lagrange's Remainder

Suppose f and its first $(n-1)$th derivatives are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a + \theta h)$$

where $\theta \in (0,1)$

Proof 3.5

For $0 \leq t \leq h$ define

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \dots \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}B$$

Where we choose B such that $\phi(h) = 0$.

We see that $\phi(0) = \phi'(0) = \dots = \phi^{(n-1)}(0) = 0$

$\phi(0) = \phi(h) = 0 \Rightarrow \phi'(h_1) = 0$ for some $0 < h_1 < h$

$\phi'(0) = \phi'(h_1) = 0 \Rightarrow \phi''(h_2) = 0$ for some $0 < h_2 < h_1$

$\vdots$

Finally $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0 \Rightarrow \phi^{(n)}(h_n) = 0$, where

$$0 < h_n < h_{n-1} < \dots < h_1 < h$$

So $h_n = \theta h$ for $\theta \in (0,1)$. $\phi^{(n)}(t) = f^{(n)}(a+t) - B \Rightarrow B = f^{(n)}(a+\theta h)$

Finally set $t = h$, $\phi(h) = 0$ and put this value for B in the second line of the proof. $\square$

Theorem 3.6: Taylor's Theorem with Cauchy Remainder

Suppose f and its first $(n-1)$th derivatives are continuous in $[0,h]$ and $f^{(n)}$ exists for $x \in (0,h)$. Then

$$f(h) = f(0) + hf'(0) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where $R_n = \dfrac{(1-\theta)^{n-1}h^n f^{(n)}(\theta h)}{(n-1)!}$, $\theta \in (0,1)$

Proof 3.6

For $t \in [0, h]$ define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \ldots - \frac{(h-t)^{n-1} f^{(n-1)}(t)}{(n-1)!}$$

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) + \ldots - \frac{(h-t)^{n-1} f^{(n)}(t)}{(n-1)!}$$

$$\Rightarrow F'(t) = -\frac{(h-t)^{n-1} f^{(n)}(t)}{(n-1)!}$$

Set $\phi(t) = F(t) - \left(\frac{h-t}{h}\right)^p F(0)$

where $p \in \mathbb{Z}, 1 \leq p \leq n$. Then $\phi(0) = \phi(h) = 0$. By Rolle's Theorem, there exists $\theta \in (0, 1)$ such that $\phi'(\theta h) = 0$. But

$$\phi'(\theta h) = F'(\theta h) + p\frac{(1-\theta)^{p-1}}{h} F(0) = 0$$

By subsituting the expressions for F and F' we get

$$f(h) = f(0) + hf'(0) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{h^n (1-\theta)^{n-1} f^{(n)}(\theta h)}{(n-1)! p (1-\theta)^{p-1}}$$

Letting $p = n$ gives Lagrange's remainder and $p = 1$ gives Cauchy's. $\square$

4 Power Series

Theorem 4.1: Comparison of Power Series

If $\sum_0^\infty a_n z_1^n$ converges and $|z| < |z_1|$ then $\sum_0^\infty a_n z^n$ converges absolutely.

Proof 4.1

We must have $a_n z_1^n \to 0$ so for all n, $a_n z_1^n \leq K$ for some K. Hence

$$a_n z^n \leq K|z/z_1|^n$$

So by comparison with the geometric series $\sum_0^\infty |z/z_1|^n$ the sum converges. $\square$

Theorem 4.2: Radius of Convergence

A power series either:

1. Converges absolutely for all z
2. Converges absolutely for all z in a circle $|z| = R$ and diverges outside it
3. Converges for $z = 0$ only

Proof 4.2

Let $S = \{x \in \mathbb{R} : x \geq 0 \text{ and } \sum a_n x^n \text{ converges}\}$

Clearly $0 \in S$. By the previous theorem, we have $x_1 \in S \implies [0, x_1] \in S$. We now consider the following cases

- If S is not bounded, then $[0, \infty) \in S$ and we have case (1)
- If S is bounded by some non-zero value, $R = \sup(S)$ exists and for all $|z| < R$, the series converges which is case (2)
- If $S = \{0\}$ then we have case (3)

$\square$

Theorem 4.3: The Exponential Function

Define $\exp : \mathbb{R} \to \mathbb{R}$ by $\exp(z) = \sum_0^\infty \frac{z^n}{n!}$. Then

1. $\exp(x)$ is everywhere differentiable and $\exp'(x) = \exp(x)$
2. $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$
3. $\exp(x) > 0$ for all $x \in \mathbb{R}$
4. exp is strictly increasing
5. $\exp(x) \to \infty$ as $x \to \infty$ and $\exp(x) \to 0$ as $x \to -\infty$
6. $\exp : \mathbb{R} \to (0, \infty)$ is a bijection

