1 Measures

Let E be any set. A collection $\mathcal{E}$ of subsets of E is called a *σ-algebra* if the following holds:

1. $\emptyset \in \mathcal{E}$.
2. If $A \in \mathcal{E}$, then $A^c = E \setminus A \in \mathcal{E}$.
3. If $(A_n : n \in \mathbb{N})$, $A_n \in \mathcal{E}$, then $\bigcup_n A_n \in \mathcal{E}$.

Examples.

- $\mathcal{E} = \{\emptyset, E\}$
- $\mathcal{E} = \mathcal{P}(E)$, the set of all subsets of E.

Note that $\bigcap_n A_n = (\bigcup_n A_n^c)^c$, any σ-algebra $\mathcal{E}$ is also closed under countable intersection of its elements. Also $B \setminus A = B \cap A^c \in \mathcal{E}$ whenever $A, B \in \mathcal{E}$.

Any set E with a choice of σ-algebra $\mathcal{E}$ is called a *measurable* space, and the elements of $\mathcal{E}$ are called *measurable sets*.

A *measure* μ is a set-function $\mu : \mathcal{E} \to [0, \infty]$ such that $\mu(\emptyset) = 0$, and for any $(A_n : n \in \mathbb{N})$, $A_n \in \mathcal{E}$ pairwise disjoint ($A_n \cap A_m = \emptyset$ for all $n \neq m$) then

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n) \qquad \text{(countable additivity of } \mu\text{)}$$

If $\mathcal{E}$ is countable, then for any $A \in \mathcal{P}(E)$ and a measure μ

$$\mu(A) = \mu\left(\bigcup_{x \in A} \{x\}\right) = \sum_{x \in A} \mu(\{x\})$$

hence there is a one-to-one correspondence between measures and mass functions on E.

For any collection $\mathcal{A}$ of subsets of E, we define the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ as

$$\sigma(\mathcal{A}) = \{A \subseteq E : A \in \mathcal{E} \ \forall \sigma\text{-algebras } \mathcal{E} \supseteq \mathcal{A}\}$$

which equals (Example sheet)

$$\sigma(\mathcal{A}) = \bigcap_{\sigma\text{-algebras } \mathcal{E} \supseteq \mathcal{A}} \mathcal{E}$$

To construct good 'generators' we define

1. $\mathcal{A}$ is called a *ring over* E if $\emptyset \in \mathcal{A}$; if $A, B \in \mathcal{A}$ then $B \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

2. $\mathcal{A}$ is called an *algebra over* E if $\emptyset \in \mathcal{A}$; if $A, B \in \mathcal{A}$ then $A^c \in \mathcal{A}$, $A \cup B \in \mathcal{A}$.

Notice that in a ring $A \Delta B = (B \setminus A) \cup (A \setminus B) \in \mathcal{A}$ and $A \cap B = (A \cup B) \setminus (A \Delta B) \in \mathcal{A}$. Also, $B \setminus A = B \cap A^c = (B^c \cup A)^c \in \mathcal{A}$, so an algebra is a ring.

Fact: If $\bigcup_n A_n$, $A_n \in \mathcal{E}$, $\mathcal{E}$ some σ-algebra (or a ring if the union is finite) - then we can find $B_n \in \mathcal{E}$ <u>disjoint</u> such that $\bigcup_n A_n = \bigcup_n B_n$. Indeed, define $\tilde{A}_n = \bigcup_{j \leq n} A_j$, and set $B_n = \tilde{A}_n \setminus \tilde{A}_{n-1}$, then the fact follows. ["disjointification of countable unions"]

Definition. A *set function* on any collection $\mathcal{A}$ of subsets of E (where $\emptyset \in \mathcal{A}$) is a map $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$. We say μ is

1. *increasing* if $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$; $A, B \in \mathcal{A}$
2. *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$; $A \cup B \in \mathcal{A}$ and $A \cap B = \emptyset$.
3. *countably additive* if $\mu\left(\cup_n A_n\right) = \sum_n \mu(A_n)$ for any $(A_n : n \in \mathbb{N})$ where $A_n \in \mathcal{A}$ disjoint and $\cup_n A_n \in \mathcal{A}$.
4. *countably sub-additive* if $\mu(\cup_n A_n) \leq \sum_n \mu(A_n)$ for all $(A_n : n \in \mathbb{N})$ such that $\cup_n A_n \in \mathcal{A}$

Remark: one can show that a measure μ on a σ-algebra satisfies 1-4 above.

