Methods Lecture Notes

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Part I: self-adjoint ODEs

1 Fourier Series

1.1 Periodic Functions

A function $f(x)$ is periodic if

$$f(x+T) = f(x) \ \forall x$$

where T is the period.

Example: Simple harmonic motion

$$y = A \sin \omega t$$

where A is the amplitude and period $T = \frac{2\pi}{\omega}$, with angular frequency ω. In space, refer to wavelength $\lambda = \frac{2\pi}{k}$ and (angular) wavenumber $k\frac{2\pi}{\lambda}$.

Properties of sine and cosine functions:

Consider the set of functions

$$g_n(x) = \cos \frac{n\pi x}{L}, \ h_n(x) = \sin \frac{n\pi x}{L}$$

which are periodic on the interval $0 \leq x < 2L$. Recall the identites

$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

Define an inner product

$$\langle f, g \rangle = \int_0^{2L} f(x)g(x)\mathrm{d}x \qquad (*)$$

The functions g_n, h_n are mutually orthogonal on the interval $0 \le x < 2L$ with respect to $(*)$.

$$\begin{aligned}\langle h_n, h_m\rangle &= \int_0^{2L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\mathrm{d}x \\ &= \frac{1}{2}\int_0^{2L}\left(\cos(n-m)\frac{\pi x}{L} - \cos(n+m)\frac{\pi x}{L}\right)\mathrm{d}x \\ &= \frac{1}{2}\frac{L}{\pi}\left[\frac{\sin(n-m)\frac{\pi x}{L}}{n-m} - \frac{\sin(n+m)\frac{\pi x}{L}}{n+m}\right]_0^{2L} \\ &= 0 \text{ for all } n \neq m\end{aligned}$$

For $n = m$,

$$\langle h_n, h_n\rangle = \int_0^{2L} \sin^2\frac{n\pi x}{2}\mathrm{d}x = L$$

Hence

$$\langle h_n, h_m\rangle = \begin{cases} L\delta_{nm} & \forall n, m \neq 0 \\ 0 & m = 0\end{cases} \tag{1.1}$$

Similarly

$$\langle g_n, g_m\rangle = \int_0^{2L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\mathrm{d}x = \begin{cases} L\delta_{nm} & \forall n, m \neq 0 \\ 2L\delta_{0n} & m = 0\end{cases} \tag{1.2}$$

$$\langle h_n, g_m\rangle = \int_0^{2L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\mathrm{d}x = 0,\ \forall n, m \tag{1.3}$$

We assert that the functions g_n, h_n form a complete orthogonal set, i.e they span the space of 'well behaved' periodic functions on $0 \le x < 2L$ and they are linearly independent.

1.2 Definition of a Fourier Series

We can express any 'well behaved' periodic function with period $2L$ as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{1.4}$$

where a_n, b_n are constants such that the RHS is convergent for all x where f is continuous at x. At a discontinuous point, the series approaches the midpoint

$$\frac{1}{2}(f(x_+) + f(x_-))$$

Consider $\langle h_m, f\rangle$ and substitute (1.4) i.e

$$\int_0^{2L} \sin\frac{m\pi x}{L} f(x)\mathrm{d}x = \sum_{n=1}^{\infty} Lb_N\delta_{nm} = Lb_m$$

By the orthogonality relations $(1.1 - 1.3)$. Hence we find

$$\begin{aligned} b_n &= \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \\ a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\mathrm{d}x \end{aligned} \tag{1}$$

Notes:

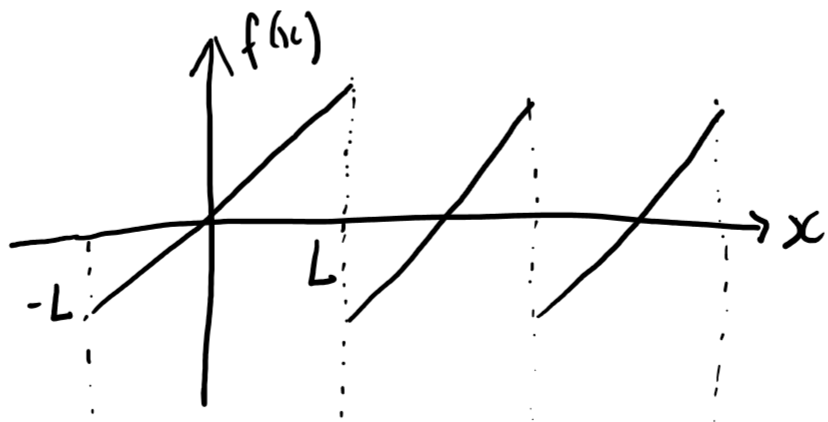
(i) a_n includes $n = 0$ since $\frac{1}{2}a_0$ is the average. $\langle f(x)\rangle = \frac{1}{2L}\int_0^{2L} f(x)\mathrm{d}x$

(ii) Range of integration is one period so $\int_0^{2L}\mathrm{d}x \equiv \int_{-L}^{L}\mathrm{d}x$ etc

(iii) Can think of Fourier series as a decomposition into harmonics. Simplest Fourier series are sine and cosine functions.

Classic example: "sawtooth" wave

Consider $f(x) = x$ for $-L \leq x < L$ and periodic elsewhere.



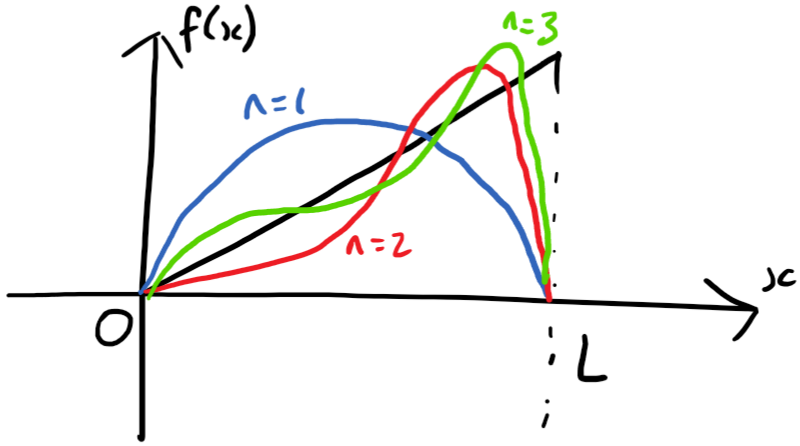
Here, we have $a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{2}\mathrm{d}x = 0$ for all n and have

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\mathrm{d}x \\ &= -\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{L}\right]_0^L + \frac{2}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\mathrm{d}x \\ &= -\frac{2L}{n\pi}\cos n\pi + \frac{2L}{(n\pi)^2}\sin n\pi \\ &= \frac{2L}{n\pi}(-1)^{n+1} \end{aligned}$$

So the "sawtooth" Fourier series is

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{L}$$

which is clearly convergent.



Recall the 'sawtooth' FS:

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}$$

Comments: As n increases

(i) FS approximation improves (convergent where continuous)

(ii) FS $\to 0$ at $x + L$, i.e midpoint of dicontinuity

(iii) FS has a persistent 'overshoot' at $x = L$ (approximately 9% known as the Gibbs phenomenon)

1.3 The Dirichlet Conditions (Fourier's Theorem)

Sufficiency conditions for a "well behaved" function to have a unique FS.

If $f(x)$ is a bounded function (with period $2L$) with a finite number of minima, maxima and discontinuities in $0 \le x < 2L$, then the FS converges to $f(x)$ at all points where its continuous, and at discontinuous points converges to midpoint $\frac{1}{2}\left[f(x_+) + f(x_-)\right]$

Notes:

(i) These are weak conditions (contrast to Taylor series) but pathological functions are excluded

(ii) Converse is not true (e.g $\sin 1/x$ has a FS)

(iii) Proof is difficult (see Jeffreys&Jeffreys)

1.4 Convergence of FS

Rate of convergence depends on 'smoothness'

Theorem 1.1. *If $f(x)$ has continuous derivatives up to a pth derivative which is discontinuous, then FS converges as $\mathcal{O}(n^{-(p+1)})$ as $n \to \infty$.*

Example ($p = 0$): 'Square wave'. If

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ -1 & -1 \le x < 0 \end{cases}$$

Then FS is

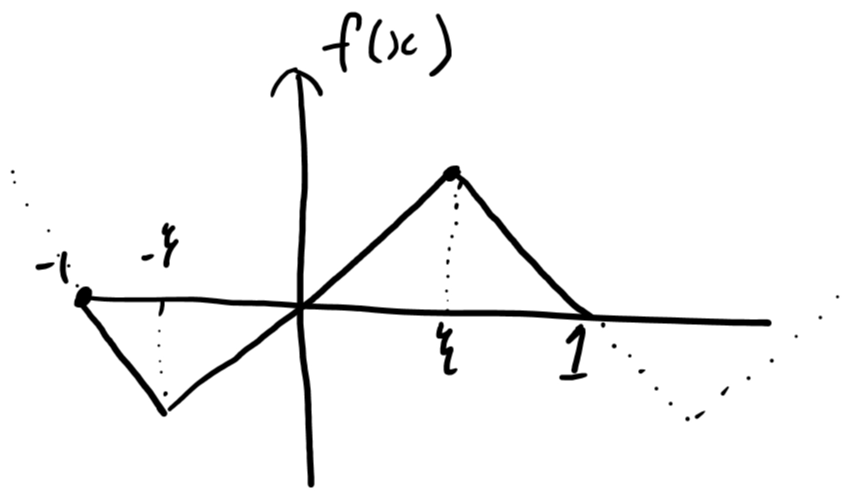
$$f(x) = 4 \sum_{m=1}^{\infty} \frac{\sin(2m-1)\pi x}{(2m-1)\pi} \tag{1.7}$$

Exercise ($p = 1$): General "see-saw" wave. If

$$f(x) = \begin{cases} x(1-\xi) & 0 \le x < \xi \\ \xi(1-x) & \xi \le x < 1 \end{cases}$$

and $f(-x) = -f(x)$ for $0 < x \le 1$. Can show the FS is

$$f(x) = 2\sum_{m=1}^{\infty} \frac{\sin(n\pi\xi)\sin(n\pi x)}{(n\pi)^2} \tag{1.8}$$



For $\xi = 1/2$, show

$$f(x) = 2\sum_{m=1}^{\infty} (-1)^{m+1} \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^2}$$

Exercise ($p = 2$): Take $f(x) = \frac{1}{2}x(1-x)$, $0 \le x < 1$ and odd for $-1 \le x < 0$. Show FS is

$$f(x) = 4\sum_{n=1}^{\infty} \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^3} \tag{1.9}$$

Example ($p = 3$): $f(x) = (1-x^2)^2$ with FS $a_n = \mathcal{O}(1/n^4)$.

1.5 Integration of FS

It is always valid to integrate the FS (1.4) of $f(x)$ term-by-term to obtain

$$F(x) = \int_{-L}^{x} f(x)\mathrm{d}x$$

because $F(x)$ satisfies Dirichlet conditions if $f(x)$ does.

1.6 Differentiation of FS

Take care with term-by-term differentiation.

Conterexample: Take "square wave" FS (1.7) and find

$$f'(x) =^{?} 4 \sum_{m=1}^{\infty} \cos(2m-1)\pi x$$

which is unbounded!

Theorem 1.2. *If $f(x)$ is continuous and satisfies Dirichlet conditions and $f'(x)$ satisfies Dirichlet conditions, then $f'(x)$ can be found by term-by-term differentation of FS (1.4) of $f(x)$.*

Exercise: Differentiate "seesaw" FS (1.8) with $\xi = 1/2$ to find offset "square wave" FS (1.7)

1.7 Pareseval's Theorem

Relation between the integral of the squate of a function and the square of Fourier coefficients:

$$\begin{aligned}
\int_0^{2L} [f(x)]^2 \mathrm{d}x &= \int_0^{2L} \left[\frac{1}{2}a_0 + \sum_n a_n \cos\frac{n\pi x}{L} + \sum_n b_n \sin\frac{n\pi x}{L}\right]^2 \mathrm{d}x \\
&= \int_0^{2L} \left[\frac{1}{4}a_0^2 + \sum_n a_n^2 \cos^2\frac{n\pi x}{L} + \sum_n b_n^2 \sin^2\frac{n\pi x}{L}\right] \mathrm{d}x \\
&= L\left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right] \qquad (1.10)
\end{aligned}$$

Also called the completeness relation because LHS$\geq$RHS if any basis functions are missing.

Example: Sawtooth wave $f(x) = x$, $-L \leq x \leq L$.

$$\text{LHS} = \int_{-L}^{L} x^2 \mathrm{d}x = \frac{2}{3}L^3$$

$$\text{RHS} = L\sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2} = \frac{4L^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

And so $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Aside: Pareseval for functions $\langle f, f\rangle$ is equivalent to Pythagoras for vectors.

1.8 Alternative Fourier Series

Half-range series

Consider $f(x)$ defined only on $0 \leq x < L$. We can extend its range over $-L \leq x < L$ in two simple ways:

(i) Require it to be odd $f(-x) = -f(x)$. Then $a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \tag{1.11}$$

So $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ which is a fourier sine series.

(ii) Require it to be even $f(-x) = f(x)$, Then $b_n = 0$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\mathrm{d}x \tag{1.12}$$

So $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

Complex representation

Recall

$$\cos\frac{n\pi x}{L} = \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right),\ \sin\frac{n\pi x}{L} = \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)$$

So FS (1.4) becomes

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{npix}{L} \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n - ib_n)e^{\frac{in\pi x}{L}} + \sum_{n=1}^{\infty}(a_n + ib_n)e^{-\frac{in\pi x}{L}} \\ &= \sum_{m=-\infty}^{\infty} c_m e^{\frac{im\pi x}{L}} \end{aligned} \tag{1.13}$$

where for $m > 0$, $m = n$, $c_m = \frac{1}{2}(a_n - ib_n)$ and $m < 0, n = -m$, $c_m = \frac{1}{2}(a_{-m} + ib_{-m})$ and $m = 0, c_0 = \frac{1}{2}a_0$. Equivalently

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{im\pi x}{L}}\,\mathrm{d}x \tag{1.14}$$

with inner product $(*)$ upgraded to

$$\langle f, g\rangle = \int f^* g\,\mathrm{d}x$$

Orthogonality:
$$\int_{-L}^{L} e^{-\frac{im\pi x}{L}} e^{\frac{in\pi x}{L}} \mathrm{d}x = 2L\delta_{mn} \tag{1.15}$$
Pareseval:
$$\int_{-L}^{L} |f(x)|^2 \mathrm{d}x = 2L \sum_{m=-\infty}^{\infty} |c_m|^2$$

1.9 Some FS motivation

Self-adjoint matrices

Suppose $\mathbf{u}, \mathbf{v}$ are complex N-vectors with inner product
$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\dagger \mathbf{v} \tag{1.16}$$
The $N \times N$ matrix A is self-adjoint (or Hermitian) uf
$$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, A\mathbf{v} \rangle, \ \forall \mathbf{u}, \mathbf{v}$$
i.e $A^\dagger = A$. The eigenvalues λ_n and eigenvectors $\mathbf{v}_n$ satisfy
$$A\mathbf{v}_n = \lambda_n \mathbf{v}_n \tag{1.17}$$
and have the following properties

(i) Evals are real $\lambda_n^* = \lambda_n$

(ii) If $\lambda_n \neq \lambda_m$ then evecs are orthogonal
$$\langle \mathbf{v}_n, \mathbf{v}_m \rangle = 0$$

(iii) We can rescale to make an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$.

Given $\mathbf{b}$, we can solve for $\mathbf{x}$ in
$$A\mathbf{x} = \mathbf{b} \tag{1.18}$$
Express $\mathbf{b} = \sum_{n=1}^{N} b_n \mathbf{v}_n$ (b_n's known). Seek a solution $\mathbf{x} = \sum_{n=1}^{N} c_n \mathbf{v}_n$ (c_n's unknown). Substituting into (1.18)
$$A\mathbf{x} = \sum_n Ac_n \mathbf{v}_n = \sum_n c_n \lambda_n \mathbf{v}_n = \sum_n b_n \mathbf{v}_n$$
By orthogonality $c_n \lambda_n = b_n \implies c_n = \frac{b_n}{\lambda_n}$, so solution is
$$\mathbf{x} = \sum_{n=1}^{N} \frac{b_n}{\lambda_n} \mathbf{v}_n \tag{1.19}$$

Solving inhomogeneous ODE with FS

We wish to find $y(x)$ given $f(x)$ for

$$\mathcal{L}y \equiv -\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(x) \tag{1.20}$$

with boundary conditions $y(0) = y(L) = 0$. The related eigenvalue problem is

$$\mathcal{L}y_n = \lambda_n y_n,\ y_n(0) = y_n(L) = 0$$

(iu.e $y_n'' = \lambda_n y_n$) has eigenvalues and evals

$$y_n(x) = \sin\frac{n\pi x}{L},\ \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{1.21}$$

Try $y(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$

Expand $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ with (1.11)

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x$$

Substitute in (1.20):

$$\mathcal{L}y = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\sum_n c_n \sin\frac{n\pi x}{L}\right) = \sum_n c_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} = \sum_n b_n \sin\frac{n\pi x}{L}$$

By orthogonality (1.11), we have

$$c_n\left(\frac{n\pi}{L}\right)^2 = b_n,\ c_n = \frac{b_n}{\left(\frac{n\pi}{L}\right)^2}$$

So solution is

$$y(x) = \sum_n \frac{b_n}{\left(\frac{n\pi}{L}\right)^2}\sin\frac{n\pi x}{L} = \sum_n \frac{b_n}{\lambda_n} y_n \tag{1.22}$$

Example: 'square wave' source ($L = 1$) $f(x) = 1, 0 \leq x < 1$ (odd function). This has FS (1.7) $f(x) = 4\sum_m \frac{\sin(2m-1)\pi x}{(2m-1)\pi}$ so the solutions (1.22) should be

$$y(x) = \sum_n \frac{b_n}{\lambda_n} y_n = 4\sum_m \frac{\sin(2m-1)\pi x}{((2m-1)\pi)^3}$$

But this is FS (1.9) for

$$y(x) = \frac{1}{2}x(1-x) \tag{1.23}$$

Exercise: Integrate $\mathcal{L}y = 1$ directly with BCs to verify soln (1.23)

2 Sturm-Liouville Theory

2.1 Review of second-order linear ODEs

We wish to solve the general inhomogeneous ODE

$$\mathcal{L}y \equiv \alpha(x)y'' + \beta(x)y' + \gamma(x)y = f(x) \tag{2.1}$$

- The homogeneous equation
$$\mathcal{L}y = 0 \tag{2.2}$$
has two independent solutions, $y_1(x), y_2(x)$, the complementary function $y_c(x)$ is the general sol of (2.2)
$$y_c(x) = Ay_1(x) + By_2(x) \tag{2.3}$$
where A, B are constants

- The inhomogeneous equation
$$\mathcal{L} = f(x) \tag{2.4}$$
has a special solution $y_p(x)$ called the particular integral. The general solution of (2.4) is then
$$y(x) = y_p(x) + Ay_1(x) + By_2(x) \tag{2.5}$$

- Two boundary or initial conditions are required to determine A, B

  (a) Boundary conditions. Solve (2.4) on $a \leq x \leq b$ given y at $x = a, b$ (Dirichlet) or given y' at $x = a, b$ (Neumann) or mixed $y + ky'$ etc. Homogeneous boundary conditions are often assumed ($y(a) = y(b) = 0$) and these admit the trivial solution $y \equiv 0$. Can always make the b.c's homogeneous by adding complementary function y_c
  $$\tilde{y} = y + Ay_1 + By_2 \text{ such that } \tilde{y}(a) = \tilde{y}(b) = 0$$

  (b) initial data solve (2.4) for $x \geq a$, given y, y' at $x = a$

General eigenvalue problems

To solve (2.1) employing eigenfunction expansions, we must first solve the related eigenvalue problem

$$\alpha(x)y'' + \beta(x)y' + \gamma(x)y = -\lambda\rho(x)y \tag{2.6}$$

with specified boundary conditions. This fowm often occurs after seperation of variables for PDEs in several dimensions.

2.2 Self-adjoint operators

Inner product: For two (complex valued) functions f, g on $a \le x \le b$ define

$$\langle f, g \rangle = \int_a^b f^*(x) g(x) \mathrm{d}x$$

(Later, will assume real f, g) Norm $||f|| = \langle f, f \rangle^{1/2}$.

Sturm-Liouville equation

The eigenvalue problem (2.6) greatly simplifies if $\mathcal{L}$ is self-adjoint, i.e it can be expressed in Sturm-Liouville form

$$\mathcal{L}y \equiv -(py')' + qy = \lambda wy \tag{2.7}$$

where the weight function $w(x)$ is non-negative.

