Introduction

Course structure:

(i) Preliminary toolbox: inequalities

(ii) Normed vector spaces (NVS)

(iii) (Recalls on) finite-dimensional case

(iv) Hahn-Banach Theorems (how big is the dual?)

(v) Completeness: Baire's Theorem & consequences for NVS

(vi) Detailed study of the topology of $C(K)$

(vii) The Hilbert space

(viii) Projection & duality

(ix) Introduction to operators and spectral theory

1 Preliminary toolbox: Young's, Hölder's & Minkowski's inequalities for vectors & sequences

Proposition (Young's inequality for products)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\forall a, b \geq 0,\ ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. The result is clear for $a = 0$ or $b = 0$. Assume $a, b > 0$ and note $L : (0, \infty) \to \mathbb{R}$, $t \mapsto \ln t$ is strictly concave: $L''(t) = -\frac{1}{t^2} < 0$.

Therefore for all $A, B > 0$, $\lambda \in (0, 1)$

$$\ln(\lambda A + (1 - \lambda)B) \geq \lambda \ln A + (1 - \lambda) \ln B$$

with equality iff $A = B$. Apply this to $A = a^p$, $B = b^q > 0$ and $\lambda = \frac{1}{p}$. This gives

$$\ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) = \ln(ab)$$

so applying exp to both sides gives the result and furthermore we have equality iff $a^p = b^q$. □

Proposition (Hölder's inequality for vectors & sequences)**.** Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

(i) for any $n \in \mathbb{N} \setminus \{0\} = \mathbb{N}^*$, $\forall x, y \in \mathbb{R}^n$

$$\sum_{k=1}^{n} |x_k y_k| \leq ||x||_p ||y||_q \tag{$*$}$$

with $||x||_p = (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$ and similarly for $||y||_q$.

(ii) define

$$\ell^p = \{x \in \mathbb{R}^{\mathbb{N}^*} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$$

then $\forall x \in \ell^p$, $y \in \ell^q$

$$\sum_{k=1}^{\infty} |x_k y_k| \leq ||x||_{\ell^p} ||y||_{\ell^q}$$

where $||x||_{\ell^p} = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ and similar for $||y||_{\ell^q}$.

Proof. To show (i) implies (ii): take $n \to \infty$ in (i) so

$$\sum_{k=1}^{n} |x_k|^p \to ||x||_{\ell^p}^p$$

and similarly

$$\sum_{k=1}^{n} |y_k|^q \to ||y||_{\ell^q}^q$$

By (i)

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}$$

so

$$\begin{aligned}\sum_{k=1}^{\infty} |x_k y_k| = \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k y_k|\right) &\leq \lim_{n\to\infty} \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q} \\ &= ||x||_{\ell^p} ||y||_{\ell^q}\end{aligned}$$

Proof of (i): if $||x||_{\ell^p}$ or $||y||_{\ell^q} = 0$, result is clear. Otherwise define $\tilde{x}$, $\tilde{y}$ sequences in ℓ^p and ℓ^q by

$$\tilde{x}_k = \frac{x_k}{||x||_{\ell^p}},\ \tilde{y}_k = \frac{y_k}{||y||_{\ell^q}}$$

Then $||\tilde{x}||_{\ell^p} = 1$, $||\tilde{y}||_{\ell^q} = 1$. Then $(*)$ is equivalent to showing

$$\sum_{k=1}^{n} |\tilde{x}_k \tilde{y}_k| \leq 1 \tag{$**$}$$

Apply Young's inequality on each $k = 1, \ldots, n$ so

$$|\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}|\tilde{x}_k|^p + \frac{1}{q}|\tilde{y}_k|^q$$

Summing over k:

$$\sum_{k=1}^{n} |\tilde{x}_k \tilde{y}_k| \leq \frac{1}{p}\left(\sum_{k=1}^{n} |\tilde{x}_k|^p\right) + \frac{1}{q}\left(\sum_{k=1}^{n} |\tilde{y}_k|^q\right) \leq \frac{1}{p} + \frac{1}{q} = 1$$

□

Remark: Equality in $(*)$ is equivalent to equality in $(**)$ which is equivalent to equality in Young's for each k so $|\tilde{x}_k|^p = |\tilde{y}_k|^q$ for $k = 1, \ldots, n$. Also, the $p = 1$, $q = \infty$ case is easy.

