Introduction

The course is split into two parts:

- Logic: syntax and semantics.
- Set theory: what does the universe of sets look like?

Course structure

(I) Propositional logic (logic)

(II) Well-orderings & ordinals (set theory)

(III) Posets & Zorn's lemma (set theory)

(IV) Predicate logic (logic)

(V) Set theory (set theory)

(VI) Cardinals (set theory)

Books:

1. Johnstone, *Notes on Logic & Set Theory*
2. Van Dalen, *Logic & Structure* (Chapter 4 and what 'goes next')
3. Hajnal & Hamburger, *Set Theory* (Chapters 2 and 6)
4. Forster, *Logic, Induction & Sets*

1 Propositional Logic

Let P be a set of *primitive propositions*. Unless otherwise stated, $P = \{p_1, p_2, \ldots\}$. The *language* L or $L(P)$ is defined inductively by

1. If $p \in P$, then $p \in L$
2. $\perp \in L$ ($\perp$ is read 'false')
3. If $p, q \in L$ then $(p \Rightarrow q) \in L$.

e.g $((p_1 \Rightarrow p_2) \Rightarrow (p_1 \Rightarrow p_3)), (p_4 \Rightarrow \perp), (\perp \Rightarrow \perp)$.

Notes.

1. Each proposition (member of L) is a finite string of symbols from language: $\vdash, \Rightarrow, \perp, p_1, p_2, \ldots$ (for clarity often omit outer brackets, use other types of bracket, etc).
2. 'L is defined inductively' means, more precisely, the following

- Put $L_1 = P \cup (\perp)$;
- Having defined L_n, put $L_{n+1} = L_n \cup \{(p \Rightarrow q) : p, q \in L_n\}$;
- Set $L = \bigcup_{n \geq 1} L_n$.

3. Every $p \in L$ is uniquely built up from steps 1,2 using 3. For example, $((p_1 \Rightarrow p_2) \Rightarrow (p_1 \Rightarrow p_3))$ can from $(p_1 \Rightarrow p_2)$ and $(p_1 \Rightarrow p_3)$.

We can now introduce $\neg p$ ('not p') as an abreviation for $(p \Rightarrow \perp)$; $p \vee q$ ('p or q') as an abreviation for $(\neg p) \Rightarrow q$; $p \wedge q$ ('p and q') as an abreviation for $\neg(p \Rightarrow (\neg q))$.

1.1 Semantic Implication

Definition. A *valuation* is a function $v : L \to \{0, 1\}$ (thinking of 0 as 'False' and 1 as 'True') such that

(i) $v(\perp) = 0$

(ii) $v(p \Rightarrow q) = \begin{cases} 0 & \text{if } v(p) = 1,\ v(q) = 0 \\ 1 & \text{otherwise} \end{cases}$.

Remark. On $\{0, 1\}$, could define a constant $\perp = 0$ and an operation $\Rightarrow$ by

$$(a \Rightarrow b) = \begin{cases} 0 & \text{if } a = 1, b = 0 \\ 1 & \text{otherwise} \end{cases}.$$

Then a valuation is precisely a mapping $L \to \{0, 1\}$ that preserves ($\perp$ and $\Rightarrow$).

Proposition 1.1.

(i) If v, v' are valuations with $v(p) = v(p')$ for all $p \in P$, then $v = v'$.

(ii) For any function $w : P \to \{0, 1\}$, there exists a valuation v with $v(p) = w(p)$ for all $p \in P$.

Proof.

(i) Have $v(p) = v'(p)$ for all $p \in L_1$. But if $v(p) = v'(p)$ and $v(q) = v'(q)$, then $v(p \Rightarrow q) = v'(p \Rightarrow q)$, so $v(p) = v'(p)$ for all $p \in L_2$. Continuing inductively we obtain $v(p) = v'(p)$ for all $p \in L_n$ for each n.

(ii) Set $v(p) = w(p)$ for all $p \in P$ and $v(\perp) = 0$ to obtain v on L_1. Now put

$$v(p \Rightarrow q) = \begin{cases} 0 & v(p) = 1, v(q) = 0 \\ 1 & \text{otherwise} \end{cases}$$

to obtain v on L_2, then induction.

□

Example. Let v be the valuation with $v(p_1) = v(p_3) = 1$, $v(p_n) = 0$ for all $n \neq 1, 3$. Then $v((p_1 \Rightarrow p_2) \Rightarrow p_3) = 0$.

Definition. A *tautology* is an element $t \in L$ such that $v(t) = 1$ for any valuation v. We write $\vDash t$.

Examples.

1. $p \Rightarrow (q \Rightarrow p)$

$v(p)$	$v(q)$	$v(p \Rightarrow q)$	$v(p \Rightarrow (q \Rightarrow p))$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

So this is a tautology.

2. $(\neg\neg p) \Rightarrow p$, i.e $((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p$ ('law of excluded middle')

$v(p)$	$v(p \Rightarrow \perp)$	$v((p \Rightarrow \perp) \Rightarrow \perp)$	$v(((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p)$
0	1	0	1
1	0	1	1

3. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ ("how implication chains"). Suppose this is not a tautology. Then we have a v with $v(p \Rightarrow (q \Rightarrow r)) = 1$ and $v((p \Rightarrow q) \Rightarrow (q \Rightarrow r)) = 0$. Then $v(p \Rightarrow q) = 1$ and $v(p \Rightarrow r) = 0$. Hence $v(p) = 1$ and $v(r) = 0$, so $v(q) = 1$. Hence $v(p \Rightarrow (q \Rightarrow r)) = 0$, contradiction.

Definition. For $S \subseteq L$, $t \in L$, we say S *entails* or *semantically implies* t, written $S \vDash t$ if every valuation with $v(s) = 1$ for all $s \in S$ has $v(t) = 1$.

Example. $\{p \Rightarrow q, q \Rightarrow r\}$ entails $p \Rightarrow r$. Indeed, suppose we have v with $v(p \Rightarrow q)$, $v(q \Rightarrow r) = 1$ but $v(p \Rightarrow r)$. Then $v(p) = 1$, $v(r) = 0$. Hence $v(q) = 1$, contradicting $v(q \Rightarrow r) = 1$.

Definition. We say v is a *model* of $S \subseteq L$ or S is *true* in v, if $v(s) = 1$ for all $s \in S$. Thus S entails t means: every model of S is also a model of $\{t\}$.

Remark. $\vDash t$ says $\emptyset \vDash t$.

1.2 Syntatic implication

For a notion of proof, we'll need axioms and deduction rules. As axioms, we'll take:

1. $p \Rightarrow (q \Rightarrow p)$ for all $p, q \in L$;
2. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ for all $p, q \in L$;
3. $(\neg\neg p) \Rightarrow p$ for all $p \in L$.

Notes.

1. Sometimes we call these 'axiom schemes' since each is actually a set of axioms.
2. Each of these are tautologies.

For deduction rules, we'll have only *modus ponens*: from each p and $p \Rightarrow q$ we can deduce q.

Definition. For $S \subseteq L$, and $t \in S$, say S *proves* or *syntactically implies* t, written $S \vdash t$ if there exists a sequence $t_1, \dots, t_n$ in L with $t_n = t$ such that every t_i is either

(i) An axiom; or

(ii) A member of S; or

(iii) Such that there exist $j, k < i$ with $t_k \Rightarrow (t_j \Rightarrow t_n)$ (modus ponens).

Say S consists of the *hypotheses* or *premises*, and t the *conclusion*.

Example. $\{p \Rightarrow q, q \Rightarrow r\} \vdash p \Rightarrow r$:

1. $q \Rightarrow r$ (hypothesis)
2. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ (axiom 1)
3. $p \Rightarrow (q \Rightarrow r)$ (modus ponens' on 2,3)
4. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ (axiom 2)
5. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ (modus ponens' on 3,4)
6. $p \Rightarrow q$ (hypothesis)
7. $p \Rightarrow r$ (modus ponens on 5,6)

Definition. If $\emptyset \vdash t$, say t is a *theorem*, written $\vdash t$.

Example. $\vdash (p \Rightarrow p)$. We want to try to get to $(p \Rightarrow (p \Rightarrow)) \Rightarrow (p \Rightarrow p)$ using axiom 2.

1. $[p \Rightarrow ((p \Rightarrow p) \Rightarrow p)] \Rightarrow [(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)]$ (axiom 2)
2. $p \Rightarrow ((p \Rightarrow p) \Rightarrow p)$ (axiom 1)
3. $(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$ (modus ponens on 1,2)
4. $p \Rightarrow (p \Rightarrow p)$ (axiom 1)
5. $p \Rightarrow p$ (modus ponens on 3,4)

Often, showing $S \vdash p$ is made easier by:

Proposition 1.2 (Deduction Theorem)**.** *Let $S \subseteq L$ and $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ if and only if $S \cup \{p\} \vdash q$. Informally: "provability corresponds to the connective '$\Rightarrow$' in L".*

Proof. First we show ($\Rightarrow$): given a proof of $p \Rightarrow q$ from S, write down:

1. p (hypothesis)
2. q (modus ponens)

Which is a proof of q from $S \cup \{p\}$.

Now we show ($\Leftarrow$): we have a proof $t_1, \ldots, t_n$ of q from $S \cup \{p\}$. We'll show that $S \vdash (p \Rightarrow t_i)$ for all i.

If t_i is an axiom, write down

1. t_i (axiom)
2. $t_i \Rightarrow (p \Rightarrow t_i)$ (axiom 1)
3. $p \Rightarrow t_i$ (modus ponens)

So $S \vdash (p \Rightarrow t_i)$.

If $t_i \in S$, do the same thing except step 1 will be "t_i (hypothesis)" instead of "t_i (axiom)".

If $t_i := p$, we have $S \vdash (p \Rightarrow p)$, since $\vdash (p \Rightarrow p)$.

If t_i is obtained by modus ponens, we have t_j and $t_k = (t_j \Rightarrow t_i)$ for some $j, k < n$. By induction, we can assume $S \vdash (p \Rightarrow t_j)$ and $S \vdash (p \Rightarrow (t_j \Rightarrow t_i))$. So write down

1. $[p \Rightarrow (t_j \Rightarrow t_i)] \Rightarrow [(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)]$ (axiom 2)
2. $(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)$ (modus ponens)

3. $p \Rightarrow t_i$ (modus ponens)

So $S \vdash p \Rightarrow t$. □

Example. To show $\{p \Rightarrow q, q \Rightarrow r\} \vdash (p \Rightarrow r)$, it is sufficient to show $\{p \Rightarrow q, q \Rightarrow r, p\} \vdash r$, which is just modus ponens twice.

Question: how are $\models$ and $\vdash$ related?

Aim: $S \models t \iff S \vdash t$ (Completeness Theorem).

This is made up of:

- $S \vdash t \Rightarrow S \models t$ (soundness) i.e "our axioms and deduction rule are not silly";
- $S \models t \Rightarrow S \vdash t$ (adequacy) "our axioms are strong enough to deduce from S, every semantic consequence of S".

Proposition 1.3 (Soundness)**.** *Let $S \subseteq L$, $t \in L$. Then $S \vdash t \Rightarrow S \models t$.*

Proof. We have a proof $t_1, \ldots, t_n$ of t from S. So we must show that every model of S is a model of t, i.e if v is a valuation with $v(s) = 1$ for all $s \in S$, then $v(t) = 1$. But $v(p) = 1$ for each axiom p (each axiom is a tautology), and for each $p \in S$ whenever $v(p) = v(p \Rightarrow q) = 1$, we have $v(q)$. So $v(t_i) = 1$ for all i (induction). □

One case of adequacy is: if $S \models \perp$, then $S \vdash \perp$. We say S is *constistent* if $S \nvdash \perp$. So our statement is: S has no model $\Rightarrow$ S inconsistent, i.e S consistent $\Rightarrow$ S has a model.

In fact, this implies adequacy in general. Indeed, if $S \models t$ then $S \cup \{\neg t\}$ has no model. Hence (by the special case) $S \cup \{\neg t\} \vdash \perp$. So $S \vdash (\neg t \Rightarrow \perp)$, i.e $S \vdash (\neg\neg t)$. But $S \vdash (\neg\neg t) \Rightarrow t$ (axiom 3), so $S \vdash t$.

So our task is: given S consistent, find a model of S. Could try: define

$$v(t) = \begin{cases} 1 & t \in S \\ 0 & t \notin S \end{cases}.$$

But this fails, since S might not be *deductively closed*, meaning $S \vdash p \Rightarrow p \in S$. So we could first replace S with its deductive closure $\{t \in L : S \vdash t\}$ (which is consistent, because S is). However, this still fails: if S does not 'mention' p_3, then $S \nvdash p_3$ and $S \nvdash \neg p_3$, so $v(p_3) = v(\neg p_3) = 0$ which is impossible.

Theorem 1.4 (Model Existence Theorem)**.** *Let $S \subseteq L$ be consistent. Then S has a model.*

Idea: extend S to 'swallow up', for each p, one of p and $\neg p$.

Proof. Claim: for any consistent $S \subseteq L$ and $p \in L$, $S \cup \{p\}$ or $S \cup \{\neg p\}$ is consistent.

