My Introduction to the Double Pendulum

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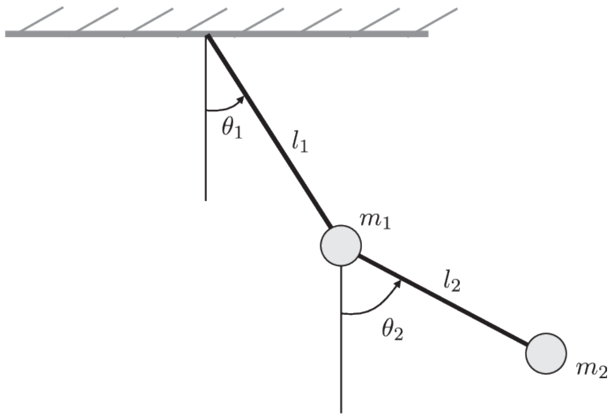
Introduction

The double pendulum is a physical system consisting of two pendulums attached end-to-end. The first pendulum is attached to a fixed pivot, and the second pendulum is attached to the free end of the first. Both pendulums are free to swing in a vertical plane, and their motion is influenced by gravity and their mutual interactions.



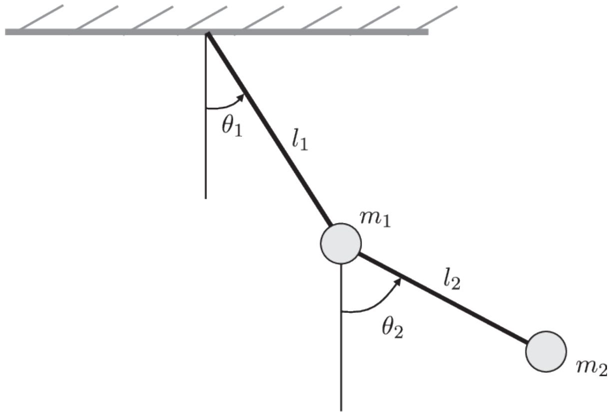
At first glance, the double pendulum may seem like a straightforward extension of the simple pendulum. However, its behavior is far more complex. While a simple pendulum exhibits regular and predictable motion, the double pendulum can move in ways that are highly sensitive to its initial conditions. Small changes in starting position or velocity can lead to drastically different motion, making the system an example of chaotic dynamics.

The double pendulum is interesting not only because of its chaotic behavior but also because it is governed by the same fundamental principles that describe many physical systems. These principles include Newton's laws of motion, conservation of energy, and the influence of forces such as gravity. Despite its simplicity, the double pendulum captures the essence of many more complicated systems found in nature and engineering.

In addition to being a fascinating system to study theoretically, the double pendulum is also a compelling subject for numerical simulation and visualization. By modeling the system with mathematical equations and solving them using a computer, we can explore its motion in detail, create animations, and analyze the factors that contribute to its chaotic behavior.

The double pendulum serves as a powerful example of how simple systems can exhibit complex behavior, and it provides an excellent opportunity to apply principles of physics and computational modeling to understand the dynamics of the natural world.

1 Deriving the Equations of Motion



So, what we want to do is to be able to describe both the motions of masses 1 and 2 here. In order to do so, we're first going to make a series of assumptions. Firstly, the lengths l1 & l2 are non-elastic, and of constant length. We're also going to confine ourselves to planar motion in x and y. The point that both masses are fixed to also is constant, and there is no air resistance.

Starting with Newton's F=ma, we can write for both masses:

$$F_1 = m_1 \frac{d^2\theta_1}{dt^2}$$

$$F_2 = m_2 \frac{d^2\theta_2}{dt^2}$$

Here we're using radians for position, where $\theta = \frac{s}{r}$, where r can be replaced with either l1 or l2, and s is the circumference about their center.

This is the method I would have thought originally used, but when you consider what forces are involved, apparently it gets hairy (apparently, according to google and all the derivations I found there). For example, you have gravity, and the velocity of each mass, which is perpendicular to l in each case. However the velocity of one mass, affects the other mass, we also must consider tension, and this makes resolving forces a nightmare. Therefore, we must take a step back.