Proposition (Minkowski's inquality for vectors & sequences)**.** Let $p \in [1, \infty)$, then

(i) for all $x, y \in \mathbb{R}^n$
$$||x + y||_p \leq ||x||_p + ||y||_p$$

(ii) for all $x, y \in \ell^p$
$$||x + y||_{\ell^p} = ||x||_{\ell^p} + ||y||_{\ell^p}$$

Proof. To show (i) implies (ii): by taking $n \to \infty$ as before

$$\sum_{k=1}^{\infty} |x_k|^p \to ||x||_{\ell^p}^p$$
$$\sum_{k=1}^{\infty} |y_k|^p \to ||y||_{\ell^p}^p$$
$$\sum_{k=1}^{n} |x_k + y_k|^p \to ||x + y||_{\ell^p}^p$$

Proof of (i): if $p = 1$ this is just the usual triangle inequality on each coordinate. So let $p \in (1, \infty)$ and

$$\sum_{k=1}^{n} |x_k + y_k|^p = \sum_{k=1}^{n} |x_k + y_k| \cdot |x_k + y_k|^{p-1}$$

$$\leq \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}$$

$$\underbrace{\leq}_{\text{Hölder: } q = \frac{p}{p-1}} ||x||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + ||y||_p \left(\sum |x_k + y_k|^{(p-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\leq (||x||_p + ||y||_p)\,||x+y||_p^{p-1}$$

so we have proved

$$||x+y||_p^p \leq (||x||_p + ||y||_p)\,||x+y||_p^{p-1}$$

If $||x+y||_p = 0$, result is clear. Otherwise divide by $||x+y||_p^{p-1}$ to get

$$||x+y||_p \leq ||x||_p + ||y||_p$$

□

Remark: equality occurs iff there is equality in the triangle inequality and Hölder's.
Remarks:

1. Equality case: $p = 1$: $|x_k + y_k| \leq |x_k| + |y_k|$, i.e the usual triangle inquality
2. For $p = 2$ there's another proof: define $\mathcal{P} : \mathbb{R} \to \mathbb{R}$, $\lambda \mapsto ||x + \lambda y||^2$. Then $\mathcal{P}(\lambda) = a\lambda^2 + 2b\lambda + c$ and $\mathcal{P} \geq 0$. So

$$\langle x, y\rangle = b^2 \leq ac = ||x||^2||y||^2, \text{ Hölder's inequality}$$

2 Normed Vector Spaces (NVS)

Remark: this is not the most general structure for linear analysis - topological vector spaces (TVS).

Recall:

Definition. A *vector space* V over a field $\mathbb{F}$ is a set (of elements called *vectors*) with two operations:

$$A : V \times V \to V,\ (v, w) \mapsto v + w \text{ addition}$$

$$M : \mathbb{F} \times V \to V,\ (\lambda, v) \mapsto \lambda v \text{ scalar multiplication}$$

such that

- $(V, +)$ is an abelian group with identity 0.
- M is compatible with $(\mathbb{F}, 0)$ in the sense that $\lambda_1(\lambda_2 v) = \lambda_1 \lambda_2 v$
- M distributes over $(V, +)$ and $(\mathbb{F}, +)$.

In this course $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$ unless stated otherwise.

Definition. Given a vector space V over $\mathbb{F}$:

- a *subspace* $W \subseteq V$ is a vector space over $\mathbb{F}$ included in V
- for a set $S \subseteq V$, a *linear combination of elements of* S is a finite sum of elements of S with coefficients in $\mathbb{F}$
- for a set $S \subseteq V$, the *span of* S, $\text{span}(S)$ is the smallest subspace of V containing S, and is also the set of linear combinations of S.