Proof of claim: if not, then $S \cup \{p\} \vdash \perp$ and $S \cup \{\neg p\} \vdash \perp$. So $S \vdash (p \Rightarrow \perp)$ (deduction theorem), i.e $S \vdash (\neg p)$. Hence from $S \cup \{\neg p\} \vdash \perp$ we obtain $S \vdash \perp$.

Now, L is countable (as each L_n is countable) so we can list L as $t_1, t_2, \ldots$. Let $S_0 = S$. Let $S_1 = S_0 \cup \{t_1\}$ or $S_1 \cup \{\neg t_1\}$ with S_1 consistent. In general, given S_{n-1} let $S_n = S_{n-1} \cup \{t_n\}$ or $S_n = S_{n-1} \cup \{\neg t_n\}$ so that S_n is consistent. Now set $\overline{S} = S_0 \cup S_1 \cup S_2 \cup \ldots$. Thus for all $t \in L$, either $t \in \overline{S}$ or $(\neg t) \in \overline{S}$.

Now $\overline{S}$ is consistent: if $\overline{S} \vdash \perp$ then, since proofs are finite, we'd have $S_n \vdash \perp$ for some n, a contradiction.

Also, $\overline{S}$ is deductively closed: if $\overline{S} \vdash p$, must have $p \in \overline{S}$, since otherwise $(\neg p) \in \overline{S}$, so $\overline{S} \vdash (p \Rightarrow \perp)$ and $\overline{S} \vdash \perp$.

Now define $v : L \to \{0, 1\}$ by

$$t \mapsto \begin{cases} 1 & t \in \overline{S} \\ 0 & \text{otherwise} \end{cases}.$$

We'll show v is a valuation (then we're done as $v = 1$ on S).

$\underline{v(\perp)}$: have $\perp \notin \overline{S}$ (since $\overline{S}$ is consistent), so $v(\perp) = 0$.

$\underline{v(p \Rightarrow q)}$: if $v(p) = 1$, $v(q) = 0$, then have $p \in \overline{S}$, $q \notin \overline{S}$. But if $(p \Rightarrow q) \in \overline{S}$, then since $p \in \overline{S}$, $q \in \overline{S}$ (since $\overline{S}$ is deductively closed). Now if $v(q) = 1$, $q \in \overline{S}$. But $\overline{S} \vdash (q \Rightarrow (p \Rightarrow q))$ (axiom 1), so $\overline{S} \vdash (p \Rightarrow q)$ hence $(p \Rightarrow q) \in \overline{S}$ ($\overline{S}$ is deductively closed). Finally, if $v(p) = 0$, we have $p \notin \overline{S}$ and want to show $(p \Rightarrow q) \in \overline{S}$. Then $(p \Rightarrow \perp) \in \overline{S}$, so it is enough to show $(p \Rightarrow \perp) \vdash (p \Rightarrow q)$. So it's enough to show $(p, p \Rightarrow \perp) \vdash q$, so enough to show $\perp \vdash q$. But $\perp \vdash (\neg\neg q)$ (axiom 1), and $(\neg\neg q) \vdash q$ (axiom 3), so $\perp \vdash q$ as required. $\square$

Remarks.

1. We used $P = (p_1, p_2, \ldots)$, in saying L is countable. In fact, it also holds if P is uncountable (see later in course).
2. Sometimes this theorem is called 'The Completeness Theorem'

By the remarks stated before this theorem, we have

Corollary 1.5 (Adequacy). *Let $S \subseteq L$, $t \in L$, with $S \models t$. Then $S \vdash t$.*

Hence we have

Theorem 1.6 (Completeness Theorem). *Let $S \subseteq L$, $t \in L$. Then $S \vdash t \iff S \models t$.*

Corollary 1.7 (Compactness Theorem). *Let $S \subseteq L$, $t \in L$ with $S \models t$. Then some finite $S' \subseteq S$ has $S' \models t$.*

Proof. This is trivial if we replace $\models$ by $\vdash$ (as all proofs are finite). $\square$

For $t = \perp$, the theorem says: if $S \models T$ then some finite $S' \subseteq S$ has $S' \vdash \perp$, i.e if every finite $S' \subseteq S$ has a model then S has a model. In fact, this is equivalent to compactness in general: $S \models t$ says $S \cup \{\neg t\}$ has no model, and $S' \models t$ says $S' \cup \{\neg t\}$ has no model.

Corollary 1.8 (Compactness Theorem equivalent form). *Let $S \subseteq L$. Then if every finite subset of S has a model, so does S.*

Another application:

Corollary 1.9 (Decidability Theorem). *Let $S \subseteq L$ be finite and $t \in L$. Then there is an algorithm to decide, in finite time, whether of not $S \vdash t$.*

Remark. This is a very surprising result.

Proof. Trivial if we replace $\vdash$ with $\models$: to check if $S \models t$ we just draw the truth table. $\square$

2 Well-ordering & Ordinals

Definition. A *total order* or *linear order* is a pair $(X, <)$ where X is a set and $<$ is a relation on X that is

(i) *irreflexive*: for all $x \in X$, not $x < x$;

(ii) *transitive*: for all $x, y, z \in X$, if $x < y$, $y < z$ then $x < z$;

(iii) *trichotomous*: for all $x, y \in X$, either $x = y$ or $x < y$ or $y < x$.

We sometimes write $x > y$ if $y < x$, and $x \leq y$ if $x < y$ or $x = y$.

We can instead define a total order in terms of $\leq$ as follows:

(i) *reflextive*: for all $x \in X$, $x \leq x$;

(ii) *transitive*: for all $x, y, z \in X$, if $x \leq y$, $y \leq z$ then $x \leq z$;

(iii) *antisymmetric*: for all $x, y \in X$, if $x \leq y$, $y \leq x$ then $x = y$;

(iv) *trichotomous*: for all $x, y \in X$ either $x \leq y$ or $y \leq x$.

Examples.

1. $\mathbb{N}, <$;
2. $\mathbb{Q}, \leq$;
3. $\mathbb{R}, \leq$;
4. $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ under 'divides' is <u>not</u> a total order, e.g 2 and 3 are not related;
5. $\mathcal{P}(S), \subseteq$ is <u>not</u> a total order - fails trichotomy.

Definition. A total order $(X, <)$ is a *well-ordering* if every (non-empty) subset has a least element, i.e for all $S \subseteq X$ if $S \neq \emptyset$ then there exists $x \in S$ such that $x \leq y$ for all $y \in S$.

Examples.

1. $\mathbb{N}, <$;
2. $\mathbb{Z}, <$ is not a well ordering;
3. $\mathbb{Q}, <$ is not a well ordering;
4. $\mathbb{R}, <$ is not a well ordering;
5. $[0, 1] \subseteq \mathbb{R}, <$ is not a well ordering, e.g $(0, 1]$ has no least element;
6. $\{1/2, 2/3, 3/4, \ldots\} \subseteq \mathbb{R}$ is well ordered;
7. $\{1/2, 2/4, 3/4, \ldots\} \cup \{1\}$ is well ordered;

8. $\{1/2, 2/4, 3/4, \ldots\} \cup \{2\}$ is well ordered;
9. $\{1/2, 2/3, 3/4, \ldots\} \cup \{1 + 1/2, 1 + 2/3, 1 + 3/4, \ldots\}$ is well ordered.

Remark. $(X, <)$ is a well ordering if and only if there is no infinite strictly decreasing sequence.

We say total orders X, Y are *isomorphic* if there exists a bijection $f : X \to Y$ such that $x < y$ if and only if $f(x) < f(y)$. For example, Examples 1&6, 7&8 above are isomorphic. However examples 1&7 are not isomorphic, since in 7 there exists a greatest element, but not in 1.

Proposition 2.1 (Proof by induction)**.** *Let X be well ordered and let $S \subseteq X$ be such that whenever $y \in S$ for all $y < x$, then $x \in S$. Then $S = X$. Equivalently, if $p(x)$ is a property such that $p(y)$ for all $y < x$ implies $p(x)$, then $p(x)$ for all $x \in X$.*

Proof. Suppose $S \neq X$ and let x be least in $X \setminus S$. THen $y \in S$ for all $y < x$ but $x \notin S$, a contradiction. □

Proposition 2.2. *Let X, Y be isomorphic well-orderings. Then there exists a unique isomorphism.*

Note. Note this is false for general total orders, for example $\mathbb{Z} \to \mathbb{Z}$ could have $x \mapsto x - t$ for any t, or $\mathbb{R} \to \mathbb{R}$ could have $x \mapsto x^3$.

Proof. Let $f, g : X \to Y$ be isomorphisms. We'll show $f(x) = g(x)$ for all x by induction on X. Given $f(y) = g(y)$ for all $y < x$, we want to show $f(x) = g(x)$. We must have $f(x) = a$ where a is the least element of $Y \setminus \{f(y) : y < x\}$ (non-empty since it contains $f(x)$). Indeed, if not then $f(x') = a$ for some $x' > x$, contradicting the fact f is order preserving. Similarly have $g(x) = a$. □

Definition. A subset I of a total order X is an *initial segment* if $x \in I$, $y < x$ implies $y \in I$ (i.e I is closed under $<$). For example $I_x = \{y \in X : y < x\}$ is an inititial segment for any $x \in X$, however not every inital segment is of this form, e.g in $\mathbb{Q}$ $\{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$.

Note. In a well-ordering, every proper initial segment I is of the form I_x, for some $x \in X$. Indeed let x be the least element of $X \setminus I$ (non-empty since I is proper). Then $I = I_x$, since if $y < x$ then $y \in I$ (by choice of x), and conversely if $y \in I$, must have $y < x$ or else $y \geq x$ implying $x \in I$ (as I is an initital segment).

Our aim is to show that every subset of a well-ordering X is isomorphic to an initital segment of X.

Note. This is false in general for total orders, e.g $\{1, 2, 3\}$ in $\mathbb{Z}$, or $\mathbb{Q}$ in $\mathbb{R}$.

Theorem 2.3 (Definition by recursion)**.** *Let X be a well-ordering and let Y be any set. Take $G : \mathcal{P}(X \times Y) \to Y$ (i.e a 'rule'). Then there exists a function $f : X \to Y$ such that $f(x) = G(f|_{I_x})$ for all $x \in X$. Moreover, f is unique.*

Note. In defining $f(x)$, we make use of f on $I_x = \{y : y < x\}$.

Proof. Say h is 'an attempt' if $h : I \to Y$ for some initital segment I of X, and for all $x \in I$ we have $h(x) = G(h|_{I_x})$. [This is the main idea].

Note that if h, h' are attempts both defined at x, then $h(x) = h'(x)$, by induction on x (if $h(y) = h'(y)$ for all $y < x$ then $h(x) = h'(y)$).

Also, for every x, there exists an attempt defined at x, also by induction. Indeed, suppose that for all $y < x$ there exists an attempt defined at y. So for all $y < x$ there exists a unique (by above) attempt h_y with domain $\{z : z \leq y\}$. Now let $h = \bigcup_{y \leq x} h_y$, this is an attempt with domain I_x (single valued by uniqueness). Thus $h \cup \{(x, G(h))\}$ is an attempt defined at x. Now define $f : X \to Y$ by setting $f(x) = y$ if there exists an attempt h defined at x such that $h(x) = y$.

Uniqueness of f: if f, f' are both such functions, then $f(x) = f'(x)$ for all x by induction ($f(y) = f'(y)$ for all $y < x$ implies $f(x) = f'(x)$). □

Proposition 2.4 (Subset collapse)**.** *Let X be a well-ordering and $Y \subseteq X$. Then Y is isomorphic to an initital segment of X. Moreover, I is unique.*

Proof. To have $f : Y \to X$ an isomorphism with an initial segment of X, we need precisely that for every $x \in Y$ we have that $f(x)$ is the minimum element of $X \setminus \{f(y) : y < x\}$. So we're done by the previous theorem. □

Note. We have $X \setminus \{f(y) : y < x\} \neq \emptyset$, since $f(y) \leq y$ for all y (induction), so $x \notin \{f(y) : y < x\}$.

In particular, X itself cannot be isomorphic to a proper intial segment (uniqueness).

How do different well-orderings relate to each other?

Definition. For well-orderings X, Y we write $X \leq Y$ if X is isomorphic to an initial segment of Y.

Example. If $X = \mathbb{N}$, $Y = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right)$, then $X \leq Y$.

Proposition 2.5. *Let X, Y be well-orderings. Then $X \leq Y$ or $Y \leq X$.*

Proof. Suppose $Y \not\leq X$, we'll show $X \leq Y$. To obtain $f : X \to Y$ an isomorphism with an initial segment of Y, we need precisely that for every $x \in X$, $f(x)$ is the least element in $Y \setminus \{f(y) : y < x\}$ [note this can only be empty if Y is isomorphic to I_x]. So we're done by recursion. $\square$

Proposition 2.6. *Let X, Y be well-orderings with $X \leq Y$ and $Y \leq X$. Then X and Y are isomorphic.*

Note. This proposition and the previous one are "the most we could ever hope for".

Proof. We have isomorphisms f from X to some initial segment of Y, and g from Y to some initial segment of X. Then $g \circ f : X \to X$ is an isomorphism from X to an initial segment of X (as initial segment of an initial segment of X is itself an initial segment). So by uniqueness $g \circ f = \mathrm{id}_X$. Similarly $f \circ g = \mathrm{id}_Y$. Hence f and g are inverses, thus bijections. $\square$

New well-ordering from old

For well-orderings X, Y, we say $X < Y$ if $X \leq Y$ and X is not isomorphic to Y. Equivalently, $X < Y$ if and only if X is isomorphic to a proper initial segement of Y.

