Theorem (Lawler,Schramm,Werner). $\xi(1,1) = \frac{5}{4}$, $\xi(2,0) = \frac{2}{3}$.

1 Conformal maps

We consider a domain $U \subseteq \mathbb{C}$ (i.e an open and connected subset of the complex plane). We say U is *simply connected* if $\mathbb{C} \setminus U$ is connected.

We say $f : U \to \mathbb{C}$ is *holomorphic* if it is complex differentiable. If f is holomorphic and injective we say it is *univalent*. If $f : U \to V$ is holomorphic and bijective we say f is a *conformal map*.

Remark. If $f : U \to V$ is conformal then

$$f(w) = f(z) + f'(z)(w - z) + o(|w - z|)$$

and $f'(z) \neq 0$. Hence f locally looks like a translation combined with a scaling and rotation.

We will work in 2d throughout this course. This gives a richness to the conformal maps, as shown by the following theorem.

Theorem (Riemann mapping theorem). *If $U \subsetneq \mathbb{C}$ is a simply connected domain and $z \in U$ then there exists a unique conformal map $f : \mathbb{D} \to U$ with $f(0) = z$ and $\arg f'(0) = 0$.*

Where we have taken $\mathbb{D} = \{z : |z| < 1\}$ to be the open unit disc. We will also take $\mathbb{H} = \{z : \Im z > 0\}$ to be the open upper half-plane.

Examples.

- Let $f(z) = \frac{z-i}{z+i}$. Then $f : \mathbb{H} \to \mathbb{D}$ is a conformal map.
- $f : \mathbb{D} \to \mathbb{D}$ is conformal if and only if $f(w) = \lambda \frac{w-z}{\bar{z}w-1}$ for some $\lambda, z \in \mathbb{C}$ with $|\lambda| = 1$, $z \in \mathbb{D}$.
- $f : \mathbb{H} \to \mathbb{H}$ is conformal if and only if $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.
- Given a simply connected domain D and disjioint subarcs $A, B \subseteq \partial D$, there is a unique conformal map from U to the rectangle such that A, B are mapped to parallel sides with length 1. The length L of the other sides is called the extremal length $\mathrm{EL}_D(A, B)$ and is unique.

Recall that if $f = u + iv$ (with u, v denoting the real/imaginary parts of f respectively) then f is holomorphic iff it satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It follows from this that if f is holomorphic,

$$\Delta u = \left(\frac{\partial}{\partial x}\right)^2 u + \left(\frac{\partial}{\partial y}\right)^2 u = \frac{\partial^2}{\partial x \partial y} v - \frac{\partial^2}{\partial x \partial y} v = 0$$

and similarly $\Delta v = 0$.

Conversely, if $u : U \to \mathbb{R}$ (for U a simply connected domain) is harmonic there exists $v : U \to \mathbb{R}$ such that $u + iv$ is holomorphic.

A consequence of this is that if u is harmonic on a bounded domain D and continuous on $\overline{D}$, for $z \in D$ and B a Brownian motion starting from z and $\tau := \inf\{t : B_t \notin D\}$, we have $u(z) = \mathbb{E}_z[u(B_\tau)]$ (see Part III Advanced Probability).

Conformal invariance of 2d Brownian motion

Let $f : D \to \tilde{D}$ be a conformal map and B be a Brownian motion starting at $z \in \mathbb{C}$. Define $\tau = \inf\{t : B_t \notin D\}$ and let $\sigma(t) = \inf\{s : \int_0^s |f'(B_r)|^2 \mathrm{d}r = t\}$. Then $f(B_{\sigma(t)})$ has the law of a Brownian motion starting from $f(z)$ until exiting $\tilde{D}$.

Proof. See Part III Stochastic Calculus. $\square$

We have seen that for u harmonic on D and continuous on $\overline{D}$ we have $u(z) = \mathbb{E}_z[u(B_{\tau_D})]$. We get the following corollary by taking a Brownian motion until it hits $\partial B(z, r)$.

Corollary (Mean value property)**.** *For $B(z, r) \subseteq D$*

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \mathrm{d}\theta.$$

Proposition (Strong maximum principle)**.** Let u be harmonic in D, D a domain. If u attains a global maximum in D then u is constant.