Converging to S-L form

Multiply (2.6) by an integrating factor $F(x)$ to find

$$F\alpha y'' + F\beta y' + F\alpha y = -\lambda F\rho y$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(F\alpha y') - F'\alpha y' - F\alpha' y + F\beta y' + F\gamma y = -\lambda F\rho y$$

Eliminating y' terms $\Rightarrow F'\alpha = F(\beta - \alpha') \Rightarrow \frac{F'}{F} = \frac{\beta - \alpha'}{\alpha}$ thus

$$F(x) = \exp \int \frac{\beta - \alpha'}{\alpha} \mathrm{d}x \tag{2.8}$$

and $(F\alpha y')' + F\gamma y = -\lambda F\rho y$ so $p(x) = F(x)\alpha(x)$, $q(x) = F(x)\gamma(x)$ and $w(x) = F(x)\rho(x)$ (note $F(x) > 0$).

Example: Put Hermite equation for Simple Harmonic Oscillator

$$y'' - 2xy' + 2ny = 0$$

into S-L form (2.7). Comparing with (2.6), $\alpha = 1, \beta = -2x, \gamma = 0, \lambda\rho = 2n$. By (2.8)

$$F = \exp \int \frac{-2x - 0}{1} \mathrm{d}x = e^{-x^2}$$

Hence

$$\mathcal{L}y \equiv -(e^{-x^2} y')' = 2ne^{-x^2} y \tag{2.9}$$

Self-adjoint definition

$\mathcal{L}$ is self-adjoint on $a \leq x \leq b$ for all pairs of functions y_1, y_2 satisfying appropriate boundary conditions if

$$\langle y_1, \mathcal{L}y_2\rangle = \langle \mathcal{L}y_1, y_2\rangle$$

or

$$\int_a^b y_1^*(x)\mathcal{L}y_2(x)\mathrm{d}x = \int_a^b (\mathcal{L}y_1(x))^* y_2(x)\mathrm{d}x \tag{2.10}$$

Boundary conditions: substitute S-L form (2.7) into (2.10) to find

$$\langle y_1, \mathcal{L}y_2\rangle - \langle \mathcal{L}y_1, y_1\rangle = \int_a^b \left[-y_1(py_2')' + y_1 q y_2 + y_2(py_1')' - y_2 q y_1\right] \mathrm{d}x$$

$$= \int_a^b \left[-(py_1y_2')' + (py_1'y_2)'\right] \mathrm{d}x = \left[-py_1y_2' + py_1'y_2\right]_a^b \tag{2.11}$$

Which equals 0 for suitible boundary conditions at $x = a, b$.

Self-adjoint compatible boundary conditions include:

- Homogeneous b.c's $y(a) = y(b) = 0$ or $y'(a) = y'(b) = 0$ or mixed $y = ky' = 0$ etc. The regular S-L problem is defined to be equivalent to homogeneous b.c's
- Periodic: $y(a) = y(b)$
- Singular points of the ODE: $p(a) = p(b) = 0$
- Combinations of the above

2.3 Properties of self-adjoint operators

1. Eigenvalues λ_n are real
2. Eigenfunctions y_n are orthogonal
3. Eigenfunctions y_n are a complete set

1. Real Eigenvalues: Given

$$\mathcal{L}y_n = \lambda_n w y_n \tag{2.12}$$

take complex conjugate

$$\mathcal{L}y_n^* = \lambda_n^* w y_n^*$$

($\mathcal{L}$ and w real). Consider

$$\int_a^b (y_n^*\mathcal{L}y_n - y_n\mathcal{L}y_n^*)\mathrm{d}x = (\lambda_n - \lambda_n^*)\int_a^b w y_n^* y_n \mathrm{d}x$$

but LHS $= 0$ by (2.10) and so $\lambda_n = \lambda_n^* \Rightarrow \lambda_n$ is real.

2. Orthogonal eigenfunctions: Consider (2.12) with 2nd eval $\lambda_m \neq \lambda_n$, $\mathcal{L}y_m = \lambda_m w y_m$. Then from (2.10)

$$\int_a^b (y_m \mathcal{L}y_n - y_n \mathcal{L}y_m)\mathrm{d}x = (\lambda_n - \lambda_m)\int_a^b wy^n y^m \mathrm{d}x = 0 \text{ by } (2.10)$$

But since $\lambda_m \neq \lambda_n$,

$$\int_a^b wy_m y_n \mathrm{d}x = 0 \ \forall n \neq m \tag{2.13}$$

so y_n, y_m are orthogonal wrt $w(x)$ on $a \leq x \leq b$

Define inner product wrt weight $w(x)$ on $a \leq x \leq b$ as

$$\langle f, g\rangle_w = \int_a^b w(x) f(x)^* g(x)\mathrm{d}x = \langle wf, g\rangle = \langle f, wg\rangle \tag{2.14}$$

So the orthogonality relation (2.13) becomes

$$\langle y_n, y_m\rangle_w = 0 \ \forall n \neq m \tag{2.15}$$

Aside: watch the weight! We can eliminate $w(x)$ by redifining $\tilde{y} = \sqrt{w}y$ and replacing $\mathcal{L}y$ by $\frac{1}{\sqrt{w}}(\frac{\tilde{y}}{\sqrt{w}})$, but it is generally simpler to keep $w(x)$!

Exercise: for the hermite equation (2.9) eliminate w with $\tilde{y} = e^{-\frac{x^2}{2}}y$ to find

$$\tilde{\mathcal{L}}y = -\tilde{y}'' + (x^2 - 1)\tilde{y} = 2n\tilde{y}$$

3. Eigenfunction expansions: Completeness (not proven here) implies we can approximate any 'well-behaved' function $f(x)$ on $a \leq x \leq b$ by the series

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x) \tag{2.16}$$

To find expansion coefficients, consider

$$\int_a^b w(x) y_m(x) f(x)\mathrm{d}x = \sum_{n=1}^{\infty} a_n \int_a^b wy_n y_m \mathrm{d}x = a_m \int_a^b wy_m^2 \mathrm{d}x$$

Hence

$$a_n = \frac{\int_a^b w(x) y_n(x) f(x)\mathrm{d}x}{\int_a^b w(x) y_m(x)^2 \mathrm{d}x} \tag{2.17}$$

Eigenfunctions are normalised for convenience, so with unit normalisation have

$$Y_n(x) = \frac{y_n(x)}{\left(\int_a^b wy_n^2 \mathrm{d}x\right)^2}$$

so

$$\langle Y_n, Y_m\rangle_w = \delta_{nm} \tag{2.18}$$

are orthonormal with $f(x) = \sum_{n=1}^{\infty} A_n Y)n(x)$ and $A_n = \int_a^b w(x) Y_n(x) f(x)\mathrm{d}x$.

2.4 Completeness & Pareseval's identity

Consider

$$\int_a^b \left[f(x) - \sum_{n=1}^{\infty} a_n y_n\right]^2 w\mathrm{d}x$$

$$= \int_a^b \left[f^2 - 2f\sum_n a_n y_N + \sum_n a_n^2 y_n^2\right] w\mathrm{d}x$$

$$= \int_a^b wf^2\mathrm{d}x - \sum_{n=1}^{\infty} a_n^2 \int_a^b wy_n^2\mathrm{d}x$$

because (2.17) $\int fwy_n\mathrm{d}x = a_n \int wy_n^2\mathrm{d}x$.

If the eigenfunctions are complete then series expansion converges.

$$\int_a^b wf^2\mathrm{d}x = \sum_{n=1}^{\infty} a_n^2 \int_a^b wy_n^2\mathrm{d}x \tag{2.19}$$

Or for unit normalise Y this equals $\sum_{n=1}^{\infty} A_n^2$.

Beseel's inequality if some efuncs missing

$$\int_a^b wf^2\mathrm{d}x \geq \sum_{n=1}^{\infty} A_n^2$$

Define partial sums $S_N(x) = \sum_{n=1}^{N} a_n y_n$ with

$$f(x) = \lim_{N\to\infty} S_N(x) \tag{2.20}$$

Convergence is defined in terms of mean square error

$$\varepsilon_N = \int_a^b w\left[f(x) - S_N(x)\right]^2 \mathrm{d}x \to 0 \text{ as } N \to \infty$$

This global definition of convergence in mean not pointwise convergence of FS.

The error in the partial sum (2.20) is minimised by a_n (2.19) for the $N = \infty$ expansion.

$$\frac{\partial \varepsilon_N}{\partial a_n} = -2\int_a^b y_n w\left[f - \sum_{n=1}^{N} a_n y_n\right]\mathrm{d}x = -2\int_a^b (wfy_n - a_n wy_n^2)\mathrm{d}x = 0$$

if a_n is given by (2.17). Also minimal since $\frac{\partial^2 \varepsilon_N}{\partial an^2} = 2\int wy_n^2 \geq 0$. Hence a_n is best possible choice at all N.

2.5 Exemplar: Legendre polynomials

Consider Legendre's equation (arising from spherical polars $x = \cos\theta$)

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{2.21}$$

on the interval $-1 \leq x \leq 1$ with y finite at $x = \pm 1$. (2.21) is in S-L form (2.7) with $p = 1 - x^2, q = 0, w = 1$. How to solve? Seek a power series about $x = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Substitute:

$$(1 - x^2)\sum n(n-1)c_n x^{n-2} - 2x\sum c_n x^{n-1} + \lambda \sum c_n x^n = 0$$

Equate powers of x^n

$$(n+2)(n+1)c_{n+2} - n(n-1)c_n - 2nc_n + \lambda c_n = 0$$

$$\implies c_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} c_n \tag{2.22}$$

So specifying c_0, c_1 gives 2 independent solutions.

$$y_{\text{even}} = c_0 \left[1 + \frac{(-\lambda)}{2!}x^2 + \frac{(6-\lambda)(-\lambda)}{4!}x^4 + \dots\right]$$

$$y_{\text{odd}} = c_1 \left[x + \frac{2-\lambda}{3!}x^3 + \dots\right]$$

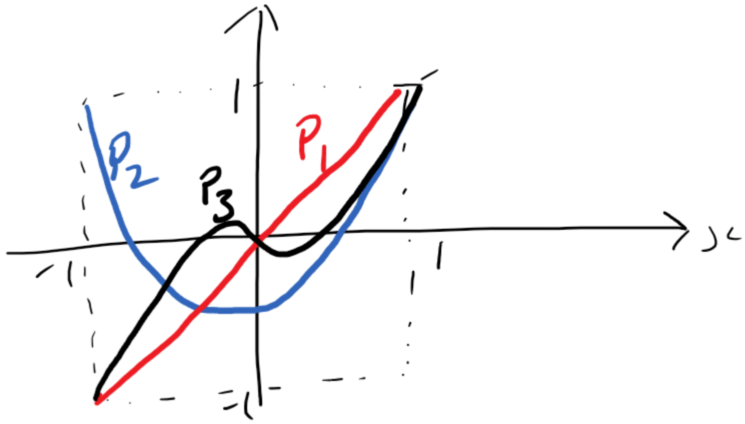
As $n \to \infty$, $\frac{c_{n+2}}{c_n} \to 1$, so radius of convergence $|x| < 1$, i.e divergent at $x = \pm 1$. What can be done? <u>Finiteness</u>. Take $\lambda = l(l+1)$ with l an integer. Then one or other series terminates, i.e $c_n = 0$, $\forall n \geq l + 2$. These <u>Legendre polynomials</u> $P_l(x)$ are eigenfunctions of (2.21) on $-1 \leq x \leq 1$ with normalisation convention $P_l(1) = 1$. Exercise:

$$l = 0, \lambda = 0, P_0(x) = 1$$

$$l = 1, \lambda = 2, P_1(x) = x$$

$$l = 2, \lambda = 6, P_2(x) = (3x^2 - 1)/2$$

$$l = 3, \lambda = 12, P_3(x) = (5x^3 - 3x)/2$$



Note:

- $P_l(x)$ has l zeros
- $P_l(x)$ is odd/even if l is odd/even

Orthogonality:

$$\int_{-1}^{1} P_n P_m \mathrm{d}x = 0, \ \forall n \neq m$$

Nomalisation:

$$\int_{-1}^{1} P_n^2 \mathrm{d}x = \frac{2}{2n+1} \tag{2.24}$$

This is called Rodrigues formula. Can use it to prove

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (x^2-1)^n$$

Generating function:

$$\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}} = 1 + \frac{1}{2}(2xt-t^2) + \frac{3}{8}(2xt-t^2)^2 + \dots$$

Recursion relations

$$l(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x)$$

$$(2l+1)P_l(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[P_{l+1}(x) - P_{l-1}(x)\right]$$

Eigenfunction expansion: Any function $f(x)$ on $-1 \leq x \leq 1$ can be expressed as

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x) \tag{2.25}$$

where

$$a_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) \mathrm{d}x \tag{2.26}$$

2.6 S-L Theory & inhomogeneous ODEs

Consider the inhomogeneous problem on $a \leq x \leq b$

$$\mathcal{L}y = f(x) \equiv w(x)F(x) \tag{2.27}$$

Given eigenfunctions $y_n(x)$ satisfying $\mathcal{L}y_n = \lambda_n w y_n$, expand as

$$y(x) = \sum_n c_n y_n(x)$$

$$F(x) = \sum_n a_n y_n(x)$$

with (2.17)

$$a_n = \frac{\int_a^b wFy_n \mathrm{d}x}{\int wy_n^2 \mathrm{d}x}$$

Substituting into (2.27)

$$\mathcal{L}y = \mathcal{L}\sum_n c_n y_n = \sum_n c_n \lambda_n y_n = w\sum_n a_n y_n$$

By orthogonality (2.13) $c_n\lambda_n = a_n \Rightarrow c_n = a_n/\lambda_n$ so solution is

$$y(x) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} y_n(x) \tag{2.28}$$

(assuming $\lambda_n \neq 0$ for all n) Recall FS (1.22).

<u>Generalisation</u>: driving forces often induce a linear response term $\tilde{\lambda} w y$

$$\mathcal{L}y - \tilde{\lambda} w y = f(x) \tag{2.29}$$

where $\tilde{\lambda}$ is fixed. The solution (2.28) becomes

$$y(x) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n - \tilde{\lambda}} y_n(x) \tag{2.30}$$

Integral solution and Green's function

Recall (2.28)

$$y(x) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} y_n(x) = \sum_N \frac{y_n(x)}{\lambda_n N_n} \int_a^b w(\xi) F(\xi) y_n(\xi) \mathrm{d}\xi$$

by (2.17) with $N_n = \int w y_n^2 \mathrm{d}x$

$$= \int_a^b \sum_{n=1}^{\infty} \frac{y_n(x) y_n(\xi)}{\lambda_n N_n} w(\xi) F(\xi) \mathrm{d}\xi$$

$$= \int_a^b G(x,\xi) f(\xi) \mathrm{d}\xi \tag{2.31}$$

where $G(x,\xi) = \sum_{n=1}^{\infty} \frac{y_n(x) y_n(\xi)}{\lambda_n N_n}$ is the eigenfunction expansion of the Green's function. $G(x,\xi)$ depends only on $\mathcal{L}$ and b.c's and not the source $f(x)$ - it acts like an inverse operator $\mathcal{L}^{-1} = \int G(x,\xi)\mathrm{d}\xi$

3 The Wave Equation

3.1 Waves on an elastic string

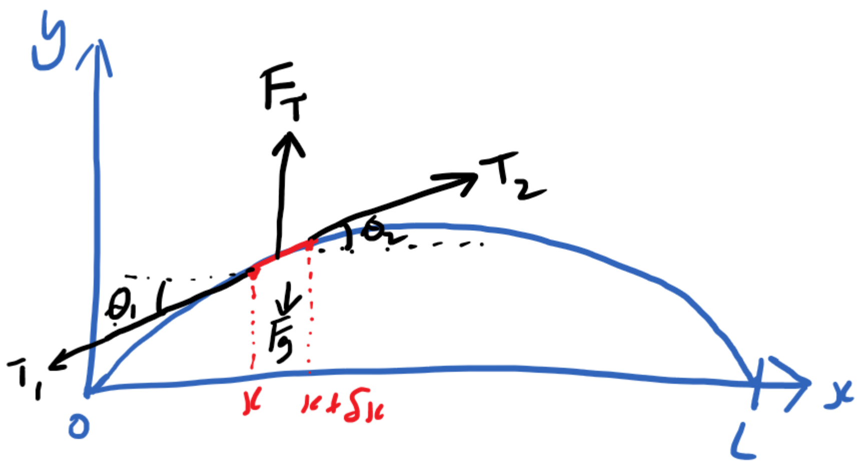
Consider small displacements $y(x,t)$ on a stretched string with fixed ends at $x = 0$ and $x = L$, that is, with boundary conditions

$$y(0,t) = y(L,t) = 0 \tag{3.1}$$

Determine the strings motion for specified initial conditions

$$y(x,0) = p(x) \text{ and } \frac{\partial y}{\partial x}(x,0) = q(x) \tag{3.2}$$

Derive equation of motion: balance forces on string segment $(x, x+\delta x)$ and take the limit $\delta x \to 0$.



Assume $\left|\frac{\partial y}{\partial x}\right| << 1$ for all x so θ_1, θ_2 are small.

Resolve in x-direction $T_1 \cos\theta_1 = T_2 \cos\theta_2$ but $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ so $\cos\theta \approx 1$ since θ is small. So $T_1 \approx T_2 \approx T$. Hence, tension T constant, independent of x up to terms of order $\theta(\left|\frac{\partial y}{\partial x}\right|^2)$.

Resolve in y-direction

$$\begin{aligned} F_T &= T_2 \sin\theta_2 - T_1 \sin\theta_1 \\ &\approx T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial y}{\partial x}\right|_{x} \right) \\ &\approx T\frac{\partial^2 y}{\partial x^2}\delta x \end{aligned}$$

Thus

$$F = ma = (\mu\delta x)\frac{\partial^2 y}{\partial t^2} = F_t + F_g = T\frac{\partial^2 y}{\partial x^2}\delta x - g\mu\delta x$$

where μ is mass per unit length. Define the wave speed $c = \sqrt{T/\mu}$ and we find

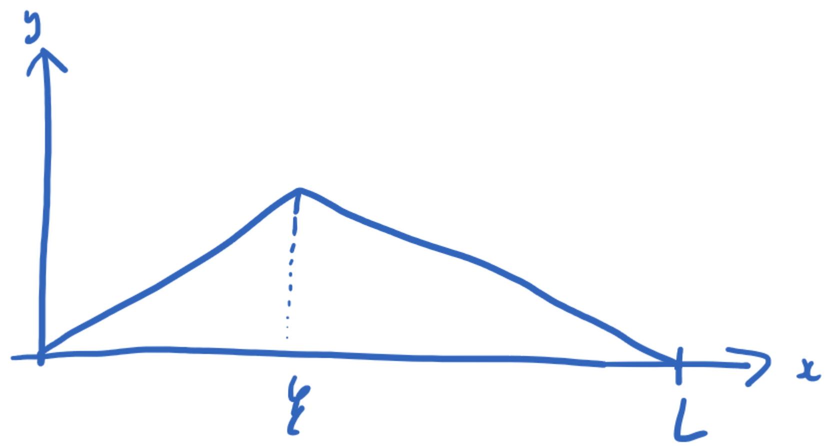
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - g = c^2\frac{\partial^2 y}{\partial x^2} - g \tag{3.3}$$

Assume gravity is negligible, we have the 1D wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \tag{3.4}$$

3.2 Seperation of variables

We wish to solve the wave equation (3.4) subject to b.c's (3.1) and i.c's (3.2)



Consider possible solutions of seperable form

$$y(x,t) = X(x)T(t) \tag{3.5}$$

Substitute in (3.4) $\frac{1}{c^2}\ddot{y} = y''$

$$\frac{1}{c^2}X\ddot{T} = X''T \implies \frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}$$

But LHS depends only on t and RHS depends only on x. So both sides must be constant, say $-\lambda$.

$$X'' + \lambda X = 0 \tag{3.6}$$

$$\ddot{T} + \lambda c^2 T = 0 \tag{3.7}$$

3.3 Boundary conditions & normal modes

Three possiblilities for λ (+,0,−) in spatial ODE (3.6) but restricted by boundary conditions (3.1)

(i) $\lambda < 0$: Take $\chi^2 = -\lambda$, then

$$X(x) = Ae^{\chi x} + Be^{-\chi x} = \tilde{A}\cosh\chi x + \tilde{B}\sinh\chi x$$

but b.c's $X(0) = X(L) = 0$ imply $\tilde{A} = \tilde{B} = 0$.

(ii) $\lambda = 0$: have $X(x) = Ax + B$ which again give $A = B = 0$.

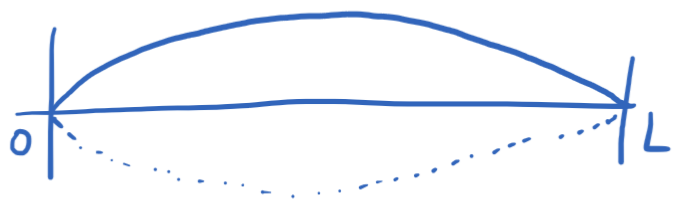
<u>(iii) $\lambda > 0$</u>: Then $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. Here b.c's imply $A = 0$, $B\sin\sqrt{\lambda}L = 0$ so $\sqrt{\lambda}L = n\pi$, so

$$X_n(x) = B_n \sin\frac{n\pi x}{L},\ \ \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{3.8}$$

i.e eigenfuncs and eigenvals of system.

These are <u>normal modes</u> of the system because spatial shape in x does not change in time.

