

Introduction

Course structure:

- (i) Preliminary toolbox: inequalities
- (ii) Normed vector spaces (NVS)
- (iii) (Recalls on) finite-dimensional case
- (iv) Hahn-Banach Theorems (how big is the dual?)
- (v) Completeness: Baire's Theorem & consequences for NVS
- (vi) Detailed study of the topology of $C(K)$
- (vii) The Hilbert space
- (viii) Projection & duality
- (ix) Introduction to operators and spectral theory

1 Preliminary toolbox: Young's, Hölder's & Minkowski's inequalities for vectors & sequences

Proposition (Young's inequality for products). Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\forall a, b \geq 0, \quad ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The result is clear for $a = 0$ or $b = 0$. Assume $a, b > 0$ and note $L : (0, \infty) \rightarrow \mathbb{R}, t \mapsto \ln t$ is strictly concave: $L''(t) = -\frac{1}{t^2} < 0$.

Therefore for all $A, B > 0, \lambda \in (0, 1)$

$$\ln(\lambda A + (1 - \lambda)B) \geq \lambda \ln A + (1 - \lambda) \ln B$$

with equality iff $A = B$. Apply this to $A = a^p, B = b^q > 0$ and $\lambda = \frac{1}{p}$. This gives

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) = \ln(ab)$$

so applying exp to both sides gives the result and furthermore we have equality iff $a^p = b^q$. \square

Proposition (Hölder's inequality for vectors & sequences). Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

(i) for any $n \in \mathbb{N} \setminus \{0\} = \mathbb{N}^*$, $\forall x, y \in \mathbb{R}^n$

$$\sum_{k=1}^n |x_k y_k| \leq \|x\|_p \|y\|_q \quad (*)$$

with $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ and similarly for $\|y\|_q$.

(ii) define

$$\ell^p = \{x \in \mathbb{R}^{\mathbb{N}^*} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

then $\forall x \in \ell^p, y \in \ell^q$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|x\|_{\ell^p} \|y\|_{\ell^q}$$

where $\|x\|_{\ell^p} = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ and similar for $\|y\|_{\ell^q}$.

Proof. To show (i) implies (ii): take $n \rightarrow \infty$ in (i) so

$$\sum_{k=1}^n |x_k|^p \rightarrow \|x\|_{\ell^p}^p$$

and similarly

$$\sum_{k=1}^n |y_k|^q \rightarrow \|y\|_{\ell^q}^q$$

By (i)

$$\sum_{k=1}^n |x_k y_k| \leq \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \left(\sum_{k=1}^n |y_k|^q \right)^{1/q}$$

so

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k y_k| &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k y_k| \right) \leq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n |x_k|^p \right)^{1/p} \left(\sum_{k=1}^n |y_k|^q \right)^{1/q} \\ &= \|x\|_{\ell^p} \|y\|_{\ell^q} \end{aligned}$$

Proof of (i): if $\|x\|_{\ell^p}$ or $\|y\|_{\ell^q} = 0$, result is clear. Otherwise define \tilde{x}, \tilde{y} sequences in ℓ^p and ℓ^q by

$$\tilde{x}_k = \frac{x_k}{\|x\|_{\ell^p}}, \quad \tilde{y}_k = \frac{y_k}{\|y\|_{\ell^q}}$$

Then $\|\tilde{x}\|_{\ell^p} = 1, \|\tilde{y}\|_{\ell^q} = 1$. Then (*) is equivalent to showing

$$\sum_{k=1}^n |\tilde{x}_k \tilde{y}_k| \leq 1 \quad (**)$$

Apply Young's inequality on each $k = 1, \dots, n$ so

$$|\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p} |\tilde{x}_k|^p + \frac{1}{q} |\tilde{y}_k|^q$$

Summing over k :

$$\sum_{k=1}^n |\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p} \left(\sum_{k=1}^n |\tilde{x}_k|^p \right) + \frac{1}{q} \left(\sum_{k=1}^n |\tilde{y}_k|^q \right) \leq \frac{1}{p} + \frac{1}{q} = 1$$

□

Remark: Equality in (*) is equivalent to equality in (**) which is equivalent to equality in Young's for each k so $|\tilde{x}_k|^p = |\tilde{y}_k|^q$ for $k = 1, \dots, n$. Also, the $p = 1$, $q = \infty$ case is easy.

Proposition (Minkowski's inequality for vectors & sequences). Let $p \in [1, \infty)$, then

(i) for all $x, y \in \mathbb{R}^n$

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

(ii) for all $x, y \in \ell^p$

$$\|x + y\|_{\ell^p} = \|x\|_{\ell^p} + \|y\|_{\ell^p}$$

Proof. To show (i) implies (ii): by taking $n \rightarrow \infty$ as before

$$\begin{aligned} \sum_{k=1}^{\infty} |x_k|^p &\rightarrow \|x\|_{\ell^p}^p \\ \sum_{k=1}^{\infty} |y_k|^p &\rightarrow \|y\|_{\ell^p}^p \\ \sum_{k=1}^n |x_k + y_k|^p &\rightarrow \|x + y\|_{\ell^p}^p \end{aligned}$$

Proof of (i): if $p = 1$ this is just the usual triangle inequality on each coordinate. So let $p \in (1, \infty)$ and

$$\begin{aligned} \sum_{k=1}^n |x_k + y_k|^p &= \sum_{k=1}^n |x_k + y_k| \cdot |x_k + y_k|^{p-1} \\ &\leq \sum_{k=1}^n |x_k| |x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k| |x_k + y_k|^{p-1} \\ &\leq \|x\|_p \left(\sum_{k=1}^n |x_k + y_k|^{(p-1) \frac{p}{p-1}} \right)^{\frac{p-1}{p}} + \|y\|_p \left(\sum_{k=1}^n |x_k + y_k|^{(p-1) \frac{p}{p-1}} \right)^{\frac{p-1}{p}} \end{aligned}$$

Hölder: $q = \frac{p}{p-1}$

$$\leq (||x||_p + ||y||_p) ||x + y||_p^{p-1}$$

so we have proved

$$||x + y||_p^p \leq (||x||_p + ||y||_p) ||x + y||_p^{p-1}$$

If $||x + y||_p = 0$, result is clear. Otherwise divide by $||x + y||_p^{p-1}$ to get

$$||x + y||_p \leq ||x||_p + ||y||_p$$

□

Remark: equality occurs iff there is equality in the triangle inequality and Hölder's.

Remarks:

1. Equality case: $p = 1$: $|x_k + y_k| \leq |x_k| + |y_k|$, i.e the usual triangle inequality
2. For $p = 2$ there's another proof: define $\mathcal{P} : \mathbb{R} \rightarrow \mathbb{R}$, $\lambda \mapsto ||x + \lambda y||^2$. Then $\mathcal{P}(\lambda) = a\lambda^2 + 2b\lambda + c$ and $\mathcal{P} \geq 0$. So

$$\langle x, y \rangle = b^2 \leq ac = ||x||^2 ||y||^2, \text{ Hölder's inequality}$$

2 Normed Vector Spaces (NVS)

Remark: this is not the most general structure for linear analysis - topological vector spaces (TVS).

Recall:

Definition. A *vector space* V over a field \mathbb{F} is a set (of elements called *vectors*) with two operations:

$$A : V \times V \rightarrow V, (v, w) \mapsto v + w \text{ addition}$$

$$M : \mathbb{F} \times V \rightarrow V, (\lambda, v) \mapsto \lambda v \text{ scalar multiplication}$$

such that

- $(V, +)$ is an abelian group with identity 0.
- M is compatible with $(\mathbb{F}, 0)$ in the sense that $\lambda_1(\lambda_2 v) = \lambda_1 \lambda_2 v$
- M distributes over $(V, +)$ and $(\mathbb{F}, +)$.

In this course \mathbb{F} will be \mathbb{R} or \mathbb{C} unless stated otherwise.

Definition. Given a vector space V over \mathbb{F} :

- a *subspace* $W \subseteq V$ is a vector space over \mathbb{F} included in V
- for a set $S \subseteq V$, a *linear combination of elements of S* is a finite sum of elements of S with coefficients in \mathbb{F}
- for a set $S \subseteq V$, the *span of S* , $\text{span}(S)$ is the smallest subspace of V containing S , and is also the set of linear combinations of S .

