Introduction

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1 Some basic aspects of continuous-time Markov Chains

Definition. A sequence of random variables is called a *stochastic process* or *process*. The process $X = (X_n)_{n\geq 1}$ is called a discrete-time Markov Chain with state space I if for all $x_0, x_1, \dots, x_n \in I$

$$\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}).$$

If $\mathbb{P}(X_{n+1} = y | X_n = x)$ is independent of n, the chain is called *time-homogeneous*. We then write $P = (P_{x,y})_{x,y\in I}$ for the *transition matrix* where $P_{x,y} = \mathbb{P}(X_1 = y | X_0 = x)$. The data associated to every time-homogeneous Markov Chain is the transition matrix P and the initial distribution μ, i.e $\mathbb{P}(X_0 = x_0) = \mu(x_0)$.

From now on:

- I denotes a countable (or finite) state space.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which all the relevant random variables are defined.

Definition. $X = (X(t) : t \geq 0)$ is a (right-continuous) continuous-time random process with values in I if

(a) for all $t \geq 0$, $X(t) = X_t$ is a random variable such that $X(t) : \Omega \to I$;

(b) for all $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous (right-continuous sample path). In our case this means for all $\omega \in \Omega$, for all $t \geq 0$, there exists $\varepsilon > 0$ (depending on ω, t) such that

$$X_t(\omega) = X_s(\omega) \ \forall s \in [t, t + \varepsilon].$$

Fact. A right-continuous random process is defined by its finite-dimensional distributions

$$\mathbb{P}(X_{t_0=i}, X_{t_1=i_1}, \dots, X_{t_n} = i_{t_n}),\ n \geq 0,\ t_k \geq 0,\ i_k \in I.$$

For every $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ of a right-continuous process stays constant for a while. So there are 3 possibilities:

(i) The path makes infinitely many jumps overall but only finitely many in a given interval $[0, t]$.

(ii) The path makes finitely many jumps & then gets absorbed in some state.

(iii) The path makes infinitely jumps in a finite time interval. After the 'explosion time' ζ, the process starts up again.

Write $J_0 = 0, J_1, J_2, \ldots$ for the jump times and $S_1, S_2, \ldots$ for the holding times, defined by

$$J_0 = 0,\ J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\},$$
$$S_n = \begin{cases} J_n - J_{n-1} & J_{n-1} < \infty \\ \infty & \text{otherwise} \end{cases}.$$

By right-continuity, $S_n > 0$ for all n. If $J_{n+1} = \infty$ for some n, we define $X_\infty = X_{J_n}$ as the final value, otherwise X_∞ is not defined. The explosion time ζ is defined by

$$\zeta = \sup(J_n) = \sum_{n=1}^{\infty} S_n.$$

We are not going to consider what happens to a chain after explosion. We thus set $X_t = \infty$ for all $t \geq \zeta$ (adjoining a new state '∞'). We call such a chain *minimal*.

Definition. We define the *jump chain* Y_n of $(X_t)_{t\geq 0}$ by setting $Y_n = X_{J_n}$ for all n.

Definition. A right-continuous random process $X = (X_t)_{t\geq 0}$ has the Markov property (and is called a continuous-time markov chain) if for all $i_1, i_2, \ldots, i_n \in I$ and $0 \leq t_1 < t_2 < \ldots < t_n$,

$$\mathbb{P}(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}, \ldots, X_{t_0} = i_0) = \mathbb{P}(X_{t_n} = i_n | X_{t_0} = i_0).$$

Remark. For all $h > 0$, $Y_n = X(hn)$ defines a discrete-time Markov Chain.

Definition. The transition probabilities are $P_{ij}(s,t) = \mathbb{P}(X_t = j | X_s = i)$, $s \leq t$, $i, j \in I$. It is called *time-homogeneous* if it depends on $t - s$ only, i.e

$$P_{ij}(s,t) = P_{i,j}(0, t-s).$$

In this case we just write $P_{ij}(t-s)$. As in the case of discrete time, a (time-homogeneous) Markov process is characterised by

1. Its initial distribution $\lambda_i = \mathbb{P}(X_0 = i)$, $i \in I$;
2. Its *family of transition matrices* $(P(t))_{t\geq 0} = (P_{ij}(t))_{t\geq 0}$.

The family $(P(t))_{t\geq 0}$ is called the *transition subgroup* of the MC.

A (time-homogeneous) Markov process is characterised by

- its initial distribution;
- its transition subgroup $(P(t))_{t\geq 0}$

$$(P(t))_{t\geq 0} = (P(t))_{\substack{i,j\in I\\ t\geq 0}} = (\mathbb{P}(X_t = j|X_0 = i))_{\substack{i,j\in I\\ t\geq 0}}$$

It is easy to see that

- $P(0)$ is the identity
- $P(t)$ is a stochastic matrix for all t (i.e rows sum to 1)
- $P(t+s) = P(t)P(s)\ \forall s,t$ (Chapman-Kolmogorov equation)

$$\begin{aligned}
P_{xz}(t+s) &= \mathbb{P}(X_{t+s} = z|X_0 = x)\\
&= \sum_{y\in I}\mathbb{P}(X_{t+s} = z|X_0 = x, X_t = y)\mathbb{P}(X_t = y|X_0 = x)\\
&= \sum_{y\in I}\mathbb{P}(X_s = z|X_0 = y)\mathbb{P}(X_t = y|X_0 = x)\\
&= \sum_{y\in I} P_{yz}(s)P_{xy}(t) = P_{x\cdot}(t)P_{\cdot z}(s)
\end{aligned}$$

Holding times

Let X be a (right-continuous continuous-time time-homogeneous) Markov Chain on a countable state-space I.

