Introduction

Quadratics (Babylonians):

$$X^2 + bX = c = (X + \frac{1}{2}b)^2 + c - \frac{b^2}{4}$$

$$= (X - x_1)(X - x_2) \implies x_1 x_2 = c, x_1 + x_2 = -b$$

$$x_1 = \frac{1}{2}\left[(x_1 + x_2) + (x_1 - x_2)\right] = \frac{1}{2}\left[-b + \sqrt{b^2 - 4c}\right]$$

Cubics (Italy, 16th Century):

$$X^3 + aX^2 + bX + c = (X - x_1)(X - x_2)(X - x_3)$$

$$\implies x_1 + x_2 + x_3 = -a, x_1x_2 + x_1x_3 + x_2x_3 = b, x_1x_2x_3 = -c$$

WLOG $X \to X - a/3$ and $a = 0$

$$x_1 = \frac{1}{3}\left[(x_1 + x_2 + x_3) + \underbrace{(x_1 + \omega x_2 + \omega^2 x_3)}_{=u} + \underbrace{(x_1 + \omega^2 x_2 + \omega x_3)}_{=v}\right]$$

where $\omega = e^{2\pi i/3}$ so $\omega^2 + \omega + 1 = 0$. Cyclic permutation of x_1, x_2, x_3 gives $u \to \omega u \to \omega^2 u$ and $v \to \omega v \to \omega^2 v$ which implies u^3 and v^3 are invariant under cyclic permutations of the roots.

Also $u \leftrightarrow v$ under $x_2 \leftrightarrow x_3$. So $u^3 + v^3$, u^3v^3 are invariant under permutations of roots.

In fact,

$$u^3 + v^3 = 27x_1x_2x_3 = -27c$$

$$u^3v^3 = -27b^2$$

So u^3, v^3 are roots of $Y^2 + 27cY - 27b^2$. This gives a formula for x_1 (Cardano's formula).

Can follow a similar method for quartics - auxilliary cubic equation. Unfortunately it doesn't work for quintics - the reason being group theory.

1 Polynomials

In this course, all rings are commutative and non-zero. Let R be a ring, then $R[X]$ denotes the ring of polynomials $\sum_{i=0}^{n} a_i X^i$, $a_i \in R$. A polynomial $f \in R[X]$ determines a function $R \to R$, $r \mapsto f(r)$.

The polynomial is not in general determined by this function, e.g let $R = \mathbb{Z}/p\mathbb{Z}$ (p prime). Then for all $a \in R$, $a^p = a$ so the polynomials X^p and X represent the same function.

In the case when $R = K$ (a field), $K[X]$ is a Euclidean domain. The "division algorithm" says that if $f, g \in K[X]$, $g \neq 0$ then there exists unique $q, r \in K[X]$ such that $f = gq + r$ and $\deg r < \deg g$ (define $\deg(0) = -\infty$).

In particular, if $g = X - a$ is linear then $f = (X - a)q + f(a)$ ("remainder theorem"). So $K[X]$ is also a PID and a UFD - every polynomial is a product of irreducible polynomials, and there are GCD's, computable via Euclids algorithm in the usual way.

Proposition 1.1. *If K is a field, $0 \neq f \in K[X]$, then f has at most $\deg f$ roots in K.*

Proof. If f has no roots then we are done. Otherwise, suppose $f(a) = 0$ for $a \in K$. Then

$$f = (X - a)g$$

for some $g \in K[X]$ and $\deg g = \deg f - 1$. If $b \in K$ is a root of f then either $b = a$ or $g(b) = 0$ so the number of roots of f is at most one more than the number of roots of g. Now done by induction. □

2 Symmetric polynomials

Let R be a ring, consider $R[X_1, \ldots, X_n]$ for $n \geq 1$.

Definition. A polynomial $f \in R[X_1, \ldots, X_n]$ is *symmetric* if for every $\sigma \in S_n$, $f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) = f$.

The set of symmetric polynomials is a subring of $R[X_1, \ldots, X_n]$.

Example. $X_1 + \ldots + X_n$, or more generally, $p_k = X_1^k + \ldots + X_n^k = \sum_{i=1}^n X_i^k$.

Alternative definition: if $f \in R[X_1, \ldots, X_n]$, define $f\sigma = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$. This is an action (on the right) of S_n on $R[X_1, \ldots, X_n]$. A polynomial f is symmetric if and only if it is fixed by this action.

Definition. *The elementary symmetric polynomials* are

$$s_r(X_1, \ldots, X_n) = \sum_{1 \leq i_1 < \ldots < i_r \leq n} X_{i_1} X_{i_2} \ldots X_{i_r}$$

Example. When $n = 3$ we have

$$\begin{aligned} s_1 &= X_1 + X_2 + X_3 \\ s_2 &= X_1X_2 + X_1X_3 + X_2X_3 \\ s_3 &= X_1X_2X_3 \end{aligned}$$

Theorem 2.1.

(i) *Every symmetric polynomial over R can be expressed as a polynomial in $\{s_r : 1 \leq r \leq n\}$, with coefficients in R.*

(ii) *There are no non-trivial relations between $s_1, \ldots, s_n$.*

Remark:

(a) Consider the ring homomorphism

$$\theta : R[Y_1, \dots, Y_n] \to R[X_1, \dots, X_n],\ Y_r \mapsto s_r$$

then (i) says the image of θ is the set of symmetric polynomials. (ii) says that θ is injective.

(b) Equivalent definition of the s_r's is

$$\prod_{i=1}^{n}(T + X_i) = T^n + s_1 T^{n-1} + \ldots + s_{n-1}T + s_n$$

If we need to specify the number of variables, write $s_{r,n}$ instead of s_r.

Proof. Terminology:

- A *monomial* is some $X_I = X_1^{i_1} \dots X_n^{i_n}$ for $I \in \mathbb{N}^n = \{0, 1, 2, \dots\}^n$. Its (total) degree is $\sum_\alpha i_\alpha$.
- A *term* is some cX_I, for $0 \neq c \in R$. So a polynomial is uniquely a sum of terms.
- *Total degree* of f is the maximum degree over its terms

Lexicographical ordering on monomials X_I: write $X_I > X_J$ if either $i_1 > j_1$ or, for some $1 \leq r < n$, $i_1 = j_1, \dots, i_r = j_r$ and $i_{r+1} > j_{r+1}$.

This is a total ordering: for each pair $I \neq J$, exactly one of $X_I > X_J$ or $X_J > X_I$ holds.

First we prove (ii):

Let d be the total degree of some symmetric polynomial f, and let X_I be the largest (in lexicographical order) monomial which occurs in f, with coefficient $c \in R$. As f is symmetric, we must have $i_1 \geq i_2 \geq \ldots \geq i_n$ (otherwise we could exchange variables to get a larger monomial).

So

$$X_I = X_1^{i_1 - i_2} (X_1 X_2)^{i_2 - i_3} \dots (X_1, \dots X_n)^{i_n}$$

consider

$$g = s_1^{i_1 - i_2} s_2^{i_2 - i_3} \dots s_{n-1}^{i_{n-1} - i_n} s_n^{i_n}$$

the leading monomial (i.e largest in lexicographical order) of g is X_I, and g is symmetric. So $f - cg$ is symmetric of total degree $\leq d$, and its leading monomial term is smaller (lexicographical) than X_I. As the set of monomials of degree at most d is finite, this process terminates.

To prove (ii): induct on n. Suppose we have $G \in R[Y_1, \dots, Y_n]$ with $G(s_{n,1}, \dots, s_{n,n}) = 0$. We want to show $G = 0$. If $n = 1$, this is trivial ($s_{1,1} = X_1$). If $G = Y_n^k H$, with $Y_n \nmid H$, then $s_{n,n}^k H(s_{n,1}, \dots, s_{n,n}) = 0$. As $s_{n,n} = X_1 \dots X_n$, $s_{n,n}$ is not a zero divisor in $R[X_1, \dots, X_n]$ so $H(s_{n,1}, \dots, s_{n,n}) = 0$.

So we may assume G is not divisible by Y_n. Replace X_n by 0. Then

$$s_{n,r}(X_1, \dots, X_{n-1}, 0) = \begin{cases} s_{n-1,r}(X_1, \dots, X_{n-1}) & \text{if } r < n \\ 0 & \text{if } r = n \end{cases}$$

and so $G(s_{n-1,1}, \dots, s_{n-1,n-1}, 0) = 0$. So by induction, $G(Y_1, \dots, Y_{n-1}, 0) = 0$, i.e $Y_n \mid G$, a contradiction.

$\square$

Example. $f = \sum_{i \neq j} X_i^2 X_j$ for $n \geq 3$. The leading term is $X_1^2 X_2 = X_1(X_1 X_2)$. Then compute

$$s_1 s_2 = \sum_i \sum_{j<k} X_i X_j X_k = \sum_{i \neq j} X_i^2 X_j + 3 \sum_{i<j<k} X_i X_j X_k$$

so $f = s_1 s_2 - 3 s_3$.

Computing say $\sum X_i^5$ by hand is tedious. But there are alternative formulae.

Recall $p_k = \sum_{i=1}^n X_i^k$ for $k \geq 1$.

Theorem 2.2 (Newton's formulae)**.** *Let $n \geq 1$. Then for all $k \geq 1$*

$$p_k - s_1 p_{k-1} + \dots + (-1)^{k-1} s_{k-1} p_1 + (-1)^k k s_k = 0$$

by convention, $s_0 = 1$, and $s_r = 0$ if $r > n$.

Proof. We may assume $R = \mathbb{Z}$ (or $\mathbb{R}$). Generating function

$$F(T) = \prod_{i=1}^n (1 - X_i T) = \sum_{r=0}^n (-1)^r s_r T^r$$

Take logarithmic derivative with respect to T:

$$\frac{F'(T)}{F(T)} = \sum_{i=1}^n \frac{-X_i}{1 - X_i T} = -\frac{1}{T} \sum_{i=1}^n \sum_{r=1}^\infty X_i^r T^r = -\frac{1}{T} \sum_{r=1}^\infty p_r T^r$$

So

$$-TF'(T) = s_1 T - 2 s_2 T^2 + \dots + (-1)^{n-1} n s_n T^n$$

$$= F(T) \sum_{r=1}^\infty p_r T^r = (s_0 - s_1 T + \dots + (-1)^n s_n T^n) \left(p_1 T + p_2 T^2 + \dots\right)$$

comparing coefficients of T^k gives the result. $\square$

Definition. The *discriminant polynomial* is

$$D(X_1, \dots, X_n) = \Delta(X_1, \dots, X_n)^2$$

where $\Delta = \prod_{i<j}(X_i - X_j)$. (Recall from IA Groups that applying $\sigma \in S_n$ to Δ multiplies Δ by $\mathrm{sgn}(\sigma)$, so D is symmetric.)

So $D(X_1, \dots, X_n) = d(s_1, \dots, s_n)$ for some polynomial d ($\mathbb{Z}$-coefficients). For example, when $n = 2$, $D = (X_1 - X_2)^2 = s_1^2 - 4s_2$.

Definition. Let $f = T^n + \sum_{i=0}^{n-1} a_{n-i}T^i \in R[T]$. Its *discriminant* is $\mathrm{Disc}(f) = d(-a_1, a_2, -a_3, \dots, (-1)^n a_n) \in R$.

Observe that if $f = \prod_{i=1}^n (T - x_i)$, $x_i \in R$, then $a_r = (-1)^r s_r(x_1, \dots, x_n)$, so

$$\mathrm{Disc}(f) = \prod_{i<j}(x_i - x_j)^2 = D(x_1, \dots, x_n)$$

If moreover $R = K$ is a field, then $\mathrm{Disc}(f) = 0$ iff f has a repeated root (i.e $x_i = x_j$ for some $i \neq j$). E.g when $n = 2$, $\mathrm{Disc}(T^2 + bT + c) = b^2 - 4c$.

3 Fields

Recall:

Definition. A *field* is a ring K (commutative with a 1) in which every non-zero element has a multiplicative inverse. The set of non-zero elements of K is a group under multiplication, written $K^\times$ or K^*, called the *multiplicative group of* K.

Definition. The *characteristic of a field* K is the least positive integer p (if it exists) such that $p \cdot 1_K = 0_K$, or is said to be 0 if no such p exists.

Example. $\mathbb{Q}$ has characteristic 0 and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ has characteristic p (p prime).

The characteristic $\mathrm{char}(K)$ of K is either 0 or a prime. Inside K, there is a smaller subfield, called the *prime subfield* of K. It is either isomorphic to $\mathbb{Q}$ (if characteristic is 0), or to $\mathbb{F}_p$ (if $\mathrm{char}(K) = p$).

Proposition 3.1. *Let $\varphi : K \to L$ be a homomorphism of fields. Then φ is an injection.*

Proof. $\varphi(1_K) = 1_L \neq 0$, so $\mathrm{Ker}(\varphi) \subsetneq K$ is a proper ideal of K, so $\mathrm{Ker}(\varphi) = (0)$ □

Definition. Let $K \subseteq L$ be fields (where the field operations on K are the same as those on L). We say K is a *subfield of* L, and L is *an extension of* K, denoted L/K.

Remarks:

(i) The notation L/K has nothing to do with the quotient (some write $L \mid K$)

(ii) It is useful to be more general - if $i : K \to L$ is a homomorphism of fields, then Proposition 3.1 says that K is isomorphic to its image $i(K) \subseteq L$. In this situation, also say L is an extension of K.

Example. Some extensions include

- $\mathbb{C}/\mathbb{R}$
- $\mathbb{R}/\mathbb{Q}$
- $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}/\mathbb{Q}$

Definition. $K \subseteq L$, $x \in L$. Define $K[x] = \{p(x) : p \in K[T]\}$ (a subring of L). Define $K(x) = \{\frac{p(x)}{q(x)} : p, q \in K[T], q(x) \neq 0\}$ (a subfield of L) "K adjoin x". For $x_1, \ldots, x_n \in L$, define

$$K(x_1, \ldots, x_n) = \left\{ \frac{p(x_1, \ldots, x_n)}{q(x_1, \ldots, x_n)} : p, q \in K[T_1, \ldots, T_n], q(x_1, \ldots, x_n) \neq 0 \right\}$$

(Easy to check $K(x_1, \ldots, x_{n-1})(x_n) = K(x_1, \ldots, x_n)$). Likewise $K[x_1, \ldots, x_n]$ is defined analagously.

Definition. Suppose L/K is a field extension. Then L is naturally a vector space over its subfield K (forget multiplication by elements of L). We can ask if it is a finite-dimensional vector space, if so we say that L/K is a *finite extension* and write $[L : K] = \dim_K(L)$ for the dimension. The dimension is called the *degree of the extension L over K*. If the dimension is infinite write $[L : K] = \infty$.

$\dim_K$ denotes the dimension as a K-vector space. Of course L has dimension 1 over itself. As a K-vector space, $L \cong K^{[L:K]}$.

Example.

(i) $\mathbb{C}/\mathbb{R}$, $[\mathbb{C} : \mathbb{R}] = 2$

(ii) For any field K, $K(X)$ = field of rational functions in X = field of fractions of polynomial ring $K[X] = \{\frac{p}{q} : p, q \in K[X], q \neq 0\}$. Then $[K(X) : K] = \infty$ since $1, X, X^2, \ldots$ are linearly independent.

(iii) $\mathbb{R}/\mathbb{Q}$, $[\mathbb{R} : \mathbb{Q}] = \infty$. This follows from countability - every finite dimensional vector space over $\mathbb{Q}$ is countable.

This course is largely about properties (and symmetries) of finite extensions of fields.

Definition. We say an extension L/K is *quadratic* (*cubic*,...) if $[L : K] = 2(, 3, \ldots)$

Proposition 3.2. *Suppose K is a finite field (necessarily of characteristic $p > 0$). Then $|K|$ is a power of p.*

Proof. Certainly $K/\mathbb{F}_p$ is finite, so $K \cong (\mathbb{F}_p)^n$ (as a vector space), where $n = [K : \mathbb{F}_p]$, so $|K| = p^n$. □

Later on we will see that every prime power $q = p^n$ admits a field $\mathbb{F}_q$ with q elements.

Here is a simple but powerful fact:

Theorem 3.3 ("Tower Law")**.** *Suppose M/L and L/K are field extensions. Then M/K is a finite extension if and only if both M/L and L/K are finite. If so, then $[M : K] = [M : L][L : K]$.*

In fact, a slightly more general statement holds:

Theorem 3.4. *Let L/K be an extension, V an L-vector space. Then $\dim_K(V) = [L : K] \dim_L(V)$ (and obvious conclusions if any quantities are infinite).*

Example. If $V = \mathbb{C}^n$ then $V \cong \mathbb{R}^{2n}$.

Proof. Let $\dim_L(V) = d < \infty$. Then $V \cong L \oplus \ldots \oplus L = L^d$ as an L-vector space, so also as a K-vector space. If $[L : K] = n < \infty$, then $L \cong K^n$ as a K-vector space, so

$$V \cong \underbrace{K^n \oplus \ldots \oplus K^n}_{d \text{ times}} = K^{nd}$$

so $\dim_K(V) = [L : K] \dim_L(V)$. If V is finite-dimensional over K, then a K-basis for V certainly spans V over L. So if $\dim_L(V) = \infty$ then $\dim_K(V) = \infty$. Likewise, if $[L : K] = \infty$ and $V \neq \{0\}$, then V has an infinite linearly independent subset, so $\dim_K(V) = \infty$. □

Another important fact:

Proposition 3.5.