Definition. Given V a vector space over \mathbb{F} and a set $S \subseteq V$:

- S is *linearly independent* if for all $m \in \mathbb{N}^*$ and for all $\alpha_1, \dots, \alpha_m \in \mathbb{F}$, for all $s_1, \dots, s_m \in S$, $\sum_{i=1}^m \alpha_i s_i = 0$ if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$.
- S is a *basis* of V if it is linearly independent and $\text{span}(S) = V$.
- If there exists a finite basis S of V , then V has finite dimension, otherwise it is infinite-dimensional.

Remark: later we'll prove with Zorn's lemma that any vector space has a basis.

Definition. A *normed vector space* (NVS) V over \mathbb{F} is a vector space over \mathbb{F} together with a function $N : V \rightarrow \mathbb{R}_+$, $v \mapsto \|v\|$ (the *norm*), with

1. $\|v\| \geq 0$ for all $v \in V$, with equality only at $v = 0$ (*positive definiteness*)
2. For all $\lambda \in \mathbb{F}$, $v \in V$ $\|\lambda v\| = |\lambda| \|v\|$ (compatibility between N and M)

3. For all $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$ (compatibility between N and A)

Example. $V = \mathbb{R}^n$, $v = (v_1, \dots, v_n)$, $\|v\| = (v_1^2 + \dots + v_n^2)^{1/2}$ or

$$\begin{cases} \|v\|_p = (|v_1|^p + \dots + |v_n|^p)^{1/p} & \text{for } p \in [1, \infty) \\ \|v\|_\infty = \sup_{i=1}^n |v_i| & \text{for } p = \infty \end{cases}$$

Definition. Given a set X , a *topology* τ on X is a collection of subsets of X (“open sets”) such that

- $\emptyset \in \tau$, $X \in \tau$
- τ is stable under any union
- τ is stable under finite intersections

Definition.

- For (X, d) a metric space, the *induced topology* is the smallest topology that contains open balls in d
- For a NVS $(V, \|\cdot\|)$, the induced topology is that associated with $d(v, w) = \|v - w\|$

Natural question: \mathbb{F} field, V vector space over \mathbb{F} . Norm on V , $\tau_{\|\cdot\|}$. Continuity of operations M and A ?

Proposition. Let $(V, \|\cdot\|)$ be a NVS over \mathbb{F} (\mathbb{F} either \mathbb{R} or \mathbb{C}), then

- (i) A, M are continuous for the following topologies: $\tau_{\|\cdot\|}$ on V , then product topology of it on $V \times V$, $\tau_{|\cdot|}$ over \mathbb{F} , then product topology of $\tau_{|\cdot|}$ and $\tau_{\|\cdot\|}$ on $\mathbb{F} \times V$
- (ii) Translations $T_{v_0} : V \rightarrow V$, $v \mapsto v + v_0$, $v_0 \in V$ and dilations $D_{\lambda_0} : V \rightarrow V$, $v \mapsto \lambda_0 v$, $\lambda_0 \in \mathbb{F}^*$ are homeomorphisms

Proof.

- (i) Let us prove that $A : V \times V \rightarrow V$ is continuous: consider an open set $\emptyset \neq U \subseteq V$ and $(v_1, v_2) \in A^{-1}(U)$, i.e $v_1 + v_2 \in U$. Since U is open, there is $\varepsilon > 0$ such that $\underbrace{B_V(v_1 + v_2, \varepsilon)}_{\text{open ball}} \subseteq U$.

We have that $A(B(v_1, \varepsilon/2), B(v_2, \varepsilon/2)) \subseteq B_V(v_1 + v_2, \varepsilon)$ (triangle inequality). Note also that $B(v_1, \varepsilon/2) \times B(v_2, \varepsilon/2)$ is open (product topology), so $A^{-1}(U)$ is open and A is continuous.

Now we show $M : \mathbb{F} \times V \rightarrow V$ is continuous. Consider an open set $U \neq \emptyset$ in V , $(\lambda, v) \in M^{-1}(U)$. Since U is open, there exists $\varepsilon > 0$ such that $B_V(\lambda v, \varepsilon) \subseteq U$ (WLOG $\varepsilon < 1$). Then (check)

$$M\left(B_{\mathbb{F}}\left(\lambda, \frac{\varepsilon}{3 \max(1, \|v\|)}\right), B_V\left(v, \frac{\varepsilon}{3 \max(1, |\lambda|)}\right)\right) \subseteq B_V(\lambda v, \varepsilon)$$

- (ii) T_{v_0} and D_{λ_0} are linear, continuous with inverses T_{-v_0} and $D_{\lambda_0^{-1}}$ respectively, so are homeomorphisms.

□

3 Characterisation of NVS

Idea: in order to better understand the topology of NVS's, we ask how special is a "normable" topology among topologies compatible with vector space operations?

Definition (TVS). A *topological vector space* (TVS) over \mathbb{F} is a vector space over \mathbb{F} together with a topology τ such that

- (i) A and M are continuous
- (ii) every singleton $\{x_0\}$ is closed

Remark:

- 1. (i) says that T_{v_0} and D_{λ_0} , $\lambda_0 \neq 0$ are homeomorphisms
- 2. (ii) is called T_1 in the classification of separation properties, and implies Hausdorff for TVS

Definition. Given V a TVS

- $C \subseteq V$ is *convex* if $C = \{\lambda c_1 + (1 - \lambda)c_2 : c_1, c_2 \in C, \lambda \in [0, 1]\}$
- V is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0
- $B \subseteq V$ is *bounded* if for any U open around 0, there exists $t_0 > 0$ such that $\forall t > t_0, B \subseteq tU$
- V is *locally bounded* if there is $U \in \tau$ containing 0 and bounded

Example. Let $(V, \|\cdot\|)$ be a NVS, then for all $r > 0$, $U = B(0, r)$ (open ball) is open, bounded and convex. Indeed

- Convexity follows from the triangle inequality
- Boundedness: any other \tilde{U} open around 0 contains some open $\tilde{U}_0 = B(0, r_0) \in \tilde{U}$. Then for any $t > \frac{r}{r_0}$, $U \subseteq t\tilde{U}_0 \subseteq t\tilde{U}$.

Question: can we reverse-engineer the norm if we have these two properties?

Theorem (Kolmogorov 1934). *Let (V, τ) be a TVS such that there is a bounded convex neighborhood of 0, say C . Then V is "normable" - there is a norm $\|\cdot\|$ on V that induces the topology τ .*

Proof. Step 1: there is $\tilde{C} \subseteq C$ which is a *balanced* convex bounded neighborhood of 0. "Balanced" means that for all $\lambda \in \mathbb{F}$ such that $|\lambda| \leq 1$, $\lambda\tilde{C} \subseteq \tilde{C}$.

$M : \mathbb{F} \times V \rightarrow V$ is continuous so $M^{-1}(C)$ is a neighbourhood of $(0, 0)$. So there exists $B_{\mathbb{F}}(0, \varepsilon) \times U$ with $\varepsilon > 0$ and U open around 0 such that $M(B_{\mathbb{F}}(0, \varepsilon), U) \subseteq C$.

Define \tilde{C} to be the convex hull (i.e smallest convex set superset) of $M(B_{\mathbb{F}}(0, \varepsilon), U)$.

Then \tilde{C} is clearly convex, is a subset of C since C is convex and $M(B_{\mathbb{F}}(0, \varepsilon), U) \subseteq C$. \tilde{C} is also bounded since $\tilde{C} \subseteq C$ and C is bounded (obvious that boundedness is inherited by inclusion). Finally \tilde{C} is balanced since $\lambda B_{\mathbb{F}}(0, \varepsilon) \subseteq B_{\mathbb{F}}(0, \varepsilon)$ for $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$ and

$$\underbrace{\lambda M(B_{\mathbb{F}}(0, \varepsilon), U)}_{=M(\lambda B_{\mathbb{F}}(0, \varepsilon), U)} \subseteq M(B_{\mathbb{F}}(0, \varepsilon), U)$$

Notice $\lambda[\text{Convex Hull}(S)] = \text{Convex Hull}(\lambda S)$ (exercise). So deduce $\lambda\tilde{C} \subseteq \tilde{C}$.

Step 2: define the *Minkowski guage* (functional) of \tilde{C}

$$\mu_{\tilde{C}} : V \rightarrow \mathbb{R}_+, v \mapsto \inf\{t \geq 0 : v \in t\tilde{C}\}$$

$\mu_{\tilde{C}}$ is well-defined in $[0, \infty)$ since: any v satisfies $\frac{v}{t} \rightarrow 0$ as $t \rightarrow \infty$ by continuity of M . So $\frac{v}{t}$ must “enter” the neighborhood \tilde{C} of 0 for t large enough.