We can 'make a bigger one': given a well-ordering X, pick some $x \notin X$ and well-order $X \cup \{x\}$ by setting $y < x$ for all $y \in X$. This is a well-ordering and is $> X$. Call this the *successor* of X, written X^+.

We can 'put some together': given $\{X_i\}_{i \in I}$ well-orderings, seek X with $X \geq X_i$ for all i. For well-orderings $(X, <_X), (Y, <_Y)$ we say Y *extends* X if $X \subseteq Y$, $<_Y |_X = <_X$, and X is an intitial segment of $(Y, <_Y)$. Say well-ordeings $\{X_i\}_{i \in I}$ ar *nested* if for all i, j, X_i extends X_j or X_j extends X_i.

Proposition 2.7. *Let $\{X_i\}_{i \in I}$ be a nested set of well-orderings. Then there exists a well-ordering X such that $X \geq X_i$ for all i.*

Proof. Let $X = \bigcup_{i \in I} X_i$, with ordering $<_X = \bigcup_{i \in I} <_i$, i.e $x < y$ in X if there exists i such that $x, y \in X_i$ and $x <_i y$. Given $S \subseteq X$ non-empty, we have $S \cap X_i$ non-empty for some $i \in I$. Let x be the least element of $S \cap X_i$ (under $<_i$). Then x is the least element of S in X since X_i is an initial segment of X, by nestedness. So X is a well-ordering, and $X \geq X_i$ for all i. $\square$

Remark. The above proposition also holds if we don't know the X_i are nested.

Ordinals

"Does the collection of all well-orderings itself form a well-ordering?"

Definition. An *ordinal* is a well-ordered set, with two well-ordered sets regarded as the same if they are isomorphic. [1]

Definition. For a well-ordering X, corresponding to an ordinal α, say X has *order-type* α.

For any $k \in \mathbb{N}$, write k for the order-type of the (unique up to isomorphiam) well-ordering on a set of size k. Write ω for the order-type of $\mathbb{N}$.

Example. In $\mathbb{R}$:

- $\{-2, 3, \pi, 5\}$ has order-type 4;
- $\{1/2, 2/3, 3/4, \ldots\}$ has order-type ω.

[1] Just as a rational is an expression m/n with two regarded as the same if $mn' = m'n$. However, cannot formalise this using equivalence classes in the case of ordinals, see later chapter.

Write $\alpha \leq \beta$ if $X \leq Y$, where X has order-type α and Y has order-type β (note this is well defined since it doesn't depend on the choice of X, Y). Similarly define $\alpha < \beta$, α^+ etc.

Hence for all ordinals α, β, $\alpha \leq \beta$ or $\beta \leq \alpha$. Also, if $\alpha \leq \beta$ and $\beta \leq \alpha$, $\alpha = \beta$.

Proposition 2.8. *For any ordinal α, the ordinals $< \alpha$ form a well-ordered set of order-type α.*

Proof. Let X have order-type α. Then the well-ordered sets $< X$ are precisely (up to isomorphism) the proper initial segments of X, i.e they are I_x for $x \in X$. These order biject with X itself, via $I_x \leftrightarrow x$. $\square$

So for any α, have $I_\alpha = \{\beta : \beta < \alpha\}$ a well-ordered set of order-type α.

Proposition 2.9. *Every non-empty set S of ordinals has a least element.*

Proof. Choose $\alpha \in S$. If α is minimal in S, we're done. Otherwise, $S \cap I_\alpha$ is non-empty, so has a least element in I_α since I_α is well-ordered, and this element is least in all of S. $\square$

However:

Theorem 2.10 (Burali-Forti Paradox)**.** *The ordinals do not fom a set.*

Proof. Suppose X was the set of all ordinals. Then X is a well-ordered set, so has an order type, say α. Thus X is order-isomorphic to I_α, so X is order-isomorphic to a proper initial segment of itself, contradiction. $\square$

Note. Given a set $S = \{\alpha_i\}_{i \in I}$ of ordinals, there exists an upper bound α for S, by applying proposition 2.7 to the nested family of the $\{I_{\alpha_n}\}_{i \in I}$. Hence by proposition 2.9 it has a least upper bound. We write $\sup S$.

Example. $\sup\{2, 4, 6, \ldots\} = \omega$.

We'll give some examples of ordinals

Examples.

- $0, 1, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots, \omega + \omega$ (really $\omega + 1$ is ω^+ and $\omega + \omega = \omega \cdot 2 = \sup\{\omega, \omega + 1, \ldots\}$).
- Continuing with $\omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot 4, \ldots, \omega \cdot \omega^2$ where $\omega^2 = \omega \cdot \omega = \sup\{\omega, \omega \cdot 2, \omega \cdot 3, \ldots\}$.
- Now $\omega^2, \omega^2 + 1, \ldots, \omega^2 + \omega$ and $\omega^2 + \omega \cdot 2, \omega^2 + \omega \cdot 3, \ldots, \omega^2 + \omega^2 = \omega^2 \cdot 2$.
- $\omega^2 \cdot 2, \omega^2 \cdot 3, \ldots, \omega^3$.
- $\omega^3, \ldots, \omega^3 + \omega^2 \cdot 7 + \omega \cdot 4 + 13$.
- $\omega^\omega = \sup\{\omega, \omega^2, \omega^3, \ldots\}$
- $\omega^{\omega+1} = \sup\{\omega^\omega + 1, \omega^\omega + 2, \ldots\}$
- $\omega^{\omega \cdot 2}, \omega^{\omega \cdot 3}, \ldots, \omega^{\omega^2}$.
- ω^{ω^ω}
- $\omega^{\omega^{\omega^2}}$, $\omega^{\omega^{\omega^4}}$
- $\omega^{\omega^{\omega}} = \varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$.
- $\varepsilon_0, \varepsilon_0 + 1, \ldots, \varepsilon_0 + \omega, \ldots, \varepsilon_0 + \varepsilon_0$
- $\varepsilon_0 \cdot \omega, \ldots, \varepsilon_0^2$
- $\varepsilon_0^{\varepsilon_0} = \sup\{\varepsilon_0^\omega, \varepsilon_0 \omega^\omega, \varepsilon_0^{\omega^{\omega^\omega}}\}$
- $\varepsilon_1 = \sup\{\varepsilon_0, \varepsilon_0^{\varepsilon_0}, \varepsilon_0^{\varepsilon_0^{\varepsilon_0}}\}$.

All of the above are countable (e.g countable union of countable sets). Is there an uncountable ordinal? i.e is there is there an uncountable well-ordering. e.g can well-order $\mathbb{N}$, can well-order $\mathbb{Q}$ (biject with $\mathbb{N}$), can we well-order $\mathbb{R}$? Amazingly, we can prove we can.

Theorem 2.11. *There is an uncountable ordinal.*

Proof. Let $A = \{R \in \mathcal{P}(\omega \times \omega) : R \text{ is a well-ordering of a subset of } \omega\}$. Let $B = \{\text{order-type}(R) : R \in A\}$. So $\alpha \in B$ if and only if α is a countable ordinal. Let $\omega_1 = \sup B$. We must have ω_1 uncountable - if it was countable, then it would be the greatest element of B, contradicting $\omega_1 < \omega_1^+$ since ω_1^+ is countable. $\square$

Remark. Alternatively having the set B, could say that B isn't all ordinals since the set of ordinals is not a set (Burali-Forti), so there exists an uncountable ordinal.

Note. ω_1 is the least uncountable ordinal by definition of B.

The ordering ω_1 has some remarkable properties, e.g

1. ω_1 is uncountable but $\{\beta : \beta < \alpha\}$ is countable for all $\alpha < \omega_1$.
2. Any sequence $\alpha_1, \alpha_2, \ldots$ in I_{ω_1} is bounded. Namely, by $\sup\{\alpha_1, \alpha_2, \ldots\}$ which is countable as a countable union of countable sets.

The same argument shows:

Theorem (Hartogs' Lemma)**.** *For every set X, there exists an ordinal α that does not inject into X.*

We call the least such ordinal as in Hartogs' Lemma $\gamma(X)$, e.g $\gamma(\omega) = \omega_1$.

Definition. Say α is a *successor* if there exists β such that $\alpha = \beta^+$. Otherwise we say α is a *limit*.

Note that α has a greatest element if and only if it is a successor. So α is a limit if and only if α has no greatest element, i.e for all $\beta < \alpha$ there exists $\gamma < \alpha$ with $\beta < \gamma$.

Example. 5 is a successor: 4^+. $\omega+2$ is a successor: $(\omega^+)^+$. ω is not a successor: no greatest element. 0 is also a limit.

Ordinal Arithmetic

We define $\alpha + \beta$ by induction on β (α fixed) by:

- $\alpha + 0 = \alpha$;
- $\alpha + (\beta^+) = (\alpha + \beta)^+$;
- $\alpha + \lambda = \sup\{\alpha + \gamma : \gamma < \lambda\}$ for λ a non-zero limit.

Examples.

- $\omega + 1 = \omega + 0^+ = (\omega + 0)^+ = \omega^+$;
- $\omega + 2 = \omega + 1^+ = (\omega + 1)^+ = \omega^{++}$;
- $1 + \omega = \sup\{1 + \gamma : \gamma < \omega\} = \omega$ - so $+$ is not commutative.

Remark. Officially (as the ordinals do not form a set), this means: to define $\alpha + \beta$ we actually define $\alpha + \gamma$ on $\{\gamma : \gamma \leq \beta\}$, which is a set; plus uniqueness. Similarly, for proof by induction: if for some α we have $p(\alpha)$ false, then on $\{\gamma : \gamma \leq \alpha\}$, p is not everywhere true.

Proposition 2.12. *We have $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ for all ordinals α, β, γ.*

Proof. We proceed by induction on γ (α, β fixed). If $\gamma = 0$: $\alpha + (\beta + 0) = \alpha + \beta = (\alpha + \beta) + 0$.

Successors:

$$\begin{aligned} \alpha + (\beta + \gamma^+) &= \alpha + (\beta + \gamma)^+ \\ &= (\alpha + (\beta + \gamma))^+ \\ &= ((\alpha + \beta) + \gamma)^+ \\ &= (\alpha + \beta) + \gamma^+ \end{aligned}$$

λ a non-zero limit:

$$\begin{aligned} (\alpha + \beta) + \lambda &= \sup\{(\alpha + \beta) + \gamma : \gamma < \lambda\} \\ &= \sup\{\alpha + (\beta + \gamma) : \gamma < \lambda\}. \end{aligned}$$

We claim that $\beta + \lambda$ is a limit. Indeed, have $\beta + \lambda = \sup\{\beta + \gamma : \gamma < \lambda\}$. But for every $\gamma < \lambda$, there exists $\gamma' < \lambda$ with $\gamma < \gamma'$ (λ a limit), so $\beta + \gamma < \beta + \gamma'$. Thus there is no greatest element of $\{\beta + \gamma : \gamma < \lambda\}$, so $\beta + \lambda = \sup\{\beta + \gamma : \gamma < \lambda\}$ is a limit.

Therefore $\alpha + (\beta + \lambda) = \sup\{\alpha + \delta : \delta < \beta + \lambda\}$. So need to show $\sup\{\alpha + (\beta + \gamma) : \gamma < \lambda\} = \sup\{\alpha + \delta : \delta < \beta + \lambda\}$. Indeed, $\gamma < \lambda$ implies $\beta + \gamma < \beta + \lambda$ so $\{\alpha + (\beta + \gamma) : \gamma < \lambda\} \subseteq \{\alpha + \delta : \delta < \beta + \lambda\}$. Conversely, for all $\delta < \beta + \lambda$, we have $\delta \leq \beta + \gamma$ for some $\gamma < \lambda$ (definition of $\beta + \lambda$) so $\alpha + \delta \leq \alpha + (\beta + \gamma)$. So each member of right hand set is at most some member of the left hand set. $\square$

Notes.

1. We used: $\beta \leq \gamma \Rightarrow \alpha + \beta \leq \alpha + \gamma$ (trivial by induction on γ)
2. $\beta < \gamma \Rightarrow \alpha+\beta < \alpha+\gamma$ since $\beta < \gamma \Rightarrow \beta^+ \leq \gamma$ which implies $\alpha+\beta^+ \leq \alpha+\gamma$ so $\alpha + \beta < (\alpha + \beta)^+ = \alpha + \beta^+ \leq \alpha + \beta$.
3. However $1 < 2$, but $1 + \omega = 2 + \omega = \omega$. So "stuff on the right always works as expected".

The above is the inductive definition of $+$. There is also a synthetic definition of $+$: $\alpha+\beta$ is the order type of $\alpha \sqcup \beta$ (disjoint union, e.g $(\alpha \times \{0\}) \cup (\beta \times \{1\})$), with all of α coming before all of β.

Example.

- $\omega + 1$ is the order type of $\underbrace{\;\omega\;}\underbrace{\;\bullet\;}$;
- $1 + \omega$ is the order type of $\underbrace{\;\bullet\;}\underbrace{\;\omega\;}$;
- $\alpha + (\beta + \gamma)$ is the order type of $\underbrace{\;\alpha\;}\underbrace{\;\beta\;}\underbrace{\;\gamma\;}$.

