Introduction

The course is split into two parts:

- Logic: syntax and semantics.
- Set theory: what does the universe of sets look like?

Course structure

(I) Propositional logic (logic)

(II) Well-orderings & ordinals (set theory)

(III) Posets & Zorn's lemma (set theory)

(IV) Predicate logic (logic)

(V) Set theory (set theory)

(VI) Cardinals (set theory)

Books:

1. Johnstone, *Notes on Logic & Set Theory*
2. Van Dalen, *Logic & Structure* (Chapter 4 and what 'goes next')
3. Hajnal & Hamburger, *Set Theory* (Chapters 2 and 6)
4. Forster, *Logic, Induction & Sets*

1 Propositional Logic

Let P be a set of *primitive propositions*. Unless otherwise stated, $P = \{p_1, p_2, \ldots\}$. The *language* L or $L(P)$ is defined inductively by

1. If $p \in P$, then $p \in L$
2. $\perp \in L$ ($\perp$ is read 'false')
3. If $p, q \in L$ then $(p \Rightarrow q) \in L$.

e.g $((p_1 \Rightarrow p_2) \Rightarrow (p_1 \Rightarrow p_3)), (p_4 \Rightarrow \perp), (\perp \Rightarrow \perp)$.

Notes.

1. Each proposition (member of L) is a finite string of symbols from language: $\vdash, \Rightarrow, \perp, p_1, p_2, \ldots$ (for clarity often omit outer brackets, use other types of bracket, etc).
2. 'L is defined inductively' means, more precisely, the following

- Put $L_1 = P \cup (\perp)$;
- Having defined L_n, put $L_{n+1} = L_n \cup \{(p \Rightarrow q) : p, q \in L_n\}$;
- Set $L = \bigcup_{n \geq 1} L_n$.

3. Every $p \in L$ is uniquely built up from steps 1,2 using 3. For example, $((p_1 \Rightarrow p_2) \Rightarrow (p_1 \Rightarrow p_3))$ can from $(p_1 \Rightarrow p_2)$ and $(p_1 \Rightarrow p_3)$.

We can now introduce $\neg p$ ('not p') as an abreviation for $(p \Rightarrow \perp)$; $p \vee q$ ('p or q') as an abreviation for $(\neg p) \Rightarrow q$; $p \wedge q$ ('p and q') as an abreviation for $\neg(p \Rightarrow (\neg q))$.

1.1 Semantic Implication

Definition. A *valuation* is a function $v : L \to \{0,1\}$ (thinking of 0 as 'False' and 1 as 'True') such that

(i) $v(\perp) = 0$

(ii) $v(p \Rightarrow q) = \begin{cases} 0 & \text{if } v(p) = 1,\ v(q) = 0 \\ 1 & \text{otherwise} \end{cases}$.

Remark. On $\{0,1\}$, could define a constant $\perp = 0$ and an operation $\Rightarrow$ by

$$(a \Rightarrow b) = \begin{cases} 0 & \text{if } a = 1, b = 0 \\ 1 & \text{otherwise} \end{cases}.$$

Then a valuation is precisely a mapping $L \to \{0,1\}$ that preserves ($\perp$ and $\Rightarrow$).

Proposition 1.1.

(i) If v, v' are valuations with $v(p) = v(p')$ for all $p \in P$, then $v = v'$.

(ii) For any function $w : P \to \{0,1\}$, there exists a valuation v with $v(p) = w(p)$ for all $p \in P$.

Proof.

(i) Have $v(p) = v'(p)$ for all $p \in L_1$. But if $v(p) = v'(p)$ and $v(q) = v'(q)$, then $v(p \Rightarrow q) = v'(p \Rightarrow q)$, so $v(p) = v'(p)$ for all $p \in L_2$. Continuing inductively we obtain $v(p) = v'(p)$ for all $p \in L_n$ for each n.

(ii) Set $v(p) = w(p)$ for all $p \in P$ and $v(\perp) = 0$ to obtain v on L_1. Now put

$$v(p \Rightarrow q) = \begin{cases} 0 & v(p) = 1, v(q) = 0 \\ 1 & \text{otherwise} \end{cases}$$

to obtain v on L_2, then induction.