Theorem (Caratheodory)**.** *Let μ be a countably additive set function on a ring $\mathcal{A}$ of subsets of E. Then there exists a measure μ^* on $\sigma(\mathcal{A})$ such that $\mu^*|_{\mathcal{A}} = \mu$.*

Proof. For $B \subseteq E$ define the *outer measure* μ^* as

$$\mu^*(B) = \inf\left\{\sum_{n\in\mathbb{N}} \mu(A_n) : A_n \in \mathcal{A}, B \subseteq \bigcup_n A_n\right\}$$

and set $\mu^*(B) = \infty$ if the set within the infimum is empty.

Define

$$\mathcal{M} = \{A \subseteq E : \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)\ \forall B \subseteq E\}$$

the "μ^*-measurable" sets.

Step 1: μ^* is countably sub-additive on $\mathcal{P}(E)$. For any $B \subseteq E$ and $B_n \subseteq E$ such that $B \subseteq \bigcup_n B_n$ we have

$$\mu^*(B) \leq \sum_n \mu^*(B_n) \tag{†}$$

WLOG we assume $\mu^*(B_n) < \infty$ for all n so for all $\varepsilon > 0$, there exists A_{nm} such that $B_n \subseteq \bigcup_m A_{nm}$ and

$$\mu^*(B_n) + \frac{\varepsilon}{2^n} \geq \sum_m \mu(A_{nm})$$

Now since μ^* and since $B \subseteq \bigcup_n B_n \subseteq \bigcup_{n,m} A_{nm}$, hence

$$\mu^*(B) \leq \mu^*\left(\bigcup_{n,m} A_{nm}\right) \leq \sum_{n,m} \mu(A_{nm}) \leq \sum_n \mu^*(B_n) + \underbrace{\sum_n \frac{\varepsilon}{2^n}}_{=\varepsilon}$$

so (†) follows since ε was arbitrary.

Step 2: μ^* extends μ. Let $A \in \mathcal{A}$. Clearly $A = A \cup \emptyset \cup \ldots \cup \emptyset$, so by definition of μ^*, $\mu^*(A) \leq \mu(A) + 0 + \ldots + 0$. So we need to prove $\mu(A) \leq \mu^*(A)$. Again, assume $\mu^*(A) < \infty$ WLOG, and let $A_n \in \mathcal{A}$ be such that $A \subseteq \bigcup_n A_n$. Then $A = \bigcup_n \underbrace{(A \cap A_n)}_{\in\mathcal{A}}$, and since μ is countably sub-additive on $\mathcal{A}$, we have

$$\mu(A) = \mu\left(\bigcup_n (A \cap A_n)\right) \leq \sum_n \mu(\underbrace{A \cap A_n}_{\subseteq A_n}) \leq \sum_n \mu(A_n)$$

so since the (A_n) were arbitrary, by taking infima, we have $\mu(A) \leq \mu^*(A)$.

