Introduction

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1 Some basic aspects of continuous-time Markov Chains

Definition. A sequence of random variables is called a *stochastic process* or *process*. The process $X = (X_n)_{n\geq 1}$ is called a discrete-time Markov Chain with state space I if for all $x_0, x_1, \ldots, x_n \in I$

$$\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}).$$

If $\mathbb{P}(X_{n+1} = y | X_n = x)$ is independent of n, the chain is called *time-homogeneous*. We then write $P = (P_{x,y})_{x,y\in I}$ for the *transition matrix* where $P_{x,y} = \mathbb{P}(X_1 = y | X_0 = x)$. The data associated to every time-homogeneous Markov Chain is the transition matrix P and the initial distribution μ, i.e $\mathbb{P}(X_0 = x_0) = \mu(x_0)$.

From now on:

- I denotes a countable (or finite) state space.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which all the relevant random variables are defined.

Definition. $X = (X(t) : t \geq 0)$ is a (right-continuous) continuous-time random process with values in I if

(a) for all $t \geq 0$, $X(t) = X_t$ is a random variable such that $X(t) : \Omega \to I$;

(b) for all $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous (right-continuous sample path). In our case this means for all $\omega \in \Omega$, for all $t \geq 0$, there exists $\varepsilon > 0$ (depending on ω, t) such that

$$X_t(\omega) = X_s(\omega)\ \forall s \in [t, t + \varepsilon].$$

Fact. A right-continuous random process is defined by its finite-dimensional distributions

$$\mathbb{P}(X_{t_0=i}, X_{t_1=i_1}, \ldots, X_{t_n} = i_{t_n}),\ n \geq 0,\ t_k \geq 0,\ i_k \in I.$$

For every $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ of a right-continuous process stays constant for a while. So there are 3 possibilities:

(i) The path makes infinitely many jumps overall but only finitely many in a given interval $[0, t]$.

(ii) The path makes finitely many jumps & then gets absorbed in some state.

(iii) The path makes infinitely jumps in a finite time interval. After the 'explosion time' ζ, the process starts up again.

Write $J_0 = 0, J_1, J_2, \ldots$ for the jump times and $S_1, S_2, \ldots$ for the holding times, defined by

$$J_0 = 0,\ J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\},$$

$$S_n = \begin{cases} J_n - J_{n-1} & J_{n-1} < \infty \\ \infty & \text{otherwise} \end{cases}.$$

By right-continuity, $S_n > 0$ for all n. If $J_{n+1} = \infty$ for some n, we define $X_\infty = X_{J_n}$ as the final value, otherwise X_∞ is not defined. The explosion time ζ is defined by

$$\zeta = \sup(J_n) = \sum_{n=1}^{\infty} S_n.$$

We are not going to consider what happens to a chain after explosion. We thus set $X_t = \infty$ for all $t \geq \zeta$ (adjoining a new state '∞'). We call such a chain *minimal*.

Definition. We define the *jump chain* Y_n of $(X_t)_{t\geq 0}$ by setting $Y_n = X_{J_n}$ for all n.

Definition. A right-continuous random process $X = (X_t)_{t\geq 0}$ has the Markov property (and is called a continuous-time markov chain) if for all $i_1, i_2, \ldots, i_n \in I$ and $0 \leq t_1 < t_2 < \ldots < t_n$,

$$\mathbb{P}(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}, \ldots, X_{t_0} = i_0) = \mathbb{P}(X_{t_n} = i_n | X_{t_0} = i_0).$$

Remark. For all $h > 0$, $Y_n = X(hn)$ defines a discrete-time Markov Chain.

Definition. The transition probabilities are $P_{ij}(s,t) = \mathbb{P}(X_t = j | X_s = i)$, $s \leq t$, $i, j \in I$. It is called *time-homogeneous* if it depends on $t - s$ only, i.e

$$P_{ij}(s,t) = P_{i,j}(0, t-s).$$

In this case we just write $P_{ij}(t-s)$. As in the case of discrete time, a (time-homogeneous) Markov process is characterised by

1. Its initial distribution $\lambda_i = \mathbb{P}(X_0 = i)$, $i \in I$;
2. Its *family of transition matrices* $(P(t))_{t\geq 0} = (P_{ij}(t))_{t\geq 0}$.