Proof 4.3

1. Easy to check $R = \infty$ and $\exp(x) = \exp'(x)$ from the series, so claim follows
2. Let $F(z) = \exp(x + y - z)\exp(z)$ then $F'(z) = 0$. Hence F is constant and by comparing $F(0)$ and $F(y)$ we get $\exp(x)\exp(y) = \exp(x+y)$
3. For $x > 0$ clearly $\exp(x) > 0$ by comparison with the power series. Furthermore from (2) for $x > 0$ we have $\exp(x)\exp(-x) = \exp(0) = 1 \Rightarrow \exp(-x) > 0$
4. $\exp'(x) = \exp(x) > 0$
5. $\exp(x) > 1 + x$ so $\exp(x) \to \infty$ as $x \to \infty$. Similarly by noting $\exp(-x) = (\exp(x))^{-1}$ by (2) we have $\exp(x) \to 0$ as $x \to -\infty$
6. Follows from (4) and (5)

$\square$

5 Integration

Theorem 5.1: Monotonic Functions are Integrable

Let $f : [a, b] \to \mathbb{R}$ be monotonic. Then it is integrable.

Proof 5.1

Wlog f is increasing. Then

$$\sup_{x\in[x_{j-1},x_j]} f(x) = f(x_j), \quad \inf_{x\in[x_{j-1},x_j]} f(x) = f(x_{j-1})$$

And thus $S(f, D) - s(f, D) = \sum_{j=1}^{n}(x_j - x_{j-1})(f(x_j) - f(x_{j-1}))$. Consider the dissection D with points equally spaced at a distance $(b - a)/n$ apart. Then

$$S(f, D) - s(f, D) = \frac{b - a}{n}(f(b) - f(a))$$

Clearly this $\to 0$ as $n \to \infty$. $\square$

Theorem 5.2: Uniform continuity

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)|$$

Proof 5.2

Suppose the claim is false. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$ we can find $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ but $f(x) - f(y) \geq \varepsilon$.

Take $\delta < 1/n$ to get x_n, y_n with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. By Bolzano-Weirstrass there is a subsequence $x_{n_k} \to c$. Then

$$|y_{n_k} - c| \leq |x_{n_k} - y_{n_k}| + |x_{n_k} - c| \to 0$$

But $|f(x_{n_k})-f(y_{n_k})| \geq \varepsilon$. Now let $k \to \infty$ and we get $f(x_{n_k})-f(y_{n_k}) \to f(c)-f(c) = 0$ by continuity of f at c. Contradiction. $\square$

Theorem 5.3: Continuous Functioons are Integrable

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable.

Proof 5.3

By the previous theorem, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Let D be the dissection with points equally spaced at a distance $(b-a)/n$ apart. Choose n large so that $(b-a)/n < \delta$. Furthermore

$$\sup_{x \in [x_{j-1}, x_j]} f(x) = f(p_j), \inf_{x \in [x_{j-1}, x_j]} f(x) = f(q_j)$$

For some $p_j, q_j \in [x_{j-1}, x_j]$ since continuous functions achieve their bounds. Hence

$$S(f, D) - s(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1})(f(p_j) - f(q_j)) < \varepsilon \sum_{j=1}^{n} (x_j - x_{j-1}) = \varepsilon(b - a)$$

$\square$

Theorem 5.4: Elementary Integral Properties

Let f, g be be bounded and integrable on $[a, b]$.

1. If $f \leq g$ on $[a, b]$, then $\int_a^b f \leq \int_a^b g$
2. $f + g$ is integrable on $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$
3. For any constant K, Kf is integrable and $\int_a^b Kf = K \int_a^b f$
4. $|f|$ is integrable and $|\int_a^b f| \leq \int_a^b |f|$
5. The product fg is integrable

Proof 5.4

1. $\int_a^b f = I^*(f) \leq S(f,D) \leq S(g,D)$ hence $\int_a^b f = I^*(f) \leq I^*(g) = \int_a^b g$ by taking the infimum over D

2. Clearly $\sup(f+g) \leq \sup(f) + \sup(g)$ on every interval. Hence

   $$I^*(f+g) \leq S(f+g, D_1 \cup D_2) \leq S(f, D_1 \cup D_2) + S(g, D_1 \cup D_2) \leq S(f, D_1) + S(g, D_2)$$

   Now take infimum over D_2 while keeping D_1 fixed to get

   $$I^*(f+g) \leq S(f, D_1) + I^*(g)$$

   Taking infimum over D_1

   $$\int_a^b (f+g) = I^*(f+g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g$$

   Similarly $I_*(f) + I_*(g) \leq I_*(f+g)$ so $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

3. Follows immediately from noting $S(Kf, D) - s(Kf, D) = K(S(f,D) - s(f,D))$

4. Consider $f_+(x) = \max(f(x), 0)$ then $\sup(f_+) - \inf(f_+) \leq \sup(f) - \inf(f)$ on every interval. Hence

   $$S(f,D) - s(f,D) < \varepsilon \implies S(f_+, D) - s(f_+, D) < \varepsilon$$

   So f_+ is integrable since f is. By noting that $|f| = 2f_+ - f$ and applying (2) and (3) we have $|f|$ is integrable. Since $-|f| \leq f \leq |f|$ (1) also tells us that

   $$\left| \int_a^b f \right| \leq \int_a^b |f|$$

5. Suppose $f \geq 0$. Then $\sup(f)^2 = \sup(f^2)$ on every interval, and similarly for inf. f is bounded since it is continuous so let K be such that $|f(x)| \leq K$ for all $x \in [a.n]$. Then

   $$S(f^2, D) - s(f^2, D) = \sum_{j=1}^{n} (x_j - x_{j-1})(M_j^2 - m_j^2)$$

   $$= (M_j + m_j) \sum_{j=1}^{n} (x_j - x_{j-1})(M_j - m_j) \leq 2K(S(f,D) - s(f,D))$$