Step 3: $\mathcal{M} \supseteq \mathcal{A}$. Let $A \in \mathcal{A}$, then $B \subseteq (B \cap A) \cup (B \cap A^c) \cup \emptyset \ldots \cup \emptyset = \bigcup_n B_n$ so by (†) we have

$$\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c) + 0 \ldots + 0$$

so we need to prove $\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c)$. Again, WLOG assume $\mu^*(B) < \infty$, and so for all $\varepsilon > 0$ there exist $A_n \in \mathcal{A}$ such that $B \subseteq \bigcup_n A_n$ and

$$\mu^*(B) + \varepsilon \geq \sum_n \mu(A_n) \tag{$\circ$}$$

now $B \cap A \subseteq \bigcup_n (\underbrace{A_n \cap A}_{\in \mathcal{A}})$ and $B \cap A^c \subseteq \bigcup_n (\underbrace{A_n \cap A^c}_{A_n \setminus A \in \mathcal{A}})$. Therefore by definition of inf in μ^* and additivity of μ

$$\begin{aligned}
\mu^*(B \cap A) + \mu^*(B \cap A^c) &\leq \sum_n \left(\mu(A \cap A_n) + \mu(A^c \cap A_n)\right) \\
&= \sum_n \mu(A_n) \\
&\underbrace{\leq}_{\circ} \mu^*(B) + \varepsilon
\end{aligned}$$

since ϵ was arbitrary, $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$, so $A \in \mathcal{M}$.

Step 4: $\mathcal{M}$ is an algebra. Clearly $\emptyset \in \mathcal{M}$, and by the definition of $\mathcal{M}$ its obvious that $A^c \in \mathcal{M}$ whenever $A \in \mathcal{M}$. So let $A_1, A_2 \in \mathcal{M}$

$$\mu^*(B) = \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c), \text{ since } A_1 \in \mathcal{M}$$

$$= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c), \text{ since } A_2 \in \mathcal{M}$$

Clearly $A_1 \cap A_2^c = (A_1 \cap A_2^c) \cap A_1$ and $A_1^c = (A_1 \cap A_2)^c \cap A_1^c$ so

$$\begin{aligned}
&\mu^*(B) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1^c) \\
&= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap (A_1 \cap A_2)^c), \text{ since } A_1 \in \mathcal{M}
\end{aligned}$$

so $A_1 \cap A_2 \in \mathcal{M}$, and $\mathcal{M}$ is an algebra.

Step 5: Let $A = \bigcup_n A_n$, $A_n \in \mathcal{M}$, WLOG A_n disjoint (disjointification). Want $A \in \mathcal{M}$ and $\mu^*(A) = \sum_n \mu^*(A_n)$. By (†) we clearly have

$$\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c) + 0 + \ldots + 0$$

and

$$\mu^*(A) \leq \sum_n \mu^*(A_n)$$

so we only need two converse inequalities. Similar to before

$$\begin{aligned}
&\mu^*(B) \\
&= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
&= \mu^*(B \cap A_1) + \mu^*(B \cap \underbrace{A_1^c \cap A_2}_{=A_2 \text{ as disjoint}}) + \mu^*(B \cap A_1^c \cap A_2^c) \\
&= \sum_{n \le N} \mu^*(B \cap A_n) + \mu^*(B \cap A_1^c \cap \ldots \cap A_N^c)
\end{aligned}$$

since $\bigcup_{n \le N} \subseteq A$ so $\bigcap_{n \le N} A_n^c \supseteq A^c$, taking limits

$$\mu^*(B) \ge \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \mu^*(B \cap A^c)$$

and by (†)

$$\mu^*(B) \ge \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

so $A \in \mathcal{M}$. Applying the previous with $B = A$, we see

$$\mu^*(A) \ge \sum_{n=1}^{\infty} \mu^*(A \cap A_n) + \mu^*(A \cap A^c) = \sum_n \mu^*(A_n)$$

□

Definition. A collection $\mathcal{A}$ of subsets of E is called a π-*system* if $\emptyset \in \mathcal{A}$ and if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Definition. $\mathcal{A}$ is called a *d-system* if $E \in \mathcal{A}$, and if $B_1, B_2 \in \mathcal{A}$ such that $B_1 \subseteq B_2$, then $B_2 \setminus B_1 \in \mathcal{A}$, and if $A_n \in \mathcal{A}$, $A_n \uparrow \bigcup_n A_n = A$, then $A \in \mathcal{A}$.