Step 3: let us prove $v \mapsto \mu_{\tilde{C}}(v)$ is a norm:

- $\mu_{\tilde{C}}(v) \geq 0$ by construction
- if $\mu_{\tilde{C}} = 0$, then (assume $v \neq 0$ for contradiction) there exists U open around 0 with $v \notin U$ (since $V \setminus \{v\}$ is open). Since \tilde{C} is bounded, there exists $t_1 > 0$ such that $\tilde{C} \subseteq t_1 U$. Since $\mu_{\tilde{C}}(v) = 0$, there exists $t_2 \in (0, t_1^{-1})$ such that $v \in t_2 \tilde{C}$, then $v \in t_2 \tilde{C} \subseteq t_1^{-1} \tilde{C} \subseteq U$, a contradiction.
- Want to show $\mu_{\tilde{C}}(\lambda v) = |\lambda| \mu_{\tilde{C}}(v)$ for $\lambda \in \mathbb{F}^\times$, $v \in V$. Use \tilde{C} balanced: for all $t > 0$ such that $\lambda v \in t\tilde{C}$, we have

$$\frac{\lambda}{|\lambda|} v \in \frac{t}{|\lambda|} \tilde{C} \implies v \in \frac{t}{|\lambda|} \tilde{C} \implies \mu_{\tilde{C}}(v) \leq \frac{1}{|\lambda|} \mu_{\tilde{C}}(\lambda v)$$

The inequality in the other direction follows by reasoning with λ^{-1} . So $|\lambda| \mu_{\tilde{C}}(v) = \mu_{\tilde{C}}(\lambda v)$.

- Want to show $\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$ for all $v_1, v_2 \in V$. Indeed, given $t_1, t_2 > 0$ such that $v_1 \in t_1 \tilde{C}$, $v_2 \in t_2 \tilde{C}$, we have

$$v_1 + v_2 \in t_1 \tilde{C} + t_2 \tilde{C} = (t_1 + t_2) \left[\frac{t_1}{t_1 + t_2} \tilde{C} + \frac{t_2}{t_1 + t_2} \tilde{C} \right] \subseteq (t_1 + t_2) \tilde{C} \text{ (convexity)}$$

so $\mu_{\tilde{C}}(v_1 + v_2) \leq t_1 + t_2$. By taking infima over t_1, t_2 :

$$\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$$

Step 4: prove $\mu_{\tilde{C}}$ induces the topology τ .

- Want to prove

$$\underbrace{B(v_0, \varepsilon)}_{\text{open ball for } \mu_{\tilde{C}}} = \{v \in V : \mu_{\tilde{C}}(v - v_0) < \varepsilon\} \in \tau$$

Take $v \in B(v_0, \varepsilon)$ then by the triangle inequality

$$B(v, \varepsilon - |v|) \subseteq B(v_0, \varepsilon)$$

and $B(v, \varepsilon') \supseteq v + \frac{\varepsilon'}{2}\tilde{C}$ by definition of the ball for $\mu_{\tilde{C}}$. And (since translations, dilations continuous) $v + \frac{\varepsilon'}{2}\tilde{C}$ is a neighborhood of v .

$B(v_0, \varepsilon)$ open (in τ) around its points, so is in τ .

- Take $U \in \tau$, and (wlog) $0 \in U$. Let us prove $0 \in B(0, \varepsilon_0) \subseteq U$ for some $\varepsilon_0 > 0$. Indeed \tilde{C} is bounded so there exists $\varepsilon_0 > 0$ such that $\tilde{C} \subseteq \varepsilon_0^{-1}U$ hence $U \supseteq \varepsilon_0\tilde{C}$ and so $U \supseteq \varepsilon\tilde{C} \forall \varepsilon < \varepsilon_0$ and thus $U \supseteq B(0, \varepsilon_0)$.

□

Remarks:

1. $B(0, \varepsilon_0) \subseteq \bigcup_{0 \leq \varepsilon < \varepsilon_0} \varepsilon \tilde{C}$
2. T_1 implies Hausdorff (T_2). Consider $v_0 \neq v_1$ in V : so $0 \neq v_1 - v_0$, T_1 implies there is U open around 0 with $v_1 - v_0 \notin U$. Then (since A, M continuous) $(v, w) \mapsto v - w$ is continuous and there exists \tilde{U} open around 0 such that $\tilde{U} - \tilde{U} \subseteq U$. Then $v_0 + \tilde{U}$ and $v_1 + \tilde{U}$ are open disjoint neighborhoods of v_0 and v_1 respectively (disjoint since otherwise $v_1 - v_0 \in \tilde{U} - \tilde{U} \subseteq U$).

4 Some examples of NVS'

Definition. Let $(V, \|\cdot\|)$ be an NVS (over $\mathbb{F} = \mathbb{R}$ or \mathbb{C}). If (V, d) , d distance induced by $\|\cdot\|$ is a complete metric space, then $(V, \|\cdot\|)$ is called a *Banach space*.

Example. $\mathbb{R}^n, \mathbb{C}^n, n \geq 1$ are Banach spaces, for $\|\cdot\|_p, p \in [1, \infty)$.

Example. Given (X, τ) a general topological space, define

$$B_{\mathbb{F}}(X) = \{\text{functions } : X \rightarrow \mathbb{F} \text{ bounded}\}$$

$$C_{\mathbb{F}}(X) = \{\text{functions } : X \rightarrow \mathbb{F} \text{ continuous}\}$$

$$C_{\mathbb{F},b}(X) = C_{\mathbb{F}}(X) \cap B_{\mathbb{F}}(X)$$

If $X = K$ is compact, $C_{\mathbb{F}}(X) = C_{\mathbb{F},b}(X)$. These are vector spaces over \mathbb{F} with addition $(f + g)(x) = f(x) + g(x)$ and multiplication $(fg)(x) = f(x)g(x)$.

Norm on $C_{\mathbb{F},b}(X)$: the supremum norm, $\|f\|_{\infty} = \sup_{x \in X} |f(x)|$

Proposition. $(C_{\mathbb{F},b}, \|\cdot\|_{\infty})$ is a Banach space over \mathbb{F} .

Proof.

- $\|f\|_{\infty}$ is well defined in \mathbb{R}^+ since f is bounded.
- $\|f\|_{\infty} = 0$ means $f(x) = 0$ for all $x \in X$ and so $f = 0$.
- Homogeneity and triangle inequality: inherited from $|\cdot|$ in \mathbb{F} (exercise).
- Completeness: let $(f_k)_{k \geq 1}$ be a Cauchy sequence under $\|\cdot\|_{\infty}$. For each $x \in X$ we have $|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_{\infty} \rightarrow 0$ as $n, m \rightarrow \infty$. So $(f_k(x))_{k \geq 1}$ is Cauchy in \mathbb{F} , so (since \mathbb{F} is complete) there exists a limit $f(x) = \lim_{k \rightarrow \infty} f_k(x)$. This defines a function $f : X \rightarrow \mathbb{F}$.
- For all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\forall m, n \geq n_0, \forall x \in X$,

$$|f_m(x) - \underbrace{f_n(x)}_{\rightarrow f(x)}| \leq \varepsilon$$

so for all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\forall m \geq n_0, \forall x \in X$ we have

$$|f_m(x) - f(x)| \leq \varepsilon$$

so $\|f_m - f\|_\infty \leq \varepsilon$ and $f_m \rightarrow f$ uniformly, so $f \in C_{\mathbb{R},b}$ by properties of the uniform limit.

□

Example. Given $U \subseteq \mathbb{R}^n$ open, bounded and non-empty; $m \in \mathbb{N}^*$, consider

$$\begin{aligned} C^m(\overline{U}) = \{f : U \rightarrow \mathbb{R} : f \text{ is } m \text{ times differentiable on } U, \forall \alpha \in \mathbb{N}^n \\ \text{s.t. } |\alpha| = \alpha_1 + \dots + \alpha_m \leq m \\ , \partial^\alpha f \text{ is continuous and bounded on } U\} \end{aligned}$$

Then $(C^m(\overline{U}), \|\cdot\|_{C^m})$ is a Banach space where

$$\|f\|_{C^m} = \sup_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} \underbrace{\sup_{x \in U} |\partial^\alpha f(x)|}_{\|\partial^\alpha f\|_\infty}$$

Exercise: check that this is complete and $\partial^\alpha f, \alpha \leq m-1$, extends continuously to \tilde{U} .

