Disclaimer

These notes have been written with the intention of being used by people who have already taken previous analysis courses in the Tripos (beyond IA Analysis). For this reason, some of the results which have been seen already in Tripos are omitted.

1 Basic Analysis

Definition. For $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, define

$$d(x, A) = \inf\{d(x, a) = a \in A\}.$$

Remark. If A is closed, $d(x, A) = 0$ if and only if $x \in A$. Indeed if $d(x, A) = 0$, there exists a sequence $(a_n)_{n\geq 1}$ such that $d(x, a_n) \to 0$ by the definition of infima. This is the same as $a_n \to x$, and since A is closed $x \in A$. The converse is obvious.

Proposition. The map $x \mapsto d(x, A)$ is continuous.

Proof. It is easy to show $|d(x, A) - d(y, A)| \leq d(x, y)$ and then the claim follows. $\square$

Theorem. *If $E \subseteq \mathbb{R}^n$ is compact and $f : E \to f(E)$ is continuous, then E is compact.*

Proof. Take a sequence $(f(x_n))_{n\geq 1}$ in $f(E)$. Then $(x_n)_{n\geq 1}$ has a convergent subsequence $(x_{n_j})_{j\geq 1}$, and $f(x_{n_j}) \to f(x) \in f(E)$. $\square$

Theorem (Fundamental Theorem of Algebra)**.** *If $P(z)$ is a polynomial with coefficients in $\mathbb{C}$, then P has a root.*

Proof. Write $P(z) = \sum_{j=0}^{n} a_j z^j$ (wlog $a_n = 1$). Note

$$|P(z)| \geq |z|^n(|a_n| - |a_{n-1}||z|^{-1} - \ldots - |a_0||z|^{-n}).$$

Hence if $R = 2\left(2 + \sum_{j=0}^{n-1} |a_j|\right)$, whenever $|z| \geq R$ we have $|P(z)| > |z|^n/2$. Then $\overline{B}_R(0)$ is compact so $|P(z)|$ achieves a minimum on $B_R(0)$ at some $p_0 \in B_R(0)$. Since $|P(z)| \geq |z|^n/2 \geq R^n/2 > |a_0|$ for $|z| \geq R$, p_0 is a global minimum.

Now by considering $P(z + p_0)$, we can assume $p_0 = 0$. By considering $e^{i\theta}P(z)$ we can assume $a_0 \in \mathbb{R}$ and $a_0 > 0$. By considering $P(e^{i\theta}z)$ we can assume $P(z) = a_0 + a_m z^m + \sum_{m+1}^{n} a_j z^j$ for $a_0, a_m \in \mathbb{R}$, $a_0 > 0$ and $a_m < 0$. Then $\frac{P(\delta)-P(0)}{\delta^m} \to a_m$ as $\delta \to 0$.

So the real part of $\frac{P(\delta)-P(0)}{\delta^m}$ tends to a_m and the imaginary part tends to 0. Hence if δ is sufficiently small and positive $|P(\delta)| < |P(0)|$. $\square$

Remark. Clearly this implies every complex polynomial of degree n has exactly n roots with multiplicity (division algorithm).

2 Laplace's Equation

Definition. *Laplace's equation* is $\nabla^2\phi = 0$.

Definition (Boundary)**.** Let (X, d) be a metric space, $E \subseteq X$.

1. The *closure* $\overline{E} = \mathrm{Cl}(E)$ is the collection of points $x \in E$ such that there exists a sequence $(x_n)_{n\geq 1}$ in E with $x_n \to x$.
2. The *interior* $\mathrm{int}(E)$ is the set of points $x \in E$ such that there exists $\delta > 0$ with $B(x, \delta) \subseteq E$.
3. The *boundary* ∂E is defined by $\partial E = \mathrm{Cl}(E) \setminus \mathrm{int}(E)$.

For the remainder of this section:

- We work in $\mathbb{R}^m$;
- Let $\Omega \neq \emptyset$ be a bounded open set;
- Let $\phi : \mathrm{Cl}(\Omega) \to \mathbb{R}$ be a continuous function which is twice differentiable on Ω;
- Let $f : \partial\Omega \to \mathbb{R}$ be a continuous function.

We want to show that the problem $\nabla^2\phi = 0$ on Ω, $\phi = f$ on $\partial\Omega$ has at most one solution.

Lemma. *If $\nabla^2\phi > 0$ on Ω, then ϕ achieves its maximum on $\partial\Omega$.*

Remark. The maximum must exist since $\mathrm{Cl}(\Omega)$ is closed and bounded.

Proof. Suppose ϕ attains its maximum at $x^* \in \Omega$. Then there exists $\delta > 0$ such that

$$(x_1^* - \delta, x^* + \delta) \times \ldots \times (x_n^* - \delta, x^* + \delta) \subseteq \Omega.$$

Define $f_j(t) = \phi(x_1^*, \ldots, x_{j-1}^*, x_j^* + t, x_{j+1}^*, \ldots, x_m^*)$. Then f_j has a maximum at 0 and f_j is twice differentiable with $f_j''(0) \leq 0$. Hence $\frac{\partial\phi(x^*)}{\partial x_j^2} \leq 0$ so $\nabla^2\phi(x^*) \leq 0$, a contradiction. $\square$

Theorem. *If $\nabla^2\phi = 0$ on Ω, then the maximum is achieved at $\partial\Omega$.*

Proof. Set $\phi_n(x) = \phi(x) + \frac{1}{n}\sum_{j=1}^m x_j^2$. Then $\nabla^2\phi_n = \nabla^2\phi + \frac{2m}{n} = \frac{2m}{n}$. Then by the previous lemma, ϕ_n has a maximum at $x_n^* \in \partial\Omega$. The boundary is closed and bounded so compact, hence we can find x^* and a subsequence $(x_{n(j)}^*)_{j\geq 1}$ such that $x_{n(j)}^* \to x^*$. Then

$$\phi(x_{n(j)}^*) + \frac{1}{n(j)}\sum_{i=1}^m (x_k^*)^2 \geq \phi(x) + \frac{1}{n(j)}\sum_{i=1}^m x_k^2 \geq \phi(x)\ \forall x \in \mathrm{Cl}(\Omega).$$

So let $n(j) \to \infty$ and use continuity of ϕ to obtain $\phi(x^*) \geq \phi(x)$ for all $x \in \mathrm{Cl}(\Omega)$. $\square$

Remark. Now if $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$ on Ω and $\phi_1 = \phi_2 = f$ on $\partial\Omega$, then $\phi_1 - \phi_2 = 0$ on $\partial\Omega$ so $\phi_1 - \phi_2 \leq 0$ on $\mathrm{Cl}(\Omega)$ by the previous theorem. Similarly $\phi_2 - \phi_1 \leq 0$ on $\mathrm{Cl}(\omega)$ so $\phi_1 = \phi_2$.

What about existence?

Zaremba counterexample

Consider $\Omega = D \setminus \{0\} = \{x \in \mathbb{R}^2 : 0 < ||x|| < 1\}$. Take $f(x) = 0$ for $||x|| = 1$ and $f(0) = 1$. We'll show $\nabla^2\phi = 0$, $\phi = f$ on $\partial\Omega$ has no solution.