Definition. Given V a vector space over $\mathbb{F}$ and a set $S \subseteq V$:

- S is *linearly independent* if for all $m \in \mathbb{N}^*$ and for all $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$, for all $s_1, \ldots, s_m \in S$, $\sum_{i=1}^m \alpha_i s_i = 0$ if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_m$.
- S is a *basis* of V if it is linearly independent and $\text{span}(S) = V$.
- If there exists a finite basis S of V, then V has finite dimension, otherwise it is infinite-dimensional.

Remark: later we'll prove with Zorn's lemma that any vector space has a basis.

Definition. A *normed vector space* (NVS) V over $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a function $N : V \to \mathbb{R}_+$, $v \mapsto ||v||$ (the *norm*), with

1. $||v|| \geq 0$ for all $v \in V$, with equality only at $v = 0$ (*positive definiteness*)
2. For all $\lambda \in \mathbb{F}$, $v \in V$ $||\lambda v|| = |\lambda| ||v||$ (compatibility between N and M)

3. For all $v, w \in V$, $||v+w|| \leq ||v|| + ||w||$ (compatibility between N and A)

Example. $V = \mathbb{R}^n$, $v = (v_1, \ldots, v_n)$, $||v|| = (v_1^2 + \ldots + v_n^2)^{1/2}$ or

$$\begin{cases} ||v||_p = (|v_1|^p + \ldots + |v_n|^p)^{1/p} & \text{for } p \in [1, \infty) \\ ||v||_\infty = \sup_{i=1}^n |v_i| & \text{for } p = \infty \end{cases}$$

Definition. Given a set X, a *topology* τ on X is a collection of subsets of X ("open sets") such that

- $\emptyset \in \tau$, $X \in \tau$
- τ is stable under any union
- τ is stable under finite intersections

Definition.

- For (X, d) a metric space, the *induced topology* is the smallest topology that contains open balls in d
- For a NVS $(V, ||\cdot||)$, the induced topology is that associated with $d(v, w) = ||v - w||$

Natural question: $\mathbb{F}$ field, V vector space over $\mathbb{F}$. Norm on V, $\tau_{||\cdot||}$. Continuity of operations M and A?

Proposition. Let $(V, ||\cdot||)$ be a NVS over $\mathbb{F}$ ($\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$), then

(i) A, M are continuous for the following topologies: $\tau_{||\cdot||}$ on V, then product topology of it on $V \times V$, $\tau_{|\cdot|}$ over $\mathbb{F}$, then product topology of $\tau_{|\cdot|}$ and $\tau||\cdot||$ on $\mathbb{F} \times V$

(ii) Translations $T_{v_0} : V \to V$, $v \mapsto v + v_0$, $v_0 \in V$ and dilations $D_{\lambda_0} : V \to V$, $v \mapsto \lambda_0 v$, $\lambda_0 \in \mathbb{F}^*$ are homeomorphisms

Proof.

(i) Let us prove that $A : V \times V \to V$ is continuous: consider an open set $\emptyset \neq U \subseteq V$ and $(v_1, v_2) \in A^{-1}(U)$, i.e $v_1 + v_2 \in U$. Since U is open, there is $\varepsilon > 0$ such that $\underbrace{B_V(v_1 + v_2, \varepsilon)}_{\text{open ball}} \subseteq U$.

We have that $A(B(v_1, \varepsilon/2), B_V(v_2, \varepsilon/2)) \subseteq B_V(v_1+v_2, \varepsilon)$ (triangle inequality). Note also that $B(v_1, \varepsilon/2) \times B(v_2, \varepsilon/2)$ is open (product topology), so $A^{-1}(U)$ is open and A is continuous.