Example. $C_{\mathbb{R}}([0,1])$, the set of continuous functions from $[0,1]$ to \mathbb{R} . This is a vector space over \mathbb{R} .

- $(C_{\mathbb{R}}([0,1]), \|\cdot\|_\infty)$ is a Banach space (Example sheet)
- Could take another norm such that

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx \right)^{1/p}, \quad p \in [1, \infty)$$

Study of $(C_{\mathbb{R}}([0,1]), \|\cdot\|_p)$:

- $\|\cdot\|_p$ is well defined: Riemann and Lebesgue integrable.
- If $\|f\|_p = 0$ and $f \neq 0$ then there exists $\varepsilon > 0$ and $x_0 \in [0,1]$ such that $|f(x_0)| \geq \varepsilon$, so by continuity there exist $a < b \in [0,1]$ such that $\inf_{x \in [a,b]} |f(x)| \geq \frac{\varepsilon}{2}$. Then $\int_0^1 |f(x)|^p dx \geq \left(\frac{\varepsilon}{2}\right)^p (b-a) > 0$ which is impossible.
- Homogeneity is clear.
- Triangle inequality:

$$\|f + g\|_p^p = \int_0^1 |f + g|^p dx = \int_0^1 |f + g| |f + g|^{p-1} dx$$

$$\begin{aligned} &\leq \int_0^1 |f| |f+g|^{p-1} dx + \int_0^1 |g| |f+g|^{p-1} dx \\ &\underbrace{\leq}_{\text{Hölder:}} \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1} \end{aligned}$$

If $\|f+g\|_p = 0$ then it's clear. Otherwise this implies $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

- Completeness? Define

$$f_k(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{4k} \\ \left[x - \left(\frac{1}{2} - \frac{1}{4k}\right)\right] 4k & \frac{1}{2} - \frac{1}{4k} \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

then $(f_k)_{k \geq 1}$ is Cauchy for $\|\cdot\|_p$, and the limit is $1_{[1/2, 1]}$ which is not continuous. So not complete.

Remark: what about the completion? In general, abstract completions are often not very useful; however in this case, it is: Lebesgue space $L^p([0, 1])$, defined as equivalence classes for the “almost everywhere” equality.

Example. Take functions from $X = \mathbb{N} \rightarrow \mathbb{R}$ or \mathbb{C} , get $\ell_{\mathbb{F}}^p$ for $p \in [1, \infty]$, with norm $\|(x_k)\|_p = \left(\sum_{k \geq 1} |x_k|^p\right)^{1/p}$ for $p < \infty$ and $\|(x_k)\|_{\infty} = \sup_{k \geq 1} |x_k|$. Exercise: show this is indeed a norm and this is complete, hence Banach.

Remark: for $p \in (0, 1)$, ℓ^p is similarly defined.

***Non-examinable example of TVS*:**

- Define for $U \subseteq \mathbb{R}^n$ open & non-empty, $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , $C_{\mathbb{F}}(U)$ the set of continuous functions $U \rightarrow \mathbb{F}$.
- TVS for the topology τ defined by the translations of the following basis of neighborhoods around 0: take $(K_n)_{n \geq 1}$ a sequence of increasing compact sets, $\bigcup_{n \geq 1} K_n = U$. Define

$$U_n = \left\{ f \in C_{\mathbb{F}}(U) : \sup_{K_n} |f| \leq \frac{1}{n} \right\}$$

- Exercise: show this indeed a TVS and τ does not depend on the choice of the (K_n) .
- Proposition: $(C(U), \tau)$ is a locally convex, not locally bounded TVS (therefore not normable). Furthermore, it is metrizable with $d(f, g) = \sum_{k \geq 1} \frac{1}{2^k} \left(\frac{\sup_{K_k} |f - g|}{1 + \sup_{K_k} |f - g|} \right)$. Also $(C(U), d)$ is complete (Frechet space).

Remarks:

1. Not locally bounded: suppose there exists B bounded neighborhood of 0, then there exists $n_0 \geq 1$ such that $U_{n_0} \subseteq B$. B is bounded so there exists $t > 0$ such that $B \subseteq tU_{n_0+1}$ so $U_{n_0} \subseteq tU_{n_0+1}$. But this is impossible since we can always construct $f \in U_{n_0}$ such that $\sup_{K_{n_0+1}} |tf| > 1/n$
2. Let $C_c(U)$ be the set of continuous functions with compact support. Then V is a neighborhood of 0 if and only if $V \cap C(K_n)$ is a neighborhood of 0 in $C(K_n)$. This is a non-countable topology.

5 Bounded linear maps & duality

Definition. Given (V, τ_V) and (W, τ_W) TVS', $T : V \rightarrow W$ linear is *bounded* if it maps bounded sets to bounded sets: for any $B_V \subseteq V$ bounded, then $T(B_V)$ is bounded in W .

Proposition. Given (V, τ_V) , (W, τ_W) TVS' which are locally bounded (note this includes NVS'), and $T : V \rightarrow W$ is linear, then T is bounded if and only if T is continuous.

Proof.

Step 1: T bounded $\implies T$ continuous at 0. Let U_W be an open neighborhood of 0 in W , and U_V an open bounded neighborhood of 0 in V . Then $T(U_V)$ is bounded, so there exists $t > 0$ such that $T(U_V) \subseteq tU_W$. So $T^{-1}(U_W) \supseteq t^{-1}U_V$ and $t^{-1}U_V$ is open around 0 in V (using the fact dilations are continuous).

Step 2: T continuous at 0 $\implies T$ is continuous everywhere. Let $w \in W$, U_W open around w , $v \in V$ such that $T(v) = w$. Then $U_W - w$ is open around 0 in W (translation continuous), so by Step 1, $T^{-1}(U_W - w)$ is a neighborhood of 0 in V . So

$$\begin{aligned} T^{-1}(U_W) &= T^{-1}(\{w\}) + T^{-1}(U_W - w) \\ &= \bigcup_{v' \in T^{-1}(\{w\})} (v' + T^{-1}(U_W - w)) \\ &\supseteq \underbrace{v + T^{-1}(U_W - w)}_{\text{ngbd around } v} \end{aligned}$$

Step 3: T continuous $\implies T$ bounded. Let $B_V \subseteq V$ be bounded, and U_W an open neighborhood of 0 in W . Then $T^{-1}(U_W)$ is open around 0 in V . So (since B_V bounded) there exists $t > 0$ such that $B_V \subseteq tT^{-1}(U_W)$ and so $T(B_V) \subseteq tU_W$.

We have proved that $T(B_V)$ is covered by a dilation of any neighborhood of 0, so is bounded. \square

Definition. Given $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ NVS' on \mathbb{F} , and $T : V \rightarrow W$ linear, T is bounded iff T is continuous iff there exists $t > 0$ such that $T(B_V(0, 1)) \subseteq B_W(0, t)$. The infimum of such t 's is denoted $\|T\|$.

Remark: can check that $\|T\|$ is equivalently defined as

$$\|T\| = \sup_{\|v\|_V \leq 1} \|Tv\|_W = \sup_{\|v\|_V < 1} \|Tv\|_W = \sup_{\|v\|_V = 1} \|Tv\|_W \quad (*)$$

Definition. Given $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ NVS', denote

$$\mathcal{L}(V, W) = \{T : V \rightarrow W \text{ linear map}\}$$

$$\mathcal{B}(V, W) = \{T : V \rightarrow W \text{ linear bounded map}\}$$

Proposition. $(\mathcal{B}(V, W), \|\cdot\|)$ is an NVS.

Proof.

- $\mathcal{L}(V, W)$ is a vector space via $(\lambda_1 T_1 + \lambda_2 T_2)(v) = \lambda_1 T_1(v) + \lambda_2 T_2(v)$.
- $\mathcal{B}(V, W)$: dilation/(finite) sums of bounded sets are bounded. So T bounded implies λT is bounded and T_1, T_2 bounded implies $T_1 + T_2$ bounded.

- $|||T|||$ is well-defined in \mathbb{R}_+ for T bounded, $|||0||| = 0$ and if $|||T||| = 0$ then $T(B_V(0, 1)) \subseteq B_W(0, t)$ for all $t > 0$ and so by continuity of dilation, $T(B_V(0, 1)) = \{0\}$. By linearity, this implies $T = 0$.
- $|||\lambda T||| = |\lambda| |||T|||$ and $|||T_1 + T_2||| \leq |||T_1||| + |||T_2|||$ follows from (*)

□

Proposition. Let $(V, \|\cdot\|_V)$ be a NVS and $(W, \|\cdot\|_W)$ a Banach space. Then $(\mathcal{B}(V, W), |||\cdot|||)$ is a Banach space.