The family $(P(t))_{t\geq 0}$ is called the *transition subgroup* of the MC.

A (time-homogeneous) Markov process is characterised by

- its initial distribution;
- its transition subgroup $(P(t))_{t\geq 0}$

$$(P(t))_{t\geq 0} = (P(t))_{\substack{i,j\in I\\ t\geq 0}} = (\mathbb{P}(X_t = j|X_0 = i))_{\substack{i,j\in I\\ t\geq 0}}$$

It is easy to see that

- $P(0)$ is the identity
- $P(t)$ is a stochastic matrix for all t (i.e rows sum to 1)
- $P(t+s) = P(t)P(s)\ \forall s,t$ (Chapman-Kolmogorov equation)

$$\begin{aligned}
P_{xz}(t+s) &= \mathbb{P}(X_{t+s} = z|X_0 = x)\\
&= \sum_{y\in I} \mathbb{P}(X_{t+s} = z|X_0 = x, X_t = y)\mathbb{P}(X_t = y|X_0 = x)\\
&= \sum_{y\in I} \mathbb{P}(X_s = z|X_0 = y)\mathbb{P}(X_t = y|X_0 = x)\\
&= \sum_{y\in I} P_{yz}(s)P_{xy}(t) = P_{x\cdot}(t)P_{\cdot z}(s)
\end{aligned}$$

Holding times

Let X be a (right-continuous continuous-time time-homogeneous) Markov Chain on a countable state-space I.

Suppose X starts from $x \in I$. Question: how long does X stay in the state x?

Definition. We call S_x the *holding time* at state x ($S_x > 0$ by right-continuity).

Let $s,t \geq 0$. Then

$$\begin{aligned}
\mathbb{P}(S_x > t+s|S_x > s) &= \mathbb{P}(X_u = x\ \forall u \in [0,t+s]|X_u = x\ \forall u \in [0,s])\\
&= \mathbb{P}(X_u = x\ \forall u \in [s,t+s]|X_u = x\ \forall u \in [0,s])\\
&= \mathbb{P}(X_u = x\ \forall u \in [s,t+s]|X_s = x)\\
&= \mathbb{P}(X_u = x\ \forall u \in [0,t]|X_0 = x)\\
&= \mathbb{P}(S_x > t).
\end{aligned}$$

Thus S_x has the memoryless property.

By the next theorem, we will get that S_x has the exponential distribution, say with parameter q_x.

Theorem 1.1 (Memoryless property)**.** *Let S be a positive random variable. Then S has the memoryless property, i.e $\mathbb{P}(S > t + s | S > s) = \mathbb{P}(S > t)$ for all $s, t \geq 0$ if and only if S has the exponential distribution.*

Proof. It is easy to see the exponential distribution is memoryless. So we prove the other direction. Set $F(t) = \mathbb{P}(S > t)$. Then $F(s + t) = F(s)F(t)$ for all $s, t \geq 0$.

Since S is a positive random variable, there exists $n \in \mathbb{N}$ large such that $F(1/n) = \mathbb{P}(S > 1/n) > 0$. Then $F(1) = F(1/n)^n > 0$. So we can set $F(1) = e^{-\lambda}$ for some $\lambda \geq 0$.

For $k \in \mathbb{N}$, $F(k) = F(1)^k = e^{-\lambda k}$. For p/q rational, $F(p/q) = F(1/q)^p = (F(1/q)^q)^{p/q} = F(1)^{p/q} = e^{-\lambda \frac{p}{q}}$.

For any $t \geq 0$, for any $r, s \in \mathbb{Q}$ such that $r \leq t \leq s$, since F is decreasing

$$e^{-\lambda s} = F(s) \leq F(t) \leq F(r) = e^{-\lambda r}.$$

So taking sequences of rationals approaching t, we have $F(t) = e^{-\lambda t}$. $\square$

Poisson Process'

We are now going to look at the simplest (and most important) example of continuous time Markov Chains - the Poisson process.

Definition. Suppose $S_1, S_2, \ldots$ are iid random variables with $S_1 \sim \text{Exp}(\lambda)$. Define the *jump times* $J_0 = 0, J_1 = S_1$, $J_n = S_1 + \ldots + S_n$ for all n, and set $X_t = i$ if $J_i \leq t < J_{i+1}$. Then $I = \{0, 1, 2, \ldots\}$ and note that X is right-continuous and increasing. X is called a *Poisson process* of parameter/intensity λ. We sometimes refer to the jump times $(J_i)_{i \geq 1}$ as the *points* of the Poisson process, then X =number of points in $[0, t]$.

