Introduction

Course structure:

(i) Preliminary toolbox: inequalities

(ii) Normed vector spaces (NVS)

(iii) (Recalls on) finite-dimensional case

(iv) Hahn-Banach Theorems (how big is the dual?)

(v) Completeness: Baire's Theorem & consequences for NVS

(vi) Detailed study of the topology of $C(K)$

(vii) The Hilbert space

(viii) Projection & duality

(ix) Introduction to operators and spectral theory

1 Preliminary toolbox: Young's, Hölder's & Minkowski's inequalities for vectors & sequences

Proposition (Young's inequality for products)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\forall a, b \geq 0,\ ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The result is clear for $a = 0$ or $b = 0$. Assume $a, b > 0$ and note $L : (0, \infty) \to \mathbb{R}$, $t \mapsto \ln t$ is strictly concave: $L''(t) = -\frac{1}{t^2} < 0$.

Therefore for all $A, B > 0$, $\lambda \in (0, 1)$

$$\ln(\lambda A + (1 - \lambda)B) \geq \lambda \ln A + (1 - \lambda) \ln B$$

with equality iff $A = B$. Apply this to $A = a^p$, $B = b^q > 0$ and $\lambda = \frac{1}{p}$. This gives

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) = \ln(ab)$$

so applying exp to both sides gives the result and furthermore we have equality iff $a^p = b^q$. $\square$

Proposition (Hölder's inequality for vectors & sequences)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

(i) for any $n \in \mathbb{N} \setminus \{0\} = \mathbb{N}^*$, $\forall x, y \in \mathbb{R}^n$

$$\sum_{k=1}^{n} |x_k y_k| \leq ||x||_p ||y||_q \tag{$*$}$$

with $||x||_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ and similarly for $||y||_q$.

(ii) define

$$\ell^p = \{x \in \mathbb{R}^{\mathbb{N}^*} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

then $\forall x \in \ell^p$, $y \in \ell^q$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq ||x||_{\ell^p} ||y||_{\ell^q}$$

where $||x||_{\ell^p} = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ and similar for $||y||_{\ell^q}$.

Proof. To show (i) implies (ii): take $n \to \infty$ in (i) so

$$\sum_{k=1}^{n} |x_k|^p \to ||x||_{\ell^p}^p$$

and similarly

$$\sum_{k=1}^{n} |y_k|^q \to ||y||_{\ell^q}^q$$

By (i)

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}$$

so

$$\begin{aligned}\sum_{k=1}^{\infty} |x_k y_k| = \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k y_k|\right) &\leq \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q} \\ &= ||x||_{\ell^p} ||y||_{\ell^q}\end{aligned}$$

Proof of (i): if $||x||_{\ell^p}$ or $||y||_{\ell^q} = 0$, result is clear. Otherwise define $\tilde{x}$, $\tilde{y}$ sequences in ℓ^p and ℓ^q by

$$\tilde{x}_k = \frac{x_k}{||x||_{\ell^p}},\ \tilde{y}_k = \frac{y_k}{||y||_{\ell^q}}$$

Then $||\tilde{x}||_{\ell^p} = 1$, $||\tilde{y}||_{\ell^q} = 1$. Then $(*)$ is equivalent to showing

$$\sum_{k=1}^{n} |\tilde{x}_k \tilde{y}_k| \leq 1 \tag{$**$}$$

Apply Young's inequality on each $k = 1, \ldots, n$ so

$$|\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}|\tilde{x}_k|^p + \frac{1}{q}|\tilde{y}_k|^q$$

Summing over k:

$$\sum_{k=1}^n |\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}\left(\sum_{k=1}^n |\tilde{x}_k|^p\right) + \frac{1}{q}\left(\sum_{k=1}^n |\tilde{y}_k|^q\right) \leq \frac{1}{p} + \frac{1}{q} = 1$$

□

Remark: Equality in $(*)$ is equivalent to equality in $(**)$ which is equivalent to equality in Young's for each k so $|\tilde{x}_k|^p = |\tilde{y}_k|^q$ for $k = 1, \ldots, n$. Also, the $p = 1$, $q = \infty$ case is easy.

Proposition (Minkowski's inquality for vectors & sequences)**.** Let $p \in [1, \infty)$, then

(i) for all $x, y \in \mathbb{R}^n$

$$||x + y||_p \leq ||x||_p + ||y||_p$$

(ii) for all $x, y \in \ell^p$

$$||x + y||_{\ell^p} = ||x||_{\ell^p} + ||y||_{\ell^p}$$

Proof. To show (i) implies (ii): by taking $n \to \infty$ as before

$$\sum_{k=1}^\infty |x_k|^p \to ||x||_{\ell^p}^p$$
$$\sum_{k=1}^\infty |y_k|^p \to ||y||_{\ell^p}^p$$
$$\sum_{k=1}^n |x_k + y_k|^p \to ||x + y||_{\ell^p}^p$$

Proof of (i): if $p = 1$ this is just the usual triangle inequality on each coordinate. So let $p \in (1, \infty)$ and

$$\sum_{k=1}^n |x_k + y_k|^p = \sum_{k=1}^n |x_k + y_k| \cdot |x_k + y_k|^{p-1}$$

$$\leq \sum_{k=1}^n |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k||x_k + y_k|^{p-1}$$

$$\underbrace{\leq}_{\text{Hölder: } q = \frac{p}{p-1}} ||x||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + ||y||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\leq \left(||x||_p + ||y||_p\right) ||x+y||_p^{p-1}$$

so we have proved

$$||x+y||_p^p \leq \left(||x||_p + ||y||_p\right) ||x+y||_p^{p-1}$$

If $||x+y||_p = 0$, result is clear. Otherwise divide by $||x+y||_p^{p-1}$ to get

$$||x+y||_p \leq ||x||_p + ||y||_p$$

□

Remark: equality occurs iff there is equality in the triangle inequality and Hölder's.
Remarks:

1. Equality case: $p = 1$: $|x_k + y_k| \leq |x_k| + |y_k|$, i.e the usual triangle inquality
2. For $p = 2$ there's another proof: define $\mathcal{P} : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto ||x + \lambda y||^2$. Then $\mathcal{P}(\lambda) = a\lambda^2 + 2b\lambda + c$ and $\mathcal{P} \geq 0$. So

$$\langle x, y \rangle = b^2 \leq ac = ||x||^2 ||y||^2, \text{ Hölder's inequality}$$

2 Normed Vector Spaces (NVS)

Remark: this is not the most general structure for linear analysis - topological vector spaces (TVS).

Recall:

Definition. A *vector space* V over a field $\mathbb{F}$ is a set (of elements called *vectors*) with two operations:

$$A : V \times V \to V,\ (v, w) \mapsto v + w \text{ addition}$$

$$M : \mathbb{F} \times V \to V,\ (\lambda, v) \mapsto \lambda v \text{ scalar multiplication}$$

such that

- $(V, +)$ is an abelian group with identity 0.
- M is compatible with $(\mathbb{F}, 0)$ in the sense that $\lambda_1(\lambda_2 v) = \lambda_1\lambda_2 v$
- M distributes over $(V, +)$ and $(\mathbb{F}, +)$.

In this course $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$ unless stated otherwise.

Definition. Given a vector space V over $\mathbb{F}$:

- a *subspace* $W \subseteq V$ is a vector space over $\mathbb{F}$ included in V
- for a set $S \subseteq V$, a *linear combination of elements of* S is a finite sum of elements of S with coefficients in $\mathbb{F}$
- for a set $S \subseteq V$, the *span of* S, $\text{span}(S)$ is the smallest subspace of V containing S, and is also the set of linear combinations of S.

Definition. Given V a vector space over $\mathbb{F}$ and a set $S \subseteq V$:

- S is *linearly independent* if for all $m \in \mathbb{N}^*$ and for all $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$, for all $s_1, \ldots, s_m \in S$, $\sum_{i=1}^m \alpha_i s_i = 0$ if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_m$.
- S is a *basis* of V if it is linearly independent and $\text{span}(S) = V$.
- If there exists a finite basis S of V, then V has finite dimension, otherwise it is infinite-dimensional.

Remark: later we'll prove with Zorn's lemma that any vector space has a basis.

Definition. A *normed vector space* (NVS) V over $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a function $N : V \to \mathbb{R}_+$, $v \mapsto ||v||$ (the *norm*), with

1. $||v|| \geq 0$ for all $v \in V$, with equality only at $v = 0$ (*positive definiteness*)
2. For all $\lambda \in \mathbb{F}$, $v \in V$ $||\lambda v|| = |\lambda|||v||$ (compatibility between N and M)

3. For all $v, w \in V$, $||v+w|| \leq ||v|| + ||w||$ (compatibility between N and A)

Example. $V = \mathbb{R}^n$, $v = (v_1, \dots, v_n)$, $||v|| = (v_1^2 + \ldots + v_n^2)^{1/2}$ or

$$\begin{cases} ||v||_p = (|v_1|^p + \ldots + |v_n|^p)^{1/p} & \text{for } p \in [1, \infty) \\ ||v||_\infty = \sup_{i=1}^n |v_i| & \text{for } p = \infty \end{cases}$$

Definition. Given a set X, a *topology* τ on X is a collection of subsets of X ("open sets") such that

- $\emptyset \in \tau$, $X \in \tau$
- τ is stable under <u>any</u> union
- τ is stable under finite intersections

Definition.

- For (X, d) a metric space, the *induced topology* is the smallest topology that contains open balls in d
- For a NVS $(V, ||\cdot||)$, the induced topology is that associated with $d(v, w) = ||v - w||$

Natural question: $\mathbb{F}$ field, V vector space over $\mathbb{F}$. Norm on V, $\tau_{||\cdot||}$. Continuity of operations M and A?

Proposition. Let $(V, ||\cdot||)$ be a NVS over $\mathbb{F}$ ($\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$), then

(i) A, M are continuous for the following topologies: $\tau_{||\cdot||}$ on V, then product topology of it on $V \times V$, $\tau_{|\cdot|}$ over $\mathbb{F}$, then product topology of $\tau_{|\cdot|}$ and $\tau||\cdot||$ on $\mathbb{F} \times V$

(ii) Translations $T_{v_0} : V \to V$, $v \mapsto v + v_0$, $v_0 \in V$ and dilations $D_{\lambda_0} : V \to V$, $v \mapsto \lambda_0 v$, $\lambda_0 \in \mathbb{F}^*$ are homeomorphisms

Proof.

(i) Let us prove that $A : V \times V \to V$ is continuous: consider an open set $\emptyset \neq U \subseteq V$ and $(v_1, v_2) \in A^{-1}(U)$, i.e $v_1 + v_2 \in U$. Since U is open, there is $\varepsilon > 0$ such that $\underbrace{B_V(v_1 + v_2, \varepsilon)}_{\text{open ball}} \subseteq U$.

We have that $A(B(v_1, \varepsilon/2), B_V(v_2, \varepsilon/2)) \subseteq B_V(v_1+v_2, \varepsilon)$ (triangle inequality). Note also that $B(v_1, \varepsilon/2) \times B(v_2, \varepsilon/2)$ is open (product topology), so $A^{-1}(U)$ is open and A is continuous.

Now we show $M : \mathbb{F} \times V \to V$ is continuous. Consider an open set $U \neq \emptyset$ in V, $(\lambda, v) \in M^{-1}(U)$. Since U is open, there exists $\varepsilon > 0$ such that $B_V(\lambda v, \varepsilon) \subseteq U$ (WLOG $\varepsilon < 1$). Then (check)

$$M\left(B_{\mathbb{F}}\left(\lambda, \frac{\varepsilon}{3\max(1, ||v||)}\right), B_V\left(v, \frac{\varepsilon}{3\max(1, |\lambda|)}\right)\right) \subseteq B_V(\lambda v, \varepsilon)$$

(ii) T_{v_0} and D_{λ_0} are linear, continuous with inverses T_{-v_0} and $D_{\lambda_0^{-1}}$ respectively, so are homeomorphisms.

$\square$

3 Characterisation of NVS

Idea: in order to better understand the topology of NVS's, we ask how special is a "normable" topology among topologies compatible with vector space operations?

