1 Motivation

This section is motivation and will not be rigorous. We have a 'Dirac delta function' such that for all 'nice' functions f

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) \mathrm{d}x = f(x_0).$$

Can we define $\delta'(x - x_0)$? Could try

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x &= \lim_{h \to 0} \int_{-\infty}^{\infty} \left[\frac{\delta(x - x_0 + h) - \delta(x - x_0)}{h}\right] f(x) \\ &= \lim_{h \to 0} \frac{1}{h} \left[f(x_0 - h) - f(x_0)\right] \\ &= -f'(x_0).\end{aligned}$$

i.e

$$\int_{-\infty}^{\infty} \delta'(x - x_0) f(x) \mathrm{d}x = -\int_{-\infty}^{\infty} \delta(x - x_0) f'(x) \mathrm{d}x$$

which looks like some form of integration-by-parts. We can make this rigorous using distribution theory.

Fourier transform of polynomials

If $f \in L^1(\mathbb{R})$ then

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) \mathrm{d}x.$$

How could we take the Fourier transform of something like $f(x) = x^n$? May recall identity

$$\delta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x$$

and then get

$$\begin{aligned}\hat{f}(\lambda) &= \int_{-\infty}^{\infty} x^n e^{-\lambda x} \mathrm{d}x \\ &= \left(i \frac{\partial}{\partial \lambda}\right)^n \int_{-\infty}^{\infty} e^{-i\lambda x} \mathrm{d}x \\ &= i^n 2\pi \delta^{(n)}(\lambda).\end{aligned}$$

Recall Parseval's theorem: for suitable f, g

$$\int_{-\infty}^{\infty} \hat{g}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} g(x) \hat{f}(x) \mathrm{d}x.$$

Could define the Fourier transform of $g(x) = x$ to be the function $\lambda \mapsto \hat{x}(\lambda)$ such that

$$\int_{-\infty}^{\infty} \hat{x}(\lambda) f(\lambda) \mathrm{d}\lambda = \int_{-\infty}^{\infty} x \hat{f}(x) \mathrm{d}x$$

for all 'nice' functions f. We can make this rigorous using distributions.

Discontinuous solutions to PDEs

From linear acoustics, air pressure $p = p(x,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0. \tag{$*$}$$

Could introduce a 'nice' $f = f(x,t)$, say $f \in C_c^\infty(\mathbb{R}^2)$. Then $(*)$ implies

$$\int\int (p_{xx} - p_{tt}) f(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

So we can integrate by parts to interpret this as

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0.$$

We say that $p = p(x,t)$ is a *weak solution* to $(*)$ if

$$\int\int (f_{xx} - f_{tt}) p(x,t) \mathrm{d}x \mathrm{d}t = 0$$

for all $f \in C_c^\infty(\mathbb{R}^2)$. In each case, to extend a definition to a larger domain of applicability, we had to introduce a space of "nice" functions.

This is the theme of distribution theory: functions get replaced by linear maps on some auxilliary space of test functions V. A *distribution* is a linear map $u : V \to \mathbb{C}$, i.e we study the topological dual of V. Let $\langle \cdot, \cdot \rangle$ denote pairing between v and V^*, i.e for $u \in V^*$, $f, g \in V$, $\alpha, \beta \in \mathbb{C}$

$$\langle u, \alpha f + \beta g \rangle = \alpha \langle u, f \rangle + \beta \langle u, g \rangle.$$

The topological dual V^* consists of linear $u : V \to \mathbb{C}$ such that whenever $f_n \to f$ in V, we have $\langle u, f_n \rangle \to \langle u, f \rangle$ in $\mathbb{C}$. For example we could take $V = C^\infty(\mathbb{R})$ equipped with the topology of uniform convergence (i.e $f_n \to f$ in V if for all compact $K \subseteq \mathbb{R}$ and all $n \geq 0$, $\left| \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^n (f_n - f) \right| \to 0$) then $\delta_{x_0} : V \to bbC$ defined by $\langle \delta_{x_0}, f \rangle = f(x_0)$. Note that this is indeed continuous.

2 Distributions

2.1 Notation & Preliminaries

Throughout (unless otherwise specified) X, Y denote open subsets of $\mathbb{R}^n$, K a compact subset of $\mathbb{R}^n$. Integrals over $X, \mathbb{R}^n$ are written as $\int_X [\cdot]\mathrm{d}x$, $\int [\cdot]\mathrm{d}x$ respectively.

