Introduction

The course is split into two parts:

- Logic: syntax and semantics.
- Set theory: what does the universe of sets look like?

Course structure

(I) Propositional logic (logic)

(II) Well-orderings & ordinals (set theory)

(III) Posets & Zorn's lemma (set theory)

(IV) Predicate logic (logic)

(V) Set theory (set theory)

(VI) Cardinals (set theory)

Books:

1. Johnstone, *Notes on Logic & Set Theory*
2. Van Dalen, *Logic & Structure* (Chapter 4 and what 'goes next')
3. Hajnal & Hamburger, *Set Theory* (Chapters 2 and 6)
4. Forster, *Logic, Induction & Sets*

1 Propositional Logic

Let P be a set of *primitive propositions*. Unless otherwise stated, $P = \{p_1, p_2, \ldots\}$. The *language* L or $L(P)$ is defined inductively by

1. If $p \in P$, then $p \in L$
2. $\perp \in L$ ($\perp$ is read 'false')
3. If $p, q \in L$ then $(p \Rightarrow q) \in L$.

e.g $((p_1 \Rightarrow p_2) \Rightarrow (p_1 \Rightarrow p_3)), (p_4 \Rightarrow \perp), (\perp \Rightarrow \perp)$.

Notes.

1. Each proposition (member of L) is a finite string of symbols from language: $\vdash, \Rightarrow, \perp, p_1, p_2, \ldots$ (for clarity often omit outer brackets, use other types of bracket, etc).
2. 'L is defined inductively' means, more precisely, the following

- Put $L_1 = P \cup (\perp)$;
- Having defined L_n, put $L_{n+1} = L_n \cup \{(p \Rightarrow q) : p, q \in L_n\}$;
- Set $L = \bigcup_{n \geq 1} L_n$.

3. Every $p \in L$ is uniquely built up from steps 1,2 using 3. For example, $((p_1 \Rightarrow p_2) \Rightarrow (p_1 \Rightarrow p_3))$ can from $(p_1 \Rightarrow p_2)$ and $(p_1 \Rightarrow p_3)$.

We can now introduce $\neg p$ ('not p') as an abreviation for $(p \Rightarrow \perp)$; $p \vee q$ ('p or q') as an abreviation for $(\neg p) \Rightarrow q$; $p \wedge q$ ('p and q') as an abreviation for $\neg(p \Rightarrow (\neg q))$.

1.1 Semantic Implication

Definition. A *valuation* is a function $v : L \to \{0,1\}$ (thinking of 0 as 'False' and 1 as 'True') such that

(i) $v(\perp) = 0$

(ii) $v(p \Rightarrow q) = \begin{cases} 0 & \text{if } v(p) = 1,\ v(q) = 0 \\ 1 & \text{otherwise} \end{cases}$.

Remark. On $\{0,1\}$, could define a constant $\perp = 0$ and an operation $\Rightarrow$ by

$$(a \Rightarrow b) = \begin{cases} 0 & \text{if } a = 1, b = 0 \\ 1 & \text{otherwise} \end{cases}.$$

Then a valuation is precisely a mapping $L \to \{0,1\}$ that preserves ($\perp$ and $\Rightarrow$).

Proposition 1.1.

(i) If v, v' are valuations with $v(p) = v(p')$ for all $p \in P$, then $v = v'$.

(ii) For any function $w : P \to \{0,1\}$, there exists a valuation v with $v(p) = w(p)$ for all $p \in P$.

Proof.

(i) Have $v(p) = v'(p)$ for all $p \in L_1$. But if $v(p) = v'(p)$ and $v(q) = v'(q)$, then $v(p \Rightarrow q) = v'(p \Rightarrow q)$, so $v(p) = v'(p)$ for all $p \in L_2$. Continuing inductively we obtain $v(p) = v'(p)$ for all $p \in L_n$ for each n.