One shows (Example sheet) that a d-system which is also a π-system is a σ-algebra.

Lemma (Dynkin)**.** *Let $\mathcal{A}$ be a π-system. Then any d-system that conatins $\mathcal{A}$ also contains $\sigma(\mathcal{A})$.*

Proof. Define
$$\mathcal{D} = \bigcap_{\mathcal{D}' \supseteq \mathcal{A} \text{ a d-system}} \mathcal{D}'$$
which is again a d-system (Example sheet). We show that $\mathcal{D}$ is a π-system, hence a σ-algebra containing $\mathcal{A}$. Define
$$\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \ \forall A \in \mathcal{A}\}$$
which contains $\mathcal{A}$ as $\mathcal{A}$ is a π-system. Next we show $\mathcal{D}'$ is a d-system. Clearly $E \cap A = A \in \mathcal{A} \subseteq \mathcal{D}$, so $E \in \mathcal{D}'$. Next let $B_1, B_2 \in \mathcal{D}'$ such that $B_1 \subseteq B_2$ then $(B_2 \setminus B_1) \cap A = (\underbrace{B_2 \cap A}_{\in \mathcal{D}}) \setminus (\underbrace{B_1 \cap A}_{\in \mathcal{D}}) \in \mathcal{D}$ and so $B_2 \setminus B_1 \in \mathcal{D}'$.

Next take $B_n \uparrow B$, $B_n \in \mathcal{D}'$ then $\underbrace{B_n \cap A}_{\in \mathcal{D}} \uparrow B \cap A \in \mathcal{D}$ so $B \in \mathcal{D}'$.

Hence $\mathcal{D}'$ is a d-system containing $\mathcal{A}$, so by minimality of $\mathcal{D}'$, $\mathcal{D} \subseteq \mathcal{D}'$. Conversely, by construction $\mathcal{D}' \subseteq \mathcal{D}$, so $\mathcal{D}' = \mathcal{D}$.

Next define
$$\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \ \forall A \in \mathcal{D}\}$$
which by the preceding step ($\mathcal{D}' = \mathcal{D}$) contains $\mathcal{A}$. Just as before, one shows that $\mathcal{D}'' = \mathcal{D}$ and so $\mathcal{D}$ is a π-system (as $\mathcal{D}''$ is by construction). □

Theorem (Uniqueness of extension)**.** *Let μ_1, μ_2 be measures on $(E, \mathcal{E})$ such that $\mu_1(E) = \mu_2(E) < \infty$, and suppose $\mu_1 = \mu_2$ on a π-system $\mathcal{A}$ such that $\mathcal{E} \subseteq \sigma(\mathcal{A})$. Then $\mu_1 = \mu_2$ on $\mathcal{E}$.*

Proof. Define
$$\mathcal{D} = \{A : \mu_1(A) = \mu_2(A)\}$$
which contains $\mathcal{A}$ by hypothesis. We show that $\mathcal{D}$ is a d-system, and hence by Dynkin's Lemma, contains $\sigma(\mathcal{A})$, so the theorem follows.

To see this, note first that $E \in \mathcal{D}$ by hypothesis. Next, by additivity and finiteness of μ_1, μ_2, for $B_1 \subseteq B_2$, $B_1, B_2 \in \mathcal{D}$.
$$\mu_1(B_2 \setminus B_1) = \mu_1(B_2) - \mu_1(B_1) = \mu_2(B_2) - \mu_2(B_1) = \mu_2(B_2 \setminus B_1)$$

so $B_2 \setminus B_1 \in \mathcal{D}$. Finally take $B_n \uparrow B$, $B_n \in \mathcal{D}$. This implies $B \setminus B_n \downarrow \emptyset$ and (by Example sheet) $\mu_i(B \setminus B_n) \to \mu_i(\emptyset) = 0$ for $i = 1, 2$. This implies for $\mu_i(B) < \infty$ that $\mu_i(B_n) \to \mu_i(B)$ as $n \to \infty$ for both $i = 1, 2$. But then