Theorem 1.2 (Markov property)**.** *Let $(X_t)_{t \geq 0}$ be a Poisson process of intensity λ. Then for all $s \geq 0$, the process $(X_{s+t} - X_s)_{t \geq 0}$ is also a Poisson process of intensity λ, and is independent of $(X_t)_{0 \leq t \leq s}$.*

Proof. Set $Y_t = X_{t+s} - X_s$ for all $t \geq 0$. Let $i \in \{0, 1, 2, \ldots\}$ and condition on $\{X_s = i\}$, Then the jump times for the process Y are $J_{n+1} - s, J_{n+2} - s, \ldots$ and the holding times are

$$\begin{aligned}
T_1 &= J_{n+1} - s = S_{i+1} - (s - J_i) \\
T_2 &= S_{i+2} \\
T_3 &= S_{i+3} \\
&\vdots
\end{aligned}$$

Since $\{X_s = i\} = \{J_i \leq s\} \cap \{s < J_{i+1}\} = \{J_i \leq s\} \cap \{S_{i+1} > s - J_i\}$, conditional on $\{X_s, i\}$, by the memoryless property of the exponential distribution (and independence of S_{i+1} and J_i) we see that $T_1 \sim \text{Exp}(\lambda)$. Moreover the times J_j, $j \geq 2$ are independent of S_k, $k \leq i$ and hence independent of $(X_r)_{r \leq s}$, and they have iid $\text{Exp}(\lambda)$ distribution. Thus $((X_{s+t} - X_s))_{t \geq 0}$ is a Poisson process of parameter λ and is independent of $(X_t)_{0 \leq t \leq s}$. $\square$

Similar to this, one can show the Strong Markov property for a Poisson process of parameter λ. Recall a random variable $T \in [0, \infty]$ is called a *stopping time* if for all t, the event $\{T \leq t\}$ depends only on $(X_s)_{s \leq t}$.

Theorem 1.3 (Strong Markov property)**.** *Let $(X_t)_{t \geq 0}$ be a Poisson process of parameter λ and T a stopping time. Then conditional on $T < \infty$, the process $(X_{T+t} - X_T)_{t \geq 0}$ is a Poisson process of parameter λ and independent of $(X_s)_{s \leq T}$.*

Theorem 1.4. *Let $(X_t)_{t\geq 0}$ be an increasing right-continuous process taking values in $\{0, 1, 2, \dots\}$ with $X_0 = 0$. Let $\lambda > 0$. Then the following are equivalent*

(a) The holding times $S_1, S_2, \dots$ are iid $\mathrm{Exp}(\lambda)$ and the jump chain is given by $Y_n = n$ (i.e X is a poisson process of intensity λ)

(b) (Infinitesimal def) X has independent increments and as $h \downarrow 0$ uniformly in t we have

$$\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h),$$
$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h).$$

(c) X has independent and stationary increments and for all $t \geq 0$, $X_t \sim \mathrm{Poi}(\lambda t)$.

Proof. First we show (a)$\Rightarrow$(b). If (a) holds, then by the Markov property, the increments are independent and stationary ($(X_{t+s} - X_s)_{t\geq 0} =^d (X_t - X_0)_{t\geq 0}$). Using stationarity we have (uniformly in t) as $h \to 0$,

$$\mathbb{P}(X_{t+h} - X_t = 0) = \mathbb{P}(X_h = 0) = \mathbb{P}(S_1 > h) = e^{-\lambda h} = 1 - \lambda h + o(h),$$

$$\mathbb{P}(X_{t+h} - X_t \geq 1) = \mathbb{P}(X_h \geq 1) = \mathbb{P}(S_1 \leq h) = 1 - e^{-\lambda h} = \lambda h + o(h),$$

$$\begin{aligned}\mathbb{P}(X_{t+h} - X_t \geq 2) = \mathbb{P}(X_h \geq 2) &= \mathbb{P}(S_1 + S_2 \leq h) \\ &\leq \mathbb{P}(S_1 \leq h, S_2 \leq h) \\ &= \mathbb{P}(S_1 \leq h)^2 \\ &= (\lambda h + o(h))^2 = o(h).\end{aligned}$$