Definition (TVS)**.** A *topological vector space* (TVS) over $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a topology τ such that

(i) A and M are continuous

(ii) every singleton $\{x_0\}$ is closed

Remark:

1. (i) says that T_{v_0} and D_{λ_0}, $\lambda_0 \neq 0$ are homeomorphisms
2. (ii) is called T_1 in the classification of seperation properties, and implies Hausdorff for TVS

Definition. Given V a TVS

- $C \subseteq V$ is *convex* if $C = \{\lambda c_1 + (1-\lambda)c_2 : c_1, c_2 \in C, \lambda \in [0,1]\}$
- V is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0
- $B \subseteq V$ is *bounded* if for any U open around 0, there exists $t_0 > 0$ such that $\forall t > t_0$, $B \subseteq tU$
- V is *locally bounded* if there is $U \in \tau$ containing 0 and bounded

Example. Let $(V, ||\cdot||)$ be a NVS, then for all $r > 0$, $U = B(0,r)$ (open ball) is open, bounded and convex. Indeed

- Convexity follows from the triangle inequality
- Boundedness: any other $\tilde{U}$ open around 0 contains some open $\tilde{U}_0 = B(0, r_0) \in \tilde{U}$. Then for any $t > \frac{r}{r_0}$, $U \subseteq t\tilde{U}_0 \subseteq t\tilde{U}$.

Question: can we reverse-engineer the norm if we have these two properties?

Theorem (Kolmogorov 1934)**.** *Let (V, τ) be a TVS such that there is a bounded convex neighborhood of 0, say C. Then V is "normable" - there is a norm $||\cdot||$ on V that induces the topology τ.*

Proof. Step 1: there is $\tilde{C} \subseteq C$ which is a *balanced* convex bounded neighborhood of 0. "Balanced" means that for all $\lambda \in \mathbb{F}$ such that $|\lambda| \leq 1$, $\lambda\tilde{C} \subseteq \tilde{C}$.

$M : \mathbb{F} \times V \to V$ is continuous so $M^{-1}(C)$ is a neighourhood of $(0,0)$. So there exists $B_{\mathbb{F}}(0,\varepsilon) \times U$ with $\varepsilon > 0$ and U open around 0 such that $M(B_{\mathbb{F}}(0,\varepsilon), U) \subseteq C$.

Define $\tilde{C}$ to be the convex hull (i.e smallest convex set supserset) of $M(B_{\mathbb{F}}(0,\varepsilon), U)$.

Then $\tilde{C}$ is clearly convex, is a subset of C since C is convex and $M(B_{\mathbb{F}}(0,\varepsilon), U) \subseteq C$. $\tilde{C}$ is also bounded since $\tilde{C} \subseteq C$ and C is bounded (obvious that boundedness is inherited by inclusion). Finally $\tilde{C}$ is balanced since $\lambda B_{\mathbb{F}}(0,\varepsilon) \subseteq B_{\mathbb{F}}(0,\varepsilon)$ for $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$ and

$$\underbrace{\lambda M(B_{\mathbb{F}}(0,\varepsilon), U)}_{=M(\lambda B_{\mathbb{F}}(0,\varepsilon), U)} \subseteq M(B_{\mathbb{F}}(0,\varepsilon), U)$$

Notice $\lambda[\text{Convex Hull}(S)] = \text{Convex Hull}(\lambda S)$ (exercise). So deduce $\lambda\tilde{C} \subseteq \tilde{C}$.

Step 2: define the *Minkowski guage* (functional) of $\tilde{C}$

$$\mu_{\tilde{C}} : V \to \mathbb{R}_+, \ v \mapsto \inf\{t \geq 0 : v \in t\tilde{C}\}$$

$\mu_{\tilde{C}}$ is well-defined in $[0,\infty)$ since: any v satisfies $\frac{v}{t} \to 0$ as $t \to \infty$ by continuity of M. So $\frac{v}{t}$ must "enter" the neighborhood $\tilde{C}$ of 0 for t large enough.

Step 3: let us prove $v \mapsto \mu_{\tilde{C}}(v)$ is a norm:

- $\mu_{\tilde{C}}(v) \geq 0$ by construction
- if $\mu_{\tilde{C}} = 0$, then (assume $v \neq 0$ for contradiction) there exists U open around 0 with $v \notin U$ (since $V \setminus \{v\}$ is open). Since $\tilde{C}$ is bounded, there exists $t_1 > 0$ such that $\tilde{C} \subseteq t_1 U$. Since $\mu_{\tilde{C}}(v) = 0$, there exists $t_2 \in (0, t_1^{-1})$ such that $v \in t_2\tilde{C}$, then $v \in t_2\tilde{C} \subseteq t_1^{-1}\tilde{C} \subseteq U$, a contradiction.
- Want to show $\mu_{\tilde{C}}(\lambda v) = |\lambda|\mu_{\tilde{C}}(v)$ for $\lambda \in \mathbb{F}^\times$, $v \in V$. Use $\tilde{C}$ balanced: for all $t > 0$ such that $\lambda v \in t\tilde{C}$, we have

  $$\frac{\lambda}{|\lambda|} v \in \frac{t}{|\lambda|}\tilde{C} \implies v \in \frac{t}{|\lambda|}\tilde{C} \implies \mu_{\tilde{C}}(v) \leq \frac{1}{|\lambda|}\mu_{\tilde{C}}(\lambda v)$$

  The inequality in the other direction follows by reasoning with λ^{-1}. So $|\lambda|\mu_{\tilde{C}}(v) = \mu_{\tilde{C}}(\lambda v)$.
- Want to show $\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$ for all $v_1, v_2 \in V$. Indeed, given $t_1, t_2 > 0$ such that $v_1 \in t_1\tilde{C}$, $v_2 \in t_2\tilde{C}$, we have

  $$v_1+v_2 \in t_1\tilde{C}+t_2\tilde{C} = (t_1+t_2)\left[\frac{t_1}{t_1+t_2}\tilde{C} + \frac{t_2}{t_1+t_2}\tilde{C}\right] \subseteq (t_1+t_2)\tilde{C} \text{ (convexity)}$$

  so $\mu_{\tilde{C}}(v_1 + v_2) \leq t_1 + t_2$. By taking infima over t_1, t_2:

  $$\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$$

Step 4: prove $\mu_{\tilde{C}}$ induces the topology τ.

- Want to prove

$$\underbrace{B(v_0,\varepsilon)}_{\text{open ball for } \mu_{\tilde{C}}} = \{v \in V : \mu_{\tilde{C}}(v - v_0) < \varepsilon\} \in \tau$$

  Take $v \in B(v_0, \varepsilon)$ then by the triangle inequality

$$B(v, \varepsilon - |v|) \subseteq B(v_0, \varepsilon)$$

  and $B(v, \varepsilon') \supseteq v + \frac{\varepsilon'}{2}\tilde{C}$ by definition of the ball for $\mu_{\tilde{C}}$. And (since translations, dilations continuous) $v + \frac{\varepsilon'}{2}\tilde{C}$ is a neighborhood of v.

  $B(v_0, \varepsilon)$ open (in τ) around its points, so is in τ.

- Take $U \in \tau$, and (wlog) $0 \in U$. Let us prove $0 \in B(0, \varepsilon_0) \subseteq U$ for some $\varepsilon_0 > 0$. Indeed $\tilde{C}$ is bounded so there exists $\varepsilon_0 > 0$ such that $\tilde{C} \subseteq \varepsilon_0^{-1} U$ hence $U \supseteq \varepsilon_0 \tilde{C}$ and so $U \supseteq \varepsilon \tilde{C}$ $\forall \varepsilon < \varepsilon_0$ and thus $U \supseteq B(0, \varepsilon_0)$.

□

Remarks:

1. $B(0, \varepsilon_0) \subseteq \bigcup_{0 \leq \varepsilon < \varepsilon_0} \varepsilon \tilde{C}$
2. T_1 implies Hausdorff (T_2). Consider $v_0 \neq v_1$ in V: so $0 \neq v_1 - v_0$, T_1 implies there is U open around 0 with $v_1 - v_0 \notin U$. Then (since A, M continuous) $(v, w) \mapsto v - w$ is continuous and there exists $\tilde{U}$ open around 0 such that $\tilde{U} - \tilde{U} \subseteq U$. Then $v_0 + \tilde{U}$ and $v_1 + \tilde{U}$ are open disjoint neighborhoods of v_0 and v_1 respectively (disjoint since otherwise $v_1 - v_0 \in \tilde{U} - \tilde{U} \subseteq U$).

4 Some examples of NVS'

Definition. Let $(V, ||\cdot||)$ be an NVS (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). If (V, d), d distance induced by $||\cdot||$ is a complete metric space, then $(V, ||\cdot||)$ is called a *Banach space.*

Example. $\mathbb{R}^n$, $\mathbb{C}^n$, $n \geq 1$ are Banach spaces, for $||\cdot||_p$, $p \in [1, \infty)$.

Example. Given (X, τ) a general topological space, define

$$B_{\mathbb{F}}(X) = \{\text{functions } : X \to \mathbb{F} \text{ bounded}\}$$

$$C_{\mathbb{F}}(X) = \{\text{functions } : X \to \mathbb{F} \text{ continuous}\}$$

$$C_{\mathbb{F},b} = C_{\mathbb{F}}(X) \cap B_{\mathbb{F}}(X)$$

If $X = K$ is compact, $C_{\mathbb{F}}(X) = C_{\mathbb{F},b}(X)$. These are vector spaces over $\mathbb{F}$ with addition $(f+g)(x) = f(x) + g(x)$ and multiplication $(fg)(x) = f(x)g(x)$.

Norm on $C_{\mathbb{F},b}(X)$: the supremum norm, $||f||_\infty = \sup_{x \in X} |f(x)|$

Proposition. $(C_{\mathbb{F},b}, ||\cdot||_\infty)$ is a Banach space over $\mathbb{F}$.

Proof.

- $||f||_\infty$ is well defined in $\mathbb{R}^+$ since f is bounded.
- $||f||_\infty = 0$ means $f(x) = 0$ for all $x \in X$ and so $f = 0$.
- Homogeneity and triangle inequality: inherited from $|\cdot|$ in $\mathbb{F}$ (exercise).
- Completeness: let $(f_k)_{k \geq 1}$ be a Cauchy sequence under $||\cdot||_\infty$. For each $x \in X$ we have $|f_m(x) - f_n(x)| \leq ||f_m - f_n||_\infty \to 0$ as $n, m \to \infty$. So $(f_k(x))_{k \geq 1}$ is Cauchy in $\mathbb{F}$, so (since $\mathbb{F}$ is complete) there exists a limit $f(x) = \lim_{k \to \infty} f_k(x)$. This defines a function $f : X \to \mathbb{F}$.
- For all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\forall m, n \geq n_0$, $\forall x \in X$,

$$|f_m(x) - \underbrace{f_n(x)}_{\to f(x)}| \leq \varepsilon$$

so for all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\forall m \geq n_0$, $\forall x \in X$ we have

$$|f_m(x) - f(x)| \leq \varepsilon$$

so $||f_m - f||_\infty \leq \varepsilon$ and $f_m \to f$ uniformly, so $f \in C_{\mathbb{F},b}$ by properties of the uniform limit.

$\square$

Example. Given $U \subseteq \mathbb{R}^n$ open, bounded and non-empty; $m \in \mathbb{N}^*$, consider

$$\begin{aligned} C^m(\overline{U}) = \{f : U \to \mathbb{R} : &f \text{ is } m \text{ times differentiable on } U, \forall \alpha \in \mathbb{N}^n \\ &\text{s.t } |\alpha| = \alpha_1 + \ldots + \alpha_m \leq m \\ &, \partial^\alpha f \text{ is continuous and bounded on } U\} \end{aligned}$$

Then $(C^m(\overline{U}), ||\cdot||_{C^m})$ is a Banach space where

$$||f||_{C^m} = \sup_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} \underbrace{\sup_{x \in U} |\partial^\alpha f(x)|}_{||\partial^\alpha f||_\infty}$$

Exercise: check that this is complete and $\partial^\alpha f$, $\alpha \leq m-1$, extends continuously to $\tilde{U}$.

Example. $C_{\mathbb{R}}([0,1])$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$. This is a vector space over $\mathbb{R}$.