   And therefore f^2 is integrable. For general f, we have $|f|$ is integrable by (4) and so $f^2 = |f|^2$ is integrable too. Finally, to show the product is integrable note

   $$4fg = (f+g)^2 - (f-g)^2$$

   And apply previous properties. $\square$

Theorem 5.5: Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{R}$ be bounded and integrable. Then define $F : [a, b] \to \mathbb{R}$

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

F is continuous. Furthermore, if f is continuous at x, F is differentiable at x with $F'(x) = f(x)$.

Proof 5.5

First we show that F is continuous. Note $F(x+h) - F(x) = \int_x^{x+h} f(t)\mathrm{d}t$ so $|F(x+h) - F(x)| = |\int_x^{x+h} f(t)\mathrm{d}t| \leq K|h|$ for some K (since f is bounded). Then letting $h \to 0$ shows F is indeed continuous.

To show $F'(x) = f(x)$, consider $x + h \in [a, b]$ with $h \neq 0$. Then

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \frac{1}{|h|} \left| \int_x^{x+h} (f(t) - f(x))\mathrm{d}t \right|$$

If f is continuous at x, given $\varepsilon > 0$ we have $\delta > 0$ such that $|t - x| < \delta$ implies $|f(t) - f(x)| < \varepsilon$. If $|h| < \delta$ we can write

$$\frac{1}{|h|} \left| \int_x^{x+h} (f(t) - f(x))\mathrm{d}t \right| \leq \frac{1}{|h|}(|h|\varepsilon) = \varepsilon$$

□

Theorem 5.6: Taylor's Theorem with Integral Remainder

Let $f^{(n)}(x)$ be continuous for $x \in [0, h]$. Then

$$f(h) = f(0) + \ldots + \frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + R_n$$

where $R_n = \dfrac{h^n}{(n-1)!} \displaystyle\int_0^1 (1-t)^{n-1} f^{(n)}(th)\mathrm{d}t$

Proof 5.6

By substituting $u = th$ we have $R_n = \frac{1}{(n-1)!}\int_0^h (h-u)^{n-1} f^{(n)}(u)\mathrm{d}u$

Intergating by parts we get $R_n = -\frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} + R_{n-1}$. So by applying integration by parts $n-1$ times, we get

$$R_n = -\frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} - \ldots - hf'(0) + \underbrace{\int_0^h f'(u)\mathrm{d}u}_{f(h)-f(0)}$$

As required. $\square$

Theorem 5.7: Mean Value Theorem for Integrals

Let $f, g : [a, b] \to \mathbb{R}$ be continuous with $g(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = f(c)\int_a^b g(x)\mathrm{d}x$$

Proof 5.7

Apply Cauchy's MVT to $F(x) = \int_a^x fg\ \mathrm{d}x$ and $G(x) = \int_a^x g\ \mathrm{d}x$

Then for some $c \in (a, b)$, $(F(b) - F(a))G'(c) = F'(c)(G(b) - G(a))$ and therefore

$$\left(\int_a^b fg\ \mathrm{d}x\right) g(c) = f(c)g(c)\int_a^b g\ \mathrm{d}x$$

Since $g(c) \neq 0$ we are done. $\square$

Theorem 5.8: Integral Test

Let f be a positive decreasing function for $x \geq 1$. Then

1. The integral $\int_1^\infty f(x)\mathrm{d}x$ and the series $\sum_1^\infty f(n)$ either both converge or both diverge
2. As $n \to \infty$, $\sum_{r=1}^n f(r) - \int_1^n f(x)\mathrm{d}x$ tends to a limit l with $0 \leq l \leq f(1)$

Proof 5.8

Since f is decreasing, it is integrable on every bounded subinterval of $[1, \infty)$. If $n-1 \leq x \leq n$ then

$$f(n-1) \geq f(x) \geq f(n) \implies f(n-1) \geq \int_{n-1}^{n} f(x)\mathrm{d}x \geq f(n)$$

By summing the inequality over n we get

$$\sum_{r=1}^{n-1} f(r) \geq \int_{1}^{n} f(x)\mathrm{d}x \geq \sum_{r=2}^{n} f(r)$$

And from this claim (1) follows immediately. To prove (2) set $\phi(n) = \sum_1^n f(r) - \int_1^n f(x)\mathrm{d}x$. Then $0 \leq \phi(n) \leq f(1)$ and

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^{n} f(x)\mathrm{d}x \leq 0$$

Hence ϕ is decreasing and bounded below, so tends to a limit l with $0 \leq l \leq f(1)$. $\square$