(ii) Set $v(p) = w(p)$ for all $p \in P$ and $v(\perp) = 0$ to obtain v on L_1. Now put

$$v(p \Rightarrow q) = \begin{cases} 0 & v(p) = 1, v(q) = 0 \\ 1 & \text{otherwise} \end{cases}$$

to obtain v on L_2, then induction.

□

Example. Let v be the valuation with $v(p_1) = v(p_3) = 1$, $v(p_n) = 0$ for all $n \neq 1, 3$. Then $v((p_1 \Rightarrow p_2) \Rightarrow p_3) = 0$.

Definition. A *tautology* is an element $t \in L$ such that $v(t) = 1$ for any valuation v. We write $\models t$.

Examples.

1. $p \Rightarrow (q \Rightarrow p)$

$v(p)$	$v(q)$	$v(p \Rightarrow q)$	$v(p \Rightarrow (q \Rightarrow p))$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

So this is a tautology.

2. $(\neg\neg p) \Rightarrow p$, i.e $((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p$ ('law of excluded middle')

$v(p)$	$v(p \Rightarrow \perp)$	$v((p \Rightarrow \perp) \Rightarrow \perp)$	$v(((p \Rightarrow \perp) \Rightarrow \perp) \Rightarrow p)$
0	1	0	1
1	0	1	1

3. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ ("how implicatino chains"). Suppose this is not a tautology. Then we have a v with $v(p \Rightarrow (q \Rightarrow r)) = 1$ and $v((p \Rightarrow q) \Rightarrow (q \Rightarrow r)) = 0$. Then $v(p \Rightarrow q) = 1$ and $v(p \Rightarrow r) = 0$. Hence $v(p) = 1$ and $v(r) = 0$, so $v(q) = 1$. Hence $v(p \Rightarrow (q \Rightarrow r)) = 0$, contradiction.

Definition. For $S \subseteq L$, $t \in L$, we say S *entails* or *semantically implies* t, written $S \vDash t$ if every valuation with $v(s) = 1$ for all $s \in S$ has $v(t) = 1$.

Example. $\{p \Rightarrow q, q \Rightarrow r\}$ entails $p \Rightarrow r$. Indeed, suppose we have v with $v(p \Rightarrow q)$, $v(q \Rightarrow r) = 1$ but $v(p \Rightarrow r)$. Then $v(p) = 1$, $v(r) = 0$. Hence $v(q) = 1$, contradicting $v(q \Rightarrow r) = 1$.

Definition. We say v is a *model* of $S \subseteq L$ or S is *true* in v, if $v(s) = 1$ for all $s \in S$. Thus S entails t means: every model of S is also a model of $\{t\}$.

Remark. $\vDash t$ says $\emptyset \vDash t$.

1.2 Syntatic implication

For a notion of proof, we'll need axioms and deduction rules. As axioms, we'll take:

1. $p \Rightarrow (q \Rightarrow p)$ for all $p, q \in L$;
2. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ for all $p, q \in L$;
3. $(\neg\neg p) \Rightarrow p$ for all $p \in L$.

Notes.

1. Sometimes we call these 'axiom schemes' since each is actually a set of axioms.
2. Each of these are tautologies.

For deduction rules, we'll have only *modus ponens*: from each p and $p \Rightarrow q$ we can deduce q.

Definition. For $S \subseteq L$, and $t \in S$, say S *proves* or *syntactically implies* t, written $S \vdash t$ if there exists a sequence $t_1, \ldots, t_n$ in L with $t_n = t$ such that every t_i is either

(i) An axiom; or

(ii) A member of S; or

(iii) Such that there exist $j, k < i$ with $t_k \Rightarrow (t_j \Rightarrow t_n)$ (modus ponens).

Say S consists of the *hypotheses* or *premises*, and t the *conclusion*.