(i) Let K be a field, $G \subseteq K^\times$ a finite subgroup. Then G is cyclic

(ii) If K is finite, then $K^\times$ is cyclic

Proof. We prove (i) ((ii) follows immediately): (recall from IB GRM) we can write

$$G \cong \frac{\mathbb{Z}}{m_1\mathbb{Z}} \oplus \ldots \oplus \frac{\mathbb{Z}}{m_k\mathbb{Z}}$$

where $1 < m_1 \mid m_2 \mid \ldots \mid m_k = m$. So for all $x \in G$, $x^m = 1$. As K is a field, the polynomial $T^m - 1$ has at most m roots. So $|G| < m$. Hence $k = 1$ and G is cyclic.

□

Remark: Let $K = F = \mathbb{Z}/p\mathbb{Z}$. The above says there exists $a \in \{1, \ldots, p-1\}$ such that $\mathbb{Z}/pZ = \{0\} \cup \{a, a^2, \ldots, a^{p-1}\}$. a is called a primitive root modulo p.

Proposition 3.6. *Let R be a ring, p a prime such that $p \cdot 1_R = 0_R$ (e.g R a field of characteristic p). Then the map*

$$\varphi_p : R \to R,\ \varphi_p(x) = x^p$$

is a homomorphism from R to itself (called the Frobenius endomorphism of R).

Proof. Have to show:

- $\varphi_p(1) = 1$
- $\varphi_p(xy) = \varphi_p(x)\varphi_p(y)$
- $\varphi_p(x + y) = \varphi_p(x) + \varphi_p(y)$

The first two are obvious. For the last one,

$$\begin{aligned}\varphi_p(x + y) &= x^p + \sum_{i=1}^{p-1} \underbrace{\binom{p}{i}}_{\equiv 0 \pmod p} x^i y^{p-i} + y^p \\ &= \varphi_p(x) + \varphi_p(y)\end{aligned}$$

□

Example. This gives another proof of Fermat's Little Theorem: $x^p \equiv x \pmod p$ (induction on x: $(x + 1)^p = x^p + 1$).

4 Algebraic elements and extensions

Definition. Have L/K an extension, $x \in L$. We say x is *algebraic over K* if there exists a non-zero polynomial $f \in K[T]$ such that $f(x) = 0$. Otherwise we say x is *transcendental over K*.

Suppose $f \in K[T]$; evaluation $f(x) \in L$. This gives a map $\mathrm{ev}_x : K[T] \to L$, $f \mapsto f(x)$. This is obviously a homomorphism of rings.

$I = \mathrm{Ker}(\mathrm{ev}_x) \subseteq K[T]$ is an ideal (the set of polynomials which vanish at x). As $\mathrm{Im}(\mathrm{ev}_x)$ is a subring of L, it is an integral domain. So I is a prime ideal. Two possibilities:

(i) $I = \{0\}$. Then the only f with $f(x) = 0$ is $f = 0$. Hence x is transcendental over K.

(ii) $I \neq \{0\}$. As $K[T]$ is a PID, there exists a unique monic irreducible $g \in K[T]$ such that $I = (g)$. So $f(x) = 0$ if and only if f is a multiple of g. So x is algebraic over K; we call g the *minimal polynomial* of x over K. written $m_{x,K}$. It is the unique monic irreducible polynomial such that x is a root (and the monic polynomial of least degree with this property). [Depends on K as well as x]

Example.

- $x \in K$, $m_{x,K} = T - x$
- p prime, $d \geq 1$. Then $T^d - p \in \mathbb{Q}[T]$ is irreducible (Eisenstein's criterion) so it is the minimal polynomial of $\sqrt[d]{p} = x \in \mathbb{R}$ over $\mathbb{Q}$.
- $z = e^{2\pi i/p}$ (p prime) is a root of $T^p - 1$ and of $\frac{T^p-1}{T-1} = g(T) = T^{p-1} + \ldots + T + 1 \in \mathbb{Q}[T]$. As

$$g(T+1) = \frac{(T+1)^p - 1}{T} = T^{p-1} + \binom{p}{1} T^{p-2} + \ldots + \binom{p}{2} T = \binom{p}{1}$$

  which is irreducible by Eisenstein, so g is irreducible and g is the minimal polynomial of z over $\mathbb{Q}$.

Definition. The *degree of x over K* (x algebraic over K) is the degree of $m_{x,K}$, written $\deg_K(x)$ or $\deg(x/K)$.

Ring/field characterisation of algebraicity:

Proposition 4.1. *Let L/K be a field extension, $x \in L$. The following are equivalent*

(i) *x is algebraic over K*

(ii) $[K(x) : K] < \infty$

(iii) $\dim_K K[x] < \infty$

(iv) $K[x] = K(x)$

(v) *$K[x]$ is a field*

If these hold, then $\deg_K(x) = [K(x) : K]$.

Note: recall $K[x] = \{p(x)\}$, $K(x) = \left\{\frac{p(x)}{q(x)} | q(x) \neq 0, p, q \in K[T]\right\}$.

Proof. (ii) $\iff$ (iii), (iv) $\iff$ (v) are obvious.

Show (iii) $\Rightarrow$ (v),(iv) and (ii): let $0 \neq y = g(x) \in K[x]$. Consider $K[x] \to K[x]$, $z \mapsto yz$. It is a K-linear transformation, injective as $y \neq 0$, and since $\dim_K K[x] < \infty$, it is a bijection. So there exists z such that $yz = 1$. So $K[x]$ is a field, equal to $K(x)$ and $[K(x) : K] < \infty$.

Show (v)$\Rightarrow$(i): wlog $x \neq 0$, then $x^{-1} = a_0 + a_1 x + \ldots + a_n x^n \in K[x]$. Then $a_n x^{n-1} + \ldots + a_0 x - 1 = 0$, so x is algebraic over K.

Show (i)$\Rightarrow$(iii) and degree formula: The image of $\mathrm{ev}_x : K[T] \to L$ is $K[x] \subseteq L$. x is algebraic over K so the kernel of this map is $(m_{x,K})$, which is a maximal ideal ($m_{x,K}$ is irreducible). Applying the first isomorphism theorem gives

$\underbrace{K[T]/(m_{x,K})}_{\text{field}} \cong K[x]$. $m_{x,K}$ is monic of degree $d = \deg_K(x)$. So $K[T]/(m_{x,K})$ has basis $1, T, \ldots, T^{d-1}$. So $\dim_K K[x] = d < \infty$. Furthermore $\deg_K(x) = [K(x) : K] = d$. $\square$

Corollary 4.2.

(i) *$x_1, \ldots, x_n$ are algebraic over K if and only if $L = K(x_1, \ldots, x_n)$ is a finite extension over K. If so, every element of L is algebraic in K*

(ii) *If x, y are algebraic over K, then so are $x \pm y$, xy and $1/x$ (if $x \neq 0$).*

(iii) *Let L/K any extension. Then $\{x \in L : x \text{ algebraic over } K\}$ is a subfield of L*

Proof.

(i) If x_n is algebraic over K, it's certainly algebraic over $K(x_1, \ldots, x_{n-1})$, so $[L : K(x_1, \ldots, x_{n-1})]$. So by induction on n and the Tower Law, $[L : K] < \infty$. Conversely, if $[L : K] < \infty$, then the subfield $K(y)$ is finite over K for all $y \in L$, so y is algebraic over K by Proposition 4.1.

(ii) $x + y, xy, \frac{1}{x} \in K(x, y)$. So algebraic by (i).

(iii) Trivial from (ii).

$\square$

Example. $z = e^{2\pi i/p}$, p prime. z has degree $p-1$. Let $x = 2\cos 2\pi/p = z+z^{-1} \in \mathbb{Q}(z)$. So x is algebraic over $\mathbb{Q}$. Note $\mathbb{Q}(z) \supseteq \mathbb{Q}(x) \supseteq \mathbb{Q}$, $z^2 - xz + 1 = 0$. Hence the degree of z over $\mathbb{Q}(x)$ is at most 2. We have $[\mathbb{Q}(z) : \mathbb{Q}] = p-1$ so $[\mathbb{Q}(z) : \mathbb{Q}(x)] = 2$ or 1. But $z \notin \mathbb{Q}(x) \subseteq \mathbb{R}$. So $[\mathbb{Q}(z) : \mathbb{Q}(x)] = 2$ and by the tower law $\deg_{\mathbb{Q}}(x) = \frac{p-1}{2}$.

We have
$$z^{\frac{p-1}{2}} + z^{\frac{p-3}{2}} + \ldots + z^{-\frac{p-1}{2}} = 0$$
$z + z^{-1} = x$. So can express this polynomial as a polynomial in $z + z^{-1} = x$ of degree $\frac{p-1}{2}$.

Example. Let $x = \sqrt{m} + \sqrt{n}$, $m, n \in \mathbb{Z}$ such that m, n, mn are not squares. We have
$$(x - \sqrt{m})^2 = n = x^2 - 2\sqrt{m}x + m$$
So $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{m})] \leq 2$, since the above is a quadratic with coefficients in $\mathbb{Q}(\sqrt{m})$. In the exact same way we have $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{n})] \leq 2$. The quadratic also implies $\sqrt{m} \in \mathbb{Q}(x)$. So by the tower law either $[\mathbb{Q}(x) : \mathbb{Q}] = 4$ or $[\mathbb{Q}(x) : \mathbb{Q}] = 2$ and $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{n})$ (since m, n not squares, $[\mathbb{Q}(\sqrt{m}) : \mathbb{Q}] = 2$).

$\mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{n})$ implies $\sqrt{m} = a + b\sqrt{n}$, $a, b \in \mathbb{Q}$. This implies $m = a^2 + b^2 n + 2ab\sqrt{n}$. $b = 0$ implies $m = a^2$ and $a = 0$ implies $mn = b^2 n^2$, a contradiction. So $\deg_{\mathbb{Q}}(x) = 4$.

Definition. An extension L/K is *algebraic* if every $x \in L$ is algebraic over K.

Proposition 4.3.

(i) Finite extensions are algebraic

(ii) $K(x)$ is algebraic over K if and only if x is algebraic over K

(iii) Let $M/L/K$ be a series of extensions. Then M/K is algebraic if and only if both M/L and L/K are algebraic

Proof.

(i) If $[L : K] < \infty$ then $\forall x \in L$, $[K(x) : K] < \infty$, so x is algebraic over K.

(ii) $(\Rightarrow)$ is by definition, $(\Leftarrow)$ follows from (i).

(iii) Assume M/K is algebraic. Then for all $x \in M$, x is algebraic over K, so certainly x is algebraic over L. So M/L is algebraic. Since $L \subseteq M$, L/K must be algebraic as M/K is.

The other direction follows from the below Lemma.

$\square$

Lemma 4.4. *Let $M/L/K$ be a series of extensions, where L/K is algebraic. Let $x \in M$. Suppose x is algebraic over L. Then x is algebraic over K.*

Proof. There exists $f = T^n + a_{n-1}T^n + \ldots + a_0 \in L[T]$ with $f \neq 0$ and $f(x) = 0$. Let $L_0 = K(a_0, \ldots, a_{n-1})$, then as each $a_i \in L$ is algebraic over K, by Corollary 4.2, $[L_0 : K]$ is finite. As $f \in L_0[T]$, x is algebraic over L_0. So $[L_0(x) : L_0] < \infty$, so $[L_0(x) : K] < \infty$ by the tower law, and so $[K(x) : K] < \infty$ and x is algebraic over K. $\square$

Example. Let $K = \mathbb{Q}$, $L = \{x \in \mathbb{C} : x \text{ is algebraic over } \mathbb{Q}\} = \overline{\mathbb{Q}}$. This is a field by Corollary 4.2. Obviously $L/\mathbb{Q}$ is algebraic, but the extension is <u>not</u> finite. Indeed, for all $n \geq 1$, $\sqrt[n]{2} \in L$ and $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$ (as $T^n - 2$ is irreducible over $\mathbb{Q}$). So as this holds for any n, L can't be finite. We'll see other fields like $\overline{\mathbb{Q}}$ later on (algebraically closed fields).

5 Algebraic numbers in $\mathbb{R}$ and $\mathbb{C}$

Traditionally, $x \in \mathbb{C}$ is said to be *algebraic* if it's algebraic over $\mathbb{Q}$, and otherwise said to be *transcendental*. $\overline{\mathbb{Q}}$ is a subfield of $\mathbb{C}$. It is a proper subfield since $\mathbb{Q}[T]$ is countable, and each polynomial has countably (finitely) many roots, so there are countably many elements of $\overline{\mathbb{Q}}$.

However $\mathbb{C}$ is uncountable. So there are "lots" of transcendental numbers. This argument is non-constructive - it is harder to write a transcendental number explicitly, or to show some given number is transcendental.

Liouville showed that $\sum_{n\geq 1} \frac{1}{10^{n!}}$ is transcendental ("algebraic numbers can't be very well approximated by rationals").

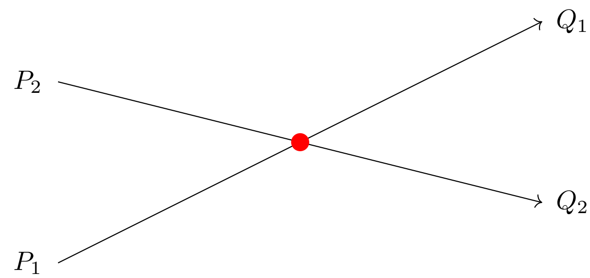
Hermite, Lindermann showed that e and π are transcendental.

In the 20th Century: <u>Gelfond-Schneider Theorem</u>: if x, y are algebraic ($x \neq 1$), then x^y is algebraic if and only if y is rational. For example, this implies $\sqrt{2}^{\sqrt{3}}$ is transcendental. Also $e^\pi = (-1)^{-i/2}$ is transcendental.

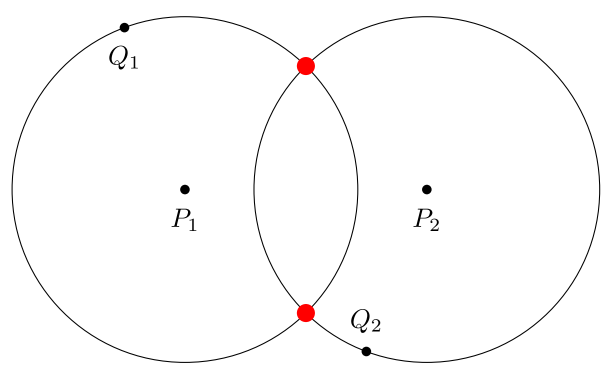
Ruler & compass constructions

We have 3 basic geometric operations (in plane geometry).

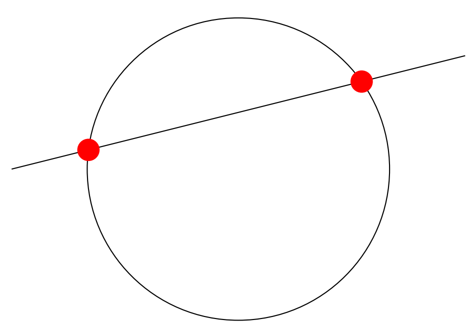
(A) Given $P_1, P_2, Q_1, Q_2 \in \mathbb{R}^2$ with $P_i \neq Q_i$, we can construct (with a ruler) the point of intersection of the lines P_1Q_1, P_2Q_2 (assuming they intersect properly).



(B) Given P_1, P_2, Q_1, Q_2 with $P_i \neq Q_i$, we can construct the intersection points of the circles with centres P_i passing through Q_i.



(C) Can intersect lines with circles.



Definition. We say $(x, y) \in \mathbb{R}^2$ is *constructable from*

$$\{(x_1, y_1), \dots, (x_n, y_n)\}$$

if it can be obtained by a finite sequence of constructions of type A,B,C, each involving only the starting points $\{(x_i, y_i) : 1 \leq i \leq n\}$ and any produced in a previous step.

Definition. We say $x \in \mathbb{R}$ is *constructable* if $(x, 0)$ is constructable from $\{(0, 0), (1, 0)\}$.

Note: every $x \in \mathbb{Q}$ is constructable, and so is $\sqrt{2}$.

Definition. Let $K \subseteq \mathbb{R}$ be a subfield. We say K is *constructable* if there exists some $n \geq 0$ and some sequence of fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \mathbb{R}$ and $a_i \in F_i$ (for $1 \leq i \leq n$) such that

(i) $K \subseteq F_n$

(ii) $F_i = F_{i-1}(a_i)$

(iii) $a_i^2 \in F_{i-1}$

Note: (ii) and (iii) imply that $[F_i : F_{i-1}] \leq 2$. So by the tower law, $K/\mathbb{Q}$ is finite and $[K : \mathbb{Q}]$ is a power of 2.

Theorem 5.1. *If $x \in \mathbb{R}$ is constructable, then $K = \mathbb{Q}(x)$ is constructable.*

Corollary 5.2. *If $x \in \mathbb{R}$ is constructable, then x is algebraic over $\mathbb{Q}$ and $\deg_{\mathbb{Q}}(x)$ is a power of 2 (follows from the above note and the theorem).*

Proof of Theorem 5.1. Induction on $k \geq 1$: we prove that if $(x, y) \in \mathbb{R}^2$ can be constructed with k R&C (Ruler & Compass) constructions, then $\mathbb{Q}(x, y)$ is a constructable extension of $\mathbb{Q}$.

So assume we have

$$\mathbb{Q} = F_0 \subseteq \ldots \subseteq F_n$$

satisfying (ii),(iii) and such that the coordinates of all points obtained after $(k-1)$ constructions lie in F_n.

Elementary analytic geometry tells us that in (A) the intersection point has coordinates which are rational functions of the coordinates of the points $\{P_i, Q_i\}$ with rational coefficients.