- <u>Fundamental mode</u> ($n = 1$): $\lambda_1 = \pi^2/L^2$



- <u>Second mode</u> ($n = 2$)

- <u>Third mode</u> ($n = 3$)

3.4 Initial conditions and temporal solutions

Substitute evals $\lambda_n = (n\pi/L)^2$ into time ODE (3.7)

$$\ddot{T} + \frac{n^2\pi^2c^2}{L^2}T = 0$$

which has solutions

$$T_n(t) = C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L} \tag{3.9}$$

Thus a specific solution to (3.4) satisfying (3.1) is

$$y_n(x,t) = T_n(t)X_n(x) = \left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)\sin n\pi x2$$

Since the wave equation (3.4) is homogeneous and linear (and the b.c's are homogeneous), we can add solutions together to find the general string solution:

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)\sin n\pi x2 \tag{3.10}$$

By construction (3.10) satisfies b.c's (3.1) so now impose intitial conditions (3.2)

$$y(x,0) = p(x) = \sum_{n=1} C_n \sin n\pi xL$$

$$\frac{\partial y}{\partial t}(x,0) = q(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}D_n \sin n\pi xL$$

So the coefficients are those for the Fourier sine series given by (1.12).

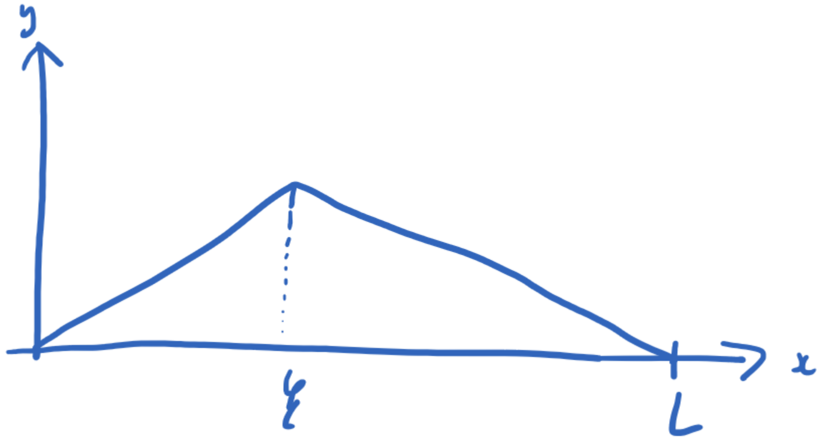
$$C_n = \frac{2}{L}\int_0^L p(x)\sin n\pi xL\mathrm{d}x \quad D_n = \frac{2}{n\pi c}\int_0^L q(x)\sin n\pi xL\mathrm{d}x \tag{3.11}$$

Hence (3.10-11) is the solution to (3.4) satisfying (3.1-2).

Example: Pluck string at $x = \xi$, drawing it back as

$$y(x,0) = p(x) = \begin{cases} x(1-\xi) & 0 \le x \le \xi \\ \xi(1-x) & \xi \le x \le 1 \end{cases}$$

$$\frac{\partial y}{\partial x}(x,0) = q(x) = 0$$



Then with FS (1.8) $C_n = \frac{2\sin n\pi\xi}{(n\pi)^2}$, $D_n = 0$, so we have general solution

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \sin n\pi\xi \sin n\pi x \cos n\pi ct$$

3.5 Oscillation energy

A vibrating string has kinetic energy due to its motion (e.g particle $(1/2)mv^2$).

$$KE = \frac{1}{2}\mu \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x$$

and potential energy due to stretching by Δx

$$PE = T\Delta x T \int_0^L \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right) \mathrm{d}x \approx \frac{1}{2}T \int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x$$

for $\left|\frac{\partial y}{\partial x}\right| << 1$.

The total summed energy becomes ($c^2 = T/\mu$)

$$E = \frac{1}{2}\mu \int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right] \mathrm{d}x \tag{3.13}$$

Substitute solution (3.10) and use orthogonality (1.1)

$$\begin{aligned} E = \frac{1}{2}\mu \sum_{n=1}^{\infty} \int_0^L & \left[\left(\frac{n\pi c}{L} C_n \sin\frac{n\pi ct}{L} + \frac{n\pi c}{L} D_n \cos\frac{n\pi ct}{L}\right)^2 \sin^2\frac{n\pi x}{L}\right. \\ & \left. + c^2\left(C_n\cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)^2 \frac{n^2\pi^2}{L^2}\cos^2\frac{n\pi x}{L}\right] \mathrm{d}x \\ &= \frac{1}{4}\mu \sum_{n=1}^{\infty} \frac{n^2\pi^2c^2}{L}(C_n^2 + D_n^2) \qquad (3.14) \\ &= \sum_{\text{normal modes}} [\text{energy in } n\text{th modes}] \end{aligned}$$

3.6 Wave reflection and transmission

Recalling the travelling wave solution (3.12)

A simple harmonic travelling wave is

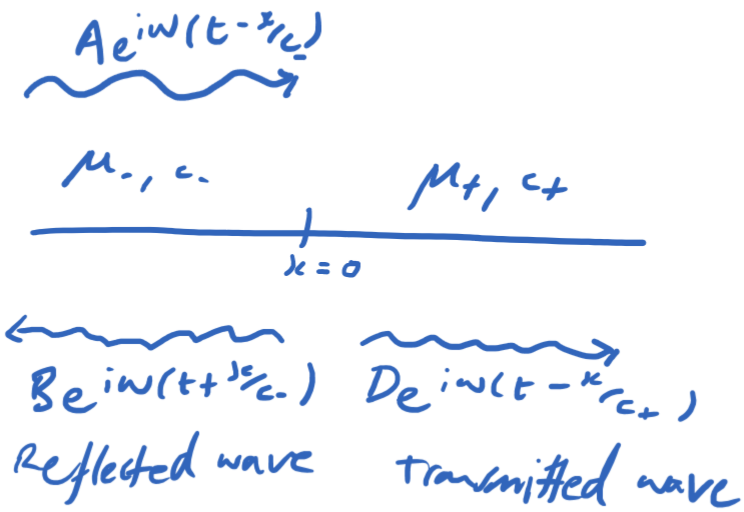
$$y = \Re[Ae^{iw(t-x/c)}] = |A|\cos[w(t-x/c)+\phi] \tag{3.15}$$

where the phase is $\phi = \arg A$ and wavelenth is $\frac{2\pi c}{w}$.

Consider a density discontinuity on a string at $x = 0$ with

$$\begin{cases} \mu = \mu_- & \text{for } x < 0 \\ \mu = \mu_+ & \text{for } x > 0 \end{cases}$$

assuming constant tension T.



Boundary (or junction) conditions at $x = 0$

- string does not break i.e y is continuous for all t. Hence $A + B = D$ $(*)$.
- Forces balance

$$T\frac{\partial y}{\partial x}\bigg|_{x=0_-} = T\frac{\partial y}{\partial x}\bigg|_{x=0_+}$$

  i.e $\frac{\partial y}{\partial x}$ is continuous for all t so

$$-iwA/c_- + iwB/c_- = -iwD/c_+ \tag{$\dagger$}$$

So solving $(*)$ using $(\dagger)$

$$"A = D + D\frac{c_-}{c_+} = \frac{D}{c_+}(c_+ + c_-)$$

So given A, we have solution

$$D = \frac{2c_+}{c_- + c_+}A \qquad B = \frac{c_+ - c_-}{c_- + c_+}A \tag{3.16}$$

In general different phase shifts ϕ possible

Limiting cases

1. Continuity $c_+ = c_- \implies D = A, B = 0$ so no reflection
2. Dirichlet b.c's $\frac{\mu_+}{\mu_-} \to \infty$ then $\frac{c_+}{c_-} \to 0$ and $D = 0$, $B = -A$ so get total reflection.
3. Neumann b.c's $\frac{\mu_+}{\mu_-} \to 0$ then $\frac{c_+}{c_-} \to \infty$ and $D = 2A$, $B = A$

3.7 Wave equations in 2D plane polar coordinates

The 2D wave equation for $u(r, \theta, t)$ is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \tag{3.17}$$

with b.c's at $r = 1$ on a unit disk

$$u(1, \theta, t) = 0 \tag{3.18}$$

and i.c's for $t = 0$

$$u(r, \theta, 0) = \phi(r, \theta), \quad \frac{\partial u}{\partial t}(r, \theta, 0) = \psi(r, \theta) \tag{3.19}$$

<ins>Temporal separation</ins>: substitute

$$u(r, \theta, t) = T(t)V(r, \theta) \tag{3.20}$$

into (3.17) to find

$$\ddot{T} + \lambda c^2 T = 0 \tag{3.21}$$

$$\nabla^2 V + \lambda V = 0 \tag{3.22}$$

which in plane polars is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \lambda V = 0$$

<ins>Spatial seperation</ins>: now try $V(r, \theta) = R(r)\Theta(\theta)$ in (3.22)

$$\Theta'' + \mu\Theta = 0 \tag{3.24}$$

$$r^2 R'' + rR' + (\lambda r^2 - \mu)R = 0 \tag{3.24}$$

where λ, μ are seperation constants.
<ins>Polar solution</ins>: Configuration implies periodic boundary conditions $\Theta(0) = \Theta(2\pi)$ with $\mu > 0$, so the eigenvalue is $\mu = m^2$ for $m \in \mathbb{Z}$ with solution

$$\Theta_m(\theta) = A_m \cos m\theta + B_m \sin m\theta \ (m > 0) \tag{3.25}$$

<ins>Radial equation</ins>: Divide (3.24) by τ to bring into S-L form (2.7)

$$\frac{\mathrm{d}}{\mathrm{d}r}(rR') = -\frac{m^2}{r} = -\lambda r R, \ (0 \leq r < 1) \tag{3.26}$$

where $p(r) = r$, $q(r) = m^2/r$ and weight $w(r) = r$, with self-adjoint b.c's with $R(1) = 0$ and bounded at $R(0)$. Since $p(0) = 0$, we have a regular singular point.

3.8 Bessel's equation

Substitute $z = \sqrt{\lambda} r$ in (3.26) to get

$$z^2 \frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + z \frac{\mathrm{d}R}{\mathrm{d}z} + (z^2 - m^2)R = 0 \tag{3.27}$$

which is Bessel's equation $(zR')' + (z - m^2/z)R = 0$.

<u>Frobenius solution</u>: Substitute the power series $R = z^p \sum_{n=0}^{\infty} a_n z^n$ to obtain

$$\sum_n \left[a_n(n+p)(n+p-1)z^{n+p} + (n+p)z^{n+p} + z^{n+p+2} + m^2 z^{n+p}\right] = 0$$

Equate powers of z: indicial equation for z^p is $p^2 - m^2 = 0$ so $p = m, -m$.

Regular solution $p = m$ has recursion relation

$$(n+m)^2 a_n + a_{n-2} - m^2 a_n = 0 \implies a_n = -\frac{1}{n(n+2m)} a_{n-2}$$

Stepping up from a_0 we have

$$a_{2n} = a_0 \frac{(-1)^n}{2^{2n} n!(n+m)(n+m-1)\ldots(m+1)}$$

Take $a_0 = \frac{1}{2^m m!}$ (for convenience) to find the Bessel function (of the first kind)

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{z}{2}\right)^{2n} \tag{3.28}$$

Second solution with $p = -m$ is the Neumann function (Bessel function of the second kind)

$$Y_m(z) = \lim_{\gamma \to m} \frac{J_\gamma(z)\cos(\gamma\pi) - J_{-\gamma}(z)}{\sin(\gamma\pi)}$$

Exercise: Use (3.28) to show that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^m J_m(z)) = z^m J_{m-1}(z)$$

and hence

$$J_m'(z) = \frac{m}{z} J_m(z) = J_{m-1}(z) \tag{3.29}$$

Repeat with z^{-m} to find recursion relations

$$\begin{cases} J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z) \\ J_{m-1}(z) - J_{m+1}(z) = 2J_m'(z) \end{cases} \tag{3.30}$$

Asymptotic behaviour of $J_m(z), Y_m(z)$

- As $z \to 0$,

$$\begin{cases} J_0(z) \to 1 \\ J_m(z) \to \frac{1}{m!}\left(\frac{z}{2}\right)^m \\ Y_0(z) \to \frac{2}{\pi}\log(z/2) \\ Y_m(z) \to -\frac{(m-1)!}{\pi}(2/z)^m \end{cases} \tag{3.31}$$

- As $z \to \infty$, oscillatory

$$\begin{cases} J_m(z) \approx \sqrt{\frac{2}{\pi z}}\cos(z - \frac{m\pi}{2} - \frac{\pi}{2}) \\ Y_m(z) \approx \sqrt{\frac{2}{\pi z}}\sin(z - \frac{m\pi}{2} - \frac{\pi}{4}) \end{cases} \tag{3.31}$$

Zeros of Bessel function $J_m(z)$

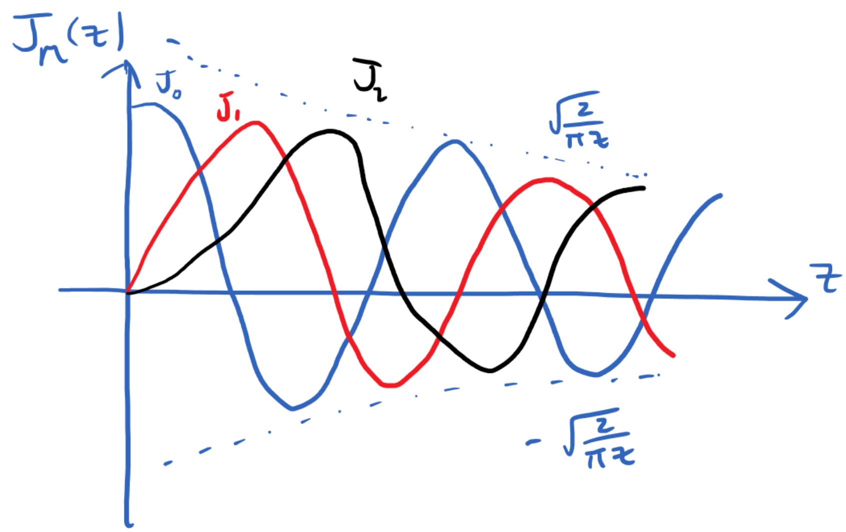
Given (3.32) there are zeros out to $z = \infty$. Define j_{mn} to be the nth zero, $J_m(j_{mn}0) = 0$. From (3.32) this occurs approximately when

$$\cos(z - \frac{m\pi}{2} - \frac{\pi}{4}) = 0$$

i.e $z - \frac{m\pi}{2} - \frac{\pi}{4} = n\pi - \frac{\pi}{2}$ (modal point) So zero

$$z \approx n\pi + \frac{m\pi}{2} - \frac{\pi}{4} \equiv \tilde{j}_{mn} \tag{3.33}$$

(accuracy $\left|\frac{j_{mn} - \tilde{j}_{mn}}{j_{mn}}\right| < \frac{0.1}{n}$ for $n > m^2/2$)



3.9 2D wave equation (continued): vibrating drum

From section 3.8 radial solution to (3.26) become

$$R_m(z) = R_m(\sqrt{\lambda}r) = AJ_m(\sqrt{\lambda}r) + BY_m(\sqrt{\lambda}r)$$

Impose b.c's:

- Regularity at $r = 0 \Rightarrow B = 0$ by (3.31)
- Unit disc $r = 1$ with $R = 0$ implies $J_m(\sqrt{\lambda}) = 0$. But these zeros occur at j_{mn} so out eigenvalues must be

$$\lambda_{mn} = j_{mn}^2 \tag{3.34}$$

With the polar mode (3.26) the spatial solution

$$V_{mn}(r,\theta) = \Theta_m(\theta)R_{mn}(\sqrt{\lambda_{mn}}r) = (A_{mn}\cos m\theta + B_{mn}\sin m\theta)J_m(j_{mn}r) \tag{3.35}$$

The temporal solution to (3.21) $\ddot{T} = -\lambda c T$ are

$$T_{mn}(t) = \cos(j_{mn}ct) \text{ and } \sin(j_{mn}ct)$$

For our linear PDE (3.17) we can sum together to obtain the general solution for the drum.

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=1}^{\infty} J_0(j_{0n}r)(A_{0n}\cos(j_{0n}ct) + C_{0n}\sin(j_{0n}ct)) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)\cos j_{mn}ct \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(C_{mn}\cos m\theta + D_{mn}\sin m\theta)\sin j_{mn}ct \end{aligned} \tag{3.36}$$

Now impose intial conditions (3.19)

$$u(r,\theta,0) = \phi(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) \tag{3.37}$$

$$\frac{\partial u}{\partial \theta}(r,\theta,0) = \psi(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} j_{mn}cJ_m(j_{mn}r)(C_{mn}\cos m\theta + D_{mn}\sin m\theta)$$

Orthogonality: Find coefficients by multiplying by J_m, cos, sin and exploiting orthogonality and Sheet 1 Q8 gives

$$\int_0^1 J_m(j_{mn})J_m(j_{mn}r)r\mathrm{d}r = \frac{1}{2}[J'_m(j_{mn})]^2\delta_{nk} \tag{3.38}$$

$$= \frac{1}{2}[J_{m+1}(j_{mn})]^2\delta_{nk} \tag{3.29}$$

Now integrate to obtain A_{mn}

$$\int_0^{2\pi} \cos p\theta \mathrm{d}\theta \int_0^1 J_p(j_{pq}r)\phi(r,\theta)r\mathrm{d}r = \frac{\pi}{2}[J_{p+1}(j_{pq})]^2 A_{pq}$$

Example: Initial radial profile

$$u(r,\theta,0) = \phi(r) = 1 - r^2 \implies m = 0,\ B_{mn} = 0\ \forall m,\ A_{mn} = 0\ \forall m \neq 0$$

$$\frac{\partial u}{\partial t}(r,0,0) = 0 \implies C_{mn} = D_{mn} = 0$$

We need to find

$$A_{0n} = \frac{2}{J_1(j_{0n})^2}\int_0^1 J_0(j_{0n}r)(1-r^2)r\mathrm{d}r = \frac{2}{J_1(j_{0n})^2}\frac{J_2(j_{0n})}{j_{0n}^2} \approx \frac{J_2(j_{0n})}{n} \text{ as } n\to\infty$$

Solution is

$$u(r,\theta,t) = \sum_{n=1}^{\infty} A_{0n}J_0(j_{0n}r)\cos(j_{0n}ct)$$

Fundamental frequency $w_d = j_{01}c\cdot\frac{2}{d} \approx 4.8(c/d)$

4 The Diffusion Equation

4.1 Physical origin of heat equation

Applies to processes that "diffuse" due to spatial gradients. An early example was Fick's law with flux

$$J = -D\nabla c$$

with concentration c and diffusion coefficient D. For heat flow, we have Fourier's law

$$q = -k\nabla\theta \tag{4.1}$$

Where q, k and θ are the heat flux, thermal conductivity and temperature respectively. In a volume V, the overall heat energy Q is

$$Q = \int_V c_v \rho\theta \mathrm{d}V \tag{4.2}$$

Where c_v and ρ are the specific heat and mass density respectively.
so rate of change due to heat flow

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_V cv\rho\frac{\partial\theta}{\partial t}\mathrm{d}V \tag{$*$}$$

Now integrate (4.1) over surface S enclosing V

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_S q\cdot\hat{n}\mathrm{d}S = \int_S(-k\nabla\theta)\cdot\hat{n}\mathrm{d}S = \int_V(-k\nabla^2\theta)\mathrm{d}V \tag{$\dagger$}$$

Equating $*$ and $\dagger$ we find

$$\int_V c_v\rho\frac{\partial\theta}{\partial t} - k\nabla^2\theta)\mathrm{d}V = 0$$

Since this is true for all V, the integrand must equal zero:

$$\frac{\partial\theta}{\partial t} - \frac{k}{c_v\rho}\nabla^2\theta = 0$$

So with $D = \frac{k}{c_v\rho}$ we have

$$\frac{\partial\theta}{\partial t} - D\nabla^2\theta = 0 \tag{4.3}$$

Brownian motion (random walks)

Gas particles diffuse by scattering every Δt with probability (PDF) $p(\xi)$ of moving a distance ξ. Average is $\langle\xi\rangle = \int p(\xi)\xi\mathrm{d}\xi = 0$.