Example. $\{p \Rightarrow q, q \Rightarrow r\} \vdash p \Rightarrow r$:

1. $q \Rightarrow r$ (hypothesis)
2. $(q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$ (axiom 1)
3. $p \Rightarrow (q \Rightarrow r)$ (modus ponens' on 2,3)
4. $[p \Rightarrow (q \Rightarrow r)] \Rightarrow [(p \Rightarrow q) \Rightarrow (p \Rightarrow r)]$ (axiom 2)
5. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$ (modus ponens' on 3,4)
6. $p \Rightarrow q$ (hypothesis)
7. $p \Rightarrow r$ (modus ponens on 5,6)

Definition. If $\emptyset \vdash t$, say t is a *theorem*, written $\vdash t$.

Example. $\vdash (p \Rightarrow p)$. We want to try to get to $(p \Rightarrow (p \Rightarrow)) \Rightarrow (p \Rightarrow p)$ using axiom 2.

1. $[p \Rightarrow ((p \Rightarrow p) \Rightarrow p)] \Rightarrow [(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)]$ (axiom 2)
2. $p \Rightarrow ((p \Rightarrow p) \Rightarrow p)$ (axiom 1)
3. $(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$ (modus ponens on 1,2)
4. $p \Rightarrow (p \Rightarrow p)$ (axiom 1)
5. $p \Rightarrow p$ (modus ponens on 3,4)

Often, showing $S \vdash p$ is made easier by:

Proposition 1.2 (Deduction Theorem)**.** *Let $S \subseteq L$ and $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ if and only if $S \cup \{p\} \vdash q$. Informally: "provability corresponds to the connective '$\Rightarrow$' in L".*

Proof. First we show ($\Rightarrow$): given a proof of $p \Rightarrow q$ from S, write down:

1. p (hypothesis)
2. q (modus ponens)

Which is a proof of q from $S \cup \{p\}$.

Now we show ($\Leftarrow$): we have a proof $t_1, \ldots, t_n$ of q from $S \cup \{p\}$. We'll show that $S \vdash (p \Rightarrow t_i)$ for all i.

If t_i is an axiom, write down

1. t_i (axiom)
2. $t_i \Rightarrow (p \Rightarrow t_i)$ (axiom 1)
3. $p \Rightarrow t_i$ (modus ponens)

So $S \vdash (p \Rightarrow t_i)$.

If $t_i \in S$, do the same thing except step 1 will be "t_i (hypothesis)" instead of "t_i (axiom)".

If $t_i := p$, we have $S \vdash (p \Rightarrow p)$, since $\vdash (p \Rightarrow p)$.

If t_i is obtained by modus ponens, we have t_j and $t_k = (t_j \Rightarrow t_i)$ for some $j, k < n$. By induction, we can assume $S \vdash (p \Rightarrow t_j)$ and $S \vdash (p \Rightarrow (t_j \Rightarrow t_i))$. So write down

1. $[p \Rightarrow (t_j \Rightarrow t_i)] \Rightarrow [(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)]$ (axiom 2)
2. $(p \Rightarrow t_j) \Rightarrow (p \Rightarrow t_i)$ (modus ponens)

3. $p \Rightarrow t_i$ (modus ponens)

So $S \vdash p \Rightarrow t$. $\square$

Example. To show $\{p \Rightarrow q, q \Rightarrow r\} \vdash (p \Rightarrow r)$, it is sufficient to show $\{p \Rightarrow q, q \Rightarrow r, p\} \vdash r$, which is just modus ponens twice.

Question: how are $\vDash$ and $\vdash$ related?

Aim: $S \vDash t \iff S \vdash t$ (Completeness Theorem).

This is made up of:

- $S \vdash t \Rightarrow S \vDash t$ (soundness) i.e "our axioms and deduction rule are not silly";
- $S \vDash t \Rightarrow S \vdash t$ (adequacy) "our axioms are strong enough to deduce from S, every semantic consequence of S".

