1 Differential Geometry

1.1 Parameterised curves & arc length

- A curve $C : [a, b] \ni t \mapsto \mathbf{x}(t)$ in $\mathbb{R}^3$ is called **differentiable** if each component x_i of $\mathbf{x}$ is differentiable

- A curve is called **regular** if $|\mathbf{x}'(t)| \neq 0$ for all t

- If a curve $C : [a, b] \ni t \mapsto \mathbf{x}(t)$ is both differentiable and regular, we can find the **arc length** of C using

$$l(C) = \int_a^b |x'(t)| \mathrm{d}t = \int_S \mathrm{d}s$$

- From this we deduce that $\mathrm{d}s = |\mathbf{x}'(t)|\mathrm{d}t$. Note that for regular curves we have $|\mathbf{x}'(t)| \neq 0$ and so we can invert this relationship between s and t to parameterise $t = t(s)$ and we get $\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{|\mathbf{x}'(t(s))|}$

1.2 Curvature & Torsion

- Define the **tangent vector** $\mathbf{t}(s) = \mathbf{r}'(s)$. This is a unit vector so the second derivative $\mathbf{r}''(s) = \mathbf{t}'(s)$ only measures change in direction.

- Hence we define **curvature** $\kappa(s) = |\mathbf{r}''(s)| = |\mathbf{t}'(s)|$

- Since $|\mathbf{t}| = 1$, we have $\mathbf{t}' \cdot \mathbf{t} = 0$. Define **principal normal** $\mathbf{n} = \mathbf{t}'/\kappa$

- We extend $\mathbf{t}$ and $\mathbf{n}$ to an orthonormal basis by defining the **binormal** by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$

- Can show $\mathbf{b}'$ is parallel to $\mathbf{n}$ so define the **torsion** such that $\mathbf{b}' = -\tau\mathbf{n}$

- This gives us the equations

$$\mathbf{t}' = \kappa\mathbf{n}, \ \mathbf{b}' = -\tau\mathbf{n}$$

- The **radius of curvature** is the required radius of a circle such that the circle "best fits" the curve at a point. It is given by $R = 1/\kappa$

2 Coordinates, Differentials & Gradients

2.1 Differentials & first order changes

- For a function $f = f(u_1, u_2, \ldots, u_n)$ we define the **differential** of f by

$$\mathrm{d}f = \frac{\partial f}{\partial u_i}\mathrm{d}u_i$$

- Similarly, for a vector $\mathbf{x} = \mathbf{x}(u_1, u_2, \ldots, u_n)$ we have

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_i}\mathrm{d}u_i$$

2.2 Coordinates & line elements

- We say that (u, v, w) are a set of **orthogonal curvilinear coordinates** if the vectors

$$\mathbf{e}_u = \frac{\frac{\partial \mathbf{x}}{\partial u}}{\left|\frac{\partial \mathbf{x}}{\partial u}\right|}, \ \mathbf{e}_v = \frac{\frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial v}\right|}, \ \mathbf{e}_w = \frac{\frac{\partial \mathbf{x}}{\partial w}}{\left|\frac{\partial \mathbf{x}}{\partial w}\right|}$$

  form a right handed, orthonormal basis

- We also write $h_u = \left|\frac{\partial \mathbf{x}}{\partial u}\right|$ and similarly for v and u. This gives the line element

$$\mathrm{d}\mathbf{x} = h_u\mathbf{e}_u\mathrm{d}u + h_v\mathbf{e}_v\mathrm{d}v + h_w\mathbf{e}_w\mathrm{d}w$$

  Which tells us how small changes in coordinates "scale-up" to changes in position $\mathbf{x}$

2.3 The gradient operator

- For $f : \mathbb{R}^3 \to \mathbb{R}$ define the **gradient** of f, written ∇f, by
$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + o(\mathbf{h})$$
- Define the **directional derivative** in direction $\mathbf{v}$ by
$$D_{\mathbf{v}} f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x})}{t}$$
- Comparing the two gives the relation $D_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f$ and hence ∇f points in the direction of greatest increase of f
- For a surface defined implicitly by $S = \{\mathbf{x} \in \mathbb{R}^3 : f(\mathbf{x}) = 0\}$, $\nabla f(\mathbf{x})$ is normal to the surface at $\mathbf{x}$
- It can be shown that $\nabla f \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$ and from this we get
$$\frac{1}{h_{\mathbf{u}}} \frac{\partial f}{\partial u} \mathbf{e}_u + \frac{1}{h_{\mathbf{v}}} \frac{\partial f}{\partial v} \mathbf{e}_v + \frac{1}{h_{\mathbf{w}}} \frac{\partial f}{\partial w} \mathbf{e}_w$$

