1 Motivation

This section is motivation and will not be rigorous. We have a 'Dirac delta function' such that for all 'nice' functions f

$$\int_{-\infty}^{\infty} \delta(x-x_0)f(x)\mathrm{d}x = f(x_0).$$

Can we define $\delta'(x-x_0)$? Could try

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x-x_0)f(x)\mathrm{d}x &= \lim_{h\to 0}\int_{-\infty}^{\infty}\left[\frac{\delta(x-x_0+h)-\delta(x-x_0)}{h}\right]f(x) \\ &= \lim_{h\to 0}\frac{1}{h}\left[f(x_0-h)-f(x_0)\right] \\ &= -f'(x_0).\end{aligned}$$

i.e

$$\int_{-\infty}^{\infty} \delta'(x-x_0)f(x)\mathrm{d}x = -\inf_{-\infty}^{\infty} \delta(x-x_0)f'(x)\mathrm{d}x$$

which looks like some form of integration-by-parts. We can make this rigorous using distribution theory.

Fourier transform of polynomials

If $f \in L^1(\mathbb{R})$ then

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}f(x)\mathrm{d}x.$$

How could we take the Fourier transform of something like $f(x) = x^n$? May recall identity

$$\delta(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\mathrm{d}x$$

and then get

$$\begin{aligned}\hat{f}(\lambda) &= \int_{-\infty}^{\infty} x^n e^{-\lambda x}\mathrm{d}x \\ &= \left(i\frac{\partial}{\partial\lambda}\right)^n\int_{-\infty}^{\infty} e^{-i\lambda x}\mathrm{d}x \\ &= i^n 2\pi\delta^{(n)}(\lambda).\end{aligned}$$

Recall Parseval's theorem: for suitable f, g

$$\int_{-\infty}^{\infty} \hat{g}(\lambda)f(\lambda)\mathrm{d}\lambda = \int_{-\infty}^{\infty} g(x)\hat{f}(x)\mathrm{d}x.$$

Could define the Fourier transform of $g(x) = x$ to be the function $\lambda \mapsto \hat{x}(\lambda)$ such that

$$\int_{-\infty}^{\infty} \hat{x}(\lambda)f(\lambda)\mathrm{d}\lambda = \int_{-\infty}^{\infty} x\hat{f}(x)\mathrm{d}x$$

for all 'nice' functions f. We can make this rigorous using distributions.

Discontinuous solutions to PDEs

From linear acoustics, air pressure $p = p(x,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0. \tag{$*$}$$

Could introduce a 'nice' $f = f(x,t)$, say $f \in C_c^\infty(\mathbb{R}^2)$. Then $(*)$ implies

$$\int\int (p_{xx} - p_{tt}) f(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

So we can integrate by parts to interpret this as

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

We say that $p = p(x,t)$ is a *weak solution* to $(*)$ if

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. In each case, to extend a definition to a larger domain of applicability, we had to introduce a space of "nice" functions.

This is the theme of distribution theory: functions get replaced by linear maps on some auxilliary space of test functions V. A *distribution* is a linear map $u : V \to \mathbb{C}$, i.e we study the topological dual of V. Let $\langle \cdot, \cdot \rangle$ denote pairing between v and V^*, i.e for $u \in V^*$, $f, g \in V$, $\alpha, \beta \in \mathbb{C}$

$$\langle u, \alpha f + \beta g \rangle = \alpha \langle u, f \rangle + \beta \langle u, g \rangle.$$

The topological dual V^* consists of linear $u : V \to \mathbb{C}$ such that whenever $f_n \to f$ in V, we have $\langle u, f_n \rangle \to \langle u, f \rangle$ in $\mathbb{C}$. For example we could take $V = C^\infty(\mathbb{R})$ equipped with the topology of uniform convergence (i.e $f_n \to f$ in V if for all compact $K \subseteq \mathbb{R}$ and all $n \geq 0$, $\left| \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (f_n - f) \right| \to 0$) then $\delta_{x_0} : V \to bbC$ defined by $\langle \delta_{x_0}, f \rangle = f(x_0)$. Note that this is indeed continuous.