1 Lebesgue Integration Theory

1.1 Review of measure theory

Definition. Given a set E, a *σ-algebra* on E is a collection $\mathcal{E}$ of subsets of E such that:

(i) $E \in \mathcal{E}$;

(ii) $A \in \mathcal{E} \Rightarrow A^c = E \setminus A \in \mathcal{A}$;

(iii) $A_n \in \mathcal{E}$, $n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$.

$(E, \mathcal{E})$ is called a *measurable space*, and any $A \in \mathcal{E}$ is called a *measurable set.*

Given a collection $\mathcal{A}$ of subsets of E, $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$.

Definition. A *measure* on $(E, \mathcal{E})$ is a function $\mu : \mathcal{E} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$;

(ii) $A_n \in \mathcal{E}$, $n \in \mathbb{N}$ disjoint $\Rightarrow \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

$(E, \mathcal{E}, \mu)$ is called a *measure space.*

Definition (Borel measure)**.** If (E, τ) is a topological space, then $\sigma(\tau)$ is called a *Borel algebra*, denoted $\mathcal{B}(E)$, and a measure on $(E, \mathcal{B}(E))$ is called a *Borel measure.*

Example. $E = \mathbb{R}^n$, μ the Lebesgue measure satisfying $\mu((a_1, b_1) \times \ldots \times (a_n, b_n)) = (b_1 - a_1) \ldots (b_n - a_n)$.

Notation: we write $\mu(\mathrm{d}x) = \mathrm{d}x$ and $\mu(A) = |A|$ when μ is the Lebesgue measure.

Definition (Measurable function)**.** Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Then $f : E \to F$ is *measurable* if $f^{-1}(A) \in \mathcal{E}$ whenever $A \in \mathcal{F}$. If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are Borel algebras, a measurable function is called a *Borel function.* Special case: $(F, \mathcal{F}) = ([0, \infty], \mathcal{B}([0, \infty]))$, then $f : E \to F$ is called a *non-negative measurable function.*

Fact. The class of measurable functions is closed under addition, multiplication and taking (pointwise) limits.

Definition. $f : E \to F$ ($F = [0, \infty]$ or $\mathbb{R}^n$ or $\mathbb{C}^n$) is a *simple function* if $f = \sum_{k=1}^{K} a_k \mathbb{1}_{A_k}$ for some $K \in \mathbb{N}$, $a_k \in F$, $A_k \in \mathcal{E}$. For a simple function, the integral is

$$\int f \mathrm{d}\mu = \int f(x) \mathrm{d}\mu(x) = \sum_{k=1}^{K} a_k \mu(A_k) \ (0 \cdot \infty := 0).$$

For a non-negative measurable f, we define

$$\int f \mathrm{d}\mu = \sup \left\{ \int g \mathrm{d}\mu : g \text{ simple }, 0 \leq g \leq f \right\}.$$

Definition. A measurable function $f : E \to \mathbb{R}$ is said to be *integrable* if $\int |f|\mathrm{d}\mu < \infty$. Write $f = f_+ - f_-$ with $f_\pm$ non-negative, measurable, $\int f_\pm \mathrm{d}\mu < \infty$, and then $\int f\mathrm{d}\mu = \int f_+\mathrm{d}\mu - \int f_-\mathrm{d}\mu$. For $f : E \to \mathbb{R}^n$, this is applied in each component.

Theorem (Monotone convergence theorem)**.** *Let $(E, \mathcal{E}, \mu)$ be a measure space, and let (f_n) be a (pointwise) increasing sequence of non-negative functions on E converging to f. Then*

$$\lim_{n\to\infty} \int_E f_n \mathrm{d}\mu = \int_E f \mathrm{d}\mu.$$

Theorem (Dominated convergence theorem)**.** *Let (f_n) be a sequence of measurable functions on a measure space $(E, \mathcal{E}, \mu)$ such that:*

(i) $f_n \to f$ *pointwise almost everywhere;*

(ii) $|f_n| \leq g$ *almost everywhere for some integrable* g*.*

Then

$$\lim_{n\to\infty} \int_E f_n \mathrm{d}\mu = \int_E f \mathrm{d}\mu.$$

1.2 L^p spaces

Definition. Let $(E, \mathcal{E}, \mu)$ be a measure space. For $p \in [1, \infty)$ and $f : E \to \mathbb{R}$ define

$$||f||_{L^p} = \left(\int_E |f|^p \mathrm{d}\mu \right)^{1/p}$$

and

$$||f||_{L^\infty} = \mathrm{esssup}|f| = \inf\{K : |f| \leq K \text{ a.e}\}.$$

The space L^p, $p \in [1, \infty]$ is defined by

$$L^p = L^p(E, \mathcal{E}, \mu) = \{f : E \to \mathbb{R} \text{ measurable} : ||f||_{L^p} < \infty\}/\sim .$$

Where $f \sim g$ if $f = g$ a.e.

Theorem (Riesz-Fisher theorem)**.** *L^p is a Banach space for all $p \in [1, \infty]$.*

Notation: when $E = \mathbb{R}^n$, μ the Lebesgue measure, write $L^p(E, \mu) = L^p(\mathbb{R}^n)$.

Fact. For $p \in [1, \infty)$, the simple functions f with $\mu(\{x : f(x) \neq 0\}) < \infty$ are dense in L^p. For $p = \infty$ we can drop the condition on the measure of the support.

Definition. For $f, g : \mathbb{R}^n \to \mathbb{R}$, the *convolution* $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x - y)\mathrm{d}y,$$

provided the integral exists. Note that $f * g = g * f$, convolution is associative, and $\mu(f * g) = \mu(f)\mu(g)$.

Theorem. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$.*

Before we prove the theorem, we will need some preliminary results.

Remark. This theorem is false for $p = \infty$.

Notation: a multiindex is $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Set $|\alpha| = \alpha_1 + \ldots + \alpha_n$; $\alpha! = \alpha_1! \ldots \alpha_n!$; $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ for $X \in \mathbb{R}^n$; $\nabla^\alpha f = D^\alpha f = \frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$.

Definition. We say $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ if $f \mathbb{1}_K \in L^p(\mathbb{R}^n)$ for any $K \subseteq \mathbb{R}^n$ compact.

Proposition. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $g \in C^k_c(\mathbb{R}^n)$, some $k \geq 0$. Then $f * g \in C^k(\mathbb{R}^n)$ and $\nabla^\alpha (f * g) = f * (\nabla^\alpha g)$ for all $|\alpha| \leq k$.

Proof. First we check for $k = 0$. Set $T_z f(x) = f(x - z)$, $z \in \mathbb{R}^n$. Then $T_z(f * g) = f * (T_z g)$. Also $T_z g(x) \to g(x)$ for all x as $z \to 0$ (continuity of g). Furthermore $|T_z g(x)| \leq ||g||_{L^\infty} \mathbb{1}_{B_R(0)}(x)$ if $|x| + 1 \leq R$, $|z| < 1$ (we can just take R large enough so it holds everywhere since g has compact support). Then $|f(y) T_z g(x - y)| \leq C |f(y)| \mathbb{1}_{B_R(0)}(x - y)$, for $C := ||g||_{L^\infty}$.

Since $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $|f(y)| \mathbb{1}_{B_R(0)}(x - y)$ is integrable in y, so by the dominated convergence theorem,

$$T_z(f * g) = (f * T_z g)(x) = \int_{\mathbb{R}^n} f(y) T_z g(x - y) \mathrm{d}y \xrightarrow{z \to 0} \int_{\mathbb{R}^n} f(y) g(x - y) \mathrm{d}y = (f * g)(x).$$

And so $f * g \in C^0$. Now let $k = 1$. Let $\nabla_i^h g(x) = \frac{g(x + h e_i) - g(x)}{h}$, where e_i is the ith unit vector. Then $\nabla_i^h g(x) \to \nabla_i g(x)$ as $h \to 0$.

By the mean value theorem, there exists $t \in [-h, h]$ such that

$$\nabla_i^h g(x) = \nabla_i g(x + t e_i) \Rightarrow |\nabla_i^h g(x)| \leq ||\nabla_i g||_{L^\infty} \mathbb{1}_{B_R(0)}(x).$$

Again by the dominated convergence theorem, $\nabla_i^h (f * g) = f * (\nabla_i^h g) \to f * \nabla_i g$. Thus $f * g \in C^1$. The case $k > 1$ is similar, with induction. □

Proposition (Minkowski's integral inequality)**.** Let $p \in [1, \infty)$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ Borel. Then

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x, y) \mathrm{d}x \right|^p \mathrm{d}y \right]^{1/p} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x, y)|^p \mathrm{d}y \right]^{1/p} \mathrm{d}x.$$

Proof. Example sheet 1. □

Proposition. Let $p \in [1, \infty)$, $g \in L^p(\mathbb{R}^n)$. Then

$$||T_z g - g||_{L^p} \to 0 \text{ as } |z| \to 0.$$

Remark. This is not true for $p = \infty$. Let $\theta(x) = \mathbb{1}_{x \geq 0}$. Then $||T_z \theta - \theta||_{L^\infty} = 1$ if $z \neq 0$.

Proof. Consider first $g = \mathbb{1}_R$, R a rectangle. Then the result is clear. Hence it also follows for any finite union of rectangles. If B is a Borel set, $|B| < \infty$, then for every $\varepsilon > 0$, there exists a finite union of rectangles R such that

$$||\mathbb{1}_B - \mathbb{1}_R||_{L^p} = |B \triangle R|^{1/p} < \varepsilon.$$

Hence

$$||T_z\mathbb{1}_B - \mathbb{1}_B||_{L^p} \leq \underbrace{||T_z\mathbb{1}_B - T_z\mathbb{1}_R||_{L^p}}_{=||\mathbb{1}_B-\mathbb{1}_R||_{L^p}<\varepsilon} + \underbrace{||T_z\mathbb{1}_R - \mathbb{1}_R||_{L^p}}_{<\varepsilon \text{ for } |z| \text{ small}} + \underbrace{||\mathbb{1}_R - \mathbb{1}_B||_{L^p}}_{<\varepsilon}.$$

Thus the result holds for $g = \mathbb{1}_B$, $B \in \mathcal{B}(\mathbb{R}^n)$. Thus the result holds for simple functions g. Finally, for any $g \in L^p$, there is a $\tilde{g}$ simple such that $||g - \tilde{g}||_{L^p} < \varepsilon$. Then

$$||T_z g - g||_{L^p} \leq \underbrace{||T_z g - T_z\tilde{g}||_{L^p}}_{=||g-\tilde{g}||_{L^p}<\varepsilon} + \underbrace{||T_z\tilde{g} - \tilde{g}||_{L^p}}_{<\varepsilon \text{ for } |z| \text{ small}} + \underbrace{||g - \tilde{g}||_{L^p}}_{<\varepsilon}.$$

□

Theorem. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi \mathrm{d}x = 1$ and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then for any $g \in L^p$, $p \in [1, \infty)$, it follows that $\varphi_\varepsilon * g \in C^\infty(\mathbb{R}^n)$ and $\varphi_\varepsilon * g \to g$ in L^p.*

Proof. We have

$$\begin{aligned}
|\varphi_\varepsilon * g(x) - g(x)| &= \left| \int_{\mathbb{R}^n} [\varphi_\varepsilon(y) g(x-y) - g(x)] \, \mathrm{d}y \right| \\
&\underbrace{=}_{z:=y/\varepsilon} \left| \int_{\mathbb{R}^n} \varphi(z) \, [g(x - \varepsilon z) - g(x)] \, \mathrm{d}z \right| \\
&\leq \int_{\mathbb{R}^n} \varphi(z) \, |T_{\varepsilon z} g(x) - g(x)| \, \mathrm{d}z.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\varphi_\varepsilon * g - g\|_{L^p} &= \left(\int_{\mathbb{R}^n} \underbrace{|\varphi_\varepsilon * g - g|^p}_{\int_{\mathbb{R}^n} \varphi(z) |T_{\varepsilon z} g - g| \mathrm{d}z} \, \mathrm{d}x \right)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi(z)^p |T_{\varepsilon z} g(x) - g(x)|^p \mathrm{d}x \right)^{1/p} \mathrm{d}z \\
&= \int_{\mathbb{R}^n} \varphi(z) \underbrace{||T_{\varepsilon z} g - g||_{L^p}}_{\to 0 \text{ as } \varepsilon \to 0} \mathrm{d}z
\end{aligned}$$

where the inequality comes from Minkowski's inequality. Also the final term tends to 0 as $\varepsilon \to 0$ by the DCT since $\varphi(z)||T_{\varepsilon z} g - g||_{L^p}|| \leq 2\varphi(z)||g||_{L^p}$ and φ is integrable. ☐

Definition. φ as above is called a (smooth) mollifier.

Corollary. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $p \in [1, \infty)$.*

Proof. The previous theorem implies $C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is dense in L^p. Since $||f - f\mathbb{1}_{B_R(0)}||_{L^p} \to 0$ as $R \to \infty$ by the DCT, for $f \in L^p$, applying the theorem with $g = f\mathbb{1}_{B_R(0)}$ it follows that $C_c^\infty(\mathbb{R}^n)$ is dense in L^p. ☐

1.3 Lebesgue Differentiation Theorem

Recall:

Theorem (Fundamental Theorem of Calculus)**.** *For $f : \mathbb{R} \to \mathbb{R}$ continuous, $F(x) := \int_0^x f(t)\mathrm{d}t$ is differentiable with $F'(x) = f(x)$.*

We actually have a stronger result:

Theorem (Lebesgue Differentiation Theorem)**.** *For $f : \mathbb{R}^n \to \mathbb{R}$ integrable,*

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \mathrm{d}y = 0 \text{ for almost all } x.$$

The x for which this holds are called Lebesgue points.

We will need a few preliminary results and definitions before we can prove this.

Corollary. *If $g \in L^1(\mathbb{R})$ and $G(x) = \int_{-\infty}^{x} g(t)\mathrm{d}t$, then G is differentiable for almost every x with $G'(x) = g(x)$.*

Corollary. *If φ is a smooth mollifier and $g \in L^p(\mathbb{R}^n)$, then $\varphi_\varepsilon * g \xrightarrow{\varepsilon \to 0} g$ almost everywhere.*

Definition. For $f : \mathbb{R}^n \to \mathbb{R}$ integrable, the *Hardy-Littlewood Maximal Function* $\mathrm{Mf} : \mathbb{R}^n \to [0, \infty]$ is defined by

$$\mathrm{Mf}(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\mathrm{d}y.$$

Remark. We sometimes write $⨍_{B_r(x)} |f(y)|\mathrm{d}y$ for $\frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\mathrm{d}y$.

Lemma (Wiener's covering lemma)**.** *If K is compact and $K \subseteq \bigcup_{i=1}^N B_i$ for open balls $(B_i)_{i=1}^N$, there exists a subcollection $(B_{i_k})_k$ of disjoint balls such that*

$$\left| \bigcup_{i=1}^N B_i \right| \leq 3^n \sum_k |B_{i_k}|.$$

Proof. Example sheet. □

Proposition. Take $f \in L^1(\mathbb{R}^n)$. Then Mf is a Borel function, finite almost everywhere, and

$$|\underbrace{\{\mathrm{Mf} > \lambda\}}_{:=A_\lambda}| \leq \frac{3^n}{\lambda} ||f||_{L^1}.$$

Proof. For each $x \in A_\lambda$, there exists $r_x > 0$ such that

$$\frac{1}{B_{r_x}(x)} \int_{B_{r_x}(x)} |f(y)|\mathrm{d}y > \lambda.$$

We claim that A_λ is open. Then we will have shown Mf is Borel as the $A_\lambda = (\mathrm{Mf})^{-1}((\lambda, \infty])$ are open, and the sets $(\lambda, \infty]$ generate the Borel σ-algebra.

We'll actually show A_λ^c is closed. Suppose $(x_k)_{k \geq 1}$ is a sequence in A_λ^c with $x_k \to x$. Suppose $x \in A_\lambda$. By the Dominated Convergence Theorem,

$$\frac{1}{B_{r_x}(x_k)} \int_{B_{r_x}(x_k)} |f(y)|\mathrm{d}y \to \frac{1}{B_{r_x}(x)} \int_{B_{r_x}(x)} |f(y)|\mathrm{d}y.$$

Since $x_k \notin A_\lambda$, the LHS is $\leq \lambda$ for all k, but the RHS is $> \lambda$ which is impossible. Hence $x \in A_\lambda^c$ and A_λ^c is closed.

To prove the inequality, let $K \subseteq A_\lambda$ be compact. Since $\{B_{r_x}(x)\}_{x \in A_\lambda}$ is an open cover of K, there exists a finite subcover $K \subseteq \bigcup_{i=1}^N B_i$, where $B_i = B_{r_x}(x)$ for some $x \in A_\lambda$. Now take a subcollection $(B_{i_k})_k$ of disjoint balls as in Wiener's covering lemma.

Since $\frac{1}{|B_i|}\int_{B_i}|f(y)|\mathrm{d}y > \lambda$, it follows that $|B_i| < \frac{1}{\lambda}\int_{B_i}|f(y)|\mathrm{d}y$. Hence

$$|K| \leq 3^n \sum_k |B_{i_k}| < \frac{3^n}{\lambda}\sum_k \int_{B_{i_k}} |f(y)|\mathrm{d}y \leq \frac{3^n}{\lambda}\int_{\mathbb{R}^n}|f(y)|\mathrm{d}y.$$

Since this holds for any $K \subseteq A_\lambda$ compact, by regularity of the Lebesgue measure, it also holds for A_λ. In particular, $|\{\mathrm{Mf} = \infty\}| \leq |\{\mathrm{Mf} > \lambda\}| \xrightarrow{\lambda\to\infty} 0$, i,e $\mathrm{Mf} < \infty$ almost everywhere. $\square$

Now we are ready to prove:

Theorem (Lebesgue Differentiation Theorem)**.** *For $f : \mathbb{R}^n \to \mathbb{R}$ integrable,*

$$\lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \mathrm{d}y = 0 \text{ for almost all } x.$$

The x for which this holds are called Lebesgue points.