2.2 Distributions & Test Functions

Definition. The space $\mathcal{D}(X)$ consists of smooth functions $\varphi : X \to \mathbb{C}$ of compact support. We say a sequence $(\varphi_m)_{m\geq 0}$ in $\mathcal{D}(X)$ converges to 0 in $\mathcal{D}(X)$ if there exists $K \subseteq X$ compact such that $\mathrm{supp}(\varphi_m) \subseteq K$ and $\sup_K |\partial^\alpha \varphi_m| \to 0$ for all multi-indices α.

Functions in $\mathcal{D}(X)$ have nice properties. For example, if $\varphi \in \mathcal{D}(X)$ then $\varphi = 0$ before you reach the boundary of X. This means integration-by-parts is easy since

$$\int_X \varphi \partial^\alpha \psi \mathrm{d}x = (-1)^{|\alpha|} \int_X \psi \partial^\alpha \varphi \mathrm{d}x.$$

Since $\varphi \in \mathcal{D}(X)$ is smooth we have

$$\varphi(x+h) = \sum_{|\alpha|\leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h)$$

where R_N is $o(|h|^N)$ uniformly in x.

Definition. A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is called a *distribution* if for all $K \subseteq X$ compact there exist $C, N \geq 0$ such that

$$|u(\varphi)| \leq C \sum_{|\alpha|\leq N} \sup_K |\partial^\alpha \varphi| \tag{$*$}$$

for all $\varphi \in \mathcal{D}(X)$ with $\mathrm{supp}(\varphi) \subseteq K$. The space of such linear maps is denoted by $\mathcal{D}'(X)$, i.e "distributions on X". If the same N can be used in $(*)$ for all compact $K \subseteq X$, say the least such N is the order of u, written $\mathrm{ord}(u)$.

For $x_0 \in X$ define $\delta_{x_0}(\varphi) = \varphi(x_0)$ for $\varphi \in \mathcal{D}(X)$. Then $\delta_{x_0} : \mathcal{D}(X) \to \mathbb{C}$ is linear and

$$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq \sup |\varphi|$$

so we can take $C = 1, N = 0$ in $(*)$, so $\mathrm{ord}(\delta_{x_0}) = 0$.

For $\{f_\alpha\}$ in $C(X)$, define $T : \mathcal{D}(X) \to \mathbb{C}$ by

$$T(\varphi) = \sum_{|\alpha|\leq M} \int_X f_\alpha \partial^\alpha \varphi \mathrm{d}x.$$

Take $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$. Then

$$\begin{aligned} |T(\varphi)| &\leq \sum_{|\alpha| \leq M} \int_K |f_\alpha||\partial^\alpha \varphi| \mathrm{d}x \\ &\leq \left(\max_\alpha \int_K |f_\alpha| \mathrm{d}x \right) \sum_{|\alpha| \leq M} \sup |\partial^\alpha \varphi| \end{aligned}$$

so $(*)$ holds with $C = \max_\alpha \int_K |f_\alpha| \mathrm{d}x$, $N = M$. Hence $T \in \mathcal{D}'(X)$.

Note this estimate would hold if the $\{f_\alpha\}$ were only assumed locally integrable, written $f_\alpha \in L^1_{\mathrm{loc}}(X)$.

Remark. For $f \in L^1_{\mathrm{loc}}$ we have a corresponding distribution $T_f : \mathcal{D}(X) \to \mathbb{C}$ defined by $T_f(\varphi) = \int_X f\varphi \mathrm{d}x$. We often simply write $T_f = f$.

Lemma. *A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is a distribution if and only if $u(\varphi_m) \to 0$ for all sequences $\varphi_m \to 0$ in $\mathcal{D}(X)$.*

Proof. Suppose $u \in \mathcal{D}'(X)$ and $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then $\operatorname{supp}(\varphi_m) \subseteq K$ for some K independent of m and there exist $C, N \geq 0$

$$|\varphi_m(u)| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi_m| \to 0$$

for all α.

Suppose not, i.e $u : \mathcal{D}(X) \to \mathbb{C}$ is linear and $u(\varphi_m) \to 0$ whenever $\varphi_m \to 0$ in $\mathcal{D}(X)$, but u is not a distribution. Then there is a compact set $K \subseteq X$ such that for all C, N, $(*)$ fails on some φ with support contained in K. So there must be some $\varphi_m \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi_m) \subseteq K$ and

$$|u(\varphi_m)| > m \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Now replace φ_m with $\varphi'_m = \frac{\varphi_m}{u(\varphi_m)}$. So we may assume $u(\varphi_m) = 1$ WLOG. Hence

$$1 > m \sum_{|\alpha| \leq m} \sup_K |\partial^\alpha \varphi_m|.$$

Therefore $\sup_K |\partial^\alpha \varphi_m| < \frac{1}{m}$ for all $|\alpha| \leq m$. Hence $\varphi_m \to 0$ in $\mathcal{D}(X)$, giving a contradiction since $u(\varphi_m) \not\to 0$. $\square$

2.3 Limits in $\mathcal{D}'(X)$

We often have some sequence (u_m) in $\mathcal{D}'(X)$. If there is some $u \in \mathcal{D}'(X)$ such that $\varphi(u_m) \to \varphi(u)$ for all φ we say $u_m \to u$ in $\mathcal{D}'(X)$.