Proposition 2.13. *The two definitions of $+$ are equivalent.*

Proof. We write $+$ for the inductively defined one, and $+'$ for the synthetic one. We'll show $\alpha + \beta = \alpha +' \beta$ for all $\alpha + \beta$ by induction on β (α fixed).
Zero: $\alpha + 0 = \alpha +' = 0 = \alpha$.

Successors: $\alpha + (\beta^+) = (\alpha + \beta)^+ = (\alpha +' \beta)^+$ which is the order type of $\underbrace{\;\alpha\;}\underbrace{\;\beta\;}\underbrace{\;\bullet\;}$ which is $\alpha +' \beta^+$.

λ a non-zero limit: $\alpha + \lambda = \sup\{\alpha + \gamma : \gamma < \lambda\} = \sup\{\alpha +' \gamma : \gamma < \lambda\} = \alpha +' \lambda$ (since sup is a union as sets are nested) $\square$

Moral: synthetic definition beats the inductive one, if we do have a synthetic definition.

Definition. Define $\alpha\beta$ (α fixed, recursion on β) by:

- $\alpha 0 = 0$;
- $\alpha(\beta^+) = \alpha\beta + \alpha$;
- $\alpha\lambda = \sup\{\alpha\gamma : \gamma < \lambda\}$ for λ a non-zero limit.

Examples.

- $\omega 2 = \omega 1 + \omega = (\omega 0 + \omega) + \omega = \omega + \omega$;

- $\omega 3 = \omega + \omega + \omega$;
- $\omega\omega = \sup\{0, \omega, \omega + \omega, \ldots\}$;
- $2\omega = \sup\{0, 2, 4, 6, 8, \ldots\} = \omega$, so again this is <u>not</u> commutative.

Can show that $\alpha(\beta\gamma) = (\alpha\beta)\gamma$, etc.

We also have a synthetic definition (which can be shown to be equivalent): $\alpha\beta$ is equal to the order type of

$$\underbrace{\underbrace{\alpha}\underbrace{\alpha}\underbrace{\alpha}\cdots\underbrace{\alpha}}_{\beta\text{ times}},$$

ordered by: $(x, y) < (z, w)$ if $y < w$ or $y = w$ and $x < z$.

Example. $\omega 2$ is the order type of $\underbrace{\omega}\underbrace{\omega}$ which is $\omega + \omega$. Also 2ω is the order type of

$$\underbrace{\underbrace{\bullet\bullet}\underbrace{\bullet\bullet}\underbrace{\bullet\bullet}\cdots\underbrace{\bullet\bullet}}_{\omega\text{ times}},$$

which is ω.

We can also do exponentiation, towers etc similarly. For example, define α^β by

- $\alpha^0 = 1$;
- $\alpha^{(\beta^+)} = \alpha^\beta \alpha$;
- $\alpha^\lambda = \sup\{\alpha^\gamma : \gamma < \lambda\}$ for λ a non-zero limit.

For example, $\omega^2 = \omega^1\omega = (\omega^0\omega)\omega = \omega\omega$; $2^\omega = \sup\{2^0, 2^1, \ldots\} = \omega$.

3 Posets and Zorn's Lemma

Definition. A *partially ordered set* or *poset* is a pair $(X, \leq)$, where X is a set and $\leq$ is a relation on X that is

(i) Reflexive: $x \leq x$ for all $x \in X$;

(ii) Transitive: $x \leq y, y \leq z$ implies $x \leq z$ for all $x, y, z \in X$;

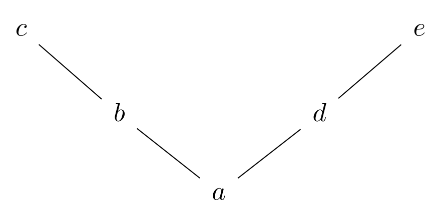
(iii) Antisymmetric: $x \leq y, y \leq x$ implies $x = y$ for all x, y.

We write $x < y$ if $x \leq y$ and $x \neq y$. In terms of $<$, a poset is:

(i) Irreflexive: $x \not< x$ for all $x \in X$;

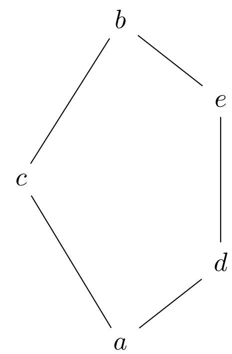
(ii) Transitive: $x < y, y < z$ implies $x < z$ for al $x, y, z \in X$.

Examples.

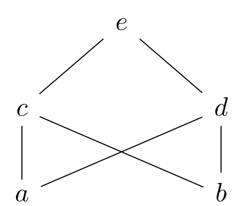
1. Any total order.
2. $\mathbb{N}^+$ with $x \leq y$ if $x|y$.
3. $(\mathcal{P}(S), \subseteq)$ for any set S.
4. $X \subseteq \mathcal{P}(S)$ under $\subseteq$. For example, V a vector space, X the set of all subspaces.
5. 

   Consider a tree graph, with all edges pointing upwards. Then we say $x \leq y$ for vertices x, y if the xy is a directed edge pointing upwards, and extend $\leq$ by transitivity. In general the *Hasse diagram* of a poset is a drawing of its posets,with an upwards line from x to y if y covers x (meaning $y > x$ and no z has $y > z > x$). Hasse diagrams can be useful: e.g $(\mathbb{N}, \leq)$, or useless: e.g for $(\mathbb{Q}, \leq)$ there are no covers.

6.



7.



8.

$a \quad b \quad c \quad d \quad e$

A subset S of a poset X is a *chain* if it is totally ordered. E.g in example 2 above, $\{1, 2, 4, 8, 16, \ldots\}$. Or in example 5, $\{a, b, c\}$ or $\{a, c\}$.

A subset S is an *antichain* if no two elements are related. E.g in 2, $\{n : n \text{ prime}\}$, in 5, $\{c, e\}$, or in 8 take whole poset.
For $S \subseteq X$, an *upper bound* for S is an $x \in X$ such that $x \geq y$ for all $y \in S$. We say x is a *least upper bound* for S if x is an upper bound, and if y is an upper bound of S, $x \leq y$.

Examples.

- In $\mathbb{R}$: if $S = \{x : x < \sqrt{2}\}$ then 7 is an upper bound, and $\sqrt{2}$ is the least upper bound. We write $\sqrt{2} = \sup S$, or $\bigvee S$.
- In $\mathbb{Q}$: $\{x : x^2 < 2\}$ has 7 as an upper bound, but there is no least upper bound.
- In example 5 from before, $\{a, b\}$ has upper bounds b and c, so least upper bound b. $\{b, d\}$ has no upper bound.
- From example 7 from before, $\{a, b\}$ has upper bounds c, d, e, so does not have a least upper bound.

We say X is *complete* if every $S \subseteq X$ has a least upper bound. For example, $\mathbb{R}$ is not complete, e.g $\mathbb{Z}$ has no upper bound. $(0,1)$ is not complete since $(0,1)$ itself has no upper bound.

$X = \mathcal{P}(S)$ is always complete: sup of $\{A_i : i \in I\}$ is $\bigcup_{i \in I} A_i$.

Note. Every complete poset X has a greatest element x, namely $\sup X$, and also a least element y, namely $\sup\{\emptyset\}$.

For $f : X \to Y$ where X, Y are posets, we say f is *order preserving* if $x \leq y$ implies $f(x) \leq f(y)$.

Examples.

1. $f : \mathbb{N} \to \mathbb{N}$, $x \mapsto x + 1$.
2. $f : [0,1] \to [0,1]$, $x \mapsto \frac{1+x}{2}$.
3. $f : \mathcal{P}(S) \to \mathcal{P}(S)$, $A \mapsto A \cup \{i\}$ for some fixed $i \in S$.

Not every order preserving $f : X \to X$ has a fixed point, e.g example 1 above. However:

Theorem 3.1 (Knaster-Tarski Fixed Point Theorem)**.** *Let X be a complete poset. Then any order preserving $f : X \to X$ has a fixed point.*

Proof. Let $E = \{x \in X : x \leq f(x)\}$, and let $s = \sup E$. We'll show that $f(s) = s$.

We'll first show $s \leq f(s)$. It is enough to show $f(s)$ is an upper bound for E, then done since s is a least upper bound for E. Indeed, if $x \in E$, then $x \leq s$ so $f(x) \leq f(s)$. Now since $x \in E$, $x \leq f(x) \leq f(s)$.

Now we show $f(s) \leq s$. It is enough to show $f(s) \in E$, then done since s is an upper bound for E. We have $s \leq f(s)$, so $f(s) \leq f(f(s))$. i.e $f(s) \in E$. $\square$

Remark. We need to show $s \leq f(s)$ before $f(s) \leq s$ since $s \leq f(s)$ says $s \in E$.

An application of this is:

Corollary 3.2 (Schröder-Bernstein)**.** *Let $f : A \to B$ and $g : B \to A$ be injections. Then there exists a bijection $h : A \to B$.*

Proof. We seek partitions $A = P \cup Q$, $P \cap Q = \emptyset$, $B = R \cup S$, $R \cap S = \emptyset$ such that $f(P) = R$ and $g(S) = Q$ (then set $h = f$ on P and $h = g^{-1}$ on R). Thus we seek exactly a fixed point of $\Theta : \mathcal{P}(A) \to \mathcal{P}(A)$ given by $P \mapsto A \setminus g(B \setminus f(P))$. But $\mathcal{A}$ is complete, and Θ is order preserving, so done by Knaster-Tarski. $\square$

Zorn's Lemma

For a poset X, $x \in X$ is said to be *maximal* if no $y \in X$ has $y > x$. For example, in $[0,1]$, 1 is maximal. We've seen many posets without any maximal element, for example $(\mathbb{R}, \leq)$ or $(\mathbb{N}^+, |)$. In each case, there exists a chain with no upper bound (e.g in $(\mathbb{N}^+, |)$ take powers of 2).

Theorem 3.3 (Zorn's Lemma)**.** *Let X be a (non-empty) poset in which every chain has an upper bound. Then there exists a maximal element of X.*

Proof. Suppose not. So for each $x \in X$ have $x' \in X$ with $x' > x$, and for each chain we C we have an upper bound $u(C)$. Fix some $x \in X$ and define x_α for each $\alpha < \gamma(X)$ by recursion: $x_0 = x$, $x_{\alpha+1} = x'_\alpha$ and $x_\lambda = U(\{x_\beta : \beta < \lambda\})$ for λ a non-zero limit [note that the x_β, $\beta < \lambda$ do form a chain, by induction]. Then we have injected $\gamma(X)$ into X, a contradiction. $\square$

Remark. The proof was easy, given well-orderings; recursion and Hartogs' from Chapter 2.

A typical application: does every vector space have a basis?

Examples.

- $\mathbb{R}^3$ has a basis $\{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$.
- Space of all real polynomials has basis $\{1, X, X^2, X^3, \ldots\}$.
- Space S of all real sequences - $\{e_i\}_{i\in\mathbb{N}}$ is <u>not</u> a basis - span doesn't contain $(1,1,1,1,\ldots)$. In fact S has no countable basis (and can actually show there's no explicit basis).
- $\mathbb{R}$ as a vector space over $\mathbb{Q}$ - no explicit basis. A basis in this case is called a *Hamel basis*.

Theorem 3.4. *Every vector space V has a basis.*

Proof. Let $X = \{A \subseteq V : A \text{ is linearly independent}\}$, ordered by $\subseteq$. We seek a maximal element of X [then done since if a maximal element doesn't span, could extend by adding something not in span]. Have $X \neq \emptyset$ since $\emptyset \in X$. Given a chain $\{A_i : n \in \mathbb{I}\}$, let $A = \bigcup_{i\in I} A_i$. Certainly $A \supseteq A_i$ for every i, so we just need to show $A \in X$, i.e A is linearly independent. Suppose A is not linearly independent. Then we have $x_1, \ldots, x_n \in A$ which are linearly dependent. We have $x_1 \in A_{i_1}, \ldots x_n \in A_{i_n}$ for some $i_1, \ldots, i_n \in I$, but some A_{i_k} contains all of $A_{i_1}, \ldots, A_{i_n}$ (since the A_i form a chain), so $x_1, \ldots, x_n \in A_{i_k}$, contradicting the fact A_{i_k} is linearly independent.

Hence by Zorn's Lemma, there exists a maximal element of X. $\square$

Notes.

1. The only actual linear algebra we did was in the 'then done' part.

2. In the statement of Zorn's Lemma, the hypothesis $X \neq \emptyset$ is not strictly needed since $\emptyset$ is not a chain so has an upper bound.

Another application: completeness theorem for propositional logic with no restriction on P.

Theorem 3.5. *Let $S \subseteq L = L(P)$ (for any set P) be a set that is consistent. Then S has a model.*

Proof. We'll extend S to $\overline{S}$ (consistent) such that for all $t \in L$, $t \in \overline{S}$ or $(\neg t) \in \overline{S}$ - then done as in Chapter 1 by setting $v = 1$ on $\overline{S}$, $v = 0$ elsewhere.