- $(C_{\mathbb{R}}([0,1]), ||\cdot||_\infty)$ is a Banach space (Example sheet)
- Could take another norm such that

$$||f||_p = \left(\int_0^1 |f(x)|^p \mathrm{d}x\right)^{1/p}, \ p \in [1, \infty)$$

Study of $(C_{\mathbb{R}}([0,1]), ||\cdot||_p)$:

- $||\cdot||_p$ is well defined: Riemann and Lebesgue integrable.
- If $||f||_p = 0$ and $f \neq 0$ then there exists $\varepsilon > 0$ and $x_0 \in [0,1]$ such that $|f(x_0)| \geq \varepsilon$, so by continuity there exist $a < b \in [0,1]$ such that $\inf_{x\in[a,b]} |f(x)| \geq \frac{\varepsilon}{2}$. Then $\int_0^1 |f(x)|^p \mathrm{d}x \geq \left(\frac{\varepsilon}{2}\right)^p (b-a) > 0$ which is impossible.
- Homogeneity is clear.
- Triangle inequality:

$$||f+g||_p^p = \int_0^1 |f+g|^p \mathrm{d}x = \int_0^1 |f+g||f+g|^{p-1} \mathrm{d}x$$

$$\leq \int_0^1 |f||f+g|^{p-1}\mathrm{d}x + \int_0^1 |g||f+g|^{p-1}\mathrm{d}x$$
$$\underbrace{\leq}_{\text{Hölder:}} ||f||_p||f+g||_p^{p-1} + ||g||_p||f+g||_p^{p-1}$$

If $||f+g||_p = 0$ then its clear. Otherwise this implies $||f+g||_p \leq ||f||_p + ||g||_p$.

- Completeness? Define

$$f_k(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{4k} \\ \left[x - \left(\frac{1}{2} - \frac{1}{4k}\right)\right] 4k & \frac{1}{2} - \frac{1}{4k} \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

then $(f_k)_{k\geq 1}$ is Cauchy for $|| \cdot ||_p$, and the limit is $1_{[1/2,1]}$ which is not continuous. So not complete.

Remark: what about the completion? In general, abstract completions are often not very useful; however in this case, it is: Lebesgue space $L^p([0,1])$, defined as equivalence classes for the "almost everywhere" equality.

Example. Take functions from $X = \mathbb{N} \to \mathbb{R}$ or $\mathbb{C}$, get $\ell^p_{\mathbb{F}}$ for $p \in [1, \infty]$, with norm $||(x_k)||_p = \left(\sum_{k\geq 1} |x_k|^p\right)^{1/p}$ for $p < \infty$ and $||(x_k)||_\infty = \sup_{k\geq 1} |x_k|$. Exercise: show this is indeed a norm and this is complete, hence Banach.

Remark: for $p \in (0, 1)$, ℓ^p is similarly defined.

Non-examinable example of TVS:

- Define for $U \subseteq \mathbb{R}^n$ open & non-empty, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $C_{\mathbb{F}}(U)$ the set of continuous functions $U \to \mathbb{F}$.
- TVS for the topology τ defined by the translations of the following basis of neighborhoods around 0: take $(K_n)_{n\geq 1}$ a sequence of increasing compact sets, $\bigcup_{n\geq 1} K_n = U$. Define

$$U_n = \left\{ f \in C_{\mathbb{F}}(U) : \sup_{K_n} |f| \leq \frac{1}{n} \right\}$$

- Exercise: show this indeed a TVS and τ does not depend on the choice of the (K_n).
- Proposition: $(C(U), \tau)$ is a locally convex, not locally bounded TVS (therefore not normable). Furthermore, it is metrizable with $d(f, g) = \sum_{k\geq 1} \frac{1}{2^n} \left(\frac{\sup_{K_n} |f-g|}{1+\sup_{K_n} |f-g|} \right)$. Also $(C(U), d)$ is complete (Frechet space).

Remarks:

1. Not locally bounded: suppose there exists B bounded neighborhood of 0, then there exists $n_0 \geq 1$ such that $U_{n_0} \subseteq B$. B is bounded so there exists $t > 0$ such that $B \subseteq tU_{n_0+1}$ so $U_{n_0} \subseteq tU_{n_0+1}$. But this is impossible since we can always construct $f \in U_{n_0}$ such that $\sup_{K_{n_0+1}} |tf| > 1/n$
2. Let $C_c(U)$ be the set of continuous functions with compact support. Then V is a neighborhood of 0 if and only if $V \cap C(K_n)$ is a neighborhood of 0 in $C(K_n)$. This is a non-countable topology.

5 Bounded linear maps & duality

Definition. Given (V, τ_V) and (W, τ_W) TVS', $T : V \to W$ linear is *bounded* if it maps bounded sets to bounded sets: for any $B_V \subseteq V$ bounded, then $T(B_V)$ is bounded in W.

Proposition. Given (V, τ_V), (W, τ_W) TVS' which are locally bounded (note this includes NVS'), and $T : V \to W$ is linear, then T is bounded if and only if T is continuous.

Proof.

Step 1: T bounded $\implies$ T continuous at 0. Let U_W be an open neighborhood of 0 in W, and U_V an open bounded neighborhood of 0 in V. Then $T(U_V)$ is bounded, so there exists $t > 0$ such that $T(U_V) \subseteq tU_W$. So $T^{-1}(U_W) \supseteq t^{-1}U_V$ and $t^{-1}U_V$ is open around 0 in V (using the fact dilations are continuous).

Step 2: T continuous at 0 $\implies$ T is continuous everywhere. Let $w \in W$, U_W open around w, $v \in V$ such that $T(v) = w$. Then $U_W - w$ is open around 0 in W (translation continuous), so by Step 1, $T^{-1}(U_W - w)$ is a neighborhood of 0 in V. So

$$\begin{aligned} T^{-1}(U_W) &= T^{-1}(\{w\}) + T^{-1}(U_W - w) \\ &= \bigcup_{v' \in T^{-1}(\{w\})} \big(v' + T^{-1}(U_W - w)\big) \\ &\supseteq \underbrace{v + T^{-1}(U_W - w)}_{\text{ngbd around } v} \end{aligned}$$

Step 3: T continuous $\implies$ T bounded. Let $B_V \subseteq V$ be bounded, and U_W an open neighborhood of 0 in W. Then $T^{-1}(U_W)$ is open around 0 in V. So (since B_V bounded) there exists $t > 0$ such that $B_V \subseteq tT^{-1}(U_W)$ and so $T(B_V) \subseteq tU_W$.

We have proved that $T(B_V)$ is covered by a dilation of any neighborhood of 0, so is bounded. □

Definition. Given $(V, ||\cdot||_V)$, $(W, ||\cdot||_W)$ NVS' on $\mathbb{F}$, and $T : V \to W$ linear, T is bounded iff T is continuous iff there exists $t > 0$ such that $T(B_V(0,1)) \subseteq B_W(0,t)$. The infimum of such t's is denoted $|||T|||$.

Remark: can check that $|||T|||$ is equivalently defined as

$$|||T||| = \sup_{||v||_V \leq 1} ||Tv||_W = \sup_{||v||_V < 1} ||Tv||_W = \sup_{||v||_V = 1} ||Tv||_W \qquad (*)$$

Definition. Given $(V, ||\cdot||_V)$,$(W, ||\cdot||_W)$ NVS', denote

$$\mathcal{L}(V,W) = \{T : V \to W \text{ linear map}\}$$

$$\mathcal{B}(V,W) = \{T : V \to W \text{ linear bounded map}\}$$

Proposition. $(\mathcal{B}(V,W), |||\cdot|||)$ is an NVS.

Proof.

- $\mathcal{L}(V,W)$ is a vector space via $(\lambda_1 T_1 + \lambda_2 T_2)(v) = \lambda_1 T_1(v) + \lambda_2 T_2(v)$.
- $\mathcal{B}(V,W)$: dilation/(finite) sums of bounded sets are bounded. So T bounded implies λT is bounded and T_1, T_2 bounded implies $T_1 + T_2$ bounded.

- $|||T|||$ is well-defined in $\mathbb{R}_+$ for T bounded, $|||0||| = 0$ and if $|||T||| = 0$ then $T(B_V(0,1)) \subseteq B_W(0,t)$ for all $t > 0$ and so by continuity of dilation, $T(B_V(0,1)) = \{0\}$. By linearity, this implies $T = 0$.
- $|||\lambda T||| = |\lambda|\ |||T|||$ and $|||T_1 + T_2||| \leq |||T_1||| + |||T_2|||$ follows from $(*)$

$\square$

Proposition. Let $(V, ||\cdot||_V)$ be a NVS and $(W, ||\cdot||_W)$ a Banach space. Then $(\mathcal{B}(V,W), |||\cdot||||)$ is a Banach space.

Proof. We have proved that $(\mathcal{B}(V,W), |||\cdot|||)$ is an NVS above. So we prove completeness. Let $(T_k)_{k\geq 1}$ be a Cauchy sequence in $(\mathcal{B}(V,W), |||\cdot|||)$. Then

$$\sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \to 0 \text{ as } k_0 \to \infty \tag{$**$}$$

$$\forall v \in V,\ \sup_{k_1,k_2\geq k_0} ||T_{k_1}(v) - T_{k_2}(v)||_W \leq ||v||_V |||T_{k_1} - T_{k_2}||| \xrightarrow{k_0\to\infty} 0 \tag{$***$}$$

so $(T_k(v))_{k\geq 1}$ is a Cauchy sequence in W. Since W is complete, can let the associated limit be $T(v)$.

Then T is linear by pointwise limits:

$$\begin{aligned} T(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{k\to\infty} T_k(\lambda_1 v_1 + \lambda_2 v_2) &= \lim_{k\to\infty} [\lambda_1 T_k(v_1) + \lambda_2 T_k(v_2)] \\ &= \lambda_1 T(v_1) + \lambda_2 T(v_2) \end{aligned}$$

Use $(***)$, take $k_2 \to \infty$ so

$$\forall v \in V,\ \sup_{k_1\geq k_0} ||T_{k_1}(v) - T(v)||_W \leq ||v||_V \left(\sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \right) \to 0 \text{ as } k_0 \to \infty$$

Hence for $v \in V$ such that $||v|| \leq 1$ we have

$$\sup_{k_1\geq k_0} ||T_{k_1}(v) - T(v)||_W \leq \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \tag{$\dagger$}$$

Then (for $v \in V$ with $||v|| \leq 1$) by the triangle inequality

$$||T(v)||_W \leq ||\underbrace{T_{k_0}(v)}_{\text{bounded}}|| + \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}|||$$

$$\sup_{||v||\leq 1} ||T(v)||_W \leq |||T_{k_0}||| + \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}|||$$

So T is bounded. Now $(\dagger)$ implies

$$\sup_{k_1\geq k_0} |||T_{k_1} - T||| \leq \sup_{k_1,k_2\geq k_0} |||T_{k_1} - T_{k_2}||| \xrightarrow{k_0\to\infty} 0$$

So $T_{k_1} \xrightarrow{|||\cdot|||} T$. $\square$

Remark: can deduce from (†) that for all $v \in V$ with $||v|| \leq 1$,

$$||T_k(v)||_W - |||T_k - T||| \leq ||T(v)||_W \leq ||T_k(v)||_W + |||T_k - T|||$$

Then taking supremum over $||v|| \leq 1$

$$\left| \sup_{||v|| \leq 1} ||Tv||_W - \sup_{||v|| \leq 1} ||T_k(v)||_W \right| \leq |||T_k - T||| \xrightarrow{k \to \infty} 0$$

So $|||T_k||| \xrightarrow{k\to\infty} |||T|||$.

Definition. Let $(V, ||\cdot||_V)$ be a NVS over $\mathbb{F}$. Let

$$\mathcal{L}(V, \mathbb{F}) = \{\text{linear maps } V \to \mathbb{F}\}, \text{ the } \textit{algebraic dual}$$

$$\mathcal{B}(V, \mathbb{F}) = \{\text{bounded linear maps } V \to \mathbb{F}\} \text{ denoted } (V^*, ||\cdot||_{V^*})$$

Note that by the previous proposition $\mathcal{B}(V, \mathbb{F})$ is Banach (since $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is complete).

Definition. Let $(V, ||\cdot||_V)$, $(W, ||\cdot||_W)$ be NVS', $T \in \mathcal{B}(V, W)$. Then T^* (the *adjoint* of T) defined as $T^* : W^* \to V^*$, $\psi \mapsto \varphi = \psi \circ T$. i.e $T^*(\psi)(v) = \psi(T(v))$.

Proposition. T^* is well-defined $W^* \to V^*$, linear and bounded (for $||\cdot||_{W^*}$ and $||\cdot||_{V^*}$) with $|||T^*||| \leq |||T|||$.

Remark: soon, with the help of the Hahn-Banach Theorem, we'll prove that the duals are "big enough" so that $|||T^*||| = |||T|||$.

Proof.