Proof. Let

$$A_\lambda = \left\{ x \in \mathbb{R}^n : \limsup_{r\to 0} ⨍_{B_r(x)} |f(y) - f(x)| \mathrm{d}y > 2\lambda \right\}$$

Then it suffices to show $|A_\lambda| = 0$ for any $\lambda > 0$. Indeed, the non-Lebesgue points are then $\bigcup_n A_{1/n}$, a countable union of sets of measure 0.

Given $\varepsilon > 0$, let $g \in C_c^\infty(\mathbb{R}^n)$ be such that $||f - g||_{L^1} < \varepsilon$. Then

$$⨍_{B_r(x)} |f(y) - f(x)| \mathrm{d}y \le \underbrace{⨍_{B_r(x)} |f(y) - g(y)| \mathrm{d}y}_{\le M(f-g)(x)} + |f(x) - g(x)| + \underbrace{⨍_{B_r(x)} |g(x) - g(y)| \mathrm{d}y}_{\to 0 \text{ since } g \in C^\infty}.$$

$$\implies \limsup_{r\to 0} ⨍_{B_r(x)} |f(y) - f(x)| \mathrm{d}y \le M(f-g)(x) + |f(x) - g(x)|.$$

If $x \in A_\lambda$, then either $M(f-g)(x) > \lambda$ or $|f(x) - g(x)| > \lambda$. The Hardy-Littlewood maximal inequality says $|\{M(f-g) > \lambda\}| \le \frac{3^n}{\lambda} ||f-g||_{L^1}$. Then by Markov's inequality $|\{|f-g| > \lambda\}| \le \frac{1}{\lambda}||f-g||_{L^1}$. Hence

$$|A_\lambda| \le \frac{3^n + 1}{\lambda} ||f - g||_{L^1} < \frac{3^{n+1} + 1}{\lambda} \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $|A_\lambda| = 0$. □

1.4 Littlewood's Principles

Theorem (Egorov)**.** *Let $E \subseteq \mathbb{R}^n$, $|E| < \infty$, and $f_k : E \to \mathbb{R}$, $k \ge 1$ be a sequence of measurable functions such that $f_k \to f$ almost everwhere. Then for every $\varepsilon > 0$, there is a closed subset $A_\varepsilon \subseteq E$ such that $|E \setminus A_\varepsilon| < \varepsilon$ and $f_k \to f$ uniformly on A_ε.*

Proof. Without loss of generality, $f_k(x) \to f(x)$ for all $x \in E$ (otherwise restrict to a subset of E of full measure). Let

$$E_k^n = \left\{ x \in E : |f_j(x) - f(x)| < \frac{1}{n} \ \forall j > k \right\}.$$

Then $E^n_{k+1} \supseteq E^n_k$, $\bigcup_k E^n_k = E$, hence $|E^n_k| \uparrow |E|$ as $k \to \infty$. Let k_n be such that $|E \setminus E^n_{k_n}| < 2^{-n}$ and for $N \in \mathbb{N}$ set

$$A_N = \bigcap_{n \geq N} E^n_{k_n} \implies |E \setminus A_N| \leq \sum_{n \geq N} |E \setminus E^n_{k_n}| \leq 2^{-N+1} < \varepsilon \text{ for } N = N_\varepsilon.$$

Now it suffices to show $f_j \to f$ uniformly on A_N. Indeed, for $x \in A_N$ and any $n \geq N$, $|f_j(x) - f(x)| < \frac{1}{n}$ for all $j > k_n$. Hence $\limsup_{j\to\infty} \sup_{A_N} |f_j - f| \leq \frac{1}{n}$ for all $n \geq N$, hence $\lim_{j\to\infty} \sup_{A_N} |f_j - f| = 0$. □

Theorem (Lusin)**.** *Let $f : E \to \mathbb{R}$ be a Borel function, where $E \subseteq \mathbb{R}^n$ and $|E| < \infty$. Then for every $\varepsilon > 0$, there exists $F_\varepsilon \subseteq E$ closed such that $|E \setminus F_\varepsilon| < \varepsilon$ and $f|_{F_\varepsilon}$ is continuous.*

Remark. Careful: this does <u>not</u> mean that f is continuous at $x \in F_\varepsilon$ in the topology of $\mathbb{R}^n$.

Proof. First we show that the statement holds for simple functions f. Let $f = \sum_{m=1}^M a_m \mathbb{1}_{A_m}$ with the A_m disjoint and $\bigcup_m A_m = E$. Then there are compact sets $K_m \subseteq A_m$ with $|A_m \setminus K_m| < \frac{\varepsilon}{M}$ by regularity of the Lebesgue measure. Then if $F_\varepsilon = \bigcup_m K_m$, $|E \setminus F_\varepsilon| < \varepsilon$. Since f is constant on each K_m, and the distance between K_m and $K_{m'}$ is strictly positive for $m \neq m'$ (compactness), this implies $f|_{F_\varepsilon}$ is continuous.

Now we show the statement holds for any measurable f. Let f_n be simple functions such that $f_n \to f$ almost everywhere, and $C_n \subseteq E$ be such that $|C_n| < 2^{-n}$ and $f_n|_{E \setminus C_n}$ is continuous for all n. By Egorov's Theorem, there exists A_ε such that $f_n \to f$ uniformly on A_ε and $|E \setminus A_\varepsilon| < \varepsilon$. Set $F'_\varepsilon = A_\varepsilon \setminus \bigcup_{n \geq N} C_n$ so $|E \setminus F'_\varepsilon| < 2\varepsilon$ for $N = N_\varepsilon$ sufficiently large. Since $f_n|_{F'_\varepsilon}$, $n \geq N$ is continuous $f_n \to f$ uniformly on F'_ε, $f|_{F'_\varepsilon}$ is continuous.

By regularity of the Lebesgue measure, there exists $F_\varepsilon \subseteq F'_\varepsilon$ closed with $|F'_\varepsilon \setminus F_\varepsilon| < \varepsilon$ so $|E \setminus F_\varepsilon| < 3\varepsilon$ and we are done. □

2 Banach and Hilbert space analysis

2.1 The Hilbert space L^2

For any measure space $(E, \mathcal{E}, \mu)$, $L^2(E, \mu)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{L^2} = \int_E \overline{f} g \mathrm{d}\mu.$$

Definition. A subset $S = \{u_j\}_{j \in J} \subseteq H$ of a Hilbert space H is

- *Orthogonal* if $\langle u_j, u_k \rangle = 0$ for all $j \neq k$;
- *Orthonormal* if it is orthogonal and $\langle u_j, u_j \rangle = 1$ for all j;
- *Complete* if $\overline{\operatorname{span}\{u_j\}} = H$.

A complete orthonormal set is called a *Hilbert basis.*

Fact. A Hilbert space is separable (i.e there is a countable dense subset) if and only if there is a countable orthonormal (Hilbert) basis.

Examples.

(i) $L^2([-\pi, \pi]), S = \left\{ \frac{1}{\sqrt{2\pi}} e^{-inx} \right\}_{n \in \mathbb{Z}}$. Then S is a Hilbert basis; the Fourier basis (completeness follows from the Stone-Weierstrass theorem & density of C^∞).

(ii) $L^2(\mathbb{R})$, $S = \{\psi_{n,k}\}_{n,k \in \mathbb{Z}}$ where

$$\psi_{n,k}(x) = 2^{n/2} \psi(2^n x - k),$$

$$\psi(x) = \begin{cases} 1 & x \in [0, 1/2) \\ -1 & x \in [1/2, 1) \\ 0 & \text{otherwise} \end{cases}.$$

S is a Hilbert basis; the *Haar system.*

(iii) $L^2(\mathbb{R}, \mu(\mathrm{d}x))$, where $\mu(\mathrm{d}x) = (2\pi)^{-1/2} \exp(x^2/2)\mathrm{d}x$; the Gauss measure. Then take $\{H_n\}_{n \in \mathbb{Z}_{\geq 0}}$, where the H_n are obtained by applying Gram-Schmidt to $\{1, x, x^2, \ldots\}$; the Hermite polynomials. Then $\{H_n\}_{n \in \mathbb{Z}_{\geq 0}}$ is a Hilbert basis.

Theorem (Reisz representation theorem)**.** *For any bounded linear functional $\Lambda : H \to \mathbb{R}$ (respectively $\mathbb{C}$), there is a unique $w \in H$ such that $\Lambda(u) = \langle w, u \rangle$ for all $u \in H$.*

2.2 Radon-Nikodym Theorems

Definition. Let $(E, \mathcal{E})$ be a measurable space and let μ, ν be two measures on $(E, \mathcal{E})$. Then ν is said to be *absolutely continuous* with respect to μ, written $\nu \ll \mu$, if for all $A \in \mathcal{E}$, $\nu(A) = 0$ whenever $\mu(A) = 0$. Two measures μ, ν are *mutually singular*, written $\mu \perp \nu$ if there is $B \in \mathcal{E}$ such that $\mu(B) = 0 = \nu(B^c)$.

Theorem (Radon-Nikodym)**.** *Let μ and ν be finite measures on $(E, \mathcal{E})$ with $\nu \ll \mu$. Then there exists $\omega \in L^1(E, \mathcal{E})$ such that for all $A \in \mathcal{E}$,*

$$\nu(A) = \int_A \omega \mathrm{d}\mu.$$

Equivalently, for all $h : E \to [0, \infty]$ Borel,

$$\int h \mathrm{d}\nu = \int h\omega \mathrm{d}\mu.$$

Proof. Set $\alpha = \mu + 2\nu$ and $\beta = 2\mu + \nu$. Define

$$\Lambda(f) = \int_E f \mathrm{d}\beta.$$

Then

$$|\Lambda(f)| \leq \int_E |f| \mathrm{d}\beta \leq 2 \int_E |f| \mathrm{d}\alpha \leq 2\sqrt{\alpha(E)} ||f||_{L^2(E,\alpha)}.$$

So $\Lambda : L^2(E, \alpha) \to \mathbb{R}$ is bounded and linear. So by the Riesz representation theorem, there is $g \in L^2(E, \alpha)$ such that $\Lambda(f) = \langle g, f \rangle_{L^2(E,\alpha)}$ for all $f \in L^2(E, \alpha)$. Hence $\int f \mathrm{d}\beta = \int gf \mathrm{d}\alpha$, and

$$\int f(2\mathrm{d}\mu + \mathrm{d}\nu) = \int gf(\mathrm{d}\mu + 2\mathrm{d}\nu) \iff \int f(2 - g)\mathrm{d}\mu = \int f(2g - 1)\mathrm{d}\nu. \quad (*)$$

We claim that g takes values in $[1/2, 2]$ μ-a.e and ν-a.e, and that $g \neq 1/2$ μ-a.e (this implies $g \neq 1/2$ ν-a.e since $\nu \ll \mu$). Assuming the claim, the proof is completed as follows; by the monotone convergence theorem, $(*)$ can be extended to all $f : E \to [0, \infty]$. Given $h : E \to [0, \infty]$ measurable, set

$$f(x) = \frac{h(x)}{2g(x) - 1}, \ \omega(x) = \frac{2 - g(x)}{2g(x) - 1}, \ x \in \{g \neq 1/2\}.$$

Then

$$\int h \mathrm{d}\nu = \int f(2g - 1)\mathrm{d}\nu = \int f(2 - g)\mathrm{d}\mu = \int h\omega \mathrm{d}\mu.$$

In particular, taking $h = 1$, we see $\omega \in L^1(E, \mu)$.

Now we prove the claim: let $f = \mathbb{1}_{A_j}$, with $A_j = \left\{x \in E : g(x) < \frac{1}{2} - \frac{1}{j}\right\}$. Then we have

$$\int f(2g - 1)\mathrm{d}\nu \leq -\frac{2}{j}\nu(A_j),$$

$$\int f(2-g)\mathrm{d}\mu \geq \frac{3}{2}\mu(A_j),$$
$$\implies \frac{3}{2}\mu(A_j) \leq -\frac{2}{j}\nu(A_j) \implies \mu(A_j) = \nu(A_j) = 0.$$

Implying $g \geq 1/2$ both μ-a.e and ν-a.e. To show $g \leq 2$ μ-a.e and ν-a.e the proof is analogous, instead with $A_j = \{x \in E : g(x) \geq 2 + 1/j\}$. To show $\mu(\{g = 1/2\}) = 0$, set $f = \mathbb{1}_Z$, $Z = \{g = 1/2\}$ in $(*)$, giving

$$\frac{3}{2}\int \mathbb{1}\{g = 1/2\}\mathrm{d}\mu = 0.$$

$\square$

2.3 The dual of L^p

Definition. A *topological vector space* (TVS) X is a vector space together with a topology in which $(x,y) \mapsto x+y$ and $(\lambda, x) \mapsto \lambda x$ are continuous. The *dual space* X' is the linear space of continuous linear maps $\Lambda : X \to \mathbb{R}$ (or $\mathbb{C}$).

If X is a normed vector space equipped with the topology induced by the norm, then linear maps on X are bounded if and only if they are continuous. We can define a norm on X' by

$$||\Lambda||_{X'} = \sup_{\substack{x \in X \\ ||x|| \leq 1}} |\Lambda(x)|.$$

Then X' is a Banach space (even if X isn't).

We aim to identify $L^p(\mathbb{R}^n)'$ with $L^q(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$, if $p \in [1, \infty)$.

Proposition. Let $q \in [1, \infty]$. For every $g \in L^q(\mathbb{R}^n)$,

$$\Lambda_g(f) = \int \bar{f} g \mathrm{d}x$$

defines $\Lambda_g \in L^p(\mathbb{R}^n)'$ with $||\Lambda_g|| = ||g||_{L^q}$.

Proof. By Hölder's inequality, $|\Lambda_g(f)| \leq ||f||_{L^p} ||g||_{L^q}$. Hence $\Lambda_g \in L^p(\mathbb{R}^n)'$ and $||\Lambda_g|| \leq ||g||_{L^q}$. Equality: see Example sheet 1. $\square$

Corollary. *The map $J : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)'$, $g \mapsto \Lambda_g$ is a linear isometry. Thus we can identify $L^q(\mathbb{R}^n)$ as a subspace of $L^p(\mathbb{R}^n)'$.*

Remark. When $p = 2$ then $L^2(\mathbb{R}^n)' = L^2(\mathbb{R}^n)$, i.e J is surjective (Riesz representation theorem).

Theorem. *Let $p \in [1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then J is surjective, i.e $L^p(\mathbb{R}^n)' = L^q(\mathbb{R}^n)$.*

Remarks.

1. $L^1(\mathbb{R}^n)' = L^\infty(\mathbb{R}^n)$, but $L^\infty(\mathbb{R}^n)' \neq L^1(\mathbb{R}^n)$;
2. The same is true if $\mathbb{R}^n$ is replaced by $U \subseteq \mathbb{R}^n$ open.

Definition. $\Lambda \in L^p(\mathbb{R}^n)'$ is *positive* if

$$\Lambda(f) \geq 0 \text{ for all } f \in L^p(\mathbb{R}^n) \text{ such that } f \geq 0 \text{ a.e.}$$

Lemma. *Let $\Lambda \in L^p(\mathbb{R}^n)'$ be positive. Then there is $g \in L^q(\mathbb{R}^n)$ non-negative with*

$$\Lambda(f) = \int_{\mathbb{R}^n} f g \mathrm{d}x \text{ for all } f \in L^p(\mathbb{R}^n).$$

Furthermore $||g||_{L^q} = ||\Lambda||$.

Proof. Let $\mu(\mathrm{d}x) = e^{-|x|^2}\mathrm{d}x$. Then $\mu(\mathbb{R}^n) < \infty$. Define

$$\nu(A) = \Lambda\left(e^{-|x|^2/p}\mathbb{1}_A\right) \text{ for } A \in \mathcal{B}(\mathbb{R}^n).$$

First we show that ν is a finite measure on $\mathbb{R}^n$. Clearly $\nu(\emptyset) = 0$ and $\nu(A) \in [0,\infty)$ since Λ is positive. Let $A_k \in \mathcal{B}(\mathbb{R}^n)$ be a sequence of disjoint sets and $B_m = \bigcup_{k=1}^m A_k$. Then

$$\begin{aligned}|\nu(B_\infty) - \nu(B_m)| &\le \|\Lambda\| \left\|e^{-|x|^2/p}(\mathbb{1}_{B_\infty} - \mathbb{1}_{B_m})\right\|_{L^p} \\ &= \|\Lambda\|\mu(B_\infty \setminus B_m)^{1/p} \to 0.\end{aligned}$$

So ν is countably additive, and thus a measure. Now we claim $\nu \ll \mu$. Indeed if $\mu(A) = 0$, $\nu(A) \le ||\Lambda||\mu(A)^{1/p}$. Thus by the Radon-Nikodym theorem, there is $\omega \in L^1(\mathbb{R}^n, \mu)$ non-negative such that

$$\nu(A) = \int_A \omega \mathrm{d}\mu = \int_A \omega e^{-|x|^2}\mathrm{d}x \text{ for all } A \in \mathcal{B}(\mathbb{R}^n).$$

Now let $f = e^{-|x|^2/p}\tilde{f}$ where $\tilde{f}$ is simple. Then by linearity of Λ,

$$\begin{aligned}\Lambda(f) = \int \tilde{f}\mathrm{d}\nu &= \int \tilde{f}\omega e^{-|x|^2}\mathrm{d}x \\ &= \int f \underbrace{\omega e^{-\left(1-\frac{1}{p}\right)|x|^2}}_{\tilde{\omega}=\omega e^{-\frac{1}{q}|x|^2}}\mathrm{d}x.\end{aligned}$$

Hence $\Lambda(f) = \int f\tilde{\omega}\mathrm{d}x$ for all f as above. Exercise: functions of the form $f = e^{-|x|^2/p}\tilde{f}$ for $\tilde{f}$ are dense in $L^p(\mathbb{R}^n)$. Then we have $\Lambda(f) = \int f\tilde{\omega}\mathrm{d}x$ for all $f \in L^p(\mathbb{R}^n)$ since Λ is continuous.