Suppose X starts from $x \in I$. Question: how long does X stay in the state x?

Definition. We call S_x the *holding time* at state x ($S_x > 0$ by right-continuity).

Let $s,t \geq 0$. Then

$$\begin{aligned}
\mathbb{P}(S_x > t+s|S_x > s) &= \mathbb{P}(X_u = x\ \forall u\in[0,t+s]|X_u = x\ \forall u \in [0,s])\\
&= \mathbb{P}(X_u = x\ \forall u\in[s,t+s]|X_u = x\ \forall u \in [0,s])\\
&= \mathbb{P}(X_u = x\ \forall u\in[s,t+s]|X_s = x)\\
&= \mathbb{P}(X_u = x\ \forall u\in[0,t]|X_0 = x)\\
&= \mathbb{P}(S_x > t).
\end{aligned}$$

Thus S_x has the memoryless property.

By the next theorem, we will get that S_x has the exponential distribution, say with parameter q_x.

Theorem 1.1 (Memoryless property). *Let S be a positive random variable. Then S has the memoryless property, i.e $\mathbb{P}(S > t + s | S > s) = \mathbb{P}(S > t)$ for all $s, t \geq 0$ if and only if S has the exponential distribution.*

Proof. It is easy to see the exponential distribution is memoryless. So we prove the other direction. Set $F(t) = \mathbb{P}(S > t)$. Then $F(s + t) = F(s)F(t)$ for all $s, t \geq 0$.

Since S is a positive random variable, there exists $n \in \mathbb{N}$ large such that $F(1/n) = \mathbb{P}(S > 1/n) > 0$. Then $F(1) = F(1/n)^n > 0$. So we can set $F(1) = e^{-\lambda}$ for some $\lambda \geq 0$.

For $k \in \mathbb{N}$, $F(k) = F(1)^k = e^{-\lambda k}$. For p/q rational, $F(p/q) = F(1/q)^p = (F(1/q)^q)^{p/q} = F(1)^{p/q} = e^{-\lambda \frac{p}{q}}$.

For any $t \geq 0$, for any $r, s \in \mathbb{Q}$ such that $r \leq t \leq s$, since F is decreasing

$$e^{-\lambda s} = F(s) \leq F(t) \leq F(r) = e^{-\lambda r}.$$

So taking sequences of rationals approaching t, we have $F(t) = e^{-\lambda t}$. $\square$

Poisson Process'

We are now going to look at the simplest (and most important) example of continuous time Markov Chains - the Poisson process.

Definition. Suppose $S_1, S_2, \ldots$ are iid random variables with $S_1 \sim \text{Exp}(\lambda)$. Define the *jump times* $J_0 = 0, J_1 = S_1$, $J_n = S_1 + \ldots + S_n$ for all n, and set $X_t = i$ if $J_i \leq t < J_{i+1}$. Then $I = \{0, 1, 2, \ldots\}$ and note that X is right-continuous and increasing. X is called a *Poisson process* of parameter/intensity λ. We sometimes refer to the jump times $(J_i)_{i \geq 1}$ as the *points* of the Poisson process, then X =number of points in $[0, t]$.

Theorem 1.2 (Markov property). *Let $(X_t)_{t \geq 0}$ be a Poisson process of intensity λ. Then for all $s \geq 0$, the process $(X_{s+t} - X_s)_{t \geq 0}$ is also a Poisson process of intensity λ, and is independent of $(X_t)_{0 \leq t \leq s}$.*

Proof. Set $Y_t = X_{t+s} - X_s$ for all $t \geq 0$. Let $i \in \{0, 1, 2, \ldots\}$ and condition on $\{X_s = i\}$, Then the jump times for the process Y are $J_{n+1} - s, J_{n+2} - s, \ldots$ and the holding times are

$$\begin{aligned}
T_1 &= J_{n+1} - s = S_{i+1} - (s - J_i) \\
T_2 &= S_{i+2} \\
T_3 &= S_{i+3} \\
&\vdots
\end{aligned}$$

Since $\{X_s = i\} = \{J_i \leq s\} \cap \{s < J_{i+1}\} = \{J_i \leq s\} \cap \{S_{i+1} > s - J_i\}$, conditional on $\{X_s, i\}$, by the memoryless property of the exponential distribution (and independence of S_{i+1} and J_i) we see that $T_1 \sim \text{Exp}(\lambda)$. Moreover the times J_j, $j \geq 2$ are independent of S_k, $k \leq i$ and hence independent of $(X_r)_{r \leq s}$, and they have iid $\text{Exp}(\lambda)$ distribution. Thus $((X_{s+t} - X_s))_{t \geq 0}$ is a Poisson process of parameter λ and is independent of $(X_t)_{0 \leq t \leq s}$. $\square$

Similar to this, one can show the Strong Markov property for a Poisson process of parameter λ. Recall a random variable $T \in [0, \infty]$ is called a *stopping time* if for all t, the event $\{T \leq t\}$ depends only on $(X_s)_{s \leq t}$.

Theorem 1.3 (Strong Markov property)**.** *Let $(X_t)_{t \geq 0}$ be a Poisson process of parameter λ and T a stopping time. Then conditional on $T < \infty$, the process $(X_{T+t} - X_T)_{t \geq 0}$ is a Poisson process of parameter λ and independent of $(X_s)_{s \leq T}$.*