Suppose the PDF after $N\Delta t$ steps is $P_{N\Delta t}(x)$, then for $(N+1)\Delta t$ steps:

$$\begin{aligned}
P_{(N+1)\Delta t}(x) &= \int_{-\infty}^{\infty} p(\xi) P_{N\Delta}(x-\xi)\mathrm{d}\xi \\
&\approx \int_{-\infty}^{\infty} p(\xi)\left[P_{N\Delta t}(x) + P'_{N\Delta t}(x)(-\xi) + P''_{N\Delta t}(x)\frac{\xi^2}{2} + \dots\right]\mathrm{d}\xi \\
&\approx P_{N\Delta t}(x) - P'_{N\Delta t}(x)\underbrace{\langle\xi\rangle}_{=0} + P''_{N\Delta t}(x)\frac{\langle\xi^2\rangle}{2} + \dots
\end{aligned}$$

Identifying $P_{N\Delta t}(x) = P(x, N\Delta t)$, we have $P(x,(N+1)\Delta t) - P(x, N\Delta t) = \frac{\partial^2}{\partial x^2}(P(x,N\Delta t))\frac{\langle\xi^2\rangle}{2}$. Assuming $\frac{\langle\xi^2\rangle}{2} = D\Delta t$, then for small $\Delta t \to 0$ we find

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} \tag{4.4}$$

4.2 Similarity solutions

The characteristic relation between variance and time, suggesting we seek solutions with dimensionless parameter

$$\eta \equiv \frac{x}{2\sqrt{Dt}} \tag{4.5}$$

Can we find solutions $\theta(x,t) = \theta(\eta)$? Change variables in (4.3):

$$\text{LHS: } \frac{\partial\theta}{\partial t} = \frac{\partial\eta}{\partial t}\frac{\partial\theta}{\partial\eta} = -\frac{1}{2}\frac{x}{\sqrt{D}t^{3/2}}\theta' = -\frac{1}{2}\frac{\eta}{t}\theta'$$

$$\text{RHS: } D\frac{\partial^2\theta}{\partial x^2} = D\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial x}\frac{\partial\theta}{\partial\eta}\right) = D\frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{Dt}}\theta'\right) = \frac{1}{4t}\theta''$$

Equating:

$$\theta'' = -2\eta\theta' \tag{4.6}$$

Take $\psi = \theta'$, $\frac{\psi'}{\psi} = -2\eta$

$$\Rightarrow \log\psi = -\eta^2 + const \Rightarrow \psi = \theta' = (const)e^{-\eta^2}$$

So

$$\theta(\eta) = C\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-u^2}\mathrm{d}u = C\mathrm{erf}(\eta) = C\mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right) \tag{4.7}$$

where the error function is

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^{z} e^{-u^2}\mathrm{d}u$$

This describes discontinuous initial conditions that spread over time.

4.3 Heat conduction in a finite bar

Suppose we have a bar of length $2L$ with $-L \leq x \leq L$ and initial temperature

$$\theta(x,0) = H(x) = \begin{cases} 1 & 0 \leq x \leq L \\ 0 & -L \leq x \leq 0 \end{cases} \tag{4.8}$$

with b.c's

$$\theta(L,t) = 1 \text{ and } \theta(-L,t) = 0 \tag{4.9}$$

Transforming boundary conditions

:
The b.c's (4.9) are not homogeneous. Can we identify steady state solution (time independent) that reflects the late-time behaviour?

Try $\theta_s(x) = Ax + B$, this satisfies $\frac{\partial^2\theta}{\partial x^2} = 0$. To satisfy (4.9) $A = \frac{1}{2L}$, $B = \frac{1}{2}$.

$$\theta_s \frac{x+L}{2L} \tag{4.10}$$

Transform and solve for

$$\hat{\theta}(x,t) = \theta(x,t) - \theta_s(x)$$

with homogeneous b.c's $\hat{\theta}(-L,t) = \hat{\theta}(L,t) = 0$ and i.c's

$$\hat{\theta}(x,0) = H(x) - \frac{x+L}{2L} \tag{4.11}$$

Seperation of variables

Try $\hat{\theta}(x,t) = X(x)T(t)$ which by (4.3) gives

$$X'' = -\lambda X, \quad \dot{T} = -D\lambda T \tag{4.12}$$

B.c's imply $\lambda > 0$ with

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

For $\cos\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda_m} = \frac{m\pi}{2L}$, $m = 1, 3, 5, \ldots$

$\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L}$, $n = 1, 2, 3 \ldots$

But i.c's are odd so take

$$X_n = B_n \sin\frac{n\pi x}{L}, \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

Put λ_n into (4.12) to find

$$T_n(t) = C_n \exp(-\frac{Dn^2\pi^2}{L^2}t)$$

So general solution is

$$\hat{\theta}(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\frac{Dn^2\pi^2}{L^2}t} \tag{4.13}$$

Impose i.c's (4.11) at $t = 0$ in (4.13)

$$b + n = \frac{1}{L}\int_{-L}^{L} \hat{\phi}(x,0)\sin\frac{n\pi x}{L}\mathrm{d}x$$

where $\hat{\phi}(x,0) = H(x) - \frac{x+L}{2L}$

$$= \underbrace{\frac{2}{L}\int_0^L (H(x) - \frac{1}{2})\sin\frac{n\pi x}{L}\mathrm{d}x}_{\text{square wave (1.7)}} - \underbrace{\frac{2}{L}\int_0^L \frac{x}{2L}\sin\frac{n\pi x}{L}\mathrm{d}x}_{\text{sawtooth (1.6)}}$$

$$= \underbrace{\frac{2}{n\pi}}_{\text{or }=0\text{ if }n\text{ even}} - \frac{(-1)^{n+1}}{n\pi} = \frac{1}{n\pi}$$

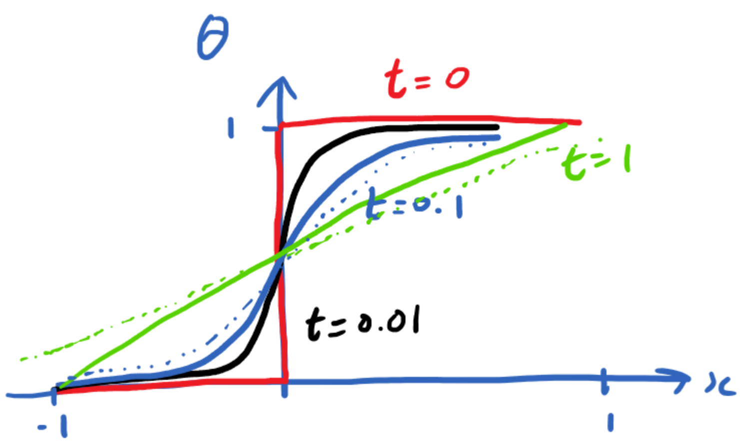
Solution

$$\hat{\theta}(x,t) = \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin\frac{n\pi x}{L}e^{-D\frac{n^2\pi^2}{L^2}t}$$

or with original b.c's (4.9)

$$\theta(x,t) = \frac{x+L}{2L} + \hat{\theta}(x,t) \tag{4.14}$$

Plot with $L = 1$ and $D = 1$



Approx solution (4.7) $\frac{1}{2}(1+\mathrm{erf}(x/2\sqrt{D}t))$ dashed lines - excellent fit for $t << 1$.

5 The Laplace Equation

Laplace's equation

$$\nabla^2\phi = 0 \tag{5.1}$$

has wide application in mathematical physics, applied maths and pure maths (harmonic analysis).

Examples include:

- Steady state heat flow
- Potential theory $F = -\nabla\phi$
- Incompressible fluid flow $v = \nabla\phi$ (zero curl)

We solve (5.1) in a domain D subject to b.c's:

Dirichlet: ϕ given on boundary surface ∂D

Neumann: $\hat{n} \cdot \nabla\phi$

5.1 3D Cartesian Coordinates

Equation (5.1) becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{5.2}$$

Seek seperable solution $\phi(x, y, z) = X(x)Y(y)Z(z)$

$$X''YZ + XY''Z + XYZ'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda_l$$

and $Y''/Y = -\lambda_m$ so

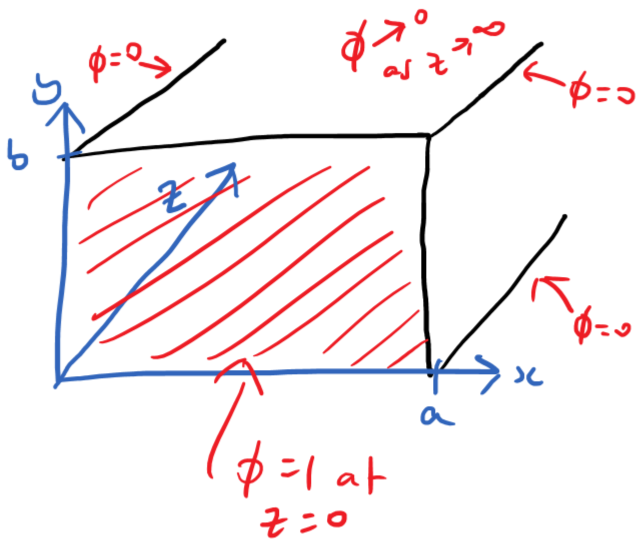
$$Z''/Z = -\lambda_n = \lambda_l + \lambda_m \tag{5.3}$$

General solution from eigenmodes

$$\phi(x, y, z) = \sum_{l,m,n} a_{lmn} X_l(x) Y_m(y) Z_n(z) \tag{5.4}$$

Example: Steady heat conduction

(i.e (4.3) with $\frac{\partial\phi}{\partial t} = 0 \Rightarrow$ (5.1)) Consider a semi infinite rectangular bar with b.c's $\phi = 0$ at $x = 0, a$ and $y = 0, b$, $\phi = 1$ at $z = 0$, $\phi \to 0$ as $z \to \infty$.



Solve for eigenmodes successively:

- $X'' = -\lambda_l X$ with $X(0) = X(a) = 0$ so $\lambda_l = \frac{l^2\pi^2}{a^2}$, $X_l = \sin\frac{l\pi x}{a}$. $l = 1, 2, 3, \ldots$
- $Y'' = -\lambda_m Y$, $\lambda_m = \frac{m^2\pi^2}{b^2}$, $Y_m = \sin\frac{m\pi y}{b}$ $m = 1, 2, 3, \ldots$
- $Z'' = -\lambda_n Z = (\lambda_l + \lambda_m)Z = \pi^2(\frac{l^2}{a^2} + \frac{m^2}{b^2})Z$ with b.c's $Z \to 0$ as $z \to \infty$ eliminating any growing exponential so $Z_{lm} = \exp\left[-\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}\pi z\right]$

So our general solution (5.4) becomes

$$\phi(x, y, z) = \sum_{l,m} a_{lm} \sin\frac{l\pi x}{L} \sin\frac{m\pi y}{L} \exp\left[-\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}\pi z\right]$$

Now fix a_{lm} using $\phi(x, y, 0) = 1$ using Fourier sine b_n (1.12):

$$\begin{aligned}
a_{lm} &= \frac{2}{b}\int_0^b \frac{2}{a}\int_0^a \underbrace{1\sin\frac{l\pi x}{a}}_{\text{square wave}} \underbrace{\sin\frac{m\pi y}{b}}_{\text{FS (1.7)}} \mathrm{d}x\mathrm{d}y \\
&= \frac{4a}{a(2k-1)\pi}\frac{4b}{b(2p-1)\pi)} = \frac{16}{\pi^2(2k-1)(2p-1)} \\
&= \frac{16}{\pi^2 lm}
\end{aligned}$$

Where $l = 2k-1,\ m = 2p-1$ odd. So heat flow solution is

$$\phi(x,y,z) = \sum_{l,m\text{ odd}} \frac{16}{\pi^2 lm} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{6} \exp\left[-\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)^{1/2}\pi z\right]$$

Due to the exponential term, lower l, m terms dominate heavily for large z.

5.2 2D Plane Polar Coordinates

Recall

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{5.6}$$

and try $\phi(r,\theta) = R(r)\Theta(\theta)$ to find

$$\Theta'' + \mu\Theta = 0$$

and

$$r(rR')' - \mu R = 0$$

- Polar equation: periodic b.c's give $\mu = m^2$ and $\Theta_m(\theta) = \cos m\theta$ and $\Theta_m(\theta) = \sin m\theta$
- Radial equation:

  $$r(rR')' - m^2 R = 0 \tag{5.7}$$

  Try $R = \alpha r^\beta \Rightarrow \beta^2 - m^2 = 0 \Rightarrow \beta = \pm m$ so get solutions $R_m = r^m$ and $R_m = r^{-m}$

  If $m = 0$, $(rR')' = 0 \Rightarrow rR' = const \Rightarrow R = \log(r)$ so get solutions $R_0 = const$ and $R_0 = \log(r)$

General solution:

$$\phi(r,\theta) = \frac{a_0}{2} + c_0\log r + \sum_{m=1}^{\infty}\left[(a_m\cos m\theta + b_m\sin m\theta)r^m + (c_m\cos m\theta + d_m\sin m\theta)r^{-m}\right] \tag{5.8}$$

Example: soap film on a unit disk

Solve (5.6) with a distorted circular wire of radius $r = 1$, with given b.c's $\phi(1,\theta) = f(\theta)$, to find $\phi(r,\theta)$ for $r < 1$

Regularity at $r = 0$ implies $c_m = d_m = 0$ for all m. So (5.8) becomes

$$\phi(r,\theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m\cos m\theta + b_m\sin m\theta)r^m$$

At $r = 1$

$$\phi(1,\theta) = f(\theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos m\theta + b_m \sin m\theta)$$

so the FS coefficients (1.5) are

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos m\theta \mathrm{d}\theta$$
$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta \mathrm{d}\theta$$

5.3 3D Cylindrical Polar Coordinates

Here,

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{5.9}$$

With $\phi = R(r)\Theta(\theta)Z(z)$ we have

$$\Theta'' = -\mu\Theta, \quad Z'' = \lambda Z, \quad r(rR')' + (\lambda r^2 - \mu)R = 0$$

- Polar (as before) $\mu_m = m^2$, $\Theta_m(\theta) = \cos m\theta$ and $\sin m\theta$
- Radial (Bessel's equation (3.26)): with solutions $R = J_m(kr)$ and $Y_m(kr)$. Setting b.c's $R = 0$ at $r = a$ means $J_m(ka) = 0 \Rightarrow k = \frac{j_{mn}}{a}$ where j_{mn} is the nth zero of J_m. So radial solution

$$R_{mn}(r) = J_m(\frac{j_{mn}}{a}r) \tag{5.10}$$

(eliminate Y_m as $Y_m \to -\infty$ as $r \to 0$)

- Z equation: $Z'' = k^2 Z$ implies $Z = e^{-kz}$ and e^{kz} (usually eliminate e^{kz} with b.c's)

Hence general solution is

$$\phi(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{mn}\cos m\theta + b_{mn}\sin m\theta)\, J_m(\frac{j_{mn}}{a}r)e^{-j_{mn}r/a} \tag{5.11}$$

5.4 3D Spherical Polar Coordinates

Recall that

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \\ \mathrm{d}V &= r^2\sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \end{aligned}$$

With $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$

Laplace equation (5.1) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial\Phi^2}{\partial\phi^2} = 0 \tag{5.12}$$

Axisymmetric case

(no ϕ dependence): Seek seperable solutions $\Phi(r,\theta) = R(r)\Theta(\theta)$

$$\begin{cases} (\sin\theta\Theta')' + \lambda\sin\theta\Theta = 0 \\ (r^2R')' - \lambda R = 0 \end{cases} \tag{5.13}$$

- Polar equation: substitute $x = \cos\theta$ with $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta \Rightarrow \frac{\mathrm{d}\Theta}{\mathrm{d}\theta} = -\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}x}$. Substituting:

$$-\sin\theta\frac{\mathrm{d}}{\mathrm{d}x}\left[-\sin^2\theta\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right] + \lambda\sin\theta\Theta = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right] + \lambda\Theta = 0$$

  Which is Legendre's equation (2.21) with eigenvalues $\lambda_l = l(l+1)$ and eigenfunctions (2.3)

$$\Theta_l(\theta) = P_l(x) = P_l(\cos\theta) \tag{5.14}$$

- Radial equation: $(r^2R')' - l(l+1)R = 0$. Seek solutions $R = \alpha r^\beta$:

$$\beta(\beta+1) - l(l+1) = 0$$

- with two solutions $\beta = l$ or $\beta = -l-1$ so

$$R_l = r^l \text{ and } r^{-l-1}$$

So general axisymmetric solution is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}(a_lr^l + b_lr^{-l-1})P_l(\cos\theta) \tag{5.15}$$

where $a_l, b_,$ determined by b.c's usually at fixed $r = r_0$. Use orthogonality of P_l's to obtain coefficients.

Unit sphere solution

Solve $\nabla^2\Phi = 0$ with axisymmetric b.c's at $r = 1$, $\Phi(1,\theta) = f(\theta)$. Regularity implies $b_l = 0$, so we have

$$f(\theta) = \sum_{l=0} a_lP_l(\cos\theta)$$

Or with $f(\theta) = F(\cos\theta)$

$$F(x) = \sum_{l=0}^{\infty} a_lP_l(x)$$

so by (2.25)

$$a_l = \frac{2l+1}{2}\int_{-1}^{1} F(x)P_l(x)\mathrm{d}x$$

Generating function for $P_l(x)$

Consider a charge on z-axis at $r_0 = (0, 0, 1)$, then the potential at P becomes

$$\begin{aligned}\Phi(r) = \frac{1}{|r - r_0|} &= \frac{1}{(x^2 + y^2 + (z-1)^2)^{1/2}} \\ &= \frac{1}{(r^2 \sin^2\theta + r^2\cos^2\theta - 2r\cos\theta + 1)^{1/2}} \\ &= \frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}} = \frac{1}{\sqrt{r^2 - 2rx + 1}}\end{aligned}$$

We can represent any axisymmetric solution (5.12) as a sum (5.15) (with $b_n = 0$ for $r < 1$)

$$\frac{1}{\sqrt{r^2 - 2rx + 1}} = \sum_{l=0}^{\infty} a_l P_l(x) r^l$$

With normalisation condition $P_l(1) = 1$ at $x = 1$, we get

$$\frac{1}{1 - r} = \sum_{l=0}^{\infty} a_l r^l$$

so $a_l = 1$ (by geometric series). Thus the generating function is

$$\frac{1}{\sqrt{r^2 - 2rx + 1}} = \sum_{l=0}^{\infty} P_l(x) r^l \tag{5.16}$$

Expand LHS with binomial theorem to find $P_l(x)$.

6 The Dirac Delta Function

6.1 Definition of $\delta(x)$

Define a generalised function $\delta(x-\xi)$ with the following properties

$$\begin{cases} \delta(x-\xi) = 0 & \forall x \neq \xi \\ \int_{-\infty}^{\infty} \delta(x-\xi)\mathrm{d}x = 1 \end{cases} \tag{6.1}$$

This acts as a linear operator $\int \mathrm{d}x\delta(x-\xi)$ on an arbitrary function $f(x)$ to produce a number $f(\xi)$, that is,

$$\int_{-\infty}^{\infty} \mathrm{d}x\delta(x-\xi)f(x) = f(\xi) \tag{6.2}$$

provided $f(x)$ is 'well-behaved' at $x = \xi, \pm\infty$.
Notes:

- The delta function $\delta(x)$ is classified as a distribution (not a function)
- $\delta(x)$ always appears in an integrand as a linear operator, where it is well-defined.
- Represents a unit point source (e.g mass/charge) or an impulse.

Some limiting approximations

Discrete: define

$$\delta_n(x) = \begin{cases} 0 & x > 1/n \\ n/2 & |x| \leq 1/n \\ 0 & x < -1/n \end{cases}$$

Then $\delta_n(x) \to \delta(x)$
Continuous: define

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2} \tag{6.3}$$

Then take $\varepsilon \to 0$ to get $\delta(x)$. Verify (6.2):

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2} f(x)\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} f(\varepsilon y)\mathrm{d}y \\ &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}y \frac{1}{\sqrt{\pi}} e^{-y^2} \left[f(0) + \varepsilon y f'(0) + \dots\right] \\ &= f(0) \end{aligned}$$

For all 'well-behaved' f at $x = 0, \pm\infty$.

Further examples:

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ikx}\mathrm{d}k \tag{6.4}$$

$$\delta_n(x) = \frac{n}{2}\operatorname{sech}^2 nx \tag{6.5}$$

6.2 Properties of $\delta(x)$

Heaviside function $H(x)$

The unit step function

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{6.6}$$

is the integral of $\delta(x)$

$$H(x) = \int_{-\infty}^{x} \delta(x)\mathrm{d}x \tag{6.7}$$

and we can identify $H'(x) = \delta(x)$.

