1 Conditional Expectation

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(X_i)_{i \in I}$ be a collection of random variables defined on this space. Then we define $\sigma(X_i : i \in I) \subseteq \mathcal{F}$ to be the smallest σ-algebra such that all of the X_i are measurable, i.e

$$\sigma(X_i : i \in I) = \sigma(X_i^{-1}(B) : i \in I,\ B \in \mathcal{B}(\mathbb{R})).$$

Definition. If $B \in \mathcal{F}$ has $\mathbb{P}(B) > 0$ then we define

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

for any $A \in \mathcal{F}$. Furthermore, if X is an integrable random variable we define

$$\mathbb{E}[X|B] = \frac{\mathbb{E}[X \mathbb{1}(B)]}{\mathbb{P}(B)}.$$

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say a σ-algebra $\mathcal{G}$ is countably generated if there exist $(B_i)_{i \in I}$ pairwise disjoint (with I countable) such that $\bigcup_{i \in I} B_i = \Omega$ and $\mathcal{G} = \sigma(B_i : i \in I)$.

Let X be an integrable random variable and $\mathcal{G}$ a countably generated σ-algebra. We want to define $X' = \mathbb{E}[X|\mathcal{G}]$. So define

$$X'(\omega) = \mathbb{E}[X|B_i] \text{ whenever } \omega \in B_i.$$

Or equivalently,

$$X'(\omega) = \sum_{i \in I} \mathbb{E}[X|B_i] \mathbb{1}(\omega \in B_i)$$

where we use the convention that $\mathbb{E}[X|B_i] = 0$ if $\mathbb{P}(B_i) = 0$. Then X' is indeed $\mathcal{G}$-measurable (note $\mathcal{G}$ is the set of $\bigcup_{j \in J} B_j$ for $J \subseteq I$).

Note that for any $G \in \mathcal{G}$ we have $\mathbb{E}[X \mathbb{1}(G)] = \mathbb{E}[X' \mathbb{1}(G)]$. Also

$$\mathbb{E}[|X'|] \leq \mathbb{E}\left[\sum_{i \in I} \mathbb{E}[|X||B_i] \mathbb{1}(B_i)\right] = \sum_{i \in I} \mathbb{E}[|X||B_i] \mathbb{P}(B_i) = \mathbb{E}|X| < \infty$$

so X' is integrable.

Theorem (Monotone convergence theorem)**.** *Let $(X_n)_{n \geq 1}$ be a sequence of non-negative random variables with $X_n \uparrow X$ as $n \to \infty$ almost surely. Then $\mathbb{E}X_n \uparrow \mathbb{E}X$ as $n \to \infty$.*

Proof. See Part II Probability & Measure. □

Theorem (Dominated convergence theorem)**.** *Let $(X_n)_{n \geq 1}$ be a sequence of random variables with $X_n \to X$ as $n \to \infty$ almost surely and $|X_n| \leq Y$ almost surely for some Y integrable. Then $\mathbb{E}X_n \to \mathbb{E}X$ as $n \to \infty$.*

Proof. See Part II Probability & Measure. □

Definition (L^p)**.** Let $p \in [1, \infty]$ and f be a measurable function. Define the *L^p-norm*

$$||f||_p = (\mathbb{E}[|f|^p])^{1/p} \text{ for } p \in [1, \infty)$$
$$||f||_\infty = \inf\{\lambda : |f| \leq \lambda \text{ a.e}\}.$$

Furthermore write $f \sim g$ if $f = g$ almost everywhere. Then define the L^p-space $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) = \{f : ||f||_p < \infty\}/\sim$.

Theorem ($\mathcal{L}^2$ is a Hilbert space)**.** *$\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space with inner product $\langle U, V \rangle = \mathbb{E}[UV]$. For a closed subspace $\mathcal{H}$, if $f \in \mathcal{L}^2$ there exists a unique $g \in \mathcal{H}$ with $||f - g||_2 = \inf\{||f - h||_2 : h \in \mathcal{H}\}$ and $\langle f - g, h \rangle = 0$ for all $h \in \mathcal{H}$. g is called the orthogonal projection of f on $\mathcal{H}$.*

Proof. See Part II Probability & Measure. □

Theorem (Conditional expectation)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ a sub σ-algebra, $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists an integrable random variable Y satisfying*

(a) Y is $\mathcal{G}$-measurable;

(b) for all $A \in \mathcal{G}$, $\mathbb{E}[X\mathbb{1}(A)] = \mathbb{E}[Y\mathbb{1}(A)]$.