- Well-defined: follows since linearity and boundedness are stable under composition, i.e if $T : V \to W$ is linear and bounded, $\psi : W \to \mathbb{F}$ is linear and bounded, so is $\psi \circ T : V \to \mathbb{F}$. So $\psi \circ T \in V^*$

- Linearity:

$$\begin{aligned} T^*\left(\lambda_1\psi_1 + \lambda_2\psi_2\right)(v) &= \left(\lambda_1\psi_1 + \lambda_2\psi_2\right)(Tv) \\ &= \lambda_1\left[\psi_1(Tv)\right] + \lambda_2\left[\psi_2(Tv)\right] \\ &= \lambda_1 T^*(\psi_1)(v) + \lambda_2 T^*(\psi_2)(v) \end{aligned}$$

- Boundedness:

$$|||T^*||| = \sup_{||\psi||_{W^*}} ||T^*(\psi)||_{V^*} = \sup_{||\psi||_{W^*} \leq 1} \sup_{||v||_V \leq 1} |T^*(\psi)(v)|$$

$$\leq \sup_{||\psi||_{W^*} \leq 1} \sup_{||v||_V \leq 1} |\psi(Tv)| \leq \sup_{||\psi||_{W^*} \leq 1} \sup_{||v||_V \leq 1} ||\psi||_{W^*} |||T||| \cdot ||v||_V \leq |||T|||$$

$\square$

Definition. Let $(V, ||\cdot||_V)$ be an NVS. Since $(V^*, ||\cdot||_{V^*})$ is a NVS (Banach), we can define its dual, denoted $(V^{**}, ||\cdot||_{V^{**}})$ the *bidual* of V (again Banach).

Proposition. Define $\Phi : V \to V^{**}$, $v \mapsto \Phi(v)$ by

$$\forall \varphi \in V^*,\ \Phi(v)(\varphi) = \varphi(v)$$

Then Φ is well-defined, linear and bounded with $|||\Phi||| \leq 1$. Φ is called the *canonical bi-dual embedding.*

Remark: with the Hahn-Banach Theorem, we'll prove Φ is an isometry. In particular, $|||\Phi||| = 1$ and Φ is injective. However, Φ is not always surjective. In fact, V and V^{**} are not always isomorphic.

Proof.

- Well-defined: given $v \in V$, $\phi \in V^*$ is linear, and bounded since

$$\sup_{||\varphi||_{V^*} \leq 1} |\varphi(v)| \leq ||v||_V$$

- Linearity:

$$\begin{aligned}\Phi(\lambda_1 v_1 + \lambda_2 v_2)(\varphi) &= \varphi(\lambda_1 v_1 + \lambda_2 v_2)\\ &= \lambda_1 \varphi(v_1) + \lambda_2 \varphi(v_2)\\ &= \lambda_1 \Phi(v_1)(\varphi) + \lambda_2 \Phi(v_2)(\varphi)\end{aligned}$$

- Boundedness:

$$|||\Phi||| = \sup_{||v||_V \leq 1} ||\Phi(v)||_{V^{**}} = \sup_{||v||_V \leq 1} \sup_{||\varphi||_{V^*} \leq 1} |\underbrace{\Phi(v)(\varphi)}_{\varphi(v)}|$$

$$= \sup_{||v||_V \leq 1} \sup_{||\varphi_{V^*}|| \leq 1} \underbrace{|\varphi(v)|}_{\leq ||\varphi||_{V^*}||v||_V} \leq 1$$

$\square$

Example. Let V, W be finite-dimensional NVS' with bases $(v_i)_{i=1}^m$ and $(w_j)_{j=1}^n$ respectively. Let $T : V \to W$ be linear (and thus bounded as finite dimensional). Take $(v_i^*)_{i=1}^m$ defined by $v_i^*(v_{i'}) = \delta_{ii'}$ and $(w_j^*)_{j=1}^n$ defined by $w_j^*(w_{j'}) = \delta_{jj'}$. Then V^*,W^* are finite-dimensional NVS' with bases (v_i^*) and (w_j^*) respectively. If T has a matrix $A = (a_{ij})_{i=1,j=1}^{i=m,j=n}$ in with respect to the bases (v_i) and (w_j), then

$$Tv_i = \sum_{j=1}^n a_{ij} w_j$$

and T^* has matrix $A^T = (a_{ji})_{j=1,i=1}^{j=n,i=m}$ with respect to the bases (w_j^*) and (v_i^*).

Example. Space of square summable spaces $\ell^2(\mathbb{F})$ (as usual $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is infinite dimensional. There are linear maps on this space that are

- Bounded, injective but not surjective: $T(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ a "right shift" of the sequence
- Bounded, surjective but not injective: $T(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$ a "left shift" of the sequence
- Linear but not bounded: find a basis $(e_i)_{i \in I}$, extract $(e_n)_{n \geq 1}$ a countable subset. Then define $T : e_n \mapsto ne_n$, $e_i \mapsto 0$ for $i \notin \mathbb{N}$.

Duality: $(\ell^2)^* = \ell^2$ (Hilbert representation theorem)

Example. For ℓ^p, $p \in (1, \infty)$, $p \neq 2$, we have duals

$$\ell^p \to (\ell^p)^* = \ell^q \to (\ell^q)^* = \ell^p \text{ where } \frac{1}{p} + \frac{1}{q}$$

$$\ell^1 \to (\ell^1)^* = \ell^\infty \to (\ell^\infty)^* \neq \ell^1$$

Example. (Question 8 Example sheet 1) $\left(C^1([0,1]), ||\cdot||_{C^0}\right) \to \left(C^1([0,1]), ||\cdot||_{C^1}\right)$, $f \mapsto f$ is unbounded.

Zorn's Lemma

In a finite-dimensional NVS V, we have a "simple" dual V^*. In infinite-dimension, we have not even proved that if V is non-trivial (i.e not $\{0\}$) then V^* is non-trivial.

The Hahn-Banach Theorem will answer several questions:

- $V \neq \{0\} \implies V^* \neq \{0\}$
- V^* seperates points of V
- Φ (the bidual embedding) is isometric, $|||\Phi||| = 1$
- $|||T^*||| = |||T|||$

Idea of Hahn-Banach: extend linear bounded maps already defined on a subspace.

Strategy:

1. "Co-dimension 1" extension: any linear bounded map $V \to \mathbb{F}$ has an extension to $W \to \mathbb{F}$ where $V \subseteq W$ with codimension 1.
2. Transfinite induction: Zorn's Lemma (or equivalently the Axiom of Choice)

Remark: if $V = \bigcup_{n \geq 1} V_n$, V_n subspace, $V_n \subseteq V_{n+1}$, $\dim(V_n) = n$, could use step 1 above and standard (countable) induction. However, no Banach spaces are like this.

Definition. A set S is *partially ordered* (poset) if there is a binary relation "$\leq$" such that

- $\forall x, y \in S$, $x \leq y$ or not (partial order)
- $\forall x \in S$, $x \leq x$ (reflexive)
- $\forall x, y, z \in S$, if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive)
- $\forall x, y \in S$, if $x \leq y$ and $y \leq x$ then $x = y$ (non-ambiguous)

Definition. A poset S is *totally ordered* if $\forall x, y \in S$, if $x \not\leq y$ then $x \geq y$.

Definition. Given $S' \subseteq S$ (where $(S, \leq)$ is a poset), we say $l \in S$ is a *upper bound* of S' if $\forall x \in S'$, $x \leq l$. l is a *least upper bound* of S' if it is an upper bound and any other upper bound $l' \in S$ satisfies $l' \geq l$.

Definition. A subset S' of S ($(S, \leq)$ a poset) that is totally ordered is called a *chain*.

Definition. A poset $(S, \leq)$ has the *least upper bound property* if any non-empty chain has a least upper bound.

Definition. Given a poset $(S, \leq)$, $m \in S$ is said to be *maximal* if $\forall x \in S$, $x \geq m$ implies $x = m$.

Theorem (Zorn's Lemma). *Any non-empty poset $(S, \leq)$ with the least upper bound property has (at least one) maximal element.*

Remarks:

1. In fact Zorn's Lemma is true just with "upper bound" property on chains.
2. Zorn's Lemma is equivalent to the Axiom of Choice

5.1 Finite dimension

Definition. Let V be a NVS with two norms $||\cdot||_1$ and $||\cdot||_2$. Then these norms are said to be *equivalent*, denoted $||\cdot||_1 \sim ||\cdot||_2$ if there are two constants, $c, c' > 0$ such that

$$\forall v \in V,\ C||v||_1 \leq ||v||_2 \leq C'||v||_1$$

Remarks:

1. This defines equivalence classes on norms.
2. $|| \cdot ||_1 \sim || \cdot ||_2$ implies that their induced topologies are the same. The converse is also true: indeed $B_{||\cdot||_1}(0, 1)$ is open around 0 for τ_2, so there exists $\varepsilon > 0$ such that $B_{||\cdot||_2}(0, \varepsilon) \subseteq B_{||\cdot||_1}(0, 1)$, which implies that for all $v \in V \setminus \{0\}$

$$\frac{\varepsilon v}{2||v||_2} \in B_{||\cdot||_2}(0, \varepsilon) \subseteq B_{||\cdot||_1}(0, 1) \implies ||v||_1 \leq \frac{2}{\varepsilon}||v||_2$$

   and similarly for the opposite bound.

3. When 2 norms are equivalent, they generate te same notion of bounded linear maps, converging spaces & Cauchy sequences.

Proposition.

(i) All norms are equivalent in finite-dimension

(ii) Given $(V, || \cdot ||_V)$ a finite-dimensional NVS, $(W, || \cdot ||_W)$ a NVS, any linear map $T : V \to W$ is bounded

(iii) Given $(V, || \cdot ||_V)$ an NVS, if $\overline{B}_V(0, 1)$ is compact, then V is finite dimensional.

Proof.

(i) Let us prove all norms are equivalent to $||\cdot||_\infty$, defined for a basis $(e_i)_{i=1}^n$ as $||v||_\infty = \sup_{1\leq i\leq n}|v_i|$ for $v = \sum v_i e_i$.

Let $||\cdot||$ be a norm on V

$$||v|| = \left|\left|\sum_{i=1}^n v_i e_i\right|\right| \leq \sum_{i=1}^n |v_i|||e_i|| \leq \underbrace{\left(\sum_{i=1}^n ||e_i||\right)}_{=C'}||v||_\infty$$

Consider $\varphi : (V, ||\cdot||_\infty) \to \mathbb{R}_+$ defined by $v \mapsto ||v||$. Then φ is continuous:

$$|\varphi(v) - \varphi(w)| = |||v|| - ||w||| \leq ||v-w|| \leq C'||v-w||_\infty$$

Define $S_{||\cdot||_\infty}(0,1) = \{v \in V : ||v||_\infty = 1\}$. Then $\varphi : S_{||\cdot||_\infty}(0,1) \to \mathbb{R}_+$ continuous, so attains its minimum: there exists $v_0 \in S_{||\cdot||_\infty}(0,1)$ such that $\forall v \in S_{||\cdot||_\infty}(0,1)$, $\varphi(v) \geq \varphi(v_0)$.

Then $v_0 \neq 0$ since $||v_0||_\infty = 1$ and so $\varphi(v_0) = ||v_0|| = C > 0$. This implies

$$\left|\left|\frac{v}{||v||_\infty}\right|\right| \geq C,\ \forall v \in V \setminus \{0\} \implies \forall v \in V,\ ||v|| \geq C||v||_\infty$$

(ii) Completeness and the fact closed bounded sets are compact follows from (i) since true with $(\mathbb{F}^n, ||\cdot||_\infty)$.

$$||T(v)||_W = \left|\left|\sum_{i=1}^n v_i T(e_i)\right|\right|_W \leq \sum_{i=1}^n |v_i|||T(e_i)||_W$$

$$\leq ||v||_\infty \left(\sum_{i=1}^n ||T(e_i)||_W\right) \leq \frac{1}{C}||v||_V \left(\sum_{i=1}^n ||T(e_i)||_W\right)$$

so T is bounded

□

Theorem (Riesz). *If $(V, \|\cdot\|)$ is an NVS, $\overline{B}(0,1)$ compact then V finite dimensional.*

Proof. $\overline{B}(0,1) \subseteq \bigcup_{v \in \overline{B}(0,1)} B(v, 1/2)$ open covering. Then compactness implies there exist $v_1, \dots, v_n$ in $\overline{B}(0,1)$ such that $\overline{B}(0,1) \subseteq \bigcup_{i=1}^n B(v_i, 1/2)$. Denote $W = \text{span}(v_1, \dots, v_n)$ a subspace of V. Then $\overline{B}(0,1) \subseteq \bigcup_{i=1}^n (v_i + B(0, 1/2))$.