Proposition 1.3 (Soundness)**.** *Let $S \subseteq L$, $t \in L$. Then $S \vdash t \Rightarrow S \vDash t$.*

Proof. We have a proof $t_1, \ldots, t_n$ of t from S. So we must show that every model of S is a model of t, i.e if v is a valuation with $v(s) = 1$ for all $s \in S$, then $v(t) = 1$. But $v(p) = 1$ for each axiom p (each axiom is a tautology), and for each $p \in S$ whenever $v(p) = v(p \Rightarrow q) = 1$, we have $v(q)$. So $v(t_i) = 1$ for all i (induction). $\square$

One case of adequacy is: if $S \vDash \perp$, then $S \vdash \perp$. We say S is *constistent* if $S \nvdash \perp$. So our statement is: S has no model $\Rightarrow$ S inconsistent, i.e S consistent $\Rightarrow$ S has a model.

In fact, this implies adequacy in general. Indeed, if $S \vDash t$ then $S \cup \{\neg t\}$ has no model. Hence (by the special case) $S \cup \{\neg t\} \vdash \perp$. So $S \vdash (\neg t \Rightarrow \perp)$, i.e $S \vdash (\neg\neg t)$. But $S \vdash (\neg\neg t) \Rightarrow t$ (axiom 3), so $S \vdash t$.

So our task is: given S consistent, find a model of S. Could try: define

$$v(t) = \begin{cases} 1 & t \in S \\ 0 & t \notin S \end{cases}.$$

But this fails, since S might not be *deductively closed*, meaning $S \vdash p \Rightarrow p \in S$. So we could first replace S with its deductive closure $\{t \in L : S \vdash t\}$ (which is consistent, because S is). However, this still fails: if S does not 'mention' p_3, then $S \nvdash p_3$ and $S \nvdash \neg p_3$, so $v(p_3) = v(\neg p_3) = 0$ which is impossible.

Theorem 1.4 (Model Existence Theorem)**.** *Let $S \subseteq L$ be consistent. Then S has a model.*

Idea: extend S to 'swallow up', for each p, one of p and $\neg p$.

Proof. Claim: for any consistent $S \subseteq L$ and $p \in L$, $S \cup \{p\}$ or $S \cup \{\neg p\}$ is consistent.

Proof of claim: if not, then $S \cup \{p\} \vdash \perp$ and $S \cup \{\neg p\} \vdash \perp$. So $S \vdash (p \Rightarrow \perp)$ (deduction theorem), i.e $S \vdash (\neg p)$. Hence from $S \cup \{\neg p\} \vdash \perp$ we obtain $S \vdash \perp$.

Now, L is countable (as each L_n is countable) so we can list L as $t_1, t_2, \ldots$. Let $S_0 = S$. Let $S_1 = S_0 \cup \{t_1\}$ or $S_1 \cup \{\neg t_1\}$ with S_1 consistent. In general, given S_{n-1} let $S_n = S_{n-1} \cup \{t_n\}$ or $S_n = S_{n-1} \cup \{\neg t_n\}$ so that S_n is consistent. Now set $\overline{S} = S_0 \cup S_1 \cup S_2 \cup \ldots$. Thus for all $t \in L$, either $t \in \overline{S}$ or $(\neg t) \in \overline{S}$.

Now $\overline{S}$ is consistent: if $\overline{S} \vdash \perp$ then, since proofs are finite, we'd have $S_n \vdash \perp$ for some n, a contradiction.

Also, $\overline{S}$ is deductively closed: if $\overline{S} \vdash p$, must have $p \in \overline{S}$, since otherwise $(\neg p) \in \overline{S}$, so $\overline{S} \vdash (p \Rightarrow \perp)$ and $\overline{S} \vdash \perp$.

Now define $v : L \to \{0, 1\}$ by

$$t \mapsto \begin{cases} 1 & t \in \overline{S} \\ 0 & \text{otherwise} \end{cases}.$$

We'll show v is a valuation (then we're done as $v = 1$ on S).

$\underline{v(\perp)}$: have $\perp \notin \overline{S}$ (since $\overline{S}$ is consistent), so $v(\perp) = 0$.