Example sheet 1 gives that

$$||\tilde{\omega}||_{L^q} = \sup\left\{\int |f\tilde{\omega}|\mathrm{d}x : ||f||_{L^p} \le 1\right\}.$$

Thus

$$||\tilde{\omega}||_{L^p} \le ||\Lambda|| \text{ since } \int |f\tilde{\omega}|\mathrm{d}x = \int |f|\tilde{\omega}\mathrm{d}x = \Lambda(|f|) \le ||\Lambda||||f||_{L^p}.$$

Conversely, $\Lambda(f) \le ||f||_{L^p}||\tilde{\omega}||_{L^q}$ by Hölder's inequality, so $||\Lambda|| \le ||\tilde{\omega}||_{L^q}$ and $||\Lambda|| = ||\tilde{\omega}||_{L^q}$. $\square$

Theorem. *Let $p \in [1, \infty)$. Then $\int : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)'$, $g \mapsto \Lambda_g$ where $\Lambda_g(f) = \int fg$ is a linear isometry and surjective.*

Proof. First consider the real case. In Example Sheet 2 its shown that if $\Lambda \in L^p(\mathbb{R}^n)'$ is real-values, there are Λ_+ and Λ_- both bounded and positive such that $\Lambda = \Lambda_+ - \Lambda_-$. The claim follows from the previous lemma.

In the complex case, if $\Lambda \in L(\mathbb{R}^n, \mathbb{C})'$ then $\Lambda_r(f) = \Re\Lambda(f)$ and $\Lambda_i(f) = \Im\Lambda(f)$ define two $\mathbb{R}$-linear $\Lambda \in L^p(\mathbb{R}^n, \mathbb{R})$ such that

$$\Lambda(f_r + if_i) = \Lambda_r(f_r) - \Lambda_i(f_i) + i\Lambda_r(f_i) + i\Lambda_i(f_r).$$

The claim then follows by the real-valued case. □

2.4 Riesz-Markov Theorem

Fact. For any finite (positive) regular Borel measure on $\mathbb{R}^n$, $\Lambda_\mu(f) = \int f \mathrm{d}\mu$ defines a positive bounded linear functional on $C_c(\mathbb{R}^n, ||\cdot||_\infty)$.

Lemma. *Λ uniquely determines μ and for any $U \in \mathbb{R}^n$ open*

$$\mu(U) = \sup\{\Lambda_\mu(g) : g \in C_c(\mathbb{R}^n), 0 \leq g \leq \mathbb{1}_U\}. \tag{$*$}$$

Proof sketch. We would like to take $f = \mathbb{1}_A$ for $A \in \mathcal{B}(\mathbb{R}^n)$, but this is not continuous. So we approximate by continuous functions: assume $U \in \mathbb{R}^n$ is open, set $U_k = U \cap \{|x| < k\}$, abd

$$\chi_k(x) = \begin{cases} 1 & x \in U_k,\ d(x, U_k^c) \geq \frac{1}{k} \\ 0 & x \notin U_k \\ kd(x, U_k^c) & x \in U_k,\ d(x, U_k^c) < 1/k \end{cases}.$$

Then $\chi_k \in C_c(\mathbb{R}^n)$ and $\chi_k \uparrow \mathbb{1}_U$. So by the Monotone Convergence Theorem,

$$\mu(U) = \lim_{k\to\infty} \int \chi_k \mathrm{d}\mu = \lim_{k\to\infty} \Lambda(\chi_k).$$

And $(*)$ also follows. Since μ is regular, this determines μ on all Borel sets. □

Definition. A *signed measure* is the difference of two mutually singular finite positive measures.

Theorem (Riesz-Markov Theorem)**.** *Given $\Lambda : C_c(\mathbb{R}^n) \to \mathbb{R}$ linear positive and bounded, there is a unique finite Borel measure μ on $\mathbb{R}^n$ such that*

$$\Lambda(f) = \int_{\mathbb{R}} f \mathrm{d}\mu,\ \forall f \in C_c(\mathbb{R}^n).$$

The dual space $C_c(\mathbb{R}^n)$ is the space of signed measures.

2.5 Strong, weak & weak-$*$ topologies

Example Sheet 2: if X is a Banach space, then the closed unit ball is compact iff X is finite dimensional.

Goal: recover some form of compactness by considering a weaker topology.

Definition. A *seminorm* p on a vector space X (over $\mathbb{R}$ or $\mathbb{C}$) is a map $p : X \to \mathbb{R}$ such that

(i) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$;

(ii) $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$;

(iii) $p(x) \geq 0$ for all $x \in X$.

(Note: it is not necessarily positive semidefinite)

Definition. A family $\mathcal{P}$ of seminorms is *separating* if for every $x \in X$ with $x \neq 0$ there is $p \in \mathcal{P}$ such that $p(x) \neq 0$.

Definition. The topology $\tau_{\mathcal{P}}$ induced by a family of seminorms $\mathcal{P}$ is generated by

$$\beta = \{x + B : x \in X,\ B \in \dot{\beta}\}.$$

Where $\dot{\beta}$ consists of finite intersections of $V(p, n) = \{x \in X : p(x) < 1/n\}$ for $p \in \mathcal{P}$, $n \in \mathbb{N}$. $(X, \tau_{\mathcal{P}})$ is a locally convex topological vector space (LCTVS).

Theorem. *β is a neighbourhood base for the topology $\tau_{\mathcal{P}}$ (every open set $U \in \tau_{\mathcal{P}}$ is a union of sets in β), and the vector space operations $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda x$ are continuous, as is every seminorm $p \in \mathcal{P}$.*

Example Sheet 2: for $(x_k)_{k \geq 1}$ in X, $x_k \to x$ in $\tau_{\mathcal{P}}$ if and only if $p(x - x_k) \to 0$ for all $p \in \mathcal{P}$.

Fact. If $\mathcal{P} = (p_k)_{k \in \mathbb{N}}$ is countable, then the topology is induced by the metric

$$d_{\mathcal{P}}(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x-y)}{1 + p_k(x-y)}.$$

Definition. If $\mathcal{P}$ is as above with metric as above, if the metric $d_{\mathcal{P}}$ is complete, $(X, d_{\mathcal{P}})$ is called a *Fréchet space.*

Examples.

(i) X a Banach space, $\mathcal{P}_s = \{||\cdot||\}$: the corresponding topology $\tau_s = \tau_{\mathcal{P}_s}$ is called *norm* or *strong topology.* We have $x_k \to x$ in τ_s if and only if $||x_k - x|| \to 0$.

(ii) X a Banach space, $\mathcal{P}_w = \{p_\Lambda : \Lambda \in X'\}$ where $p_\Lambda(x) = |\Lambda(x)|$. Each p_Λ is a seminorm and the Hahn-Banach theorem implies $\mathcal{P}_w$ is separating. (For $X = L^p(\mathbb{R}^n)$ this can be verified directly.) The topology $\tau_w = \tau_{\mathcal{P}_w}$ is called the *weak topology.* We have $x_k \to x$ in τ_w if and only if $\Lambda(x_k) \to \Lambda(x)$ for all $\Lambda \in X'$. We write $x_k \rightharpoonup^w x$. Also $x_k \to x$ implies $x_k \rightharpoonup^w x$.

(iii) X a Banach space, then X' is also a Banach space. Hence we have a strong and weak topology on X'. The *weak-$*$* topology τ_{w^*} is generated by $\mathcal{P}_{w^*} = \{p_x : x \in X\}$ where $p_x(\Lambda) = |\Lambda(x)|$. Then $\Lambda_k \to \Lambda$ in τ_{w^*} if and only if $\Lambda_k(x) \to \Lambda(x)$ for every $x \in X$. We write $\Lambda_k \rightharpoonup^{w^*} \Lambda$.

Remark. If X is reflexive, i.e $X'' = X$, then $\tau_w = \tau_{w^*}$.

Example. Let $p \in [1, \infty)$ and $(f_k)_{k\geq 1}$ be a sequence in $L^p(\mathbb{R}^n)$. Then

$$\begin{aligned} f_k \to f \text{ in } L^p &\iff \int |f_k - f|^p \mathrm{d}x \to 0 \\ f_k \rightharpoonup^w f \text{ in } L^p &\iff \int g(f_k - f)\mathrm{d}x \to 0 \text{ for all } g \in L^q \\ f_k \rightharpoonup^{w^*} f \text{ in } L^p &\iff f_k \rightharpoonup^w f \text{ in } L^p \end{aligned}$$

On the other hand, if $(f_k)_{k\geq 1}$ is in $L^\infty(\mathbb{R}^n)$,

$$\begin{aligned} f_k \to f \text{ in } L^\infty &\iff \text{esssup}|f_k - f| \to 0 \\ f_k \xrightarrow{w^*} \text{ in } L^\infty &\iff \int g(f_k - f)\mathrm{d}x \to 0 \text{ for all } g \in L^1 \\ f_k \xrightarrow{w^*} \text{ in } L^\infty &\iff f_k \xrightarrow{w} \text{ in } L^\infty \end{aligned}$$

2.6 Compactness

Theorem (Arzela-Ascoli Theorem)**.** *Let $I = [0, 1]$ (or a compact Hausdorff space). Suppose a sequence of continuous functions $f_k : I \to \mathbb{R}$ is*

- *Bounded:* $\sup_k \sup_{x\in I} |f_k(x)| < \infty$
- *Equicontinuous: for all $\varepsilon > 0$ there exists δ such that* $\sup_k \sup_{x\in I} \sup_{y\in B(x,\delta)} |f_k(x) - f_k(y)| < \varepsilon$.

Then there is a subsequence (i_k) such that f_{i_k} converges to some continuous f.

Application: $C^{0,\alpha}(I)$ embeds compactly into $C^0(I)$, where $C^{0,\alpha}(I) = \{f \in C^0(I) : ||f||_{C^{0,\alpha}} < \infty\}$,

$$||f||_{C^{0,\alpha}} = \sup_{x\in I} |f'(x)| + \sup_{\substack{x,y\in I \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

The identity map id : $C^{0,\alpha}(I) \to C^0(I)$ is compact, i.e any sequence $(f_i)_{i\geq 1}$ in $C^{0,\alpha}$ that is bounded in $C^{0,\alpha}$ has a convergent subsequence in $C^0(I)$.

Theorem (Banach-Alaoglu)**.** *Let X be a separable Banach space, and let $(\Lambda_j)_{j\geq 1}$ be a bounded sequence in X', say $\sup_j ||\Lambda_j||_{X'} \leq 1$. Then there is a subsequence (j_i) and $\Lambda \in X'$ such that $\Lambda_{j_i} \to^{w^*} \Lambda$.*

Example. Let $p \in (1,\infty]$ and $(f_j)_{j\geq 1}$ be a sequence in $L^p(\mathbb{R}^n)$ such that $||f_j||_{L^p} \leq K$ for all j. Then there is $f \in L^p$ with $||f||_{L^p} \leq K$ and a subsequence (j_i) such that for every $g \in L^q(\mathbb{R}^n)$, $\int f_{j_i} g \mathrm{d}x \to \int fg\mathrm{d}x$. (Just apply Banach-Alaoglu noting $L^q(\mathbb{R}^n)' = L^p(\mathbb{R}^n)$ for $p \in [1,\infty)$ and L^q is separable for such q.)

Proof. Step 1: construction. Let $D = \{x_k\}_{k=1}^\infty \subseteq X$ be dense (can do this by separability). Since $(\Lambda_j(x_1))_{j\geq 1}$ is a bounded sequence, there is a subsequence $J_1 \subseteq \mathbb{N}$ and $\Lambda(x_1) \in \mathbb{R}$ (or $\mathbb{C}$) such that $\Lambda_j(x_1) \to \Lambda(x_1)$ for $j \in J_1$, $j \to \infty$. Iterating, there are nested subsequences $J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots$ and $\Lambda(x_k) \in \mathbb{R}$ (or $\mathbb{C}$) such that $\Lambda_j(x_k) \to \Lambda(x_k)$ for $j \in J_l$, $l \geq k$.

Now take the 'diagonal subsequence' J of $J_1 \supseteq J_2 \supseteq \ldots$ defined by $J = (j_n)_{n\geq 1}$ where j_n is the first element of J_n. i.e it has first element which is the first element of J_1, second element which is the first element of J_2, etc. Then $\Lambda_j(x_k) \to \Lambda(x_k)$ for $j \in J, j \to \infty$.

Step 2: we'll show $\Lambda : D \to \mathbb{R}$ is uniformly continuous so can be extended uniquely to $\Lambda : X \to \mathbb{R}$ continuous. For each $x, y \in D$ such that $||x-y|| < \varepsilon$, there is $j \in J$ such that $|\Lambda_j(x) - \Lambda(x)| < \varepsilon$, $|\Lambda_j(y) - \Lambda(y)| < \varepsilon$. Hence

$$|\Lambda(x) - \Lambda(y)| \leq |\Lambda(x) - \Lambda_j(x)| + |\Lambda(y) - \Lambda_j(y)| + |\Lambda_j(x-y)| \leq 3\varepsilon.$$

Step 3: we show $\Lambda : X \to \mathbb{R}$ (or $\mathbb{C}$) is linear. For $x, y \in X$, $a \in \mathbb{R}$ (or $\mathbb{C}$), let $x', y', z' \in D$ be such that $||x-x'|| < \varepsilon, ||y-y'|| < \varepsilon, ||x+ay-z'|| < \varepsilon$. Then take $j \in J$ such that $|\Lambda(x') - \Lambda_j(x')| < \varepsilon$, $|\Lambda(y') - \Lambda_j(y')| < \varepsilon$, $|\Lambda(z') - \Lambda_j(z')| < \varepsilon$. Then

$$\begin{aligned}
|\Lambda(x+ay) - \Lambda(x) - a\Lambda(y)| &\leq |\Lambda(x+ay) - \Lambda(z')| + |\Lambda(x) - \Lambda(x')| + |a||\Lambda(y) - \Lambda(y')| \\
&\quad + |\Lambda(z') - \Lambda_j(z')| + |\Lambda(x') - \Lambda_j(x')| + |a||\Lambda(y') - \Lambda_j(y')| \\
&\quad + |\Lambda_j(x') - \Lambda_j(x') - a\Lambda_j(y')| \\
&\leq C\varepsilon + ||\Lambda_j|| ||z' - x' - ay'|| \leq C'\varepsilon
\end{aligned}$$

so $\Lambda(x+ay) = \Lambda(x) + a\Lambda(y)$.

Step 4: $||\Lambda|| \leq 1$. We have

$$||\Lambda|| = \sup_{\substack{x \in X \\ ||x|| \leq 1}} |\Lambda(x)| = \sup_{\substack{x \in D \\ ||x|| \leq 1}} |\Lambda(x)| \leq 1 \text{ by density.}$$

Step 5: $\Lambda_j \to^{w^*} \Lambda$. For $x' \in D$ take $x \in X$ with $||x - x||'$. Then we have

$$|\Lambda_j(x) - \Lambda(x)| \leq |\Lambda_j(x - x')| + |\Lambda_j(x') - \Lambda(x')| + |\Lambda(x - x')| < 3\varepsilon.$$

So $\Lambda_j(x) \to \Lambda(x)$ for all $x \in X$. $\square$

2.7 Hahn-Banach Theorem

Suppose $\Lambda : M \to \mathbb{R}$ (or $\mathbb{C}$) is a bounded linear functional on a subspace $M \subseteq X$ of a Banach space. Goal: extend Λ to $\tilde{\Lambda} : X \to \mathbb{R}$ (or $\mathbb{C}$) with $||\tilde{\Lambda}||_{X'} = ||\Lambda||_{M'}$.

Definition. Let X be a real vector space. Then $p : X \to \mathbb{R}$ is *sublinear* if

(i) $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$;

(ii) $p(tx) = tp(x)$ for all $x \in X$, $t \ge 0$.

Examples.

- $p(x) = |l(x)|$ for $l : X \to \mathbb{R}$ linear.
- Any seminorm.

Note. If p is sublinear, l is linear, $l(x) \le p(x)$ for all $x \in M$, then

$$-p(-x) \le l(x) \le p(x).$$

Lemma (Bounded extension lemma). *Let X be a real vector space, $p : X \to \mathbb{R}$ sublinear, $M \subseteq X$ a subspace. Assume $l : M \to \mathbb{R}$ is linear and $l(y) \le p(y)$ for all $y \in M$. For $x \in X \setminus M$, let $\tilde{M} = span\{x, M\}$. Then there is an extension $\tilde{l} : \tilde{M} \to \mathbb{R}$ linear such that $\tilde{l}(y) = l(y)$ for all $y \in M$ and $\tilde{l}(z) \le p(z)$ for all $z \in \tilde{M}$.*

Proof. If $z \in \tilde{M}$, there are unique $y \in M$ and $\lambda \in \mathbb{R}$ such that $z = y + \lambda x$. Define $\tilde{l}(x) = a$ for some a to be defined, and $\tilde{l}(y) = l(y)$ for $y \in M$ and then $l(z)$ is defined by linearity.

Claim: $a = \sup\{l(y) - p(y-x) : y \in M\}$ works. For each $y, z \in M$,

$$l(y) + l(z) = l(y+z) \le p(y+z) \le p(y-x) + p(x+z).$$

Hence

$$l(y) - p(y-x) \le p(z+x) - l(z). \tag{$*$}$$

Note this implies $a < \infty$. Also $(*)$ implies

$$l(y) - a \le p(y-x) \text{ for all } y \in M. \tag{$*'$}$$

and

$$l(z) + a \le p(z+x) - (l(y) - p(y-x)) + a \text{ for all } y \in M. \tag{$*''$}$$

So taking the infimum of $(*'')$ over $y \in M$:

$$l(z) + a \le p(z+x) - a + a = p(z+x).$$

Now

$$\tilde{l}(y+\lambda x) = l(y) + a\lambda \le p(y+\lambda x) \text{ for all } y \in M, \lambda > 0$$

by taking $z = \lambda^{-1} y$ in $(*'')$ and multiplying across by λ. Also

$$\tilde{l}(y+\lambda x) = l(y) + a\lambda \le p(y+\lambda x) \text{ for all } y \in M, \lambda < 0$$

by replacing y with $|\lambda|^{-1} y$ in $(*')$ and multiplying across by $|\lambda|$. Hence $l(z) \le p(z)$ for all $z \in \tilde{M}$. □

Corollary. *If M has finite codimension in X, then any $l : M \to \mathbb{R}$ satisfying $l(y) \leq p(y)$ for all $y \in M$ can be extended to $\tilde{l} : X \to \mathbb{R}$ linear with $l(x) \leq p(x)$ for all $x \in X$.*

Proof. Apply lemma repeatedly. □

Definition. Let S be a set. A *partial order* is a binary relation $\leq$ on S such that

(i) $a \leq a$ for all $a \in S$ (reflexive);

(ii) $a \leq b, b \leq c \Rightarrow a \leq c$ (transitive);

(iii) $a \leq b, b \leq a \Rightarrow a = b$ (antisymmetry).