$\overline{B}(0,1) \subseteq W + B^{\cdot}(0, 1/2) \subseteq W + \overline{B}(0, 1/2)$

Iterate on $\overline{B}(0,1/2) = \frac{1}{2}\overline{B}(0,1)$: $\overline{B}(0,1/2) \subseteq W + \overline{B}(0,1/4)$.

$$\overline{B}(0,1) \subseteq \bigcap_{k=1}^{K} \left(W + \overline{B}(0, 2^{-k})\right), \ \forall K \geq 1$$

Then

$$\overline{B}(0,1) \subseteq \bigcap_{k \geq 1} \left(W + \overline{B}(0, 2^{-k})\right) \subseteq \overline{W} = W$$

$\overline{B}(0,1) \subseteq W$ implies $V = W$. □

Back to (Zorn's Lemma) and the Hahn-Banach Theorem

Construction of basis:

Proposition. Let $V \neq \{0\}$ be a vector space over $\mathbb{F}$ and $S \subseteq V$ subset which is linearly independent. Then there exists a subset $B \subseteq V$ linearly independent such that $S \subseteq B$ and $\text{span}(B) = V$ (i.e a basis).

Proof. Let $\mathcal{F} = \{\text{linearly independent subsets } S' \subseteq V \text{ such that } S \subseteq S'\}$. Then $\mathcal{S} \neq \emptyset$ since $S \in \mathcal{F}$.

$(\mathcal{F}, \subseteq)$ is a poset (easy check).

If $\Theta \subseteq \mathcal{F}$ is a chain (totally ordered for $\subseteq$) then it has a least upper bound: $\overline{S} = \bigcup_{S' \in \Theta} S'$.

Properties of $\overline{S}$:

- $\overline{S} \supseteq S'$, for all $S' \in \Theta$ so S' is an upper bound for Θ
- An upper bound for Θ will include each $S' \in \Theta$ so $\overline{S}$ is a least upper bound.
- $\overline{S} \supseteq S$ since $\overline{S} = \bigcup_{S' \in \Theta} S'$ and each $S' \supseteq S$.
- $\overline{S}$ is linearly independent: let $(v_1, \dots, v_n) \in \overline{S}$ be distinct elements. Then for all $i = 1, \dots, n$ there exists $S'_i \in \Theta$ such that $v_i \in S'_i$. Chain structure (total order) means there exists $i_0 \in \{1, \dots, n\}$ such that $S'_j \subseteq S'_{i_0}$ for all $j = 1, \dots, n$. So $\{v_1, \dots, v_n\} \subseteq S'_{i_0}$ is linearly independent, and so $\overline{S}$ is.

Now Zorn's Lemma says that there exists a maximal element in $\mathcal{F}$: $B \supseteq S$, B linearly independent and maximal. Assume $\text{span}(B) \subsetneq V$, then we have $v_0 \in V \setminus \text{span}(B)$ and $B' = B \cup \{v_0\}$ is a strictly larger element of $\mathcal{F}$, a contradiction. Hence $V = \text{span}(B)$. $\square$

Note that the statement of the geometric form of Hahn-Banach below is ***non-examinable***

Theorem (Hahn-Banach "algebraic" form)**.**

(i) Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $p : V \to \mathbb{R}_+$ such that for all $v_1, v_2 \in V$, $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ and for all $\lambda \in \mathbb{F}$, $v \in V$ we have $p(\lambda v) = |\lambda| p(v)$.

Let $W \subseteq V$ be a subspace of V and $f : W \to \mathbb{F}$ linear with $|f(w)| \leq p(w)$ for all $w \in W$. Then there exists $\tilde{f} : V \to \mathbb{F}$ linear, with $\tilde{f}|_W = f$ and $|f(v)| \leq p(v)$ on all of V.

(ii) Let V be a vector space over $\mathbb{F} = \mathbb{R}$ and $p : V \to \mathbb{R}_+$ such that for all $v_1, v_2 \in V$, $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ and for all $\lambda > 0$, $v \in V$ we have $p(\lambda v) = \lambda p(v)$.

Let $W \subseteq V$ be a subspace of V and $f : W \to \mathbb{F}$ be linear with $f \leq p$ on W. Then there exists $\tilde{f} : V \to \mathbb{F}$ linear with $\tilde{f}|_W = f$, and $\tilde{f} \leq p$ on V.

Proof. Step 1: (i) in $\mathbb{R}$ implies (ii) in $\mathbb{C}$. Start from $f : W \to \mathbb{F} = \mathbb{C}$. Note that a vector space V over $\mathbb{C}$ can be seen as a vector space over $\mathbb{R}$. Indeed if $(e_i)_{i \in I}$ is a basis over $\mathbb{C}$, and $V_0 = \text{span}_{\mathbb{R}}((e_i)_{i \in I})$, $V = V_0 \oplus (iV_0)$ (same with W).

Define $g = \Re(f)$, this satisfies $|g| \leq p$. Then (i) on $\mathbb{R}$ implies there exists $\tilde{g} : V \to \mathbb{R}$ linear extending g such that $|\tilde{g}| \leq p$.

Define $\tilde{f}(v) := \tilde{g}(v) - i\tilde{g}(iv)$. Then $\tilde{f}(\lambda v) = \lambda \tilde{f}$ for all $\lambda \in \mathbb{R}$ (f linear). Also $\tilde{f}(iv) = i\tilde{f}(v)$. Hence f is linear over $\mathbb{C}$. This extends g to all of V.

Also for all $v \in V$, there exists $\theta \in [0, 2\pi)$ such that $|f(v)| = \Re(\tilde{f}(e^{i\theta} v)) = \tilde{g}(e^{i\theta} v) \leq p(e^{i\theta} v) = p(v)$.

Step 2: (ii) in $\mathbb{R}$ implies (i) in $\mathbb{R}$. If $W \subseteq V$ is a subspace, $p : V \to \mathbb{R}_+$ such that $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ for all $v_1, v_2 \in V$ and $p(\lambda v) = |\lambda| p(v)|$ for all $\lambda \in \mathbb{R}, v \in V$, and $f : W \to \mathbb{R}$ is linear such that $|f(v)| \leq p(v)$ for all $v \in W$ then (ii) can be applied to obtain $\tilde{f} : V \to \mathbb{R}$ linear extending f such that $\tilde{f}(v) \leq p(v)$ for all $v \in V$ (no modulus a priori in this conclusion).

We also deduce $\tilde{f}(-v) = p(-v) = p(v)$, so $|\tilde{f}(v)| \leq p(v)$.

Step 3: proof of (ii) in $\mathbb{R}$.

(a) Co-dimension 1 case: consider $V = W \oplus (\mathbb{R}v_0)$, $v_0 \neq 0$. We have $f : W \to \mathbb{R}$ linear, $f \leq p$ on W. To extend f it is enough to prescribe $\tilde{f}$ at v_0, then linearity does the rest: for $w \in W$, $\tilde{f}(w+av_0) = \tilde{f}(w)+a\tilde{f}(v_0) = f(w)+a\tilde{f}(v_0)$.

The value of $\tilde{f}(v_0)$ must satisfy:

$$\tilde{f}(w + av_0) \leq p(w + av_0),\ a > 0 \text{ and for } a < 0$$

This gives

$$\underbrace{-p\left(-\frac{w}{a} - v_0\right) + f\left(-\frac{w}{a}\right)}_{A(w')} \underbrace{\leq}_{a<0} \tilde{f}(v_0) \underbrace{\leq}_{a>0} \underbrace{p\left(\frac{w}{a} + v_0\right) - f\left(\frac{w}{a}\right)}_{B(w'')}$$

where $w' = -\frac{w}{a}$ and $w'' = \frac{w}{a}$. Then for all $w', w'' \in W$, $\tilde{f}(v_0) \in [A(w'), B(w'')]$. Set $\beta = \tilde{f}(v_0)$. Then a consistent value of β exists if and only if

$$\sup_{w' \in W} A(w') \leq \inf_{w'' \in W} B(w'')$$

This is indeed satisfied since

$$f(w') + f(w'') = f(w' + w'') \leq p(w' + w'') \leq p(w' - v_0) + p(w'' + v_0)$$

(b) Transfinite induction: define

$$\mathcal{S} = \{(\tilde{f}, \tilde{W}) : \tilde{f} : \tilde{W} \to \mathbb{R} \text{ linear },\ \tilde{f} \leq p \text{ and } \tilde{W} \supseteq W,\ \tilde{f}|_W = f\}$$

Now $\mathcal{S}$ is a poset under $(f_1, W_1) \subseteq (f_2, W_2)$ if $W_1 \subseteq W_2$ and $f_2|_{W_1} = f_1$. Also $\mathcal{S}$ has the least upper bound property: indeed consider $\Theta \subseteq \mathcal{S}$ a chain (totally ordered subset). Then for $(\bar{f}, \bar{W})$ defined by

$$\bar{W} = \bigcup_{W' : (f', W') \in \Theta} W'$$

and $\bar{f}(v) = f'(v)$ for all $v \in \bar{W}$, for $(f', W') \in \Theta$ such that $v \in W'$. Also $\bar{f}$ is well defined since Θ is totally ordered: so if $v \in W_1' \cap W_2'$ then wlog $W_1' \subseteq W_2'$, $f_2'|_{W_1'} = f_1'$ so $\bar{f}(v) = f_2'(v) = f_2'(v)$.

$\bar{f}$ is linear as Θ is totally ordered: $\bar{f}(\lambda v) = f'(\lambda v) = \lambda f'(v) = \lambda \bar{f}(v)$ for $(f', W') \in \Theta$ with $v \in W'$. Also

$$\bar{f}(v_1 + v_2) = f_2'(v_1 + v_2) = f_2'(v_1) + f_2'(v_2) = \bar{f}(v_1) + \bar{f}(v_2)$$

Finally $\bar{f} \leq p$ since for all $v \in \bar{W}$, $v \in W'$, $(f', W') \in \Theta$, $\bar{f}(v) = f'(v) \leq p(v)$.

So by Zorn's Lemma, there is a maximal element $(\tilde{f}, \tilde{W})$ in $\mathcal{S}$. If $\tilde{W} \subsetneq V$, then there exists $v_0 \in V \setminus \tilde{W}$ and the previous step applied to $\tilde{W} \subseteq \tilde{W} \oplus \mathbb{R}v_0$ and $\tilde{f} : \tilde{W} \to \mathbb{R}$ linear with $\tilde{f} \leq p$, gives the existence of a

$$\tilde{f}' : \underbrace{\tilde{W} \oplus \mathbb{R}v_0}_{\tilde{W}'} \to \mathbb{R}$$

linear with $\tilde{f}'|_{\tilde{W}} = \tilde{f}$. But then $(\tilde{f}', \tilde{W}')$ is strictly larger than $(\tilde{f}, \tilde{W})$, a contradiction.

$\square$

Theorem (Geometric form of Hahn-Banach)**.**

(i) *Let $(V, ||\cdot||)$ be an NVS over $\mathbb{R}$, $A \subseteq V$ open, convex and non-empty; $B \subseteq V$ convex and non-empty; $A \cap B = \emptyset$. Then there is a closed hyperplane weakly seperating A and B: there exists $f \in V^* \setminus \{0\}$, $\alpha \in \mathbb{R}$ such that $\sup_A f \leq \alpha \leq \inf_B f$ (the hyperplane is $f^{-1}(\{\alpha\})$)*

(ii) *Let $(V, ||\cdot||)$ be an NVS over $\mathbb{R}$, $A \subseteq V$ closed, conevex and non-empty; $B \subseteq V$ compact, convex and non-empty; $A \cap B = \emptyset$. Then there is a closed hyperplane strictly seperating A and B: there exists $f \in V^* \setminus \{0\}$, $\alpha_1 < \alpha_2 \in \mathbb{R}$ such that $\sup_A f \leq \alpha_1 < \alpha_2 \leq \inf_B f$.*

Proof.

(i) Let $C_0 = A - B = \{a - b : a \in A, b \in B\}$. Then $C_0 \neq \emptyset$ since A and B are non-empty, convex as

$$\lambda(a-b) + (1-\lambda)(a'-b') = \underbrace{(\lambda a + (1-\lambda a'))}_{\in A} - \underbrace{(\lambda b + (1-\lambda)b')}_{\in B}$$

Also C_0 is open since $C_0 = \bigcup_{b \in B} (\underbrace{A-b}_{\text{open}})$.