Let $X = \{T \supseteq S : T \text{ consistent}\}$, ordered by $\subseteq$. We seek a maximal element of X [then done: let $\overline{S}$ be maximal, then if $t \notin \overline{S}$, we must have $\overline{S} \cup \{t\} \vdash \perp$ (maximality of $\overline{S}$), so $\overline{S} \vdash (\neg t)$ (deduction theorem), so $(\neg t) \in \overline{S}$ by maximality of $\overline{S}$].

Now, $X \neq \emptyset$ since $S \in X$. Given a non-empty chain $\{T_i : i \in I\}$, put $T = \bigcup_{i \in I} T_i$. Have $T \supseteq T_i$ for all i, so we just need to show $T \in X$. Indeed $S \subseteq T$ (chain is non-empty). Also, we claim T is consistent. Suppose not, i.e $T \vdash \perp$. Then some $\{t_1, \ldots, t_n\} \vdash \perp$ for some $t_1, \ldots, t_n \in T$ (as proofs are finite). Now, $t_1 \in T_{i_1}, \ldots, t_n \in T_{i_n}$ for some $i_1, \ldots, i_n \in I$. Then $t_1, \ldots, t_n \in T_{i_k}$ for some k since the T_i form a chain. But this contradicts the fact T_{i_k} is consistent.

Hence by Zorn's Lemma, X has a maximal element. □

Theorem 3.6 (Well-ordering principle)**.** *Every set S can be well-ordered.*

Note. This is a very surprising result, e.g for $S = \mathbb{R}$ - until one has met Hartogs' lemma.

Proof. Let $X = \{(A, R) : A \subseteq S, R \text{ is a well-ordering of } A\}$, ordered by $(A, R) \leq (A', R')$ if the latter extends the former (i.e $R'|_A = R$ and A is an initial segment of R'). We have $X \neq \emptyset$ (e.g $(\emptyset, \emptyset) \in X$). Given a chain $\{(A_i, R_i) : i \in I\}$, have an upper bound $\left(\bigcup_{i \in I} A_i, \bigcup_{i \in I} R_i\right)$, since the family is nested.

So by Zorn's Lemma, there exists a maximal element (A, R). Must have $A = S$, if not we can take $x \in S \setminus A$ and 'take the successor': well-order $A \cup \{x\}$ by making $x > y$ for all $y \in A$ - contradicting maximality of (A, R). □

Zorn's Lemma & The Axiom of Choice

In our proof of Zorn's Lemma, we made infinitely many arbitrary choices - when selecting the x'. We also did this in IA, when showing a countable union of countable sets is countable: have $A_1, A_2, \ldots$, each having a listing, and we fixed, all at once, a listing for each of them.

In terms of 'rules for building sets', we are appealing to the *Axiom of Choice* which states that, given a family of non-empty sets, one can choose an element fom each one. More precisely: for any family $\{A_i : i \in I\}$ of non-empty sets, there is a *choice function* $f : I \to \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

This is different in character from the other 'rules for building sets' (e.g 'given A, B can form $A \cup B$' or 'given A, can form $\mathcal{P}(A)$') in that the object whose existence is asserted is not uniquely specified by its properties. So the use of the Axiom of Choice gives rise to non-constructive proofs. [Many proofs in maths, even without AC, are non-constructive - e.g the proof by countability argument that there exists a transcendental number, or proof that in $\mathbb{Q}[X_1, \ldots, X_n]$ every ideal is finitely generated].

So it is often nice to know: did a proof <u>need</u> AC.

Did our proof of Zorn's Lemma <u>need</u> AC? Answer: yes, we can actually deduce AC from Zorn's (using only the other set-building rules). Indeed, AC follows from well-ordering (the previous theorem): given our family $\{A_i : i \in I\}$, just well-order $\bigcup_{i \in I} A_i$ and now set $f(i)$ to be the least element of A_i for each $i \in I$.

<u>Conclusion</u>: AC $\iff$ ZL $\iff$ WO (in the presence of the other set-building rules).

Remark. AC is trivial if $|I| = 1$ ($A \neq \emptyset$ means there exists $xinA$), also easy to prove for all I finite (by induction on $|I|$). But, in general it turns out that AC cannot be deduced from the other set-building rules.

Notes.

1. ZL is hard from first principles because it needed ordinals, recursion and Hartogs' - not because it's equivalent to AC.

2. No theorem in Chapter 2 used AC. Indeed, AC was used only in two remarks in Chapter 2: the fact that in a non-well-ordering there exists an infinite decreasing sequence; and the fact that ω_1 is not a countable supremum.

4 Predicate Logic

Recall that a group is a set A equipped with functions $M : A^2 \to A$ (of arity 2), $i : A^1 \to A$ (of arity 1), and a constant $e \in A$, (i.e a function $A^0 \to A$, i.e arity 0).

Also recall a poset is a set A equipped with a relation $\leq \subseteq A^2$ (arity 2) such that certain axioms hold.

Let Ω ('set of all function symbols') and Π ('set of all relation symbols') be disjoint sets, and $\alpha : \Omega \cup \Pi \to \mathbb{N}$ ('arity'). The *language* $L = L(\Omega, \Pi, \alpha)$ is the set of all *formulae*, defined as follows:

Variables: $x_1, x_2, x_3, \ldots$.

Terms: defined inductively by

(i) Each variable is a term;

(ii) For $f \in \Omega$, $\alpha(f) = n$, and terms $t_1, \ldots, t_n$, $ft_1 \ldots t_n$ is a term (can insert brackets,commas etc for readability).

Example. Language of groups: $\Omega = \{M, i, e\}$ (arities $2, 1, 0$ respectively), $\Pi = \emptyset$. Some terms: $M(x_1, x_2)$, $M(x_1, i(x_2))$, e, $M(e, e)$, $M(e, x_1)$.

Atomic formulae:

(i) $\perp$ is an atomic formula;

(ii) For terms s and t, $(s = t)$ is an atomic formula;

(iii) For $\phi \in \Pi$, $\alpha(\phi) = n$, and terms $t_1, \ldots, t_n$, $\phi(t_1, \ldots, t_n)$ is an atomic formula.

Example. In language of groups: $e = M(e, e)$, $M(x, y) = M(y, x)$ are atomic formulae.

Example. Language of posets: $\Omega = \emptyset$, $\Pi = \{\leq\}$ (arity 2). Some terms: $x = y$, $x \leq y$ (officially '$\leq (xy)$').

Formulae: defined inductively by

(i) Each atomic formula is a formula;

(ii) If p, q are formulae, then $(p \Rightarrow q)$ is a formula;

(iii) If p a formula, x a variable then $(\forall x)p$ is a formula.

Example. In the language of groups: $(\forall x)(M(x, x) = e)$, $(M(x, x) = e) \Rightarrow (\exists)(M(y, y) = x)$.

Notes.

1. A formula is a finite string of symbols;
2. Can now define $(\neg p), p \wedge q, p \vee q$, etc and also $(\exists x)p$ as $\neg(\forall x)(\neg p)$.

A term is *closed* if it contains no variables. For example, $e, M(e, i(e))$ - but not $M(x, e)$ or $M(x, i(x))$.

An occurrence of a variable x in a formula p is *bound* if it is inside the brackets of a '$(\forall x)$' quantifier; otherwise it is *free*.

Example. In $(\forall x)(M(x,x) = e)$, each occurrence of x is bound. In $(M(x,x) = e) \Rightarrow (\exists y)(M(y,y) = x)$, each x is free but each y is bound.

Note. Consider $M(x,x) = e \Rightarrow (\forall x)(\forall y)(M(x,y) = M(y,x))$. The first two occurrences of x are free, while all variables in the right side of '$\Rightarrow$' are bound.

Definition. A *sentence* is a formula with no free variables.

Example. $(\forall x)(M(x,x) = e)$, $(\forall x)(M(x,x) = e \Rightarrow (\exists y)(M(y,y) = x))$ are sentences. In the language of posets: $(\forall x)(\exists y)(x \leq y \wedge \neg x = y)$ is a sentence.

Definition. For a formula p, term t and variable x, the *substitution* $p[t/x]$ is obtained from p by replacing each free occurrence of x with t.

Example. If p is '$(\exists y)(M(y,y) = x)$' then $p[e/x]$ is '$(\exists y)(M(y,y) = e)$'.

Semantic Implication

Definition. Let $L = L(\Omega, \Pi, \alpha)$ be a language. An *L-structure* is a non-empty[2] set A equipped with, for each $f \in \Omega$, a function $f_A : A^n \to A$ (where $n = \alpha(f)$), and for each $\phi \in \Pi$, a subset $\phi_A \subseteq A^n$ $(n = \alpha(\phi))$.

Example. Language of groups: an L-structure is an A with $M_A : A^2 \to A$, $i_A : A \to A$, $e_A \in A$ (note: may not be a group). Language of posets: an L-structure is an A with $\leq_A \subseteq A^2$ (may not be a poset).

For an L-structure A and a sentence p, we want to define 'p holds in A'.

Example. '$(\forall x)(M(x,x) = e)$' should hold in A if and only if for each $a \in A$, have $M_A(a,a) = e_A$.

So "insert '$\in A$' after each '$\forall x$' and add subscripts 'sub A' and read it aloud".

[2] See later for why.

We now formally define what we mean in the above:

Define the *interpretation* $p_A \in \{0,1\}$ of a sentence p in an L-structure A as follows.

The *interpretation* $t_A \in A$ of a closed term in an L-structure A is defined inductively: $(ft_1, \ldots, t_n)_A = f_A((t_1)_A, \ldots, (t_n)_A)$ for $f \in \Omega$, $\alpha(f) = n$, $t_1, \ldots, t_n$ closed terms. [Note: c_A is already defined for each constant-symbol $c \in \Omega$.]

Example. $M(e, i(e))_A = M_A(l_A, i_A(e_A))$.

The interpretation of an atomic sentence:

(i) $\perp_A = 0$;

(ii)
$$(s = t)_A = \begin{cases} 1 & \text{if } s_A = t_A \\ 0 & \text{if not} \end{cases}$$
for any closed terms s, t;

(iii)
$$\phi(t_1, \ldots, t_n)_A = \begin{cases} 1 & \text{if } ((t_1)_A, \ldots, (t_n)_A) \in \phi_A \\ 0 & \text{if not} \end{cases}$$
for any $\phi \in \Pi$, $\alpha(\phi) = n$, closed terms $t_1, \ldots, t_n$.

Interpretation of sentences - inductively defined by:

(i)
$$(p \Rightarrow q)_A = \begin{cases} 0 & \text{if } p_A = 1, q_A = 0 \\ 1 & \text{otherwise} \end{cases}$$

(ii)
$$((\forall x)p)_A = \begin{cases} 1 & \text{if } p[\bar{a}/x]_A = 1 \text{ for all } a \in A \\ 0 & \text{otherwise} \end{cases}$$
where we add a constant symbol $\bar{a}$ to L (for a fixed $a \in A$), to form a language L', and make A into an L'-structure by setting $\bar{a}_A = a$.

Remark. For a formula p with free variables, can define $p_A \subseteq A^{\#\text{free variables}}$. E.g if p is $m(x,x) = e$ then $p_A = \{a \in A : m_A(a,a) = e_A\} \subseteq A^1$.

If $p_A = 1$, say p *holds* in A, or p is *true* in A, or A is a *model* of p. For a *theory* T (set of sentences), A is a *model* of T if $p_A = 1$ for all $p \in T$.

For a theory T, sentence p, say $T \vDash p$ if every model of T is a model of p. For example, the three group axioms $\vDash M(e,e) = e$.

Examples.

1. Groups: let L be the language of groups, $T = \{(\forall x)(\forall y)(\forall z)(M(x, M(y, z)) = M(M(x, y), z)), (\forall x)(M(x, e) = x \wedge M(e, x) = x), (\forall x)(M(x, i(x)) = e \wedge M(i(x), x) = e)\}$. Then an L-structure is a model of T if and only if it is a group. Say T *axiomatises* the theory of groups/the class of groups (often, the elements of T are called the *axioms* of T).

2. Posets: let L be the language of posets, T the usual poset axioms. Then T axiomatises the class of posets.

3. Fields: let L be the language of fields: $\Omega = \{0, 1, +, \times, -\}$ arities $0, 0, 2, 2, 1$ respectively; $T =$ usual field axioms - including $(\forall x)(\neg(x = 0) \Rightarrow (\exists y)(xy = 1))$. Then T axiomatises the class of fields. For example, $T \vDash$ "inverses are unique"$= (\forall x)(\neg(x = 0) \Rightarrow (\forall x)(\forall z)((yx = 1 \wedge zx = 1) \Rightarrow y = z))$.

4. Graphs: let L with $\Omega = \emptyset$, $\Pi = \{a\}$, $\alpha(a) = 2$ (a is "adjacency"). For T take $T = \{(\forall x)(\neg a(x, x)), (\forall x)(\forall y)(a(x, y) \Rightarrow a(yx))\}$. Then T axiomatises the class of graphs.

Syntactic Entailment

We'll need (logical) axioms and deduction rules. Have 7 axioms (3 usual ones, 2 for '$=$', 2 for '$\forall$'):

1. $p \Rightarrow (q \Rightarrow p)$ for all p, q formulae;

2. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ for all p, q, r formulae;

3. $(\neg\neg p) \Rightarrow p$ for each formula p:

4. $(\forall x)(x = x)$ for any variable x;

5. $(\forall x)(\forall y)(y = x \Rightarrow (p \Rightarrow p[y/x]))$ for any variables x, y, formula p with y not occurring bound;

6. $[(\forall x)p] \Rightarrow p[t/x]$ for any variable x, formula p and term t with no free variable of t occuring bound in p;

7. $(\forall x)(p \Rightarrow q) \Rightarrow (p \Rightarrow (\forall x)q)$ for any variable x and formulae p, q with x not occurring free in p.