Derivative of $\delta(x)$

Define $\delta'(x)$ using integration by parts

$$\int_{-\infty}^{\infty} \delta'(x-\xi)f(x)\mathrm{d}x = [\delta(x-\xi)f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x-\xi)f'(x)\mathrm{d}x = -f'(\xi) \tag{6.8}$$

Example: Gaussian approximation (6.3) $\delta'_\varepsilon(x) = \frac{-2x}{\varepsilon^3\sqrt{\pi}}e^{-x^2/\varepsilon^2}$

Sampling property:

$$\int_a^b f(x)\delta(x-\xi)\mathrm{d}x = \begin{cases} f(\xi) & a < \xi < b \\ 0 & \text{otherwise} \end{cases} \tag{6.10}$$

Even property:

$$\int_{-\infty}^{\infty} f(x)\delta(-(x-\xi))\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\delta(x-\xi)\mathrm{d}x = \int_{-\infty}^{\infty} f(\xi-u)\delta(u)\mathrm{d}u = f(\xi)$$

Scaling property:

$$\int_{-\infty}^{\infty} f(x)\delta(a(x-\xi))\mathrm{d}x = \frac{1}{|a|}f(\xi) \tag{6.11}$$

Advanced scaling: Suppose $g(x)$ has m isolated zeros at $x_1, x_2, \ldots, x_n$. Then (with $g'(x_i) \neq 0$):

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i))}{|g'(x_i)|} \tag{6.12}$$

Isolation property: If $g(x)$ is continuous at $x = 0$ then

$$g(x)\delta(x) = g(0)\delta(x) \tag{6.13}$$

6.3 Eigenfunction expansion of $\delta(x)$

Fourier series (complex)

For $-L \le x < L$, represent $\delta(x) = \sum_{n=-\infty}^{\infty} e^{in\pi x/L}$. FS coefficient (1.15) is

$$c_n = \frac{1}{2L}\int_{-L}^{L} \delta(x) e^{-in\pi x/L} \mathrm{d}x = \frac{1}{2L}$$

So

$$\delta(x) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{in\pi x/L} \tag{6.14}$$

Take $f(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\pi x/L}$ then

$$\int_{-L}^{L} f^*(x)\delta(x)\mathrm{d}x = \frac{1}{2L}\sum_{n=-\infty}^{\infty} d_n \int_{-L}^{L} e^{-in\pi x/L} e^{in\pi x/L}\mathrm{d}x = \sum_{n=-\infty}^{\infty} d_n = f(0)$$

The Dirac comb comes from extending periodically to all $\mathbb{R}$:

$$\sum_{m=-\infty}^{\infty} \delta(x - 2mL) = \sum_{n=-\infty}^{\infty} e^{in\pi x/L}$$

General eigenfunctions

Suppose $\delta(x-\xi) = \sum_{n=1}^{\infty} a_n y_n(x)$, $a \le x \le b$ with coefficients (2.17)

$$a_n = \frac{\int_a^b w(x) y_n(x) \delta(x-\xi)\mathrm{d}x}{\int_a^b w y_n^2 \mathrm{d}x} = \frac{w(\xi) y_n(\xi)}{\int_a^b w y_n^2 \mathrm{d}x} = w(\xi) Y_n(\xi)$$

For unit normalised Y_n. Then

$$\delta(x-\xi) = w(\xi)\sum_{n=1}^{\infty} Y_n(\xi) Y_n(x) = w(x)\sum_{n=1}^{\infty} Y_n(\xi) Y_n(x)$$

By the isolation property (6.13). hence

$$\delta(x-\xi) = w(x)\sum_{n=1}^{\infty} \frac{y_n(\xi) y_n(x)}{\mathcal{N}_n} \tag{6.15}$$

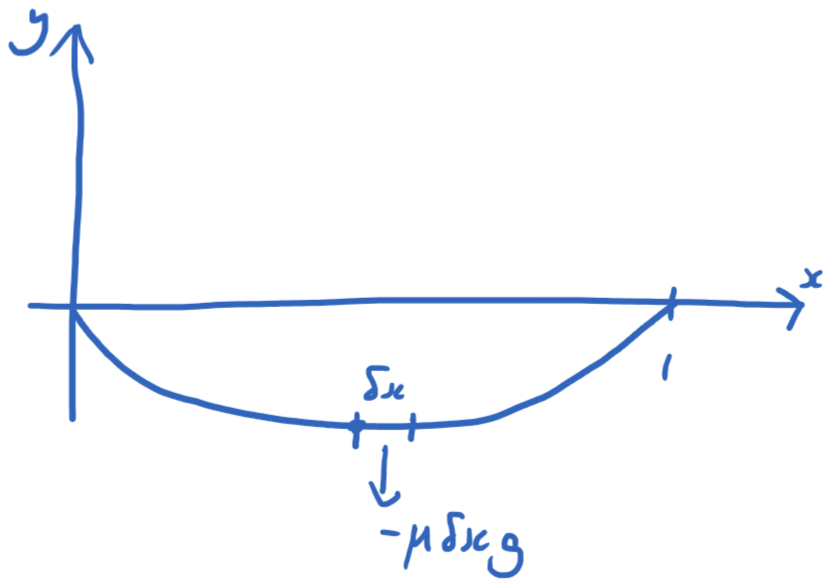
where $\mathcal{N}_n = \int_a^b w y_n^2 \mathrm{d}x$ is the normalisation factor.

Example: Consider FS for $y(0) = y(1) = 0$ with $y_n(x) = \sin n\pi x$. Here from (1.11) we have $\delta(x-\xi) = 2\sum_{n=1}^{\infty} \sin n\pi\xi \sin n\pi x$ where $0 < \xi < 1$.

7 Green's functions

7.1 Physical Motivation: Static Forces on a String

Consider a massive static string (tension T, linear mass density μ) suspended with fixed ends $y(0) = y(1) = 0$ (7.1)



By resolving forces, we have

$$T\frac{\partial^2 y}{\partial x^2} - \mu g = 0$$

So solve inhomogeneous ODE subject to (7.1)

$$-\frac{\partial^2 y}{\partial x^2} = f(x) \tag{7.2}$$

with $f(x) = -\mu g/T$.

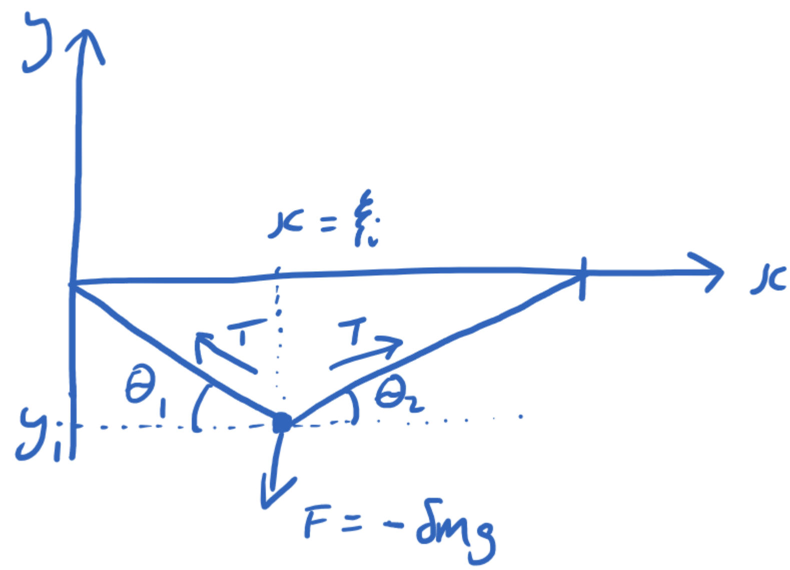
Solution 1: Direct Integration for uniform mass density. ODE (7.2) gives

$$-y = -\frac{\mu g}{2T}x^2 + k_1 x + k_2$$

BC's (7.1) give

$$y(x) = \left(-\frac{\mu g}{T}\right)\frac{1}{2}x(1-x) \tag{7.3}$$

Solution 2: Superposition of point masses on light string. Consider point mass δm $(= \mu\delta x)$ suspended at $x = \xi_i$ on a very light string.



Resolve in y-direction to find $y_i(\xi_i)$:

$$
\begin{aligned}
0 &= T(\sin\theta_1 + \sin\theta_2) - \delta mg \\
&= T\left(\frac{-y_i}{\xi_i} + \frac{-y_i}{1-\xi_i}\right) - \delta mg \\
\implies -T(y_i(1-\xi_i) + y_i\xi_i) &= \delta mg\xi_i(1-\xi_i)
\end{aligned}
$$

so

$$y_i(\xi_i) = \frac{-\delta mg}{T}\xi_i(1-\xi_i)$$

Hence solution is

$$y_i(x) = \frac{-\delta mg}{T}\begin{cases} x(1-\xi_i) & x < \xi_i \\ \xi_i(1-x) & x > \xi_i \end{cases} = f_iG(x,\xi) \tag{7.4}$$

Where $f_i = \frac{-\delta mg}{T}$ is the source and $G(x,\xi)$ is the solution for the unit point source (Green's function). Now sum N point masses δm at $x = \{\xi_i\}$ by linearity

$$y(x) = \sum_{i=1}^{N} f_iG(x,\xi_i)$$

or in the continuum limit with $f_i = \frac{-\delta mg}{T} = \frac{-\mu\delta xg}{T} = f(x)\mathrm{d}x$ with $f(x) = \frac{-\mu g}{T}$

we have

$$\begin{aligned} y(x) &= \int_0^1 f(\xi)G(x,\xi)\mathrm{d}\xi \\ &= \frac{-\mu g}{T}\left[\int_0^x \xi(1-x)\mathrm{d}\xi + \int_1^x x(1-\xi)\mathrm{d}\xi\right] \\ &= \frac{-\mu g}{T}\left[\left[\frac{\xi^2}{2}(1-x)\right]_0^x + \left[x(\xi-\frac{\xi^2}{2})\right]_x^1\right] \\ &= \frac{-\mu g}{T}\left(\frac{x^2}{2}(1-x) + \frac{x}{2} - x(x-\frac{x^2}{2})\right) \\ &= \frac{-\mu g}{T}\frac{1}{2}x(1-x) \end{aligned} \tag{7.5}$$

7.2 Definition of Green's function

We wish to solve the inhomogeneous ODE on $a \leq x \leq b$

$$\mathcal{L} \equiv \alpha(x)y'' + \beta(x)y' + \gamma(x)y = f(x) \tag{7.6}$$

with $\alpha \neq 0$, β, γ continuous and bounded and homogeneous b.c's $y(a) = y(b) = 0$. The Green's function $G(x,\xi)$ for the operator $\mathcal{L}$ is the solution for a unit point source at $x = \xi$:

$$\mathcal{L}G(x,\xi) = \delta(x-\xi) \tag{7.7}$$

which satisfies homogeneous b.c's $G(a,\xi) = G(b,\xi) = 0$. By linearity, we can construct solutions to (7.6) by integrating over the source $f(x)$ with Green's function $G(x,\xi)$:

$$y(x) = \int_a^b G(x,\xi)f(\xi)\mathrm{d}\xi \tag{7.8}$$

where $y(x)$ satisfies homogeneous b.c's. Can verify formally that

$$\mathcal{L}y = \int \mathcal{L}G(x,\xi)f(\xi)\mathrm{d}\xi = \int \delta(x-\xi)f(\xi)\mathrm{d}\xi = f(x)$$

so the solution (7.8) is given by the inverse operator $y = \mathcal{L}^{-1}f$, where

$$\mathcal{L}^{-1} = \int G(x,\xi)\mathrm{d}\xi$$

Defining properties (summary)

The Green's function splits into two parts:

$$G(x,\xi) = \begin{cases} G_1(x,\xi) & a \leq x < \xi \\ G_2(x,\xi) & \xi < x < b \end{cases} \tag{7.9}$$

such that:

1. Homogeneous solutions: G soves homogeneous equation $\forall x \neq \xi$ so

$$\mathcal{L}G_1 = 0, \quad \mathcal{L}G_2 = 0 \tag{7.10}$$

2. Homogeneous b.c's G satisfies homogeneous b.c's so

$$G_1(a, \xi) = 0, \quad G_2(b, \xi) = 0 \tag{7.11}$$

3. Continuity conditions: G is continuous at $x = \xi$ so

$$G_1(\xi, \xi) = G_2(\xi, \xi) \tag{7.12}$$

4. Jump conditions: Derivative is discontinuous at $x = \xi$ with

$$[G']_{\xi^i}^{\xi^+} = \left.\frac{\mathrm{d}G_2}{\mathrm{d}x}\right|_{x=\xi^+} - \left.\frac{\mathrm{d}G_1}{\mathrm{d}x}\right|_{x=\xi^-} = \frac{1}{\alpha(\xi)} \tag{7.13}$$

where $\alpha(x)$ is defined in (7.6)

7.3 Constructing $G(x,\xi)$: Boundary Value Problems

Solve $\mathcal{L}G(x,\xi) = \delta(x-\xi)$ on $a \le x \le b$ subject to homogeneous b.c's $G(a,\xi) = G(b,\xi) = 0$ (with $a < \xi < b$)

Assume 2 independent homogeneous solutions $y_1(x), y_2(x)$ are known.

For $a \le x < \xi$: $G_1(x,\xi) = Ay_1(x) + By_2(x)$ such that $Ay_1(a) + By_2(a) = 0$ (i.e choose suitable A, B). This defines a complementary function (2.3) $y_-(x)$ such that $y_-(a) = 0$. So general homogeneous solution with $G_1 = 0$ at $x = a$ is:

$$G_1 = Cy_-(x) \text{ with } y_-(a) = 0 \tag{7.14}$$

For $\xi < x \le b$: Similarly find

$$G_2 = D_+(x) \text{ with } y_+(b) = 0 \tag{7.15}$$

Why is G continuous at $x = \xi$?

Suppose G were discontinuous, so locally $G \propto H(x-\xi) + \dots$ (6.7) which implies $G' \propto \delta(x-\xi)$ and $G'' \propto \delta'(x-\xi)$.

So LHS $\mathcal{L}G \propto \alpha(x)\delta'(x-\xi) + \beta(x)\delta(x-\xi) + \gamma(x)H(x-\xi)$. But on RHS there is no $\delta'(x-\xi)$. Hence we have $[G]_{\xi^-}^{\xi^+} = 0$ so

$$Cy_-(\xi) = D_+(\xi) \tag{7.16}$$

Why the jump condition for G' at $x = \xi$?

Integrate $\mathcal{L}G(x,\xi) = \delta(x-\xi)$ across $x = \xi$: LHS is

$$\begin{aligned}\int_{\xi^-}^{\xi^+} \mathcal{L}G\mathrm{d}x &= \int_{\xi^-}^{\xi^+} (\alpha G'' + \beta G' + \gamma G)\mathrm{d}x \\ &= \alpha(\xi)[G']_{\xi^-}^{\xi^+} + (\beta - \alpha')[G]_{\xi^-}^{\xi^+} + \int_{\xi^-}^{\xi^+} (\gamma - \beta' + \alpha'')G\mathrm{d}x = \alpha(\xi)[G']_{\xi^-}^{\xi^+}\end{aligned}$$

RHS is $\int_{\xi^-}^{\xi^+} \delta(x-\xi)\mathrm{d}x = 1$. Thus $[G']_{\xi^-}^{\xi^+} = \frac{1}{\alpha(\xi)}$ so

$$Dy_+(\xi) - Cy_-'(\xi) = \frac{1}{\alpha(\xi)} \tag{7.17}$$

Wronskian $W(\xi)$

Solving (7.16) and (7.17) we find

$$C(\xi) = \frac{y_+(\xi)}{\alpha(\xi)W(\xi)}, \quad D = \frac{y_-(\xi)}{\alpha(\xi)W(\xi)} \tag{7.18}$$

where

$$W(\xi) = y_-(\xi)y_+'(\xi) - y_+(\xi)y_-'(\xi) \tag{7.19}$$

Note $W(\xi) \neq 0$ if y_+, y_- are linearly independent. Hence,

$$G(x,\xi) = \begin{cases} \frac{y_-(x)y_+(\xi)}{\alpha(\xi)W(\xi)} & a \leq x < \xi \\ \frac{y_+(x)y_-(\xi)}{\alpha(\xi)W(\xi)} & \xi < x \leq b \end{cases} \tag{7.20}$$

So the solution to (7.6) with $y(a) = y(b) = 0$ is

$$\begin{aligned} y(x) &= \int_a^b G(x,\xi)f(\xi)\mathrm{d}\xi \\ &= \int_a^x G_2(x,\xi)f(\xi)\mathrm{d}\xi + \int_x^b G_1(x,\xi)f(\xi)\mathrm{d}\xi \\ &= y_+(x)\int_a^x \frac{y_-(\xi)f(\xi)}{\alpha(\xi)W(\xi)}\mathrm{d}\xi + y_-(x)\int_x^b \frac{y_+(\xi)f(\xi)}{\alpha(\xi)W(\xi)}\mathrm{d}\xi \end{aligned} \tag{7.21}$$

Notes:

1. If $\mathcal{L}$ is in S-L form (2.7), i.e $\beta = \alpha'$, then denominator $\alpha(\xi)W(\xi)$ is a constant and G is symmetric, $G(x,\xi) = G(\xi,x)$.
2. Often take $\alpha = 1$
3. Indefinite integrals $\int_x$ in (7.21) are partincular integrals in general solution (2.5).

Example: Solve $y'' - y = f(x)$ with $y(0) = y(1) = 0$. Construct $G(x,\xi)$: homogeneous solutions $y_1 = e^x, y_2 = e^{-x}$ so imposing b.c's

$$G = \begin{cases} C\sinh x & 0 \leq x < \xi \\ D\sinh(1-x) & \xi < x \leq b \end{cases}$$

Continuity at $x = \xi$ implies $C\sinh\xi = D\sinh(1-\xi)$ so

$$C = \frac{D\sinh(1-\xi)}{\sinh\xi} \tag{$*$}$$

$[G']_{\xi^-}^{\xi^+} = 1 \implies -D\cosh(1-\xi) - C\cosh(\xi) = 1$ so $-D(\cosh(1-\xi)\sinh\xi + \sinh(1-\xi)\cosh(\xi)) = -D\sinh(1) = \sinh(\xi)$.

$$D = -\frac{\sinh\xi}{\sinh 1}, \quad C = -\frac{\sinh(1-\xi)}{\sinh 1}$$

So the solution is

$$y(x) = -\frac{\sinh(1-x)}{\sinh 1}\int_0^x \sinh\xi f(\xi)\mathrm{d}\xi - \frac{\sinh x}{\sinh 1}\int_x^1 \sinh(1-\xi)f(\xi)\mathrm{d}\xi \tag{7.22}$$

Inhomogeneous B.C's

Find y_p solution to $\mathcal{L}y = 0$ satisfying inhomogeneous b.c's $y(a), y(b) \neq 0$. Find Green's function for $\mathcal{L}y_g = f$ with $y_g(a) = y_g(b) = 0$ where $y_g = y - y_p$.

E.g: $y'' - y = f(x)$ with $y(0) = 0, y(1) = 1$. Then $y_p = A\sinh x + B\cosh x$. $y_p(0) = 0 \implies B = 0$, $y_p(1) = 1 \implies A = \frac{1}{\sinh 1}$. Solve for $y_g = y - y_p$ with $y_g(0) = y_g(1) = 0$. Solution

$$y(x) = \frac{\sinh x}{\sinh 1} + y_g$$

Higher-order ODEs

If $\mathcal{L}y = f(x)$ to nth order (coefficient $\alpha(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$) with homogeneous b.c's, then generalise Green's function $\mathcal{L}G(x,\xi) = \delta(x-\xi)$ with properties:

1. G_1, G_2 homogeneous solutions satisfying homogeneous b.c's
2. Continuity: $G_1 = G_2$, $G_1' = G_2', \ldots, G_1^{(n-2)} = G_2^{(n-2)}$ at $x = \xi$
3. Jump conditions $(n-1)$th derivative

$$[G^{(n-1)}]_{\xi^-}^{\xi^+} = G_2^{(n-1)}\Big|_{\xi^+} - G_1^{(n-1)}\Big|_{\xi^-} = \frac{1}{\alpha(\xi)}$$

Eigenfunction expansion $G(x,\xi)$

Suppose $\mathcal{L}$ is in S-L form (2.7) with eigenfunctions $y_n(x)$ and eigenvalues λ_n. Then seek $G(x,\xi) = \sum_{n=1}^{\infty} A_n y_n(x)$ satisfying $\mathcal{L}G = \delta(x-\xi)$

$$\begin{aligned}\mathcal{L}G = \sum_n A_n \mathcal{L} y_n(x) &= \sum_n A_n \lambda_n w(x) y_n(x) \\ &= \delta(x-\xi) = w(x) \sum_n y_n(\xi) \frac{y_n(x)}{\mathcal{N}_n}\end{aligned}$$

with $\mathcal{N}_n = \int w y_n^2 \mathrm{d}x$. So $A_n(\xi) = \frac{y_n(\xi)}{\lambda_n \mathcal{N}_n}$. Thus

$$G_n(x,\xi) = \sum_{n=1}^{\infty} \frac{y_n(\xi) y_n(x)}{\lambda_n \int w y_n^2 \mathrm{d}x} \tag{7.23}$$

which we obtained without $\delta(x-\xi)$ in (2.31).

7.4 Constructing $G(x,\xi)$: Initial value problem

Solve $\mathcal{L}y = f(t)$ for $t \geq a$ with $y(a) = y'(a) = 0$ (7.24) using $G(t,\tau)$ satisfying $\mathcal{L}G = \delta(x-\tau)$ with same b.c's.

For $t < \tau$: $G_1 = Ay_1(t) + By_2(t)$ with $Ay_1(a) + By_2(a) = 0$ and $Ay_1'(a) + By_2'(a) = 0$. If $A \neq B \neq 0$ then $y_1 y_2' - y_2 y_1' = 0$, so must have $A = B = 0$ as $W(a) \neq 0$.

So $G(t,\tau) \equiv 0$ for $a \leq t < \tau$, i.e no change until impulse $t = \tau$.