$0 \neq C_0$ since $A \cap B = \emptyset$. Let $v_0 \in C_0$, define $C = C_0 - v_0$. Then C is open, convex, non-empty and includes 0. Define $p = \mu_C$ (Minkowski gauge):

$$\forall v \in V,\ p(v) = \inf\{t \geq 0 : v \in tC\}$$

p satisfies (see proof of Kolmogorov)

- p is well-defined
- $p(\lambda v) = \lambda p(v),\ \forall \lambda > 0$
- $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ (using C convex)
- $p(-v)$ is not necessarily equal to $p(v)$ (C is not necessarily balanced)

Let $f : \mathbb{R}v_0 \to \mathbb{R}$ be linear defined by $f(-v_0) = 1$. Since $-v_0 \notin C$ ($0 \notin C_0$) we have $p(-v_0) \geq 1$, so $f \leq p$ ($\tilde{f}(-v_0) \leq p(-v_0)$) so $\tilde{f}(-\lambda v_0) \leq p(-\lambda v_0)$ for all $\lambda > 0$, and for $\lambda < 0$ $\tilde{f}(-\lambda v_0) \leq 0$.

The Hahn-Banach theorem (algebraic version) gives $\tilde{f} : V \to \mathbb{R}$ linear such that $\tilde{f}|_{\mathbb{R}v_0} = f$, $\tilde{f}(-v_0) = 1$. So $\tilde{f} \neq 0$, and since $p < 1$ in C, $\tilde{f}|_C < 1$, so since C is open around 0: there exists $B(0, \varepsilon) \subseteq C$ such that

$$\sup_{v \in B(0,\varepsilon)} \tilde{f}(v) \leq 1 \implies \sup_{v \in B(0,\varepsilon)} |\tilde{f}| \leq 1 \implies \tilde{f} \in V^*,\ ||\tilde{f}||_{V^*} \leq \varepsilon^{-1}$$

And

$$\tilde{f}|_C < 1 \implies \tilde{f}|_{C_0} < 0 \implies \sup_A \tilde{f} \leq \inf_B \tilde{f}$$

So there is $\alpha \in \mathbb{R}$ such that $\sup_A \tilde{f} \leq \alpha \leq \inf_B \tilde{f}$

(ii) $C_0 = B - A$ non-empty, convex, doesn't include 0, is closed: given $(a_n - b_n)_{n\geq 1}$ a sequence in C_0 with $(a_n - b_n) \to e$, we have (since B is compact), there exists a subsequence $(a_{n'} - b_{n'})_{n'\geq 1}$ such that $b_{n'}$ converges to $b \in B$, so $a_{n'}$ converges to $a \in A$ as A is closed. So $l = a - b \in C_0$.

So there exists an open ball $B(0, \varepsilon)$ such that $B(0, \varepsilon) \cap C_0 = \emptyset$. Apply (i) to $\tilde{A} = B(0, \varepsilon)$ (open, convex, non-empty) and $\tilde{B} = C_0$ (convex, non-empty). Then there exists $f : V \to \mathbb{R}$ bounded and linear, $f \neq 0$ such that

$$\sup_{B(0,\varepsilon)} f \leq \alpha \leq \inf_{C_0} f = \inf_{B} f - \sup_{A} f$$

Where $\alpha = \varepsilon ||f||_{V^*} = \sup_{v \in B(0,\varepsilon)} |f(v)| > 0$.

$\square$

Consequences of Hahn-Banach

Proposition.

(i) Given $(V, ||\cdot||)$ an NVS, W a subspace, $f \in W^*$ (linear and continuous on W), there exists $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$, and $||\tilde{f}||_{V^*} = ||f||_{W^*}$.

(ii) If $(V, ||\cdot||)$, is an NVS with $V \neq \{0\}$, then $V^* \neq \{0\}$.

(iii) Given $(V, ||\cdot||)$ an NVS with $V \neq \{0\}$, and $v, w \in V$ with $v \neq w$ then there exists $f \in V^*$ such that $f(v) \neq f(w)$.

Proof.

(i) Apply HB (algebraic form) with $p : V \to \mathbb{R}+$, $v \mapsto ||f||_{W^*}||v||$. This satisfies the assumptions trivially and $|f| \leq p$ on W, so there exists $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$ and $|\tilde{f}(v)| \leq p(v) \leq ||f||_{W^*}||v||$ for all $v \in V$. This implies $||f||_{V^*} \leq ||f||_{W^*}$ and we clearly have equality.

(ii) Consider $v_0 \in V \setminus \{0\}$. Then define ("support functional" for v_0) $f : W = \mathbb{F}v_0 \to \mathbb{F}$ the linear map such that $f(v_0) = ||v_0||$. Then (i) implies the existence of $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$ and $||f||_{W^*} = ||f||_{V^*} = 1$. Hence $\tilde{f} \neq 0$ and $V^* \neq \{0\}$.

(iii) Given $v \neq w$ in V, apply (ii) to $v_0 = v - w$. Then there is $\tilde{f} \in V^*$ such that $\tilde{f}(v_0) = \tilde{f}(v) - \tilde{f}(w) = ||v_0|| \neq 0$.

□

Proposition. Given $(V, ||\cdot||)$ an NVS, $\Phi : V \to V^{**}$ defined by $v \mapsto \Phi(v)$ where $\Phi(v)(f) = f(v)$ for any $f \in V^*$. This is an isometry (in particular $|||\Phi||| = 1$).

Proof. We have already proven that $||\Phi(v)||_{V^{**}} \leq ||v||_V$ for all $v \in V$. Let us prove this is an equality. Consider $v \in V \setminus \{0\}$, let f_v be a support functional for v, $f_v \in V^*$, $f_v(v) = ||v||_V$, $||f_v||_{V^*} = 1$ (constructed in the proof of (ii) in the previous proposition). Now $\Phi(v)(f_v) = f_v(v) = ||v||_V$. Hence

$$\sup_{\substack{f \in V^* \\ ||f||_{V^*} \leq 1}} |\Phi(V)(f)| \geq ||v||_V \implies ||\Phi(v)||_{V^{**}} \geq ||v||_V$$

□

Proposition. Let V, W be NVS', $T : V \to W$ linear and bounded. Then $T^* : W^* \to V^*$ (the adjoint) satisfies $|||T^*||| = |||T|||$.

Proof. We already proved $|||T^*||| \leq |||T|||$. So we show the reverse inequality. Consider $v \in V$ such that $||v|| = 1$ and $w = Tv \neq 0$. Let $g_w \in W^*$ be a support functional for $w \in W$. Then $T^*(g_w)(v) = g_w(Tv) = g_w(w) = ||w||_W$. So

$$||T^*(g_w)||_{V^*} = \sup_{\substack{v' \in V \\ ||v'||=1}} |T^*(g_w)(v')| \geq ||w||_W$$

so

$$|||T^*||| = \sup_{\substack{g \in W^* \\ ||g||_{W^*}=1}} ||T^*(g)||_{V^*} \geq ||T^*(g_w)|| \geq ||w||_W$$

so

$$|||T^*||| \geq ||w||_W = ||Tv|_W$$

So take the supremum over $v \in V, ||v|| = 1$ to get

$$|||T^*||| \sup_{\substack{v \in V \\ ||v||=1}} ||Tv||_W = |||T|||$$

□

6 The Baire Category Theorem

Hahn Banach: uses sublinearity of gauges/norms (convexity of associated unit ball) to study the dual space and build linear forms.

Baire: use completeness to prove that complete NVS' are necessarily "big" - used for existence of objects and local-to-global estimates.

The following theorem was proved by Osgood (1897) in $\mathbb{R}$ and by Baire (1899) in general.

Definition. Let (X, τ) be a topological space.

(i) A subset $B \subseteq X$ is *rare* (or *nowhere dense*) if $\overline{B}$ has empty interior, i.e for all $U \in \tau$, $B \cap U$ is not dense in U.

(ii) A subset $B \subseteq X$ is *meagre* (first category) in X if it can be written as a countable union of rare sets. Otherwise B is *non-meagre* (second category) in X.

(iii) (X, τ) is *meagre/non-meagre* (first/second category) if it is as a subset of itself.

Proposition. Given (X, τ) a topological space, the following are equivalent

(i) X is non-meagre

(ii) For all $(C_n)_{n \geq 1}$ a countable collection of closed sets covering X, at least one C_n has non-empty interior

(iii) For all $(O_n)_{n \geq 1}$ a countable collection of open sets which are all dense in X, $\bigcap_{n \geq 1} O_n \neq \emptyset$

Proof. (ii) implies (i): if $X = \bigcup_n A_n$, with A_n rare, then $C_n := \bar{A}_n$ are closed with empty interior, and $X = \bigcup_n C_n$.

(i) implies (ii): if $X = \bigcup_n C_n$, C_n closed with empty interior, then $A_n := C_n$ are rare.

(ii) implies (iii): given $(O_n)_{n\geq 1}$ open dense sets, $C_n = O_n^c$ are closed with empty interior: otherwise there exists $U \in \tau$, $U \subseteq C_n$ such that $U \cap O_n = \emptyset$ (contadicting density). Also $\bigcap_n O_n \neq \emptyset \iff \bigcup_n C_n \supsetneq X$.

(iii) implies (ii): Given $(C_n)_{n\geq 1}$ closed sets with $U_{n\geq 1} C_n = X$, if all C_n have empty interiors, then $O_n := C_n^c$ contradicts (iii) so at least one C_n has non empty interior □

Theorem (Baire's Theorem)**.** *Let (X, d) be a complete metric space. Then X is non-meagre. In fact it is a Baire space, a space in which countable intersections of dense open sets are dense.*

Proof. It is enough to prove that (X, d) is a Baire space. Consider $(O_n)_{n\geq 1}$ a sequence of open dense sets, and U an arbitrary open set. We will show $U \cap (\bigcap_n O_n) \neq \emptyset$.

Induction: since O_1 is dense, $O_1 \cap U$ is non-empty and open. Pick $x_1 \in O_1 \cap U$, with $B(x_1, r_1) \subseteq O_1 \cap U$ for some $r_1 > 0$. Then $O_2 \cap B(x_1, r_1/2) \neq \emptyset$ (density of O_2) and open. So there exists $x_2 \in O_2$ and $r_2 > 0$ such that $B(x_2, r_2) \subseteq O_2 \cap B(x_1, r_2/2)$.

General step: there exists $B(x_{k+1}, r_{k+1}) \subseteq O_{k+1} \cap B(x_k, r_k/2)$ for $x_{k+1} \in X$, $r_{k+1} > 0$. This builds a sequence $(x_k)_{k\geq 1}$ in X which is Cauchy: for all $k \geq k_0 \geq 1$, $x_k \in B(x_{k_0}, r_{k_0}/2)$ and inclusion of balls implies $r_{k+1} \leq r_k/2$, for $k \geq 1$. So $r_k \leq 2^{-k+1} r_1 \to 0$, so it is indeed Cauchy. Hence $x_k \to e$ for some $e \in X$ and $e \in \overline{B}(x_{k_0}, r_{k_0}/2)$ for all $k_0 \geq 1$. So $e \in O_{k+1} \cap B(x_k, r_k/2)$ for all k, and so $e \in (\bigcap_n O_n) \cap U$ (contained in U since $B(x_1, r_1)$ is). □

Theorem (Baire)**.** *If (X, τ) is a compact and Hausdorff space, then X is:*

(i) Normal: for all C_1, C_2 disjoint non-empty closed sets, there exist $U_1, U_2 \in \tau$ disjoint such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$.

(ii) X is a Baire space.

Proof.

(i) Let C_1, C_2 be as in the statement. For all $x \in C_1, y \in C_2$ there exist $U^1_{x,y}, U^2_{x,y} \in \tau$ such that $x \in U^1_{x,y}$, $y \in U^2_{x,y}$ and $U^1_{x,y} \cap U^2_{x,y} = \emptyset$. Fix $y \in C_2$, so $C_1 \subseteq \bigcup_{x \in C_1} U^1_{x,y}$ (since $x \in U^1_{x,y}$).
Since C_1 is a closed subset of a compact space X, it is compact. So extract a finite covering: take $x_1, \ldots, x_m \in C_1$ such that $C_1 \subseteq \bigcup_{i=1}^m U^1_{x_i,y}$. Denote

$V_y^1 = \bigcup_{i=1}^m U_{x_i,y}^1$ and $V_y^2 = \bigcap_{i=1}^m U_{x_i,y}^2$. Observe that V_y^1, V_y^2 are open and disjoint. Then C_2 is compact (closed in compact space), $C_2 \subseteq \bigcup_{y \in C_2} V_y^2$ (since $y \in V_y^2$). So can extract a finite covering: take $y_1, \ldots, y_n \in C_2$ such that $C_2 \subseteq \bigcup_{j=1}^m V_{y_j}^2$.

Finally denote $U^1 = \bigcap_{j=1}^n V_{y_j}^1$ and $U^2 = \bigcup_{j=1}^n V_{y_j}^2$. Then U^1, U^2 are open, disjoint and $C_1 \subseteq U_1$, $C_2 \subseteq U_2$.