Theorem (*Non-examinable*). *If (u_m) is a sequence in $\mathcal{D}'(X)$ and $u(\varphi) = \lim_{m\to\infty} u(\varphi_m)$ exists for all $\varphi \in \mathcal{D}(X)$, then $u \in \mathcal{D}'(X)$.*

Proof. Not given. □

Take $u_m \in \mathcal{D}'(\mathbb{R})$ defined by $u_m(\varphi) = \int \sin(mx)\varphi(x)\mathrm{d}x$. By integration-by-parts we have

$$|\varphi(u_m)| = \left|\frac{1}{m}\int \cos(mx)\varphi'(x)\mathrm{d}x\right| \to 0.$$

i.e $\sin(mx) \to 0$ in $\mathcal{D}'(\mathbb{R})$.

2.4 Basic Operations

2.4.1 Differentiation & Multiplication by Smooth Functions

For $u \in C^\infty(X) \subseteq L^1_{\mathrm{loc}}(X)$, $\partial^\alpha u \in D'(X)$ by

$$\begin{aligned}\langle \partial^\alpha u, \phi \rangle &= \int_X \phi \partial^\alpha u \mathrm{d}x \\ &= (-1)^{|\alpha|} \int_X u \partial^\alpha \phi \mathrm{d}x \\ &= (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle.\end{aligned}$$

This leads to

Definition. For $u \in D'(X)$, $f \in C^\infty(X)$ define

$$\langle \partial^\alpha (fu), \phi \rangle := (-1)^\alpha \langle u, f \partial^\alpha \phi \rangle$$

for $\phi \in \mathcal{D}(X)$ [note $\partial^\alpha(fu) \in \mathcal{D}'(X)$]. We call $\partial^\alpha u$ the *distributional derivatives* of u.

For δ_x we have

$$\begin{aligned}\langle \partial^\alpha \delta_x, \phi \rangle &= (-1)^{|\alpha|} \langle \delta_x, \partial^\alpha \phi \rangle \\ &= (-1)^{|\alpha|} \partial^\alpha \phi(x).\end{aligned}$$

Define the *Heaviside function*

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Then $H \in L^1_{\mathrm{loc}}(\mathbb{R})$ so

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x) \mathrm{d}x = \phi(0) = \langle \delta_0, \phi \rangle.$$

Hence $H' = \delta_0$. Generally we say $u = v$ in $\mathcal{D}'(X)$ if $\langle u, \cdot \rangle = \langle v, \cdot \rangle$.

Lemma. *If $u \in \mathcal{D}'(\mathbb{R})$ and $u' = 0$ in $\mathcal{D}'(\mathbb{R})$ then u is constant.*

Proof. Fix $\theta \in \mathcal{D}(\mathbb{R})$ with $\langle 1, \theta \rangle = \int_{\mathbb{R}} \theta \mathrm{d}x = 1$. For $\phi \in \mathcal{D}(\mathbb{R})$ write

$$\phi = \underbrace{(\phi - \langle 1, \phi \rangle \theta)}_{:= \phi_A} + \underbrace{\langle 1, \phi \rangle \theta}_{:= \phi_B}.$$

Note that $\langle 1, \phi_A \rangle = \int_{\mathbb{R}} \phi_A \mathrm{d}x = 0$ so we have

$$\Phi_A(x) := \int_{-\infty}^x \phi_A(t) \mathrm{d}t$$

defines $\Phi_A \in \mathcal{D}(\mathbb{R})$ with $\Phi_A' = \phi_A$. So

$$\begin{aligned}\langle u, \phi\rangle &= \langle u, \phi_A\rangle + \langle u, \phi_B\rangle \\ &= \langle u, \Phi_A'\rangle + \langle 1, \phi\rangle\langle u, \theta\rangle \\ &= \underbrace{-\langle u', \phi_A\rangle + \langle 1, \phi\rangle}_{=0} + \underbrace{\langle 1, \phi\rangle\langle u, \theta\rangle}_{:=c \text{ constant}}\end{aligned}$$

so u is constant in $\mathcal{D}'(\mathbb{R})$. □

2.4.2 Translation & Reflection

If $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ define *reflection* and *translation* by

$$\check{\phi}(x) = \phi(-x),\ (\tau_h\phi)(x) = \phi(x-h).$$

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ we define

$$\langle \check{u}, \phi\rangle = u, \check{\phi} \qquad \text{(reflection)}$$

and

$$\langle \tau_h u, \phi\rangle = \langle u, \tau_{-h}\phi\rangle \qquad \text{(translation)}$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