Proof. We have proved that $(\mathcal{B}(V, W), |||\cdot|||)$ is an NVS above. So we prove completeness. Let $(T_k)_{k \geq 1}$ be a Cauchy sequence in $(\mathcal{B}(V, W), |||\cdot|||)$. Then

$$\sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}||| \rightarrow 0 \text{ as } k_0 \rightarrow \infty \quad (**)$$

$$\forall v \in V, \sup_{k_1, k_2 \geq k_0} \|T_{k_1}(v) - T_{k_2}(v)\|_W \leq \|v\|_V |||T_{k_1} - T_{k_2}||| \xrightarrow{k_0 \rightarrow \infty} 0 \quad (***)$$

so $(T_k(v))_{k \geq 1}$ is a Cauchy sequence in W . Since W is complete, can let the associated limit be $T(v)$.

Then T is linear by pointwise limits:

$$\begin{aligned} T(\lambda_1 v_1 + \lambda_2 v_2) &= \lim_{k \rightarrow \infty} T_k(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{k \rightarrow \infty} [\lambda_1 T_k(v_1) + \lambda_2 T_k(v_2)] \\ &= \lambda_1 T(v_1) + \lambda_2 T(v_2) \end{aligned}$$

Use (**), take $k_2 \rightarrow \infty$ so

$$\forall v \in V, \sup_{k_1 \geq k_0} \|T_{k_1}(v) - T(v)\|_W \leq \|v\|_V \left(\sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}||| \right) \rightarrow 0 \text{ as } k_0 \rightarrow \infty$$

Hence for $v \in V$ such that $\|v\| \leq 1$ we have

$$\sup_{k_1 \geq k_0} \|T_{k_1}(v) - T(v)\|_W \leq \sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}||| \quad (\dagger)$$

Then (for $v \in V$ with $\|v\| \leq 1$) by the triangle inequality

$$\|T(v)\|_W \leq \underbrace{\|T_{k_0}(v)\|_W}_{\text{bounded}} + \sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}|||$$

$$\sup_{\|v\| \leq 1} \|T(v)\|_W \leq |||T_{k_0}||| + \sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}|||$$

So T is bounded. Now (\dagger) implies

$$\sup_{k_1 \geq k_0} |||T_{k_1} - T||| \leq \sup_{k_1, k_2 \geq k_0} |||T_{k_1} - T_{k_2}||| \xrightarrow{k_0 \rightarrow \infty} 0$$

So $T_{k_1} \xrightarrow{|||\cdot|||} T$.

□

Remark: can deduce from (\dagger) that for all $v \in V$ with $\|v\| \leq 1$,

$$\|T_k(v)\|_W - \|T_k - T\| \leq \|T(v)\|_W \leq \|T_k(v)\|_W + \|T_k - T\|$$

Then taking supremum over $\|v\| \leq 1$

$$\left| \sup_{\|v\| \leq 1} \|Tv\|_W - \sup_{\|v\| \leq 1} \|T_k(v)\|_W \right| \leq \|T_k - T\| \xrightarrow{k \rightarrow \infty} 0$$

So $\|T_k\| \xrightarrow{k \rightarrow \infty} \|T\|$.

Definition. Let $(V, \|\cdot\|_V)$ be a NVS over \mathbb{F} . Let

$$\mathcal{L}(V, \mathbb{F}) = \{\text{linear maps } V \rightarrow \mathbb{F}\}, \text{ the algebraic dual}$$

$$\mathcal{B}(V, \mathbb{F}) = \{\text{bounded linear maps } V \rightarrow \mathbb{F}\} \text{ denoted } (V^*, \|\cdot\|_{V^*})$$

Note that by the previous proposition $\mathcal{B}(V, \mathbb{F})$ is Banach (since $\mathbb{F} = \mathbb{R}$ or \mathbb{C} is complete).

Definition. Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be NVS', $T \in \mathcal{B}(V, W)$. Then T^* (the *adjoint* of T) defined as $T^* : W^* \rightarrow V^*$, $\psi \mapsto \varphi = \psi \circ T$. i.e $T^*(\psi)(v) = \psi(T(v))$.

Proposition. T^* is well-defined $W^* \rightarrow V^*$, linear and bounded (for $\|\cdot\|_{W^*}$ and $\|\cdot\|_{V^*}$) with $\|T^*\| \leq \|T\|$.

Remark: soon, with the help of the Hahn-Banach Theorem, we'll prove that the duals are "big enough" so that $\|T^*\| = \|T\|$.

Proof.

- Well-defined: follows since linearity and boundedness are stable under composition, i.e if $T : V \rightarrow W$ is linear and bounded, $\psi : W \rightarrow \mathbb{F}$ is linear and bounded, so is $\psi \circ T : V \rightarrow \mathbb{F}$. So $\psi \circ T \in V^*$
- Linearity:

$$\begin{aligned} T^*(\lambda_1 \psi_1 + \lambda_2 \psi_2)(v) &= (\lambda_1 \psi_1 + \lambda_2 \psi_2)(Tv) \\ &= \lambda_1 [\psi_1(Tv)] + \lambda_2 [\psi_2(Tv)] \\ &= \lambda_1 T^*(\psi_1)(v) + \lambda_2 T^*(\psi_2)(v) \end{aligned}$$

- Boundedness:

$$\begin{aligned} \|T^*\| &= \sup_{\|\psi\|_{W^*}} \|T^*(\psi)\|_{V^*} = \sup_{\|\psi\|_{W^*} \leq 1} \sup_{\|v\|_V \leq 1} |T^*(\psi)(v)| \\ &\leq \sup_{\|\psi\|_{W^*} \leq 1} \sup_{\|v\|_V \leq 1} |\psi(Tv)| \leq \sup_{\|\psi\|_{W^*} \leq 1} \sup_{\|v\|_V \leq 1} \|\psi\|_{W^*} \|Tv\|_W \leq \|T\| \end{aligned}$$

□

Definition. Let $(V, \|\cdot\|_V)$ be an NVS. Since $(V^*, \|\cdot\|_{V^*})$ is a NVS (Banach), we can define its dual, denoted $(V^{**}, \|\cdot\|_{V^{**}})$ the *bi-dual* of V (again Banach).

Proposition. Define $\Phi : V \rightarrow V^{**}$, $v \mapsto \Phi(v)$ by

$$\forall \varphi \in V^*, \Phi(v)(\varphi) = \varphi(v)$$

Then Φ is well-defined, linear and bounded with $\|\Phi\| \leq 1$. Φ is called the *canonical bi-dual embedding*.

Remark: with the Hahn-Banach Theorem, we'll prove Φ is an isometry. In particular, $\|\Phi\| = 1$ and Φ is injective. However, Φ is not always surjective. In fact, V and V^{**} are not always isomorphic.

Proof.

- Well-defined: given $v \in V$, $\phi \in V^*$ is linear, and bounded since

$$\sup_{\|\varphi\|_{V^*} \leq 1} |\varphi(v)| \leq \|v\|_V$$

- Linearity:

$$\begin{aligned} \Phi(\lambda_1 v_1 + \lambda_2 v_2)(\varphi) &= \varphi(\lambda_1 v_1 + \lambda_2 v_2) \\ &= \lambda_1 \varphi(v_1) + \lambda_2 \varphi(v_2) \\ &= \lambda_1 \Phi(v_1)(\varphi) + \lambda_2 \Phi(v_2)(\varphi) \end{aligned}$$

- Boundedness:

$$\begin{aligned} \|\Phi\| &= \sup_{\|v\|_V \leq 1} \|\Phi(v)\|_{V^{**}} = \sup_{\|v\|_V \leq 1} \sup_{\|\varphi\|_{V^*} \leq 1} \underbrace{|\Phi(v)(\varphi)|}_{\varphi(v)} \\ &= \sup_{\|v\|_V \leq 1} \sup_{\|\varphi\|_{V^*} \leq 1} \underbrace{|\varphi(v)|}_{\leq \|\varphi\|_{V^*} \|v\|_V} \leq 1 \end{aligned}$$

□

Example. Let V, W be finite-dimensional NVS' with bases $(v_i)_{i=1}^m$ and $(w_j)_{j=1}^n$ respectively. Let $T : V \rightarrow W$ be linear (and thus bounded as finite dimensional). Take $(v_i^*)_{i=1}^m$ defined by $v_i^*(v_{i'}) = \delta_{ii'}$ and $(w_j^*)_{j=1}^n$ defined by $w_j^*(w_{j'}) = \delta_{jj'}$. Then V^*, W^* are finite-dimensional NVS' with bases (v_i^*) and (w_j^*) respectively. If T has a matrix $A = (a_{ij})_{i=1, j=1}^{i=m, j=n}$ in with respect to the bases (v_i) and (w_j) , then

$$Tv_i = \sum_{j=1}^n a_{ij} w_j$$

and T^* has matrix $A^T = (a_{ji})_{j=1, i=1}^{j=n, i=m}$ with respect to the bases (w_j^*) and (v_i^*) .