A set S with a partial order is called a *poset*. If additionally $a \leq b$ or $b \leq a$ holds for all $a, b \in S$, then $\leq$ is called a *total order*. A totally ordered subset $T \subseteq S$ of a poset S is called a *chain*. An element $u \in S$ is an *upper bound* for $T \subseteq S$ if $t \leq u$ for all $t \in T$. A *maximal element* $m \in S$ is an element such that $m \leq x$ implies $m = x$.

Examples.

(i) If A is any set, $S = 2^A$ is a poset partially ordered by inclusion of sets.

(ii) $\mathbb{R}$ (with the usual ordering) is a totally ordered set with no maximal element.

(iii) The collection of open balls in $\mathbb{R}^n$ is a poset ordered by inclusion. The subset $T = \{B_r(0) : 0 < r \leq 1\}$ is a chain in S. $B_1(0)$ is a maximal element of T. $B_2(0)$ is an upper bound of T.

Lemma (Zorn's Lemma)**.** *Let $(S, \leq)$ be a poset in which every totally ordered subset has an upper bound. Then $(S, \leq)$ contains at least one maximal element.*

We will treat Zorn's Lemma as an axiom.

Theorem (Hahn-Banach)**.** *Let X be a real vector space, $p : X \to \mathbb{R}$ sublinear, $M \subseteq X$ a subspace. For any $l : M \to \mathbb{R}$ linear such that $l(x) \leq p(x)$ for all $x \in M$, there exists $\tilde{l} : X \to \mathbb{R}$ linear such that $\tilde{l}|_M = l$ and $\tilde{l}(y) \leq p(y)$ for all $y \in X$.*

Proof. Let

$$S = \{(N, \tilde{l}) : X \supseteq N \supseteq M,\ \tilde{l} : N \to \mathbb{R} \text{ linear},\ \tilde{l}(x) \leq p(x)\ \forall x \in N,\ \tilde{l}|_M = l\}$$

and define the partial order $(N_1, \tilde{l}_1) \leq (N_2, \tilde{l}_2) \iff N_1 \subseteq N_2,\ \tilde{l}_2|_{N_1} = \tilde{l}_1$. For every totally ordered subset $T \subseteq S$, we obtain an upper bound for T via

$$N_T := \bigcup_{(N,\tilde{l}) \in T} N,\ l_T(x) = \tilde{l}(x) \text{ if } x \in N \text{ for some } (N, \tilde{l}) \in T$$

which is well-defined since where the $\tilde{l}$ are defined (for $(N, \tilde{l}) \in T$), they agree since T is a total order. Further, $(N, \tilde{l}) \leq (N_T, l_T)$ for every $(N, \tilde{l}) \in T$. Thus (N_T, l_T) is an upper bound.

Applying Zorn's Lemma, there is a maximal element $(\tilde{N}, \tilde{l})$ of S. It suffices to show $\tilde{N} = X$. Suppose not, then there is $x \in X \setminus \tilde{N}$ and the bounded extension lemma gives an extension l^* to $N^* = \text{span}\{x, \tilde{N}\}$ such that $(\tilde{N}, \tilde{l}) \leq (N^*, l^*)$, contradicting maximality of $(\tilde{N}, \tilde{l})$. $\square$

Corollary. *Let X be a normed vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $M \subseteq X$ a subspace. Then every bounded linear functional $\Lambda : M \to \mathbb{K}$ can be extended to a bounded linear functional $\tilde{\Lambda} : X \to \mathbb{K}$ such that $||\tilde{\Lambda}||_{X'} = ||\Lambda||_{M'}$ and $\tilde{\Lambda}|_M = \Lambda$.*

Proof. If $\mathbb{K} = \mathbb{R}$, then $p(x) = ||\Lambda|| \cdot ||x||$ is sublinear and the result follows immediately from Hahn-Banach. If $\mathbb{K} = \mathbb{C}$, then $\Lambda(x) = l(x) - il(ix)$ with $l : X \to \mathbb{R}$, $l(x) = \Re(\Lambda(x))$ a real linear function. Since $|\Lambda(x)| = l(e^{i\theta}x)$ for suitable $\theta \in [0, 2\pi]$,

$$\sup_{\substack{||x|| \leq 1 \\ x \in N}} |\Lambda(x)| = \sup_{\substack{||x|| \leq 1 \\ x \in N}} l(x), \ N \subseteq X.$$

So apply Hahn-Banach to l and the result follows. $\square$

Corollary. *Let X be a normed vector space and $x \in X$. Then there is $\Lambda_x \in X'$ such that $||\Lambda_x|| = 1$ and $\Lambda_x(x) = ||x||$. Λ_x is called a support functional.*

Proof. Let $M = \text{span}\{x\}$ and define $l \in M'$ by $l(tx) = t||x||$, $t \in \mathbb{K}$. Clearly, $||l|| = 1$ and $l(x) = ||x||$. Apply Hahn-Banach to get the result. $\square$

Remark. For $X = L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, can construct a support functional by hand (Example Sheet 1).

Corollary. *Let X be a normed vector space and $x \in X$. Then $x = 0$ if and only if $\Lambda(x) = 0$ for all $\Lambda \in X'$.*

Corollary. *Let X be a normed vector space and $x, y \in X$ be distinct. Then there exists $\Lambda \in X'$ such that $\Lambda(x) \neq \Lambda(y)$: i.e linear functionals separate points.*

Corollary. *The map* $\Phi : X \to X''$, $\Phi(x) = \tilde{x}$ *where* $\tilde{x}(\Lambda) = \Lambda(x)$ *is an isometry.*

Proof. By definition

$$||\Phi(x)||_{X'} = \sup_{\substack{\Lambda \in X' \\ ||\Lambda|| \leq 1}} |\Phi(x)(\Lambda)| = \sup_{\substack{\Lambda \in X' \\ ||\Lambda|| \leq 1}} |\Lambda(x)| \leq \sup_{\substack{\Lambda \in X' \\ ||\Lambda|| \leq 1}} ||\Lambda|| \cdot ||x|| = ||x||$$

By choosing $\Lambda = \Lambda_x$ (the support functional), there is equality. □

Definition. X is said to be *reflexive* if Φ is surjective, i.e $X = X''$.

Example. $L^p(\mathbb{R}^n)$ is reflexive iff $p \in [1, \infty)$.

Theorem. *Let* $A, B \subseteq X$ *be disjoint, nonempty, convex subsets of a normed space* X *(real or complex). Then*

(a) If A *is open, there exists* $\Lambda \in X'$ *such that and* $\gamma \in \mathbb{R}$ *such that*

$$\Re\Lambda(x) < \gamma \leq \Re\Lambda(y) \ , \forall x \in A, \forall y \in B.$$

If B *is also open the second inequality can be made strict.*

(b) If A *is compact and* B *is closed, then there exists* $\Lambda \in X'$ *and* $\gamma_1, \gamma_2 \in \mathbb{R}$ *such that*

$$\Re\Lambda(x) < \gamma_1 < \gamma_2 < \Re\Lambda(y) \ , \forall x \in A, \forall y \in B.$$

Proof. Assume X is a vector space over $\mathbb{R}$ (otherwise just apply to real part)

(a) Fix $a_0 \in A$, $b_0 \in B$ and set

$$x_0 = b_0 - a_0, \ C = A - B + x_0 \ni 0.$$

Note C is convex (since A and B are), $x_0 \notin C$. (since $A \cap B = \emptyset$). Thus C is a convex neighbourhood of 0. Let $p(x) = \inf\{t > 0 : t^{-1}x \in C\}$. Then p is sublinear wih $p(x) \leq k||x||$ for some k, and $p(x) < 1$ if and only if $x \in C$ (Example Sheet 2). Define $M = \{tx_0 : t \in \mathbb{R}\}$, and define $l : M \to \mathbb{R}$ by $l(tx_0) = t$.

We claim that $l(x) \leq p(x)$ for all $x \in M$. If $t > 0$, $l(tx_0) = t \leq tp(x_0)$ since $x_0 \notin C$. If $t < 0$, $l(tx_0) = t \leq 0 \leq p(tx_0)$. By Hahn-Banach, l can be extended to $\Lambda : X \to \mathbb{R}$ with $\Lambda(x) \leq p(x)$ for all $x \in X$. Moreover, $-k||x|| \leq -p(-x) \leq \Lambda(x) \leq p(x)$ so $|\Lambda(x)| \leq k||x||$ and $\Lambda \in X'$.

We claim that $\Lambda(a) < \Lambda(b)$ for all $a \in A$ and all $b \in B$. Indeed

$$\underbrace{\Lambda(a - b + x_0)}_{\Lambda(a) - \Lambda(b) + 1} \leq p(a - b + x_0) < 1.$$

Since non-zero elements of the dual are open maps (Example Sheet 2), $\Lambda(A)$ is an open interval (since A is open). Take γ to be the right endpoint of $\Lambda(A)$. Then $\Lambda(x) < \gamma \leq \Lambda(x)$. If B is also open, the inequality is strict.

(b) Since A is compact, B is closed and $A \cap B$,

$$d = \inf\{||a - b|| : a \in A, b \in B\} > 0.$$

Let $V = B_{1/2}(0)$. Then $A + V$ is open and disjoint from B. By (a), there is a $\Lambda \in X'$ such that $\Lambda(A + V)$ and $\Lambda(B)$ are disjoint intervals of $\mathbb{R}$. These intervals are also a positive distance apart so there exist $\gamma_1 < \gamma_2$ between them.

□

Corollary. *Let X be a Banach space, $M \subseteq X$ a subspace and $x_0 \in X$. If $x_0 \notin \overline{M}$ then there is $\Lambda \in X'$ such that $\Lambda(x_0) = 1$ and $\Lambda(x) = 0$ for all $x \in \overline{M}$.*

Proof. Apply (b) of the previous theorem with $A = \{x_0\}$, $B = \overline{M}$. Thus there exists $\Lambda \in X'$ such that $\Lambda(x_0) \notin \Lambda(M)$. Thus $\Lambda(\overline{M})$ must be a proper subsapce of $\mathbb{K}$, so $\{0\}$. Also $\Lambda(x_0) \neq 0$, so $\frac{\Lambda}{\Lambda(x_0)}$ is the required element of X'. □

3 Distributions

Distributions are generalised functions.

Example. $G(x) = \frac{1}{4\pi|x|}$, $x \in \mathbb{R}^3$ solves $-\nabla^2 G = \delta$ as distributions. What this means is that for all sufficiently nice $f : \mathbb{R}^3 \to \mathbb{R}$, $\int(-\nabla^2 f)G\mathrm{d}x = f(0)$.

3.1 Distributions, the space $\mathcal{D}(U)$ and $\mathcal{D}'(U)$

For $U \subseteq \mathbb{R}^n$ open, $C_c^\infty(U) = \{\phi : U \to \mathbb{R} \text{ smooth and } \operatorname{supp}\phi \subseteq U \text{ is compact}\}$.

Theorem. *There is a topology on $C_c^\infty(U)$ such that*

(i) The vector space operations are continuous;

(ii) A sequence $(\phi_j)_{j\geq 1}$ in $C_c^\infty(U)$ converges to 0 if and only if there is $K \subseteq U$ compact such that $\operatorname{supp}\phi_j \subseteq K$ for all j and for all α, $\sup_K |\nabla^\alpha \phi_j| \to 0$;

(iii) If Y is a LCTVS (Locally Compact TVS) and $\Lambda : C_c^\infty(U) \to Y$ is linear then Λ is sequentially continuous if and only if it is continuous.

Proof. Not given. □

Definition. $C_c^\infty(U)$ with the above topology is called the space of *test functions* and is denoted $\mathcal{D}(U)$.

Examples. Let $\phi \in C_c^\infty(\mathbb{R})$.

(a) If $\phi_j(x) = e^{-j}\phi(jx)$, then $\phi_j \to 0$ in $\mathcal{D}(\mathbb{R})$;

(b) If $\phi_j(x) = j^{-100}\phi(jx)$, then ϕ_j does not necessarily converge to 0 in $\mathcal{D}(\mathbb{R})$;

(c) If $\phi_j(x) = e^{-j}\phi(x-j)$ then ϕ_j does not necessarily converge to 0 in $\mathcal{D}(\mathbb{R})$.

Definition. The *space of distributions* $\mathcal{D}'(U)$ is the dual space of $\mathcal{D}(U)$ with the weak-$*$ topology.

In practice, $u \in \mathcal{D}'(U)$ if and only if $u(\phi_j) \to u(\phi)$ whenever $\phi_j \to \phi$ in $\mathcal{D}(U)$. Also, $u_j \to u$ in $\mathcal{D}'(U)$ if and only if $u_j(\phi) \to u(\phi)$ for all $\phi \in \mathcal{D}(U)$.

Examples.

(a) For any $x \in U$, define $\delta_x : \mathcal{D}(U) \to \mathbb{R}$ by $\delta_x(\phi) = \phi(x)$. This is called the *Dirac* or *δ distribution*.

(b) If $f \in L^1_{\mathrm{loc}}(U)$ then $T_f : \mathcal{D}(U) \to \mathbb{R}$, $T_f(\phi) = \int_U f\phi\mathrm{d}x$ defines a $T_f \in \mathcal{D}'(U)$.

Fact. $T_f = T_g \iff \int_U (f-g)\phi\mathrm{d}x = 0$ for all $\phi \in C_c^\infty(U) \iff f = g$ almost everywhere. Hence the map $T : L^1_{\mathrm{loc}}(U) \to \mathcal{D}'(U)$, $f \mapsto T_f$ is an injection.

Example. If $\alpha \in C^\infty(U)$, then $T_{\alpha f}(\phi) = \int_U f\alpha\phi\mathrm{d}x = T_f(\alpha\phi)$ for all $\phi \in \mathcal{D}(U)$.

Definition. If $u \in \mathcal{D}'(U)$ is a distribution we define $\alpha u \in \mathcal{D}'(U)$ by $\alpha u(\phi) = u(\alpha\phi)$ for all $\phi \in \mathcal{D}(U)$.

Example. If $f \in C^1(U)$ then

$$T_{\nabla_i f}(\phi) = \int_U (\nabla_i f)\phi \mathrm{d}x = -\int_U f(\nabla_i \phi)\mathrm{d}x = -T_f(\nabla_i\phi) \ \forall \phi \in C_c^\infty.$$

Definition. If $u \in \mathcal{D}'(U)$ define $\nabla^\alpha u \in \mathcal{D}'(U)$ by

$$\nabla^\alpha u(\phi) = (-1)^{|\alpha|} u(\nabla^\alpha \phi) \ \forall \phi \in C_c^\infty.$$

Example. Define $H : \mathbb{R} \to \mathbb{R}$ by $H(x) = 1$ for all $x \geq 0$ and $H(x) = 0$ for all $x < 0$ (Heaviside function). Then for $\phi \in \mathcal{D}(\mathbb{R})$, $\nabla T_H(\phi) = -\int H\phi' \mathrm{d}x = -\int_0^\infty \phi'(x)\mathrm{d}x = \phi(0) = \delta_0(\phi)$. Hence $\nabla T_H = \delta_0$ or $H' = \delta_0$ in the sense of distributions.

3.2 Compactly supported distributions: $\mathcal{E}(U)$ and $\mathcal{E}'(U)$

Now consider $C^\infty(U) = \{\phi : U \to \mathbb{R} \text{ smooth}\}$.

Let $(K_i \subseteq U : i \in \mathbb{N})$ be compact sets such that $K_i \subseteq \mathrm{int}(K_{i+1})$, $U = \bigcup_i K_i$. For $\phi \in C^\infty$ define $p_N(\phi) = \sup_{x \in K_N} \sup_{|\alpha| \leq N} |\nabla^\alpha \phi(x)|$. Then $\mathcal{P} = \{p_N\}_{N \geq 1}$ is a separating family of seminorms.

Definition. The space $C^\infty(U)$ with the locally convex topology induced by $\mathcal{P}$ is denoted $\mathcal{E}(U)$.

Remark. Since $\mathcal{P}$ is countable, $\mathcal{E}(U)$ is a metric space. It is also complete, i.e a Fréchet space.

A sequence $(\phi_j)_{j \geq 1}$ in $\mathcal{E}(U)$ converges to 0 if and only if for all $K \subseteq U$ compact and all α, $\sup_{x \in K} |\nabla^\alpha \phi_j(x)| \to 0$.

Fact. $\mathcal{D}(U) \subseteq \mathcal{E}(U)$ so $\mathcal{E}'(U) \subseteq \mathcal{D}'(U)$.

Example. If $\phi \in C_c^\infty$, then $\phi_j(x) = e^{-j}\phi(x - j)$ converges to 0 in $\mathcal{E}(\mathbb{R})$ but not in $\mathcal{D}(\mathbb{R})$.

Definition. $u \in \mathcal{D}'(U)$ has *support* in $S \subseteq U$ if $u(\phi) = 0$ for all $\phi \in C_c^\infty(U \setminus S)$. If S can be taken compact, say u has *compact support*.