$$\mu_1(B) = \lim_{n\to\infty} \mu_1(B_n) = \lim_{n\to\infty} \mu_2(B_n) = \mu_2(B)$$

and so $B \in \mathcal{D}$, and thus $\mathcal{D}$ is a d-system. □

Remark: the above theorem applies to finite measures μ such that $\mu(E) < \infty$. The above theorem extends (as we will see) to σ-finite measures μ for which $E = \bigcup_{n\in\mathbb{N}} E_n$ such that $\mu(E_n) < \infty$.

Borel-σ-algebras

Definition. Let E be a topological space (Hausdorff, or metric space). The σ-algebra generated by $\mathcal{A} = \{B \subseteq E : B \text{ is open}\}$ is called the *Borel-σ-algebra*, denoted by $\mathcal{B}(E)$, or just $\mathcal{B}$ when $E = \mathbb{R}$. Elements of $\mathcal{B}(E)$ are the Borel subsets of E. A measure μ on $(E, \mathcal{B}(E))$ is called a *Borel measure on E*. A *Radon* measure μ is a Borel measure such that $\mu(K) < \infty$ for all $K \subseteq E$ compact (closed in Hausdorff spaces, hence measurable).

Construction of Lebesgue measure

We will (eventually) construct a unique Borel measure μ on $\mathbb{R}^d$ such that

$$\mu\left(\prod_{i=1}^{d}[a_i, b_i]\right) = \prod_{i=1}^{d}|b_i - a_i|,\ a_i < b_i,\ i = 1, \dots, d$$

We will do $d = 1$ first.

Theorem. *There exists a unique Borel measure (called the Lebesgue measure) μ on $\mathbb{R}$ such that*

$$\mu((a, b]) = b - a,\ \forall a < b \tag{†}$$

Proof. Consider the collection $\mathcal{A}$ of subsets of $\mathbb{R}$ of the form

$$A = (a_1, b_1] \cup \ldots \cup (a_n, b_n]$$

(intervals pairwise disjoint) which form a ring ($\emptyset = ((a, a])$, unions and differences are clear), which generates (Example sheet) generates the same σ-algebra on the open such intervals, and open intervals with rational endpoints generate $\mathcal{B}$, so $\sigma(\mathcal{A}) \supseteq \mathcal{B}$.

Define a set function μ on $\mathcal{A}$ by

$$\mu(A) = \sum_{i=1}^{n} (b_i - a_i)$$

μ is clearly additive, and well-defined since if $A = \bigcup_j C_j$ and $A = \bigcup_k D_k$ for distinct disjoint unions, then $C_j = \bigcup_k (C_j \cap D_k)$ and $D_k = \bigcup_j (D_K \cap C_k)$, so

$$\mu(A) = \mu\left(\bigcup_j C_j\right) = \sum_j \mu(C_j) = \sum_j \mu\left(\bigcup_k (C_j \cap D_k)\right)$$

$$= \sum_{j,k} \mu(C_j \cap D_k) = \ldots = \mu\left(\bigcup_k D_k\right) = \mu(A)$$

by additivity of μ. Now to prove existence of μ, we apply Caratheodory's theorem and need to check that μ is countably additive on $\mathcal{A}$. By the Example sheet, it suffices to show that for all $A_n \in \mathcal{A}$ such that $A_n \downarrow \emptyset$ we have $\mu(A_n) \to 0$.

Assume this is not the case, so there exists some $\varepsilon > 0$ and $B_n \in \mathcal{A}$ such that $B_n \downarrow \emptyset$ but $\mu(B_n) \geq 2\varepsilon$ for all n. We can approximate B_n from within by $C_n = \bigcup_{i=1}^{N_n} \left(a_{n_i} + \frac{\varepsilon 2^{-n}}{N_n}, b_{n_i}\right] \in \mathcal{A}$ such that $\mu(B_n \setminus C_n) = \varepsilon 2^{-n} \cdot \frac{N_n}{N_n} = \varepsilon 2^{-n}$.