Example. Space of square summable spaces $\ell^2(\mathbb{F})$ (as usual $\mathbb{F} = \mathbb{R}$ or \mathbb{C}) is infinite dimensional. There are linear maps on this space that are

- Bounded, injective but not surjective: $T(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ a “right shift” of the sequence
- Bounded, surjective but not injective: $T(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$ a “left shift” of the sequence
- Linear but not bounded: find a basis $(e_i)_{i \in I}$, extract $(e_n)_{n \geq 1}$ a countable subset. Then define $T : e_n \mapsto ne_n, e_i \mapsto 0$ for $i \notin \mathbb{N}$.

Duality: $(\ell^2)^* = \ell^2$ (Hilbert representation theorem)

Example. For $\ell^p, p \in (1, \infty), p \neq 2$, we have duals

$$\ell^p \rightarrow (\ell^p)^* = \ell^q \rightarrow (\ell^q)^* = \ell^p \text{ where } \frac{1}{p} + \frac{1}{q}$$

$$\ell^1 \rightarrow (\ell^1)^* = \ell^\infty \rightarrow (\ell^\infty)^* \neq \ell^1$$

Example. (Question 8 Example sheet 1) $(C^1([0, 1]), \|\cdot\|_{C^0}) \rightarrow (C^1([0, 1]), \|\cdot\|_{C^1})$, $f \mapsto f$ is unbounded.

Zorn's Lemma

In a finite-dimensional NVS V , we have a “simple” dual V^* . In infinite-dimension, we have not even proved that if V is non-trivial (i.e not $\{0\}$) then V^* is non-trivial.

The Hahn-Banach Theorem will answer several questions:

- $V \neq \{0\} \implies V^* \neq \{0\}$
- V^* separates points of V
- Φ (the bidual embedding) is isometric, $\|\Phi\| = 1$
- $\|T^*\| = \|T\|$

Idea of Hahn-Banach: extend linear bounded maps already defined on a subspace.

Strategy:

1. “Co-dimension 1” extension: any linear bounded map $V \rightarrow \mathbb{F}$ has an extension to $W \rightarrow \mathbb{F}$ where $V \subseteq W$ with codimension 1.
2. Transfinite induction: Zorn's Lemma (or equivalently the Axiom of Choice)

Remark: if $V = \bigcup_{n \geq 1} V_n$, V_n subspace, $V_n \subseteq V_{n+1}$, $\dim(V_n) = n$, could use step 1 above and standard (countable) induction. However, no Banach spaces are like this.

Definition. A set S is *partially ordered* (poset) if there is a binary relation “ \leq ” such that

- $\forall x, y \in S$, $x \leq y$ or not (partial order)
- $\forall x \in S$, $x \leq x$ (reflexive)
- $\forall x, y, z \in S$, if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive)
- $\forall x, y \in S$, if $x \leq y$ and $y \leq x$ then $x = y$ (non-ambiguous)

Definition. A poset S is *totally ordered* if $\forall x, y \in S$, if $x \not\leq y$ then $x \geq y$.

Definition. Given $S' \subseteq S$ (where (S, \leq) is a poset), we say $l \in S$ is a *upper bound* of S' if $\forall x \in S'$, $x \leq l$. l is a *least upper bound* of S' if it is an upper bound and any other upper bound $l' \in S$ satisfies $l' \geq l$.

Definition. A subset S' of S ((S, \leq) a poset) that is totally ordered is called a *chain*.

Definition. A poset (S, \leq) has the *least upper bound property* if any non-empty chain has a least upper bound.

Definition. Given a poset (S, \leq) , $m \in S$ is said to be *maximal* if $\forall x \in S$, $x \geq m$ implies $x = m$.

Theorem (Zorn's Lemma). *Any non-empty poset (S, \leq) with the least upper bound property has (at least one) maximal element.*

Remarks:

1. In fact Zorn's Lemma is true just with "upper bound" property on chains.
2. Zorn's Lemma is equivalent to the Axiom of Choice

5.1 Finite dimension

Definition. Let V be a NVS with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then these norms are said to be *equivalent*, denoted $\|\cdot\|_1 \sim \|\cdot\|_2$ if there are two constants, $c, c' > 0$ such that

$$\forall v \in V, C\|v\|_1 \leq \|v\|_2 \leq C'\|v\|_1$$

Remarks:

1. This defines equivalence classes on norms.
2. $\|\cdot\|_1 \sim \|\cdot\|_2$ implies that their induced topologies are the same. The converse is also true: indeed $B_{\|\cdot\|_1}(0, 1)$ is open around 0 for τ_2 , so there exists $\varepsilon > 0$ such that $B_{\|\cdot\|_2}(0, \varepsilon) \subseteq B_{\|\cdot\|_1}(0, 1)$, which implies that for all $v \in V \setminus \{0\}$

$$\frac{\varepsilon v}{2\|v\|_2} \in B_{\|\cdot\|_2}(0, \varepsilon) \subseteq B_{\|\cdot\|_1}(0, 1) \implies \|v\|_1 \leq \frac{2}{\varepsilon}\|v\|_2$$

and similarly for the opposite bound.

3. When 2 norms are equivalent, they generate the same notion of bounded linear maps, converging spaces & Cauchy sequences.

Proposition.

- (i) All norms are equivalent in finite-dimension
- (ii) Given $(V, \|\cdot\|_V)$ a finite-dimensional NVS, $(W, \|\cdot\|_W)$ a NVS, any linear map $T : V \rightarrow W$ is bounded
- (iii) Given $(V, \|\cdot\|_V)$ an NVS, if $\overline{B}_V(0, 1)$ is compact, then V is finite dimensional.

Proof.

- (i) Let us prove all norms are equivalent to $\|\cdot\|_\infty$, defined for a basis $(e_i)_{i=1}^n$ as $\|v\|_\infty = \sup_{1 \leq i \leq n} |v_i|$ for $v = \sum v_i e_i$.

Let $\|\cdot\|$ be a norm on V

$$\|v\| = \left\| \sum_{i=1}^n v_i e_i \right\| \leq \sum_{i=1}^n |v_i| \|e_i\| \leq \underbrace{\left(\sum_{i=1}^n \|e_i\| \right)}_{=C'} \|v\|_\infty$$

Consider $\varphi : (V, \|\cdot\|_\infty) \rightarrow \mathbb{R}_+$ defined by $v \mapsto \|v\|$. Then φ is continuous:

$$|\varphi(v) - \varphi(w)| = |||v| - |w|| \leq \|v - w\| \leq C' \|v - w\|_\infty$$

Define $S_{\|\cdot\|_\infty}(0, 1) = \{v \in V : \|v\|_\infty = 1\}$. Then $\varphi : S_{\|\cdot\|_\infty}(0, 1) \rightarrow \mathbb{R}_+$ continuous, so attains its minimum: there exists $v_0 \in S_{\|\cdot\|_\infty}(0, 1)$ such that $\forall v \in S_{\|\cdot\|_\infty}(0, 1)$, $\varphi(v) \geq \varphi(v_0)$.