Lemma. *For $u \in \mathcal{D}'(\mathbb{R}^n)$ define*

$$v_h = \frac{\tau_{-h}u - u}{h}.$$

fIf $\frac{h}{|h|} \to m \in \mathbb{S}^{n-1}$ as $|h| \to 0$ then $v_h \to m \cdot \partial u$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. For $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\langle v_h, \phi\rangle = \langle u, \frac{\tau_h\phi - \phi}{h}\rangle.$$

By Taylor's theorem

$$(\tau_h\phi - \phi)(x) = \phi(x-h) - \phi(x) = -\sum_i h_i \frac{\partial\phi}{\partial x_i}(x) + R_1(x,h)$$

where $R_1 = o(|h|)$ in $\mathcal{D}(\mathbb{R}^n)$ [see Example Sheet 1] so by sequential continuity

$$\begin{aligned}\langle v_h, \phi\rangle &= -\sum_i \frac{h_i}{|h|}\langle u, \frac{\partial\phi}{\partial x_i}\rangle + o(1) \\ &= \langle \sum_i \frac{h_i}{|h|}\frac{\partial u}{\partial x_i}, \phi\rangle + o(1) \\ &\to \langle m \cdot \partial u, \phi\rangle \text{ as } |h| \to 0.\end{aligned}$$

□

2.4.3 Convolution in $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^{\ltimes})$

For $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(\tau_x \check{\phi})(y) = \check{\phi}(y-x) = \phi(x-y).$$

If $u \in C^\infty(\mathbb{R}^n)$ define convolution with $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned} u * \phi(x) &= \int_{\mathbb{R}^n} u(x-y)\phi(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \phi(x-y)u(y)\mathrm{d}y \\ &= \langle u, \tau_x \check{\phi}\rangle. \end{aligned}$$

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ define

$$(u * \phi)(x) = \langle u, \tau_x \check{\phi}\rangle.$$

How regular is $u * \phi$?

Lemma. *For $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ write $\Phi_x(y) = \phi(x,y)$. If for each $x \in \mathbb{R}^n$ there exists a neighbourhood $N_x \subseteq \mathbb{R}^n$ of x and compact set $K \subseteq \mathbb{R}^n$ such that*

$$\operatorname{supp}(\phi|_{N_x \times \mathbb{R}^n}) \subseteq N_x \times K$$

then $\partial_x^\alpha \langle u, \Phi_x\rangle = \langle u, \partial_x^\alpha \Phi\rangle$ for $u \in \mathcal{D}'(\mathbb{R}^n)$.

Proof. By Taylor's theorem

$$\Phi_{x+h}(y) - \Phi_x(y) = \sum_i h_i \frac{\partial \phi}{\partial x_i}(x,y) + R_1(x,y,h).$$

For $|h|$ sufficienly small we have $x + h \in N_x$ so $\operatorname{supp}(R_1(x,\cdot,h)) \subseteq K$ and also

$$sup_y |\partial_y^\alpha R(x,y,h)| = o(|h|)$$

so $R_1(x,\cdot,h) = o(|h|)$ in $\mathcal{D}(\mathbb{R}^n)$. By sequential continuity

$$\langle u, \Phi_{x+h}\rangle - \langle u, \Phi_x\rangle = \sum_i h_i \langle u, \frac{\partial}{\partial x_i}\Phi_x\rangle + o(|h|)$$

and so $\frac{\partial}{\partial x_i}\langle u, \Phi_x\rangle = \langle u, \frac{\partial}{\partial x_i}\Phi_x\rangle$ and the result follows by induction. □

Corollary. *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ then $u * \phi \in C^\infty(\mathbb{R}^n)$ and*

$$\partial^\alpha(u * \phi) = u * \partial^\alpha \phi.$$

Proof. Have $(u * \phi)(x) = \langle u, \tau_x \check{\phi}\rangle$ so take $\Phi_x = \tau_x \check{\phi}$ in previous lemma. □