So if the kth construction is of type (A), then $x, y \in F_n$. For constructions (B) and (C), the coordinates of the two intersections can be written as $a \pm b\sqrt{e}$, $c \pm d\sqrt{e}$, where a, e are rational functions of the coordinates of $\{P_i, Q_i\}$. So for the two newly constructed points $x, y \in F_n(\sqrt{e})$, which is a constructable extension of $\mathbb{Q}$. $\square$

Remark: it is not hard to show that the converse is true, i.e if $\mathbb{Q}(x)/\mathbb{Q}$ is constructable then x is constructable.

Examples of classical problems:

1. "Squaring the circle" - construct a square whose area is that of a given circle, i.e have to construct $\sqrt{\pi}$. But since π is transcendental, it (and therefore $\sqrt{\pi}$) is not constructable.

2. "Duplicating the cube" - Construct a cube with volume twice that of a given cube, i.e construct $\sqrt[3]{2}$. But $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ is not a power of two, so $\mathbb{Q}(\sqrt[3]{2})$ (and so $\sqrt[3]{2}$) is not constructable.

3. "Trisect the angle" - say we are trying to trisect $2\pi/3$, which is certainly constructable. So if we can trisect $2\pi/3$, we can construct the angle $2\pi/9$, i.e the real numbers $\cos(2\pi/9), \sin(2\pi/9)$ are constructable. By the formula
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
we note $\cos(2\pi/9)$ is a root of $8X^3 - 6X + 1$, and $2\cos(2\pi/9) - 2$ is a root of $X^3 + 6X^2 + 9X + 3$ which is irreducible over $\mathbb{Q}$ by Eisenstein's criterion. So $\deg_{\mathbb{Q}}(\cos(2\pi/9)) = 3$ (not a power of two) so not constructable.

Later in the course we will see the following theorem

Theorem (Gauss). *A regular n-gon is constructable if and only if n is the product of a power of 2 and distinct primes of the form $2^{2^k} + 1$ ("Fermat primes").*

6 Splitting fields

Problem: we have a field K, $f \in K[T]$ - find an extension L/K (preferably as small as possible) such that f factors in $L[T]$ as a product of linear polynomials.

Example. Let $K = \mathbb{Q}$. By the Fundamental Theorem of Algebra, we can factor any monic $f \in \mathbb{Q}[T]$ as
$$f = \prod_{i=1}^{n}(T - x_i),\ x_i \in \mathbb{C}$$
(Later we will give another proof of the FTA.) So the "best" L would be $\mathbb{Q}(x_1, \ldots, x_n)$, a finite extension of $\mathbb{Q}$.

Example. Let $K = \mathbb{F}_p$. Let f be irreducible of degree $d > 1$. How to find L?

First step: find an extension in which f has at least one root.

Key construction: suppose $f \in K[T]$ is (monic and) irreducible. Let $L_f = K[T]/(f)$. As f is irreducible, (f) is maximal and so L_f is a field. By construction, if $x = T \pmod{(f)} \in L_f$ (the coset $T + (f)$), then $f(x) = 0$. Hence L_f/K

is a field extension in which f has a root.

Questions:

- Is L_f unique?
- What about the remaining roots?

Theorem 6.1. *Let $f \in K[T]$ be irreducible and monic. Let $L_f = K[T]/(f)$, $t \in L_f$ the residue class $T + (f)$. Then L_f/K is a finite extension of fields, $[L_f : k] = \deg(f)$ and f is the minimal polynomial of t over K.*

Proof. See previous example. □

So we have an extension of K in which f has a root. To what extent is this unique?

Also recall that if x is algebraic over K, then $K(x) \cong K[T]/(m_{x,K})$, where $m_{x,K}$ is the minimal polynomial of x over K.

Definition. Suppose K is a field, L/K and M/K extensions of K. A K-*homomorphism* from L to M is a field homomorphism $\sigma : L \to M$ such that $\sigma|_K = \mathrm{id}_K$. We also sometimes call this a K-*embedding*, since σ is an injection.

Theorem 6.2. *Let $f \in K[T]$ be irreducible, L/K be an arbitrary extension. Then*

(i) If $x \in L$ is a root of f, then there exists a unique K-homomorphism $\sigma : L_f \to L$ sending $T + (f)$ to x.

(ii) Every K-homomorphism $L_f \to L$ arises as in (i). So there is a bijection between

$$\{K\text{-homomorphisms } L_f \xrightarrow{\sigma} L\} \leftrightarrow \{\text{roots of } f \text{ in } L\}$$

In particular, there are at most $\deg(f)$ such σ.

Proof. Note

$$\begin{aligned} f(x) = 0 &\iff \mathrm{ev}_x(f) = 0 \\ &\iff \mathrm{Ker}(\mathrm{ev}_x) = (f) \\ &\iff \mathrm{ev}_x \text{ comes from a homomorphism } \sigma : K[T]/(f) \to L \\ &\qquad \text{which is the identity on } K \end{aligned}$$

where $\mathrm{ev}_x : K[T] \to L$ is the homomorphism $g \mapsto g(x)$. □

Corollary 6.3. *If $L = K(x)$ for x algebraic over K, then there exists a unique isomorphism $\sigma : L_f \to K(x)$ such that $\sigma(t) = x$, with $f = m_{x,K}$.*

Proof. Take $L = K(x)$ in the above Theorem. □

Definition. Let x, y be algebraic over K. We say x, y are K-*conjugate* if they have the same minimal polynomial.

Then by the last corollary, both $K(x)$ and $K(y)$ are isomorphic to L_f (where f is their common minimal polynomial).

Corollary 6.4. *x, y are K-conjugate if and only if there exists a K-isomorphism $\sigma : K(x) \to K(y)$ with $\sigma(x) = y$.*

Proof. ($\Rightarrow$) follows by corollary 6.3.

($\Leftarrow$) follows since for all g in $K[T]$ we have $\sigma(g(x)) = g(\sigma(x)) = g(y)$ so x, y have the same minimal polynomial. $\square$

Moral: "the roots of an irreducible polynomial are algebraically indistinguishable".

It is useful (for inductive arguments) to have a generalisation of Theorem 6.2.

Definition. Let L/K, L'/K' be field extensions. Let $\sigma : K \to K'$ be a homomorphism of fields. If $\tau : L \to L'$ is a homomorphism such that $\tau(x) = \sigma(x)$ whenever $x \in K$, we say τ is a *σ-homomorphism* from L to L'. We also say τ *extends* σ or that σ is *the restriction of* τ to K. We write $\sigma = \tau|_K$.

From this definition we have the following variant of Theorem 6.2:

Theorem 6.5. *Let $f \in K[T]$ be irreducible, and $\sigma : K \to L$ be any homomorphism of fields. Let σf be the polynomial given by applying σ to the coefficients of f. Then*

(i) If $x \in L$ is a root of σf, there exists a unique σ-homomorphism $\tau : L_f \to L$ such that $\tau(t) = \tau(T + (f)) = x$

(ii) Every σ-homomorphism $L_f \to L$ is of the form arising from (i), so we have a bijection

$$\{\sigma\text{-homomorphisms } L_f \to L\} \leftrightarrow \{\text{roots of } \sigma f \text{ in } L\}$$

Example. σ might not be the "obvious" homomorphism. Indeed take $K = \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$, and take $L = \mathbb{C}$. There is a homomorphism $\sigma : K \to L$ given by $x + y\sqrt{2} \mapsto x - y\sqrt{2}$. Now take $f = T^2 - (1 + \sqrt{2})$. The map $L_f \xrightarrow{\tau} \mathbb{C}$ must take $t = T + (f)$ to $\pm\sqrt{1 - \sqrt{2}} = \pm i\sqrt{\sqrt{2} - 1} \in \mathbb{C}$.

If instead we took σ to be the inclusion τ takes t to $\pm\sqrt{\sqrt{2} + 1}$.

What about all roots?

Definition. Let $f \in K[T]$ be a non-zero polynomial (not necessarily irreducible). An extension L/K is a *splitting field* for f over K if

(i) f splits into linear factors in $L[T]$.

(ii) $L = K(x_1, \ldots, x_n)$ where $\{x_1, \ldots, x_n\}$ are the roots of f in L.

Remark: (ii) says that f doesn't split into linear factors over any field L' with $K \subseteq L' \subsetneq L$. Furthermore, any splitting field is necessarily finite since the $\{x_1, \ldots, x_n\}$ are algebraic.

Theorem 6.6. *Every non-zero polynomial in $K[T]$ has a splitting field.*

Proof. Induction on $\deg(f)$ (for all K). If $\deg(f) = 0$ or 1, then K is a splitting field. So assume that for all fields K' and all polynomials of degree less than $\deg(f)$, there is a splitting field.

Consider g, an irreducible factor of f. Consider $K' = L_g = K[T]/(g)$. Let $x_1 = T + (g)$. Then $g(x_1) = 0$, so $f(x_1) = 0$ and $f = (T - x_1)f_1$, for some $f_1 \in K'[T]$ and $\deg(f_1) < \deg(f)$. So by induction there is a splitting field L for f_1 over K'. Let $x_2, \ldots, x_n \in L$ be the roots of f_1 in L. Then f splits into linear factors in L, with roots $x_2, \ldots, x_n$, and $L = K'(x_2, \ldots, x_n) = K(x_1, \ldots, x_n)$. So L is a splitting field for f over K. $\square$

Theorem 6.7 ("Splitting fields are unique")**.** *Let $f \in K[T]$ be non-zero, let L/K be a splitting field for f. Let $\sigma : K \to M$ be an extension such that $\sigma f \in M[T]$ splits [into linear factors] in $M[T]$. Then*

(i) σ can be extended to a homomorphism $\tau : L \to M$.

(ii) If M is a splitting field for σf over $\sigma(K)$, then any τ as in (i) is an isomorphism. In particular, any two splitting fields for f are K-isomorphic.

Remarks:

- It is not obvious without this theorem that two splitting fields have the same degree, because of the choices we had in the construction.
- Typically there will be more than one τ.

Proof.

(i) Induction on $n = [L : K]$. If $n = 1$ then $L = K$ and we are done.

Let $x \in L \setminus K$ be a root of an irreducible factor $g \in K[T]$ of f, with $\deg(g) > 1$. Let $y \in M$ be a root of $\sigma g \in M[T]$ (since σf splits in M this exists). Theorem 6.4 implies there exists $\sigma_1 : K(x) \to M$ such that $\sigma_1(x) = y$ and σ_1 extends σ.

Now $[L : K(x)] < [L : K]$ and L is certainly a splitting field for f over $K(x)$ and $\sigma_1 f = \sigma f$ splits in M. So by induction we can extend σ_1 to a homomorphism $\tau : L \to M$.

(ii) Assume M is a splitting field for σf over $\sigma(K)$. Let τ be as in (i) and $\{x_i\}$ the roots of f in L. Then the roots of σf in M are just $\{\tau(x_i)\}$. Since M is a splitting field, $M = \sigma K(\tau(x_1), \ldots, \tau(x_n)) = \tau(L)$. So τ is an isomorphism. If $K \subseteq M$ and σ is the inclusion, τ is a K-isomorphism from L to M.

$\square$

Example.

(i) $f = T^3 - 2 \in \mathbb{Q}[T]$. In $\mathbb{C}$, $f = (T - \sqrt[3]{2})(T - \omega\sqrt[3]{2})(T - \omega^2\sqrt[3]{2})$ where $\omega = \exp(2\pi i/3)$. So a splitting field for f over $\mathbb{Q}$ is $L = \mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2})$. Then $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$, but $\omega \notin \mathbb{R}$, $\omega^2 + \omega + 1 = 0$, so $[L : \mathbb{Q}(\sqrt[3]{2})] = 2$ and $[L : \mathbb{Q}] = 6$.

(ii) $f = \frac{T^5 - 1}{T - 1} = T^4 + T^3 + T^2 + T + 1 \in \mathbb{Q}[T]$. Let $z = \exp(2\pi i/5)$. Then $f = \prod_{1 \le a \le 4}(T - z^a)$. So $\mathbb{Q}(z)$ is already a splitting field over $\mathbb{Q}$ and $[\mathbb{Q}(z) : \mathbb{Q}] = 4$.

(iii) $f = T^3 - 2 \in \mathbb{F}_7[T]$. This is irreducible since 2 is not a cube modulo 7. Consider the field $L = \mathbb{F}_7[X]/(X^3 - 2) = \mathbb{F}_7(x)$. Then $x^3 = 2$. Now $2^3 = 1 = 4^3$ in $\mathbb{F}_7$. So $(2x)^3 = (4x)^3 = 2$ and so $f = (T - x)(T - 2x)(T - 4x) \in L[T]$

7 Normal extensions

Philosophy: pass from polynomials to fields generated by their roots.

Here we will see an "intrinsic" characterisation of splitting fields.

Definition. An extension L/K is said to be *normal* if L/K is algebraic and for every $x \in L$, $m_{x,K}$ splits into linear factors over L.

Note: this condition is equivalent to: for every $x \in L$, L contains a splitting field for $m_{x,K}$. Or again, for every $f \in K[T]$ irreducible, if f has a root in L, then it splits over L.

Theorem 7.1 ("Splitting fields are normal")**.** *Let L/K be a finite extension. Then L is normal over K if and only if L is the splitting field for some $f \in K[T]$ (not necessarily irreducible).*

Proof. Suppose L/K is normal, and write $L = K(x_1, \ldots, x_n)$. Then $m_{x_i,K}$ splits in L, and L is generated by the roots of $f = \prod_i m_{x_i,K}$. So L is a splitting field for f.

Conversely, if L is the splitting field for $f \in K[T]$. Let $x \in L$, $m_{x,K} = g$ its minimal polynomial - we want to show g splits in f. Let M be a splitting field for g over L, and $y \in M$ some root of g. We want to show $y \in L$. Since L is a splitting field for f over K, L is a splitting field for f over $K(x)$, and $L(y)$ is a splitting field for f over $K(y)$.

Now there exists a K-isomorphism between $K(x)$ and $K(y)$ as x, y are both roots of the same irreducible polynomial $g \in K[T]$. So $[L : K(x)] = [L(y) : K(y)]$ by uniqueness of splitting fields. Hence multiply both sides by $[K(x) : K] = [K(y) : K] = \deg(g)$, and use the tower law to see $[L : K] = [L(y) : K] = [L(y) : L][L : K]$. So $L(y) = L$, i.e $y \in L$. $\square$

There is a "field-theoretic" version of a splitting field:

Corollary 7.2 ("Normal closure")**.** *Let L/K be a finite extension. Then there exists a finite extension M/L such that*

(i) M/K is normal

(ii) If $L \subseteq M' \subseteq M$ and M'/K is normal, then $M' = M$

Moreover, any two such extensions M are L-isomorphic.

Proof. Say $L = K(x_1, \ldots, x_k)$. Let $f = \prod_i m_{x_i,K}$. Let M be a splitting field for f over L. Then as the x_i's are roots of f, M is also a splitting field for f over K. So M/K is normal. Let M' be as in (ii); then as $x_i \in M'$, $m_{x_i,K}$ splits in M' (as M'/K is normal). So $M' = M$.

For uniqueness: any M satisfying (i) must contain a splitting field for f, and by the above, (ii) implies that M is a splitting field for f. So uniqueness follows from uniqueness of splitting fields. $\square$

8 Seperability

Over $\mathbb{C}$, we can tell if f has multiple zeros by looking at its derivative. Over arbitrary fields, turns out the same is true if we replace the analytic notion of differentiation with an algebraic one.

Definition. The *(formal) derivative* of a polynomial $f = \sum_{0 \leq i \leq d} a_i T^i \in K[T]$ is $f' = \sum_{1 \leq i \leq d} i a_i T^{i-1}$.

It is easy to check that $(f+g)' = f'+g'$, $(fg)' = f'g+fg'$ and $(f^n)' = nf'f^{n-1}$.

Example. Let K be a field of characteristic $p > 0$. Then if $f = T^p + a_0$, $f' = pT^{p-1} + 0 = 0$. So it is possible to have a non-constant polynomial with zero derivative.

Proposition 8.1. *Let $f \in K[T]$, L/K an extension and $x \in L$ a root of f. Then x is a simple root if and only if $f'(x) \neq 0$.*

Proof. Write $f = (T-x)g \in L[T]$. Then $f' = g + (T-x)g'$ so $f'(x) = g(x)$ and $g(x)$ is non-zero if and only if $(T-x) \nmid g$, i.e x is a simple root of f. $\square$

Definition. We say $f \in K[T]$ is *separable* if it splits into distinct linear factors in a splitting field (i.e has $\deg(f)$ distinct roots).

Corollary 8.2. *f is separable if and only if* $\gcd(f, f') = 1$.

Note: we take $\gcd(f,g)$ to be the unique monic h such that $(h) = (f,g)$. Then $h = af + bg$ for some a, b which can be computed by Euclids algorithm. Observe that $\gcd(f,g)$ is the same in $K[T]$ or $L[T]$ for any $K \subseteq L$, since Euclids algorithm gives the same result.

Proof of Corollary. Replacing K by a splitting field for f, we may assume f has all its roots in K. Now f is separable if and only if f, f' have no common root, which holds if and only if $\gcd(f, f') = 1$. $\square$

Example. $\mathrm{char}(K) = p > 0$, $f = T^p - b$, $b \in K$. Then $f' = 0$ so $\gcd(f, f') = f \neq 1$. So f is inseparable. Let L be any extension of K containing some $a \in L$ such that $a^p = b$. Then $f = (T-a)^p = T^p + (-a)^p = T^p - b$. So f has only one root in a splitting field. In fact, if b isn't a pth power in K, then f is irreducible (Exercise).

Theorem 8.3.

(i) Let $f \in K[T]$ be irreducible. Then f is separable if and only if $f' \neq 0$.

