Theorem (Lawler,Schramm,Werner)**.** $\xi(1,1) = \frac{5}{4}$, $\xi(2,0) = \frac{2}{3}$.

1 Conformal maps

We consider a domain $U \subseteq \mathbb{C}$ (i.e an open and connected subset of the complex plane). We say U is *simply connected* if $\mathbb{C} \setminus U$ is connected.

We say $f : U \to \mathbb{C}$ is *holomorphic* if it is complex differentiable. If f is holomorphic and injective we say it is *univalent*. If $f : U \to V$ is holomorphic and bijective we say f is a *conformal map*.

Remark. If $f : U \to V$ is conformal then

$$f(w) = f(z) + f'(z)(w - z) + o(|w - z|)$$

and $f'(z) \neq 0$. Hence f locally looks like a translation combined with a scaling and rotation.

We will work in 2d throughout this course. This gives a richness to the conformal maps, as shown by the following theorem.

Theorem (Riemann mapping theorem)**.** *If $U \subsetneq \mathbb{C}$ is a simply connected domain and $z \in U$ then there exists a unique conformal map $f : \mathbb{D} \to U$ with $f(0) = z$ and* $\arg f'(0) = 0$.

Where we have taken $\mathbb{D} = \{z : |z| < 1\}$ to be the open unit disc. We will also take $\mathbb{H} = \{z : \Im z > 0\}$ to be the open upper half-plane.

Examples.

- Let $f(z) = \frac{z-i}{z+i}$. Then $f : \mathbb{H} \to \mathbb{D}$ is a conformal map.
- $f : \mathbb{D} \to \mathbb{D}$ is conformal if and only if $f(w) = \lambda \frac{w-z}{\bar{z}w-1}$ for some $\lambda, z \in \mathbb{C}$ with $|\lambda| = 1$, $z \in \mathbb{D}$.
- $f : \mathbb{H} \to \mathbb{H}$ is conformal if and only if $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
- Given a simply connected domain D and disjioint subarcs $A, B \subseteq \partial D$, there is a unique conformal map from U to the rectangle such that A, B are mapped to parallel sides with length 1. The length L of the other sides is called the extremal length $\mathrm{EL}_D(A, B)$ and is unique.

Recall that if $f = u + iv$ (with u, v denoting the real/imaginary parts of f respectively) then f is holomorphic iff it satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It follows from this that if f is holomorphic,

$$\Delta u = \left(\frac{\partial}{\partial x}\right)^2 u + \left(\frac{\partial}{\partial y}\right)^2 u = \frac{\partial^2}{\partial x \partial y} v - \frac{\partial^2}{\partial x \partial y} v = 0$$

and similarly $\Delta v = 0$.

Conversely, if $u : U \to \mathbb{R}$ (for U a simply connected domain) there exists $v : U \to \mathbb{R}$ such that $u + iv$ is holomorphic.

A consequence of this is that if u is harmonic on a bounded domain D and continuous on $\overline{D}$, for $z \in D$ and B a Brownian motion starting from z and $\tau := \inf\{t : B_t \notin D\}$, we have $u(z) = \mathbb{E}_z[u(B_\tau)]$ (see Part III Advanced Probability).