3 Line Integrals

- For a vector field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ and piecewise smooth curve $C : [a,b] \ni t \mapsto \mathbf{x}(t)$ we have line integral
$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \mathrm{d}t$$
- We say that $\mathbf{F}$ is **conservative** if $\mathbf{F} = \nabla f$ for some scalar function f. Furthermore, we say $\mathbf{F}$ is **exact** if $\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$. We can show that
$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} \text{ is exact} \iff \mathbf{F} \text{ is conservative}$$
- If $\mathbf{F}$ is conservative, for any closed curve C we have
$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \oint_C \nabla f \cdot \mathrm{d}\mathbf{x} = \int_a^b \nabla f(\mathbf{x}) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \mathrm{d}t$$
$$= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f(\mathbf{x}))\mathrm{d}t = f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = 0$$
- Therefore line integrals of conservative functions are independent of path
- With respect to an o.c.c (u, v, w) we call $\theta_i \mathrm{d}u_i$ a **closed** differential form if
$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} \text{ for each } i, j$$
- We have
$$\theta \text{ exact} \implies \theta \text{ closed}$$
and if the domain $\Omega \subset \mathbb{R}^3$ is **simply-connected** (closed loops can be shrunk to any point in the domian) we have
$$\theta \text{ exact} \iff \theta \text{ closed}$$

4 Integration in $\mathbb{R}^2$

- In cartesian coordinates we have $\mathrm{d}A = \mathrm{d}x\mathrm{d}y = \mathrm{d}y\mathrm{d}x$
- Let $x = x(u,v)$ and $y = y(u,v)$ be a smooth, invertible transformation with smooth inverse that maps D' in the (u,v) plane to D in the (x,y) plane. Then
$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{D'} f(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v$$
- $\frac{\partial(x,y)}{\partial(u,v)}$ is the **Jacobian**, often denoted J and defined as
$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$
- To show this, consider how each rectangle in the (u,v) plane is scaled under transformation to the (x,y) plane.

5 Integration in $\mathbb{R}^3$

- Similarly to in $\mathbb{R}^2$ we have
$$\iiint_D f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{D'} f(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

5.1 Integration over surfaces

- A two dimensional surface in $\mathbb{R}^3$ can be defined implicitly using a function $f:\mathbb{R}^3\to\mathbb{R}$ as
$$S = \{\mathbf{x}\in\mathbb{R}^3 : f(\mathbf{x}) = 0\}$$
- Surfaces can also be parameterised to
$$S = \{\mathbf{x} = \mathbf{x}(u,v), (u,v)\in D\}$$
- We call such a parameterisation of S **regular** if
$$\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}\neq 0 \text{ on } S$$
- If S is regular we can define the normal
$$\mathbf{n} = \frac{\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}}{\left|\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}\right|}$$
- Then the area of S is given as
$$\int_S \mathrm{d}S = \iint_D\left|\frac{\partial\mathbf{x}}{\partial u}\times\frac{\partial\mathbf{x}}{\partial v}\right|\mathrm{d}u\mathrm{d}v$$

6 Divergence, Curl & Laplacians

- We can think of ∇ as an operator with $\nabla = \mathbf{e}_i\frac{\partial}{\partial x_i}$. From this we define **divergence** and **curl** by
$$\mathrm{div}(\mathbf{F}) := \nabla\cdot\mathbf{F},\ \mathrm{curl}(\mathbf{F}) := \nabla\times\mathbf{F}$$
- We also define the **laplacian** by
$$\nabla^2 f := \nabla\cdot\nabla f = \mathrm{div}(\mathrm{grad}(f))$$
- From these we have the following identities
$$\begin{aligned}
&\nabla\cdot(fg) = (\nabla f)g + (\nabla g)f\\
&\nabla\cdot(f\mathbf{F}) = (\nabla f)\cdot\mathbf{F} + f(\nabla\cdot\mathbf{F})\\
&\nabla\times(f\mathbf{F}) = (\nabla f)\times\mathbf{F} + f(\nabla\times\mathbf{F})\\
&\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}\\
&\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}\\
&\nabla\cdot(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})
\end{aligned}$$
- We say that $\mathbf{F}$ is **irrotational** if $\nabla\times\mathbf{F} = \mathbf{0}$. If $\mathbf{F}$ is conservative, $\mathbf{F} = \nabla f$ for some f and hence $\nabla\times\mathbf{F} = 0$. Hence
$$\mathbf{F}\text{ conservative} \implies \mathbf{F}\text{ irrotational}$$
The converse is also true if the domain is simply-connected
- Similarly, if there exists a **vector potential** for $\mathbf{F}$, i.e $\mathbf{F} = \nabla\times\mathbf{A}$ then $\nabla\cdot\mathbf{F} = 0$. When $\nabla\cdot\mathbf{F} = 0$ we say $\mathbf{F}$ is **solenoidal**. Hence
$$\text{existence of vector potential for }\mathbf{F} \implies \mathbf{F}\text{ solenoidal}$$
The converse is also true if the domain is 2-connected (every sphere in Ω can be continuously shrunk to any point in the domain)