(ii) If $\mathrm{char}(K) = 0$ then every irreducible polynomial in $K[T]$ is separable.

(iii) If $\mathrm{char}(K) = p > 0$ then an irreducible $f \in K[T]$ is inseparable if and only if $f = g(T^p)$ for some $g \in K[T]$.

Proof.

(i) Assume wlog that f is monic. Then as f is irreducible, $\gcd(f, f') = f$ or 1. But $\deg(f) > \deg(f')$ so $\gcd(f, f') \neq f$ unless $f' = 0$, and converse is obvious.

(ii) Write $f = \sum_{0 \le i \le d} a_i T^i$, $f' = \sum_{1 \le i \le d} i a_i T^{i-1}$. So $f' = 0$ if and only if $i a_i = 0$ for all $1 \le i \le d$, so $a_i = 0$ for all $1 \le i \le d$ (since characteristic 0). Hence f is constant, and not irreducible.

(iii) As above get $i a_i = 0$ for all $1 \le i \le d$, and $a_i = 0$ for all i not divisible by p. Thus $f = g(T^p)$ where $g = \sum_i a_{pi} T^i$.

$\square$

Now we go from polynomials to fields:

Definition. Let L/K be an extension. Say $x \in L$ is *separable over K* if x is algebraic over K and $m_{x,K}$ is separable. Say L/K is *separable over K* if x is separable over K for all $x \in L$.

Theorem 8.4. *Let x be algebraic over K, and L/K any extension in which $m_{x,K}$ splits. Then x is separable over K if and only if there are exactly $\deg_K(x)$ K-homomorphisms from $K(x) \to L$.*

Proof. Recall (from 6.2) that the number of such homomorphisms is the number of roots of $m_{x,K}$ in L. This is equal to $\deg_K(x)$ if and only if $m_{x,K}$ splits. $\square$

Notation: write $\mathrm{Hom}_K(L, M) = \{K\text{-homomorphisms } L \to M\}$ (not to be confused with linear maps $L \to M$).

Theorem 8.5 ("Counting embeddings"). *Let $L = K(x_1, \ldots, x_k)$ be a finite extension of K, and M/K any extension. Then $|\mathrm{Hom}_K(L, M)| \leq [L : K]$ with equality if and only if*

(i) For all i, $m_{x_i,K}$ splits into linear factors over M

(ii) All the x_i are separable over K

Remarks:

1. (i) and (ii) are the same as saying $m_{x_i,K}$ splits into distinct linear factors in M

2. Obvious variant: take any homomorphism $\sigma : K \to M$ and the condition becomes that the number of σ-homomorphisms is bounded by $[L : K]$ with equality if and only if for all i, $\sigma m_{x_i,K}$ splits over M

Proof. Induction on k. If $k = 0$ we're done. For $k \geq 1$ take $K_1 = K(x_1)$, $\deg_{K_1}(x_1) = d = [K_1 : K]$. Then $|\mathrm{Hom}_K(K_1, M)| = e = |\{\text{roots of } m_{x_1,K} \text{ in } M\}| \leq d$. Let $\sigma : K_1 \to M$ be a K-homomorphism. Apply induction to L/K_1. So there exist at most $[L : K_1]$ extensions of σ to a homomorphism $L \to M$. So $|\mathrm{Hom}_K(L, M)| \leq e[L : K_1] \leq d[L : K_1] = [L : K]$.

If equality holds, then $e = d$, i.e $M_{x_1,K}$ has d distinct roots in M. But we could have taken any other x_i instead of x_1 in the above, to get (i) and (ii).

Conversely, assume (i) and (ii) hold. Then by the previous theorem $|\mathrm{Hom}_K(K_1, M)| = d$ and (i), (ii) still hold over K_1. So by induction on k, each $\sigma : K_1 \to M$ has $[L : K_1]$ extensions to $L \to M$, so $|\mathrm{Hom}_K(L, M)| = [L : K]$. □

Theorem 8.6 ("Seperably generated implies separable"). *Let $L = K(x_1, \ldots, x_k)$ be a finite extension of K. Then L/K is separable over K if and only if x_i is separable over K for all i.*

Proof. If L/K is separable, all the x_i are separable by definition. So assume all the x_i are separable over K, and let M be a normal closure (splitting field of $\prod_i m_{x_i,K}$ over L). Then in the previous theorem, both (i) and (ii) are satisfied so $|\mathrm{Hom}_K(L, M)| = [L : K]$. But if $x \in L$, then $L = (x, x_1, \ldots, x_k)$ as well. So by the previous theorem again, x is separable. □

Corollary 8.7. *Let $x, y \in L$, L/K an extension of K. If x, y are separable over K, so are $x + y, xy$ and $1/x$ (if $x \neq 0$).*

Proof. Apply previous theorem to $K(x, y)$. So $\{x \in L : x \text{ separable over } K\}$ forms a subfield of L. □

Theorem 8.8 ("Primitive element theorem for separable extensions")**.** *Let K be an infinite field, and $L = K(x_1, \ldots, x_k)$ a finite extension where $x_1, \ldots, x_k$ are separable. Then there exists $x \in L$ such that $L = K(x)$ (by the previous, x is also separable over K).*

Proof. It is enough to consider the case $k = 2$, $L = K(x, y)$ with x, y separable over K. Let $n = [L : K]$ and let M be a normal closure for L/K. Then there exist n distinct K-homomorphisms $\sigma_i : L \to M$. Let $a \in K$ and consider $z = x + ay$. We will choose a such that $L = K(z)$.

As $L = K(x, y)$, $\sigma_i(x) = \sigma_j(x)$ and $\sigma_i(y) = \sigma_j(y)$ occurs iff $\sigma_i = \sigma_j$, i.e $i = j$. Consider $\sigma_i(z) = \sigma_i(x) + a\sigma_i(y)$. If $\sigma_i(x) = \sigma_j(x)$ then $[\sigma_i(x) - \sigma_j(x)] - a[\sigma_i(y) - \sigma_j(y)] = 0$ and if $i \neq j$, at least one of these brackets is non-zero, so there exists at most one $a \in K$ for which it holds. So there is at most one a for which $\sigma_i(z) = \sigma_j(z)$. Since K is infinite, there exists a such that $\sigma_i(z)$ is distinct for all $1 \leq i \leq n$. But then $\deg_K(z) = n$, so $L = K(z)$. □

For finite fields, the result is much easier:

Theorem 8.9. *If L/K is an extension of finite fields, then $L = K(x)$ for some $x \in K$.*

Proof. The multiplicative group $L^{\times}$ is cyclic. Let x be a generator of this group. Then $L = K(x)$. □

9 Galois Theory

Automorphisms of fields: $\sigma : L \to L$ is an *automorphism* of the field L if it is a bijective homomorphism.

The set of automorphisms of L forms a group under composition of functions and is denoted $\mathrm{Aut}(L)$ (the "automorphism group of L").

If $S \subseteq \mathrm{Aut}(L)$, is a subset, let $L^S = \{x \in L : \forall \sigma \in S,\ \sigma(x) = x\}$. This is a subfield of L (since each σ is a homomorphism) and is called the *fixed field* of S.

E.g $L = \mathbb{C}$, $\sigma =$ complex conjugation. Then $L^{\{\sigma\}} = \mathbb{R}$. Let L/K be an extension. Define $\mathrm{Aut}(L/K) = \{K\text{-automorphisms of } L\} = \{\sigma \in \mathrm{Aut}(L) : \sigma(x) = x\ \forall x \in K\}$ (a subgroupf of $\mathrm{Aut}(L)$). Then $\sigma \in \mathrm{Aut}(L/K)$ if and only if $K \subseteq L^{\{\sigma\}}$.

Theorem 9.1. *Let L/K be finite. Then $|Aut(L, K)| \leq [L : K]$.*

Proof. Take $M = L$ in Theorem 8.5. Then $\mathrm{Hom}_K(L, M) = \mathrm{Aut}(L/K)$. □

Fact: If $K = \mathbb{Q}$ or $\mathbb{F}_p$ then $\text{Aut}(K) = \{1\}$ ($\sigma(1_K) = 1_K$ implies $\sigma(m1_K) = m\sigma(1_K)$ for all $m \in \mathbb{Z}$). So for any L, $\text{Aut}(L) = \text{Aut}(L/K)$ where K is the prime subfield (copy of $\mathbb{Q}$ or $\mathbb{F}_p$).

There is a notion of when L/K has "many" symmetries.

Definition. An extension L/K is said to be *Galois* if it is algebraic and $L^{\text{Aut}(L/K)} = K$, i.e automorphisms detect when an element of L is in K.

Examples:

1. $\mathbb{C}/\mathbb{R}$ is Galois (e.g complex conjugation fixes only elements of $\mathbb{R}$). Likewise $\mathbb{Q}(i)/\mathbb{Q}$ is Galois.
2. $K/\mathbb{F}_p$ a finite extension. Then K is a finite field. The Frobenius automorphism $\varphi_p : K \to K,\ x \mapsto x^p$ has $K^{\{\varphi_p\}} = \{x \in K : x \text{ root } of T^p - T\}$. $T^p - T$ has at most p roots and everything in $\mathbb{F}_p$ is a root so $K^{\{\varphi_p\}} = \mathbb{F}_p$, i.e $K/\mathbb{F}_p$ is Galois.

Definition. If L/K is Galois, write $\text{Gal}(L/K) = \text{Aut}(L/K)$, the Galois group of L/K.

Theorem 9.2 (Classification of finite Galois extensions)**.** *Let L/K be a finite extension, $G = Aut(L/K)$. The following are equivalent*

(i) L/K is Galois (i.e $L^G = K$)

(ii) L/K is normal and separable

(iii) L is the splitting field of a separable polynomial

(iv) $|Aut(L/K)| = [L : K]$.

If so then the minimal polynomial of $x \in L$ is $m_{x,K} = \prod_{i=1}^{r}(T - x_i)$, where $\{x_1, \ldots, x_r\} = \{\sigma(x) : \sigma \in G\}$ is the orbit of G on x (the x_i are distinct)

Proof. First we show (i)$\Rightarrow$(ii) and the last part. Let $x \in L$, $\{x_1, \ldots, x_r\}$ be the orbit of G on x, $f = \prod(T - x_i)$. Then $f(x) = 0$. As G permutes $\{x_i\}$, $f \in L^G[T] = K[T]$, so $m_{x,K} \mid f$. Also since $m_{x,K}(\sigma(x)) = \sigma(m_{x,K}(x)) = 0$, every x_i is a root of $m_{x,K}$. So $f = m_{x,K}$ and x is separable over K, and $m_{x,K}$ splits in L, so L/K is normal and separable.

Now we show (ii)$\Rightarrow$(iii). By Theorem 7.1, L is a splitting field for some $f \in K[T]$. Write $f = \prod q_i^{e_i}$, where q_i are irreducible and $e_i \geq 1$. Since L/K is separable, q_i are separable, so $g = \prod q_i$ is separable amd L is also a splitting field for g.

Now we show (iii)$\Rightarrow$(iv). Write $L = K(x_1, \ldots, x_k)$, the splitting field of some separable f with roots x_i. Take $M = L$ and apply Theorem 8.5 as since $m_{x_i,K} \mid f$, the conditions for equality hold. Hence $|\text{Hom}_K(L, M)| = [L : K]$

Finally we show (iv)⇒(i). Suppose $|G| = [L : K]$. Then $G \subseteq \mathrm{Aut}(L/L^G) \subseteq \mathrm{Aut}(L/K)$ so in fact $G = \mathrm{Aut}(L/L^G)$, and $[L : K] = |G| \leq [L : L^G]$. As $L^G \supseteq K$, this implies that $L^G = K$ by the tower law. □

Corollary 9.3. *Let L/K be a finite Galois extension. Then $L = K(x)$ for some x separable over K of degree $[L : K]$.*

Proof. By (ii) in the previous theorem, L/K is separable. So by the Primitive Element Theorem, $L = K(x)$ and the result follows. □

Theorem 9.4 ("The Galois correspondence")**.** *Let L/K be a finite Galois extension, $G = \mathrm{Gal}(L/K)$.*

(a) Let $F \subseteq L$ be a subfield with $F \supseteq K$. Then L/F is a Galois extension, $\mathrm{Gal}(L/F) \subseteq \mathrm{Gal}(L/K)$. The map $F \mapsto \mathrm{Gal}(L/F)$ is a bijection between $\{F \text{ field} : K \subseteq F \subseteq L\}$ and $\{\text{subgroups } H \text{ of } G\}$ whose inverse is the map taking H to the fixed field L^H. This bijection is inclusion-reversing and if $F = L^H$, $[F : K] = (G : H)$ (where $(G : H)$ denotes the index of the subgroup).

(b) Let $\sigma \in G$, $H \subseteq G$ a subgroup, $F = L^H$. Then $\sigma H \sigma^{-1}$ corresponds to σF.

(c) The following are equivalent (for a subgroup $H \subseteq G$)

- *(i) L^H/K is Galois*
- *(ii) L^H/K is normal*
- *(iii) For all $\sigma \in G$, $\sigma(L^H) = L^H$*
- *(iv) H is a normal subgroup of G*

If so, $\mathrm{Gal}(L^H/K) \cong G/H$.

Proof.

(a) Let $x \in L$. Then $m_{x,F}$ divides $m_{x,K}$ in $F[T]$. As $m_{x,K}$ splits into distinct linear factors in L, so does $m_{x,F}$. Hence L/F is normal and separable, hence is Galois. By definition $\mathrm{Gal}(L/F) \subseteq G$.

To check we have a bijection, with claimed inverse, note $F \mapsto H = \mathrm{Gal}(L/F) \mapsto L^H$. But $L^{\mathrm{Gal}(L/F)} = F$ as L/F is Galois, i.e $L^H = F$. Also $H \mapsto L^H \mapsto \mathrm{Gal}(L/L^H)$. It is enough to show $[L : L^H] \leq |H|$ since certainly $H \subseteq \mathrm{Gal}(L/L^H)$ and $|\mathrm{Gal}(L/L^H)| \leq [L : L^H]$. By Corollary 9.3, $L = L^H(x)$ for some x, and $f = \prod_{\sigma \in H}(T - \sigma(x)) \in L^H[T]$, with x a root. So $[L : L^H] = \deg_{L^H}(x) \leq \deg(f) = |H|$. So we have a bijection.

If $F \subseteq F'$, then $\mathrm{Gal}(L/F') \subseteq \mathrm{Gal}(L/F)$, so the bijection is inclusion-reversing. Finally if $F = L^H$ then

$$[F : K] = \frac{[L : K]}{[L : F]} = \frac{|\mathrm{Gal}(L/K)|}{|\mathrm{Gal}(L/F)|} = \frac{|G|}{|H|} = (G : H)$$

(b) Under (a), $\sigma H \sigma^{-1}$ corresponds to $L^{\sigma H \sigma^{-1}} = \{x \in L : \sigma\tau\sigma^{-1} = x, \ \forall \tau \in H\}$ and $\sigma\tau\sigma^{-1} = x$ if and only if $\tau\sigma^{-1}(x) = \sigma^{-1}(x)$, i.e $\tau(y) = y$ where $x = \sigma(y)$. So $x \in L^{\sigma H \sigma^{-1}}$ if and only if $x = \sigma(y)$ for $y \in L^H$, i.e $L^{\sigma H \sigma^{-1}} = \sigma F$.

(c) L/K is separable, so L^H/K is separable, so (i) is equivalent to (ii). Let $F = L^H$. Let $F = L^H$, $x \in F$. Then {roots of $m_{x,K}$} is the orbit of x under G. So $m_{x,K}$ splits in F if and only if $\forall \sigma \in G$, $\sigma(x) \in F$. As this must hold for all $x \in F$, F is normal if and only if $\sigma F \subseteq F$. As $[\sigma F : K] = [F : K]$ (K-isomorphic extensions), this means $\sigma F = F$. By (b), this is equivalent to: $\forall \sigma \in G$, $\sigma H \sigma^{-1} = H$, i.e H is a normal subgroup of G.

Last part: since $\forall \sigma \in G$, $\sigma F = F$, we have a homomorphism $G \to \mathrm{Gal}(F/K)$ given by restricting $\sigma \in G$ to F. This homomorphism has kernel H (since $F = L^H$). So $G/H \to \mathrm{Gal}(F/K)$ is an isomorphism.

□

Example. $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt[3]{2}, \omega) \subseteq \mathbb{C}$ where $\omega = \exp(2\pi i/3)$. Then L is a splitting field for $T^3 - 2$ and $[L : \mathbb{Q}] = 6$. So L/K is the splitting field of a separable polynomial, hence is Galois, and if $G = \mathrm{Gal}(L/K)$ then $|G| = 6$. Obvious subfields of L: $F_1 = \mathbb{Q}(\omega)$, $F_2 = \mathbb{Q}(\sqrt[3]{2})$. Then $[F_1 : \mathbb{Q}] = 2$ and $[F_2 : \mathbb{Q}] = 3$.

G must be isomorphic to either cyclic groups of order 6, or S_3. $F_2/\mathbb{Q}$ isn't normal, as $\omega\sqrt[3]{2} \notin F_2$. So $H_2 = \mathrm{Gal}(L/F_2)$ isn't a normal subgroup of G. So G is non-abelian and $G \cong S_3$, and $H_2 \cong \{(12), e\}$, H_1 must be $\cong A_3$. The other subgroups are $\{(13), e\}$ and $\{(23), e\}$ which are the conjugates of H_2. So the corresponding subfields are $\{\sigma F_2 : \sigma \in G\}$, which are $\mathbb{Q}(\omega\sqrt[3]{2})$, $\mathbb{Q}(\omega^2\sqrt[3]{2})$ (conjugates of $\sigma(\sqrt[3]{2})$ are the roots of the minimal polynomial). So this describes all F with $\mathbb{Q} \subseteq F \subseteq L$.