Now since $B_n \downarrow$, we have $B_N = \bigcap_{n \leq N} B_n$ and

$$B_N \setminus (C_1 \cap \ldots \cap C_N) = B_N \cap \left(\bigcup_{n \leq N} C_n^c\right) = \bigcup_{n \leq N} B_N \setminus C_n \subseteq \bigcup_{n \leq N} B_n \setminus C_n$$

Hence since μ is increasing

$$\mu\left(B_N \setminus (C_1 \cap \ldots \cap C_N)\right) \leq \mu\left(\bigcup_{n \leq N} B_n \setminus C_n\right) \leq \sum_{n \leq N} \mu(B_n \setminus C_n) \leq \varepsilon$$

Hence the "length" of what was removed $(C_1 \cap \ldots \cap C_N)$ must be at least ε, i.e

$$\mu(C_1 \cap \ldots \cap C_N) \geq \varepsilon > 0$$

This means that $C_1 \cap \ldots \cap C_N$ is non-empty for all N, and so is

$$K_N = \overline{C_1} \cap \ldots \cap \overline{C}_N$$

($\overline{C}_i$ denotes the closure of C_i) Thus K_N is a nested sequence of non-empty closed intervals, so $\emptyset \neq \bigcap_N K_N$. But $K_N \subseteq \overline{C}_N \subseteq B_N$, so $\emptyset \neq \bigcap_N K_N \subseteq \bigcap_N B_n = \emptyset$, a contradiction. So a measure μ satisfying (†) must exist.

For uniqueness, suppose μ, λ measures such that (†) holds, and define $\mu_n(A) = \mu(A \cap (n, n+1])$, $\lambda(A) = \lambda(A \cap (n, n+1])$ for $n \in \mathbb{Z}$, which are finite measures such that $\mu_n(E) = 1 = \lambda_n(E)$ and $\mu_n = \lambda_n$ on the π-system $\mathcal{A}$. So by the uniqueness theorem, we must have $\mu_n = \lambda_n$ on $\mathcal{B}$, and

$$\mu(A) = \mu\left(\bigcup_n A \cap (n, n+1]\right) = \sum_n \mu(A \cap (n, n+1]) = \sum_n \mu_n(A)$$

$$= \sum_n \lambda_n(A) = \ldots = \lambda(A)$$

so $\lambda = \mu$. $\square$

Remarks:

1. a set $B \in \mathcal{B}$ is called a Lebesgue null set if $\mu(B) = 0$. Can write $\{x\} = \bigcap_n \left(x - \frac{1}{n}, x\right]$ and so $\mu(\{x\}) = \lim_n \frac{1}{n} = 0$. In particular $\mu((a, b)) = \mu((a, b]) = \mu([a, b])$, and any countable set Q satisfies $\mu(Q) = \mu\left(\bigcup_{q \in Q} \{q\}\right) = \sum_{q \in Q} \mu(\{q\}) = 0$. But there exist C uncountable (and measurable) in $\mathcal{B}$ such that $\mu(C) = 0$ [Cantor set].

2. Translation invariance of μ: let $x \in \mathbb{R}$, then $B + x = \{b + x : b \in B\}$ is in $\mathcal{B}$ whenever $B \in \mathcal{B}$ and we can define

$$\mu_x(B) = \mu(B + x)$$

and by uniqueness in the preceding theorem

$$\mu_x((a, b]) = \mu((a + x, b + x]) = (b + x) - (a + x) = b - a$$

so $\mu_x = \mu$.

3. Lebesgue-measurable sets: in the extension theorem, μ was assigned on the class $\mathcal{M}$, which can be shown (Example sheet) to equal

$$\mathcal{M} = \{M = A \cup N : A \in \mathcal{B}, N \subseteq B \in \mathcal{B} \text{ s.t } \mu(B) = 0\}$$