Note. Each of these is a *tautology* - i.e is true in every structure.

We have 2 deduction rules:

1. Modus ponens: from p, $p \Rightarrow q$ can deduce q;

2. Generalisation: from p can deduce $(\forall x)p$, provided x does not occur free in any premise used so to prove p.

For $S \subseteq L$, $t \in L$, say S *proves* p, written $S \vdash p$ if there exists a proof of p from S, meaning a finite sequence of formulae, ending with p such that each formula is either a logical axiom or a member of S or obtained from earlier lines by a deduction rule.

Note. Suppose we allowed the empty structure A (for a language with no constants). Then $\perp$ is false in A, and $(\forall x)\perp$ is true in A. So $((\forall x)\perp) \Rightarrow \perp$ is false in A. But this has to be true by axiom 6.

Example. $\{x = y, x = z\} \vdash y = z$. Idea: go for axiom 5 to get $y = z$ from $x = z$.

1. $(\forall x)(\forall y)(x = y \Rightarrow (x = z \Rightarrow y = z))$ (axiom 5);
2. $[(\forall x)(\forall y)(x = y \Rightarrow (x - z \Rightarrow y = z))] \Rightarrow [(\forall y)(x = y \Rightarrow (x = z \Rightarrow y = z))]$ (axiom 6);
3. $(\forall y)(x = y \Rightarrow (x = z \Rightarrow y = z))$ (modus ponens);
4. $[(\forall y)(x = y \Rightarrow (x = z \Rightarrow y = z))] \Rightarrow (x = y \Rightarrow (x = z \Rightarrow y = z))$ (axiom 6);
5. $x = y \Rightarrow (x = z \Rightarrow y = z)$ (modus ponens);
6. $x = y$ (hypothesis);
7. $x = z \Rightarrow y = z$ (modus ponens);
8. $x = z$ (hypothesis);
9. $y = z$ (modus ponens).

Proposition 4.1 (Deduction Theorem)**.** *Let $S \subseteq L$ and $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ if and only if $S \cup \{p\} \vdash q$.*

Proof. As before $(\Rightarrow)$ is trivial. Indeed, given a proof of $p \Rightarrow q$ from S, just write down p by hypothesis and apply modus ponens to get a proof of $p \Rightarrow q$ from $S \cup \{p\}$.

So we show $(\Leftarrow)$: we proceed as we did with propositional logic. The only new case is deduction by'generalisation'. So in the proof of q from $S \cup \{p\}$ suppose we have

$$(\forall x)r \qquad \text{(generalisation)}$$

and have proof of $p \Rightarrow r$ from S (induction). Now, in proof of r from $S \cup \{p\}$, no hypothesis had x free, so same is true in our proof of $p \Rightarrow r$ from S. Thus $S \vdash (\forall x)(p \Rightarrow r)$ by generalisation.

If x is not free in p, get $S \vdash p \Rightarrow (\forall x)r$ by axiom 7 (+modus ponens). If x occurs free in p, proof of r from $S \cup \{p\}$ cannot have used hypothesis p, so in fact $S \vdash r$ so $S \vdash (\forall x)r$ (generalisation). Thus $S \vdash p \Rightarrow (\forall x)r$ by axiom 1 (+modus ponens). $\square$

Aim: $S \vdash p$ if and only if $S \vdash p$.

For example, if p is true in all groups, then p must have a proof from the group axioms.
***Start of non-examinable section*.**

Proposition 4.2 (Soundness)**.** *Let S be a set of sentences, and p a sentence in a language L. Then $S \vdash p$ implies that $S \vDash p$.*

Proof. Have a proof $t_1, t_2, \ldots, t_n$ of p from S, and want to know that if A is a model of S then A is a model of t_i for every i. This is easy by induction. $\square$

For adequacy, we want to show that if $S \vDash p$ then $S \vdash p$. i.e $S \cup (\neg p) \vDash \perp$ implies $S \cup (\neg p) \vdash \perp$, i.e if $S \cup (\neg p)$ is consistent, then $S \cup (\neg p)$ has a model.

Theorem 4.3 (Model existence lemma)**.** *Let S be a set of sentences in a language L. Then if S is consistent, it has a model.*

Ideas:

1. Build our structure out of the language itself - use the closed terms of L. For example, if L is the language of fields and S is the usual field axioms, take closed terms with $+, \times$ in the obvious way: e.g '$(1+1)$'+'$(1+1)$' = '$(1+1)+(1+1)$'.

2. But the closed terms $1+0$ and 1 are distinct, yet $S \vdash 1+0 = 1$ in a field. So we quotient out by the equivalence relation on closed terms given by $s \sim t$ iff $S \vdash (s = t)$. If this set is A, we define $[s] +_A [t] = [s+t]$ (can check this is well-defined).

3. Suppose S is the field axioms for fields of characteristic 2 or 3, i.e field axioms with $1+1 = 0 \vee 1+1+1 = 0$. Does $S \perp 1+1 = 0$? No. Does $S \perp 1+1+1 = 0$? Again no. Thus $[1+1] \neq [0]$ and $[1+1+1] \neq [0]$ - so A does not have satisfy char$(A) = 2$ or 3. Solution: extend S to a maximal consistent set first.

4. Suppose S is now the field axioms for fields with a $\sqrt{2}$, i.e the field axioms together with $(\exists x)(xx = 1+1)$. But no closed term t has $[tt] = [1+1]$. S 'lacks witnesses'. Solution: for each '$(\exists x)p$'$\in S$, add a new constant c to the language, and add to S the sentence $p[c/x]$ (easy to check this is still consistent).

5. But now our new S is not neccessarily maximal consistent (as we have extended L). So must loop back to step 3 then to step 4, etc. Problem: this may not terminate.

Proof of model existence lemma. Have a consistent S in language $L(\Omega, \Pi)$. Extend S to a maximal consistent S_1 in L (Zorn). So for each sentence $p \in L$ have $p \in S_1$ or $(\neg p) \in S_1$. Now add witnesses for S_1: for rach '$(\exists x)p$'$\in S$ add a new constant c to the language, and add sentence $p[c/x]$. We obtain a theory T_1 in language $L_1 = L(\Omega \cup C_1, \Pi)$ that has witnesses for S_1 (for each '$(\exists x)p$'$\in S_1$ have $p[t/x] \in T_1$ for some closed term t). It is easy to check that T_1 is consistent.

Now extend T_1 to maximal consistent S_2 in L_1, then add witnesses to form T_2 in language $L_2 = L(\Omega \cup C_1 \cup C_2, \Pi)$. Continue inductively. Let $\overline{S} = S_1 \cup S_2 \cup \dots$ in language $\overline{L} = L(\Omega \cup C_1 \cup C_2 \cup \dots, \Pi)$.

Claim: $\overline{S}$ is consistent, complete and has witnesses (for itself).

Proof of claim: if $\overline{S} \vdash \perp$ then some $S_n \vdash \perp$ (as proofs are finite), a contradiction - so $\overline{S}$ is consistent. Now show completeness: for a sentence $p \in \overline{L}$, have $p \in L_n$ for some n (as p is a finite string). So $S_{n+1} \vdash p$ or $S_{n+1} \vdash (\neg p)$, so $\overline{S} \vdash p$ or $\overline{S} \vdash (\neg p)$. Finally show it has witnesses: if '$(\exists x)p$'$\in \overline{S}$, then it is in S_n for some n. Then $p[t/x] \in T_n$, for some closed term t so $p[t/x] \in \overline{S}$.

On the closed terms of $\overline{L}$, define $s \sim t$ if $\overline{S} \vdash (s = t)$. Easy to check that this is an equivalence relation. Let A be the set of equivalence classes, made into an $\overline{L}$-structure by:

$$f_A([t_1], \dots, [t_n]) = [f(t_1, \dots, t_n)] \text{ for } f \in \Omega \cup C_1 \cup C_2 \cup \dots, \alpha(f) = n, t_1, \dots, t_n \text{ closed terms}$$
$$\phi_A = \{([t_1], \dots, [t_n]) \in A^n : \overline{S} \vdash \phi(t_1, \dots, t_n)\} \text{ for } \phi \in \Pi, \alpha(\phi) = n, t_1, \dots, t_n \text{ closed terms.}$$

Claim: for a sentence $p \in \overline{L}$, have $p_A = 1$ if and only if $\overline{S} \vdash p$ [then done as certainly $p_A = 1$ for every $p \in S$, i.e A is a model of S].

Proof of claim: easy induction. Atomic sentences:

- $\perp_A = 0$ and $\overline{S} \nvdash \perp$;
- For closed terms s, t: $\overline{S} \vdash (s = t) \iff [s] = [t] \iff s_A = t_A \iff$ '$s = t$' holds in A;
- $\phi(t_1, \dots, t_n)$: same.

Induction step:

- $\overline{S} \vdash (p \Rightarrow q) \iff \overline{S} \vdash \neg p$ or $\overline{S} \vdash q$ (($\Rightarrow$): if $\overline{S} \nvdash (\neg p)$ and $\overline{S} \vdash q$ then $\overline{S} \vdash p$, $\overline{S} \vdash (\neg q)$ as $\overline{S}$ is complete, contradicting $\overline{S}$ consistent). Then this happens $\iff p_A = 0$ or $q_A = 1$ (induction hypothesis) $\iff (p \Rightarrow q)$ is true in A;
- $\overline{S} \vdash (\exists x)p \iff \overline{S} \vdash p[t/x]$ for some closed term t (($\Rightarrow$): $\overline{S}$ has witnesses). This happens $\iff p[t/x]_A = 1$ for some closed term $t \iff (\exists x)p$ holds in A (($\Leftarrow$): A is the set of (equivalence classes of) closed terms).

□

Hence we have

Corollary 4.4 (Adequacy)**.** *For S a theory, p a sentence in a language L, we have $S \vDash p \Rightarrow S \vdash p$.*

***End of non-examinable section*.**

Theorem 4.5 (Completeness theorem/Godel's completeness theorem for first-order logic)**.** *For S a theory, p a sentence in a language L, we have $S \vdash p \iff S \vDash p$.*

Proof. ($\Rightarrow$): soundness.

($\Leftarrow$): adequacy. □

Remarks.

1. If L is countable (Ω, Π are countable) then Zorn's Lemma is not needed;
2. 'First-order' means our variables range over elements (not subsets).

Theorem 4.6 (Compactness theorem)**.** *Let S be a theory in a language L. Then if every finite subset of S has a model, then S itself has a model.*

Proof. Trivial if we replace 'has a model' with 'is consistent' - as proofs are finite. □

Note. There is no decidability theorem for first-order logic - how do we check if $S \vDash p$?

Can we axiomatise the theory of finite groups? (i.e a theory S such that a group is finite if and only if each $p \in S$ holds in the group.)

Corollary 4.7. *The class of finite groups is not axiomatisable (in the language of groups).*

Note. It is remarkable that we can prove this, as opposed to merely guessing it is true.

Proof. Suppose S axiomatises the theory of finite groups. Consider S together with: $(\exists x_1)(\exists x_2)(x_1 \neq x_2)$ (i.e '$|G| \geq 2$'), $(\exists x_1)(\exists x_2)(\exists x_3)(x_1, x_2, x_3$ distinct$)$ (i.e '$|G| \geq 3$') and so on.

Then any finite subset of our new S' has a model (e.g $\mathbb{Z}_n$ some n large enough). So S' has a model - a finite group which, for each n has $\geq n$ elements. □

Similarly

Corollary. *Let S be a theory with arbitrarily large finite models. Then S has an infinite model.*

Proof. Add sentences as above, and apply compactness as above. $\square$

"Finiteness is not a first-order property."

Theorem 4.8 (Upward Löwenheim-Sholem theorem)**.** *Let S be a theory with an infinite model. Then S has an uncountable model.*

Proof. Add constants $\{c_i\}_{i\in I}$ to the language, where I is an uncountable set, and form theory S' by adding to S the sentences '$c_i \neq c_j$' for each $i, j \in I$ with $i \neq j$. Then any finite subset of S' has a model (our infinite model of S will do), so S' has a model. $\square$

Remark. Similarly, can get a model of S that does not inject into X, for any fixed set X. Just choose $\gamma(X)$ constants, or $\mathcal{P}(X)$ constants.

Example. There exists an infinite field (e.g $\mathbb{Q}$, so there exists an uncountable field (e.g $\mathbb{R}$), and also, a field that does not inject into $\mathcal{P}(\mathcal{P}(\mathbb{R}))$).

Theorem 4.9 (Downward Löwenheim-Sholem theorem)**.** *Let S be a theory in a countable language. Then if S has a model, it also has a countable model.*

Proof. Have S consistent, and then the model constructed in the proof of theorem 4.3 is countable. $\square$

Remark. This proof is not non-examinable, even though it relies on the non-examinable theorem 4.3.

Peano Arithmetic

We try to make the usual axioms of $\mathbb{N}$ into a first-order theory.