Theorem. *$\mathcal{E}'(U) = \{u \in \mathcal{D}'(U) : u$ has compact support$\}$.*

Lemma. *Let $u : \mathcal{E}(U) \to \mathbb{R}$ be linear. Then u is continuous if and only if*

$$\exists \text{ compact } K \subseteq \mathbb{R}^n,\ N \in \mathbb{N},\ C > 0 \text{ such that } |u(\phi)| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |\nabla^\alpha \phi(x)|. \quad (*)$$

Proof. Recall that $u \in \mathcal{E}'(U)$ if and only if $u(\phi_j) \to 0$ for all sequences (ϕ_j) in $\mathcal{E}(U)$, i.e $\phi_j \xrightarrow{\mathcal{E}(U)} 0$. Now assume $(*)$ and let (ϕ_j) be a sequence in $\mathcal{E}(U)$ with $\phi_j \xrightarrow{\mathcal{E}(U)} 0$. This is equivalent to: for all $\tilde{K} \subseteq U$ compact, $\tilde{N} \in \mathbb{N}$, $\sup_{\substack{x \in \tilde{K} \\ |\alpha| \leq \tilde{N}}} |\nabla^\alpha \phi_j(x)| \to 0$. Thus taking $\tilde{K} = K$ and $\tilde{N} = N$, $(*)$ implies $u(\phi_j) \to 0$.

Now suppose $(*)$ does not hold. Let $K_i \subseteq U$ be compact, $K_j \subseteq \text{int}(K_{j+1})$, $\bigcup_j K_j = U$. Since $(*)$ does not hold, for each j we have $\phi_j \in \mathcal{E}(U)$ such that $|u(\phi_j)| > j \sup_{x \in K_j} \sup_{|\alpha| \leq j} |\nabla^\alpha \phi_j(x)|$. Set $\psi_j = \frac{\phi_j}{|u(\phi_j)|}$. We claim that $\psi_j \to 0$ in $\mathcal{E}(U)$. For any $\tilde{K} \subseteq U$ compact, $\tilde{N} \in \mathbb{N}$, there exists $J > \tilde{N}$ such that $\tilde{K} \subseteq K_j$ for all $j > J$, so

$$\sup_{\substack{x \in \tilde{K} \\ |\alpha| \leq \tilde{N}}} |\nabla^\alpha \psi(x)| \leq \sup_{\substack{x \in \tilde{K} \\ |\alpha| \leq \tilde{N}}} \frac{\nabla^\alpha \phi_j(x)}{|u(\phi_j)|} < \frac{1}{j}.$$

As claimed. But $|u(\psi_j)| = 1$, so $|u(\psi_j)| \not\to 0$, so u is not continuous. □

Proof of Theorem. If $u \in \mathcal{E}'(U)$, the lemma implies that u has support in K. Conversely, if $u \in \mathcal{D}'(U)$ has support in $K \subseteq U$ compact, define $\tilde{u} \in \mathcal{E}'(U)$ by $\tilde{u}(\phi) = u(\chi\phi)$ for all $\phi \in \mathcal{E}(U)$, where $\chi \in C_c^\infty(U)$ satisfies $\chi = 1$ on K. The extension does not depend on χ since for any other such $\tilde{\chi}$ one has $\chi - \tilde{\chi} \in C_c^\infty(U \setminus K)$. □

Examples.

(a) If $f \in L^1(U)$ vanishes almost everywhere in $U \setminus K$ for K compact, then $T_f \in \mathcal{E}'(U)$;

(b) For any $x \in U$, $\delta_x \in \mathcal{E}'(U)$;

(c) $u \in \mathcal{D}'(U)$ where $u(\phi) = \sum_{m=-\infty}^{\infty} \phi(m) \notin \mathcal{E}(\mathbb{R})$.

3.3 Tempered distributions: the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$

Definition. $\phi \in C^\infty(\mathbb{R}^n)$ is *rapidly decreasing* if

$$\sup_{x \in \mathbb{R}^n} |(1 + |x|)^N \nabla^\alpha \phi(x)| < \infty$$

for all multi-indices α and $N \in \mathbb{N}$.

Examples.

(a) $\phi(x) = e^{-|x|^a}$ is rapidly decreasing;

(b) $\phi(x) = |x|^{-2023}$ is not rapidly decreasing.

Definition. The *Schwartz space* $S(\mathbb{R}^n)$ is the space of rapidly decreasing functions with the topology generated by the separating family of seminorms

$$p_N(\phi) = \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \le N} |(1+|x|)^N \nabla^\alpha \phi(x)|.$$

Remark. There are other equivalent families of seminorms such as

$$\sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \le N} |(1+|x|^2)^N \nabla^\alpha \phi(x)|$$
$$\sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \le N} |\nabla^\alpha (1+|x|^2)^N \phi(x)|.$$

Fact. $S(\mathbb{R}^n)$ is a Fréchet space, $\mathcal{D}(\mathbb{R}^n) \subseteq S(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n)$ continuously and $\mathcal{E}'(\mathbb{R}^n) \subseteq S'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$.

Definition. $S'(\mathbb{R}^n)$ is called the space of *tempered distributions* or *Schwartz distributions.*

Examples.

(a) If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} (1+|x|)^{-N}|f(x)|\mathrm{d}x < \infty$ for some $N \in \mathbb{N}$, then $T_f \in S'(\mathbb{R}^n)$. Indeed, if $\phi \in S(\mathbb{R}^n)$, then

$$\begin{aligned} |T_f(\phi)| &= \left| \int f(x)\phi(x)\mathrm{d}x \right| \\ &= \underbrace{\left(\int (1+|x|)^{-N}|f(x)|\mathrm{d}x \right)}_{\le C} \underbrace{\sup_{x \in \mathbb{R}^n} (1+|x|)^N |\phi(x)|}_{\to 0 \text{ if } \phi \xrightarrow{S(\mathbb{R}^n)} 0} \end{aligned}$$

so if $\phi_j \xrightarrow{S(\mathbb{R}^n)} 0$ then $T_f(\phi_j) \to 0$.

(b) If $f(x) = e^{|x|^2}$ then $T_f \in \mathcal{D}'(\mathbb{R}^n)$ but $T_f \notin S'(\mathbb{R}^n)$.

(c) $u(\phi) = \sum_{m=-\infty}^{\infty} |m|^{2023}\phi(m)$ belongs to $S'(\mathbb{R})$ but not $\mathcal{E}'(\mathbb{R})$.

3.4 Convolution

Example. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$f * \phi(x) = \int f(y)\phi(x-y)\mathrm{d}y = T_f(\tau_x \check{\phi})$$

where $\tau_x \check{\phi}(y) = \check{\phi}(y-x) = \phi(x-y)$, $\check{\phi}(y) = \phi(-y)$.

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ define

$$u * \phi(x) = u(\tau_x \check{\phi}).$$

Facts.

- $(u_1 + au_2) * \phi = u_1 * \phi + au_2 * \phi$;
- $u * (\phi_1 + a\phi_2) = u * \phi_1 + au * \phi_2$;
- $u * \check{\phi}(0) = u(\phi)$ - thus $u * \phi(0)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$ determines $u \in \mathcal{D}'(\mathbb{R}^n)$.

Example. $\delta_0 * \phi(x) = \delta_0(\tau_x \check{\phi}) = \check{\phi}(-x) = \phi(x)$. Thus $\delta_0 * \phi = \phi$.

Proposition. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

(i) $u * \phi \in C^\infty(\mathbb{R}^n)$ and $\nabla^\alpha(u * \phi) = (\nabla^\alpha u) * \phi = u * \nabla^\alpha \phi$;

(ii) If $u \in \mathcal{E}'(\mathbb{R}^n)$ then $u * \phi$ has compact support, i.e $u * \phi \in \mathcal{D}(\mathbb{R}^n)$.

Proof.

(i)

$$\frac{1}{h}(u * \phi(x + he_i) - u * \phi(x)) = u\left(\frac{1}{h}(\tau_{x+e_ih}\check{\phi} - \tau_x\check{\phi})\right) \xrightarrow{h \to 0} u(\tau_x \widecheck{\nabla_i \phi}).$$

Where we used from Example Sheet 3:

$$\frac{1}{h}(\tau_{x+e_ih}\check{\phi} - \tau_x\check{\phi}) \xrightarrow{\mathcal{D}(\mathbb{R}^n)} \nabla_i \phi(x - \cdot) = \tau_x \widecheck{\nabla_i \phi}.$$

Hence $\nabla_i(u + \phi)(x)$ exists and equals $u(\tau_x \widecheck{\nabla_i \phi}) = u * \nabla_i \phi(x)$. So by induction $u * \phi \in C^\infty$ and $\nabla^\alpha u * \phi = u * \nabla^\alpha \phi$ for all α. Also; $\nabla^\alpha \tau_x \check{\phi}(y) = \nabla_y^\alpha \phi(x - u) = (-1)^{|\alpha|} \nabla_x^\alpha \phi(x - y) = (-1)^{|\alpha|} \tau_x \widecheck{\nabla^\alpha \phi}(y)$. Thus $u * \nabla^\alpha \phi = \nabla^\alpha u * \phi$.

(ii) Assume $u(\phi) = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n \setminus K)$ for some K compact. Then for any $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{supp} \tau_x \check{\phi} \cap K = \emptyset$ for $|x|$ large enough, i.e $u * \phi$ has compact support.

$\square$

Definition. For $u_1 \in \mathcal{D}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{E}'(\mathbb{R}^n)$, define $u_1 * u_2$ to be the unique distribution such that

$$(u_1 * u_2) * \phi = u_1 * (u_2 * \phi).$$

[Note that $u_2 * \phi \in \mathcal{D}(\mathbb{R}^n)$ by the previous proposition so this makes sense.]

Example. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then $u * \delta_0 = u$. Indeed, $(u * \delta_0) * \phi = u * (\delta_0 * \phi) = u * \phi$.

Proposition. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $u_2 \in \mathcal{E}'(\mathbb{R}^n)$. Then $\nabla^\alpha(u_1 * u_2) = u_1 * (\nabla^\alpha u_2) = (\nabla^\alpha u_1) * u_2$.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then by the previous proposition

$$\begin{aligned}\underbrace{\nabla^\alpha(u_1 * u_2)}_{\in\mathcal{D}'} * \underbrace{\phi}_{\in\mathcal{D}} &= (u_1 * u_2) * (\nabla^\alpha \phi) \\ &= u_1 * (u_2 * (\nabla^\alpha \phi)) = (u_1 * \nabla^\alpha u_2) * \phi.\end{aligned}$$

□

Definition. Call $L = \sum_{|\alpha|\leq k} a_\alpha \nabla^\alpha$, $a_\alpha \in \mathbb{R}$, $\nabla^\alpha u_2 * \phi$ a *constant coefficient partial differential operator* of order k. A *fundamental solution* of L is a distribution G such that $LG = \delta_0$.

Theorem. *If $G \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of L and $f \in \mathcal{E}'(\mathbb{R}^n)$ then $u = G * f$ solves $Lu = f$. Moreover, if $f \in \mathcal{D}(\mathbb{R}^n)$ then $u = G * f \in C^\infty(\mathbb{R}^n)$ solves $Lu = f$ in the classical sense.*

Proof.

$$L(G * f) = \sum_{|\alpha|\leq k} a_\alpha \nabla^\alpha (G * f) = \left(\sum_{|\alpha|\leq k} a_\alpha \nabla^\alpha G\right) * f = \delta_0 * f = f.$$

□

Example. $L = -\nabla^2 = -\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^3$. Define $g(x) = \frac{1}{4\pi|x|} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Then $G = T_g$ is a fundamental solution for L. In particular, if $f \in C_c^\infty(\mathbb{R}^n)$ then

$$u(x) = \int_{\mathbb{R}^3} \frac{f(y)}{4\pi|x-y|} \mathrm{d}y$$

solves $Lu = f$.

3.5 Fourier Transform

Definition. If $f \in L^1(\mathbb{R}^n)$ then the *Fourier transform* of f is $\hat{f} = \mathcal{F}(f) : \mathbb{R}^n \to \mathbb{C}$, $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\mathrm{d}x$.

Example. $(n = 1)$

(i)
$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

The Fourier transform of f is $\hat{f}(\xi) = 2\frac{\sin\xi}{\xi}$;

(ii) $f(x) = e^{-|x|}$ has Fourier transform $\hat{f}(\xi) = \frac{2}{1+\xi^2}$;

(iii) $f(x) = \frac{1}{1+x^2}$ has Fourier transform $\hat{f}(\xi) = e^{-|x|}\pi$;

(iv) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ has Fourier transform $\hat{f}(\xi) = e^{-|\xi|^2/2}$.

Upshot: f regular $\leftrightarrow$ $\hat{f}$ decays.

Theorem (Riemann-Lebesgue Lemma)**.** *Let $f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^0(\mathbb{R}^n)$ and $\sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \leq ||f||_{L^1}$, $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

Proof. Assume $\xi_k \to \xi$. Then for $x \in \mathbb{R}^n$, $f(x)e^{-i\xi_k\cdot x} \to f(x)e^{-i\xi\cdot x}$ pointwise and $|f(x)e^{-i\xi_k\cdot x}| \in |f(x)| \in L^1$ so by the DCT, $\hat{f}(\xi_k) \to \hat{f}(\xi)$. Hence $\hat{f} \in C^0$. We also have
$$|\hat{f}(\xi)| = \left|\int f(x)e^{-i\xi\cdot x}\mathrm{d}x\right| \leq ||f||_{L^1}.$$

To show $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$, let $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be such that $||f - f_\varepsilon||_{L^1} < \varepsilon$. Then
$$\hat{f}_\varepsilon(\xi) = \int_{\mathbb{R}^n} f_\varepsilon(x) \underbrace{e^{-i\xi\cdot x}}_{-\frac{1}{|\xi|^2}\nabla_x^2(e^{-i\xi\cdot x})} \mathrm{d}x = -\frac{1}{|\xi|^2}\underbrace{\int_{\mathbb{R}^n}(\nabla^2 f_\varepsilon)e^{-i\xi\cdot x}\mathrm{d}x}_{\leq||\nabla^2 f_\varepsilon||_{L^1}}.$$

Hence $\limsup_{|\xi|\to\infty} |\hat{f}_\varepsilon(\xi)| = 0$. Finally note
$$|\hat{f}(\xi)| \leq |\hat{f}_\varepsilon(\xi)| + \underbrace{|\hat{f}(\xi) - \hat{f}_\varepsilon(\xi)|}_{\leq||f-f_\varepsilon||_{L^1}} \leq |\hat{f}_\varepsilon(\xi)| + \varepsilon.$$

So $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. □

Notation: $\tau_y f(x) = f(x - y)$ and $e_y(x) = e^{ix\cdot y}$.

Proposition.

(i) Let $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\lambda > 0$, and set $f_\lambda(x) = \lambda^{-n} f(x/\lambda)$. Then $\hat{f}_\lambda(\xi) = \hat{f}(\lambda\xi)$, $\widehat{e_y f}(\xi) = \tau_y \hat{f}(\xi)$, $\widehat{\tau_y f}(\xi) = e_{-y}(\xi)\hat{f}(\xi)$;

(ii) Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

Proof. Change of variables and Fubini. □

Proposition.

(i) If $f \in C^1(\mathbb{R}^n)$ and $f, \nabla_i f \in L^1(\mathbb{R}^n)$ for all $1 \le i \le n$, then

$$\widehat{\nabla_j f}(\xi) = i\xi_j \hat{f}(\xi).$$

(ii) Assume $(1 + |x|)f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^1(\mathbb{R}^n)$ and

$$\nabla_j \hat{f}(\xi) = -i\widehat{x_j f}(\xi).$$

Proof.

(i) Let $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be such that $||f_\varepsilon - f||_{L^1} + \sum_j ||\nabla_j f - \nabla_j f_\varepsilon||_{L^1} < \varepsilon$ (Exercise: show we can do this). Then (IBP)

$$\widehat{\nabla_j f_\varepsilon}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \nabla_j f_\varepsilon(x) \mathrm{d}x = i\xi_j \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f_\varepsilon(x) \mathrm{d}x = i\xi_j \hat{f}_\varepsilon(\xi).$$

Hence

$$|\widehat{\nabla_j f}(\xi) - i\xi_j \hat{f}(\xi)| \le ||\nabla_j f - \nabla_j f_\varepsilon||_{L^1} + |\xi| ||f - f_\varepsilon||_{L^1} \le (1 + |\xi|)\varepsilon \xrightarrow{\varepsilon \to 0} 0.$$

(ii) Since $x_j f \in L^1$, $-i\widehat{x_j f} \in C^0$. Need to show $\nabla_j \hat{f}$ exists and equals $-i\widehat{x_j f}$.

$$\frac{\hat{f}(\xi + he_j) - \hat{f}(\xi)}{h} = \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} \underbrace{\left(\frac{e^{-ih\cdot x_j} - 1}{h}\right)}_{\substack{\to -ix_j \\ \le |x_j|}} \mathrm{d}x \xrightarrow{h \to 0} -i\widehat{x_j f}(\xi)$$

by the DCT, using $|x_j| f \in L^1$.

□

Corollary. *The Fourier Transform maps $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ continuously.*

Proof. For any $f : \mathbb{R}^n \to \mathbb{C}$,

$$||f||_{L^1} \le \sup_{x \in \mathbb{R}^n} (1 + |x|)^{n+1} |f(x)| \underbrace{\int_{\mathbb{R}^n} \frac{\mathrm{d}y}{(1 + |y|)^{n+1}}}_{<\infty}. \qquad (*)$$

Hence if $f \in \mathcal{S}(\mathbb{R}^n)$, then $\nabla^\alpha(x^\beta f(x)) \in L^1(\mathbb{R}^n)$ for any multi-indices α, β. Thus by the previous proposition (applied repeatedly),

$$|\widehat{\nabla^\alpha(x^\beta f)}(\xi)| = |\xi^\alpha \nabla^\beta \hat{f}(\xi)|.$$

So in particular (using $(*)$),

$$\sup_\xi |\xi^\alpha \nabla^\beta \hat{f}(\xi)| \leq C \sup_{\substack{x \in \mathbb{R}^n \\ |\gamma| \leq \alpha}} \Big[(1+|x|)^{|\beta|+n+1} |\nabla^\gamma f(x)|\Big] \to 0 \text{ if } f \to 0 \text{ in } \mathcal{S}(\mathbb{R}^n).$$

Therefore $\hat{f} \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ if $f \to 0$ in $\mathcal{S}(\mathbb{R}^n)$. Hence $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is well-defined and continuous. $\square$

Theorem (Fourier inversion)**.** *Let $f \in L^1(\mathbb{R}^n)$ and assume also $\hat{f} \in L^1(\mathbb{R}^n)$. Then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x} \mathrm{d}\xi \text{ for almost all } x.$$

Thus writing $\check{f}(x) = f(-x)$ we have $\mathcal{F}^2(f) = (2\pi)^n \check{f}$.