Proof. Follows fom mean value property and compactness of paths connecting points. $\square$

Proposition (Maximum modulus principle)**.** Let $f : D \to \mathbb{C}$ holomorphic, D a domain. Then if $|f|$ attains a global maximum in D, f is constant.

Proof. Let $K \subseteq D$ be compact. By considering $f + M$ for $M > 0$ large enough we may assume $|f| > 0$ on K. Thus $\log |f|$ is harmonic. So we can apply the strong maximum principle to see $\log |f|$ is constant on K, i.e f takes values on a circle. But this is impossible unless $f' = 0$ on K. $\square$

Proposition (Schwarz lemma)**.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. Furthermore if $|f(z)| = |z|$ for some $z \neq 0$ then $f(w) = we^{i\theta}$ for some $\theta \in \mathbb{R}$.

Proof. Define the holomorphic function $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \frac{f(z)}{z} & \text{for } z \neq 0 \\ f'(0) & \text{for } z = 0 \end{cases}.$$

Then $|z| = 1$ on $\partial\mathbb{D}$, implying $|g| \leq 1$ on $\partial\mathbb{D}$. Thus $|g| \leq 1$ on $\mathbb{D}$ by the maximum modulus principle.

If $|g(z)| = 1$ for some $z \in \mathbb{D}$ then g is constant since this is a maximum. $\square$

Distortion theorems for conformal maps

Let $\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} \text{ univalent} : f(0) = 0, f'(0) = 1\}$.

Remark. We can write such f as $f(z) = z + a_2 z^2 + a_3 z^2 + \ldots$.

Goal: for $f \in \mathcal{S}$

- Koebe 1/4-theorem: $f(\mathbb{D}) \supseteq B(0, 1/4)$;
- Koebe distortion theorem: $\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}$.

Corollary. *If $f : D \to \tilde{D}$ is conformal then*

$$\frac{\operatorname{dist}(f(z), \partial\tilde{D})}{4\operatorname{dist}(z, \partial D)} \le |f'(z)| \le \frac{4\operatorname{dist}(f(z), \partial\tilde{D})}{\operatorname{dist}(z, D)}.$$

Corollary. *If f univalent in D, $B(z, R) \subseteq D$ then for $r < 1$ we have $|f'(u)| \le c(r)|f'(v)|$ for all $u, v \in B(z, rR)$.*

Define

$$\Sigma = \{g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} : g \text{ univalent},\ g(\infty) = \infty,\ g'(\infty) = 1\}.$$

Theorem (Area theorem)**.** *Let $g : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$ be univalent with $g(z) \to \infty$ as $z \to \infty$ and $g'(z) \to 1$ as $z \to \infty$. Write $g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z} + \dots$ for g near ∞. Then*

$$\sum_{n\geq 1} n|b_n|^2 \leq 1$$

and moreover

$$\operatorname{area}(\mathbb{C} \setminus g(\mathbb{C} \setminus \overline{\mathbb{D}})) = \pi \left(1 - \sum_{n\geq 1} n|b_n|^2\right).$$

Proof. Let $r > 1$ and define $C_r = g(\partial D(0, r))$. Let E_r be the inner component of $\mathbb{C} \setminus C_r$. By Green's theorem

$$\begin{aligned}
\frac{1}{2i}\int_{C_r} \overline{w}\mathrm{d}w &= \frac{1}{2i}\int_{C_r} (x - iy)(\mathrm{d}x + i\mathrm{d}y) \\
&= \frac{1}{2i}\int_{C_r} ((x - iy)\mathrm{d}x + (ix + y)\mathrm{d}y) \\
&= \frac{1}{2i}\int_{E_r} 2i\mathrm{d}x\mathrm{y} && \text{(Green's thm)} \\
&= \operatorname{area}(E_r).
\end{aligned}$$

while we also have

$$\begin{aligned}
\frac{1}{2i}\int_{C_r} \overline{w}\mathrm{d}w &= \frac{1}{2i}\int_{\partial B(0,r)} \overline{g(z)}g'(z)\mathrm{d}z \\
&= \frac{1}{2}\int_0^{2\pi} \left(re^{-i\theta} + \sum_{n\geq 1} \overline{b_n}r^{-n}e^{in\theta}\right)\left(1 - \sum_{n\geq 1} b_n r^{-n-1}e^{i(n+1)\theta}\right) re^{i\theta}\mathrm{d}\theta \\
&= \pi\left(r^2 - \sum_{n\geq 1} n|b_n|^2 r^{-2n}\right).
\end{aligned}$$