In fact, we could have seen at once that $G \cong S_3$: $f \in K[T]$ separable polynomial, $x_1, \dots, x_n$ roots in splitting field L. $G = \mathrm{Gal}(L/K)$ permutes $\{x_i\}$ as $f(\sigma x_i) = \sigma f(x_i) = 0$ and if $\sigma(x_1) = x_i$ for all i, then since $L = K(x_1, \dots, x_n)$, $\sigma = \mathrm{id}$. This gives a homomorphism $G \to S_n$ which is injective (where $n = \deg f$).

$$\begin{tikzcd}[column sep=small]
 & L \arrow[dl, dash] \arrow[dr, dash] & \\
F_1 \arrow[dr, dash] & & F_2 \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \{1\} \arrow[dl, dash] \arrow[dr, dash] & \\
H_1 \cong A_3 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\
 & G \cong S_3 &
\end{tikzcd}$$

Definition. The subgroup $\mathrm{Gal}(f/K) \subseteq S_n$ given by the image of G is the Galois group of f over K. Note that $[L : K] = |\mathrm{Gal}(L/K)| = |\mathrm{Gal}(f/K)|$ so divides $n!$.

There exist several methods for determining $\mathrm{Gal}(f/K)$.

Proposition 9.5. *A polynomial f is irreducible if and only if $\mathrm{Gal}(f/K)$ is transitive (recall that a subgroup $G \subseteq S_n$ is transitive if $\forall i, j \in \{1, \dots, n\}$, there exists $\sigma \in G$ with $\sigma(i) = j$, i.e there is only one orbit).*

Proof. Let x be a root of f in a splitting field L. Then its orbit under $G = \mathrm{Gal}(f/K)$ is the set of roots of $m_{x,K}$ (by 9.2). As $m_{x,K} \mid f$, have $m_{x,K} = f$ if and only if f is irreducible. And $m_{x,K} = f$ if and only if every root of f is in the orbit of x, i.e iff G acts transitively on the roots of f. $\square$

Remark: if $G \subseteq S_n$ is transitive, then by the orbit-stabiliser theorem, $n \mid |G|$.

Recall (from section 2) the discriminant: if $f \in K[T]$ is monic, $f = \prod_{1 \leq i \leq n}(T - x_i)$ in L (splitting field) then $\mathrm{Disc}(f) = \Delta^2 \in K$ where $\Delta = \prod_{1 \leq i \leq j \leq n}(x_i - x_j)$. $\mathrm{Disc}(f) \neq 0$ if and only if f is separable.

Proposition 9.6. *Assume $\mathrm{char}(K) \neq 2$. The fixed field of $G \cap A_n$ is $K(\Delta)$. In particular, $\mathrm{Gal}(f/K) \subseteq A_n$ if and only if $\mathrm{Disc}(f)$ is a square in K.*

Proof. If $\pi \in S_n$, the sign of π is an element of $\{\pm 1\}$, and

$$\prod_{1 \le i \le j \le n} (T_{\pi(i)} - T_{\pi(j)}) = \mathrm{sgn}(\pi) \prod_{1 \le i \le j \le n} (T_i - T_j)$$

So if $\sigma \in G$, then $\sigma(\Delta) = \mathrm{sgn}(\sigma)\Delta$. Since $\mathrm{char}(K) \neq 2$, $-1 \neq 1$ so as $\Delta \neq 0$, this implies $\Delta \in K$ if and ony if $G \subseteq A_n$ and Δ lies in the fixed field F of $G \cap A_n$. As

$$[F : K] = (G : G \cap A_n) = \begin{cases} 1 & \text{if } G \subseteq A_n \\ 2 & \text{otherwise} \end{cases}$$

we have $F = K(\Delta)$. □

Example. Let $f = T^3 + aT + b$, say $f = \prod_{i=1}^{3}(T - x_i)$, $x_3 = -x_1 - x_2$ and

$$\begin{aligned} a &= x_1x_2 - (x_1 + x_2)^2 \\ b &= x_1x_2(x_1 + x_2) \end{aligned}$$

So (plugging in) $\mathrm{Disc}(f) = -4a^3 - 27b^2$. So $\mathrm{Gal}(f/K) \subseteq A_3$ if and only if $-4a^3 - 27b^2$ is a square in K. Suppose $a = -21$, $b = -7$. Then $f \in \mathbb{Q}[T]$ is irreducible. We have $\mathrm{Disc}(f) = 4 \cdot 21^3 - 27 \cdot 7^2 = (27 \cdot 7)^2$. So $\mathrm{Gal}(f/\mathbb{Q}) \subseteq A_3$. As f is irreducible, the Galois group is transitive, so $\mathrm{Gal}(f/\mathbb{Q}) = A_3$. Thus this method computes the Galois group of any cubic polynomial (when $\mathrm{char}(K) \neq 2, 3$).

10 Finite fields

Let p be prime, and write $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We aim to describe all finite fields of characteristic p (i.e all finite extensions F of $\mathbb{F}_p$), and their Galois theory. Recall:

- $|F| = p^n$, where $n = [F : \mathbb{F}_p]$.
- $F^\times$ is cyclic of order $p^n - 1$.
- $\varphi_p : F \to F$, $x \mapsto x^p$ is an automorphism of F.

Theorem 10.1. *Let $n \geq 1$. Then there exists a field with $q = p^n$ elements. Any such field is a splitting field of the polynomial $T^q - T$ over $\mathbb{F}_p$. In particular, any two finite fields of the same order are isomorphic.*

Proof. Let F be a field with $q = p^n$ elements. Then if $x \in F^\times$, $x^{q-1} = 1$. So for all $x \in F$, $x^q = x$. So $f = T^q - T = \prod_{x \in F}(T - x)$ splits into linear factors in F, and not in any proper subfield of F. So F is a splitting field for f over $\mathbb{F}_p$. So by uniqueness of splitting fields, F is unique up to isomorphism.

To show the existence of such an F, given n, let $L/\mathbb{F}_p$ be a splitting field of $f = T^q - T$ where $q = p^n$. Let $F \subseteq L$ be the fixed field of $\varphi_p^n : x \mapsto x^q$. So F is the set of roots of f in L. So $|F| = q$ (and $F = L$). □

Notation: write $\mathbb{F}_q$ for any finite field with q elements (by the above theorem, any two such fields are isomorphic, although there is no canonical isomorphism).

Theorem 10.2. *$\mathbb{F}_{p^n}/\mathbb{F}_p$ is Galois with Galois group cyclic of order n, generated by φ_p.*

Proof. $T^{p^n} - T = \prod_{x \in \mathbb{F}_{p^n}} (T - x)$ is separable, so $\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_p$ (as the splitting field of a separable polynomial). Let $G \subseteq \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the subgroup generated by φ_p. Then $\mathbb{F}_{p^n}^G = \{x : x^p = x\} = \mathbb{F}_p$. So by the Galois correspondence $G = \mathrm{Gal}(\mathbb{F}_{p^n}, \mathbb{F}_p)$. $\square$

Theorem 10.3. *$\mathbb{F}_{p^n}$ has a unique subfield of order p^m for each $m \mid n$, and no others. If $m \mid n$ then $\mathbb{F}_{p^m} \subseteq \mathbb{F}_{p^n}$ is the fixed field of φ_p^m.*

Proof. $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$. The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are the $m\mathbb{Z}/n\mathbb{Z}$ for $m \mid n$, $m \geq 1$. So by Galois correspondence, the subfields of $\mathbb{F}_{p^n}$ are the fixed fields of these subgroups, i.e of the subgroups $\langle \varphi_p^n \rangle$, which have degree equal to the indices $(\mathbb{Z}/n\mathbb{Z} : m\mathbb{Z}/n\mathbb{Z}) = m$. $\square$

Remark: if $m \mid n$, then $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^m}) = \langle \varphi_p^m \rangle$.

Theorem 10.4. *Let $f \in \mathbb{F}_p[T]$ be separable of degree $n \geq 1$, whose irreducible factors have degrees $n_1, \ldots, n_r$, $\sum n_i = n$. Then $\mathrm{Gal}(f/\mathbb{F}_p) \subseteq S_n$ is cyclic, generated by an element of cycle type $(n_1, \ldots, n_r)$. In particular, $|\mathrm{Gal}(f/\mathbb{F}_p)|$ is equal to the lowest common multiple of $\{n_i\}$.*

Proof. Let L be a splitting field for f over $\mathbb{F}_p$, with roots $x_1, \ldots, x_n \in L$. Then $\mathrm{Gal}(L/F)$ is cyclic, generated by φ_p. As the irreducible factors of f are the minimal polynomials of the x_i's, and the set of roots of the minimal polynomial of x_i is the orbit of φ_p on x_i, the cycle type of φ_p is $(n_1, \ldots, n_r)$. The order of any such permutation is then $\mathrm{lcm}(n_1, \ldots, n_r)$. $\square$

Theorem 10.5 ("Reduction mod p"). *Let $f \in \mathbb{Z}[T]$ be a monic, separable polynomial, p prime, $n = \deg(f) \geq 1$. Suppose the reduction $\bar{f} \in \mathbb{F}_p[T]$ is also separable. Then $\mathrm{Gal}(\bar{f}/\mathbb{F}_p) \subseteq \mathrm{Gal}(f/\mathbb{Q})$, as subgroups of S_n.*

Corollary 10.6. *With the same assumptions as in the above theorem, suppose that $\bar{f} = g_1 \dots g_r$ where $g_i \in \mathbb{F}_p[T]$ are irreducible of degree n_i. Then $\mathrm{Gal}(f/\mathbb{Q})$ contains an element of cycle type $(n_1, \dots, n_r)$.*

Proof. Combine the previous two theorems. □

Example. $f = T^4 - 3T + 1$

- $p = 2$: $f = T^4 + T + 1 \pmod 2$ is irreducible (no roots and not divisible by $T^2 + T^1$ the only reducible quadratic)
- $p = 5$: $f = (T + 1)(T^3 - T^2 + T + 1) \pmod 5$ where the latter factor is irreducible

So by the Corollary, $\mathrm{Gal}(f/\mathbb{Q})$ contains a 4-cycle and a 3-cycle. So $12 \mid |G|$, so G is either S_4 or A_4. As 4-cycles are odd, G must be S_4.

Remark: if $\bar{f}$ is separable, then $\mathrm{Disc}(\bar{f}) \neq 0$, so $p \nmid \mathrm{Disc}(f)$ and f is separable. If f is separable, then $\bar{f}$ is separable for all but the finite set of primes p which divide $\mathrm{Disc}(f)$.

Remark: the identification of $\mathrm{Gal}(f/\mathbb{Q})$ with a subgroup of S_n depends on fixing an ordering of the roots. Taking a different ordering corresponds to conjugation of the Galois group $\mathrm{Gal}(f/\mathbb{Q})$ in S_n. So $\mathrm{Gal}(\bar{f}/\mathbb{F}_p) \subseteq \mathrm{Gal}(f/\mathbb{Q})$ really means that $\mathrm{Gal}(\bar{f}/\mathbb{F}_p)$ is conjugate to a subgroup of $\mathrm{Gal}(f/\mathbb{Q})$.

The following proof is ***non-examinable***:

Proof of Theorem 10.5. Let $L = \mathbb{Q}(x_1, \dots, x_n)$ be a splitting field of $f = \prod(T - x_i)$, with degree $N = [L : \mathbb{Q}]$. Consider $R = \mathbb{Z}[x_1, \dots, x_n]$. As $f(x_i) = 0$, f monic, so every element of R is a $\mathbb{Z}$-linear combination of $x_1^{a_1} \dots x_n^{a_n}$, $0 \leq a_i < n$. So R is finitely generated as an abelian group. As $R \subseteq L \cong \mathbb{Q}^N$, we must have $R \cong \mathbb{Z}^M$ for $M \leq N$. Then $\bar{R} = R/pR$ has p^M elements. Let $\bar{P}$ be a maximal ideal of $\bar{R}$, corresponding to an ideal P of R containing pR. Then $F = R/P \cong \bar{R}/\bar{P}$ is a finite field with p^d elements say.

$F = \mathbb{F}_p(\bar{x}_1, \dots, \bar{x}_n)$ where $\bar{x}_i = x_i + P \in F$ and $\bar{f} = \prod(T - \bar{x}_i)$. As $\bar{f}$ is separable, the $\bar{x}_i$ are distinct and F is a splitting field for $\bar{f}$. $G = \mathrm{Gal}(f/\mathbb{Q})$ takes R to itself (permutes x_i's). Let $H \subseteq G$ be the stabiliser of P, i.e $\{\sigma \in G : \sigma P = P\}$. THen H acts on $R/P = F$, permuting the $\bar{x}_i$'s in the same way as it permutes the x_i's. So we have an injective homomorphism $H \to \mathrm{Gal}(F/\mathbb{F}_p)$.

Now we just need to show this is an isomorphism. Let $\{P_1 = P, P_2, \dots, P_r\}$ be the orbit of P under G. The P_i are all maximal ideals, $R/P_i \cong R/P$ has p^d elements. As P_i are maximal, $P_i + P_j = R$ if $i \neq j$. So by the Chinese Remainder

Theorem, $R/(P_1 \cap \ldots \cap P_r) \cong R/P_1 \times \ldots \times R/P_r$. As $p \in P_i$, $pR \subseteq P_1 \cap \ldots \cap P_r$, so

$$p^N \geq p^M = |R/pR| \geq |R/(P_1 \cap \ldots \cap P_r)| = \prod_{i=1}^{r} |R/P_i| = p^{rd}$$

Orbit-Stabiliser Theorem implies $r = (G : H) = |G|/|H| = N/|H|$ and as $H \to \mathrm{Gal}(F/\mathbb{F}_p)$ is an injection, $H \leq d$ with equality if and only if the injection is an isomorphism. So $N = rd$, so combined with previous inequality, $N = rd$ so $H \cong \mathrm{Gal}(\bar{f}/\mathbb{F}_p)$. □

Remark: if $\mathrm{Gal}(f/\mathbb{Q})$ contains an element of cycle type $(n_1, \ldots, n_r)$ then it is a (hard) fact that there exist infinitely primes p such that $\bar{f}$ factors into irreducibles with degrees $n_1, \ldots, n_r$ ("Cebotarev density theorem" - generalisation of Dirichlet's theorem on primes in arithmetic progression).

11 Cyclotomic extensions

We will look at polynomials of the form $T^n - 1$ (later $T^n - a$).

Lemma 11.1. *Let C be a cyclic group of order $n > 1$ (written multiplicatively). If $a \in \mathbb{Z}$, $(a, n) = 1$, then the map $[a] : C \to C$, $[a](g) = g^a$ is an automorphism of C, and $(\mathbb{Z}/n\mathbb{Z})^\times \to Aut(C)$, $a \mapsto [a]$ is an isomorphism.*

Proof. Clearly $[a]$ is a homomorphism, and since $(a, n) = 1$, it is an automorphism as there exists b with $ab \equiv 1 \pmod{n}$. So have injective map $(\mathbb{Z}/n\mathbb{Z})\times \to \mathrm{Aut}(C)$ with $a \mapsto [a]$ which is obviously a homomorphism. If $\varphi \in \mathrm{Aut}(C)$ and g is a generator of C, then $\varphi(g) = g^a$ for some $a \in (\mathbb{Z}/n\mathbb{Z})^\times$, so $\varphi = [a]$. So we have an isomorphism. □

Let K be a field, $n \geq 1$. Define $\mu_n(K) = \{x \in K : x^n = 1\}$, the group of nth roots of unity in K. The group $\mu_n(K)$ is finite, hence cyclic (Proposition 3.5), hence of order dividing n. Say $\zeta \in \mu_n(K)$ is a primitive nth root of 1 if ζ has order n in $K^\times$.

Such a ζ exists if and only if $\mu_n(K)$ has n elements, in which case ζ is a generator. In particular, $f = T^n - 1$ has n distinct roots, so is separable.

In general, $f = T^n - 1$ is separable if and only if $(f, f') = 1$, and since $f' = nT^{n-1}$, this holds iff $n1_K \neq 0$.

Until the end of this section we assume $\mathrm{char}(K) > 0$ or $\mathrm{char}(K) = p > 0$ and $p \nmid n$, i.e $T^n - 1$ is separable.
Let L/K be a splitting field for $f = T^n - 1$, and $G = \mathrm{Gal}(L/K)$ (since f is separable, this is Galois). Then $|\mu_n(L)| = n$ and there exists a primitive nth root of 1, $\zeta = \zeta_n \in L$. L/K is called a *cyclotomic extension.*

Proposition 11.2.

(i) $L = K(\zeta)$

(ii) *There exists an injective homomorphism* $\chi = \chi_n : G \to (\mathbb{Z}/n\mathbb{Z})^\times$ *such that if* $\chi(\sigma) = a \pmod n$*, then* $\sigma(\zeta) = \zeta^a$*. In particular,* G *is abelian.*

(iii) χ *is an isomorphism if and only if* G *acts transitively on the set of primitive roots of unity in* L *(*χ *is called the cyclotomic character).*

Proof.

(i) $\mu_n(L) = \langle \zeta \rangle$ so the roots of $T^n - 1$ are the powers of ζ, so $L = K(\zeta)$.

(ii) Consider the action of G on L. It permutes $\mu_n(L)$ and if $\zeta, \zeta' \in \mu_n(L)$¡ $\sigma \in G$, then $\sigma(\zeta\zeta') = \sigma(\zeta)\sigma(\zeta')$. So σ acts as an automorphism of $\mu_n(L)$, and $\sigma(\zeta_n) = \zeta_n$ if and only if $\sigma = \mathrm{id}$ (as $L = K(\zeta_n)$). So we have an injective homomorphism $G \to \mathrm{Aut}(\mu_n(L))$ and $\mathrm{Aut}(\mu_n(L)) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ by lemma 11.1.