$\underline{v(p \Rightarrow q)}$: if $v(p) = 1$, $v(q) = 0$, then have $p \in \overline{S}$, $q \notin \overline{S}$. But if $(p \Rightarrow q) \in \overline{S}$, then since $p \in \overline{S}$, $q \in \overline{S}$ (since $\overline{S}$ is deductively closed). Now if $v(q) = 1$, $q \in \overline{S}$. But $\overline{S} \vdash (q \Rightarrow (p \Rightarrow q))$ (axiom 1), so $\overline{S} \vdash (p \Rightarrow q)$ hence $(p \Rightarrow q) \in \overline{S}$ ($\overline{S}$ is deductively closed). Finally, if $v(p) = 0$, we have $p \notin \overline{S}$ and want to show $(p \Rightarrow q) \in \overline{S}$. Then $(p \Rightarrow \perp) \in \overline{S}$, so it is enough to show $(p \Rightarrow \perp) \vdash (p \Rightarrow q)$. So it's enough to show $(p, p \Rightarrow \perp) \vdash q$, so enough to show $\perp \vdash q$. But $\perp \vdash (\neg\neg q)$ (axiom 1), and $(\neg\neg q) \vdash q$ (axiom 3), so $\perp \vdash q$ as required. ☐

Remarks.

1. We used $P = (p_1, p_2, \ldots)$, in saying L is countable. In fact, it also holds if P is uncountable (see later in course).
2. Sometimes this theorem is called 'The Completeness Theorem'

By the remarks stated before this theorem, we have

Corollary 1.5 (Adequacy). *Let $S \subseteq L$, $t \in L$, with $S \vDash t$. Then $S \vdash t$.*

Hence we have

Theorem 1.6 (Completeness Theorem). *Let $S \subseteq L$, $t \in L$. Then $S \vdash t \iff S \vDash t$.*

Corollary 1.7 (Compactness Theorem). *Let $S \subseteq L$, $t \in L$ with $S \vDash t$. Then some finite $S' \subseteq S$ has $S' \vDash t$.*

Proof. This is trivial if we replace $\vDash$ by $\vdash$ (as all proofs are finite). ☐

For $t = \perp$, the theorem says: if $S \vDash T$ then some finite $S' \subseteq S$ has $S' \vdash \perp$, i.e if every finite $S' \subseteq S$ has a model then S has a model. In fact, this is equivalent to compactness in general: $S \vDash t$ says $S \cup \{\neg t\}$ has no model, and $S' \vDash t$ says $S' \cup \{\neg t\}$ has no model.

Corollary 1.8 (Compactness Theorem equivalent form). *Let $S \subseteq L$. Then if every finite subset of S has a model, so does S.*

Another application:

Corollary 1.9 (Decidability Theorem). *Let $S \subseteq L$ be finite and $t \in L$. Then there is an algorithm to decide, in finite time, whether of not $S \vdash t$.*

Remark. This is a very surprising result.

Proof. Trivial if we replace $\vdash$ with $\vDash$: to check if $S \vDash t$ we just draw the truth table. ☐

2 Well-ordering & Ordinals

Definition. A *total order* or *linear order* is a pair $(X, <)$ where X is a set and $<$ is a relation on X that is

(i) *irreflexive*: for all $x \in X$, not $x < x$;

(ii) *transitive*: for all $x, y, z \in X$, if $x < y$, $y < z$ then $x < z$;

(iii) *trichotomous*: for all $x, y \in X$, either $x = y$ or $x < y$ or $y < x$.

We sometimes write $x > y$ if $y < x$, and $x \leq y$ if $x < y$ or $x = y$.

We can instead define a total order in terms of $\leq$ as follows:

(i) *reflextive*: for all $x \in X$, $x \leq x$;

(ii) *transitive*: for all $x, y, z \in X$, if $x \leq y$, $y \leq z$ then $x \leq z$;

(iii) *antisymmetric*: for all $x, y \in X$, if $x \leq y$, $y \leq x$ then $x = y$;

(iv) *trichotomous*: for all $x, y \in X$ either $x \leq y$ or $y \leq x$.