Then $v_0 \neq 0$ since $\|v_0\|_\infty = 1$ and so $\varphi(v_0) = \|v_0\| = C > 0$. This implies

$$\left\| \frac{v}{\|v\|_\infty} \right\| \geq C, \forall v \in V \setminus \{0\} \implies \forall v \in V, \|v\| \geq C \|v\|_\infty$$

- (ii) Completeness and the fact closed bounded sets are compact follows from
(i) since true with $(\mathbb{F}^n, \|\cdot\|_\infty)$.

$$\begin{aligned} \|T(v)\|_W &= \left\| \sum_{i=1}^n v_i T(e_i) \right\|_W \leq \sum_{i=1}^n |v_i| \|T(e_i)\|_W \\ &\leq \|v\|_\infty \left(\sum_{i=1}^n \|T(e_i)\|_W \right) \leq \frac{1}{C} \|v\|_V \left(\sum_{i=1}^n \|T(e_i)\|_W \right) \end{aligned}$$

so T is bounded

□

Theorem (Riesz). *If $(V, \|\cdot\|)$ is an NVS, $\overline{B}(0, 1)$ compact then V finite dimensional.*

Proof. $\overline{B}(0, 1) \subseteq \bigcup_{v \in \overline{B}(0, 1)} B(v, 1/2)$ open covering. Then compactness implies there exist v_1, \dots, v_n in $\overline{B}(0, 1)$ such that $\overline{B}(0, 1) \subseteq \bigcup_{i=1}^n B(v_i, 1/2)$. Denote $W = \text{span}(v_1, \dots, v_n)$ a subspace of V . Then $\overline{B}(0, 1) \subseteq \bigcup_{i=1}^n (v_i + B(0, 1/2))$.

$$\overline{B}(0, 1) \subseteq W + B(0, 1/2) \subseteq W + \overline{B}(0, 1/2)$$

Iterate on $\overline{B}(0, 1/2) = \frac{1}{2}\overline{B}(0, 1)$: $\overline{B}(0, 1/2) \subseteq W + \overline{B}(0, 1/4)$.

$$\overline{B}(0, 1) \subseteq \bigcap_{k=1}^K (W + \overline{B}(0, 2^{-k})), \quad \forall K \geq 1$$

Then

$$\overline{B}(0, 1) \subseteq \bigcap_{k \geq 1} (W + \overline{B}(0, 2^{-k})) \subseteq \overline{W} = W$$

$\overline{B}(0, 1) \subseteq W$ implies $V = W$. □

Back to (Zorn's Lemma) and the Hahn-Banach Theorem

Construction of basis:

Proposition. Let $V \neq \{0\}$ be a vector space over \mathbb{F} and $S \subseteq V$ subset which is linearly independent. Then there exists a subset $B \subseteq V$ linearly independent such that $S \subseteq B$ and $\text{span}(B) = V$ (i.e a basis).

Proof. Let $\mathcal{F} = \{\text{linearly independent subsets } S' \subseteq V \text{ such that } S \subseteq S'\}$. Then $S \neq \emptyset$ since $S \in \mathcal{F}$.

(\mathcal{F}, \subseteq) is a poset (easy check).

If $\Theta \subseteq \mathcal{F}$ is a chain (totally ordered for \subseteq) then it has a least upper bound: $\overline{S} = \bigcup_{S' \in \Theta} S'$.

Properties of \overline{S} :

- $\overline{S} \supseteq S'$, for all $S' \in \Theta$ so \overline{S} is an upper bound for Θ
- An upper bound for Θ will include each $S' \in \Theta$ so \overline{S} is a least upper bound.
- $\overline{S} \supseteq S$ since $\overline{S} = \bigcup_{S' \in \Theta} S'$ and each $S' \supseteq S$.
- \overline{S} is linearly independent: let $(v_1, \dots, v_n) \in \overline{S}$ be distinct elements. Then for all $i = 1, \dots, n$ there exists $S'_i \in \Theta$ such that $v_i \in S'_i$. Chain structure (total order) means there exists $i_0 \in \{1, \dots, n\}$ such that $S'_j \subseteq S'_{i_0}$ for all $j = 1, \dots, n$. So $\{v_1, \dots, v_n\} \subseteq S'_{i_0}$ is linearly independent, and so \overline{S} is.

Now Zorn's Lemma says that there exists a maximal element in \mathcal{F} : $B \supseteq S$, B linearly independent and maximal. Assume $\text{span}(B) \subsetneq V$, then we have $v_0 \in V \setminus \text{span}(B)$ and $B' = B \cup \{v_0\}$ is a strictly larger element of \mathcal{F} , a contradiction. Hence $V = \text{span}(B)$. \square

Note that the statement of the geometric form of Hahn-Banach below is ***non-examinable***

Theorem (Hahn-Banach “algebraic” form).

- (i) Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and $p : V \rightarrow \mathbb{R}_+$ such that for all $v_1, v_2 \in V$, $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ and for all $\lambda \in \mathbb{F}$, $v \in V$ we have $p(\lambda v) = |\lambda|p(v)$.

Let $W \subseteq V$ be a subspace of V and $f : W \rightarrow \mathbb{F}$ linear with $|f(w)| \leq p(w)$ for all $w \in W$. Then there exists $\tilde{f} : V \rightarrow \mathbb{F}$ linear, with $\tilde{f}|_W = f$ and $|\tilde{f}(v)| \leq p(v)$ on all of V .

- (ii) Let V be a vector space over $\mathbb{F} = \mathbb{R}$ and $p : V \rightarrow \mathbb{R}_+$ such that for all $v_1, v_2 \in V$, $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ and for all $\lambda > 0$, $v \in V$ we have $p(\lambda v) = \lambda p(v)$.

Let $W \subseteq V$ be a subspace of V and $f : W \rightarrow \mathbb{F}$ be linear with $f \leq p$ on W . Then there exists $\tilde{f} : V \rightarrow \mathbb{F}$ linear with $\tilde{f}|_W = f$, and $\tilde{f} \leq p$ on V .

Proof. Step 1: (i) in \mathbb{R} implies (ii) in \mathbb{C} . Start from $f : W \rightarrow \mathbb{F} = \mathbb{C}$. Note that a vector space V over \mathbb{C} can be seen as a vector space over \mathbb{R} . Indeed if $(e_i)_{i \in I}$ is a basis over \mathbb{C} , and $V_0 = \text{span}_{\mathbb{R}}((e_i)_{i \in I})$, $V = V_0 \oplus (iV_0)$ (same with W).

Define $g = \Re(f)$, this satisfies $|g| \leq p$. Then (i) on \mathbb{R} implies there exists $\tilde{g} : V \rightarrow \mathbb{R}$ linear extending g such that $|\tilde{g}| \leq p$.

Define $\tilde{f}(v) := \tilde{g}(v) - i\tilde{g}(iv)$. Then $\tilde{f}(\lambda v) = \lambda \tilde{f}$ for all $\lambda \in \mathbb{R}$ (\tilde{f} linear). Also $\tilde{f}(iv) = i\tilde{f}(v)$. Hence \tilde{f} is linear over \mathbb{C} . This extends g to all of V .

Also for all $v \in V$, there exists $\theta \in [0, 2\pi)$ such that $|f(v)| = \Re(\tilde{f}(e^{i\theta}v)) = \tilde{g}(e^{i\theta}v) \leq p(e^{i\theta}v) = p(v)$.

Step 2: (ii) in \mathbb{R} implies (i) in \mathbb{R} . If $W \subseteq V$ is a subspace, $p : V \rightarrow \mathbb{R}_+$ such that $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ for all $v_1, v_2 \in V$ and $p(\lambda v) = |\lambda|p(v)$ for all $\lambda \in \mathbb{R}, v \in V$, and $f : W \rightarrow \mathbb{R}$ is linear such that $|f(v)| \leq p(v)$ for all $v \in W$ then (ii) can be applied to obtain $\tilde{f} : V \rightarrow \mathbb{R}$ linear extending f such that $\tilde{f}(v) \leq p(v)$ for all $v \in V$ (no modulus a priori in this conclusion).

We also deduce $\tilde{f}(-v) = p(-v) = p(v)$, so $|\tilde{f}(v)| \leq p(v)$.

Step 3: proof of (ii) in \mathbb{R} .

- (a) Co-dimension 1 case: consider $V = W \oplus (\mathbb{R}v_0)$, $v_0 \neq 0$. We have $f : W \rightarrow \mathbb{R}$ linear, $f \leq p$ on W . To extend f it is enough to prescribe \tilde{f} at v_0 , then linearity does the rest: for $w \in W$, $\tilde{f}(w + av_0) = \tilde{f}(w) + a\tilde{f}(v_0) = f(w) + a\tilde{f}(v_0)$.