(iii) ζ_n^a is primitive if and only if $(a, n) = 1$. So the set of primitive nth roots of 1 is $\{\zeta^a : a \in (\mathbb{Z}/n\mathbb{Z})^\times\}$, which by (ii) is the orbit of ζ under G. So χ is surjective iff there is one orbit.

$\square$

Example: $K = \mathbb{Q}$, can take $L = \mathbb{Q}(e^{2\pi i/n})$. What is the minimal polynomial of $e^{2\pi i/n}$?

Definition. (K satisfying earlier hypothesis) The *nth cyclotomic polynomial* is

$$\Phi_n(T) = \prod_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} (T - \zeta_n^a)$$

which has roots precisely the set of primitive nth roots of 1 in the splitting field L of $T^n - 1$. As G permutes the primitive nth roots of 1 in L, $\Phi_n \in L^G[T] = K[T]$. Also χ is surjective if and only if $\Phi_n \in K[T]$ is irreducible.

Φ_n doesnt really depend on K. In fact, $x \in L$ satisfies $x^n = 1$ if and only if x is a primitive dth root of unity for some (unique) $d \mid n$. So $T^n - 1 = \prod_{d|n} \Phi_d$.

So $\Phi_n = (T^n - 1)/\prod_{\substack{d|n \\ d\neq n}} \Phi_d$, giving an induction definition of Φ_n, and showing that Φ_n is the image of in $K[T]$ of a polynomial in $\mathbb{Z}[T]$ which doesn't depend on K. e.g $\Phi_p = (T^p - 1)/(T - 1) = T^{p-1} + \ldots + T + 1$. $\Phi_1 = T - 1$ and $\Phi_{p^n} = (T^{p^n} - 1)/(T^{p^{n-1}} - 1) = \Phi_p(T^{p^{n-1}})$.

Have $\deg(\Phi_n) = |(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n)$, where φ is the Euler φ-function.

We have 2 special cases

Theorem 11.3 ("Irreducibility of cyclotomic polynomials")**.** *Let $K = \mathbb{Q}$. Then χ_n is an isomorphism for $n > 1$. In particular, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$, and Φ_n is irreducible over $\mathbb{Q}$.*

Proof. By 11.2, all of these statements are equivalent. So it suffices to show Φ_n is irreducible over $\mathbb{Q}$. If n is prime (or a prime power) can prove this using Eisenstein, but this doesn't work for general n.

χ_n is an isomorphism if for all primes p with $(p, n) = 1$, the class of $p \in (\mathbb{Z}/n\mathbb{Z})^\times$ lies in the image of χ (factor a with $(a, n) = 1$ as a product of primes).

Let f be the minimal polynomial of ζ over $\mathbb{Q}$. Let G be the minimal polynomial of ζ^p over $\mathbb{Q}$. If $f = g$ then ζ^p lies in the orbit of G on ζ, i.e p lies in the image of χ, so we're done.

If $f \neq g$, then $(f, g) = 1$ and $f, g \mid T^n - 1$, so $fg \mid T^n - 1$. As ζ is a root of $g(T^p)$, $f \mid g(T^p)$. Reduce modulo p to get $\bar{f} \in \mathbb{F}_p[T]$ and $\overline{g(T^p)} = \bar{g}(T^p)$ (as we're in characteristic p) and as both $\bar{f}$ and $\bar{g}$ divide $T^n - 1 \in \mathbb{F}_p[T]$, which is separable (as $p \nmid n$), which implies $\bar{f} \mid \bar{g}$, hence $\bar{f}^2 \mid \bar{f}\bar{g} \mid T^n - 1$, contradicting seperability of $\bar{f}$. □

So the minimal polynomial of $e^{2\pi i/n}$ over $\mathbb{Q}$ is $\Phi_n(T)$.

Now we look at $K = \mathbb{F}_p$.

Proposition 11.4. *$K = \mathbb{F}_p$, $(n, p) = 1$. Then*

(i) $\chi_n : G \to \langle p \rangle \subseteq (\mathbb{Z}/n\mathbb{Z})^\times$ *is an isomorphism. Also* $\chi_n(\varphi_p) = p \pmod n$

(ii) $[L : K] = r$, *the order of p modulo n.*

(iii) φ_p *has cycle type* $(r, \dots, r)$ *as a permutation of the roots of* Φ_n *(the primitive nth roots of unity in L)*

[Recall $\varphi_p \in G$ is $x \mapsto x^p$ (Frobenius map), and φ_p generates G]

Proof. $\varphi_p(\zeta) = \zeta^p$ and $L = K(\zeta)$ so $\chi_n(\varphi_p) = p$, hence $\chi_n(G) = \langle p \rangle$ and $[L : K] = |G| = |\langle p \rangle|$ which is the order of p modulo n.

If $(a, n) = 1$, $\varphi_p^k(\zeta^a) = \zeta^a \iff \varphi_p^k(\zeta) = \zeta \iff r \mid k$. So the orbits of φ_p on $\{\zeta^a : (a, n) = 1\}$ (the set of roots of Φ_n) all have length r. $\square$

Remarks:

1. This 'almost' gives another proof of the irreducibility of Φ_n over $\mathbb{Q}$. By the theorem about reduction mod p, $\mathrm{Gal}(\Phi_n/\mathbb{Q}) \supseteq \mathrm{Gal}(\Phi_n/\mathbb{F}_p)$ as subgroups (up to conjugacy) of the symmetric group $S_{\varphi(n)}$. It is not hard to show that $\chi_n(\mathrm{Gal}(\Phi_n/\mathbb{Q})) \supseteq \chi_n(\mathrm{Gal}(\Phi_n/\mathbb{F}_p)) = \langle p \rangle$. So letting $p \nmid n$ vary, we have $\mathrm{Gal}(\Phi_n/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^\times$.

2. (iii) implies that the factorisation of Φ_n over $\mathbb{F}_p$ is a product of irreducibles of degree r, which depends only on $p \pmod n$. For a general polynomial $f \in \mathbb{Z}[T]$, the factorisation of f modulo p doesn't follow any obvious pattern. Trying to answer this question is part of the "Langlands Programme"; the case when there is a pattern is a (congruence) pattern is when $\mathrm{Gal}(f/\mathbb{Q})$ is abelian, ("Class Field Theory").

Application 1

Quadratic reciprocity. Recall: for p odd prime, $a \in \mathbb{Z}$, $(a, p) = 1$, Legendre symbol

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square mod } p \\ -1 & \text{if not} \end{cases}$$

Euler's formula: $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod p$. Let $q \neq p$ another odd prime, let $n = q$ in the above so $L = K(\zeta_q)$, splitting field for $f = T^q - 1 = (T-1)\Phi_q$. So on the roots of f in L, Frobenius map φ_p has cycle type $(1, r, \dots, r)$. and there are $(q-1)/r$ r-cyles. So its sign is $\mathrm{sgn}(\varphi_p) = (-1)^{(r-1)(q-1)/r} = (-1)^{(q-1)/r}$ (q is odd). Also $2 \mid \frac{q-1}{r} \iff r \mid \frac{q-1}{2} \iff p^{\frac{q-1}{2}} \equiv 1 \pmod q$ (as r is the order of p mod q). So $\mathrm{sgn}(\varphi_p) = \left(\frac{p}{q}\right)$ by Euler's formula.

As $G = \langle \varphi_p \rangle$, $\mathrm{sgn}(\varphi_p) = 1 \iff G \subseteq A_q$. This holds iff $\mathrm{Disc}(f)$ is a square in $\mathbb{F}_p$.

Lemma 11.5. *Let $f = \prod(T - x_i)$ over any field. Then* $\mathrm{Disc}(f) = (-1)^{d(d-1)/2} \prod f'(x_i)$, $d = \deg(f)$.

Proof. Example Sheet 3. □

Continuing with the previous, $f = T^q - 1 = \prod_{a=0}^{q-1}(T - \zeta_q^a)$, $f' = qT^{q-1}$. So

$$\mathrm{Disc}(f) = (-1)^{q(q-1)/2} \prod_{a=0}^{q-1} q\zeta_q^{a(q-1)} = (-1)^{(q-1)/2} q^q \zeta_q^{(q-1)q(q-1)/2} = (-1)^{\frac{q-1}{2}} q^q$$

So

$$\left(\frac{p}{q}\right) = \left(\frac{\mathrm{Disc}(f)}{p}\right) = \left(\frac{(-1)^{(q-1)/2} q}{p}\right) = \left(\frac{q}{p}\right) (-1)^{(p-1)(q-1)/4}$$

Since q is odd and $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$. So we have obtained the quadratic reciprocity law.

Application 2

Construction of regular polygons. Ruler-and-compass construction of regular n-gon, $n \geq 3$ is equivalent to constructing the real number $\cos(2\pi/n)$.

Theorem 11.6 (Gauss)**.** *A regular n-gon is constructable iff n is a power of* 2 *times a product of distinct primes, each of which are of the form* $2^{2^k} + 1$.

Remark: when is $2^{2^k} + 1 = F_k$ (Fermat numbers) prime? $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$ are all prime. Fermat conjectured that all F_k are prime. However, we have the following result of Euler

Theorem 11.7 (Euler, 1732)**.** $F_5 = 641 \times 6700417$.

Since then, many F_k's are known to be composite, none have been seen to be prime for $k \geq 5$.

Lemma 11.8. *If m is a positive integer such that $2^m + 1$ is prime, then m is a power of 2.*

Proof. $2^{qr} + 1 = (2^r + 1)(2^{qr-r} - 2^{qr-2r} + \ldots + 1)$ if q is odd. □

Proof of Theorem 11.6. Recall $x \in \mathbb{R}$ is constructible if and only if there exists a sequence of fields $\mathbb{Q} = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m \ni x$ and $[K_{i+1} : K_i] = 2$ for all $0 \le i \le m-1$. In particular, a necessary condition is that $\deg_{\mathbb{Q}}(x)$ is a power of 2.

In our case, $x = \cos(2\pi/n) = \frac{1}{2}\left(\zeta_n + \zeta^{-1}\right)$, $\zeta_n = e^{2\pi i/n}$ so $\zeta_n^2 - 2x\zeta_n + 1 = 0$. Also $x \in \mathbb{R}$, $\zeta_n \notin \mathbb{R}$ $(n \ge 3)$ so $[\mathbb{Q}(\zeta_n) : \mathbb{Q}(x)] = 2$. So if x is constructible, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]$ is a power of 2. But $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n) = \prod_i p_i^{e_i - 1}(p-1)$, where $n = \prod_{i=1}^r p_i e^i$. So this is a power of 2 if and only if for all odd p_i, $e_i = 1$ and $p_i - 1$ is a power of 2. By the above lemma, $\varphi(n)$ is a power of 2 if and only if it is of the required form.

Now we show the other direction. Suppose $\varphi(n) = 2^m$. Then $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois, with Galois group $G \cong (\mathbb{Z}/n\mathbb{Z})^\times$, $|G| = 2^m$.

Observe that there exist subgroups $G = H_0 \supseteq H_1 \supseteq \ldots \supseteq H_m = \{1\}$ such that $[H_i : H_{i+1}] = 2$ for all $0 \le i \le m-1$. Indeed since $2 \mid |G|$ (assuming $G \neq \{1\}$), there exists $\sigma \in G$ of order 2 (Cauchy's theorem). Hence take $H_{m-1} = \langle \sigma \rangle$, and repeat on G/H_{m-1} and continue to construct all the H_i's. Then $K_i = \mathbb{Q}(\zeta_n)^{H_i}$ satisfy $[K_{i+1} : K_i] = (H_i : H_{i+1}) = 2$. □

12 Kummer extensions

We consider extensions of the form $L = K(x)$, $x^n = a \in K$ (not necessarily $a = 1$). These extensions are not necessarily Galois, e.g $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$.

First we prove a result of independent interest:

Theorem 12.1 (Linear independence of field embeddings)**.** *Let K, L be fields, $\sigma_1, \ldots, \sigma_n : K \to L$ distinct field homomorphisms, $n \ge 1$. Then if $y_1, \ldots, y_n \in L$ are such that for all $x \in K$, $y_n\sigma_1(x) + \ldots + y_n\sigma_n(x) = 0$ then $y_1 = \ldots = y_n = 0$. i.e $\sigma_1, \ldots, \sigma_n$ are L-linearly independent elements of the set of functions $K \to L$, which is an L-vector space.*

This is a special case ($G = K^\times$) of

Theorem 12.2 (Linear independence of characters)**.** *Let G be a group, L a field, and $\sigma_1, \ldots, \sigma_n : G \to L^\times$ distinct group homomorphisms. Then $\sigma_1, \ldots, \sigma_n$ are linearly independent over L.*

Proof. Induction on n: if $n = 1$ the result is obvious. Suppose $n > 1$, $y_1, \ldots, y_n \in L$ such that for all $g \in G$

$$y_1\sigma_1(g) + \ldots + y_n\sigma_n(g) = 0 \qquad (*)$$

Then there exists $h \in G$ such that $\sigma_1(h) \neq \sigma_n(h)$ (since the σ_i are distinct). As the σ_i are homomorphisms, putting hg into $(*)$, we get

$$y_1\sigma_1(h)\sigma_1(g) + \ldots + y_n\sigma_n(h)\sigma_n(g) = 0$$

Multiplying $(*)$ by $\sigma_n(h)$ and subtracting from the above

$$y_1'\sigma_1(g) + \ldots + y_{n-1}'\sigma_{n-1}(g) = 0$$

where $y_i' = y_i(\sigma_i(h) - \sigma_n(h))$. Hence by induction $y_i' = 0$ for all i, and since $\sigma_1(h) \neq \sigma_n(h)$, $y_1 = 0$. Now $(*)$ becomes a linear dependence between $\sigma_2, \ldots, \sigma_n$, hence $y_2 = y_3 = \ldots = y_n = 0$ by induction. □

Assume that $n > 1$, $n1_K \neq 0$.

Theorem 12.3. *Assume that K contains a primitive nth root of 1, $\zeta = \zeta_n$. Suppose L/K is an extension with $L = K(x)$ and $x^n = a \in K^\times$. Then*

(i) L/K is a splitting field for $f = T^n - a$, and is Galois with cyclic Galois group.

(ii) $[L : K]$ is the least $m \geq 1$ such that $x^m \in K$.

Proof.

(i) As $\mu_n(K) = \{\zeta_n^i : 0 \leq i < n\}$, has n elements, the polynomial f has n distinct roots $\{\zeta^i x\}$ in L, so L/K is a splitting field for the separable polynomial f, thus is Galois. Let $\sigma \in \mathrm{Gal}(L/K) = G$. Then $f(\sigma(x)) = 0$, so $\sigma(x) = \zeta^i x$ for some i, which is unique mod n. Define

$$\Theta : G \to \mu_n(K) = \{\zeta^i\} \cong \mathbb{Z}/n\mathbb{Z}$$

by $\Theta(\sigma) = \frac{\sigma(x)}{x}$ (which must be equal to some ζ^i). We claim this is a homomorphism: let $\sigma, \tau \in G$, then because $\zeta \in K$, $\tau(\Theta(\sigma)) = \Theta(\sigma)$, so

$$\Theta(\tau\sigma) = \frac{\tau\sigma(x)}{x} = \tau\left(\frac{\sigma(x)}{x}\right) \cdot \frac{\tau(x)}{x} = \tau(\Theta(\sigma)) \cdot \Theta(\tau) = \Theta(\sigma)\Theta(\tau)$$

So Θ is a homomorphism, and injective since $\Theta(\sigma) = 1$ if and only if $\sigma(x) = x$, i.e $\sigma = \mathrm{id}$. So G is isomorphic to a subgroup of a cyclic group, thus is cyclic.

(ii) If $m > 1$, since L/K is Galois, $x^m \in K$ if and only if for all $\sigma \in G$, $\sigma(x^m) = x^m$. This is the same as: for all $\sigma \in G$, $\Theta(\sigma)^m = 1$, i.e $|G| = [L : K] \mid m$.

□

Corollary 12.4. *Assume K contains a primitive nth root of unity, let $a \in K^\times$. Then $f = T^n - a$ is irreducible in $K[T]$ iff a is not a dth power in K, for any $1 \neq d \mid n$.*

Proof. Let $L = K(x)$ for $x^n = a$. Then minimal polynomial of x divides f, so f is irreducible if and only if $G = [L : K] = n$.

Suppose $n = md$, $d \neq 1$. Then a is a dth power in K if and only if $x^m \in K$ (as $\zeta_n \in K$), which by the previous theorem holds iff $|G| \mid m$. $\square$

Remark: this doesn't hold in general if $\zeta_n \notin K$. For example, take $K = \mathbb{Q}$, $T^4 + 4$.

Terminology: extensions of the form $L = K(\sqrt[n]{a})$, where $\zeta_n \in K$, are called *Kummer extensions.*

Example. $n = 2$, $\mathrm{char}(K) \neq 2$, $\zeta_2 = -1 \in K$. $K(\sqrt{a})/K$ is quadratic if $a \notin (K^\times)^2$. Conversely (Example sheet 1), every quadratic L/K is $L = K(\sqrt{a})$ for some a.