Language L: $\Omega = \{0, S, +, \cdot\}$ (arities $0, 1, 2, 2$), $\Pi = \emptyset$.

Axioms:

1. $(\forall x)(s(x) \neq 0)$;
2. $(\forall x)(\forall y)([s(x) = s(y)] \Rightarrow [x = y])$;
3. $(\forall y_1)\ldots(\forall y_n)[(p[0/x] \wedge (\forall x)(p \Rightarrow p[s/x])] \Rightarrow (\forall x)p]$, each formula p with free variables $y_1, \ldots, y_n, x$ (the $y_1, \ldots, y_n$ are called *parameters*);
4. $(\forall x)(x + 0 = x)$;
5. $(\forall x)(\forall y)(x + s(y) = s(x + y))$;

6. $(\forall x)(x \cdot 0 = 0)$;
7. $(\forall x)(\forall y)(x \cdot s(y) = (x \cdot y) + x)$.

These axioms are called *Peano arithmetic* or *PA* or *formal number theory.*

Note. For axiom 3, first guess would be be the same, without the parameters. But then we'd be missing sets such as $\{x : x \geq y\}$, where y is a variable.

Now, PA has an infinite model (e.g $\mathbb{N}$), so by upper Löwenheim-Sholem (ULS) it has an uncountable model - which in particular is not isomorphic to $\mathbb{N}$. Doesn't this contradict the fact that the usual axioms for $\mathbb{N}$ characterise $\mathbb{N}$ uniquely (up to isomorphism)?

Answer: 3 is not 'true' induction (over *all* subsets) - even in $\mathbb{N}$ itself, 3 applies to only countably many subsets.

Definition. We say $S \subseteq \mathbb{N}$ is *definable* or *definable in the language of PA* if there exists a formula p and free variable x such that for every $m \in \mathbb{N}$: $m \in S \iff p[m/x]$ holds in $\mathbb{N}$ (officially by m we mean $s(s(s(\dots s(0)))))$.

So only countably many sets are definable.

Examples.

- Set of squares: p is $(\exists y)(y \cdot y = x)$;
- Set of primes: p is $(x \neq 0 \wedge x \neq 1) \wedge [(\forall y)(y \mid x \Rightarrow y = 1 \vee y = x)]$;
- Set of powers of 2: p is $(\forall y)(y \text{ is prime } \wedge y \mid x \Rightarrow y = 2)$;
- Exercise: powers of 4;
- Challenge: powers of 6.

Is PA complete (i.e PA$\vdash p$ or PA$\vdash (\neg p)$ for all p)?

Theorem 4.10 (Gödel's incompleteness theorem)**.** *PA is not complete.*

So we have a sentence p such that PA$\nvdash p$ and PA$\nvdash (\neg p)$. But one of $p, \neg p$ holds in $\mathbb{N}$. So we conclude that there must be a sentence p which is true in the naturals, but PA doesn't prove it. This doesn't contradict the Completeness Theorem, which would tell us that if p is true in *every* model of PA then PA$\vdash p$.

5 Set Theory

Goal: "what does the Universe of sets look like?"

Liberating viewpoint: view set theory as 'just another first-order theory'.

Zermelo-Fraenkel Set Theory

Language of ZF: $\Omega = \emptyset$, $\Pi = \{\in\}$ (arity 2) and a 'universe of sets' is a model $(V, \in_V)$ of the 'ZF axioms'.

There are 9 axioms (2 to get started, 4 to build things, and 3 you might not think of at first).

Could view this chapter as a worked example from the previous chapter. But it is much scarier, since (hopefully) every model of ZF will contain 'all of mathematics', and so will be very complicated.

Axioms of ZF

1. Axiom of extension: "sets with the same members are equal"

$$(\forall x)(\forall y)[(\forall z)(z \in x \iff z \in y) \Rightarrow x = y].$$

   Note: converse is an instance of a logical axiom.

2. Axiom of separation (also 'comprehension' or 'subset selection'): "can form subsets of a set", or more precisely. "for a set x and property y, can form $\{z \in x : p(z)\}$"

$$(\forall t_1) \dots (\forall t_n)(\forall x)(\exists y)(\forall z)(z \in y \iff z \in x \wedge p)$$

   for each formula p with free variables $t_1, \dots, t_n, z$. Note: we do need parameters, as e.g might want to form $\{z \in x : z \in t\}$, for some variable t.

3. Empty set axiom: "there is an empty set"

$$(\exists x)(\forall y)[\neg y \in x].$$

   We write $\emptyset$ for the (unique by extension) set guaranteed by this axiom. This is (as usual) an abbreviation: so $p(\emptyset)$ means $(\exists x)(x$ has no members $\wedge$ $p(x))$. Similarly, write $\{z \in x : p(z)\}$ for the set guaranteed by the axiom of separation.

4. Pair-set axiom: "can form $\{x, y\}$"

$$(\forall x)(\forall y)(\exists z)(\forall t)(t \in z \iff t = x \vee t = y).$$

   We write $\{x, y\}$ for this z. Write $\{x\}$ for $\{x, x\}$.

Definition. We can now define the *ordered pair* $(x, y) = \{\{x\}, \{x, y\}\}$. Clearly have $(x, y) = (z, t)$ if and only if $x = z$ and $y = t$.

Definition. Say x is an *ordered pair* if $(\exists y)(\exists z)(x = (y, z))$ and say f is a *function* if

$$(\forall x)(x \in f \Rightarrow x \text{ is an ordered pair}) \wedge (\forall x)(\forall y)(\forall z)[((x, y) \in f \wedge (x, z) \in f) \Rightarrow y = z].$$

Call x the *domain* of f, written $x = \mathrm{dom}(f)$ if $(f$ if a function$) \wedge (\forall y)(y \in x \iff (\exists z)((y, z) \in f))$. Then $f : x \to y$ means

$$(f \text{ is a function }) \wedge (x = \mathrm{dom}(f)) \wedge (\forall z)(\forall t)((z, t) \in f \Rightarrow t \in y)$$

Back to the axioms:

5. Union axiom: "can form unions"

$$(\forall x)(\exists y)(\forall z)(z \in y \iff (\exists t)(z \in t \wedge t \in x)).$$

   So we think of $A \cup B \cup C$ really as $\bigcup\{A, B, C\}$.

6. Power-set axiom: "can form power-sets"

$$(\forall x)(\exists y)(\forall z)(z \in y \iff z \subseteq x)$$

where $z \subseteq x$ means $(\forall t)(t \in z \Rightarrow t \in x)$.

Notes.

1. Write $\bigcup x$ and $\mathcal{P}(x)$ for the sets guaranteed by these axioms. Can write $x \cup y$ for $\bigcup\{x, y\}$ etc.

2. No new axiom needed for $\bigcap$: can form $\bigcap x$ (for x any set, $x \neq \emptyset$) as a subset of y, any $y \in x$ - so done by separation.

3. Can form $x \times y$, as a subset of $\mathcal{P}(\mathcal{P}(x \cup y))$ - because if $t \in x, z \in y$ then $(t, z) \in \mathcal{P}(\mathcal{P}(x \cup y))$.

4. Can form the set of all functions from $x \to y$ as $\mathcal{P}(x \times y)$.

Back to the axioms:

7. Axiom of infinity: so far, any model V must be infinite. For example, writing x^+ for $x \cup \{x\}$, the successor of x, have $\emptyset, \emptyset^+, \emptyset^{++}, \dots$ distinct:

$$\emptyset^+ = \{\emptyset\},\ \emptyset^{++} = \{\emptyset, \{\emptyset\}\},\ \emptyset^{+++} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots.$$

We often write 0 for $\emptyset$, 1 for $\emptyset^+$, 2 for $\emptyset^{++}$, etc. For example, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$ etc.

Does V have an infinite set? In the 'world of maths': V is infinite. But no $x \in V$ has all $y \in V$ as members: $(\forall x)\neg(\forall y)(y \in x)$ by Russell's paradox.

So we say x is a successor set if $(\emptyset \in x) \wedge (\forall y)(y \in x \Rightarrow y^+ \in x)$.

So the axiom of infinity says: "there exists an infinite set/a successor set":

$$(\exists x)(x \text{ is a successor set}).$$

Note that any intersection of successor sets is a successor set - so there exists a least successor set, namely the intersection of all successor sets. Call this ω (this will be our copy in V, of $\mathbb{N}$). Thus

$$(\forall x)[x \in \omega \iff (\forall y)(y \text{ a sucessor set } \Rightarrow x \in y)].$$

E.g $3 = \emptyset^{+++} \in \omega$.

In particular, if $x \subseteq \omega$ is a successor set then $x = \omega$ (by definition of ω):

$$(\forall x)((x \subseteq \omega \wedge \emptyset \in x \wedge (\forall y)(y \in x \Rightarrow y^+ \in x)) \Rightarrow x = \omega)$$

(this is full induction (in V), over <u>all</u> subsets of ω - as opposed to e.g in PA from Chapter 4). It is easy to check $(\forall x)(x \in \omega \Rightarrow x^+ \neq \emptyset)$ and $(\forall x)(\forall y)((x \in \omega \wedge y \in \omega \wedge x^+ = y^+) \Rightarrow x = y)$, so ω satisfies (in V) the usual axioms for $\mathbb{N}$.

Can now define 'x is finite' for $(\exists y)(y \in \omega \wedge x$ bijects with $y)$ and 'x is countable' for (x is finite) $\vee$ (x bijects with ω).

8. Axiom of foundation: "sets are built out of simpler sets". We want to disallow $x \in x$. Similarly, want to disallow $x \in y \wedge y \in x$, and also sets $x_0, x_1, \ldots$ with $x_1 \in x_0$, $x_2 \in x_1$, $x_3 \in x_2, \ldots$. The axiom of foundation says: "every (non-empty) set has an $\in$-minimal element":

$$(\forall x)(x \neq \emptyset \Rightarrow (\exists y)(y \in x \wedge (\forall z)(z \in x \Rightarrow z \notin y))).$$

9. Axiom of replacement: often we say "have a set A_i for each $i \in I$; take $\{A_i : i \in I\}$". But why should that be a set? Why should $i \mapsto A_i$ be a function? i.e why should there be a set $\{(i, A_i) : i \in I\}$? We'd want "the image of a set, under something that looks like a function is a set".

Digression on classes

Let $(V, \in)$ be an L-structure. A *class* is a collection C of elements of V such that for some formula p, free variables x (and maybe more), we have that x belongs to C if and only if $p(x)$ holds in V.

For example, V is a class: take p to be '$x = x$'. All infinite $x \in V$ is a class: take p to be 'x is not finite'. Or the collection of all x such that $t \in x$ for some fixed t (the 'maybe more' variables refers to parameters like t here).

Note that every set $y \in V$ is a class: take p to be $x \in y$. Say C is a *proper class* if it is not a set in V, i.e

$$\neg(\exists y)(\forall x)(x \in y \iff p(x)).$$

Similarly, a *function-class* F is a collection of ordered pairs from V such that for some formula p, free variables x, y (and maybe more), we have that (x, y) belongs to F if and only if $p(x, y)$, and if $(x, y), (x, z)$ belong to F then $y = z$.

For example, the mapping $x \mapsto \{x\}$ is a function class: take $p(x, y)$ to be '$y = \{x\}$'. Note this is not a function - e.g every f has a domain (obtained as a suitable subset of $\bigcup\bigcup f$), and this f would have domain V which is not a set.

Back to the axioms:

9. Axiom of replacement: "the image of a set under a function-class is a set":

$$\underbrace{(\forall t_1)\ldots(\forall t_n)}_{\text{parameters}}[\underbrace{(\forall x)(\forall y)(\forall z)(p \wedge p[z/y] \Rightarrow y = z)}_{p \text{ is a function class}}$$
$$\implies (\forall x)\underbrace{(\exists y)(\forall z)(z \in y \iff (\exists t)(t \in x \wedge p[t/x.z/y]))}_{y \text{ is image of } x})]$$

E.g for any set x, can form $\{\{t\} : t \in x\}$ - the function class is $t \mapsto \{t\}$. This is a bad example however, as we could have formed this directly via power-sets and separation. See later for a good example.

The above are the axioms of ZF: write ZFC for ZF with AC, or Axiom of Choice: every family of non-empty sets has a choice function:

$(\forall f)((f \text{ is a function} \wedge (\forall x)(x \in \text{dom}(f) \Rightarrow x \neq \emptyset))$
$\implies (\exists g)((g \text{ is a function}) \wedge (\text{dom}(g) = \text{dom}(f)) \wedge (\forall x)(x \in \text{dom}(f) \Rightarrow g(x) \in f(x))))$

Definition. Say x is *transitive* if each member of a member of x is again a member of x:

$$(\forall y)[(\exists z)(y \in z \wedge z \in x) \Rightarrow y \in x]$$

i.e $\bigcup x \subseteq x$.

Examples. $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$ - and in general each $x \in \omega$ is transitive. Formally, this is because $\emptyset$ is transitive, and if y is transitive, so is $y^+ = y \cup \{y\}$, so done by ω-induction.

Lemma 5.1. *Every set x is contained in a transitive set.*

Remarks.

1. Officially, this says: "let $(V, \in)$ be a model of ZF. Then [statement]". Equivalently: ZF $\vdash$ [statement] (by the Completeness theorem);

2. Once we know lemma 1, we'll know that any x is contained in a least transitive set, the *transitive closure* of x, written TC(x) - because any intersection of transitive sets is transitive.