7 The Integral Theorems

7.1 Green's Theorem

- If $P = P(x,y)$, $Q = Q(x,y)$ are continuously differentiable functions on $A\cup\partial A$ and ∂A is piecewise smooth then
$$\oint_{\partial A} P\mathrm{d}x + Q\mathrm{d}y = \iint_A\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$
- Important: the orientation of ∂A is such that A lies to the left as you traverse it

7.2 Stoke's Theorem

- If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is a continuously differentiable vector field and S is an orientable piecewise regular surface with piecewise smooth boundary ∂S then

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

- The "orientable" condition means there is a consistent choice of normal vector at each point of S (i.e S has 2 sides)

7.3 Divergence Theorem

- If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is a continuously differentiable vector field and V is a volume with piecewise regular boundary ∂V then

$$\int_V \nabla \cdot \mathbf{F} \mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

- Important: the normal $\mathbf{n}$ to ∂V points out of V

8 Maxwell's Equations

- Denote the magnetic field by $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ and the electric field $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$. Similarly have the current and charge densities $\mathbf{J}(\mathbf{x}, t)$ and $\rho = \rho(\mathbf{x}, t)$ respectively
- Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{3}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \tag{4}$$

- ε_0 and μ_0 are the permittivity and permeability of free space, which obey

$$\frac{1}{\mu_0 \varepsilon_0} = c^2$$

- These equations have an equivalent integral form:
- In the case where all terms are independent of t, the equations decouple to $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$
  - If we are working on $\mathbb{R}^3$ (i.e the domain is 2-connected) these imply $\mathbf{E} = -\nabla \mathbf{\Phi}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ for vector fields $\mathbf{\Phi}, \mathbf{A}$
  - Then we get the equations $-\nabla^2 \mathbf{\Phi} = \rho/\varepsilon_0$ and $\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$

9 Poisson's & Laplace's Equations

- **Poisson's equation is**

$$\nabla^2 \varphi = F$$

  If $F \equiv 0$ it is called **Laplace's equation**
- The **Dirichlet Problem** is

$$\begin{cases} \nabla^2 \varphi = f & \text{in } \Omega \\ \varphi = f & \text{on } \partial\Omega \end{cases}$$

- The **Neumann Problem** is

$$\begin{cases} \nabla^2 \varphi = f & \text{in } \Omega \\ \frac{\partial \varphi}{\partial \mathbf{n}} = g & \text{on } \partial\Omega \end{cases}$$

  where $\frac{\partial \varphi}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla \varphi$ is the normal derivative
- By considering the corresponding homogenous problems, it may be shown that the solution to the Dirichlet problem is unique. The solution to the Neumann problem is unique up to the addition of a constant

9.1 Harmonic Functions

- **Harmonic functions** are solutions to the Laplace equation $\nabla^2\varphi = 0$
- Harmonic functions φ have the **mean value property**, that is for a sphere S_r defined as $S_r = \{\mathbf{x} : |\mathbf{x}-\mathbf{a}| < r\}$

$$\varphi(\mathbf{a}) = \frac{1}{4\pi r^2}\int_{S_r} \varphi(\mathbf{x})\mathrm{d}S$$

  For $\mathbf{a} \in \Omega \subset \mathbb{R}^3$ and r sufficiently small

  - To show this, consider the function $F(\mathbf{r})$ defined as the RHS of the identity, and show that $F'(\mathbf{r}) = 0$ and $F(\mathbf{r}) \to \varphi(\mathbf{a})$ as $r \to 0$

- If φ is harmonic on $\Omega \subset \mathbb{R}^3$ then it has the **maxiumum value property**, i.e it cannot have a maximum in any interior point of Ω unless φ is constant

  - To show this, note

$$0 = \frac{1}{4\pi\varepsilon^2}\int_{S_\varepsilon} (\varphi(\mathbf{a}) - \varphi(\mathbf{x}))\,\mathrm{d}S$$

    by the mean value property, so if $\varphi(\mathbf{a}) \geq \varphi(\mathbf{x})$ for all $\mathbf{x}$ we would have a contradiction, unless φ is constant

10 Cartesian Tensors

- A rank n tensor $T\underbrace{pq\ldots r}_{n \text{ indices}}$ is a mathematical object which transforms

$$T'_{pq\ldots r} = R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r}$$

  where $R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$ are the components of a rotation matrix and the $\mathbf{e}'_i$ are the basis vectors of the new basis