Now we show (b)$\Rightarrow$(c). If X satisfies (b), then $(X_{t+s} - X_s)_{t\geq 0}$ also satisfies (b). So X has independent and stationary increments. Now set $p_j(t) = \mathbb{P}(X_t = j)$. Then since increments are independent and X is increasing,

$$\begin{aligned}p_j(t+h) = \mathbb{P}(X_{t+h} = j) &= \sum_{i=0}^{j} \mathbb{P}(X_t = j - i)\mathbb{P}(X_{t+h} - X_t) \\ &= p_j(t)(1 - \lambda h + o(h)) + p_{j-1}(t)(\lambda h + o(h)) + o(h).\end{aligned}$$

Thus, $\frac{p_j(t+h) - p_j(t)}{h} = -\lambda p_j(t) + \lambda p_{j-1}(t) + o(1)$. Setting $s = t + h$, we get

$$\frac{p_j(s) - p_j(s-h)}{h} = -\lambda p_j(s-h) + \lambda p_{j-1}(s-h) + o(1).$$

In particular, $p_j(t)$ is continuous and differentiable with

$$p_j'(t) = -\lambda p_j(t) + \lambda p_{j-1}(t).$$

Differentiating

$$\left(e^{\lambda t} p(t)\right)' = \lambda e^{\lambda t} p_j(t) + e^{\lambda t} p_j'(t) = \lambda e^{\lambda t} p_{j-1}(t).$$

For $j = 0$ we have $p_0(t+h) = p_0(t)(1 - \lambda h + o(h))$, i.e $p_0'(t) = -\lambda p_0(t)$ so $p_0(t) = e^{-\lambda t}$. Thus

$$p_1'(t) = -\lambda p_1(t) + \lambda e^{-\lambda t}, \text{ i.e } p_1(t) = \lambda t e^{-\lambda t}.$$

And by induction

$$p_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!},$$

i.e $X_t \sim \text{Poi}(\lambda t)$.

Finally we show (c)⇒(a). We know X has independent stationary increments, We have for $t_1 \leq \ldots \leq t_k,\ n_1 \leq \ldots \leq n_k$,

$$\begin{aligned}
&\mathbb{P}(X_{t_1} = n_1, \ldots, X_{t_k} = n_k) \\
&= \mathbb{P}(X_{t_1} = n_1)\underbrace{\mathbb{P}(X_{t_2} - X_{t_1} = n_2 - n_1)}_{\sim \text{Poi}(\lambda t_1)} \ldots \underbrace{\mathbb{P}(X_{t_k} - X_{t_{k-1}} = n_k - n_{k-1})}_{\sim \text{Poi}(\lambda(t_2 - t_1))}.
\end{aligned}$$

So (c) determines the finite-dimensional distributions (f.d.d) of a right-continuous process X, hence (c) determines X. So (c)⇒(a).

Question: can we show (a)⇒(c) directly? Indeed note

$$\begin{aligned}
\mathbb{P}(X_t = n) &= \mathbb{P}(S_1 + \ldots + S_n \leq t < S_1 + \ldots + S_{n+1}) \\
&= \mathbb{P}(S_1 + \ldots + S_n \leq t) - \mathbb{P}(S_1 + \ldots + S_{n+1} \leq t) \\
&= \int_0^t \lambda e^{-\lambda x}\frac{(\lambda x)^{n-1}}{(n-1)!}\mathrm{d}x - \int_0^t \lambda e^{-\lambda x}\frac{(\lambda x)^n}{n!}\mathrm{d}x \\
&= e^{-\lambda t}\frac{(\lambda t)^n}{n!} \text{ (integration by parts).}
\end{aligned}$$

□

Theorem 1.5 (Superposition)**.** *Let X and Y be two independent Poisson processes with parameters λ and μ respectively. Then $(Z_t)_{t\geq 0} = (X_t + Y_t)_{t \geq 0}$ is a Poisson process with parameter $\lambda + \mu$.*

Proof. We use (c) from the previous theorem. So Z has stationary independent increments. Also $Z_t \sim \text{Poi}(\lambda t + \mu t)$. □

Theorem 1.6 (Thinning)**.** *Let X be a Poisson process with parameter λ. Let $(Z_i)_{i\geq 1}$ be a sequence of iid Bernouilli(p) random variables. Let Y be a Posisson prcess with values in $\{0, \ldots, \}$ which jumps at time t if and only if X_t jumps at time t and $Z_{X_t} = 1$.*

In other words, we keep every point of X with probability p independently. Then Y is another Poisson process, with parameter λp and $X - Y$ is an independent Poisson process with parameter $\lambda(1-p)$.