For $t > \tau$: by the continuity of G (7.12) we have $G_2(\tau,\tau) = 0$. So choose $G_2 = Dy_+(t)$ with

$$y_+(\tau) = Ay_1(\tau) + By_2(\tau) = 0$$

But by discontinuity in G' (7.13)

$$[G']_{\tau^-}^{\tau^+} = G_2'(\tau,\tau) - G_1'(\tau,\tau) = Dy_+'(\tau) = \frac{1}{\alpha(\tau)}$$

i.e $Ay_1'(\tau) + By_2'(\tau) = \frac{1}{\alpha(\tau)}$ and $D = \frac{1}{\alpha(t) y_+'(\tau)}$. Hence we have

$$G(t,\tau) = \begin{cases} 0 & t < \tau \\ \frac{y_+(t)}{\alpha(\tau) y_+'(\tau)} & t > \tau \end{cases} \tag{7.25}$$

The IVP (7.24) has solution

$$y(t) = \int_a^t G_2(t,\tau) f(\tau) \mathrm{d}\tau = \int_a^t \frac{y_+(t) f(\tau)}{y_+'(\tau)} \mathrm{d}\tau \tag{7.26}$$

Causality is “built in” so only forces prior to t affect the solution at t.

Example: Solve $y'' - y = f(t)$ with $y(0) = y'(0) = 0$.

1. Homogeneous solution and i.c's
   - $t < \tau$, $G_1 \equiv 0$
   - $t > \tau$, $G_2 = Ae^t + Be^{-t}$
2. Continuity implies $G_2(\tau, \tau) = 0 \Rightarrow G_2 = D\sinh(t - \tau)$
3. $[G'] = \frac{1}{\alpha} = 1 \Rightarrow G'(\tau, \tau) = D\cosh(0) = D = 1$

Hence, solution (7.26) is

$$y(t) = \int_0^t f(\tau)\sinh(t - \tau)\mathrm{d}\tau$$

8 Fourier Transforms

8.1 Introduction

Definition. The *Fourier transform* of a function $f(x)$ is

$$\tilde{f}(k) = \mathcal{F}(f)(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x \tag{8.1}$$

and the *inverse Fourier transform* is

$$f(x) = \mathcal{F}^{-1}(\tilde{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\mathrm{d}x \tag{8.2}$$

Theorem 8.1 (Fourier inversion theorem)**.**

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = f(x) \tag{8.3}$$

with a sufficient condition that f and $\tilde{f}$ are absolutely integrable (i.e $\int |f|\mathrm{d}x < \infty$)

Gaussian example: Find the FT of

$$f(x) = \frac{1}{\sigma\sqrt{\pi}}e^{-x^2/\sigma^2} \tag{8.4}$$

$$\tilde{f}(k) = \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}e^{-ikx}\mathrm{d}x = \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\cos(kx)\mathrm{d}x$$

Consider (Leibnitz rule for differentation under the integral sign)

$$\begin{aligned}
\frac{\mathrm{d}\tilde{f}}{\mathrm{d}k} = \tilde{f}'(k) &= -\frac{\sigma\sqrt{\pi}}{\int}{}^{\infty}_{-\infty} xe^{-x^2/\sigma^2}\sin kx\mathrm{d}x \\
&= \frac{1}{\sigma\sqrt{\pi}}\left[\frac{\sigma^2}{2}e^{-x^2/\sigma^2}\sin(kx)\right]_{-\infty}^{\infty} - \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\frac{k\sigma^2}{2}\right)e^{-x^2/\sigma^2}\cos(kx)\mathrm{d}x \\
&= -\frac{k\sigma^2}{2}\tilde{f}(k)
\end{aligned}$$

Integrate $\frac{\tilde{f}'}{\tilde{f}} = -\frac{k\sigma^2}{2}$ to find $\tilde{f}(k) = Ce^{-k^2\sigma^2/4}$. But put $k = 0$ in (8.4) and $\tilde{f}(0) = 1 \Rightarrow C = 1$ so

$$\tilde{f}(k) = e^{-k^2\sigma^2/4} \tag{8.5}$$

8.2 Fourier Transform relation to Fourier Series

We can write FS (1.13) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \tag{$*$}$$

where $k_n = \frac{n\pi}{L}$, so write $k_n = n\Delta k$ with $\Delta k = \frac{\pi}{L}$. Then

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik_n x}\mathrm{d}x = \frac{\Delta k}{2\pi}\int_{-L}^{L} f(x)e^{-ik_n x}\mathrm{d}x$$

So FS $(*)$ becomes

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} e^{ik_n x}\int_{-L}^{L} f(x')e^{-ik_n x'}\mathrm{d}x'$$

But

$$\sum_{n=-\infty}^{\infty} \Delta k_n g(k_n) \to \int_{-\infty}^{\infty} g(k)\mathrm{d}k \tag{8.6b}$$

where

$$g(k_n) = \frac{e^{ik_n x}}{2\pi}\int_{-L}^{L} f(x)e^{-ikx'}\mathrm{d}x'$$

So take limit $L \to \infty$ and we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k e^{ikx}\left[\int_{-\infty}^{\infty} f(x')e^{-ikx'}\mathrm{d}x'\right] = \mathcal{F}^{-1}(\mathcal{F}(f))(x)$$

Note that when $f(x)$ is discontinuous at x the FT gives

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \frac{1}{2}(f(x_-) + f(x_+)) \tag{8.7}$$

8.3 FT Properties

<u>1: Linearity</u>:

$$h(x) = \lambda f(x) + \mu g(x) \iff \tilde{h}(k) = \lambda\tilde{f}(k) + \mu\tilde{g}(k) \tag{8.8}$$

<u>2: Translation</u>

$$h(x) = f(x-\lambda) \iff \tilde{h}(k) = e^{-i\lambda k}\tilde{f}(k) \tag{8.9}$$

<u>3: Frequency shift</u>:

$$h(x) = e^{i\lambda x}f(x) \iff \tilde{h}(k) = \tilde{f}(k-\lambda) \tag{8.10}$$

<u>4: Scaling</u>:

$$h(x) = f(\lambda x) \iff \tilde{h}(k) = \frac{1}{|\lambda|}\tilde{f}(k/\lambda) \tag{8.11}$$

5: Multiplication by x:

$$h(x) = xf(x) \iff \tilde{h}(k) = i\tilde{f}(k) \tag{8.12}$$

6: Derivative:

$$h(x) = f'(x) \iff \tilde{h}(k) = ik\tilde{f}(k) \tag{8.13}$$

7: Duality: Consider (8.2) with $x \mapsto -x$

$$f(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{-ikx}\mathrm{d}k$$

so $k \leftrightarrow x$ gives

$$f(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(x)e^{-kx}\mathrm{d}x$$

Thus

$$g(x) = \tilde{f}(x) \iff \tilde{g}(k) = 2\pi f(-k) \tag{8.14}$$

We have $f(-x) = \frac{1}{2\pi}\mathcal{F}^2(f)(x)$, so repeating $\mathcal{F}^4(f)(x) = 4\pi^2 f(x)$

"Top hat" example:

Find FT for

$$f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x = \int_{-a}^{a} \cos kx\mathrm{d}x = \frac{2\sin ka}{k} \tag{8.15}$$

Fourier inversion theorem (8.3) implies

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{\sin ka}{k}\mathrm{d}k = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Now set $x = 0$, then take $k \to x$ to obtain the Dirichlet discontinuous formula:

$$\int_0^{\infty} \frac{\sin ax}{x}\mathrm{d}x = \begin{cases} \frac{\pi}{2} & a > 0 \\ 0 & a = 0 \\ -\frac{\pi}{2} & a < 0 \end{cases} = \frac{\pi}{2}\,\mathrm{sgn}(a) \tag{8.16}$$

Here, we allow $a < 0$, so $\sin(-ax) = -\sin(ax)$

8.4 Convolution and Parseval's Theorems

We want to multiply FTs in the frequency domains $\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$ so consider the inverse

$$\begin{aligned}
h(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}(k)e^{ikx}\mathrm{d}k \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)e^{-iky}\mathrm{d}y\right)\tilde{g}(k)e^{ikx}\mathrm{d}k \\
&= \int_{-\infty}^{\infty} f(y)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(k)e^{ik(x-y)}\mathrm{d}k\right)\mathrm{d}y \\
&= \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y \equiv f * g(x) \qquad (8.17)
\end{aligned}$$

i.e convolution definition. By duality (8.14) we also have

$$h(x) = f(x)g(x) \iff \tilde{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(p)\tilde{g}(k-p)\mathrm{d}p \qquad (8.18)$$

Parseval's Theorem

Consider $h(x) = g^*(-x)$, then

$$\tilde{h}(k) = \int_{-\infty}^{\infty} g^*(-x)e^{-ikx}\mathrm{d}x = \left[\int_{-\infty}^{\infty} g(-x)e^{ikx}\mathrm{d}x\right]^* = \left[\int_{-\infty}^{\infty} g(y)e^{-iky}\mathrm{d}y\right]^* = \tilde{g}^*(k)$$

Substitute into (8.17) $g(x) \mapsto g^*(-x)$

$$\int_{-\infty}^{\infty} f(y)g^*(y-x)\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{g}^*(k)e^{ikx}\mathrm{d}x$$

Take $x = 0$, then dummy $y \to x$ on LHS

$$\int_{-\infty}^{\infty} f(x)g^*(x)\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(k)\tilde{g}^*(k)\mathrm{d}k \qquad (8.19)$$

i.e $\langle g, f\rangle = \frac{1}{2\pi}\langle\tilde{g}, \tilde{f}\rangle$. Now set $g^* = f^*$:

$$\int_{-\infty}^{\infty} |f(x)|^2\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\tilde{f}(k)|^2\mathrm{d}k \qquad (8.20)$$

8.5 Fourier Transforms of Generalised Functions

We will apply $\mathcal{F}$ to generalised functions f (e.g $\delta(x), H(x)$). These can be treated as limiting distributions for which $\mathcal{F}(f)$ has a limiting approximation. This is shown with inner products with integrable (Schwarz) functions using Parseval's Theorem (8.19).

Dirac delta $\delta(x)$

Consider the inversion

$$\begin{aligned} f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)e^{-iku}\mathrm{d}u\right]e^{ikx}\mathrm{d}k \\ &= \int_{-\infty}^{\infty} f(u)\underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-u)}\mathrm{d}k\right]}_{\delta(x-u)}\mathrm{d}u \end{aligned}$$

so identify

$$\delta(x-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-u)}\mathrm{d}k$$

If $f(x) = \delta(x)$ then

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \delta(x)e^{ikx}\mathrm{d}x = 1 \tag{8.21}$$

If $f(x) = 1$, then

$$\tilde{f}(k) = \int -\infty^{\infty} e^{-ikx}\mathrm{d}x = 2\pi\delta(k) \tag{8.22}$$

If $f(x) = \delta(x-a)$ then

$$\tilde{f}(k) = e^{-ika} \tag{8.23}$$

Trigonometric functions

$$\begin{cases} f(x) = \cos\omega x \iff \tilde{f}(k) = \pi(\delta(k+\omega) + \delta(k-\omega)) \\ f(x) = \sin\omega x \iff \tilde{f}(k) = i\pi(\delta(k+\omega) - \delta(k-\omega)) \end{cases} \tag{8.24}$$

Heaviside function:

Subtle derivation requiring $H(0) = 1/2$; then

$$H(x) + H(-x) = 1 \ \forall x$$

By (8.22)

$$\tilde{H}(k) + \tilde{H}(-k) = 2\pi\delta(k) \tag{$*$}$$

Recall (6.7) $H'(x) = \delta(x)$, which implies

$$ik\tilde{H}(k) = 1 \tag{$\dagger$}$$

But $k\delta(k) = 0$, so $*$ and $\dagger$ consistent if

$$\tilde{H}(k) = \pi\delta(k) + \frac{1}{ik} \tag{8.25}$$

Dirichlet discontinuous formula (8.16)

Rewrite as

$$\frac{1}{2}\operatorname{sgn}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{ik}\mathrm{dk}$$

so

$$f(x) = \frac{1}{2}\operatorname{sgn}(x) \iff \tilde{f}(k) = \frac{1}{ik} \tag{8.26}$$

8.6 Applications of Fourier Transforms

Motivation I: ODE for Boundary Value Problem

Consider $y'' - y = f(x)$ with homogeneous b.c's $y \to 0$ as $x \to \pm\infty$.

Take the FT:

$$(-k^2 - 1)\tilde{y} = \tilde{f}$$

so the solution is

$$tildey(k) = -\frac{\tilde{f}(k)}{1+k^2} \equiv \tilde{f}(k)\tilde{g}(k)$$

where $\tilde{g}(k) = -\frac{1}{1+k^2}$, but this is the FT of

$$g(x) = -\frac{1}{2}e^{-|x|}$$

Thus convolution theorem (8.17) implies

$$\begin{aligned} y(x) &= \int_{-\infty}^{\infty} f(u)g(x-u)\mathrm{d}u = -\frac{1}{2}\int_{-\infty}^{\infty} f(u)e^{-|x-u|}\mathrm{d}u \\ &= -\frac{1}{2}\int_{-\infty}^{x} f(u)e^{u-x}\mathrm{du} - \frac{1}{2}\int_{x}^{\infty} f(u)e^{x-u}\mathrm{d}u \end{aligned}$$

which is in the form of a BVP Green's function (7.20).

Motivation II: Signal Processing (IVP):

Suppose (given) input $\mathcal{I}(t)$ is acted on by linear operator $\mathcal{L}$ to yield output $\mathcal{O}(t)$. The FT $\tilde{\mathcal{I}}(\omega)$ is denoted the resolution

$$\tilde{\mathcal{I}}(\omega) = \int_{-\infty}^{\infty} \mathcal{I}(t)e^{-i\omega t}\mathrm{d}t \tag{8.27}$$

In frequency domain, action of $\mathcal{L}\mathcal{I}(t)$ means $\tilde{\mathcal{I}}$ is multiplied by a transfer function $\tilde{\mathcal{R}}(\omega)$ to yield output

$$\mathcal{O}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathcal{R}}(\omega)\tilde{\mathcal{I}}(\omega)e^{i\omega t}\mathrm{d}\omega \tag{8.28}$$

with response function given by

$$\mathcal{R}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathcal{R}}(\omega)e^{i\omega t}\mathrm{d}\omega \tag{8.29}$$

By the convolution theorem (8.17), output is

$$\mathcal{O}(t) = \int_{-\infty}^{\infty} \mathcal{I}(u)\mathcal{R}(t-u)\mathrm{d}u$$

We assume no input $\mathcal{I}(t) = 0$ for $t < 0$, by causality zero output for $R(t) = 0$, $t < 0$. So we require $0 < u < t$:

$$\mathcal{O}(t) = \int_{0}^{t} \mathcal{I}(a)\mathcal{R}(t-u)\mathrm{d}u \tag{8.30}$$

General transfer functions for ODEs

Suppose input/output relation given by a linear ODE

$$\mathcal{L}\mathcal{O}(t) \equiv \left(\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i}{\mathrm{d}x^i}\right)\mathcal{O}(t) \equiv \mathcal{I}(t) \tag{8.31}$$

Take the FT:

$$\left(a_0 + a_1(i\omega) + a_2(i\omega)^2 + \ldots + a_n(i\omega)^n\right)\tilde{\mathcal{O}}(\omega) = \tilde{\mathcal{I}}(\omega)$$

so the transfer function (8.28) is

$$\tilde{\mathcal{R}}(\omega) = \frac{1}{a_0 + a_1(i\omega) + \ldots + a_n(i\omega)^n} \tag{8.32}$$

Factorise nth degree polynomial into product of n roots $(i\omega - c_j)^{k_j}$ with $j = 1, \ldots, J$ (with repeated roots if $k_j > 1$ i.e $\sum_{j=1}^{J} k_j$). Then

$$\tilde{\mathcal{R}} = \frac{1}{(i\omega - c_1)^{k_1}\ldots(i\omega - c_J)^{k_J}}$$

Recall that

$$\begin{aligned}\tilde{\mathcal{R}} &= \frac{1}{a_0 + a_1(i\omega) + \ldots + a_n(i\omega)^n} \\ &= \frac{1}{(i\omega - c_1)^{k_1}\ldots(i\omega - c_J)^{k_J}} \\ &= \sum_{j=1}^{J}\sum_{m=1}^{k_j}\frac{\Gamma_{jm}}{(i\omega - c_j)^m} \end{aligned} \tag{8.33}$$

since it can be expanded in partial fractions (constant Γ_{jm}). For repeated roots

$$\frac{1}{(i\omega - c_j)^{k_j}} \to \frac{\Gamma_{j1}}{(i\omega - c_j)} + \frac{\Gamma_{j2}}{(i\omega - c_j)^2} + \ldots + \frac{\Gamma_{jk_j}}{(i\omega - c_j)^{k_j}}$$

To solve we must invert $\frac{1}{(i\omega - a)^m}$ for $m \geq 1$. We know (8.6a)

$$\mathcal{F}^{-1}(\frac{1}{i\omega - a}) = \begin{cases} e^{at} & t > 0 \\ 0 & t < 0 \end{cases}$$

for $\Re(a) < 0$, so we assume that $\Re(c_j) < 0$ for all j to eliminate exponential growing solutions.

For $m = 2$ note $i\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1}{i\omega - a}\right) = \frac{1}{(i\omega - a)^2}$. Recal (8.12) $\mathcal{F}(tf(t)) = i\tilde{f}'(\omega)$, so

$$\mathcal{F}^{-1}\left(\frac{1}{(i\omega - a)^2}\right) = \begin{cases} te^{at} & t > 0 \\ 0 & t < 0 \end{cases}$$

By induction

$$\mathcal{F}^{-1}\left(\frac{1}{(i\omega - a)^m}\right) = \begin{cases} \frac{t^{m-1}}{(m-1)!}e^{at} & t > 0 \\ 0 & t < 0 \end{cases} \tag{8.34}$$

Thus the response function takes the form

$$\mathcal{R}(t) = \sum_j \sum_m \Gamma_{jm}\frac{t^{m-1}}{(m-1)!}e^{c_j t},\ t > 0 \tag{8.35}$$

We can solve (8.31) in Green's function form (8.30) or directly invert $\mathcal{R}(\omega)\mathcal{I}(\omega)$ for polynomial $\tilde{\mathcal{I}}(\omega)$.

<u>Example: Damped oscillator</u>:

Solve

$$\mathcal{L}y \equiv y'' + 2py' + (p^2 + q^2)y = f(t)$$

with damping $p > 0$ and homogeneous i.c's $y(0) = y'(0) = 0$. Fourier transform is

$$(i\omega)^2\tilde{y} + 2ipw\tilde{y} + (p^2 + q^2)\tilde{y} = \tilde{f}$$

So

$$\tilde{y} = \frac{\tilde{f}}{-\omega^2 + 2ip\omega + p^2 + q^2} \equiv \tilde{R}\tilde{f}$$

Inverting with convolution theorem (8.17)

$$y(t) = \int_0^t \mathcal{R}(t-\tau) f(\tau) \mathrm{d}\tau$$

with response function

$$\mathcal{R}(t-\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega(t-\tau)}}{p^2 + q^2 + 2ip\omega - \omega^2}$$

8.7 The Discrete Fourier Transform

Discrete sampling & the Nyquist frequency

Sample a signal $h(t)$ at equal times $t_n = n\Delta$ with time-sampling Δ, and values

$$h_n = h(n\Delta), \; n = \dots, -2, -1, 0, 1, 2, \dots \tag{8.36}$$

i.e with sampling freqency $\frac{1}{\Delta}$. The Nyquist frequency

$$f_c = \frac{1}{2\Delta}$$

is the highest frequency actually sampled at Δ.

Suppose we have a signal with given frequency f

$$\begin{aligned} g_f(t) &= A\cos(2\pi f t + \phi) = \Re(Ae^{2\pi i f t + \phi}) \\ &= \frac{1}{2}(A^{i\phi} e^{2\pi i f t} + A^{-i\phi} e^{-2\pi i f t}) \end{aligned} \tag{8.38}$$

i.e for complex FS, sum of poisitive frequency f and negative frequency $-f$ modes.

What happens if we sample at Nyquist $f = f_c$?

$$\begin{aligned} g_{f_c}(t_n) &= A\cos(2\pi(\frac{1}{2\Delta}))n\Delta + \phi) \\ &= A\cos\pi n \cos\phi + A\sin n\pi \sin\phi \\ &= A'\cos(2\pi f_c t_n) \end{aligned} \tag{8.39}$$

so phase/amplitude info is lost and we can identify $f_n \leftrightarrow -f_c$ i.e (8.38) and (8.39) are aliased together.

What happens if we sample $f > f_c$?

Exercise: Take $f = f_c + \delta f > f_c$ and show that

$$\begin{aligned} g_f(t_n) &= A\cos(2\pi(f_c+\delta f)t_n + \phi) \\ &= A\cos(2\pi(f_c-\delta f)t_n - \phi) \end{aligned}$$

So the effect is to alias a "ghost signal" to frequency $f_c - \delta f$.