The value of $\tilde{f}(v_0)$ must satisfy:

$$\tilde{f}(w + av_0) \leq p(w + av_0), \quad a > 0 \text{ and for } a < 0$$

This gives

$$\underbrace{-p\left(-\frac{w}{a} - v_0\right) + f\left(-\frac{w}{a}\right)}_{A(w')} \underbrace{\leq}_{a < 0} \tilde{f}(v_0) \underbrace{\leq}_{a > 0} p\left(\frac{w}{a} + v_0\right) - f\left(\frac{w}{a}\right) \underbrace{B(w'')}$$

where $w' = -\frac{w}{a}$ and $w'' = \frac{w}{a}$. Then for all $w', w'' \in W$, $\tilde{f}(v_0) \in [A(w'), B(w'')]$. Set $\beta = \tilde{f}(v_0)$. Then a consistent value of β exists if and only if

$$\sup_{w' \in W} A(w') \leq \inf_{w'' \in W} B(w'')$$

This is indeed satisfied since

$$f(w') + f(w'') = f(w' + w'') \leq p(w' + w'') \leq p(w' - v_0) + p(w'' + v_0)$$

- (b) Transfinite induction: define

$$\mathcal{S} = \{(\tilde{f}, \tilde{W}) : \tilde{f} : \tilde{W} \rightarrow \mathbb{R} \text{ linear}, \tilde{f} \leq p \text{ and } \tilde{W} \supseteq W, \tilde{f}|_W = f\}$$

Now \mathcal{S} is a poset under $(f_1, W_1) \subseteq (f_2, W_2)$ if $W_1 \subseteq W_2$ and $f_2|_{W_1} = f_1$. Also \mathcal{S} has the least upper bound property: indeed consider $\Theta \subseteq \mathcal{S}$ a chain (totally ordered subset). Then for (\bar{f}, \bar{W}) defined by

$$\bar{W} = \bigcup_{W' : (f', W') \in \Theta} W'$$

and $\bar{f}(v) = f'(v)$ for all $v \in \bar{W}$, for $(f', W') \in \Theta$ such that $v \in W'$. Also \bar{f} is well defined since Θ is totally ordered: so if $v \in W'_1 \cap W'_2$ then wlog $W'_1 \subseteq W'_2$, $f'_2|_{W'_1} = f'_1$ so $\bar{f}(v) = f'_2(v) = f'_1(v)$.

\bar{f} is linear as Θ is totally ordered: $\bar{f}(\lambda v) = f'(\lambda v) = \lambda f'(v) = \lambda \bar{f}(v)$ for $(f', W') \in \Theta$ with $v \in W'$. Also

$$\bar{f}(v_1 + v_2) = f'_2(v_1 + v_2) = f'_2(v_1) + f'_2(v_2) = \bar{f}(v_1) + \bar{f}(v_2)$$

Finally $\bar{f} \leq p$ since for all $v \in \bar{W}$, $v \in W'$, $(f', W') \in \Theta$, $\bar{f}(v) = f'(v) \leq p(v)$.

So by Zorn's Lemma, there is a maximal element (\tilde{f}, \tilde{W}) in \mathcal{S} . If $\tilde{W} \subsetneq V$, then there exists $v_0 \in V \setminus \tilde{W}$ and the previous step applied to $\tilde{W} \subseteq \tilde{W} \oplus \mathbb{R}v_0$ and $\tilde{f} : \tilde{W} \rightarrow \mathbb{R}$ linear with $\tilde{f} \leq p$, gives the existence of a

$$\tilde{f}' : \underbrace{\tilde{W} \oplus \mathbb{R}v_0}_{\tilde{W}'} \rightarrow \mathbb{R}$$

linear with $\tilde{f}'|_{\tilde{W}} = \tilde{f}$. But then (\tilde{f}', \tilde{W}') is strictly larger than (\tilde{f}, \tilde{W}) , a contradiction.

□

Theorem (Geometric form of Hahn-Banach).

- (i) Let $(V, \|\cdot\|)$ be an NVS over \mathbb{R} , $A \subseteq V$ open, convex and non-empty; $B \subseteq V$ convex and non-empty; $A \cap B = \emptyset$. Then there is a closed hyperplane weakly separating A and B : there exists $f \in V^* \setminus \{0\}$, $\alpha \in \mathbb{R}$ such that $\sup_A f \leq \alpha \leq \inf_B f$ (the hyperplane is $f^{-1}(\{\alpha\})$)
- (ii) Let $(V, \|\cdot\|)$ be an NVS over \mathbb{R} , $A \subseteq V$ closed, convex and non-empty; $B \subseteq V$ compact, convex and non-empty; $A \cap B = \emptyset$. Then there is a closed hyperplane strictly separating A and B : there exists $f \in V^* \setminus \{0\}$, $\alpha_1 < \alpha_2 \in \mathbb{R}$ such that $\sup_A f \leq \alpha_1 < \alpha_2 \leq \inf_B f$.

Proof.

- (i) Let $C_0 = A - B = \{a - b : a \in A, b \in B\}$. Then $C_0 \neq \emptyset$ since A and B are non-empty, convex as

$$\lambda(a - b) + (1 - \lambda)(a' - b') = \underbrace{(\lambda a + (1 - \lambda)a')}_{\in A} - \underbrace{(\lambda b + (1 - \lambda)b')}_{\in B}$$

Also C_0 is open since $C_0 = \bigcup_{b \in B} \underbrace{(A - b)}_{\text{open}}$.

$0 \notin C_0$ since $A \cap B = \emptyset$. Let $v_0 \in C_0$, define $C = C_0 - v_0$. Then C is open, convex, non-empty and includes 0. Define $p = \mu_C$ (Minkowski gauge):

$$\forall v \in V, p(v) = \inf\{t \geq 0 : v \in tC\}$$

p satisfies (see proof of Kolmogorov)

- p is well-defined
- $p(\lambda v) = \lambda p(v)$, $\forall \lambda > 0$
- $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ (using C convex)
- $p(-v)$ is not necessarily equal to $p(v)$ (C is not necessarily balanced)

Let $f : \mathbb{R}v_0 \rightarrow \mathbb{R}$ be linear defined by $f(-v_0) = 1$. Since $-v_0 \notin C$ ($0 \notin C_0$) we have $p(-v_0) \geq 1$, so $f \leq p$ ($\tilde{f}(-v_0) \leq p(-v_0)$) so $\tilde{f}(-\lambda v_0) \leq p(-\lambda v_0)$ for all $\lambda > 0$, and for $\lambda < 0$ $\tilde{f}(-\lambda v_0) \leq 0$.

The Hahn-Banach theorem (algebraic version) gives $\tilde{f} : V \rightarrow \mathbb{R}$ linear such that $\tilde{f}|_{\mathbb{R}v_0} = f$, $\tilde{f}(-v_0) = 1$. So $\tilde{f} \neq 0$, and since $p < 1$ in C , $\tilde{f}|_C < 1$, so since C is open around 0: there exists $B(0, \varepsilon) \subseteq C$ such that

$$\sup_{v \in B(0, \varepsilon)} \tilde{f}(v) \leq 1 \implies \sup_{v \in B(0, \varepsilon)} |\tilde{f}| \leq 1 \implies \tilde{f} \in V^*, \|\tilde{f}\|_{V^*} \leq \varepsilon^{-1}$$

And

$$\tilde{f}|_C < 1 \implies \tilde{f}|_{C_0} < 0 \implies \sup_A \tilde{f} \leq \inf_B \tilde{f}$$

So there is $\alpha \in \mathbb{R}$ such that $\sup_A \tilde{f} \leq \alpha \leq \inf_B \tilde{f}$

- (ii) $C_0 = B - A$ non-empty, convex, doesn't include 0, is closed: given $(a_n - b_n)_{n \geq 1}$ a sequence in C_0 with $(a_n - b_n) \rightarrow e$, we have (since B is compact), there exists a subsequence $(a_{n'} - b_{n'})_{n' \geq 1}$ such that $b_{n'}$ converges to $b \in B$, so $a_{n'}$ converges to $a \in A$ as A is closed. So $l = a - b \in C_0$.

So there exists an open ball $B(0, \varepsilon)$ such that $B(0, \varepsilon) \cap C_0 = \emptyset$. Apply (i) to $\tilde{A} = B(0, \varepsilon)$ (open, convex, non-empty) and $\tilde{B} = C_0$ (convex, non-empty). Then there exists $f : V \rightarrow \mathbb{R}$ bounded and linear, $f \neq 0$ such that

$$\sup_{B(0, \varepsilon)} f \leq \inf_{C_0} f = \inf_B f - \sup_A f$$

□