Examples.

1. $\mathbb{N}, <$;
2. $\mathbb{Q}, \leq$;
3. $\mathbb{R}, \leq$;
4. $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ under 'divides' is <u>not</u> a total order, e.g 2 and 3 are not related;
5. $\mathcal{P}(S), \subseteq$ is <u>not</u> a total order - fails trichotomy.

Definition. A total order $(X, <)$ is a *well-ordering* if every (non-empty) subset has a least element, i.e for all $S \subseteq X$ if $S \neq \emptyset$ then there exists $x \in S$ such that $x \leq y$ for all $y \in S$.

Examples.

1. $\mathbb{N}, <$;
2. $\mathbb{Z}, <$ is not a well ordering;
3. $\mathbb{Q}, <$ is not a well ordering;
4. $\mathbb{R}, <$ is not a well ordering;
5. $[0, 1] \subseteq \mathbb{R}, <$ is not a well ordering, e.g $(0, 1]$ has no least element;
6. $\{1/2, 2/3, 3/4, \ldots\} \subseteq \mathbb{R}$ is well ordered;
7. $\{1/2, 2/4, 3/4, \ldots\} \cup \{1\}$ is well ordered;

8. $\{1/2, 2/4, 3/4, \ldots\} \cup \{2\}$ is well ordered;
9. $\{1/2, 2/3, 3/4, \ldots\} \cup \{1 + 1/2, 1 + 2/3, 1 + 3/4, \ldots\}$ is well ordered.

Remark. $(X, <)$ is a well ordering if and only if there is no infinite strictly decreasing sequence.

We say total orders X, Y are *isomorphic* if there exists a bijection $f : X \to Y$ such that $x < y$ if and only if $f(x) < f(y)$. For example, Examples 1&6, 7&8 above are isomorphic. However examples 1&7 are not isomorphic, since in 7 there exists a greatest element, but not in 1.

Proposition 2.1 (Proof by induction)**.** *Let X be well ordered and let $S \subseteq X$ be such that whenever $y \in S$ for all $y < x$, then $x \in S$. Then $S = X$. Equivalently, if $p(x)$ is a property such that $p(y)$ for all $y < x$ implies $p(x)$, then $p(x)$ for all $x \in X$.*

Proof. Suppose $S \neq X$ and let x be least in $X \setminus S$. THen $y \in S$ for all $y < x$ but $x \notin S$, a contradiction. $\square$

Proposition 2.2. *Let X, Y be isomorphic well-orderings. Then there exists a unique isomorphism.*

Note. Note this is false for general total orders, for example $\mathbb{Z} \to \mathbb{Z}$ could have $x \mapsto x - t$ for any t, or $\mathbb{R} \to \mathbb{R}$ could have $x \mapsto x^3$.

Proof. Let $f, g : X \to Y$ be isomorphisms. We'll show $f(x) = g(x)$ for all x by induction on X. Given $f(y) = g(y)$ for all $y < x$, we want to show $f(x) = g(x)$. We must have $f(x) = a$ where a is the least element of $Y \setminus \{f(y) : y < x\}$ (non-empty since it contains $f(x)$). Indeed, if not then $f(x') = a$ for some $x' > x$, contradicting the fact f is order preserving. Similarly have $g(x) = a$. $\square$

Definition. A subset I of a total order X is an *initial segment* if $x \in I$, $y < x$ implies $y \in I$ (i.e I is closed under $<$). For example $I_x = \{y \in X : y < x\}$ is an initital segment for any $x \in X$, however not every inital segment is of this form, e.g in $\mathbb{Q}$ $\{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$.

Note. In a well-ordering, every proper initial segment I is of the form I_x, for some $x \in X$. Indeed let x be the least element of $X \setminus I$ (non-empty since I is proper). Then $I = I_x$, since if $y < x$ then $y \in I$ (by choice of x), and conversely if $y \in I$, must have $y < x$ or else $y \geq x$ implying $x \in I$ (as I is an initital segment).