Now we show $M : \mathbb{F} \times V \to V$ is continuous. Consider an open set $U \neq \emptyset$ in V, $(\lambda, v) \in M^{-1}(U)$. Since U is open, there exists $\varepsilon > 0$ such that $B_V(\lambda v, \varepsilon) \subseteq U$ (WLOG $\varepsilon < 1$). Then (check)

$$M\left(B_{\mathbb{F}}\left(\lambda, \frac{\varepsilon}{3\max(1, ||v||)}\right), B_V\left(v, \frac{\varepsilon}{3\max(1, |\lambda|)}\right)\right) \subseteq B_V(\lambda v, \varepsilon)$$

(ii) T_{v_0} and D_{λ_0} are linear, continuous with inverses T_{-v_0} and $D_{\lambda_0^{-1}}$ respectively, so are homeomorphisms.

$\square$

3 Characterisation of NVS

Idea: in order to better understand the topology of NVS's, we ask how special is a "normable" topology among topologies compatible with vector space operations?

Definition (TVS)**.** A *topological vector space* (TVS) over $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with a topology τ such that

(i) A and M are continuous

(ii) every singleton $\{x_0\}$ is closed

Remark:

1. (i) says that T_{v_0} and D_{λ_0}, $\lambda_0 \neq 0$ are homeomorphisms
2. (ii) is called T_1 in the classification of seperation properties, and implies Hausforff for TVS

Definition. Given V a TVS

- $C \subseteq V$ is *convex* if $C = \{\lambda c_1 + (1-\lambda)c_2 : c_1, c_2 \in C, \lambda \in [0,1]\}$
- V is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0
- $B \subseteq V$ is *bounded* if for any U open around 0, there exists $t_0 > 0$ such that $\forall t > t_0$, $B \subseteq tU$
- V is *locally bounded* if there is $U \in \tau$ containing 0 and bounded

Example. Let $(V, ||\cdot||)$ be a NVS, then for all $r > 0$, $U = B(0,r)$ (open ball) is open, bounded and convex. Indeed

- Convexity follows from the triangle inequality
- Boundedness: any other $\tilde{U}$ open around 0 contains some open $\tilde{U}_0 = B(0, r_0) \in \tilde{U}$. Then for any $t > \frac{r}{r_0}$, $U \subseteq t\tilde{U}_0 \subseteq t\tilde{U}$.

Question: can we reverse-engineer the norm if we have these two properties?

Theorem (Kolmogorov 1934)**.** *Let (V,τ) be a TVS such that there is a bounded convex neighborhood of 0, say C. Then V is "normable" - there is a norm $||\cdot||$ on V that induces the topology τ.*

Proof. Step 1: there is $\tilde{C} \subseteq C$ which is a *balanced* convex bounded neighborhood of 0. "Balanced" means that for all $\lambda \in \mathbb{F}$ such that $|\lambda| \leq 1$, $\lambda\tilde{C} \subseteq \tilde{C}$.

$M : \mathbb{F} \times V \to V$ is continuous so $M^{-1}(C)$ is a neighourhood of $(0,0)$. So there exists $B_{\mathbb{F}}(0,\varepsilon) \times U$ with $\varepsilon > 0$ and U open around 0 such that $M(B_{\mathbb{F}}(0,\varepsilon), U) \subseteq C$.

Define $\tilde{C}$ to be the convex hull (i.e smallest convex set supserset) of $M(B_{\mathbb{F}}(0,\varepsilon),U)$.

Then $\tilde{C}$ is clearly convex, is a subset of C since C is convex and $M(B_{\mathbb{F}}(0,\varepsilon),U) \subseteq C$. $\tilde{C}$ is also bounded since $\tilde{C} \subseteq C$ and C is bounded (obvious that boundedness is inherited by inclusion). Finally $\tilde{C}$ is balanced since $\lambda B_{\mathbb{F}}(0,\varepsilon) \subseteq B_{\mathbb{F}}(0,\varepsilon)$ for $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$ and

$$\underbrace{\lambda M(B_{\mathbb{F}}(0,\varepsilon),U)}_{=M(\lambda B_{\mathbb{F}}(0,\varepsilon),U)} \subseteq M(B_{\mathbb{F}}(0,\varepsilon),U)$$

Notice $\lambda[\text{Convex Hull}(S)] = \text{Convex Hull}(\lambda S)$ (exercise). So deduce $\lambda\tilde{C} \subseteq \tilde{C}$.

