Introduction

Schedule:

(I) Basic properties of continuous-time Markov Chains

(II) Qualitative properties of continuous time Markov Chains

(III) Queueing theory

(IV) Renewal theory

(V) Spatial Poisson processes

1 Some basic aspects of continuous-time Markov Chains

Definition. A sequence of random variables is called a *stochastic process* or *process*. The process $X = (X_n)_{n\geq 1}$ is called a discrete-time Markov Chain with state space I if for all $x_0, x_1, \dots, x_n \in I$

$$\mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_n = x_n | X_{n-1} = x_{n-1}).$$

If $\mathbb{P}(X_{n+1} = y | X_n = x)$ is independent of n, the chain is called *time-homogeneous*. We then write $P = (P_{x,y})_{x,y\in I}$ for the *transition matrix* where $P_{x,y} = \mathbb{P}(X_1 = y | X_0 = x)$. The data associated to every time-homogeneous Markov Chain is the transition matrix P and the initial distribution μ, i.e $\mathbb{P}(X_0 = x_0) = \mu(x_0)$.

From now on:

- I denotes a countable (or finite) state space.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which all the relevant random variables are defined.

Definition. $X = (X(t) : t \geq 0)$ is a (right-continuous) continuous-time random process with values in I if

(a) for all $t \geq 0$, $X(t) = X_t$ is a random variable such that $X(t) : \Omega \to I$;

(b) for all $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous (right-continuous sample path). In our case this means for all $\omega \in \Omega$, for all $t \geq 0$, there exists $\varepsilon > 0$ (depending on ω, t) such that

$$X_t(\omega) = X_s(\omega)\ \forall s \in [t, t+\varepsilon].$$

Fact. A right-continuous random process is defined by its finite-dimensional distributions

$$\mathbb{P}(X_{t_0=i}, X_{t_1=i_1}, \dots, X_{t_n} = i_{t_n}),\ n \geq 0,\ t_k \geq 0,\ i_k \in I.$$

For every $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ of a right-continuous process stays constant for a while. So there are 3 possibilities:

(i) The path makes infinitely many jumps overall but only finitely many in a given interval $[0, t]$.

(ii) The path makes finitely many jumps & then gets absorbed in some state.

(iii) The path makes infinitely jumps in a finite time interval. After the 'explosion time' ζ, the process starts up again.

Write $J_0 = 0, J_1, J_2, \ldots$ for the jump times and $S_1, S_2, \ldots$ for the holding times, defined by

$$J_0 = 0,\ J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\},$$

$$S_n = \begin{cases} J_n - J_{n-1} & J_{n-1} < \infty \\ \infty & \text{otherwise} \end{cases}.$$

By right-continuity, $S_n > 0$ for all n. If $J_{n+1} = \infty$ for some n, we define $X_\infty = X_{J_n}$ as the final value, otherwise X_∞ is not defined. The explosion time ζ is defined by

$$\zeta = \sup(J_n) = \sum_{n=1}^{\infty} S_n.$$

We are not going to consider what happens to a chain after explosion. We thus set $X_t = \infty$ for all $t \geq \zeta$ (adjoining a new state '∞'). We call such a chain *minimal*.

Definition. We define the *jump chain* Y_n of $(X_t)_{t\geq 0}$ by setting $Y_n = X_{J_n}$ for all n.

Definition. A right-continuous random process $X = (X_t)_{t\geq 0}$ has the Markov property (and is called a continuous-time markov chain) if for all $i_1, i_2, \ldots, i_n \in I$ and $0 \leq t_1 < t_2 < \ldots < t_n$,

$$\mathbb{P}(X_{t_n} = i_n | X_{t_{n-1}} = i_{n-1}, \ldots, X_{t_0} = i_0) = \mathbb{P}(X_{t_n} = i_n | X_{t_0} = i_0).$$

Remark. For all $h > 0$, $Y_n = X(hn)$ defines a discrete-time Markov Chain.

Definition. The transition probabilities are $P_{ij}(s,t) = \mathbb{P}(X_t = j | X_s = i)$, $s \leq t$, $i, j \in I$. It is called *time-homogeneous* if it depends on $t - s$ only, i.e

$$P_{ij}(s,t) = P_{i,j}(0, t-s).$$

In this case we just write $P_{ij}(t-s)$. As in the case of discrete time, a (time-homogeneous) Markov process is characterised by

1. Its initial distribution $\lambda_i = \mathbb{P}(X_0 = i)$, $i \in I$;
2. Its *family of transition matrices* $(P(t))_{t\geq 0} = (P_{ij}(t))_{t\geq 0}$.

The family $(P(t))_{t\geq 0}$ is called the *transition subgroup* of the MC.