Proof. Let

$$I_\varepsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{ix\xi} \mathrm{d}\xi.$$

Since $\hat{f} \in L^1$, by the DCT, $I_\varepsilon(x) \xrightarrow{\varepsilon\to 0} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}\mathrm{d}\xi$. On the other hand,

$$\begin{aligned} I_\varepsilon(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) e^{-i\xi y} \mathrm{d}y \right) e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{ix\cdot\xi} \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) \left(\int_{\mathbb{R}^n} e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{i(x-y)\cdot\xi} \mathrm{d}\xi \right) \mathrm{d}y \qquad \text{(Fubini)} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) (2\pi)^{n/2} \varepsilon^{-n} e^{-\frac{|x-y|^2}{2\varepsilon^2}} \mathrm{d}y \\ &= f * \psi_\varepsilon(x). \end{aligned}$$

Where $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(\varepsilon^{-1}x)$, $\psi(x) = (2\pi)^{-n/2}e^{-\frac{|x|^2}{2}}$. Since $\psi \in C^\infty(\mathbb{R}^n)$, $\psi \geq 0$ and $\int_{\mathbb{R}^n} \psi \mathrm{d}x = 1$, we have $f * \psi_\varepsilon \to f$ in L^1 as $\varepsilon \to 0$ (since ψ is a smooth mollifier). Hence $f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\xi)e^{ix\cdot\xi}\mathrm{d}\xi$ for almost all $x \in \mathbb{R}^n$. □

Remark. If f is continuous, this holds for all $x \in \mathbb{R}^n$.

Theorem (Parseval-Plancherel)**.** *Let $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$ and $(f,g)_{L^2} = \frac{1}{(2\pi)^n}(\hat{f},\hat{g})_{L^2}$.*

Proof. Suppose $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\hat{f}, \hat{g} \in \mathcal{S}(\mathbb{R}^n)$ and

$$\begin{aligned} (f,g)_{L^2} &= \int_{\mathbb{R}^n} \bar{f}(x) g(x) \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \bar{f}(x) \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) e^{ix\cdot\xi} \mathrm{d}\xi \right) \mathrm{d}x \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \bar{f}(x) e^{ix\cdot\xi} \mathrm{d}x \right) \hat{g}(\xi) \mathrm{d}\xi \qquad \text{(Fubini)} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}_n} \overline{\hat{f}(\xi)} \hat{g}(\xi) \mathrm{d}\xi. \end{aligned}$$

Given $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, let $f_j, g_j \in \mathcal{S}(\mathbb{R}^n)$ be such that $||f_i - f||_{L^1} + ||f_i - f||_{L^2} + ||g_i - g||_{L^1} + ||g_i - g||_{L^1} \to 0$. Then $\sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi) - \hat{f}_j(\xi)| + \sup_{\xi\in\mathbb{R}^n} |\hat{g}(\xi) - \hat{g}_j(\xi)| \leq ||f - g_j||_{L^1} + ||g - g_j||_{L^1} \to 0$.

So $(f_j)_{j\geq 1}$ is a Cauchy sequence in L^2. Hence $||\hat{f}_j - \hat{f}_k||_{L^2}^2 = (2\pi)^n ||f_j - f_k||_{L^2}^2$, so $(\hat{f}_j)_{j\geq 1}$ is also a Cauchy sequence in L^2. By completeness of L^2, there exists $\hat{f} \in L^2$ such that $\hat{f}_j \to \hat{f}$ in L^2 (exercise: show that this $\hat{f}$ is indeed the Fourier transform of f). Similarly there is $\hat{g} \in L^2$ such that $\hat{g}_j \to \hat{g}$ in L^2. Thus

$$(f,g)_{L^2} = \lim_{j\to\infty} (f_j, g_j) = \lim_{j\to\infty} \frac{1}{(2\pi)^n} (\hat{f}_j, \hat{g}_j)_{L^2} = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})_{L^2}.$$

$\square$

Corollary. *$f \mapsto (2\pi)^{-n/2}\hat{f}$ is an isometry from $L^1 \cap L^2 \subseteq L^2$ into L^2. Since $L^1 \cap L^2$ is dense in L^2, it extends uniquely to a linear isometry $(2\pi)^{-n/2}\mathcal{F}$ from L^2 to L^2.*

Definition. For $f \in L^2(\mathbb{R}^n)$, write $\hat{f} = \mathcal{F}(f)$ where $\mathcal{F}$ is the above extension of the usual Fourier transform to L^2.

Remark. If $f \in L^2(\mathbb{R}^n)$ then $f_R = f\mathbb{1}_{B_R(0)} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $f_R \to f$ as $R \to \infty$ in L^2. Thus $\hat{f}_R \to \hat{f}$ in L^2, i.e $\left(\xi \mapsto \int_{B_R(0)} f(x)e^{-ix\cdot\xi}\mathrm{d}x\right) \xrightarrow{L^2} \hat{f}$.

Example. Let $f \in L^1(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} T_{\hat{f}}(\phi) = \int_{\mathbb{R}^n} \hat{f}(\xi)\phi(\xi)\mathrm{d}\xi &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\mathrm{d}x\right)\phi(\xi)\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} f(x)\left(\int_{\mathbb{R}^n} \phi(\xi)e^{-ix\cdot\xi}\mathrm{d}\xi\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)\hat{\phi}(x)\mathrm{d}x \\ &= T_f(\hat{\phi}). \end{aligned}$$

Definition. For $u \in \mathcal{S}'(\mathbb{R}^n)$, define $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ by $\hat{u}(\phi) = u(\hat{\phi})$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Remark. The above is valid since the map $\mathcal{S} \to \mathcal{S}$ given by $\phi \mapsto \hat{\phi}$ is well-defined and continuous, so $\hat{u}$ is continuous as well. For $u \in \mathcal{D}'(\mathbb{R}^n)$, this definition wouldn't work as $\phi \in \mathcal{D}(\mathbb{R}^n)$ does not imply $\hat{\phi} \in \mathcal{D}(\mathbb{R}^n)$.

Examples.

(a) Fix $\xi \in \mathbb{R}^n$. Then $\hat{\delta}_\xi(\phi) = \delta_\xi(\hat{\phi}) = \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)\mathrm{d}x = T_{e_{-\xi}}(\phi)$ fo all $\phi \in \mathcal{S}(\mathbb{R}^n)$ (recall $e_y(x) := e^{ix\cdot y}$) i.e "$\hat{\delta}_\xi = e^{-i\xi\cdot(\cdot)}$".

(b) For $x \in \mathbb{R}^n$,

$$\hat{T}_{e_x}(\phi) = T_{e_x}(\hat{\phi}) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\phi}(\xi)\mathrm{d}\xi = (2\pi)^n\phi(x) = (2\pi)^n\delta_x(\phi).$$

So $\hat{T}_{e_x} = (2\pi)^n\delta_x$ or "$\widehat{e^{ix\cdot(\cdot)}} = (2\pi)^n\delta_x$".

Lemma. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then*

$$\begin{aligned}
&\widehat{e^{i\xi\cdot(\cdot)}u} = \tau_\xi \hat{u}, \quad \widehat{\tau_x u} = e^{ix\cdot(\cdot)}\hat{u}, \\
&\widehat{\nabla^\alpha u} = (i\xi)^\alpha \hat{u}, \quad \nabla^\alpha \hat{u} = (-1)^{|\alpha|}\widehat{x^\alpha u}, \\
&\hat{\hat{u}} = (2\pi)^n \check{u}.
\end{aligned}$$

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\widehat{e^{i\xi\cdot(\cdot)}u}(\phi) = e^{i\xi\cdot(\cdot)}u(\hat{\phi}) &= u(e^{i\xi\cdot(\cdot)}\hat{\phi}) \\
&= u(\widehat{\tau_{-\xi}\phi}) \\
&= \hat{u}(\tau_{-\xi}\phi) \\
&= \tau_\xi \hat{u}(\phi).
\end{aligned}$$

And

$$\begin{aligned}
\widehat{\nabla^\alpha u}(\phi) = \nabla^\alpha u(\hat{\phi}) &= (-1)^{|\alpha|}u(\nabla^\alpha \hat{\phi}) \\
&= (-1)^{|\alpha|}u((-i)^{|\alpha|}\widehat{\xi^\alpha \phi}) \\
&= i^{|\alpha|}u(\widehat{\xi^\alpha\phi}) \\
&= i^{|\alpha|}\hat{u}(\xi^\alpha \phi) \\
&= (i\xi)^\alpha \hat{u}(\phi).
\end{aligned}$$

And

$$\hat{\hat{u}}(\phi) = \hat{u}(\hat{\phi}) = u(\hat{\hat{\phi}}) = u((2\pi)^n\check{\phi}) = (2\pi)^n \check{u}(\phi).$$

The other statements are analogous. □

Proposition. $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a linear homeomorphism.

Proposition. Suppose $u_j \to u$ in $\mathcal{S}'(\mathbb{R}^n)$, i.e $u_j(\phi) \to u(\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{u}_j(\phi) = u_j(\hat{\phi}) \to u(\hat{\phi}) = \hat{u}(\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, i.e $\hat{u}_j \to \hat{u}$ in $\mathcal{S}'(\mathbb{R}^n)$. Thus $\mathcal{F}$ is continuous from $\mathcal{S}'$ to $\mathcal{S}'$. Since $\mathcal{F}^4 = (2\pi)^{2n}\mathrm{id}$, $\mathcal{F}$ is invertible with continuous inverse $\mathcal{F}^{-1} = (2\pi)^{-2n}\mathcal{F}^3$.

3.6 Periodic distributions

Recall: if $f \in L^2((0,1))$ then $f(x) = \sum_{n\in\mathbb{Z}} f_n e^{2\pi i n x}$ in L^2, where $f_n = \int_0^1 f(x)e^{-2\pi i n x}\mathrm{d}x$.

Definition. $u \in \mathcal{D}'(\mathbb{R}^n)$ is *periodic* if for any $g \in \mathbb{Z}^n$ $\tau_g u = u$ (recall $\tau_g u(\phi) = u(\tau_{-g}\phi)$, $\tau_{-g}(\phi)(x) = \phi(x+g)$).

Examples.

(a) For $k \in \mathbb{Z}^n$, the distribution "$e^{2\pi i k\cdot(\cdot)}$" $= T_{e_{2\pi k}}$ is periodic. Indeed,

$$\begin{aligned}\tau_g T_{e_{2\pi k}}(\phi) = T_{e_{2\pi k}}(\tau_{-g}\phi) &= \int_{\mathbb{R}^n} e^{2\pi i k\cdot x}\phi(x+g)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} e^{2\pi i k\cdot(x-g)}\phi(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} e^{2\pi i k\cdot x}\phi(x)\mathrm{d}x \\ &= T_{e_{2\pi k}}(\phi).\end{aligned}$$

(b) Suppose $v \in \mathcal{E}'(\mathbb{R}^n)$. Then $u = \sum_{k\in\mathbb{Z}_n} \tau_k v$ is periodic. Note that u defines a distribution in $\mathcal{D}'$ since $u(\phi)$ is a finite sum for every $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then for $g \in \mathbb{Z}^n$, $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\tau_g u(\phi) = \sum_{k\in\mathbb{Z}^n} \tau_{k+g} v(\phi) = \sum_{k\in\mathbb{Z}} \tau_k v(\phi) = u(\phi)$.

Definition. The *fundamental cell of the lattice* is

$$q = \{x \in \mathbb{R}^n : -\frac{1}{2} \le x_i < \frac{1}{2},\ i = 1, \dots, n\}.$$

[intuitively q is the hypercube with side lengths 1 centred at 0, but with some edges "open".]

Lemma. *Let $Q = \{x \in \mathbb{R}^n : -1 \le x_i < 1,\ i = 1, \dots, n\}$. Then there exists $\psi \in C^\infty(\mathbb{R}^n)$ such that*

(i) $\psi \ge 0$*;*

(ii) $\operatorname{supp}\psi \subseteq Q$*;*

(iii) $\sum_{g\in\mathbb{Z}^n} \tau_g\psi = 1$.

Such a ψ is called a periodic partition of unity (p.p.u). Suppose ψ and ψ' are both p.p.u's. Then if $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic, $u(\psi) = u(\psi')$.

Proof. Let $\psi_0 \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\psi_0 \subseteq \operatorname{int}(Q)$ and $\psi_0(x) = 1$ for $x \in q$ and $\psi_0 \ge 0$. Set $S(x) = \sum_{g\in\mathbb{Z}^n} \psi_0(x-g)$. Then $S \in C^\infty$ and $S(x) \ge 1$ for all $x \in \mathbb{R}^n$. Thus $\psi(x) = \frac{\psi_0(x)}{S(x)}$ satisfies the conditions.

Now let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic and ψ, ψ' be p.p.u's. Then

$$\begin{aligned}u(\psi) = u\left(\psi \sum_{g\in\mathbb{Z}^n} \tau_g\psi'\right) &= \sum_{g\in\mathbb{Z}^n} u(\psi(\tau_g\psi')) \\ &= \sum_{g\in\mathbb{Z}^n} \tau_{-g}\tau_g u(\psi(\tau_g\psi')) \\ &= \sum_{g\in\mathbb{Z}^n} \tau_{-g} u((\tau_{-g}\psi)\psi') \\ &= u\left(\left(\sum_{g\in\mathbb{Z}^n} \tau_{-g}\psi\right)\psi'\right) = u(\psi').\end{aligned}$$

□

Corollary. *Let ψ be a p.p.u. Then for $f \in L^1_{loc}(\mathbb{R}^n)$ periodic, $T_f(\psi) = \int_q f(x)\mathrm{d}x$.*

Proof. Choose ψ_n a p.p.u such that $\psi_n \to \mathbb{1}_q$ pointwise and ψ_n is bounded. □

Definition. For $u \in \mathcal{D}'(\mathbb{R}^n)$ periodic, the *average* of u over the fundamental cell is $M(u) = u(\psi)$ where ψ is any p.p.u.

Lemma. *Let $v \in \mathcal{E}'(\mathbb{R}^n)$. Then*

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v \tag{$*$}$$

converges (in the weak-$*$ topology) in $\mathcal{S}'(\mathbb{R}^n)$. Conversely, if $u \in \mathcal{D}'(\mathbb{R}^n)$ is periodic there exists $v \in \mathcal{E}'(\mathbb{R}^n)$ such that $(*)$ holds. Hence every periodic distribution is tempered.

Proof. Let $K = \operatorname{supp} v$. We have seen that there exist $N \in \mathbb{N}$, $C > 0$ such that $|v(\phi)| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |\nabla^\alpha \phi(x)|$ for all $\phi \in \mathcal{E}(\mathbb{R}^n)$. Now let $\phi \in \mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{E}(\mathbb{R}^n)$. Then

$$|\tau_g v(\phi)| = |v(\tau_{-g}\phi)| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq N}} |\nabla^\alpha \phi(x+g)|.$$

Since $K \subseteq B_R(0)$ for some $R > 0$, $1+|g| \leq 1+|x|+|x+g| \leq (1+R)(1+|x+g|)$. Hence $1 \leq (1+R)\frac{1+|g+x|}{1+|g|}$. Thus for any $M \geq 1$,

$$\begin{aligned} |\tau_g v(\phi)| &\leq C \left(\frac{1+R}{1+|g|}\right)^M \sup_{\substack{x \in K \\ |\alpha| \leq N}} \big((1+|x+g|)^M |\nabla^\alpha \phi(x+g)|\big) \\ &\leq \frac{C'}{(1+|g|)^{n+1}} \underbrace{\sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq N}} (1+|x|)^M |\nabla^\alpha \phi(x)|}_{<\infty} \end{aligned}$$

so $\sum_{g \in \mathbb{Z}^n} \tau_g v(\phi)$ converges for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, so converges in $\mathcal{S}'(\mathbb{R}^n)$.

For the converse, let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic. Let ψ be a p.p.u. Then for any $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned} u(\phi) = \underbrace{\left(\sum_{g \in \mathbb{Z}^n} \tau_g \psi\right)}_{1} u(\phi) &= \sum_{g \in \mathbb{Z}^n} u((\tau_g \psi)\phi) \\ &= \sum_{g \in \mathbb{Z}^n} \tau_g u((\tau_g \psi)\phi) \\ &= \sum_{g \in \mathbb{Z}^n} u(\psi(\tau_{-g}\phi)) \\ &= \sum_{g \in \mathbb{Z}^n} \psi u(\tau_{-g}\phi) \\ &= \sum_{g \in \mathbb{Z}^n} (\tau_g(\psi u))(\phi). \end{aligned}$$

Note that ψu has compact support:

$$\operatorname{supp}\phi \cap \operatorname{supp}\psi = \emptyset \implies \psi u(\phi) = u(\psi(\phi)) = 0.$$

Hence ψu extends uniquely to $v \in \mathcal{E}'(\mathbb{R}^n)$ and therefore $u(\phi) = \sum_{g\in\mathbb{Z}^n} \tau_g v(\phi)$. $\square$

Theorem. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic. Then*

$$u = \sum_{g\in\mathbb{Z}^n} u_g T_{e_{2\pi g}} = \text{“}\sum_{g\in\mathbb{Z}^n} u_g e^{2\pi i\cdot(\cdot)}\text{”}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and $u_g = M(e_{-2\pi g}u)$ “$= \int_q u(x)e^{-2\pi i g\cdot x}\mathrm{d}x$” satisfying $|u_g| \le C(1+|g|)^N$ for some $C > 0$, $N \in \mathbb{N}$.

Definition. The u_g above are the *Fourier coefficients of u*.