<u>Step 2</u>: define the *Minkowski guage* (functional) of $\tilde{C}$

$$\mu_{\tilde{C}} : V \to \mathbb{R}_+,\ v \mapsto \inf\{t \geq 0 : v \in t\tilde{C}\}$$

$\mu_{\tilde{C}}$ is well-defined in $[0,\infty)$ since: any v satisfies $\frac{v}{t} \to 0$ as $t \to \infty$ by continuity of M. So $\frac{v}{t}$ must "enter" the neighborhood $\tilde{C}$ of 0 for t large enough.

<u>Step 3</u>: let us prove $v \mapsto \mu_{\tilde{C}}(v)$ is a norm:

- $\mu_{\tilde{C}}(v) \geq 0$ by construction
- if $\mu_{\tilde{C}} = 0$, then (assume $v \neq 0$ for contradiction) there exists U open around 0 with $v \notin U$ (since $V \setminus \{v\}$ is open). Since $\tilde{C}$ is bounded, there exists $t_1 > 0$ such that $\tilde{C} \subseteq t_1 U$. Since $\mu_{\tilde{C}}(v) = 0$, there exists $t_2 \in (0, t_1^{-1})$ such that $v \in t_2\tilde{C}$, then $v \in t_2\tilde{C} \subseteq t_1^{-1}\tilde{C} \subseteq U$, a contradiction.
- Want to show $\mu_{\tilde{C}}(\lambda v) = |\lambda|\mu_{\tilde{C}}(v)$ for $\lambda \in \mathbb{F}^\times$, $v \in V$. Use $\tilde{C}$ balanced: for all $t > 0$ such that $\lambda v \in t\tilde{C}$, we have

  $$\frac{\lambda}{|\lambda|}v \in \frac{t}{|\lambda|}\tilde{C} \implies v \in \frac{t}{|\lambda|}\tilde{C} \implies \mu_{\tilde{C}}(v) \leq \frac{1}{|\lambda|}\mu_{\tilde{C}}(\lambda v)$$

  The inequality in the other direction follows by reasoning with λ^{-1}. So $|\lambda|\mu_{\tilde{C}}(v) = \mu_{\tilde{C}}(\lambda v)$.
- Want to show $\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$ for all $v_1, v_2 \in V$. Indeed, given $t_1, t_2 > 0$ such that $v_1 \in t_1\tilde{C}$, $v_2 \in t_2\tilde{C}$, we have

  $$v_1+v_2 \in t_1\tilde{C}+t_2\tilde{C} = (t_1+t_2)\left[\frac{t_1}{t_1+t_2}\tilde{C} + \frac{t_2}{t_1+t_2}\tilde{C}\right] \subseteq (t_1+t_2)\tilde{C} \text{ (convexity)}$$

  so $\mu_{\tilde{C}}(v_1 + v_2) \leq t_1 + t_2$. By taking infima over t_1, t_2:

  $$\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$$

<u>Step 4</u>: prove $\mu_{\tilde{C}}$ induces the topology τ.

- Want to prove

$$\underbrace{B(v_0, \varepsilon)}_{\text{open ball for } \mu_{\tilde{C}}} = \{v \in V : \mu_{\tilde{C}}(v - v_0) < \varepsilon\} \in \tau$$

  Take $v \in B(v_0, \varepsilon)$ then by the triangle inequality

$$B(v, \varepsilon - |v|) \subseteq B(v_0, \varepsilon)$$

  and $B(v, \varepsilon') \supseteq v + \frac{\varepsilon'}{2}\tilde{C}$ by definition of the ball for $\mu_{\tilde{C}}$. And (since translations, dilations continuous) $v + \frac{\varepsilon'}{2}\tilde{C}$ is a neighborhood of v.