Proof. We want to form $x \cup (\bigcup x) \cup (\bigcup \bigcup x) \cup \ldots$, which will be a set by the union axiom applied to $\{x, \bigcup x, \bigcup \bigcup x, \ldots\}$, which will itself be a set by replacement as the image of ω under the function-class $0 \mapsto x$, $1 \mapsto \bigcup x$, $2 \mapsto \bigcup \bigcup x, \ldots$. This is a good use of replacement - intuitively we are "going out into V, far from x".

But why is that a function-class? [Want $p(z, w)$ to be: $(z = 0 \wedge w = x) \vee ((\exists t)(\exists u)(z = t + 1 \wedge w = \bigcup u \wedge p(t, u)))$ - but this is nonsense as not a formula (it is self-referential).] Define 'f is an attempt' (this is the clever idea) to mean

$$(f \text{ is a function}) \wedge (\text{dom}(f) \in \omega) \wedge (\text{dom}(f) \neq \emptyset) \wedge (f(0) = x)$$
$$\wedge (\forall n \in \omega)(n \in \text{dom}(f) \wedge n \neq 0 \Rightarrow f(x) = \bigcup f(n-1)).$$

Then $(\forall n \in \omega)(\exists f)(f \text{ an attempt} \wedge n \in \text{dom}(f))$ by ω-induction, and

$$(\forall n \in \omega)(\forall f)(\forall g)(f \text{ an attempt} \wedge g \text{ an attempt}$$
$$\wedge n \in \text{dom}(f) \cap \text{dom}(g) \Rightarrow f(n) = g(n))$$

also by ω-induction. So our function-class $p = p(z, w)$ is

$$(\exists f)(f \text{ an attempt} \wedge z \in \text{dom}(f) \wedge f(z) = w).$$

$\square$

We want foundation to be capturing the idea of 'sets are built out of simpler sets'. So we'd want: if $p(y)$ for all $y \in x$ implies $p(x)$, then $p(x)$ for all x.

Theorem 5.2 (Principle of $\in$-induction)**.** *For each formula p with free variables $t_1, \ldots, t_n$,x:*

$$(\forall t_1) \ldots (\forall t_n)[(\forall x)((\forall y)(y \in x \Rightarrow p(y)) \Rightarrow p(x)) \Rightarrow (\forall x)(p(x))].$$

Proof. Given $t_1, \ldots, t_n$: given that $(\forall x)((\forall y)(y \in x \Rightarrow p(y)) \Rightarrow p(x))$, want $(\forall x)(p(x))$. Suppose some x has $\neg p(x)$. [Want to look at $\{t : \neg p(t)\}$ and take an $\in$-minimal element. But $\{t : \neg p(t)\}$ may not always be a set - e.g if $p(x)$ is $\neg x$.]

Let $u = \{t \in \mathrm{TC}(\{x\}) : \neg p(t)\}$: $u \neq \emptyset$ as $x \in u$. Let t be a minimal element of u. Then $\neg p(t)$ (as $t \in u$), but $p(z) \forall z \in t$ (by minimality of t - noting that each $z \in t$ does belong to $\mathrm{TC}(\{x\})$). This is a contradiction. $\square$

In fact, $\in$-induction is equivalent to foundation (in presence of all other ZF axioms). To deduce foundation: say 'x is regular' if $(\forall y)(x \in y \Rightarrow y$ has a least element). So foundation says: 'evey set is regular'. Proof by $\in$-induction: given $(\forall y \in x)(y$ regular) want to show x regular. For a set z with $x \in z$: if x minimal in z we're done; if x not minimal in z then there exists $y \in x$ such that $y \in z$. So x has a minimal element (as y regular).

How about $\in$-recursion - want to define $f(x)$ in terms of the $f(y)$, $y \in x$.

Theorem 5.3 ($\in$-recursion theorem)**.** *Let G be a function-class (recall this means: $(x, y) \in G \iff p(x, y)$ for some formula p), everywhere defined. Then there is a function-class F ($(x, y) \in F \iff q(x, y)$ for some formula q), everywhere defined, such that $(\forall x)(F(x) = G(F|_x))$. Moreover, F is unique.*

Remark. $F|_x = \{(y, F(y)) : y \in x\}$ is a set, by replacement.

Proof. Existence: say 'f is an attempt' if

$$(f \text{ is a function}) \wedge (\text{dom}(f) \text{ is transitive}) \wedge (\forall x)(x \in \text{dom}(f) \Rightarrow f(x) = G(f|_x)).$$

(Note that $f|x$ makes sense as $\text{dom}(f)$ is transitive.) Then

$$(\forall x)(\forall f)(\forall f')(f, f' \text{ attempts} \wedge x \in \text{dom} f \cap \text{dom} f' \Rightarrow f(x) = f'(x))$$

by $\in$-induction (as if $f(y) = f'(y)$ for all $y \in x$ then $f(x) = f'(x)$). Also $(\forall x)(\exists f)(f \text{ an attempt} \wedge x \in \text{dom}(f))$, also by $\in$-induction. Indeed, if for each $y \in x$ there exists an attempt defined at y, then for each $y \in x$ there is a unique attempt defined on its transitive closure $\text{TC}(\{y\})$, f_y say. Let $f = \bigcup\{f_y : y \in x\}$ - an attempt with domain $\text{TC}(x)$. Now set $f' = f \cup \{(x, G(f|_x))\}$ - an attempt defined at x. So take $q(x, y)$ to be '$(\exists f)(f \text{ an attempt} \wedge x \in \text{dom}(f) \wedge f(x) = y)$'.

Uniqueness: if F, F' are suitable then $(\forall x)(F(x) = F'(x))$ by $\in$-induction. $\square$

Note. Proofs of $\in$-induction and $\in$-recursion are very similar to what we did in Chapter 2.

What properties of the 'relation' (really a relation class) $p(x, y) =$'$x = y$' have we used in the above two proofs?

1. p is *well-founded*: every non-empty set has a p-minimal element.
2. p is *local*: for each y, $\{x : p(x, y)\}$ forms a set (used to build p-transitive closure).

So actually we have p-induction and p-recursion for any p that is well-founded and local.

Special case: if r is a relation on a set a, then trivially r is local - so we just need r to be well-founded. Thus our theorems from Chapter 2 were special cases of this.

"Can we model a relation by ε?". For example on $\{a, b, c\}$ let r be the relation: arb, brc. Put $a' = \emptyset$, $b' = \{\emptyset\}$, $c' = \{\{\emptyset\}\}$. Then the map $f : \{a, b, c\} \to \{a', b', c'\}$ $x \mapsto x'$ is a bijection with a transitive set such that $xry \iff f(x) \in f(y)$.

Definition. Say a relation r on a set a is *extensional* if $(\forall x \in a)(\forall y \in a)[(\forall z \in a)(zrx \iff zry) \Rightarrow x = y]$. For example the relation above.

The analogue of 'subset collapse' from Chapter 2 is

Theorem 5.4 (Mostowski's collapsing theorem)**.** *Let r be a relation on a set that is well-founded and extensional. Then there exists a transitive set b and a bijection $f : a \to b$ such that $(\forall x \in a)(\forall y \in a)(xry \iff f(x) \in f(y))$. Moreover, b and f are unique.*

Note. 'Well-founded' and 'extensional' are trivially necessary.

Proof. Define function f by r-recursion as follows. $f(x) = \{f(y) : yrx\}$, for each $x \in a$ (this is really the only possible choice). f is a function, not just a function-class by replacement - it is an image of a. Let b be the set $\{f(x) : x \in a\}$ - a set by replacement. Then f is surjective (definition of b), and b is transitive (definition of f). Need to check f is injective (then have $yrx \iff f(y) \in f(x)$ by definition of f). We'll show that $(\forall x \in a)(\forall x' \in a)(f(x') = f(x) \Rightarrow x' = x)$ by r-induction on x. So we are given $(\forall yrx)(\forall z \in a)(f(y) = f(z) \Rightarrow y = z)$ and we are given $f(x) = f(x')$ and want $x = x'$. Have $\{f(y) : yrx\} = \{f(z) : zrx'\}$ (as $f(x) = f(x')$) so $\{y : yrx\} = \{z : zrx'\}$ so $x = x'$ by the fact r is extensional.

Uniqueness: f is unique by r-induction (as must have $f(x) = \{f(y) : yrx\}$ for all $x \in a$). $\square$

In particular: every well-ordered set is order-isomorphic to a unique transitive set well-ordered by $\in$.

So say an *ordinal* is a transitive set well-ordered (or could say 'totally-ordered' thanks to foundation) by $\in$. For example, $\emptyset$, $\{\emptyset\}$, any $n \in \omega$, ω itself. Thus each well-ordering is order-isomorphic to a unique ordinal, called its order -type.

Remark. If x, y are in a well-ordered set a, with $y < x$ then the order-type of I_x (i.e $f(x)$) has an element $f(y)$, i.e order-type of I_y. So for ordinals α, β: $\alpha < \beta \iff \alpha \in \beta$ - so $\alpha = \{\beta : \beta < \alpha\}$. Thus $\alpha^+ = \alpha \cup \{\alpha\}$ and $\sup\{\alpha_i : i \in I\} = \bigcup\{\alpha_i : i \in I\}$.

Picture of the Universe

Hope: start with $\emptyset$, keep taking $\mathcal{P}$.

Define sets V_α, each ordinal α by recursion:

$$\begin{aligned}
&V_0 = \emptyset \\
&V_{\alpha+1} = \mathcal{P}(V_\alpha) \\
&V_\lambda = \bigcup\{V_\alpha : \alpha < \lambda\} \text{ for } \lambda \text{ a non-zero limit.}
\end{aligned}$$

Does this hit all sets?

Lemma 5.5. *Each V_α is transitive.*

Proof. We proceed by induction on α. $V_0 = \emptyset$ which is transitive. Successors: V_α transitive $\Rightarrow V_{\alpha+1}$ transitive, since x transitive $\Rightarrow \mathcal{P}(x)$ transitive (if $z \in y \in \mathcal{P}(x)$ then $z \in x$, so $z \subseteq x$ therefore $z \in \mathcal{P}(x)$). Limits: any union of transitive sets is transitive. $\square$

Lemma 5.6. *If $\alpha \leq \beta$ then $V_\alpha \subseteq V_\beta$.*

Proof. We proceed by induction on β, for α fixed. If $\beta = \alpha$, $V_\alpha \subseteq V_\alpha$. Successors: $V_\alpha \subseteq V_\beta$ and $V_\beta \subseteq \mathcal{P}(V_\beta)$ (V_β transitive), so $V_\alpha \subseteq \mathcal{P}(V_\beta) = V_{\beta+1}$. Limits: clear by definition. $\square$

Theorem 5.7. *Every set x belongs to some V_α.*

Remark. This says "$V = \bigcup_{\alpha \text{ ordinal}} V_\alpha$".

Notes.

1. $x \subseteq V_\alpha \iff x \in V_{\alpha+1}$. So it is enough to show each x is a subset of some V_α;
2. Once we know $x \subseteq V_\alpha$ for some α, then there exists a least such α, called the *rank* of x. For example, $\text{rank}(0) = 0$, $\text{rank}(1) = 1$, $\text{rank}(x) = x$ for all $x \in \omega$, $\text{rank}(\omega) = \omega$ - and in fact $\text{rank}(\alpha) = \alpha$ for any ordinal α (induction).

Proof. By $\in$-induction. Given a set x, may assume that for all $y \in x$ there exists α such that $y \subseteq V_\alpha$, i.e $y \subseteq V_{\text{rank}(y)}$. Thus for all $y \in x$, $y \in V_{\text{rank}(y)+1}$. So let $\alpha = \sup\{\text{rank}(y) + 1 : y \in x\}$. Then for all $y \in x$ we have $y \in V_\alpha$, i.e $x \subseteq V_\alpha$. $\square$

Remarks.

1. The V_α are the *Von Neumann Hierarchy*;

2. The above proof shows that for every x, $\text{rank}(x) = \sup\{\text{rank}(y) + 1 : y \in x\}$. For example $\text{rank}(\{\{2,3\},\{6\}\}) = \sup\{\sup\{2+1, 3+1\}+1, 6+1\} = 7$.

6 Cardinals

Looking at 'sizes' of sets, working in ZFC. Write $x \leftrightarrow y$ if $(\exists f)(f$ a bijection from x to $y)$. Want to define 'card(x)' or '$|x|$' such that $\text{card}(x) = \text{card}(y) \iff x \leftrightarrow y$ [cannot put $\text{card}(x) = \{y : x \leftrightarrow y\}$, as this is not a set]. For any x there exists an ordinal α such that $x \leftrightarrow \alpha$ (well-ordering theorem) so can just define card(x) to be the least α such that $x \leftrightarrow \alpha$ [if in ZF not ZFC: use the 'Scott trick': consider least α such that $\exists y \leftrightarrow x$ with $\text{rank}(y) = \alpha$ ('essential rank of x'), and let $\text{card}(x) = \{y \subseteq V_\alpha : y \leftrightarrow x\}$].

Say m 'is a cardinality' (or just a 'cardinal') if $m = \text{card}(x)$ for some set x. What are the infinite cardinalities?