Now take $r \downarrow 1$. $\square$

Theorem. *Let $f : \mathbb{D} \to \mathbb{C} \in \mathcal{S}$ write $f(z) = z + a_2z^2 + a_3z^3 + \dots$. Then $|a_2| \leq 2$.*

Proof. We claim there exists $g \in \mathcal{S}$ with $g(z)^2 = f(z^2)$ (we call g the "square-root transform" of f). Note

$$f(z^2) = z^2(\underbrace{1 + a_2z^2 + a_3z^4 + \dots}_{:=h(z)})$$

and since $h \neq 0$ (by $f(0) = 0$ and injectivity of f), we can define $g(z) = z\sqrt{h(z)}$. Also $g(0) = 0$ and $g'(0) = 1$. To show g is univalent, suppose $g(z_1) = g(z_2)$ for some $z_1, z_2 \in \mathbb{D}$. Then $f(z_1^2) = f(z_2^2)$ so $z_1^2 = z_2^2$, i.e $z_1 = \pm z_2$. But g is an odd function and only zero at $z = 0$ so we have $z_1 = z_2$.

To conlcude take $z \mapsto \frac{1}{g(1/z)} \in \Sigma$. This map is the same as

$$z \mapsto \frac{1}{\sqrt{f(1/z^2)}} = z - \frac{a_2}{2}\frac{1}{z} + \dots$$

so by the area theorem, $|a_2/2| \leq 1$. □

Theorem (Koebe 1/4-theorem)**.** *Let $f \in \mathcal{S}$. Then $f(\mathbb{D}) \supseteq B(0, 1/4)$.*

Proof. Let $w \notin f(\mathbb{D})$. Then

$$z \mapsto \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \dots$$

is in $\mathcal{S}$ so by the above $\left|a_2 + \frac{1}{w}\right| \leq 2$. Since $|a_2| \leq 2$ we must have $|1/w| \leq 4$. □

If we define

$$F(w) = \frac{f\left(\frac{w+z}{1+\overline{z}w}\right) - f(z)}{(1-|z|^2)f'(z)} = w + \frac{1}{2}\left((1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right) w^2 + \dots$$

we see

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\overline{z}\right| \leq 4.$$

Note

$$\begin{aligned} z\frac{f''(z)}{f'(z)} &= z\partial_z \log f'(z) = r\partial_r \log f'(z) \\ &= r\partial_r \log|f'(z)| + ir\partial_r \arg(f'(z)) \end{aligned}$$

and

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}$$

which implies

$$\frac{2r^2}{1-r^2} - \frac{4r}{1-r^2} \leq \Re\left(z\frac{f''(z)}{f'(z)}\right) \leq \frac{2r^2}{1-r^2} + \frac{4r}{1-r^2}.$$

Integrating from $r = 0$ to R,

$$\log\frac{1-R}{(1+R)^3} \leq \log|f'(Re^{i\theta})| \leq \log\frac{1+R}{(1-R)^3}.$$

So we get

Theorem (Kobe's distortion theorem)**.** *For $f \in \mathcal{S}$,*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

Definition. $A \subseteq \mathbb{H}$ is a *compact* $\mathbb{H}$*-hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. We write $A \in \mathcal{Q}$ for such a set.

For $A \in \mathcal{Q}$, pick $g : \mathbb{H} \setminus A \to \mathbb{H}$ conformal (possible by Riemann mapping theorem) with $g(\infty) = \infty$.

Question: when does a holomorphic function extend analytically to the boundary?