(ii) Consider $(O_n)_{n \geq 1}$ open dense sets, and $U \in \tau$. We want to show $(\bigcap_n O_n) \cap U \neq \emptyset$.

Induction:

- Since O_1 is dense, there exists $x_1 \in O_1 \cap U$ ($O_1 \cap U$ non-empty and open). We want to show there exists U_1 open around x_1 such that $\overline{U}_1 \subseteq O_1 \cap U$.
- $\{x_1\}$ is disjoint from $(O_1 \cap U)^c$, and both sets closed. So there exist $U_1, U_1' \in \tau$ such that $x_1 \in U_1$, $(O_1 \cap U_1)^c \subseteq U_1'$ and $U_1 \cap U_1' = \emptyset$. Then $\overline{U}_1 \subseteq (U_1')^c \subseteq O_1 \cap U$
- Continuing the induction: $x_k \in U_k \subseteq \overline{U}_k \subseteq O_k \cap U_{k-1}$. Then $\bigcap_k \overline{U}_k$ is non empty (X compact) so $\bigcap_k \overline{U}_k \subseteq U \cap (\bigcap_n O_n)$

□

Applications:

- Existence of irrationals in $\mathbb{R}$: $(\mathbb{R}, |\cdot|)$ is a complete metrix space, so a Baire space. Then for all $x \in \mathbb{R}$, $\{x\}$ is closed with empty interior. So if $\mathbb{Q} = \{q_n : n \geq 1\}$, then $\mathbb{R} = \bigcup_n \{q_n\}$ would contradict (ii) in the above proposition (before the last two theorems). In fact a similar argument proves a stronger result: if (X, d) is a metric space with no isolated points, then X is uncountable.

- There exists $f \in C([0,1])$ that is nowhere differentiable. To show this, we instead prove

$$\mathcal{D} = \{f \in C([0,1]) : f \text{ differentiable at some } x \in [0,1]\}$$

is meagre in the Baire space $(C([0,1]), ||\cdot||_\infty)$. Define

$$A_n = \{f \in C[0,1] : \underbrace{\exists x \in [0,1] \forall y \in [0,1] \cap [x - \frac{1}{n}, x + \frac{1}{n}], |f(x) - f(y)| \leq n|x - y|}_{*}\}$$

Properties of A_n:

1. A_n is closed: if $(f_k)_{k\geq 1}$ is a sequence in A_n, $f_k \xrightarrow{||\cdot||_\infty} f$, there exists $(x_k)_{k\geq 1}$ in $[0,1]$ such that $(*)$ is satisfied for f_k at each x_k. Then $[0,1]$ is compact so there exists a subsequence $(x_{\varphi(k)})_{k\geq 1}$ ($\varphi : \mathbb{N}^* \to \mathbb{N}^*$ strictly increasing) that converges: $x_{\varphi(k)} \to x_\infty \in [0,1]$. We prove that f satisfies $(*)$ for x_∞. Let $y \in (x_\infty - \frac{1}{n}, x_\infty + \frac{1}{n}) \cap [0,1]$, then for k large enough,
$$y \in (x_{\varphi(k)} - \frac{1}{k}, x_{\varphi(k)} + \frac{1}{k}) \cap [0,1] \qquad (**)$$
So $(*)$ on $(f_{\varphi(k)}, x_{\varphi(k)})$ gives $|f_{\varphi(k)}(x_\varphi(k)) - f_{\varphi(k)}(y)| \leq n|x_{\varphi(k)} - y|$. Take the limit $k \to \infty$, so $f_{\varphi(k)}(x_{\varphi(k)}) \to f(x_\infty)$ by uniform convergence. So $|f(x_\infty) - f(y)| \leq n|x_\infty - y|$. Then y is in the enpoints of $(**)$ by continuity of f.
2. A_n has empty interior in $(C([0,1]), ||\cdot||_|\infty)$: assume for contradiction that $B_{||\cdot||_\infty}(f_0, \varepsilon) \subseteq A_n$, for some $f_0 \in C([0,1])$ and $\varepsilon > 0$. Then there exist f_1 piecewise affine in $B_{||\cdot||_\infty}(f_0, \varepsilon/2)$ (using uniform continuity of f_0). Then add g_δ (sawtooth function with slopes δ^{-1} and height δ). Then for δ small enough, $f_1 + g_\delta \in B_{||\cdot||_\infty}(f_1, \varepsilon/2) \subseteq B_{||\cdot||_\infty}(f_0, \varepsilon)$ and $g_\delta \notin A_n$ (as δ^{-1} can be arbitrarily large).
3. $\mathcal{D} \subseteq \bigcup_{n\geq 1} A_n$ since differentiability at some $x \in [0,1]$ implies $|f(x) - f(y)| \leq n|x-y|$ for y close to x and n large enough.

Therefore $\mathcal{D}$ is meagre, so cannot be the whole space $(C([0,1]), ||\cdot||_\infty)$ since this is non-meagre (complete metric space).

- Illustration that "smallness" in the sense of Baire is not the same as being "small" in Lebesgue measure. These notions can coincide: $\{x\}$ is meagre and measure 0, $\mathbb{Q}$ is meagre and measure 0.

Proposition. There exists $\mathcal{D} \subseteq \mathbb{R}$ that is non-meagre with zero measure, and there exists $\mathcal{D}$ which is meagre with full measure.

Proof. Write $\mathbb{Q} = \{q_k\}_{k\geq 1}$, an enumeration of the rationals. Define $\mathcal{D}_n = \bigcup_k (q_k - \frac{1}{2^{n+k}}, q_k + \frac{1}{2^{n+k}})$. Then $\mathcal{D}_n$ is open and dense since $\mathbb{Q} \subseteq \mathcal{D}_n$. $\mu(\mathcal{D}_n) \leq \sum_{k\geq 1} \frac{1}{2^{n+k-1}} = 2^{-(n-1)}$. Define $\mathcal{D} = \bigcap_{n\geq 1} \mathcal{D}_n$ (decreasing sequence of open dense sets). Then $\mu(\mathcal{D}) \leq \mu(\mathcal{D}_n)$ for all n, so $\mathcal{D}$ has zero measure. Note that $\mathcal{D}^c = \bigcup_{n\geq 1} \mathcal{D}_n^c$ where $\mathcal{D}_n^c$ is closed with empty interior (since $\mathbb{Q} \cap \mathcal{D}_n^c = \emptyset$), so $\mathcal{D}^c$ is meagre, and since $\mathbb{R}$ is non-meagre, $\mathcal{D}$ is non-meagre. □

7 Combining Baire theory with linear structure

Theorem (Uniform Boundedness Principle)**.** *Let V, W be Banach spaces. Then*

(i) Let $(T_i)_{i\in I}$ be a collection (not necessarily countable) of bounded linear maps $V \to W$, that are "locally bounded": for all $v \in V$, $\sup_{i\in I} ||T_i v||_W < \infty$. Then

$$\sup_{i\in I} |||T_i||| = \sup_{i\in I} \sup_{\substack{v\in V \\ ||v||_V = 1}} ||T_i v|| < \infty$$

(ii) Let $(T_k)_{k\geq 1}$ be a sequence in $\mathcal{B}(V, W)$ (bounded linear maps $V \to W$) such that T_n converge pointwise to some $T \in \mathcal{L}(V, W)$ (linear but not necessarily bounded). Then T is in fact bounded and $|||T||| \leq \liminf_{n\to\infty} |||T_n|||$

(iii) $B \subseteq V$ is bounded if and only if for all $f \in V^*$, $f(B) \subseteq \mathbb{R}$ is bounded.

(iv) $B' \subseteq V^*$ is bounded if and only if for all $v \in V$, $\Phi(v)(B) \subseteq \mathbb{R}$ is bounded.

Proof. First we show (i) implies (ii): apply (i) to the collection $(T_n)_{n\geq 1}$ to obtain that $\sup_{n\geq 1} |||T_n||| = C < \infty$ (converges pointwise so locally bounded). Then we prove T is bounded with $|||T||| \leq C$. Have $||Tv|| = \lim_{n\to\infty} ||T_n v||$ and $||T_n v|| \leq C||v||$ so $||Tv|| \leq C||v||$. Now we prove that $|||T||| \leq \liminf_n |||T_n|||$.

Given $\varepsilon > 0$, there exist $v_\varepsilon \in V$ such that $||v_\varepsilon||_V = 1$ and $|||T||| \leq \varepsilon + ||Tv_\varepsilon||_W$. Then since $T_n v_\varepsilon \to T v_\varepsilon$, there exists $N \geq 1$ such that for $n \geq N$, $||Tv_\varepsilon|| \leq ||T_n v_\varepsilon|| + \varepsilon \leq |||T_n||| + \varepsilon$, so $|||T||| \leq |||T_n||| + 2\varepsilon$ for all $n \geq N$, which implies $|||T||| \leq 2\varepsilon + \liminf_{n\geq 1} |||T_n|||$ for all $\varepsilon > 0$ thus $|||T||| \leq \liminf_{n\geq 1} |||T_n|||$.

Now we show (i) implies (iii): if B is bounded, then for any $f \in V^*$, $f(B)$ is bounded since f is bounded. Assume $B \subseteq V$ is such that $f(B)$ is bounded for all $f \in V^*$. Apply (i) to the Banach spaces V^* and $\mathbb{R}$ and the following collection of bounded linear maps $(\Phi(v))_{v\in B}$. Then since $f(B)$ is bounded for all $f \in V^*$

$$\sup_{v\in B} |\Phi(v)(f)| = \sup_{v\in V} |f(v)| < \infty \ \forall f \in V^*$$

So the conclusion of (i) gives $\sup_{v\in V} ||\Phi(v)||_{V^{**}} < \infty$. Since Φ is an isometry, this means $\sup_{v\in B} ||v||_V < \infty$, so B is bounded.

Now show (i) implies (iv): the forward direction is trivial: B' bounded, $\Phi(v) : V^* \to \mathbb{R}$ is linear and bounded so $\Phi(v)(B')$ bounded. For the backward direction apply (i) with V and $\mathbb{R}$ to the collection $\{f : f \in B' \subseteq V^*\}$. Local boundedness of this collection follows since for all $v \in V$, $\sup_{f\in B'} |f(v)| = \sup_{f\in B'} |\Phi(v)(f)| < \infty$. So uniform boundedness gives $\sup_{f\in B'} ||f||_{V^*} < \infty$.

Now we prove (i): let $C_n := \{v \in V : \forall i \in I, ||T_i(v)||_W \leq n\}$.

1. C_n is closed: T_i are continuous so $C_n = \bigcap_{i\in I} T_i^{-1}(\overline{B}_{|\cdot|}(0, n))$.
2. Local boundedness implies that $V = \bigcup_{n\geq 1} C_n$.
3. Since V is a Baire space (complete metric space), there exists $n_0 \geq 1$ such that C_{n_0} has non-empty interior: so there exists $v_0 \in V$, $\varepsilon > 0$ such that $\forall i \in I$, $v \in B(v_0, \varepsilon)$ we have $||T_i(v)||_W \leq n_0$.
4. Now for any $v \in V$, $||T_i(v)|| \leq ||T_i(v + v_0)|| + ||T_i(v_0)|| \leq \frac{n_0}{\varepsilon}||v|| + ||T_i(v_0)||_W$, and $\sup_{i\in I} ||T_i(v_0)|| < \infty$ by local boundedness, so $\sup_{\substack{v\in V\\ ||v||=1}} ||T_i(v)|| < \infty$.

$\square$

Remarks:

1. The main result is (i)
2. (iii) generalises in infinite dimensions the intuition that boundedness is something we need only check in each coordinate
3. (iii) implies for instance that if $(v_n)_{\geq 1}$ "weakly converges" to v: $\forall f \in V^*$, $f(v_n) \xrightarrow{n\to\infty} f(v)$, then $(v_n)_{n\geq 1}$ is bounded.

Theorem (Open mapping theorem, Inverse mapping theorem, Closed graph theorem)**.** *Let V, W be Banach spaces. Then*

(i) Any $T \in \mathcal{B}(V,W)$ (bounded and linear) that is surjective, is also open: i.e it maps open sets to open sets.

(ii) Any $T \in \mathcal{B}(V,W)$ that is bijective is such that T^{-1} is bounded.

(iii) Any $T \in \mathcal{L}(V,W)$ (linear but not necessarily bounded) is bounded if and only if its graph $\{(v,T(v)) \in V \times W : v \in V\}$ is closed.