Proof. We shall use the infinitesimal definition. The independence of increments for Y is clear. Since $\mathbb{P}(X_{t+h} - X_t \geq 2) = o(h)$, we have

$$\mathbb{P}(Y_{t+h} - Y_t = 1) = p\mathbb{P}(X_{t+h} - X_t = 1) + o(h) = p\lambda h + o(h),$$

$$\begin{aligned}\mathbb{P}(Y_{t+h} - Y_t = 0) &= \mathbb{P}(X_{t+h} - X_t = 0) + (1-p)\mathbb{P}(X_{t+h} - X_t = 1) + o(h) \\ &= 1 - \lambda h + (1-p)(\lambda h + o(h)) + o(h) \\ &= 1 - \lambda p h + o(h).\end{aligned}$$

Hence Y is Poisson of paremeter λp. Clearly $X - Y$ is a thinning of X with Bernouilli parameter $1 - p$, so $X - Y$ is Poisson of parameter $\lambda(1-p)$.

Now we show Y and $X - Y$ are independent. It is enough to show that the f.d.d of Y and $X - Y$ are independent, i.e if $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k$, $n_1 \leq \ldots \leq n_k$ and $m_1 \leq \ldots \leq m_k$, then we want to prove

$$\begin{aligned}&\mathbb{P}(Y_{t_1} = n_1, \ldots, Y_{t_k} = n_k, X_{t_1} - Y_{t_1} = m_1, \ldots X_{t_k} - Y_{t_k} = m_k) \\ &= \mathbb{P}(X_{t_1} = n_1, \ldots, Y_{t_k} = n_k)\mathbb{P}(X_{t_1} - Y_{t_1} = m_1, \ldots, X_{t_k} - Y_{t_k} = m_K).\end{aligned}$$

We will only show this for fixed t $(k = 1)$ the general case follows similarly using independence of increments. We have

$$\begin{aligned}\mathbb{P}(Y_t = n, X_t - Y_t = m) &= \mathbb{P}(X_t = m + n, Y_t = n) \\ &= \mathbb{P}(X_t = m + n)\mathbb{P}(Y_t = n | X_t = m + n) \\ &= e^{-\lambda t}\frac{(\lambda t)^{m+n}}{(m+n)!}\binom{m+n}{n}p^n(1-p)^m \\ &= e^{-\lambda t p}\frac{(\lambda t p)^n}{n!}e^{-\lambda t(1-p)}\frac{(\lambda t(1-p))^m}{m!} \\ &= \mathbb{P}(X_t = n)\mathbb{P}(X_t - Y_t = m),\end{aligned}$$

as required. ☐

Theorem 1.7. *Let X be a Poisson Process. Conditional on the event $(X_t = n)$, the jump times $J_1, J_2, \dots, J_n$ are distributed as the order statistics of n iid $U[0,t]$ random variables. That is, they have joint density*

$$f(t_1, \dots, t_n) = \frac{n!}{t^n}\mathbb{1}(0 \le t_1 \le \dots \le t_n \le t).$$

Proof. Since $S_1, S_2, \dots$ are iid $\mathrm{Exp}(\lambda)$, the joint density of $(S_1, \dots, S_{n+1})$ is

$$\lambda^{n+1}e^{-\lambda(S_1+\dots+S_{n+1})}\mathbb{1}(S_i \ge 0\ foralli).$$

Then the jump times $J_1 = S_1, J_2 = S_1 + S_2, \dots, J_{n+1} = S_1 + \dots + S_{n+1}$ have joint density

$$g(t_1, \dots, t_{n+1}) = \lambda^{n+1}e^{-\lambda t_{n+1}}\mathbb{1}(0 \le t_1 \le t_2 \le \dots t_{n+1}).$$