Moreover Y is unique, in the sense that if Y' also satisfies (a) and (b), then $Y = Y'$ almost surely. We call Y a version of the conditional expectation of X given $\mathcal{G}$. We write $Y = \mathbb{E}[X|\mathcal{G}]$ almost surely. If $\mathcal{G} = \sigma(Z)$ for a random variable Z, then we write $\mathbb{E}[X|Z] = \mathbb{E}[X|\mathcal{G}]$.

Remark. (b) could be replaced by $\mathbb{E}[XZ] = \mathbb{E}[YZ]$ for all Z bounded and $\mathcal{G}$-measurable.

Proof. First we show uniqueness. Suppose Y and Y' both satisfy (a) and (b) and let $A = \{Y > Y'\} \in \mathcal{G}$. Then

$\mathbb{E}[Y\mathbb{1}(A)] = \mathbb{E}[Y'\mathbb{1}(A)] \Rightarrow \mathbb{E}[(Y - Y')\mathbb{1}(A)] = 0 \Rightarrow \mathbb{P}(Y > Y') = 0 \Rightarrow Y \leq Y'$ a.s

and similarly $Y \geq Y'$ a.s.

Now we show existence. First assume $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Then $\mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$ is a closed subspace of $\mathcal{L}^2(\mathcal{F})$. Hence

$$\mathcal{L}^2(\mathcal{F}) = \mathcal{L}^2(\mathcal{G}) \oplus \mathcal{L}^2(\mathcal{G})^\perp$$

so we can write $X = Y + Z$ for $Y \in \mathcal{L}^2(\mathcal{G})$ and $Z \in \mathcal{L}^2(\mathcal{G})^\perp$. Define $\mathbb{E}[X|\mathcal{G}] = Y$, so Y is $\mathcal{G}$-measurable and for all $A \in \mathcal{G}$

$$\mathbb{E}[X\mathbb{1}(A)] = \mathbb{E}[Y\mathbb{1}(A)] + \underbrace{\mathbb{E}[Z\mathbb{1}(A)]}_{=0} = \mathbb{E}[Y\mathbb{1}(A)].$$

We claim that if $X \geq 0$ almost surely, then $Y \geq 0$ almost surely. Indeed, let $A = \{Y < 0\} \in \mathcal{G}$ so $0 \leq \mathbb{E}[X\mathbb{1}(Y < 0)] = \mathbb{E}[Y\mathbb{1}(Y < 0)] \leq 0$ which implies $\mathbb{P}(Y < 0) = 0$.

Assume now that $X \geq 0$ almost surely. Define $X_n = X \wedge n \leq n$, so $X_n \in \mathcal{L}^2$ for all n. Let $Y_n = \mathbb{E}[X_n|\mathcal{G}]$. Then X_n is an increasing sequence and by the above claim, Y_n is also an increasing sequence almost surely. Define $Y = \limsup_{n\to\infty} Y_n$, so Y is $\mathcal{G}$-measurable. Also $Y =\uparrow \lim_{n\to\infty} Y_n$ almost surely. For any $A \in \mathcal{G}$ we have

$$\mathbb{E}[X\mathbb{1}(A)] = \lim_{n\to\infty} \mathbb{E}[X_n\mathbb{1}(A)] = \lim_{n\to\infty} \mathbb{E}[Y_n\mathbb{1}(A)] = \mathbb{E}[Y\mathbb{1}(A)]$$

by the Monotone Convergence Theorem.

Finally, for general X write $X = X^+ - X^-$ and define $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X^+|\mathcal{G}] - \mathbb{E}[X^-|\mathcal{G}]$. $\square$