  $B(v_0, \varepsilon)$ open (in τ) around its points, so is in τ.

- Take $U \in \tau$, and (wlog) $0 \in U$. Let us prove $0 \in B(0, \varepsilon_0) \subseteq U$ for some $\varepsilon_0 > 0$. Indeed $\tilde{C}$ is bounded so there exists $\varepsilon_0 > 0$ such that $\tilde{C} \subseteq \varepsilon_0^{-1}U$ hence $U \supseteq \varepsilon_0\tilde{C}$ and so $U \supseteq \varepsilon\tilde{C}$ $\forall \varepsilon < \varepsilon_0$ and thus $U \supseteq B(0, \varepsilon_0)$.

$\square$

Remarks:

1. $B(0, \varepsilon_0) \subseteq \bigcup_{0 \le \varepsilon < \varepsilon_0} \varepsilon \tilde{C}$
2. T_1 implies Hausforff (T_2). Consider $v_0 \neq v_1$ in V: so $0 \neq v_1 - v_0$, T_1 implies there is U open around 0 with $v_1 - v_0 \notin U$. Then (since A, M continuous) $(v, w) \mapsto v - w$ is continuous and there exists $\tilde{U}$ open around 0 such that $\tilde{U} - \tilde{U} \subseteq U$. Then $v_0 + \tilde{U}$ and $v_1 + \tilde{U}$ are open disjoint neighborhoods of v_0 and v_1 respectively (disjoint since otherwise $v_1 - v_0 \in \tilde{U} - \tilde{U} \subseteq U$).

4 Some examples of NVS'

Definition. Let $(V, ||\cdot||)$ be an NVS (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). If (V, d), d distance induced by $||\cdot||$ is a complete metric space, then $(V, ||\cdot||)$ is called a *Banach space.*

Example. $\mathbb{R}^n$, $\mathbb{C}^n$, $n \ge 1$ are Banach spaces, for $||\cdot||_p$, $p \in [1, \infty)$.

Example. Given (X, τ) a general topological space, define

$$B_{\mathbb{F}}(X) = \{\text{functions } : X \to \mathbb{F} \text{ bounded}\}$$

$$C_{\mathbb{F}}(X) = \{\text{functions } : X \to \mathbb{F} \text{ continuous}\}$$

$$C_{\mathbb{F},b} = C_{\mathbb{F}}(X) \cap B_{\mathbb{F}}(X)$$

If $X = K$ is compact, $C_{\mathbb{F}}(X) = C_{\mathbb{F},b}(X)$. These are vector spaces over $\mathbb{F}$ with addition $(f + g)(x) = f(x) + g(x)$ and multiplication $(fg)(x) = f(x)g(x)$.

Norm on $C_{\mathbb{F},b}(X)$: the supremum norm, $||f||_\infty = \sup_{x \in X} |f(x)|$

Proposition. $(C_{\mathbb{F},b}, ||\cdot||_\infty)$ is a Banach space over $\mathbb{F}$.

Proof.

- $||f||_\infty$ is well defined in $\mathbb{R}^+$ since f is bounded.
- $||f||_\infty = 0$ means $f(x) = 0$ for all $x \in X$ and so $f = 0$.
- Homogeneity and triangle inequality: inherited from $|\cdot|$ in $\mathbb{F}$ (exercise).
- Completeness: let $(f_k)_{k \ge 1}$ be a Cauchy sequence under $||\cdot||_\infty$. For each $x \in X$ we have $|f_m(x) - f_n(x)| \le ||f_m - f_n||_\infty \to 0$ as $n, m \to \infty$. So $(f_k(x))_{k \ge 1}$ is Cauchy in $\mathbb{F}$, so (since $\mathbb{F}$ is complete) there exists a limit $f(x) = \lim_{k \to \infty} f_k(x)$. This defines a function $f : X \to \mathbb{F}$.
- For all $\varepsilon > 0$, there exists $n_0 \ge 1$ such that $\forall m, n \ge n_0$, $\forall x \in X$,