(Noting the Jacobian of the transformation is 1.) Now take $A \subseteq \mathbb{R}^n$ so

$$\mathbb{P}((J_1, \dots, J_n) \in A | X_t = n) = \frac{\mathbb{P}((J_1, \dots, J_n) \in A | X_t = n)}{\mathbb{P}(X_t = n)}.$$

Note

$$\begin{aligned}
&\mathbb{P}((J_1, \dots, J_n) \in A, X_t = n) \\
&= \mathbb{P}((J_1, \dots, J_n) \in A, J_n \le t < J_{n+1}) \\
&= \int_{(t_1,\dots,t_{n+1}) \in A \times \mathbb{R}} g(t_1, \dots, t_n)\mathbb{1}(t_{n+1} \ge t \ge t_n)\mathrm{d}t_1 \dots \mathrm{d}t_{n+1} \\
&= \int_A \int_t^\infty \lambda^{n+1}e^{-\lambda t_{n+1}}\mathbb{1}(0 \le t_1 \le \dots \le t_n \le t)\mathrm{d}t_{n+1}\mathrm{d}t_1 \dots \mathrm{d}t_n \\
&= \int_A \lambda^n e^{-\lambda t}\mathbb{1}(0 \le t_1 \le \dots \le t_n \le t)\mathrm{d}t_1 \dots \mathrm{d}t_n.
\end{aligned}$$

Then we get

$$\mathbb{P}((J_1, \dots, J_n) \in A | X_t = n) = \int_A \frac{n!}{t^n}\mathbb{1}(0 \le t_1 \le \dots \le t_n \le t)\mathrm{d}t_1 \dots \mathrm{d}t_n.$$

As required. □

Now we look at a generalisation of a Poisson Process: called a Birth Process. For a Poisson Process, the rate of going from i to $i+1$ is λ. For a Birth Process, this is q_i (can depend on i). More precisely:

Definition (Birth Process)**.** For each i, let $S_i = \mathrm{Exp}(q_i)$ with $S_1, S_2, \dots$ independent. Set $J_i = S_1 + \dots + S_i$ and $X_t = i$ if $J_i \le t < J_{i+1}$. Then X is called a *Birth Process.*

We have some special cases:

1. Simple birth process: when $q_i = \lambda i$ for $i = 1, 2, \ldots$;
2. Poisson Proces $q_i = \lambda$ for all i.

Motivation for Simple Birth Process (SBP): at time 0 there is only one 'individual' i.e $X_0 = 1$. Each individual has an exponential clock of parameter λ independently. Then if there are i individuals, the first clock rings after $\mathrm{Exp}(\lambda i)$ time, and we jump from i to $i+1$ individuals. Indeed, by the memoryless property, the process begins afresh after each jump.

Proposition 1.8. *Let $(T_k)_{k\geq 1}$ be a sequence of independent random variables with $T_K \sim \mathrm{Exp}(q_k)$ and $\sum_k q_k < \infty$. Let $T = \inf_k T_k$. Then*

(a) $T \sim \mathrm{Exp}\left(\sum_k q_k\right)$

(b) The infimum is attained at a point T_K almost surely, and

$$\mathbb{P}(K = n) = \frac{q_n}{\sum_k q_k}.$$

(c) T and K are independent.

Proof. See example sheet. □

The main difference between a Poisson Process and a Birth Process is that there is the possibility of explosion in the Birth Process. Recall explosion occurs when $\zeta := \sum_n S_n < \infty$.

Proposition 1.9. *Let X be a Birth Process with rates q_i and $X_0 = 1$. Then*

1. *If $\sum_{i=1}^{\infty} \frac{1}{q_i} < \infty$, then X is explosive, i.e $\mathbb{P}(\zeta < \infty) = 1$;*
2. *If $\sum_{i=1}^{\infty} \frac{1}{q_i} = \infty$, then X is non-explosive, i.e $\mathbb{P}(\zeta = \infty) = 1$.*

Remark. This shows the SBP (as well as the PP) is non-explosive.

Proof.

1. If $\sum_n \frac{1}{q_N} < \infty$, then

$$\mathbb{E}\left[\sum_n S_n\right] = \sum_n \mathbb{E} S_n = \sum_n \frac{1}{q_n} < \infty.$$

Thus $\zeta = \sum_n S_n < \infty$ almost surely.

2.

□