Sampling Theorem

A signal $g(t)$ is bandwidth limited if it contains no frequencies above $\omega_{\max} = 2\pi f_{\max}$, i.e $\tilde{g}(\omega) = 0$ for $|\omega| > \omega_{\max}$. So

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(\omega)e^{i\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_{\max}}^{\omega_{\max}} \tilde{g}(\omega)e^{i\omega t}\mathrm{d}\omega \tag{8.41}$$

Set sampling to satisfy Nyquist condition $\Delta = \frac{1}{2f_{\max}}$ then

$$g_n \equiv g(t_n) = \frac{1}{2\pi}\int_{-\omega_{\max}}^{\omega_{\max}} \tilde{g}(\omega)e^{\frac{i\pi n\omega}{\omega_{\max}}}\mathrm{d}\omega$$

which is a complex FS with coefficients (1.13) $c_n \times \frac{\omega_{\max}}{\pi}$. The FS represents a periodic function

$$\tilde{g}_{\mathrm{per}}(\omega) = \frac{\pi}{\omega_{\max}}\sum_{n=-\infty}^{\infty} g_n e^{\frac{-i\pi n\omega}{\omega_{\max}}} \tag{8.42}$$

The actual FT $\tilde{g}(\omega)$ is found by multiplying by a "top hat"

$$\tilde{h}(\omega) = \begin{cases} 1 & |\omega| \le \omega_{\max} \\ 0 & \text{otherwise} \end{cases}$$

i.e

$$\tilde{g}(\omega) = \tilde{g}_{\mathrm{per}}(\omega)\tilde{h}(\omega) \tag{8.43}$$

which is an exact relation. Inverting this with (8.42)

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}_{\mathrm{per}}(\omega)\tilde{h}(\omega)e^{i\omega t}\mathrm{d}\omega \\ &= \frac{1}{2\omega_{\max}}\sum_{n=-\infty}^{\infty} g_n \int_{-\omega_{\max}}^{\omega_{\max}} \exp\left(i\omega\left(t - \frac{n\pi}{\omega_{\max}}\right)\right)\mathrm{d}\omega \\ &= \sum_{n=-\infty}^{\infty} g_n \frac{\sin(\omega_{\max}t - n\pi)}{\omega_{\max}t - \pi n} \end{aligned} \tag{8.44}$$

So $g(t)$ can be exactly represented after sampling at discrete times t_n (sampling theorem).

Suppose we have a finite number N of samples

$$h_m = h(t_m),\ t_m = m\Delta,\ m = 0, 1, 2, \ldots, N-1 \tag{8.45}$$

We want to approximate the Fourier Transform for N frequencies within Nyquist ($f_c = 1/(2\Delta)$) frequency, using equally spaced frequencies ($\Delta f = \frac{1}{N\Delta}$) in the range $-f_c \leq f \leq f_c$.

We could take $f_n = n\Delta f = \frac{n}{N\Delta}$ with $n = -\frac{N}{2}, -\frac{N}{2}+1, \ldots, -1, 0, 1, \ldots, \frac{N}{2}$. This has $N+1$ frequencies, but f_c and $-f_c$ are aliaised (8.39).

Note also that $(\frac{N}{2}+m)\Delta f = f_c + \delta f$ is aliased back to $(\frac{N}{2}-m)\Delta f = -(f_c - \delta f)$ from (8.40) so we choose instead $f_n = \frac{n}{N\Delta}$ with

$$n = 0, 1, 2, \ldots, \frac{N}{2}-1, \frac{N}{2}, \frac{N}{2}+1, \ldots, N-1 \tag{8.46}$$

The <u>discrete Fourier Transform</u> at frequency f_n becomes

$$\begin{aligned}\tilde{h}(f_n) &= \int_{-\infty}^{\infty} h(t)e^{-2\pi i f_n t}\mathrm{d}t \\ &\approx \Delta \sum_{m=0}^{N-1} h_m e^{-2\pi i f_n t_m} = \Delta \sum_{m=0}^{N-1} h_m e^{-2\pi i \frac{mn}{N}} \equiv \Delta \tilde{h}_d(f_n)\end{aligned} \tag{8.47}$$

Where $\tilde{h}_d(f_n) \equiv \tilde{h}_n$ is the discrete FT.

So the matrix

$$[DFT]_{mn} = e^{-2\pi i \frac{mn}{N}} \tag{8.48}$$

defines the discrete FT for $\mathbf{h} = \{h_m\}$ as $\tilde{\mathbf{h}}_d = [DFT]\mathbf{h}$.

The <u>inverse</u> is its adjoint $[DFT]^{-1} = \frac{1}{N}[DFT]^{\dagger}$ and it's built from roots of unity $\omega = e^{-2\pi i/N}$, e.g $N = 4, \omega = -i$

$$DFT = \begin{pmatrix} 1 & 1 & 1 & 1 & \\ 1 & -i & -1 & i & \\ 1 & -1 & 1 & -1 & \\ 1 & i & -1 & & -i \end{pmatrix}$$

So we have $\tilde{\mathbf{h}} = \frac{1}{N}[DFT]^{\dagger}\mathbf{h}$. The <u>inverse DFT</u> is

$$\begin{aligned} h_m = h(t_m) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h}(\omega)e^{i\omega t_m}\mathrm{d}\omega = \int_{-\infty}^{\infty} \tilde{h}(f)e^{2\pi i f t_m}\mathrm{d}f \\ &\approx \frac{1}{\Delta N}\sum_{n=0}^{N-1} \Delta \tilde{h}_d(f_n) e^{2\pi i \frac{mn}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{h}_n e^{2\pi i \frac{mn}{N}} \end{aligned} \tag{8.48}$$

or interpolating FS $h(t) = \frac{1}{N}\sum_{n=0}^{N} \tilde{h}_n e^{2\pi n t/N}$.

Exercise: Establish Parseval's Theorem for the DFT:

$$\sum_{m=0}^{N-1} |h_m|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |\tilde{h}_n|^2 \tag{8.49}$$

The convolution theorem for g_m, h_m is then

$$c_k = \sum_{m=0}^{N-1} g_m h_{k-m} \iff \tilde{c}_k = \tilde{g}_k \tilde{h}_k \tag{8.50}$$

9 Characteristics

9.1 Well-Posed Cauchy Problems

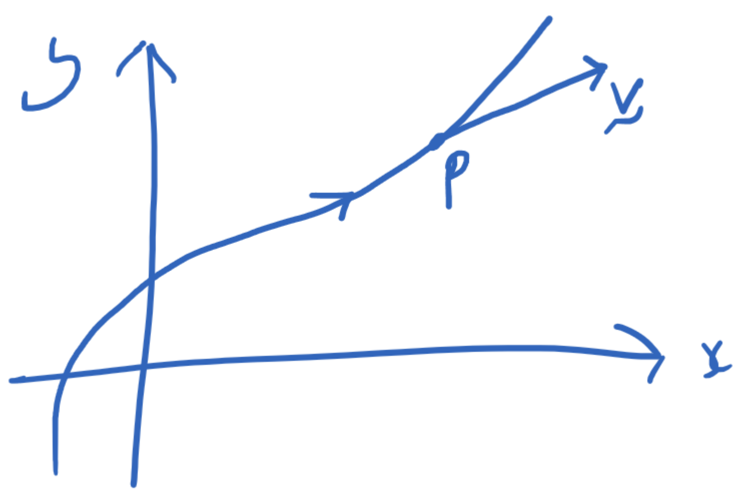
Solving PDEs depends on the nature of the equations in combination with the boundary and/or intitial data. A Cauchy problem is the PDE for ϕ together with the auxilliary data (in ϕ and its derivatives) specified on a surface (or a curve in 2D), which is called Cauchy data. A Cauchy problem is well-posed if

(i) A solution exists

(ii) The solution is unique

(iii) The solution depends continuously on the auxilliary data

9.2 Method of Characteristics



Consider a parameterised curve C given by $(x(s), y(s))$ with tangent vector

$$\mathbf{v} = \left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}, \frac{\mathrm{d}y(s)}{\mathrm{d}s}\right)$$

For a function $\phi(x, y)$ we can define a directional derivative

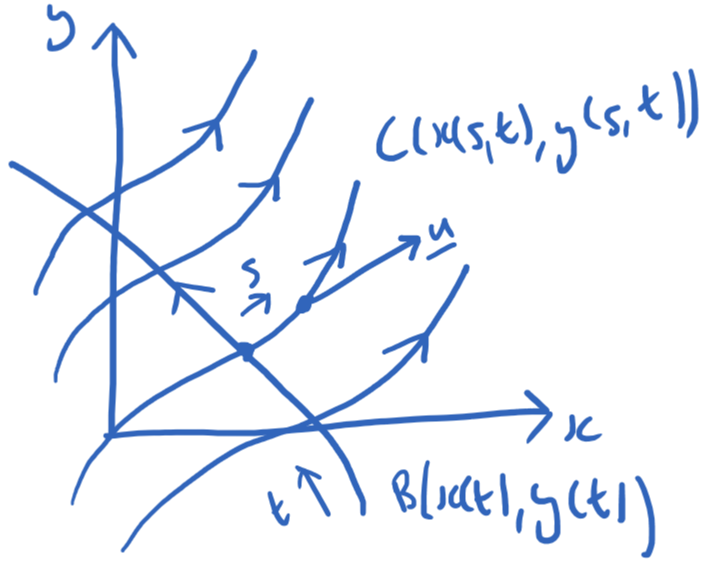
$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|_C = \frac{\mathrm{d}x(s)}{\mathrm{d}s}\frac{\partial \phi x}{\partial +}\frac{\mathrm{d}y(s)}{\mathrm{d}s}\frac{\partial \phi}{\partial y} = \mathbf{v} \cdot \nabla \left.\phi\right|_C \tag{9.1}$$

If $\mathbf{v} \cdot \nabla\phi = 0$, then $\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$ and $\phi = \textit{const}$ along C.

Now suppose we have a vector field

$$\mathbf{u} = (\alpha(x, y), \beta(x, y)) \tag{9.2}$$

with its family of integral curves C non-intersecting and filling $\mathbb{R}$, i.e at a point (x, y) the integral curve has tangent vector $\mathbf{u}(x, y)$.



Define a curve B by $(x(t), y(t))$ transverse to $\mathbf{u}$, such that its tangent vector

$$\mathbf{w} = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}, \frac{\mathrm{d}y(t)}{\mathrm{d}t}\right)$$

is nowhere parallel to $\mathbf{u}$.

Label each integral curve C of $\mathbf{u}$ using t at the intersection point with B, then use s to parameterise along the curve (i.e take $s = 0$ at B). Our integral curves $(x(s,t), y(s,t))$ satisfy

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \alpha(x,y), \ \frac{\mathrm{d}y}{\mathrm{d}s} = \beta(x,y) \tag{9.3}$$

Solve these to find a family of characteristic curves along which t remains constant.

9.3 Characteristics of a 1st order PDE

Consider a 1st order linear PDE

$$\alpha(x,y)\frac{\partial \phi}{\partial x} + \beta(x,y)\frac{\partial \phi}{\partial y} = 0 \tag{9.4}$$

with specified Cauchy data on an initial curve $B(x(t), y(t))$.

$$\phi(x(t), y(t)) = f(t) \tag{9.5}$$

Note that from (9.1) and (9.2) that

$$\alpha\phi_x + \beta\phi_y = \mathbf{u}\cdot\nabla\phi = \left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|_C$$

is the directional derivative along integral curves C of $\mathbf{u} = (\alpha, \beta)$, called the <u>characteristic curves of the PDE</u>. Now since

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \alpha\phi_x + \beta\phi_y = 0$$

from (9.4), the function $\phi(x,y)$ will be constant along the curves C. Hence the Cauchy data $f(t)$ defined on B at $s = 0$ will be propagated constantly along the curves C to give the solution

$$\phi(s,t) = \phi(x(s,t), y(s,t)) = f(t) \tag{9.6}$$

To obtain $\phi(x,y)$ transform coordinates from $\phi(s,t)$ using $s = s(x,y), t = (x,y)$ (provided Jacobian $J = x_t y_s - x_s y_t \neq 0$) to finally obtain

$$\phi(x,y) = f(t(x,y)) \tag{9.7}$$

<u>Prescription</u>: To solve (9.4) given (9.5)

1. Find characteristic equations (9.3) $\frac{\mathrm{d}x}{\mathrm{d}s} = \alpha$, $\frac{\mathrm{d}y}{\mathrm{d}s} = \beta$
2. Parameterise IC's on $B(x(t), y(t))$ (9.8)
3. Solve characteristic equations (9.3) to find

$$x = x(s,t),\ y = y(s,t)$$

   subject to IC's (9.8) at $s = 0$

$$x(0,t) = x(t),\ y(0,t) = y(t)$$

4. Solve (9.4) with (9.1)

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \alpha\phi_x + \beta\phi_y = 0$$

   i.e (9.6) $\phi(s,t) = f(t)$

5. Invert relations $s = s(x, y)$, $t = t(x, y)$
6. Change coordinates to obtain $\phi(x, y)$

Simple example: Solve $\frac{\partial \phi}{\partial x} = 0$ with $\phi(0, y) = h(y)$ given on y-axis (solution is clearly $\phi(x, y) = h(y)$).

1. $\frac{\mathrm{d}x}{\mathrm{d}s} = \alpha = 1$, $\frac{\mathrm{d}y}{\mathrm{d}s} = \beta = 0$ $(*)$
2. y-axis $(x(t), y(t)) = (0, t)$ $(\dagger)$
3. From $(*)$ $x = s + c$, $y = d$. But $s = 0$ for $x = 0 \Rightarrow c = 0$, $y = t \Rightarrow d = t$. So $x = s$ and $y = t$
4. $\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$, so $\phi = \mathit{const}$ and $\phi(s, t) = h(t)$
5. Invert $s = x$, $t = y$
6. Solution $\phi(x, y) = h(y)$

Example: Solve $e^x \phi_x + \phi_y = 0$ with $\phi(x, 0) = \cosh x$

1. Characteristic equation $\frac{\mathrm{d}x}{\mathrm{d}s} = e^x$, $\frac{\mathrm{d}y}{\mathrm{d}s} = 1$ $(*)$
2. IC's $x(t) = t, y(t) = 0$ x-axis $(\dagger)$
3. From $(*)$ $-e^{-x} = s + c,\ y = s + d$. At $s = 0$ $(x = t)$ $-e^{-t} = c$, $y = 0 = d$. Hence $e^{-x} = e^{-t} - s,\ y = s$
4. $\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0 \Rightarrow \phi(s, t) = \cosh t$
5. $s = y$, $e^{-t} = y + e^{-x} \Rightarrow t = -\log(y + e^{-x})$
6. $\phi(x, y) = \cosh\left[-\log(y + e^{-x})\right]$

Inhomogeneous 1st order PDE

Solve

$$\alpha(x, y)\phi_x + \beta(x, y)\phi_y = \gamma(x, y) \tag{9.9}$$

with Cauchy data $\phi(x(t), y(t)) = f(t)$ on curve B.

The characteristic curves C are identical to the homogeneous cae but now (9.6) implies

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right|_C = \mathbf{u} \cdot \nabla\phi = \gamma(x, y) \tag{9.10}$$

with $\phi = f(t)$ at $s = 0$ on B.

I.e $f(t)$ no longer propagated constantly & so must solve ODE (9.10). So 'upgrade' point 4 in the prescription to integrate $\phi(s, t)$ along C, before reverting to $\phi(x, y)$.

Example: Solve $\phi_x + 2\phi_y = ye^x$ with $\phi = \sin x$ on $y = x$.

1. Characteristic equations $\frac{\mathrm{d}x}{\mathrm{d}s} = 1$, $\frac{\mathrm{d}y}{\mathrm{d}s} = 2$ $(*)$
2. IC's on $y = x$, take $(x(t), y(t)) = (t, t)$ $(\dagger)$
3. From $(*)$ $x = s + c$, $y = 2s + d$. $(\dagger)$ gives that when $s = 0$, $x = t = c$ and $y = t = d$. Hence $x = s + t$ and $y = 2s + t$
4. Solve $\frac{\mathrm{d}\phi}{\mathrm{d}s} = \gamma = ye^x = (2s + t)e^{s+t}$ with $\phi = \sin t$ at $s = 0$. Note $\frac{\mathrm{d}}{\mathrm{d}s}(2se^s) = 2e^s + 2se^s$. So $\phi(s,t) = (2s - 2 + t)e^{s+t} + const$. But $\phi(0,t) = \sin t = (t-2)e^t + const$. Hence
$$\phi(s,t) = (2s - 2 + t)e^{s+t} + \sin t + (2-t)e^t$$
5. Invert $s = y - x$, $t = 2x - y$.
6. $$\phi(x,y) = (y-2)e^x + (y - 2x + 2)e^{2x-y} + \sin(2x - y)$$

9.4 Second-order PDE classification

In two dimensions, the general 2nd order linear PDE is

$$\begin{aligned}\mathcal{L}\phi \equiv & a(x,y)\frac{\partial^2\phi}{\partial x^2} + 2b(x,y)\frac{\partial^2\phi}{\partial x\partial y} + c(x,y)\frac{\partial^2\phi}{\partial y^2} + d(x,y)\frac{\partial\phi}{\partial x} \\ & + e(x,y)\frac{\partial\phi}{\partial y} + f(x,y)\phi(x,y) = 0 \end{aligned} \tag{9.11}$$

The principal part is given by

$$\sigma_p(x,y,k_x,k_y) = k^T A k = \begin{pmatrix} k_x & k_y \end{pmatrix} \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix} \begin{pmatrix} k_x \\ k_y \end{pmatrix}$$

The PDE is classified by the properties of the eigenvalues of A:

- $b^2 - ac < 0$ <u>elliptic</u> (λ_1, λ_2 same sign)
- $b^2 - ac > 0$ <u>hyperbolic</u> (λ_1, λ_2 opposite sign)
- $b^2 - ac = 0$ <u>parabolic</u> (λ_1 or $\lambda_2 = 0$)

<u>Examples</u>:

- Wave equation (3.4) $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$ $a = \frac{1}{c^2}, b = 0, c = -1$ is hyperbolic.
- Heat equation (4.3) $a = 0, b = 0, c = -D$ is parabolic.
- Laplace equation (5.1) $a = 1, b = 0, c = 1$ is elliptic.

Characteristic curves:

A curve defined by $f(x, y) = const$ will be a characteristic of

$$\begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0 \tag{9.12}$$

(Generalisation of 1st order case $\nabla f \cdot \mathbf{u} = 0$, $\mathbf{u} = (\alpha, \beta)$)

The curve can be written as $y = y(x)$ by the chain rule

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies \frac{f_x}{f_y} = -\frac{\mathrm{d}y}{\mathrm{d}x} \tag{9.13}$$

Substituting into (9.12) we obtain

$$a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2b\frac{\mathrm{d}y}{\mathrm{d}x} + c = 0$$

for which we have quadratic solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} \tag{9.14}$$

- Hyperbolic $b^2 - ac > 0$, then 2 solutions.
- Parabolic $b^2 - ac = 0$, then 1 solution.
- Elliptic $b^2 - ac < 0$, no real solutions.

Transforming to characteristic coordinates (u, v) will set $a = c = 0$ in (9.11),so the PDE will take canonical form

$$\frac{\partial^2 \phi}{\partial u \partial v} + \text{lower order terms of } \phi_u, \phi_v, \phi = 0 \tag{9.15}$$

Example: Consider $-y\phi_{xx} + \phi_{yy} = 0(*)$ with $a = -y, b = 0, c = 1, b^2 - ac = y$.

Hyperbolic for $y > 0$. Find characteristics for $y > 0$ satisfying (9.14):

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \pm \frac{1}{\sqrt{y}} \Rightarrow \sqrt{y}\mathrm{d}y = \pm \mathrm{d}x \Rightarrow \frac{2}{3} y^{3/2} \pm x = C_\pm$$

So characteristic curves are $u = \frac{2}{3}y^{3/2} + x, v = \frac{2}{3}y^{3/2} - x$.

Derivatives are $u_x = 1, u_y = y^{1/2}, v_x = -1, v_y = y^{1/2}$. Hence $\phi_x = \phi_u u_x + \phi_v v_x = \phi_u - \phi_v$. Similarly

$$\phi_y = y^{1/2}(\phi_u + \phi_v)$$

$$\phi_{xx} = \phi_{uu} - 2\phi_{uv} + \phi_{vv}$$

$$\phi_{yy} = y(\phi_{uu} + 2\phi_{uv} + \phi_{vv}) + \frac{1}{2y^{1/2}}(\phi_u + \phi_v)$$

From $(*)$

$$-y\phi_{xx} + \phi_{yy} = y\left(4\phi_{uv} + \frac{1}{2y^{3/2}}(\phi_u + \phi_v)\right)$$

So canonical form

$$\phi_{uv} + \frac{1}{6(u+v)}(\phi_u + \phi_v) = 0$$

9.5 General solution for Wave Equation

Solve

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{^2\phi}{x^2} = 0 \text{ subject to } \phi(x, 0) = f(x),\ \phi_t(x, 0) = g(x) \tag{9.16}$$

So $a = \frac{1}{c^2}, b = 0, c = -1$. Then the characteristic equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{0 + \pm\sqrt{0 + \frac{1}{c^2}}}{\frac{1}{c^2}} = \pm c$$

So choose $u = x - ct$ and $v = x + ct$, which yields simple canonical form

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0 \tag{9.17}$$

Integrate with respect to u: $\frac{\partial\phi}{\partial v} = F(v)$ and with respect to v:

$$\phi = G(u) + \int^v F(y)\mathrm{d}y \equiv G(u) + H(v)$$

Impose I.C's at $t = 0$ when $u = v = x$

$$\phi(x,0) = G(x) + H(x) = f(x) \tag{$*$}$$

$$\phi_t(x,0) = -cG'(x) + cH'(x) = g(x) \tag{$\dagger$}$$

Differentiate $(*)$:

$$G'(x) + H'(x) = f'(x) \tag{$\ddagger$}$$

So $\frac{1}{c}(\dagger) + (\ddagger)$ implies

$$H'(x) = \frac{1}{2}(f'(x) + \frac{1}{c}g(x))$$

Integrate

$$H(x) = \frac{1}{2}(f(x) - f(0)) + \frac{1}{2c}\int_0^x g(y)\mathrm{d}y$$

Then by $(*)$

$$G(x) = \frac{1}{2}(f(x) + f(0)) - \frac{1}{2c}\int_0^x g(y)\mathrm{d}y$$

Putting these together

$$\phi(x,t) = G(x-ct) + H(x+ct) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathrm{d}y \tag{9.18}$$

Domain of dependence

Waves propagate at $v = c$, so $\phi(x,t)$ is fully determined by values of f, g in interval at $t = 0$ $[x - ct, x + ct]$.