For general n:

Theorem 12.5. *Suppose K contains a primitive nth root of unity, where $n > 1$. Let L/K be a Galois extension, with $\mathrm{Gal}(L/K)$ cyclic of order n. Then $L = K(\sqrt[a]{n})$ for some $a \in K$.*

Proof. Let $G = \mathrm{Gal}(L/K) = \{\sigma^i : 0 \leq i < n\}$. For $y \in L$, let

$$x = R(y) = y + \zeta^{-1}\sigma(y) + \ldots + \zeta^{-(n-1)}\sigma^{n-1}(y) = \sum_{j=0}^{n-1} \zeta^{-j}\sigma^j(y)$$

(Lagrange resolvent). Then $\sigma(x) = \zeta x$ (since $\sigma^n = \mathrm{id}$). Hence $\sigma(x^n) = x^n$, i.e $x^n \in K$. By Theorem 12.1 (linear independence of field homomorphisms), there exists y such that $x = R(y) \neq 0$. As $\sigma^i(x) = \zeta^i x$, the $\sigma^i(x)$ are distinct. Hence $\deg_K(x) = n$ and so $L = K(x)$. $\square$

Example: suppose $L/\mathbb{Q}$ has degree 3 and is Galois. Then as $\zeta_3 \notin \mathbb{Q}$, this isn't a Kummer extension.

13 Trace and norm

Let L/K be an extension of degree n (so L is a n-dimensional K-vector space). Let $x \in L$. The map $u_x : L \to L$, $y \mapsto xy$ is obviously K-linear (since it's L-linear). So it has characteristic polynomial, determinant and trace.

Definition. The *trace* and *norm* of x (relative to L/K) are $\mathrm{Tr}_{L/K}(x) = \mathrm{tr}(U_x)$, $N_{L/K}(x) = \det U_x$ (sometimes write tr_K, $\det_K$ if it's important to know K). The *characteristic polynomial of* x is $f_{x,L/K}(T) = \det(T\,\mathrm{id} - u_x)$.

Explicitly, let $(e_i)_{i=1,\ldots,n}$ be a basis for L/K. Then there exists a unique matrix $A = (a_{ij})$ such that $xe_i = \sum_j a_{ji}e_j$. Then $\mathrm{Tr}_{L/K}(x) = \mathrm{tr}(A)$ etc.

Example. Consider $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, a quadratic extension, basis $\{1, \sqrt{d}\}$ (d not a square in $\mathbb{Q}$). Let $x = a + b\sqrt{d}$. Then $A = \begin{pmatrix} a & bd \\ b & a \end{pmatrix}$ as $x \cdot 1 = a \cdot 1 + b\sqrt{d}$ and $x\sqrt{d} = bd \cdot 1 + a\sqrt{d}$. So $\mathrm{Tr}_{L/K}(x) = 2a$, $N_{L/K}(x) = a^2 - b^2 d$.

Example. $\mathbb{C}/\mathbb{R}$, basis $\{1, i\}$. Then matrix of u_{x+iy} is $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ (usual representation of complex numbers by 2×2 real matrices - Cauchy-Riemann).

Lemma 13.1. *Let $x, y \in L$, $a \in K$, $n = [L : K]$. Then*

(i) $\mathrm{Tr}_{L/K}(x+y) = \mathrm{Tr}_{L/K}(x) + \mathrm{Tr}_{L/K}(y)$ *and* $N_{L/K}(xy) = N_{L/K}(x)N_{L/K}(y)$

(ii) $N_{L/K}(x) = 0$ *if and only if* $x = 0$.

(iii) $\mathrm{Tr}_{L/K}(1) = n$, $N_{L/K}(1) = 1$

(iv) $\mathrm{Tr}_{L/K}(ax) = a\,\mathrm{Tr}_{L/K}(x)$, $N_{L/K}(ax) = a^n N_{L/K}(x)$

So $\mathrm{Tr}_{L/K}$ *is* K*-linear,* $N_{L/K} : L^\times \to K^\times$ *is a homomorphism.*

Proof. Trivial. □

Theorem 13.2. *Let $M/L/K$ be finite extensions. Then for all $x \in M$, $\mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(x)) = \mathrm{Tr}_{M/K}(x)$ and $N_{L/K}(N_{M/L}(x)) = N_{M/K}(x)$.*

Proof. We only prove this for the trace, as this is all we'll use. Let $x \in M$. Choose bases $u_1, \dots, u_m$ for M/L and $v_1, \dots, v_n$ for L/K. Let (a_{ij}) be the matrix of $u_{x,M/L}$ with entries in L. Then $\mathrm{Tr}_{M/L}(x) = \sum_{i=1}^m a_{ii}$. For each (i,j) let the matrix of $U_{a_{ij},L/K}$ be A_{ij} (entries in K). Then $\mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(x)) = \sum_{i=1}^m \mathrm{Tr}_{L/K}(a_i i) = \sum_{i=1}^m \mathrm{tr}(A_{ii})$.

Now we consider the basis $u_1v_1, \dots, u_1v_n, \dots, u_mv_m$ for M/K. Then the matrix of $u_{x,M/K}$ is the block matrix

$$\begin{pmatrix} A_{11} & A_{12} & \dots & \\ A_{21} & A_{22} & \dots & \\ \vdots & \vdots & \ddots & \\ & & & A_{nn} \end{pmatrix}$$

so $\mathrm{Tr}_{M/K}(x) = \sum_{1 \le i \le n} \mathrm{tr}(A_{ii})$. ◻

Proposition 13.3. *Suppose $L = K(x)$, and let $f = T^n + c_{n-1}T^{n-1} + \dots + c_0 \in K[T]$ be the minimal polynomial of x/K. Then $f_{x,L/K} = f$ and $\mathrm{Tr}_{L/K}(x) = -c_{n-1}$, $N_{L/K}(x) = (-1)^n c_0$.*

Proof. It is enough to prove the first statement since the other properties are immediate from basic linear algebra. In terms of the basis $1, x, \dots, x^{n-1}$ for L/K, the matrix of u_x is

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{pmatrix}$$

Since $u_x(x^i) = x^{i+1}$ for all $0 \le i \le n-2$ and $u_x(x^{n-1}) = x^n = -\sum_j c_j x^j$. This matrix has characteristic polynomial f. ◻

Corollary 13.4. *Assume $\mathrm{char}(K) = p > 0$, $L = K(x)$, where $x \notin K$, $x^p \in K$. Then for all $y \in L$, $\mathrm{Tr}_{L/K}(y) = 0$ and $N_{L/K}(y) = y^p$.*

Proof. Recall that $[L : K] = p$ (Example sheet 2, Q7). If $y \in K$, then $\mathrm{Tr}_{L/K}(y) = py = 0$ and $N_{L/K}(y) = y^p$. Otherwise since $[L : K]$ is prime, $L = K(y)$, and if $y = \sum a_i x^i$ for $a_i \in K$, then $y^p = \sum a_i^p (x^p)^i \in K$. So the minimal polynomial of y is $T^p - y^p$. So done by 13.3. ◻

Proposition 13.5. *Let L/K be a finite separable extension of degree n. Let $\sigma_1, \dots, \sigma_n : L \to M$ be the distinct K-homomorphisms into a normal closure M for L/K. Then $\mathrm{Tr}_{L/K}(x) = \sum_{i=1}^n \sigma_i(x)$, $N_{L/K}(x) = \prod_{i=1}^n \sigma_i(x)$, and $f_{x,L/K} = \prod_{i=1}^n (T - \sigma_i(x))$.*

Remark: if L/K is a finite Galois extension, then $\mathrm{Tr}_{L/K}(x) = \sum_{\sigma \in G} \sigma(x)$ etc.

Proof. Again it is enough to show that $f_{x,L/K}$ is of that form. Let (e_i) be a basis for L/K, and let $P = (\sigma_i(e_j))_{ij} \in \mathrm{Mat}_{n\times n}(M)$. Recall that the σ_i are linearly independent (12.1), so there doesn't exist any $y_1, \ldots, y_n \in M$ such that $\sum_i u_i \sigma_i(e_j) = 0$ for all j. So P is non-singular.

Let $A = (a_{ij})$ be a matrix of u_x. Then $xe_j = \sum_r a_{rj} e_r$ hence

$$\sigma_i(x)\sigma_i(e_j) = \sum_r \sigma_i(e_r) a_{rj} \tag{$*$}$$

for all i, j. So if S is the diagonal matrix with (i,i) entry $\sigma_i(x)$, then $(*)$ says that $SP = PA$, so $S = PAP^{-1}$ as P was invertible. Therefore S, A are conjugates and have the same characteristic polynomial. S has characteristic polynomial $\prod_{i=1}^n (T - \sigma_i(x))$ and A has characteristic polynomial $f_{x,L/K}$. □

For the next theorem, note first that since $\mathrm{Tr}_{L/K} : L \to K$ is K-linear, it is either surjective or the zero map.

Theorem 13.6. *Let L/K be a finite extension. Then L/K is separable if and only if $\mathrm{Tr}_{L/K}$ is surjective.*

Remark: if $\mathrm{char}(K) = 0$, it is obvious that $\mathrm{Tr}_{L/K}(1) = n \neq 0$ so result is easy in this case.

Proof. Suppose L/K is separable and $\{\sigma_1, \ldots, \sigma_n\} = \mathrm{Hom}_K(L, M)$ for a normal closure M of L/K. Then $\mathrm{Tr}_{L/K}(x) = \sum \sigma_i(x)$. By linear independence of the σ_i, there exists x such that $\mathrm{Tr}_{L/K}(x) \neq 0$. Hence $\mathrm{Tr}_{L/K}$ is surjective.

Conversely, suppose L/K is inseparable, then there exists $x \in L$ such that $K(x) \supsetneq K(x^p)$ (Example sheet 2, Q7). By Corollary 13.4, $\mathrm{Tr}_{K(x)/K(x^p)} = 0$, so by 13.2, $\mathrm{Tr}_{M/K} = 0$. □

Example. Finite fields $\mathbb{F}_{q^n}/\mathbb{F}_q$ $(q = p^n)$ - this is separable, so there exists $x \in \mathbb{F}_{q^n}$ such that $\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x) = 1$ (exercise: prove this directly using the fact that the multiplicative group is cyclic).

Remark: can use this criterion (13.6) to give another proof that if M/L and L/K are separable, then so is M/K.

14 Algebraic closure

Definition. A field K is *algebraically closed* if every non-constant polynomial in $K[T]$ splits into linear factors over K. Equivalently, the only irreducibles in $K[T]$ are the linear polynomials.

Example. $\mathbb{C}$ is algebraically closed (FTA).

Proposition 14.1. *The following conditions on a field K are equivalent*

1. *K is algebraically closed*
2. *if L/K is an extension, and $x \in L$ is algebraic over K, then $x \in K$*
3. *if L/K is algebraic, then $L = K$*

Proof. We first show (i) implies (ii): let L, x be as in (ii). Let f be the minimal polynomial of x/K. Then f is linear, so $x \in K$.

Now show (ii) implies (iii): by definition of algebraic, all $x \in L$ are algebraic over K, so $x \in K$.

Now show (iii) implies (i): let $f \in K[T]$ be irreducible, $L = K[T]/(f)$, which is algebraic over K. Then (iii) implies $L = K$ so f is linear. □

Proposition 14.2. *Let L/K be algebraic, and suppose that every irreducible polynomial $f \in K[T]$ splits into linear factors in $L[T]$. Then L is algebraically closed (such a field is called an algebraic closure of K).*

Proof. Let M/L be an extension, $x \in M$ algebraic over L (so by 4.4) x is algebraic over K; by hypothesis $m_{x,K} \in K[T]$ splits into linear factors over L. So $x \in L$ hence by (ii) in the previous proposition, L is algebraically closed. □

Corollary 14.3. *The field $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ of all algebraic numbers is algebraically closed (and is an algebraic closure of $\mathbb{Q}$).*

Proof. We apply 14.2 to $\overline{\mathbb{Q}}/\mathbb{Q}$; this extension is algebraic and if $f \in \mathbb{Q}[T]$ is irreducible, then by the FTA, $f = \prod(T - x_i) \in C[T]$, and $x_i \in \overline{\mathbb{Q}}$ (by definition of $\overline{\mathbb{Q}}$). So the hypotheses of 14.2 hold. □

By proposition 14.2, an algebraic closure of K is the same as an algebraic extension of K which is algebraically closed. The main results are:

- every field has an algebraic closure
- it is unique up to isomorphism

The main difficulty in proving these are set-theoretic. "Morally" we are trying to find "a splitting field for all polynomials over K".

Special case: K countable - then $K[T]$ is also countable. Enumerate the monic irreducibles $\{f_i : i \geq 1\}$ in $K[T]$. Let $L_0 = K$, and for each $i \geq 1$, let L_i be a splitting field for f_i over L_{i-1} [it is possible to do this without the axiom of choice]. We may assume that $L_{i-1} \subseteq L_i$ is a subfield (if $\sigma : L_{i-1} \to L_i$ is an extension, replace L_i with $L_{i-1} \cup (L_i \setminus \sigma(L_{i-1}))$). Let $L = \bigcup_{i \geq 0} L_i$. Then every f_i splits in $L[T]$ so L is an algebraic closure of K.

Example. $\mathbb{F}_p$ has an algebraic closure.

For general (uncountable) fields, we need some set-theoretic "trick".

Lemma (Zorn's Lemma)**.** Let S be a non-empty partially ordered set. Suppose that every chain in S has an upper bound in S. Then S has a maximal element.

Definition. A binary relation $\leq$ on a set S is a *partial order* if $\forall x, y, z \in S$

(i) $x \leq x$

(ii) $x \leq y$ and $y \leq z \implies x \leq z$

(iii) $x \leq y$ and $y \leq x \implies x = y$

$(S, \leq)$ is a *partially ordered set* or *poset*. It is *totally ordered* if moreover

(iv) $\forall x, y \in S$, either $x \leq y$ or $y \leq x$

Definition. A *chain* in a poset $(S, \leq)$ is a subset $T \subseteq S$ which under $\leq$ is totally ordered. An *upper bound* for $T \subseteq S$ is an element $z \in S$ such that $\forall x \in T$, $x \leq z$. An element $y \in S$ is *maximal* if $\forall x \in S$, $y \leq x \implies y = x$.

If S is totally ordered, it is easy to see it has at most one maximal element.

Example. Let V be a vector space (over a field K). Then V has a basis.

Indeed if $V = \{0\}$ this is trivial. Otherwise let $S = \{X \subseteq V : X \text{ linearly independent}\}$. Define a partial order on S by inclusion: $X \leq X'$ if $X \subseteq X'$. S is non-empty so let $T \subseteq S$ be a chain and define $Y = \bigcup_{X \in T} X$. We need to check $Y \in S$. It is enough to check any finite subset $\{y_1, \ldots, y_k\}$ of Y is linearly independent, say $y_i \in X_i$ for $X_i \in T$. Since T is a chain, we may assume $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_k$ and therefore $\{y_1, \ldots, y_k\} \subseteq X_k$ and thus are linearly independent.

Proposition 14.4. *Let L/K be an algebraic extension and M an algebraically closed field, $\sigma : K \to M$ a homomorphism. Then there exists $\bar{\sigma} : L \to M$ extending σ.*

Proof. If $L = K(x)$, x algebraic over K with minimal polynomial $f = m_{x,K}$, then $\sigma f \in M[T]$ splits into linear factors. So there exists $\bar{\sigma} : K(x) \to M$ extending σ (in fact there is one homomorphism for each root of σf in M).

Now we prove it for general L. Assume $K \subseteq L$ is a subfield (if not consider image of K in the extension). Let

$$S = \{(F, \tau) : K \subseteq F \subseteq L,\ \tau : F \to M \text{ a homomorphism with } \tau|_K = \sigma\}$$

Write $(F, \tau) \leq (F', \tau')$ if $F \subseteq F'$ and $\tau'|_F = \tau$. Then $(S, \leq)$ is a poset, $(K, \sigma) \in S$ so $S \neq \emptyset$. Consider a chain $T = \{(F_i, \tau_i)\}_{i \in I} \subseteq S$ (I some index set). Let $F' = \bigcup_{i \in I} F_i$, since T is a chain, $\forall i, j$ either $F_i \subseteq F_j$ or $F_j \subseteq F_i$. So if $x \in F_i$, $y \in F_j$, then (assuming $F_i \subseteq F_j$) $xy, x + y \in F_j \subseteq F'$, so F' is a field. Define $\tau' : F' \to M$ by $\tau'(x) = \tau_i(x)$ for $x \in F_i$. Since $F_i \subseteq F_j$ implies $\tau_j|_{F_i} = \tau_i$, and T is a chain, τ' is well defined. So $(F', \tau') \in S$ and it is an upper bound for T.

So by Zorn's lemma, S has a maximal element (F, τ). We claim that $F = L$. If $x \in L$, then by the above applied to $F(x)/F \xrightarrow{\tau} M$, can extend τ to some $\bar{\tau} : F(x) \to M$. Then $(F(x), \bar{\tau}) \in S$ and $(F, \tau) \leq (F(x), \bar{\tau})$. So by maximality $F(x) = F$, i.e $x \in F$. $\square$

The proof of existence of algebraic closure will use the following.