Proof. First we show (i) implies (ii): if $T \in \mathcal{B}(V,W)$ is bijective, then (i) implies T is open, i.e for all $U \subseteq V$ open, $T(U)$ is open. Hence T^{-1} is continuous as $(T^{-1})^{-1}(U) = T(U)$ is open. Since T^{-1} is linear, this means T^{-1} is bounded.

Now we show (ii) implies (iii): we first show that if T is bounded, then the graph of T is closed. Assume $(v_n, T(v_n)) \xrightarrow{n\to\infty} (v,w)$ in $V \times W$. Then $v_n \to v$, and since T is bounded, T is continuous so $T(v_n) \to T(v)$ so $w = T(v)$ and (v,w) belongs to the graph. Conversely if the graph of T is closed, it is closed in the Banach space $V \times W$, so the graph of T is itself a Banach space. Define $\pi : \text{Graph}(T) \to V$, $(v,Tv) \mapsto v$. This is linear, bijective and bounded since $||\pi(v,Tv)||_V = ||v||_V \leq ||v||_V + ||Tv||_W = ||(v,Tv)||_{V\times W}$, so π^{-1} bounded by (ii) and there exists $C > 0$ such that $|v||_V + ||Tv||_W \leq C||v||_V$.

Now we prove (i): let $T \in \mathcal{B}(V,W)$ be surjective. To prove that T is open, it is enough to prove:

$$\exists \varepsilon > 0 \text{ such that } B(0,\varepsilon) \subseteq T(B(0,1)) \tag{$*$}$$

Indeed, if $(*)$ is satisfied, and if $U \subseteq V$ is an open set with $x \in U$, and $y = T(x) \in T(U)$ then $T(U) \supseteq y + \delta T(B(0,1)) \supseteq y + \delta B(0,\varepsilon) = B(y, \delta\varepsilon)$ where $\delta > 0$ is such that $B(x,\delta) \subseteq U$. So $T(U)$ is open around y, so open.

Let us prove $(*)$: since T is surjective, $W = \bigcup_{n\geq 1} T(B(0,n)) = \bigcup_{n \geq 1} \overline{T(B(0,n))}$. Since W is Banach (so meagre by Baire) and the countable union of these closed sets, there exists $n_0 \geq 1$ such that $\overline{T(B(0,n_0))}$ has non-empty interior. Since dilation is a diffeomorphism, we may assume $\overline{T(B(0,1))}$ has non-empty interior:

there exists $w_0 \in W$, $\varepsilon > 0$ such that $w_0 + B(0, 2\varepsilon) \subseteq \overline{T(B(0,1))}$. Goal: "remove this closure".

$$\overline{T(B(0,1))} \supseteq \frac{1}{2}\left(w_0 + B(0, 2\varepsilon)\right) + \frac{1}{2}\left(-w_0 + B(0, 2\varepsilon)\right)$$

since $\overline{T(B(0,1))}$ is convex and balanced. So $\overline{T(B(0,1))} \supseteq B(0, 2\varepsilon)$. Let us prove that $B(0, \varepsilon) \subseteq T(B(0,1))$

1. Let $w_1 \in B(0,\varepsilon) = \frac{1}{2}B(0, 2\varepsilon) \subseteq \frac{1}{2}\overline{T(B(0,1))} = \overline{T(B(0,1/2))}$. So there exists $v_1 \in B(0, 1/2)$ such that $||w_1 - Tv_1||_W < \varepsilon/2$.
2. Then $w_2 := w_1 - Tv_1 \in B(0, \varepsilon/2) \subseteq \overline{T(B(0,1/4))}$, and there exists $v_2 \in B(0, 1/4)$ such that $||w_2 - Tv_2||_W < \varepsilon/4$.
3. Continue this: define $w_k := w_{k-1} - Tv_{k-1} \in B(0, \varepsilon/2^k) \subseteq \overline{T(B(0, 2^{-k}))}$. Now there exists $v_k \in B(0, 2^{-k})$ such that $||w_k - Tv_k|| < \varepsilon \cdot 2^{-k}$.
4. This builds $(w_k)_{k\geq 1}$, $(v_k)_{k\geq 1}$ such that $||w_k||_W \leq \varepsilon \cdot 2^{k-1} \to 0$, $||v_k|| \leq 2^{-k} \to 0$. Then $\sum_{k=1}^n v_k \to \bar{v}$ (V complete) with $||\bar{v}||_V < 1$, and $w_k = w_1 - T\left(\sum_{l=1}^{k-1} v_l\right) \to 0$ we deduce that $w_1 = T\bar{v}$, so $w_1 \in T(B(0,1))$.

$\square$

Remark: Closed graph theorem implies: if $v_n \to v$, $Tv_n \to w$ implies $w = Tv$, then $v_n \to v$ implies $Tv_n \to Tv$.

8 Topology of $C(K)$

Define
$$C(K) = \{f : K \to \mathbb{R} : \text{ continuous}\}$$
Where K is a compact and Hausdorff topological space.

Definition. A topological space (X, τ) is

(i) T_0 if all distinct $x, y \in X$ have distinct bases of neighborhoods: there exists $U \in \tau$ such that $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$.

(ii) T_1 if for all distinct $x, y \in X$, there exist $U_1, U_2 \in \tau$ such that $x \in U_1, y \notin U_1$, $x \notin U_2, y \in U_2$ (points are closed).

(iii) T_2 (Hausdorff) if for all distinct $x, y \in X$, there exist $U_1, U_2 \in \tau$ disjoint such that $x \in U_1$, $y \in U_2$.

(iv) Normal if for all $C_1, C_2 \subseteq X$ closed, there exist $U_1, U_2 \in \tau$ disjoint such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$.

Note that Normal+T_1 implies T_2.

Lemma (Urysohn)**.** *A topological space (X, τ) is normal if and only if for all $C_1, C_2 \subseteq X$ closed and non-empty, there exists $f : X \to [0,1]$ continuous such that $f|_{C_1} = 0$, $f|_{C_2} = 1$.*

Proof. To show ($\Leftarrow$), take $U_1 = f^{-1}([0,1/2))$, $U_2 = f^{-1}((1/2,1])$. Then U_1, U_2 are open, disjoint and $C_1 \subseteq U_1$, $C_2 \subseteq U_2$.

Now we show ($\Rightarrow$).

1. Step 1: we show that given $U_0 \subseteq U_1 \subsetneq X$, non-empty and open, with $\overline{U_0} \subseteq U_1$, there is $U_{1/2}$ open such that $U_0 \subseteq \overline{U_0} \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U_1$.

   Indeed define $C_1 = \overline{U_0}$, $C_2 = U_1^c$ (non-empty and closed) so by normality there exists $U_{1/2}, U'_{1/2} \in \tau$ such that $C_1 \subseteq U_{1/2}$, $C_2 \subseteq U'_{1/2}$ and $U_{1/2} \cap U'_{1/2} = \emptyset$. Then $\overline{U_0} = C_1 \subseteq U_{1.2}$, $C_2 \subseteq U'_{1/2}$ so $U'^c_{1/2} \subseteq C_2^c = U_1$. And since $U'^c_{1/2}$ is closed, $U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U'^c_{1/2} \subseteq U_1$.

2. Step 2: induction. Let
$$D_n = \left\{ \frac{k}{2^n} : k \in \{0, 1, \ldots, 2^n\} \right\} \subseteq [0,1], \; n \geq 0$$
   Then $(D_n)_{n \geq 0}$ is an increasing sequence of sets. Induction hypothesis: given $\emptyset \neq U_0 \subseteq \overline{U_0} \subseteq U_1 \subsetneq X$, there are $(U_r)_{r \in D_n}$ open such that for all $r_1, r_2 \in D_n$, $\overline{U_{r_1}} \subseteq U_{r_2}$ whenever $r_1 < r_2$.

For $n = 0$, $D_0 = \{0, 1\}$ and there is nothing to prove. For the induction step, the idea is to fill each "gap". Let $r \in D_{n+1} \setminus D_n$, then $r = \frac{k}{2^{n+1}}$ with $k = 2k_0 + 1$ for some $k_0 \in \{0, \dots, 2^n - 1\}$. Then $U_{\frac{k_0}{2^n}}, U_{\frac{k_0+1}{2^n}}$ are already constructed with $\emptyset \neq U_{\frac{k_0}{2^n}} \subseteq \overline{U_{\frac{k_0}{2^n}}} \subseteq U_{\frac{k_0+1}{2^n}} \subseteq \overline{U_{\frac{k_0+1}{2^n}}} \subsetneq X$.

Now apply Step 1: there exists $U_{\frac{k}{2^{n+1}}}$ such that $\overline{U_{\frac{k_0}{2^n}}} \subseteq U_{\frac{k}{2^{n+1}}} \subseteq \overline{U_{\frac{k}{2^{n+1}}}} \subseteq U_{\frac{k_0+1}{2^n}}$. So induction step is done.

So we have $(U_r)_{r \in D}$ where $D = \bigcup_{n \geq 1} D_n$ such that $U_{r_1} \subseteq \overline{D_{r_1}} \subseteq D_{r_2}$ whenever $r_1 < r_2$.

3. Step 3: we now define f. Let $f(x) = \inf\{r \in D : x \in U_r\}$ for $x \in U_1$, and $f(x) = 1$ on $C_2 = U_1^c$.

4. Step 4: we show f is continuous. It is enough to check for all $a \in [0, 1)$, $f^{-1}((a, 1])$ is open and or all $b \in (0, 1]$, $f^{-1}([0, b))$ is open.

   Indeed, the open intervals are a base for the topology on $\mathbb{R}$ and for all $a < b \in \mathbb{R}$, $f^{-1}(a, b) = f^{-1}((a, 1]) \cap f^{-1}([0, b))$.

   We show $f^{-1}((a, 1])$ is open for all $a \in [0, 1)$ (the proof for $f^{-1}([0, b))$ is symmetric). Consider $x \in f^{-1}((a, 1])$. By definition $f(x) > a$, so (by the density of D) there exist $r, r' \in D$ such that $f(x) > r' > r > a$. Then $f(x) \in U_{r'}$ (as $f(x) > r'$) so $x \in U_{r'}^c$ and since $\overline{U_r} \subseteq U_{r'}$, $x \in (\overline{U_r})^c$ which is open. Finally $U_a \subseteq \overline{U_r}$ so $\overline{U_r} \subseteq U_a^c$ so $\overline{U}_r \subseteq f^{-1}((a, 1])$. Hence $x \in (\overline{U}_r)^c \subseteq f^{-1}((a, 1])$ and $f^{-1}((a, 1])$ is open as x was arbitrary.

□

Corollary. *Let (K, τ) be a topological space which is Normal and (T_1). Then $C(K)$ seperates points: for all distinct $x, y \in K$, there exists $f : K \to [0, 1]$ such that $f(x) = 0, f(y) = 1$.*

Proof. $C_1 = \{x\}$ and $C_2 = \{y\}$ are closed by (T_1). Apply previous lemma. □

Theorem (Tietze extension theorem)**.** *Let (X, τ) be a normal topological space and $C \subseteq X$ closed and non-empty. Also let $f : C \to \mathbb{R}$ be continuous and bounded. Then there exists $\tilde{f} : X \to \mathbb{R}$ continuous such that $\tilde{f}|_C = f$, and* $\sup_X |\tilde{f}| = \sup_C |f|$.

Remark: when $f : C \to \mathbb{C}$ continuous, we can extend to $\tilde{f} : X \to \mathbb{C}$ continuous such that $\tilde{f}|_C = f$, $\sup_X |\Re \tilde{f}| = \sup_C |\Re f|$ and $\sup_X |\Im \tilde{f}| = \sup_C |\Im f|$ by applying the theorem to $\Re f, \Im f$.

Proof. If f is constant the result is clear, otherwise replace f by $\frac{f - \inf f}{\sup f - \inf f}$ to deal only with $f : C \to [0, 1]$ (with in fact $\inf f = 0$, $\sup f = 1$).

Idea: define $C_1 = f^{-1}([0,1/3])$, $C_2 = f^{-1}([2/3,1])$. Then Urysohn's lemma gives $g_1 : X \to [0,1/3]$ continuous such that $g_1|_{C_1} = 0$, $g_1|_{C_2} = 1/3$. Then if $f_1 := f$, $f_2 := f_1 - g_1|_C : C \to [0 : 2/3]$. Continue this to get $f_k : C \to [0,(2/3)^{k-1}]$ continuous, then there exists $g_k : X \to [0, \frac{1}{3}(2/3)^{k-1}]$ so $f_{k+1} := f_k - g_k|_C : C \to [0,(2/3)^k]$. $\square$