Our aim is to show that every subset of a well-ordering X is isomorphic to an initital segment of X.

Note. This is false in general for total orders, e.g $\{1, 2, 3\}$ in $\mathbb{Z}$, or $\mathbb{Q}$ in $\mathbb{R}$.

Theorem 2.3 (Definition by recursion)**.** *Let X be a well-ordering and let Y be any set. Take $G : \mathcal{P}(X \times Y) \to Y$ (i.e a 'rule'). Then there exists a function $f : X \to Y$ such that $f(x) = G(f|_{I_x})$ for all $x \in X$. Moreover, f is unique.*

Note. In defining $f(x)$, we make use of f on $I_x = \{y : y < x\}$.

Proof. Say h is 'an attempt' if $h : I \to Y$ for some initital segment I of X, and for all $x \in I$ we have $h(x) = G(h|_{I_x})$. [This is the main idea].

Note that if h, h' are attempts both defined at x, then $h(x) = h'(x)$, by induction on x (if $h(y) = h'(y)$ for all $y < x$ then $h(x) = h'(y)$).

Also, for every x, there exists an attempt defined at x, also by induction. Indeed, suppose that for all $y < x$ there exists an attempt defined at y. So for all $y < x$ there exists a unique (by above) attempt h_y with domain $\{z : z \leq y\}$. Now let $h = \bigcup_{y \leq x} h_y$, this is an attempt with domain I_x (single valued by uniqueness). Thus $h \cup \{(x, G(h))\}$ is an attempt defined at x. Now define $f : X \to Y$ by setting $f(x) = y$ if there exists an attempt h defined at x such that $h(x) = y$.

Uniqueness of f: if f, f' are both such functions, then $f(x) = f'(x)$ for all x by induction ($f(y) = f'(y)$ for all $y < x$ implies $f(x) = f'(x)$). $\square$

Proposition 2.4 (Subset collapse)**.** *Let X be a well-ordering and $Y \subseteq X$. Then Y is isomorphic to an initital segment of X. Moreover, I is unique.*

Proof. To have $f : Y \to X$ an isomorphism with an initial segment of X, we need precisely that for every $x \in Y$ we have that $f(x)$ is the minimum element of $X \setminus \{f(y) : y < x\}$. So we're done by the previous theorem. $\square$

Note. We have $X \setminus \{f(y) : y < x\} \neq \emptyset$, since $f(y) \leq y$ for all y (induction), so $x \notin \{f(y) : y < x\}$.

In particular, X itself cannot be isomorphic to a proper intial segment (uniqueness).

How do different well-orderings relate to each other?

Definition. For well-orderings X, Y we write $X \leq Y$ if X is isomorphic to an initial segment of Y.

Example. If $X = \mathbb{N}$, $Y = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right)$, then $X \leq Y$.

Proposition 2.5. *Let X, Y be well-orderings. Then $X \leq Y$ or $Y \leq X$.*

Proof. Suppose $Y \not\leq X$, we'll show $X \leq Y$. To obtain $f : X \to Y$ an isomorphism with an initial segment of Y, we need precisely that for every $x \in X$, $f(x)$ is the least element in $Y \setminus \{f(y) : y < x\}$ [note this can only be empty if Y is isomorphic to I_x]. So we're done by recursion. $\square$

Proposition 2.6. *Let X, Y be well-orderings with $X \leq Y$ and $Y \leq X$. Then X and Y are isomorphic.*

Note. This proposition and the previous one are "the most we could ever hope for".

Proof. We have isomorphisms f from X to some initial segment of Y, and g from Y to some initial segment of X. Then $g \circ f : X \to X$ is an isomorphism from X to an initial segment of X (as initial segment of an initial segment of X is itself an initial segment). So by uniqueness $g \circ f = \mathrm{id}_X$. Similarly $f \circ g = \mathrm{id}_Y$. Hence f and g are inverses, thus bijections. $\square$