Theorem 14.5 ("Maximal ideal theorem")**.** *Let R be a nno-zero ring (with a 1 and commutative). Then R has a maximal ideal.*

Proof. Let $S = \{\text{proper ideals } I \subsetneq R\}$, partially ordered by inclusion. A maximal ideal is then just a maximal element of S. Let $\emptyset \neq T \subseteq S$ be a chain. Then $J := \bigcup_{I \in T} I$ is an ideal (easy to check). As $1 \notin I$, for all $I \in S$, $1 \notin J$. So J is a proper ideal and is clearly an upper bound for T. Now apply Zorn's lemma. $\square$

Theorem 14.6. *Let K be a field. Then K has an algebraic closure $\overline{K}$. If $\sigma : K \xrightarrow{\sim} K'$ is an isomorphism and $\overline{K}$, $\overline{K'}$ are algebraic closures of K and K' respectively, then there exists an isomorphism $\bar{\sigma} : \overline{K} \xrightarrow{\sim} \overline{K'}$ extending σ (so algebraic closure is unique up to isomorphism).*

Proof. First we prove existence. Let $P = \{\text{monic irreducibles in } K[T]\}$. We construct K_1 such that every $f \in P$ has a root in K_1. We first find a ring in which every $f \in P$ has a root. Let $R = K[\{T_f\}_{f \in P}]$ (i.e the finite K-linear combinations of monomials $T_{f_1}^{m_1} \dots T_{f_k}^{m_k}$, $f_i \in P$). Let I be the ideal generated by $\{f(T_f) : f \in P\}$. In R/I, $T_f + I$ is a root of f. We check $I \neq R$: if $I = R$, then $1 \in I$ so for some finite subset $Q \subseteq P$, there exist $r_f \in R$ (with $f \in Q$) with $1 = \sum_{f \in Q} r_f f(T_f)$. Enlarging Q if necessary, we may assume each r_f is a polynomial in $\{T_g : g \in Q\}$.

Let L/K be a splitting field fo $\prod_{f\in Q} f \in K[T]$, and for each $f \in Q$ take some a_f which is a root of f. Consider $\varphi : R \to L$, $\varphi|_K = \text{id}$, $\varphi(T_f) = a_f$ if $f \in Q$ and $\varphi(T_f) = 0$ if $f \notin Q$. Then $1 = \varphi(1) = \sum_{f\in Q} \varphi(r_f f(T_f)) = \sum_{f\in Q} \varphi(r_f) f(a_f) = 0$, a contradiction.

Now apply the maximal ideal theorem to R/I to get a maximal ideal $\overline{J}$, or equivalently there exists a maximal ideal $J \subseteq R$ containing I (ideals of R/I biject with ideals of R containing I). Let $K_1 = R/J$, a field; let $x_f = T_f + J \in K_1$. Then K_1/K is generated by $\{x_f\}$, $f(x_f) = 0$. So K_1/K is an algebraic extension of K in which every polynomial has a root. Apply the same procedure to K_1, $P_1 = \{\text{monic irreducibles in } K_1[T]\}$ to get K_2, and so on. Obtain $K \subseteq K_1 \subseteq K_2 \subseteq \ldots$ such that if $f \in K_n[T]$ is irreducible, it has a root in K_{n+1}.

So if $f \in K[T]$ is non constant, then $f = (T - x_1)f_1 \in K_1[T]$ then $f_1 = (T - x_2)f_2 \in K_2[T]$ so f factorises into linear factors in $K_{\deg f}[T]$. Therefore $\overline{K} = \bigcup_n K_n$ is an algebraic closure of K.

Now we show uniqueness. Let $K \subseteq \overline{K}$, $K' \subseteq \overline{K'}$ be algebraic closures, $\sigma : K \xrightarrow{\sim} K'$. Then by 14.4, σ extends to a homomorphism $\bar{\sigma} : \overline{K} \to \overline{K'}$ (as $\overline{K}/K$ is algebraic and $\overline{K'}$ is algebraically closed). But $K' \subseteq \sigma(\overline{K}) \subseteq \overline{K'}$, so $\overline{K'}/\sigma(\overline{K})$ is algebraic. As $\overline{K}$ is algebraically closed, so is $\sigma(\overline{K})$. So $\overline{K'} = \sigma(\overline{K})$ (14.1 (iii)). $\square$

15 Cubics & quartics

Let $f \in K[T]$ be a monic cubic which is separable. Then $G := \text{Gal}(f/K) \leq S_3$ acting on the roots x_1, x_2, x_3 in a splitting field L.

If f is reducible, f is the product of linear factors in K so $G = \{e\}$. If f is the product of a linear and a quadratic polynomial then $G = S_2$.

If f is irreducible (assuming $\text{char}(K) \notin \{2, 3\}$) then $G = S_3$ or A_3; and $G = A_3$ iff $\Delta^2 = \text{Disc}(f) \in (K^\times)^2$.

$$\begin{array}{ccc}
L = K(x_1, x_2, x_3) & & \{e\} \\
\Big| \, {\scriptstyle 3 \text{ if } f \text{ irred, } 1 \text{ otherwise}} & & \Big| \\
K(\Delta) = L^{G\cap A_3} & & G \cap A_3 \\
\Big| \, {\scriptstyle 2 \text{ if } \Delta^2 \in (K^\times)^2\text{, } 1 \text{ otherwise}} & & \Big| \\
K & & G
\end{array}$$

Let $K_1 = K(\Delta)$. Then $K_1 = L$ if f is reducible. If f is irreducible, then L/K_1 is Galois with Galois group $\mathbb{Z}/3\mathbb{Z}$. If $\omega \in K_1$ is a primitive 3rd root of unity, then $L = K_1(y)$ for some y with $y^3 \in K$ (Kummer extension). If not, let $L(\omega)$ be the splitting field of $f(T^3 - 1)$ over K. Then $L(\omega)/K_1(\omega)$ is Galois, with Galois group $\mathbb{Z}/3\mathbb{Z}$. So $L(\omega) = K_1(\omega, y)$ with $y^3 \in K_1(\omega)$. So the x_i lie in a field obtained by adjoining successive square and cube roots to K ($\omega = \frac{-1+\sqrt{-3}}{2}$). This is effectively Cardano's solution.

Explicitly, assume wlog that $f(T) = T^3 + bT + c$. Then $\Delta^2 = -4b^3 - 27c^2$. There are two cases:

- $b = 0$, then roots of f are $\omega^j \sqrt[3]{-c}$, so take y to be any of these
- $b \neq 0$, then y a Lagrange resolvent: if the roots of f in L are x_1, x_2, x_3, take $y = x_1 + \omega^2 x_2 + \omega x_3$ and $y' = x_1 + \omega x_2 + \omega^2 x_3$. Then $y = (1 - \omega)(x_1 - \omega x_2)$ since $x_1 + x_2 + x_3 = 0$. Then $L(\omega) = K(\Delta, \omega, y)$ if and only if $y \neq 0$ (by proof of structure of Kummer extensions). $yy' = -3b$ (direct computation) so $y \neq 0$, also $y + y' = y + y' + (x_1 + x_2 + x_3) = 3x_1$ and $y^3 = \frac{1}{2}\left(-3\sqrt{-3}\Delta + 27c\right)$ (direct computation again), so $x_1 = y - \frac{3b}{y}$.

Now let $f \in K[T]$ be a monic, separable quartic, $\mathrm{char}(K) \notin \{2,3\}$. $\mathrm{Gal}(f/K) \subseteq S_4$. Consider the action of S_4 on the partitions

$$(12 \mid 34), (13 \mid 24), (14 \mid 23) \text{ of } \{1,2,3,4\}$$

So this gives a homomorphism $S_4 \to S_3$ which has kernel
$V = \{e, (12)(34), (13)(24), (14)(23)\}$ (Klein's 4-groups). So the homomorphism is surjective. Take a splitting field L for f, roots $x_1, \dots, x_4$. Assume $x_1 + \dots + x_4 = 0$, i.e $f = T^4 + aT^2 + bT + c$.

$$\begin{array}{ccc} L & \qquad\qquad & \{e\} \\ | & & | \\ M = L^{G \cap V} & & G \cap V \\ | & & | \\ K & & G \end{array}$$

As $V \trianglelefteq S_4$, $G \cap V \trianglelefteq G$ and $\mathrm{Gal}(M/K) = G/G \cap V \to S_4/V \cong S_3$. So we should be able to write M as the splitting field of a cubic.

Let $y_{12} = x_1 + x_2 = -(x_3 + x_4) = -y_{34}$, $y_{13} = x_1 + x_3 = -y_{24}$, $y_{14} = x_1 + x_4 = -y_{23}$. Note that each element in $G \cap V$ maps y_{12} to y_{12} or $y_{32} = -y_{12}$ and similarly for y_{13} etc. So $y_{12}^2, y_{13}^2, y_{14}^2$ are fixed under $V \cap G$. If $y_{12}^2 = y_{13}^2$, either $y_{12} = y_{13}$ implying $x_2 = x_3$, or $y_{12} = -y_{13}$ implying $2x_1 + x_2 + x_3 = x_1 - x_4 = 0$, both of which are impossible (f separable).

So $\{y_{ij}^2\}$ are the roots of a separable cubic $g \in K[T]$, the *resolvent cubic*. Now $M = L^{G \cap V}$ is the splitting field of g. Note $x_1 = \frac{1}{2}(y_{12} + y_{13} + y_{14})$; likewise for x_2 etc. Hence $L = M(y_{12}, y_{13}, y_{14})$. So compute

$$g = (T - y_{12}^2)(T - y_{13}^2)(T - y_{14}^2) = T^3 + 2aT^2 + (a^2 - 4c)T - b^2$$

In particular, $y_{12}y_{13}y_{14} = b$, so $L = M(y_{12}, y_{13})$, $y_{12}^2, y_{13}^2 \in M$. So we have found a way to solve $f = 0$ by first solving the resolvent equation $g = 0$ and then taking (at most two) square roots.

16 Solubility by radicals

Suppose $f \in K[T]$ monic, $\mathrm{char}(K) = 0$. What does it mean to have a "formula" for roots of f?

Definition. An irreducible polynomial $f \in K[T]$ is *soluble by radicals over* K if there exist a sequence of fields $K = K_0 \subseteq K_1 \subseteq \dots \subseteq K_m$ with $x \in K_m$ a root of f and $K_i = K_{i-1}(y_i)$ where $y_i^{d_i} \in K_{i-1}$ $(d_i \geq 1)$.

We can adjoin "extra" roots if we want. Then we see that f is soluble by radicals over K if there exist $d \geq 1$ and $K = K_0 \subseteq K_1 \subseteq \dots \subseteq K_m$ such that

- f has a root $x \in K_m$
- $K_1 = K(\zeta_d)$, ζ_d primitive dth root of unity
- For $i \geq 1$, $K_i = K_{i-1}(y_i)$ where $a_i = y_i^d \in K_{i-1}$.

Call this condition (R).

Note that K_1/K_0 is Galois, with abelian Galois group. For $i \geq 1$, K_i/K_{i-1} is Galois, with group $\subseteq \mathbb{Z}/d\mathbb{Z}$ (Kummer extension).

To get all the roots of f, look at a normal closure M of K_m. This contains a splitting field for f over K (contains one root and f is irreducible).

To determine M, let $K_i' \subseteq M$ be the normal closure of K_i (so $K_1' = K_1 = K(\zeta_d)$).

Proposition 16.1. $K_i' = K_{i-1}'(\{\sqrt[d]{\sigma(a_i)} : \sigma \in \mathrm{Gal}(K_{i-1}'/K)\})$

Proof. Let $\sigma \in \mathrm{Gal}(K_{i-1}'/K)$, then there exists $\bar{\sigma} \in \mathrm{Gal}(K_i'/K)$ such that $\bar{\sigma}|_{K_{i-1}'} = \sigma$, $K_i' \ni \bar{\sigma}(y_i)$ as K_i'/K is Galois and $\bar{\sigma}(y_i)^d = \sigma(y_i^d) = \sigma(a_i)$. So RHS $\subseteq K_i'$.

Now it is enough to show that the RHS is normal. But the RHS is a splitting field over K_{i-1}' of $\prod_\sigma (T^d - \sigma(a_i)) = g_i \in K[T]$.

If K_{i-1}' is a splitting field of some h_{i-1} over K, then the RHS is the splitting field of $g_i h_{i-1}$ over K, so is normal. □

Proposition 16.2. $\mathrm{Gal}(K'_i/K'_{i-1})$ *is abelian.*

Proof (variant on Theorem 12.3). Let $A = \mathrm{Gal}(K'_i/K'_{i-1})$. For all $\tau \in A$, $\sigma \in \mathrm{Gal}(K'_{i-1}/K)$, $\tau(\sqrt[d]{\sigma(a_i)}) = \zeta_d^{m_\sigma} \sqrt[d]{\sigma(a_i)}$, $m_\sigma \in \mathbb{Z}/d\mathbb{Z}$. Then $\tau \mapsto (m_\sigma)_{\sigma \in \mathrm{Gal}(K'_{i-1}/K)} \in (\mathbb{Z}/d\mathbb{Z})^r$, $r = |\,\mathrm{Gal}(K'_{i-1}/K)|$ is an injective homomorphism $A \to \mathbb{Z}/d\mathbb{Z}$ (for details review 12.3).

Set $i = 1$ so $K'_1 = K_1 = K(\zeta_d)$ so the Galois group $\subseteq (\mathbb{Z}/d\mathbb{Z})^\times$. $\square$

$$\begin{tikzcd}
M = K'_m \arrow[d, dash] & & \{e\} \arrow[d, dash] \\
K'_{m-1} \arrow[d, dash] & & N_{m-1} \arrow[d, dash] \\
\vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
K = K_0 & & N_0 = G = \mathrm{Gal}(M/K)
\end{tikzcd}$$

Definition. A finite group G is soluble (or solvable) if there exist a chain of normal subgroups. i.e there exist $N_i \trianglelefteq G$ such that $G = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_m = \{e\}$ such that N_i/N_{i+1} is abelian for all i.

Example. (Simplest non-abelian soluble group) $\{e\} \trianglelefteq A_3 \trianglelefteq S_3$, $S_3/A_3 \cong \mathbb{Z}/2\mathbb{Z}$, $A_3 \cong \mathbb{Z}/3\mathbb{Z}$.

$S_4 \supseteq A_4 \supseteq V = (\mathbb{Z}/2\mathbb{Z})^2 \supseteq \{e\}$, both $V \trianglelefteq S_4$ and $A_4 \trianglelefteq S_4$, $S_4/A_4 \cong \mathbb{Z}/2\mathbb{Z}$, $A_4/V \cong \mathbb{Z}/3\mathbb{Z}$.

If $G = \mathrm{Gal}(M/K)$ as above, then $N_i/N_{i+1} = \mathrm{Gal}(K'_i/K'_{i-1})$ is abelian by 16.2. Hence G is soluble.

Theorem 16.3 (Abel-Ruffini)**.** *If $f \in K[T]$ is soluble by radicals over K, then* $\mathrm{Gal}(f/K)$ *is soluble.*

Proof. $\mathrm{Gal}(f/K) = \mathrm{Gal}(L/K) \cong \mathrm{Gal}(M/K)/\mathrm{Gal}(M/L)$. As $\mathrm{Gal}(M/K)$ is soluble, result follows from the below lemma. $\square$

Lemma 16.4. *Every subgroup and quotient of a soluble group is soluble.*

Proof. $\{e\} \trianglelefteq N_m \ldots \trianglelefteq N_1 \trianglelefteq N_0 = G$, N_i/N_{i+1} abelian, $H \subseteq G$ subgroup. Then $H \cap N_i \trianglelefteq G$ and $(H \cap N_i)/(H \cap N_{i+1}) \to N_i/N_{i+1}$ is injective. So $(H \cap N_i)/(H \cap N_{i+1})$ abelian and H is soluble.

Suppose $\pi : G \to \bar{G} = G/H$, H normal subgroup of G. Then $\pi(N_i) \trianglelefteq \bar{G}$, and $N_i/N_{i+1} \to \pi(N_i)/\pi(N_{i+1})$ is surjective. $\square$

Proposition 16.5. *If $n \geq 5$, then S_n and A_n are not soluble.*

Proof. $A_5 \subseteq A_n \subseteq S_n$, so it suffices to prove (by 16.4) that A_5 is not soluble. But A_5 is not abelian and it is simple, so is not soluble. □

Corollary 16.6. *If $\deg(f) = n \geq 5$ and $\mathrm{Gal}(f/K) \supseteq A_n$ (i.e equals A_n or S_n), then f is not soluble by radicals.*

Proof. Combine the previous. □

Remark: it is easy to show the converse of 16.3.

17 $\mathbb{C}$ is algebraically closed

What we'll use:

(i) Every polynomial of odd degree over $\mathbb{R}$ has a root (intermediate value theorem)

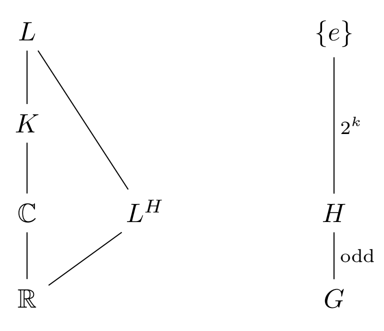
(ii) Every quadratic over $\mathbb{C}$ splits into linear factors (can take square roots)

(iii) Every finite group G has a subgroup H such that $(G : H)$ is odd and $H = 2^k$ for some k (Sylow theorem $p = 2$)

(iv) If G is a p-group then G has a subgroup of index p (since G has non-trivial centre)

Proof. Let $K/\mathbb{C}$ be a finite extension. Let $L \supseteq K$ be a normal closure over $\mathbb{R}$, so $G = \mathrm{Gal}(L/\mathbb{R})$. We'll show $L = \mathbb{C}$.

Let $H \leq G$ be a Sylow 2-subgroup. Then $[L^H : \mathbb{R}] = (G : H)$ which is odd. So if $x \in L^H$, by (i) $x \in \mathbb{R}$, i.e $L^H = \mathbb{R}$ so $G = H$ is a 2-group. Let $G \supseteq G_1 = \mathrm{Gal}(L/\mathbb{C})$ and $G_2 \subseteq G_1$ a subgroup of index 2 (exists by (iv)). Then $[L^{G_2} : \mathbb{C}] = (G_1 : G_2) = 2$ contradicting (ii).



So G has no subgroup of index 2, i.e $G_1 = \{e\}$ so $L = \mathbb{C}$. □