Theorem (Schwarz reflection principle)**.** *Let $U \subseteq \mathbb{C}$ be a domain such that $U = \{\overline{z} : z \in U\}$. Let $U^+ = U \cap \mathbb{H}$. Let $f : U^+ \to \mathbb{C}$ be holomorphic with $\lim_{\Im z \downarrow 0} \Im f(z) = 0$. Then f extends to a holomorphic function on U with $f(\overline{z}) = \overline{f(z)}$ for all $z \in U$.*

Proof. On $U^- := U \cap \{z : \Im(z) < 0\}$ set $f(z) := \overline{f(\overline{z})}$. To extend f to $U \cap \mathbb{R}$, write $f = u + iv$ for u, v harmonic and note $\lim_{\Im z \downarrow 0} v(z) = 0$. So we have extended v via

$$v(z) = \begin{cases} -v(\overline{z}) & \Im z < 0 \\ 0 & \Im z = 0 \end{cases}.$$

Then v is still harmonic as it satisfies the mean value property.

For $z \in U \cap \mathbb{R}$ pick $\varepsilon > 0$ so that $B(z, \varepsilon) \subseteq U$. Let $\tilde{u}$ be the harmonic conjugate of v on $B(z, \varepsilon)$ (unique up to an additive constant). Then $f = u + iv = \tilde{u} + iv + \text{const}$ so f extends to $B(z, \varepsilon)$. Furthermore this matches with $f(z) = \overline{f(\overline{z})}$ on U^-. For different z these extensions match so by the identity principle we are done. $\square$

Now for $A \in \mathcal{Q}$, $g : \mathbb{H} \setminus A \to \mathbb{H}$ conformal with $g(\infty) = \infty$, we can Schwarz reflect. g has a simple pole at ∞ so

$$g(z) = b_{-1} z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots.$$

Also $g(z) = \overline{g(\overline{z})} = \overline{g(z)}$ for $z \in \mathbb{R}$ which implies $b_n \in \mathbb{R}$ for all $n \geq -1$. So we can scale and then translate g so that $b_{-1} = 1$ and $b_0 = 0$.

Definition. For $A \in \mathcal{Q}$, let $g_A \colon \mathbb{H} \setminus A \to \mathbb{H}$ the conformal map with $g_A(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$.

Define the *half-plane capacity* $\operatorname{hcap}(A)$ to be equal to $b_1 \in \mathbb{R}$ as above.

For example we have $g_{[0,i]}(z) = \sqrt{z^2 + 1}$ and so $\operatorname{hcap}([0, i]) = \frac{1}{2}$ (we can see this by looking at what happens to $\mathbb{H} \setminus [0, i]$ under $z \mapsto z^2 \mapsto z^2 + 1 \mapsto \sqrt{z^2 + 1}$).

If A is instead a $\overline{\mathbb{D}} \cap \mathbb{H}$ with radius 1 centred at 0, we have $g_A(z) = z + \frac{1}{z}$ so $\operatorname{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = 1$.

It is straighforward to see $g_{rA}(z) = rg_A(z/r)$ for any $r > 0$ and so $\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A)$. Can also see that $\operatorname{hcap}(A + x) = \operatorname{hcap}(A)$ for any $x \in \mathbb{R}$.

For $A \subseteq \tilde{A}$ can also see that

$$g_{\tilde{A}} = g_{g_A(\tilde{A}\setminus A)} \circ g_A = z + \frac{\operatorname{hcap}(A)}{z} + \frac{\operatorname{hcap}(g_A(\tilde{A} \setminus A))}{z} + \dots$$

so $\operatorname{hcap}(\tilde{A}) = \operatorname{hcap}(A) + \operatorname{hcap}(g_A(\tilde{A}\setminus A))$. Thus $\operatorname{hcap}(A) \leq \operatorname{hcap}(\tilde{A})$ (after seeing later that hcap is non-negative). Also $\operatorname{hcap}(A) \leq \operatorname{hcap}(\operatorname{rad}(A) \cdot \overline{\mathbb{D}} \cap \mathbb{H}) \leq \operatorname{rad}(A)^2$ where $\operatorname{rad}(A) = \sup\{|z| : z \in A\}$.