10 Solving PDEs with Green's Functions

10.1 Diffusion equation & Fourier transform

Recall heat equation (4.3) for a conducting wire

$$\frac{\partial\theta}{\partial t}(x,t) - D\frac{\partial^2\theta}{\partial x^2}(x,t) = 0 \tag{10.1}$$

with IC's $\theta(x,0) = h(x)$ and BC's $\theta \to 0$ as $x \to \pm\infty$.

Take the FT with respect to x using (8.13)

$$\frac{\partial}{\partial t}\tilde{\theta}(k,t) = -Dk^2\tilde{\theta}(k,t)$$

Integrating gives $\tilde{\theta}(k,t) = Ce^{-Dk^2t}$ with IC's $\tilde{\theta}(k,0) = \tilde{h}(k)$ so we have

$$\tilde{\theta}(k,t) = \tilde{h}(k)e^{-Dk^2t}$$

Now invert

$$\begin{aligned}\theta(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{h}(k)e^{-Dk^2t}e^{ikx}\mathrm{d}k\\ &= \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}h(u)\exp\left(-\frac{(x-u)^2}{4Dt}\right)\mathrm{d}u\\ &\equiv \int_{-\infty}^{\infty}h(u)S_d(x-u,t)\mathrm{d}u \qquad (10.2)\end{aligned}$$

Where the <u>fundamental solution</u> is

$$S_d(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}} \qquad (10.3)$$

Also known as the diffusion kernel or source function.

<u>Note</u>: With IC's $\theta(x,0) = \theta_0\delta(x)$ then

$$\theta(x,t) = \theta_0 S_d(x,t) = \frac{\theta_0}{\sqrt{4\pi Dt}}e^{-\eta^2} \qquad (10.4)$$

where $\eta = \frac{x}{2\sqrt{Dt}}$ is the similarity parameter (4.5). I.e for $t \geq 0$ spreads smoothly as a Gaussian.

Example: Gaussian pulse

Suppose initially $f(x) = \sqrt{\frac{a}{\pi}}\theta_0 e^{-ax^2}$. Then (10.2) implies

$$\begin{aligned}
\theta(x,t) &= \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2 Dt}}\int_{-\infty}^{\infty}\exp\left[-au^2 - \frac{(x-u)^2}{4Dt}\right]\mathrm{d}u \\
&= \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2 Dt}}\int_{-\infty}^{\infty}\exp\left[-\frac{(1+4aDt)u^2 - 2xu + x^2}{4Dt}\right]\mathrm{d}u \\
&= \frac{\theta_0\sqrt{a}}{\sqrt{4\pi^2 Dt}}\int_{-\infty}^{\infty}\exp\left[-\frac{1+4aDt}{4Dt}\left(u - \frac{x}{1+4aDt}\right)^2\right]\exp\left[\frac{-ax^2}{1+4aDt}\right]\mathrm{d}u \\
&= \theta_0\sqrt{\frac{a}{\pi(1+4aDt)}}\exp\left[-\frac{ax^2}{1+4aDt}\right] \qquad (10.5)
\end{aligned}$$

Here, asymptotically the width spreads as SD $\propto \sqrt{t}$ with area constant (i.e heat energy conserved).

10.2 Forced Diffusion Equation

Consider

$$\frac{\partial}{\partial t}\theta(x,y) - D\frac{\partial^2\theta}{\partial x^2}(x,t) = f(x,t) \qquad (10.6)$$

with homogeneous IC's $\theta(x,0) = 0$.

Construct a 2D Green's function $G(x,t;\xi,\tau)$ such that

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = \delta(x-\xi)\delta(t-\tau) \qquad (10.7)$$

with $G(x,0;\xi,\tau) = 0$.

Take Fourier Transform with respect to x using (8.23)

$$\frac{\partial\tilde{G}}{\partial t} + Dk^2\tilde{G} = e^{-ik\xi}\delta(t-\tau)$$

Use multiplicative factor e^{Dk^2t}

$$\frac{\partial}{\partial t}\left[e^{Dk^2t}\tilde{G}\right] = e^{-ik\xi + Dk^2t}\delta(t-\tau)$$

Integrate with respect to t using $G = 0$ at $t = 0$

$$e^{Dk^2t}\tilde{G} = e^{-ik\xi}\int_0^t e^{Dk^2t'}\delta(t'-\tau)\mathrm{d}t' = e^{-ik\xi}e^{Dk^2\tau}H(t-\tau)$$

And so

$$\tilde{G}(k,t;\xi,\tau) = H(t-\tau)e^{-ik\xi}e^{-Dk^2(t-\tau)}$$

So inverting we get Green's function

$$\begin{aligned} G(x,t;\xi,\tau) &= \frac{H(t-\tau)}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\xi)}e^{-Dk^2(t-\tau)}\mathrm{d}k \\ &= \frac{H(t')}{2\pi}\int_{-\infty}^{\infty} e^{ikx'}e^{-Dk^2t'}\mathrm{d}k \\ &= \frac{H(t')}{\sqrt{4\pi Dt'}}e^{-\frac{x'^2}{4Dt'}} \\ &= H(t-\tau)S_d(x-\xi,t-\tau) \end{aligned} \tag{10.8}$$

where S_d is the fundamental solution (10.3). General solution is

$$\theta(x,t) = \int_0^{\infty}\int_{-\infty}^{\infty} G(x,t;\xi,\tau)f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau = \int_0^{t}\int_{-\infty}^{\infty} f(u,\tau)S_d(x-u,t-\tau)\mathrm{d}u\mathrm{d}\tau \tag{10.9}$$

This is an example of Duhamel's principle related (i) solution of forced PDE with homogeneous BC's (10.6) to (ii) solutions of homogeneous PDE with inhomogeneous BC's (10.1).

Recall solution of (10.1) with IC's at $t=\tau$

$$\theta(x,t) = \int_{-\infty}^{\infty} f(u)S_d(x-u,t-\tau)\mathrm{d}u$$

So forcing term $f(x,t)$ at $t=\tau$ acts as an initial condition for subsequent evolution.

The integral (10.9) is a superposition of all these IC effects for $0<\tau<t$.

10.3 The Forced Wave Equation

Consider

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = f(x,t) \tag{10.10}$$

with $\phi(x,0)=0$, $\phi_t(x,0)=0$. Construct Green's solution

$$\frac{\partial^2 G}{\partial t^2} - c^2\frac{\partial^2 G}{\partial x^2} = \delta(x-\xi)\delta(t-\tau)$$

with $G=0$, $G_t=0$ at $t=0$. Take Fourier Transform

$$\frac{\partial^2}{\partial t^2}\tilde{G} + c^2k^2\tilde{G} = e^{-ik\xi}\delta(t-\tau)$$

Recall §7.4 for IVP Green's function (7.26) so by inspection

$$\begin{aligned} \tilde{G} &= \begin{cases} 0 & t<\tau \\ e^{-ik\xi}\frac{\sin kc(t-\tau)}{kc} & t>\tau \end{cases} \\ &= e^{-ik\xi}\frac{\sin kc(t-\tau)}{kc}H(t-\tau) \end{aligned}$$

Invert Fourier Transform

$$\begin{aligned}
G(x,t;\xi,\tau) &= \frac{H(t-\tau)}{2\pi c}\int_{-\infty}^{\infty} e^{ik(x-\xi)}\frac{\sin kc(t-\tau)}{k}\mathrm{d}k \\
&= \frac{H(t-\tau)}{\pi c}\int_0^{\infty}\frac{\cos kA\sin kB}{k}\mathrm{d}k \\
&= \frac{H(t-\tau)}{2\pi c}\int_0^{\infty}\frac{\sin k(A+B)-\sin k(A-B)}{k}\mathrm{d}k \\
&= \frac{H(t-\tau)}{2\pi c}\left[\mathrm{sgn}(A+B)-\mathrm{sgn}(A-B)\right]
\end{aligned}$$

Now with $H(t-\tau) \Rightarrow B = c(t-\tau) > 0$, so only non-zero if $|A| < B$ i.e $|x-\xi| < c(t-\tau)$.

So Green's function or causal fundamental solution is

$$G(x,t;\xi,\tau) = \frac{1}{2c}H(c(t-\tau)-|x-\xi|) \tag{10.11}$$

The solution is

$$\begin{aligned}
\phi(x,t) &= \int_0^{\infty}\int_{-\infty}^{\infty} f(\xi,t)G(x,t;\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau \\
&= \frac{1}{2c}\int_0^{t}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau
\end{aligned} \tag{10.12}$$

10.4 Poission Equation

Solve $\nabla^2\phi = -\rho(\mathbf{r})$ (10.3) on domain D with Dirichlet BC's $\phi = 0$ on ∂D.

Fundamental solution:

The $\delta(\mathbf{r})$ function in $\mathbb{R}^3$ has the following properties

- $$\delta(\mathbf{r}-\mathbf{r}') = 0 \text{ for all } \mathbf{r} \neq \mathbf{r}'$$
- $$\int_D \delta(\mathbf{r}-\mathbf{r}')\mathrm{d}^3\mathbf{r} = \begin{cases} 1 & \mathbf{r}' \in D \\ 0 & \text{otherwise} \end{cases}$$
- $$\int_D f(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')\mathrm{d}^3 bfr = f(\mathbf{r}')$$

The free-space Green's function is defined to be

$$\nabla^2 G_{FS}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{10.15}$$

with homogeneous BC's on $\mathbb{R}^3$, $G \to 0$ as $r \to \infty$.

Consider the ball $B = \{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r} - \mathbf{r}'| < r\}$ with unit normal $\hat{\mathbf{n}}$. This is sphericall symmetric so

$$G(\mathbf{r}, \mathbf{r}') = G(|\mathbf{r} - \mathbf{r}')$$

Wlog take $\mathbf{r}' = 0$, so $G = G(r)$. Integrate (10.15) over ball B of radius r around $\mathbf{r}' = 0$.

$$\begin{aligned}\int_B \nabla^2 G_{FS} \mathrm{d}^3\mathbf{r} &= \int_S \nabla G_{FS} \cdot \hat{\mathbf{n}} \mathrm{d}S = \int_S \frac{\partial G}{\partial r} r^2 \mathrm{d}\Omega \\ &= 4\pi r^2 \frac{\partial G_{FS}}{\partial r} = \int_B \delta(\mathbf{r}) \mathrm{d}^3\mathbf{r} = 1\end{aligned}$$

Hence

$$\frac{\partial G_{FS}}{\partial r} = \frac{1}{4\pi r^2} \implies G_{FS} = -\frac{1}{4\pi r} + C$$

Since $G \to 0$ as $r \to \infty$ we have $C = 0$. So free-space Green's function is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \tag{10.16}$$

General solution in $\mathbb{R}^3$

$$\phi(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3\mathbf{r}'$$

Green's Identities
Consider two scalar functions ϕ, ψ which are twice differentiable on D:

$$\begin{aligned}\int_D \nabla\cdot(\phi\nabla\psi)\mathrm{d}^3\mathbf{r} &= \int_D (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\mathrm{d}^3 \\ &= \int_{\partial D} \phi\nabla\psi\cdot\hat{\mathbf{n}}\mathrm{d}S \qquad (10.17)\end{aligned}$$

This is Green's first identity.

Now switch $\psi \leftrightarrow \phi$ and subtract from (10.17) to get Green's second identity

$$\int_{\partial D}\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}S = \int_D (\phi\nabla^2\psi - \psi\nabla^2\phi)\mathrm{d}^3\mathbf{r} \qquad (10.18)$$

Now consider a small spherical ball $B_\varepsilon = B_\varepsilon(\mathbf{r}')$ about $\mathbf{r}'$ (wlog $\mathbf{r}' = 0$). Take ϕ in (10.18) such that $\nabla^2\phi = -\rho$ and $\psi \equiv G_{FS}(\mathbf{r}, \mathbf{r}')$ (so $\nabla^2 G = 0$, $\mathbf{r} = \mathbf{r}'$)

$$RHS = \int_{\nabla - B}\left(\phi\nabla^2 G_{FS} - G_{FS}\nabla^2\phi\right)\mathrm{d}^3\mathbf{r} = \int_{D-B_\varepsilon} G_{FS}\rho\mathrm{d}^3\mathbf{r}$$

$$LHS = \int_{\partial D}\left(\phi\frac{\partial G_{FS}}{\partial n} - G_{FS}\frac{\partial\phi}{\partial n}\right)\mathrm{d}S + \int_{S_\varepsilon}\left(\phi\frac{\partial G_{FS}}{\partial n} - G_{FS}\frac{\partial\phi}{\partial n}\right)\mathrm{d}S$$

Second integral on small sphere S_ε, so take limit as $\varepsilon \to 0$.

$$\int_{S_\varepsilon}\left(\phi\frac{\partial G_{FS}}{\partial n} - G_{FS}\frac{\partial\phi}{\partial n}\right)\mathrm{d}S = \left(\overline{\phi}\frac{-1}{4\pi\varepsilon^2} - \frac{1}{4\pi\varepsilon}\overline{\frac{\partial\phi}{\partial r}}\right)4\pi\varepsilon^2 \to -\phi(0))$$

Where $\overline{\phi}$ is the average value of ϕ on the small surface.

Combining these we get Green's third identity

$$\phi(\mathbf{r}') = \int_D G_{FS}(\mathbf{r}, \mathbf{r}')(-\rho(\mathbf{r}))\mathrm{d}^3\mathbf{r} + \int_{\partial D}\left(\phi(\mathbf{r})\frac{\partial G_{FS}}{\partial n}(\mathbf{r}, \mathbf{r}') - G_{FS}(\mathbf{r}, \mathbf{r}')\frac{\partial\phi}{\partial n}(\mathbf{r})\right)\mathrm{d}S \qquad (10.19)$$

Dirichlet Green's function
Solve $\nabla^2\phi = -\rho$ on D with inhomogeneous BC's $\phi(\mathbf{r}) = h(\mathbf{r})$ on ∂D. Dirichlet Green's function satisfies

1. $\nabla^2 G(\mathbf{r}, \mathbf{r}') = 0$ for all $\mathbf{r} \neq \mathbf{r}'$
2. $G(\mathbf{r}, \mathbf{r}') = 0$ on boundary ∂D
3. $G(\mathbf{r}, \mathbf{r}') = G_{FS}(\mathbf{r}, \mathbf{r}') + H(\mathbf{r}, \mathbf{r}')$ with $\nabla^2 H(\mathbf{r}, \mathbf{r}') = 0$ for all $\mathbf{r} \in D$

Green's second identity (10.18) with $\nabla^2\phi = -\rho$, $\nabla^2 H = 0$

$$\int_{\partial D}\left(\phi\frac{\partial H}{\partial n} - H\frac{\partial \phi}{\partial n}\right)\mathrm{d}S = \int_D H\rho\mathrm{d}^3\mathbf{r} \tag{†}$$

Now we take $G_{FS} = G - H$ in Green's third identity (10.19)

$$\phi(\mathbf{r}') = \int_D (G-H)(-\rho)\mathrm{d}^3\mathbf{r} + \int_{\partial D}\left(\phi\frac{\partial (G-H)}{\partial n} - (G-H)\frac{\partial \phi}{\partial n}\right)\mathrm{d}S$$

Subtract H terms above in (†) ($G = 0$, $\phi = h$ on ∂D)

$$\phi(\mathbf{r}') = \int_D G(\mathbf{r},\mathbf{r}')(-\rho(\mathbf{r}))\mathrm{d}^3\mathbf{r} + \int_{\partial D} h(\mathbf{r})\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}\mathrm{d}S \tag{10.20}$$

For Neumann BC's, specifying $\frac{\partial \phi}{\partial n} = k(\mathbf{r})$ on ∂D we have

$$\phi(\mathbf{r}') = \int_D G(\mathbf{r},\mathbf{r}')(-\rho(\mathbf{r}))\mathrm{d}^3\mathbf{r} + \int_{\partial D} G(\mathbf{r},\mathbf{r}')(-k(\mathbf{r}))\mathrm{d}S \tag{10.21}$$

10.5 Method of Images

For symmetric domains D we can construct Green's functions with $G = 0$ on ∂D, by cancelling the boundary potential with opposite mirror or image Green's functions placed outside D.

Laplace's equation on half-space

Solve $\nabla^2\phi = 0$ on $D = \{(x,y,z) : z > 0\}$ with $\phi(x,y,0) = h(x,y)$, $\phi \to 0$ as $|\mathbf{r}| \to \infty$.

Now $G_{FS}(\mathbf{r},\mathbf{r}') \to 0$ as $|\mathbf{r}| \to \infty$, but $G_{FS} \neq 0$ at $z = 0$. So for G_{FS} at $\mathbf{r}' = (x',y',z')$, subtract image G_{FS} at $\mathbf{r}'' = (x',y',-z')$.

$$\begin{aligned} G(\mathbf{r},\mathbf{r}') &= \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} - \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}''|} \\ &= \frac{-1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} = 0 \text{ when } z = 0 \end{aligned}$$

i.e satisfies the Dirichlet BC's on all ∂D.

We have

$$\begin{aligned} \left.\frac{\partial G}{\partial n}\right|_{z=0} &= \left.\frac{\partial G}{\partial z}\right|_{z=0} \\ &= \frac{-1}{4\pi}\left(\frac{z-z'}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{z+z'}{|\mathbf{r}-\mathbf{r}'|^3}\right) \\ &= \frac{z'}{2\pi}\left((x-x')^2+(y-y')^2+(z')^2\right)^{-3/2} \text{ at } z = 0 \end{aligned} \tag{10.22}$$

Solution is then from (10.20) (no sources)

$$\phi(x',y',z') = \frac{z'}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left((x-x')^2+(y-y')^2+z'^2\right)^{-3/2}h(x,y)\mathrm{d}x\mathrm{d}y \tag{10.23}$$

Wave equation for $x>0$

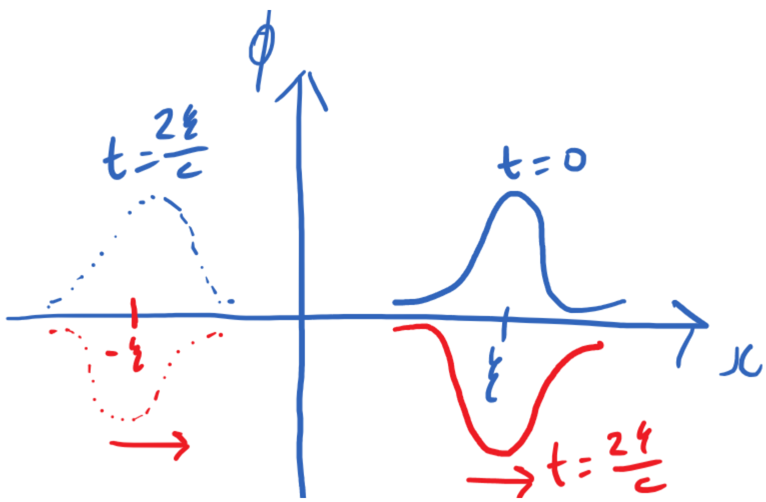
Consider $\ddot{\phi} - c^2\phi'' = f(x,t)$ with Dirichlet BC's $\phi(0,t)=0$.

Create matching Green's functions from (10.11) with opposite sign centred at $x=-\xi$

$$G(x,t,\xi,\tau) = \frac{1}{2c}H(c(t-\tau)-|x-\xi|)\underbrace{-}_{*}\frac{1}{2c}H(c(t-\tau)-|x+\xi|)$$

Where the $-$ sign $*$ can be replaced with a $+$ for Neumann BC's. Solve homogeneous problem with $f=0$ for IC's with some Gaussian pulse.

$$\phi(x,t) = \exp\left[(x-\xi+ct)^2\right] - \exp\left[(-x-\xi+ct)^2\right] \quad (x>0) \tag{10.25}$$



The solution travels left, cancels with image at $t=\xi/c$, which emerges and travels right as the reflected wave.