Lemma. *Assume $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies*

$$(e_{-k} - 1)u = 0 \ \forall k \in \mathbb{Z}^n. \tag{$*$}$$

Then $u = \sum_{g\in\mathbb{Z}^n} c_g \delta_{2\pi g}$ for $c_g \in \mathbb{C}$ satisfying $|c_g| \le C(1+|g|)^N$ for some $C > 0$, $N \in \mathbb{N}$.

Proof. We first show $\operatorname{supp} u \subseteq \Lambda^* = \{2\pi g : g \in \mathbb{Z}^n\}$. Indeed, let $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}\phi \cap \Lambda^* = \emptyset$. Then $(e_{-k}-1)^{-1}\phi \in \mathcal{S}(\mathbb{R}^n)$ since $\phi(x) = 0$ if $(e_{-k}-1)(x) = 0 \iff x \in \Lambda^*$. Thus by $(*)$

$$u(\phi) = \underbrace{(e_{-k}-1)}_{=0} u(\underbrace{(e_{-k}-1)^{-1}\phi}_{\in\mathcal{S}(\mathbb{R}^n)}) = 0.$$

Now let ψ be a p.p.u and set $\tilde{\psi}(x) = \psi(x/2\pi)$. Let $v_g = (\tau_{2\pi g}\tilde{\psi})u$ and note that $\operatorname{supp} v_g \subseteq \{2\pi g\}$, so

$$\sum_{g\in\mathbb{Z}^b} v_g = u,\ (e_{-k}-1)v_g = 0.$$

Take k an element of the standard basis of $\mathbb{R}^n$ so

$$\begin{aligned}(e^{-ix_j} - 1)v_g &= (e^{-i(x_j - 2\pi g_j)} - 1)v_g = 0\\ &= (x_i - 2\pi g_j)K(x_j)v_j\end{aligned}$$

where K is smooth and non-zero near $2\pi g$ (using $\frac{1}{t}(e^{-it}-1) = -i + \mathcal{O}(t)$). Hence $(x_j - 2\pi g_j)v_g = 0$.

Since v_g has compact support, it can be extended to $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. Since $\phi \in \mathcal{S}(\mathbb{R}^n)$, there are $\phi_j \in C^\infty(\mathbb{R}^n)$ such that (by Taylor's theorem):

$$\phi(x) = \phi(2\pi g) + \sum_{i=1}^n (x_j - 2\pi g_j)\phi_j(x).$$

So
$$v_g(\phi) = \underbrace{v_g(\phi(2\pi g))}_{\phi(2\pi g)v_g(1)} + \sum_{j=1}^{n}(x_j - 2\pi g_j)v_g(\phi_j) = \underbrace{\phi(2\pi g)}_{\delta_{2\pi g}(\phi)} u(\tau_{2\pi g}\tilde{\psi}).$$
So
$$u = \sum_{g\in\mathbb{Z}^n} v_g = \sum_{g\in\mathbb{Z}^n} c_g\delta_{2\pi g},\ c_g = u(\tau_{2\pi g}\tilde{\psi}).$$
Example sheet 3: $\forall u \in \mathcal{S}'(\mathbb{R}^n)$ there exists $N, k \in \mathbb{N}$, $C > 0$ such that
$$|u(\phi)| \leq C \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq k}} (1+|x|)^N |\nabla^\alpha \phi(x)|\ \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$
Hence
$$\begin{aligned}
|c_g| &\leq C \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq k}} (1+|x|)^N |\nabla^\alpha \tilde{\psi}(x - 2\pi g)| \\
&\leq C \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq k}} \underbrace{(1+|x+2\pi g|)^N}_{(1+|x|)^N(1+2\pi|g|)^N} |\nabla^\alpha \tilde{\psi}(x)| \\
&\leq C(1+|g|)^N \sup_{\substack{x\in\mathbb{R}^n \\ |\alpha|\leq k}} (1+|x|)^N |\nabla^\alpha \tilde{\psi}(x)|
\end{aligned}$$
So $|c_g| \leq C''(1+|g|)^N$ for some C''. $\square$

Theorem. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic. Then*

$$u = \sum_{g \in \mathbb{Z}^n} u_g T_{e_{2\pi g}} = \text{“}\sum_{g \in \mathbb{Z}^n} u_g e^{2\pi i g \cdot x}\text{”}$$

(convergence in $\mathcal{S}'(\mathbb{R}^n)$) where $u_g = M(e_{-2\pi g} u) = \text{“}\int_q u(x) e^{-2\pi i g \cdot x} \mathrm{d}x\text{”} \in \mathbb{C}$ and satisfy $|u_g| \leq C(1 + |g|)^N$ for some $C > 0$, $N \in \mathbb{N}$.

Proof. Since u is periodic, $u \in \mathcal{S}'(\mathbb{R}^n)$ and its Fourier transform $\hat{u}$ is defined. Then $\tau_k u = u$ for all $k \in \mathbb{Z}^n$ so $e_{-k}\hat{u} = \hat{u}$ for all $z \in \mathbb{Z}^n$, i.e $(e_{-k} - 1)\hat{u} = 0$. Then by the previous lemma,

$$\hat{u} = (2\pi)^n \sum_{g \in \mathbb{Z}^n} u_g \delta_{2\pi g} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Applying the inverse Fourier transform gives

$$u = \sum_{g \in \mathbb{Z}^n} u_g T_{e_{2\pi g}} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Since $e_{2\pi g} \in L^1_{\text{loc}}$, $M(e_{-2\pi k} T_{e_{2\pi g}}) = \int_q e^{2\pi i (g-k) \cdot x} \mathrm{d}x = \delta_{kg}$. Since $u \mapsto M(u)$ is continuous on $\mathcal{S}'(\mathbb{R}^n)$, it follows that $M(e_{-2\pi g} u) = u_g$. □

Examples. Let $u = \sum_{g \in \mathbb{Z}^n} \delta_g$. Then

$$u_k = M(e_{-2\pi k} u) = u(\psi e_{-2\pi k}) = \sum_{g \in \mathbb{Z}^n} \psi(g) e^{-2\pi i k \cdot g} = 1.$$

So

$$\sum_{g \in \mathbb{Z}^n} \delta_g = \sum_{g \in \mathbb{Z}^n} T_{e_{2\pi g}} \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

or

$$\text{“}\sum_{g \in \mathbb{Z}^n} f(x - g) = \sum_{g \in \mathbb{Z}^n} e^{2\pi i g \cdot x}\text{”}.$$

This is the Poisson summation formula.

Theorem. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be periodic with Fourier coefficients u_g. Then*

(i) $\nabla^\alpha u \in \mathcal{D}'(\mathbb{R}^n)$ *is periodic and* $\nabla^\alpha u = \sum_{g\in\mathbb{Z}^n}(2\pi i g)^\alpha u_g T_{e_{2\pi g}}$.

(ii) If $f \in L^1_{loc}(\mathbb{R}^n)$ is periodic, $u = T_f$, then $|c_g| \leq ||f||_{L^1(q)}$ and $|c_g| \to 0$ as $|g| \to \infty$.

(iii) If $f \in C^{n+1}(\mathbb{R}^n)$ is periodic, $u = T_f$, then $f(x) = \sum_{g\in\mathbb{Z}^n} u_g e^{2\pi i g\cdot x}$ uniformly.

(iv) If $f, h \in L^2_{loc}$ are periodic with Fourier coefficients f_g, h_g then

$$\int_q \bar{f}(x)g(x)\mathrm{d}x = \sum_{g\in\mathbb{Z}^n} \bar{f}_g h_h,\ f(x) = \sum_{g\in\mathbb{Z}^n} f_g e^{2\pi i g\cdot x} \text{ in } L^2(q).$$

Proof. Not given - similar to previous proofs. ☐

4 Sobolev spaces & applications

4.1 Sobolev spaces

Definition. Let $U \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{Z}_{\geq 0}$, $p \in [1,\infty]$. Then $f \in L^p(U)$ belongs to the *Sobolev space* $W^{k,p}(U)$ if

$$\forall |\alpha| \leq k\ \exists f^\alpha \in L^p(U) \text{ such that } \nabla^\alpha T_f = T_{f^\alpha}. \tag{$*$}$$

Then we write $f^\alpha = \nabla^\alpha f$ and say that f^α is the αth order *weak derivative* of f. The *Sobolev norm* is

$$||f||_{W^{k,p}} = \left(\sum_{|\alpha|\leq k} ||\nabla^\alpha f||^p_{L^p}\right)^{1/p} \quad \text{for } p \in [1,\infty)$$

$$||f||_{W^{k,\infty}} = \max_{|\alpha|\leq k} ||\nabla^\alpha f||_{L^\infty} \text{ for } p = \infty.$$

Fact. $W^{k,p}(U)$ is a Banach space for $p \in [1,\infty]$ and a Hilbert space if $p = 2$.

Remark. $(*)$ means $\int_U f^\alpha \phi \mathrm{d}x = (-1)^{|\alpha|}\int_U f\nabla^\alpha\phi\mathrm{d}x$ for all $\phi \in C^\infty_c(U)$.

Examples.

- Define $f \in W^{1,\infty(\mathbb{R})}$ by

$$f(x) = \begin{cases} -1 & x < -1 \\ x & x \in [-1,1] \\ 1 & x > 1 \end{cases}$$

Then

$$\nabla f(x) = \begin{cases} 0 & |x| > 1 \\ 1 & |x| \leq 1 \end{cases}$$

Indeed, for $\phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} -\int_{\mathbb{R}} f(x)\phi'(x)\mathrm{d}x &= -\int_{-\infty}^{-1} \phi'(x)\mathrm{d}x - \int_{-1}^{1} x\phi'(x)\mathrm{d}x - \int_{1}^{\infty} \phi'(x)\mathrm{d}x \\ &= \phi(-1) + \int_{-1^1} \phi(x)\mathrm{d}x - [x\phi(x)]_{-1}^{1} + \phi(1) \\ &= \int_{-1}^{1} \phi(x)\mathrm{d}x \end{aligned}$$

- The Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Then H is <u>not</u> in $W^{1,p}$ for any $p \in [1, \infty]$ since $\nabla H = \delta_0 \neq T_f$ for any $f \in L^1_{\text{loc}}$. This is despite the fact H is almost everwhere differentiable.

Notation: $\nabla^\alpha f \in L^p$ means $\exists f^\alpha \in L^p$ such that $\nabla^\alpha T_f = T_{f^\alpha}$.

Definition. For $s \in \mathbb{R}$, we say $f \in L^2(\mathbb{R}^n)$ belongs to the Sobolev space $H^s(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \mathrm{d}\xi < \infty$.

$H^s(\mathbb{R}^n)$ is a Hilbert space with inner product $(f,g)_{H^s} = (2\pi)^{-n} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\hat{g}(\xi)(1+|\xi|^2)^s \mathrm{d}\xi$.

If $s = k \in \mathbb{Z}_{\geq 0}$, then $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.

Theorem (Sobolev embedding)**.** *Let $s > \frac{n}{2} + k$ and let $f \in H^s(\mathbb{R}^n)$. Then there exists $f^* \in C^k(\mathbb{R}^n)$ such that $f = f^*$ almost everywhere. Write $f = f^*$ and $H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$.*

Proof. First assume $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\nabla^\alpha f(x) = \frac{i^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi)\mathrm{d}\xi$ (for $|\alpha| \leq k$). Hence

$$\begin{aligned} |\nabla^\alpha f(x)| &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^\alpha |\hat{f}(\xi)| \mathrm{d}\xi \\ &\leq \frac{1}{(2\pi)^n} \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \mathrm{d}\xi \right)^{1/2} \underbrace{\left(\int_{\mathbb{R}^n} |\xi|^{2\alpha} (1 + |\xi|^2)^{-s} \mathrm{d}\xi \right)^{1/2}}_{\substack{\leq C \int_{\mathbb{R}^n} (1+|\xi|^2)^{k-s} \mathrm{d}\xi < \infty \\ \text{for } s > \frac{n}{2} + k}} \end{aligned}$$

Now for $f \in H^s(\mathbb{R}^n)$, let $(f_i)_{i\geq 1}$ be a sequence in $\mathcal{S}(\mathbb{R}^n)$ such that $f_i \to f$ in H^s and $f_i \to f$ almost everywhere (can do this by Example Sheet 4). Then $(f_i)_{i\geq 1}$ is Cauchy in H^s so $(f_i)_{i\geq 1}$ is Cauchy in C^k, and $f_i \to f^*$ in $C^k(\mathbb{R}^n)$. Since $f_i \to f$ almost everywhere, we have $f^* = f$ almost everwhere. $\square$

Example. Consider (in weak sense) $-\nabla^2 u + u = f$ on $\mathbb{R}^n$ $(*)$. Then if $f \in H^s$ there is a unique $u \in H^{s+2}$ such that this holds. Indeed by Fourier transform, $(*)$ is equivalent to $(|\xi|^2+1)\hat{u}(\xi) = \hat{f}(\xi)$ almost everywhere, i.e $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{1+|\xi|^2}$ almost everywhere. Note

$$||u||_{H^{s+2}}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^{s+2} |\hat{u}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \mathrm{d}\xi = ||f||_{H^s}^2.$$

Remarks.

- This is the simplest example of *elliptic regularity*: u is more regular than f: $-\nabla^2 u + u \in H^s \Rightarrow u \in H^{s+2}$.
- If $s > n/2$ then $f \in C^0(\mathbb{R}^n)$, $u \in C^2(\mathbb{R}^n)$ and $(*)$ holds in the classical sense.

4.2 Traces of Sobolev functions

If $s > n/2$ then $H^c \subseteq C^0$ and $f|_\Sigma$ makes sense for any $f \in H^s$ and $\Sigma \subseteq \mathbb{R}^n$ a surface.

Theorem (Trace theorem)**.** *Let $s > 1/2$. Then there is a bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$ such that $Tf = f|_{\mathbb{R}^{n-1}\times\{0\}}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Tf is called the trace of f on $\Sigma = \mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n$.*

Proof. Example Sheet 4. □

Remark. By coordinate transformations, the result can be extended to sufficiently regular surfaces $\Sigma \subseteq \mathbb{R}^n$.

4.3 The space $H_0^1(U)$

Let $U \subseteq \mathbb{R}^n$ be open and let $f \in C_c^\infty(U)$. Extending f to be 0 outside U, $f \in H^1(\mathbb{R}^n)$, so $C_c^\infty(U) \subseteq H^1(\mathbb{R}^n)$.

Definition. The space $H_0^1(U)$ is the closure of $C_c^\infty(U)$ in $H^1(\mathbb{R}^n)$, with respect to norm

$$||f||_{H^1} = \left((2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)|\hat{f}(\xi)|^2 \mathrm{d}\xi \right)^{1/2} = \left(\int_{\mathbb{R}^n} \left(|\nabla f(x)|^2 + |f(x)|^2 \right) \mathrm{d}x \right)^{1/2}.$$

$H_0^1(U)$ is a Hilbert space with inner product

$$(u,v)_{H_0^1} = \int_U \left(\overline{\nabla u} \cdot \nabla v + \overline{u} v \right) \mathrm{d}x.$$

Proposition. If $u \in H_0^1(U)$ then $u = 0$ for almost all $x \notin U$.

Proof. It suffices to check that $\int_{\mathbb{R}^n} \phi u \mathrm{d}x = 0$ for all $\phi \in C_c^\infty(\operatorname{int} U^c)$. Let $\phi \in C_c^\infty(\operatorname{int} U^c)$ and set $\Lambda_\phi(v) = \int_{\mathbb{R}^n} \phi v \mathrm{d}x$. So $\Lambda_\phi(v) = 0$ for all $v \in C_c^\infty(U)$. Since also $|\Lambda_\phi(v)| \leq ||\phi||_{L^2}||v||_{L^2} \leq ||\phi||_{L^2}||v||_{H^1}$, if $u_n \in C_c^\infty(U)$, $u_n \to u \in H_0^1$ then $\Lambda_\phi(u) = 0$. i.e $\int_{\mathbb{R}^n} \phi u \mathrm{d}x = 0$ for all $u \in H_0^1$, $\phi \in C_c^\infty(\operatorname{int} U^c)$. $\square$

Remark. An analogous result holds for ∇u by the same argument.

Fact. If ∂U is sufficiently nice, any $u \in H_0^1(U)$ vanishes on ∂U in trace sense (i.e $Tu = 0$ almost everywhere on ∂U).

Proof. $T : H^1(U) \to H^{1/2}(\partial U)$ is bounded and clearly $Tu_k = 0$ for all $u_k \in C_c^\infty(Y)$. Thus if $u_k \to u$ in H^1, also $Tu = 0$. $\square$

Example (Elliptic boundary value problem)**.** Let $U \subseteq \mathbb{R}^n$ be open and consider

$$\begin{cases} -\nabla^2 u + u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}. \tag{$\dagger$}$$

We seek $u \in H_0^1(U)$ such that $-\nabla^2 u + u = f$ in the distributional sense (for $f \in L^2(U)$):

$$\int_U (-\overline{u}\nabla^2 v + \overline{u}v)\mathrm{d}x = \int_U \overline{f} v \mathrm{d}x,\ \forall v \in C_c^\infty(U). \tag{$*$}$$

Since $u \in H^1$, $-\int \overline{u}\nabla^2\phi \mathrm{d}x = \int \nabla^2\overline{u}\phi \mathrm{d}x$ for all $\phi \in C_c^\infty(\mathbb{R}^n)$, $(*)$ is equivalent to

$$\int_U (\overline{\nabla u} \cdot \nabla v + \overline{u}v)\mathrm{d}x = \int_U \overline{f} v \mathrm{d}x,\ \forall v \in C_c^\infty(U).$$

Definition. $u \in H_0^1(U)$ is a *weak solution* to $(\dagger)$ (given $f \in L^2(U)$) if

$$(u, v)_{H^1} = (f, v)_{L^2}\ \forall v \in H_0^1(U).$$

Proposition. Let $f \in L^2(U)$. Then there exists a unique weak solution u of the Elliptic boundary value problem, and $||u||_{H^1} \leq ||f||_{L^2}$. The solution operator $S : L^2(U) \to H_0^1(U)$, $f \mapsto u$, is a bounded linear operator. Viewed as an operator $S : L^2 \to L^2$, S is in fact self-adjoint.