- For example, δ_{ij} is a rank 2 tensor and ε_{ijk} is a rank 3 tensor
- For a rank n tensor $A_{ij\ldots k}$ and a rank k tensor $B_{ab\ldots c}$ can define the **tensor product**

$$(A \otimes B)_{ij\ldots kab\ldots c}$$

  this is a new tensor of rank $n+k$

- We can **contract** a rank n tensor $T_{ij\ldots k}$ to a rank $n-2$ tensor over some indices (i,j) e.g

$$T_{ii\ldots k}$$

- We say a tensor is **totally symmetric/anti-symmetric** if it is symmetric/anti-symmetric in all pairs of indices
  - Both δ_{ij} and $a_ia_ja_k$ are totally symmetric
  - ε_{ijk} is totally anti-symmetric and is the only such tensor on $\mathbb{R}^3$

10.1 Tensor Calculus

- Note that for a vector $\mathbf{x}$ we have

$$x'_i = R_{ij}x_j \iff x_j = R_{ij}x'_i$$

  - Differentiating the RHS wrt x'_k

$$\frac{\partial x_j}{\partial x'_k} = R_{ij}\frac{\partial x'_i}{\partial x'_k} = R_{ij}\delta_{ik} = R_{kj}$$

  - So by the chain rule we see that

$$\frac{\partial}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{\partial}{\partial x_j} = R_{ij}\frac{\partial}{\partial x_j}$$

    Hence "$\frac{\partial}{\partial x_i}$ transforms like a rank 1 tensor"

- If $T_{i\dots j}(\mathbf{x})$ is a tensor field of rank n then

$$\underbrace{\frac{\partial}{\partial x_p}\cdots\frac{\partial}{\partial x_q}}_{m \text{ terms}} T_{i\dots k}(\mathbf{x})$$

is a tensor field of rank $n+m$

- For a tensor field $T_{ij\dots k\dots l}(\mathbf{x})$

$$\int_V \frac{\partial}{\partial x_k} T_{ij\dots k\dots l}\mathrm{d}V = \int_{\partial V} T_{ij\dots k\dots l} n_k \mathrm{d}S$$

  - Can prove this by considering $v_k = a_i b_j \dots c_l T_{ij\dots k\dots l}$ where $a_i, b_j, \dots c_l$ are the components of constant vector fields. Then just apply the divergence theorem

10.2 Rank 2 Tensors

- Every rank 2 tensor can be written uniquely as

$$T_{ij} = S_{ij} + \varepsilon_{ijk}\omega_k$$

where $\omega_i = \frac{1}{2}\varepsilon_{ijk}T_{jk}$ and S_{ij} is symmetric

- If T_{ij} is symmetric then there exists a choice of coordinate axes s.t

$$(T_{ij}) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

  - This is just a corollary of the fact every real symmetric matrix can be diagonalised by an orthogonal transformation (see IA V&M)

10.3 Isotropic Tensors

- We say that a tensor is **isotropic** if it is invariant under changes in cartesian coordinates, i.e

$$T'_{ij\dots k} = R_{ip}R_{jq}\dots R_{kr}T_{pq\dots r} = T_{ik\dots k}$$

for <u>any</u> choice of rotation R

  - Every scalar is isotropic
  - δ_{ij} and ε_{ijk} are isotropic

- In general isotropic tensors on $\mathbb{R}^3$ are classified as

  1. All rank 0 tensors
  2. There are no rank 1 tensors
  3. General rank 2 isotropic tensor is $\alpha\delta_{ij}$
  4. General rank 3 isotropic tensor is $\beta\varepsilon_{ijk}$
  5. General rank 4 isotropic tensor is

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

  6. For higher rank tensors, general isotropic tensor is a linear combination of δ and ε

10.4 The Quotient Theorem

- The **quotient theorem** says that if $T_{i\dots jp\dots q}$ is an array of numbers defined in each cartesian coordinate system such that

$$v_{i\dots j} := T_{i\dots jp\dots q}u_{p\dots q}$$

is a tensor for <u>every</u> tensor $u_{p\dots q}$, then $T_{i\dots jp\dots q}$ is a tensor

  - Proven by considering the special case where $u_{p\dots q} = c_p \dots d_q$ for vectors $\{\mathbf{c},\dots,\mathbf{d}\}$. Then just multiply both sides by $a_i \dots b_j$ for some choice of vectors $\{\mathbf{a},\dots,\mathbf{b}\}$ and notice that we get a scalar
  - Then we have a multilinear map

$$t(\mathbf{a},\dots,\mathbf{b},\mathbf{c},\dots,\mathbf{d}) := T_{i\dots jp\dots q}a_i\dots b_j c_p \dots d_q$$

  - Then it may be shown that multilinear maps give rise to tensors (similarly to how linear transformations give rise to matrices) and we are done