$\square$

Example. Let v be the valuation with $v(p_1) = v(p_3) = 1$, $v(p_n) = 0$ for all $n \neq 1, 3$. Then $v((p_1 \Rightarrow p_2) \Rightarrow p_3) = 0$.

Definition. A *tautology* is an element $t \in L$ such that $v(t) = 1$ for any valuation v. We write $\models t$.

Examples.

1. $p \Rightarrow (q \Rightarrow p)$

$v(p)$	$v(q)$	$v(p \Rightarrow q)$	$v(p \Rightarrow (q \Rightarrow p))$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

So this is a tautology.

2. $(\neg\neg p) \Rightarrow p$, i.e $((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p$ ('law of excluded middle')

$v(p)$	$v(p \Rightarrow \perp)$	$v((p \Rightarrow \perp) \Rightarrow \perp)$	$v(((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p)$
0	1	0	1
1	0	1	1

3. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ ("how implicatino chains"). Suppose this is not a tautology. Then we have a v with $v(p \Rightarrow (q \Rightarrow r)) = 1$ and $v((p \Rightarrow q) \Rightarrow (q \Rightarrow r)) = 0$. Then $v(p \Rightarrow q) = 1$ and $v(p \Rightarrow r) = 0$. Hence $v(p) = 1$ and $v(r) = 0$, so $v(q) = 1$. Hence $v(p \Rightarrow (q \Rightarrow r)) = 0$, contradiction.

Definition. For $S \subseteq L$, $t \in L$, we say S *entails* or *semantically implies* t, written $S \vDash t$ if every valuation with $v(s) = 1$ for all $s \in S$ has $v(t) = 1$.

Example. $\{p \Rightarrow q, q \Rightarrow r\}$ entails $p \Rightarrow r$. Indeed, suppose we have v with $v(p \Rightarrow q)$, $v(q \Rightarrow r) = 1$ but $v(p \Rightarrow r)$. Then $v(p) = 1$, $v(r) = 0$. Hence $v(q) = 1$, contradicting $v(q \Rightarrow r) = 1$.

Definition. We say v is a *model* of $S \subseteq L$ or S is *true* in v, if $v(s) = 1$ for all $s \in S$. Thus S entails t means: every model of S is also a model of $\{t\}$.

Remark. $\vDash t$ says $\emptyset \vDash t$.

1.2 Syntatic implication

For a notion of proof, we'll need axioms and deduction rules. As axioms, we'll take:

1. $p \Rightarrow (q \Rightarrow p)$ for all $p, q \in L$;
2. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ for all $p, q \in L$;
3. $(\neg\neg p) \Rightarrow p$ for all $p \in L$.

Notes.

1. Sometimes we call these 'axiom schemes' since each is actually a set of axioms.
2. Each of these are tautologies.

For deduction rules, we'll have only *modus ponens*: from each p and $p \Rightarrow q$ we can deduce q.

Definition. For $S \subseteq L$, and $t \in S$, say S *proves* or *syntactically implies* t, written $S \vdash t$ if there exists a sequence $t_1, \ldots, t_n$ in L with $t_n = t$ such that every t_i is either

(i) An axiom; or

(ii) A member of S; or

(iii) Such that there exist $j, k < i$ with $t_k \Rightarrow (t_j \Rightarrow t_n)$ (modus ponens).

Say S consists of the *hypotheses* or *premises*, and t the *conclusion*.

Example. $\{p \Rightarrow q, q \Rightarrow r\} \vdash p \Rightarrow r$:

1. $q \Rightarrow r$ (hypothesis)
2. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ (axiom 1)
3. $p \Rightarrow (q \Rightarrow r)$ (modus ponens' on 2,3)
4. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ (axiom 2)
5. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ (modus ponens' on 3,4)
6. $p \Rightarrow q$ (hypothesis)
7. $p \Rightarrow r$ (modus ponens on 5,6)

Definition. If $\emptyset \vdash t$, say t is a *theorem*, written $\vdash t$.