$$|f_m(x) - \underbrace{f_n(x)}_{\to f(x)}| \le \varepsilon$$

so for all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\forall m \geq n_0$, $\forall x \in X$ we have

$$|f_m(x) - f(x)| \leq \varepsilon$$

so $||f_m - f||_\infty \leq \varepsilon$ and $f_m \to f$ uniformly, so $f \in C_{\mathbb{F},b}$ by properties of the uniform limit.

□

Example. Given $U \subseteq \mathbb{R}^n$ open, bounded and non-empty; $m \in \mathbb{N}^*$, consider

$$\begin{aligned} C^m(\overline{U}) = \{f : U \to \mathbb{R} :& f \text{ is } m \text{ times differentiable on } U, \forall \alpha \in \mathbb{N}^n \\ & \text{s.t } |\alpha| = \alpha_1 + \ldots + \alpha_m \leq m \\ & , \partial^\alpha f \text{ is continuous and bounded on } U\} \end{aligned}$$

Then $(C^m(\overline{U}), ||\cdot||_{C^m})$ is a Banach space where

$$||f||_{C^m} = \sup_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} \underbrace{\sup_{x \in U} |\partial^\alpha f(x)|}_{||\partial^\alpha f||_\infty}$$

Exercise: check that this is complete and $\partial^\alpha f$, $\alpha \leq m-1$, extends continuously to $\tilde{U}$.

Example. $C_{\mathbb{R}}([0,1])$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$. This is a vector space over $\mathbb{R}$.

- $(C_{\mathbb{R}}([0,1]), ||\cdot||_\infty)$ is a Banach space (Example sheet)
- Could take another norm such that

$$||f||_p = \left(\int_0^1 |f(x)|^p \mathrm{d}x\right)^{1/p}, \ p \in [1, \infty)$$

Study of $(C_{\mathbb{R}}([0,1]), ||\cdot||_p)$:

- $||\cdot||_p$ is well defined: Riemann and Lebesgue integrable.
- If $||f||_p = 0$ and $f \neq 0$ then there exists $\varepsilon > 0$ and $x_0 \in [0,1]$ such that $|f(x_0)| \geq \varepsilon$, so by continuity there exist $a < b \in [0,1]$ such that $\inf_{x \in [a,b]} |f(x)| \geq \frac{\varepsilon}{2}$. Then $\int_0^1 |f(x)|^p \mathrm{d}x \geq \left(\frac{\varepsilon}{2}\right)^p (b-a) > 0$ which is impossible.
- Homogeneity is clear.
- Triangle inequality:

$$||f+g||_p^p = \int_0^1 |f+g|^p \mathrm{d}x = \int_0^1 |f+g||f+g|^{p-1} \mathrm{d}x$$

$$\leq \int_0^1 |f||f+g|^{p-1}\mathrm{d}x + \int_0^1 |g||f+g|^{p-1}\mathrm{d}x$$
$$\underbrace{\leq}_{\text{Hölder:}} ||f||_p||f+g||_p^{p-1} + ||g||_p||f+g||_p^{p-1}$$

If $||f+g||_p = 0$ then its clear. Otherwise this implies $||f+g||_p \leq ||f||_p + ||g||_p$.

- Completeness? Define

$$f_k(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{4k} \\ \left[x - \left(\frac{1}{2} - \frac{1}{4k}\right)\right] 4k & \frac{1}{2} - \frac{1}{4k} \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

then $(f_k)_{k\geq 1}$ is Cauchy for $||\cdot||_p$, and the limit is $1_{[1/2,1]}$ which is not continuous. So not complete.

Remark: what about the completion? In general, abstract completions are often not very useful; however in this case, it is: Lebesgue space $L^p([0,1])$, defined as equivalence classes for the "almost everywhere" equality.