Proof. $\Lambda : H_0^1(U) \to \mathbb{C}$, $\Lambda(v) = (f, v)_{L^2}$ is a bounded linear map. Thus by the Riesz representation theorem, there exists a unique $u \in H_0^1(U)$ such that $(u, v)_{H^1} = (f, v)_{L^2}$ and $||u||_{H^1} \leq ||f||_{L^2}$ since $||u||_{H^1}^2 = (u, u)_{H^1} = (f, u)_{L^2} \leq ||f||_{L^2}||u||_{L^2}$.

Linearity of S is clear: if $f_1, f_2 \in L^2(U)$, $a \in \mathbb{C}$ and $u_1 = Sf_1$, $u_2 = Sf_2$ then $u = u_1 + au_2$ satisfies $(u, v)_{H^1} = (f_1 + af_2, v)_{L^2}$ for all $x \in H_0^1(U)$, so by uniqueness, $S(f_1 + af_2) = Sf_1 + aSf_2$.

S is self-adjoint since $(f, Sg)_{L^2} = (Sf, Sg)_{H^1} = \overline{(Sg, Sf)_{H^1}} = \overline{(g, Sf)_{L^2}} = (Sf, g)_{L^2}$. □

When is u a 'nice' function?

Definition. For $s > 0$, define $H^s_{\text{loc}}(U) = \{u \in L^2_{\text{loc}}(U) : \chi u \in H^s(\mathbb{R}^n)\ \forall \chi \in C_c^\infty(U)\}$

Proposition. Let U' be open, $\overline{U'} \subseteq U$. Then $u \in H^s_{\text{loc}}(U)$ is in $C^k(U')$ if $s > \frac{n}{2} + k$.

Proof. If U' is open, $\overline{U'} \subseteq U$ then there is $\chi \in C_c^\infty(U)$ such that $\chi(x) = 1$ for $x \in U'$. Hence by the full space Sobolev embedding, if $s > \frac{n}{2} + k$ and $u \in H^s_{\text{loc}}(U)$, $\chi u \in H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$. Since $\chi u = u$ on U', $u \in C^k(U')$. □

Proposition. If $u \in H_0^1(U)$ is the unique solution to the Elliptic boundary value problem, if $f \in L^2(U)$ then $u \in H^2_{\text{loc}}(U)$. More generally, if $f \in L^2 \cap H^k_{\text{loc}}(U)$ then $u \in H^{k+2}_{\text{loc}}(U)$. In particular, if $f \in L^2(U) \cap C^\infty$ then $u \in C^\infty(U)$ and the PDE holds in the classical sense.

Remark. This is again an example of elliptic regularity.

Proof. Fix $K \subseteq U$ compact and let $\chi_K \in C_c^\infty(U)$ be such that $\chi_K = 1$ on K. Recall the Elliptic boundary value problem means

$$\int_U (\overline{\nabla u} \cdot \nabla v + \overline{u} v) \mathrm{d}x = \int_U \overline{f} u \mathrm{d}x \ \forall v \in H_0^1(U).$$

Given $\phi \in \mathcal{S}(\mathbb{R}^n)$, let $v(x) = \chi_K(x)\phi(x)$. Then

$$\begin{aligned}
\int_U (\overline{\nabla u} \cdot \nabla v)\mathrm{d}x &= \int_U (\overline{\nabla u} \cdot (\nabla \chi_K)\phi + \chi_K \overline{\nabla u} \cdot \nabla \phi)\mathrm{d}x \\
&= -\int_U u((\nabla^2 \chi_K)\phi + \nabla \chi_K \cdot \nabla \phi + \nabla \chi_K \cdot \nabla \phi + \chi_K \nabla^2 \phi)\mathrm{d}x \\
&= -\int_U u(\nabla^2 \chi_K \phi - 2\nabla^2 \chi_K \phi - 2(\nabla \chi_K \cdot \nabla u)\phi + \chi_K \nabla^2 \phi) \\
&= -\int_U u(-\nabla^2 \chi_K \phi - 2(\nabla \chi_K \cdot \nabla u)\phi + \chi_K \nabla^2 \phi)
\end{aligned}$$

Which is equivalent to

$$\int_U \underbrace{\chi_K \overline{u}}_{\omega}(-\nabla^2 \phi + \phi)\mathrm{d}x = \int_U \underbrace{\overline{(f\chi_K - 2\nabla \chi_K \cdot \nabla u - u\nabla^2 \chi_K)}}_{\overline{g}}\phi \mathrm{d}x$$

i.e

$$\int_{\mathbb{R}^n} \omega(-\nabla^2 \phi + \phi)\mathrm{d}x = \int_{\mathbb{R}^n} \overline{g}\phi \mathrm{d}x \ \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Thus ω is a weak solution to $-\nabla^2 \omega + \omega = g$ on $\mathbb{R}^n$. Therefore $\omega \in H^2(\mathbb{R}^n)$ as seen before. For any $\psi \in C_c^\infty(U)$, we can take $K = \operatorname{supp} \psi$. Then $\psi u = \psi \omega \in H^2(\mathbb{R}^n)$ so $u \in H^2_{\mathrm{loc}}(U)$. $\square$

4.4 Rellich-Kondrachov Theorem

Theorem. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Assume $(u_j) \subseteq H_0^1(U)$ satisfies $||u_j||_{H^1} \leq 1$ and that $u_j \rightharpoonup^{w^*} u$ in $L^2(U)$, $u \in H_0^1(U)$. Then $u_j \to u$ in $L^2(U)$.*

Remark. By Banach-Alaoglu, $||u_j|| \leq 1$ allows us to restrict to a weak-$*$ convergent sunsequence in $H_0^1(U)$. Note that $u_j \rightharpoonup^{w^*} u$ in $H_0^1(U)$ implies $u_j \rightharpoonup^{w^*} u$ in $L^2(U)$. Indeed, $u_j \rightharpoonup^{w^*} u$ in H_0^1 if and only if $(u_j, v)_{H^1} \to (u, v)_{H^1}$ for all $v \in H_0^1(U) \Rightarrow (u_j, u)_{L^2} \to (u, v)_{L^2}$ for all $v \in L^2(U) \iff u_j \rightharpoonup^{w^*} u$ in $L^2(U)$.

Proof. By the Parseval identity,

$$\begin{aligned}||u_j - u||_{L^2}^2 &= (2\pi)^{-n} ||\hat{u}_j - \hat{u}||_{L^2}^2 \\ &= \underbrace{(2\pi)^{-n} \int_{|\xi|<R} |\hat{u}_j(\xi) - \hat{u}(\xi)|^2 \mathrm{d}\xi}_{\text{(I)}} + \underbrace{(2\pi)^{-n} \int_{|\xi|>R} |\hat{u}_j(\xi) - \hat{u}(\xi)|^2 \mathrm{d}\xi}_{\text{(II)}}.\end{aligned}$$

Also

$$\begin{aligned}&(2\pi)^{-n} \int_{|\xi|>R} |\hat{u}_j(\xi) - \hat{u}(\xi)|^2 \mathrm{d}\xi \\ &\leq \frac{2}{(2\pi)^n (1+R^2)} \int (1+|\xi|^2)(|\hat{u}_j(\xi)|^2 + |\hat{u}(\xi)|^2) \mathrm{d}\xi \\ &\leq \frac{2}{1+R^2} (||u_j||_{H^1}^2 + ||u||_{H^1}^2) \leq \frac{4}{R^2} < \varepsilon \text{ for } j \text{ sufficiently large and } R > R_0(\varepsilon).\end{aligned}$$

Since $\hat{u}_j(\xi) = (e_\xi, u_j)_{L^2}$ where $e_\xi(x) = e^{i\xi x} \in L^2(U)$ as U is bounded, the assumption $u_j \rightharpoonup^{w^*} u$ in $L^2(U)$ implies

$$\hat{u}_j(\xi) = (e_\xi, u_j) \to (e_\xi, u) = \hat{u}(\xi) \ \forall \xi.$$

Also,

$$\begin{aligned}|\hat{u}_j(\xi) - \hat{u}(\xi)|^2 &\leq 2(|\hat{u}_j(\xi)|^2 + |\hat{u}(\xi)|^2) \\ &\leq 2(||u_j||_{L^1}^2 + ||u||_{L^1}^2) \\ &\leq 2|U| \left(||u_j||_{L^2}^2 + ||u||_{L^2}^2\right) \leq 4|U| < \infty.\end{aligned}$$

Thus by the Dominated Convergence theorem, (I) $\to 0$ as $j \to \infty$ for every fixed R. Thus, first choosing $R > R_0(\varepsilon)$ and then $j \geq j_0(R)$ such that (I) $< \varepsilon$ we have (I) + (II) $< 2\varepsilon$ for $j > j_0$. $\square$

Corollary. *Let $U \subseteq \mathbb{R}^n$ be open and bounded, (u_j) a bounded sequence in $H_0^1(U)$. Then there is a subsequence (j_k) such that $u_{j_k} \rightharpoonup^{w^*} u$ in $H_0^1(U)$ and $u_j \to u$ in $L^2(U)$.*

Corollary. *If $S : L^2(U) \to H_0^1(U)$ is a bounded linear operator, then $S : L^2(U) \to L^2(U)$ is compact.*

Example. Let $U \subseteq \mathbb{R}^n$ be open and bounded, and let $V : U \to \mathbb{R}$ be smooth and bounded. For $f \in L^2(U)$ consider

$$\begin{cases} -\nabla^2 u + Vu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}. \tag{$*$}$$

Definition. Say $u \in H_0^1(U)$ is a weak solution to $(*)$ if

$$\int_U (\overline{\nabla u} \cdot \nabla v + V\overline{u}v)\mathrm{d}x = \int_U \overline{f} v \mathrm{d}x \ \forall v \in H_0^1(U). \tag{$\dagger$}$$

The LHS of $(\dagger)$ is not in general an inner product, so the previous proof (in the Elliptic boundary problem) for existence does not apply.

Proposition. Either there exists $\omega \in H_0^1(U) \cap C^\infty(U)$ with $\omega \neq 0$ such that $-\nabla^2\omega + V\omega = 0$, or there is a unique solution for all $f \in L^2(U)$.

Proof. $(\dagger)$ is equivalent to

$$\int_U (\overline{\nabla u} \cdot \nabla v + \overline{u}v)\mathrm{d}x = \int \overline{(f + (1-V)u)} v \mathrm{d}x \ \forall v \in H_0^1.$$

Let $S : L^2(U) \to H_0^1(U)$ be the solution operator corresponding to $V = 1$, i.e $u = Sf$ is the unique weak solution to the Elliptic boundary problem. Then $(\dagger)$ holds $\iff u = S(f + (1-V)u) \iff (\mathrm{id} - K)u = Sf$, $Ku := S((1-V)u)$. Since $K : L^2(U) \to H_0^1(U)$ is bounded, $K : L^2(U) \to L^2(U)$ is compact (by one of the above corollaries). Hence either

(a) $\mathrm{Ker}(\mathrm{id} - K) \neq 0$, so there exists $\omega \in L^2(U)$ with $(\mathrm{id} - K)\omega = 0$, or;

(b) $\mathrm{Im}(\mathrm{id} - K) = L^2(U)$, i.e there exists a unique u such that $(\mathrm{id} - K)u = Sf$.

In case (a), $\mathrm{Ker}(\mathrm{id} = K)$ is finite dimensional, so $(\mathrm{id} - K)\omega = 0 \iff \omega = S((1-V)\omega)$, implying $\omega \in H_0^1(U)$ and by iteration and Sobolev embedding, $\omega \in H_0^1(U) \cap C^\infty(U)$.

In case (b), $u = S(f + (1-V)u) \in H_0^1(U)$, so u is a weak solution to $(*)$. $\square$

Theorem. *There exists an orthonormal basis $\{\psi_k\}$ of $L^2(U)$ such that $\psi_k \in H_0^1 \cap C^\infty(U)$ and $-\nabla^2\psi_k = \lambda_k\psi_k$ in U where $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ and $\lambda_k \to \infty$.*

Proof. Since $S : L^2(U) \to H_0^1(U)$ is bounded, by Rellich-Kondrachov, $S : L^2 \to L^2$ is compact. It is also self-adjoint, so by the spectral theorem, $\sigma(S) = \{0, \mu_1, \mu_2, \ldots\}$ with $\mu_k \in \mathbb{R}$ with only one accumulation point at 0, and there is a corresponding orthonormal basis of eigenvectors $\{\psi_k\}$ of S.

Assume $S\psi_k = \mu_k\psi_k$. Then $\psi_k \in H_0^1(U)$ and

$$(\psi_k, v)_{L^2} = (S\psi_k, v)_{H^1} = \mu_k(\psi_k, v)_{H^1} \ \forall v \in H_0^1(U).$$

Setting $v = \psi_k$ we see ($\psi_k = 0$ or) $\mu_k > 0$. Thus if $\mu_k > 0$ then ψ_k is a weak solution to

$$\begin{cases} -\nabla^2\psi_k + \psi_k = \frac{1}{\mu_k}\psi_k & \text{in } U \\ \psi_k = 0 & \text{on } \partial U \end{cases} \iff \begin{cases} -\nabla^2\psi_k = -\lambda_k\psi_k & \text{in } U \\ \psi_k = 0 & \text{on } \partial U \end{cases}$$

where $\lambda_k = \frac{1}{\mu_k} - 1$. Since $\{\mu_k\}$ on has 0 as an accumulaton point, $\lambda_k \to \infty$. Elliptic regularity implies $\psi_k \in C^\infty(U)$ (iteration). □

A brief glimpse of the calculus of variations

Given $u \in H_0^1(U)$, $f \in L^2(U)$, consider

$$S(u) = \int_U \left(|\nabla u|^2 + |u|^2 - \overline{f}u - f\overline{u}\right) \mathrm{d}x = ||u||_{H^1}^2 - 2\Re((f, u)_{L^2}).$$

Theorem. *If we define $\sigma = \inf\{S(u) : u \in H_0^1(U)\}$, then $\sigma > -\infty$ and the infimum is attained by a unique $u \in H_0^1(U)$. Thus u is the weak solution to the Elliptic boundary value problem.*

Proof. By Cauchy-Schwarz,

$$S(u) \geq ||u||_{H^1}^2 - \frac{1}{2}||u||_{L^2}^2 - 2||f||_{L^2}^2 \geq -2||f||_{L^2}^2 > -\infty.$$

To see that the infimum is attained, let (u_k) be a sequence in $H_0^1(U)$ with $S(u_k) \to \sigma$. In particular, $(S(u_k))_k$ is bounded, and it follows that (u_k) is bounded in H_0^1 as

$$||u_k||_{H_0^1}^2 \leq 2S(u_k) + 4||f||_{L^2}^2.$$

By Banach-Alaoglu there exists $\tilde{u} \in H_0^1$ such that $u_k \rightharpoonup w\tilde{u}$ in $H_0^1(U)$ along a subsequence. Thus $||\tilde{u}||_{H^1} \leq \liminf_{k\to\infty} ||u_k||_{H^1}$ and

$$(f, \tilde{u})_{L^2} = \lim_{k\to\infty}(f, u_k)_{L^2} \implies S(\tilde{u}) \leq \liminf_{k\to\infty} S(u_k) = \sigma.$$

Hence $S(\tilde{u}) = \sigma$ by definition of σ.

For uniqueness it suffices to show $\tilde{u}$ solves the Elliptic boundary value problem, as we know any such solution is unique. Let $v \in H_0^1(U)$, $t \in \mathbb{R}$. Then

$$S(\tilde{u} + tv) = S(\tilde{u}) + t^2||v||_{H^1} + 2t\Re\int_U \left(\overline{\nabla\tilde{u}} \cdot \nabla v + \overline{\tilde{u}}v - \overline{f}v\right) \mathrm{d}x.$$

Since $S(\tilde{u}+tv) \geq S(\tilde{u})$ (as $\tilde{u}$ is a minimiser), we must have

$$\Re \int_U \left(\overline{\nabla \tilde{u}} \cdot \nabla v + \overline{\tilde{u}} v - \overline{f} v\right) \mathrm{d}x = 0 \; \forall v \in H_0^1$$

$$\iff \int_U \left(\overline{\nabla \tilde{u}} \cdot \nabla v + \overline{\tilde{u}} v - \overline{f} v\right) \mathrm{d}x = 0 \; \forall v \in H_0^1$$

i.e $\tilde{u}$ is a weak solution to the Elliptic boundary value problem. □

The above proof can be generalised:

Definition. Let X be a topological space. Then $S : X \to \mathbb{R}$

- is *coercive* if for any sequence (u_k) in X with $S(u_k) \leq K$ (for some $K \in \mathbb{R}$ fixed), there is a convergent subsequence of (u_k);
- is *lower semi-continuous* (LSC) if for any sequence (u_k) in X such that $u_k \to u$ in X, then $S(u) \leq \liminf_{k\to\infty} S(u_k)$.

Theorem. *Let $S : X \to \mathbb{R}$ be coercive and lower semi-continuous. Then S achieves its minimum.*

Proof. Analogous to the last theorem. □

From now, let X be a reflexive Banach space.

Proposition. If $S(u) \leq K$ then $||u|| \leq \tilde{K}$ for some $\tilde{K}$, then S is coercive with respect to the weak topology.

Proof. Banach-Alaoglu. □

Lemma. *If $U \subseteq X$ is norm-closed and convex, then it is also weakly closed.*

Proposition. If $S : X \to \mathbb{R}$ is convex and lower semi-continuous with respect to the norm topology, then it is also lower semi-continuous with respect to the weak topology.

Example. Let $U \subseteq \mathbb{R}^n$ be bounded and open, $L : \mathbb{R}^n \to \mathbb{R}$ convex and $L(z) \geq \gamma|z|^2 - C$ for some $C, \gamma > 0$ then $S : H_0^1(U) \to \mathbb{R}$ defined by $S(u) = \int_U L(\nabla u)\mathrm{d}x$ has a minimiser.