Another way of looking at the same thing is to use Lagrangian mechanics. This way of analyzing the system uses energies, rather than forces. The Lagrangian of a system is defined as:

$$L = T - U$$

Where L is the Lagrangian, T is the Kinetic, and U is the Potential energy. The idea behind the Lagrangian, is that the integral of L, gives the *action*, and for the true path of the system, this is minimized. An equation follows from this concept (apparently), called the **Euler-Lagrange Equation**:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Another handy thing about the Lagrangian, is that we can use polar coordinates for this problem. Let's get to it:

The positions of the two masses in a double pendulum can be written as follows:

For mass 1, with angle θ_1 and length l_1:

$$x_1 = l_1 \sin(\theta_1)$$
$$y_1 = -l_1 \cos(\theta_1)$$

For mass 2, with angle θ_2 and length l_2:

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$$
$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2)$$

In order to get U, we need position. In order to get K, we need velocity. The velocities of the two masses in the double pendulum system are given by:

For mass 1:

$$v_{x1} = \frac{dx_1}{dt} = l_1\dot{\theta}_1 \cos(\theta_1)$$
$$v_{y1} = \frac{dy_1}{dt} = l_1\dot{\theta}_1 \sin(\theta_1)$$

For mass 2:

$$v_{x2} = \frac{dx_2}{dt} = l_1\dot{\theta}_1 \cos(\theta_1) + l_2\dot{\theta}_2 \cos(\theta_2)$$
$$v_{y2} = \frac{dy_2}{dt} = l_1\dot{\theta}_1 \sin(\theta_1) + l_2\dot{\theta}_2 \sin(\theta_2)$$

The velocity squared for the two masses in the double pendulum system is as follows:

For mass 1:

$$V_1^2 = v_{x1}^2 + v_{y1}^2$$

Substituting v_{x1} and v_{y1}:

$$V_1^2 = l_1^2\dot{\theta}_1^2(\cos^2(\theta_1) + \sin^2(\theta_1)) = l_1^2\dot{\theta}_1^2$$

For mass 2:

$$V_2^2 = v_{x2}^2 + v_{y2}^2$$

Substituting v_{x2} and v_{y2}:

$$V_2^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

Kinetic Energy

The total kinetic energy T of the double pendulum is:

$$T = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$$

Substituting $V_1^2 = l_1^2 \dot{\theta}_1^2$ and

$$V_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

we get:

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

Potential Energy

The total potential energy V of the system is:

$$V = -m_1 g y_1 - m_2 g y_2$$

Substituting $y_1 = -l_1 \cos(\theta_1)$ and $y_2 = y_1 - l_2 \cos(\theta_2)$:

$$V = m_1 g l_1 \cos(\theta_1) + m_2 g \left(l_1 \cos(\theta_1) + l_2 \cos(\theta_2) \right)$$

Lagrangian Formulation

The Lagrangian L is given by:

$$L = T - V$$

Substituting T and V:

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$
$$- \left(m_1 g l_1 \cos(\theta_1) + m_2 g \left(l_1 \cos(\theta_1) + l_2 \cos(\theta_2) \right) \right)$$

The Euler-Lagrange equation is:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = 0$$

For θ_1

1. Compute $\frac{\partial L}{\partial \dot{\theta}_1}$:

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 \left(l_1^2 \dot{\theta}_1 + l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

2. Differentiate with respect to t:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 \left(l_1^2 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right)$$

3. Compute $\frac{\partial L}{\partial \theta_1}$:

$$\frac{\partial L}{\partial \theta_1} = -m_1 g l_1 \sin(\theta_1) - m_2 g l_1 \sin(\theta_1) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

4. Substitute into the Euler-Lagrange equation:

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 \left(l_1^2 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right)$$

$$+ m_1 g l_1 \sin(\theta_1) + m_2 g l_1 \sin(\theta_1) = 0$$

For θ_2

1. Compute $\frac{\partial L}{\partial \dot{\theta}_2}$:

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 \left(l_2^2 \dot{\theta}_2 + l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \right)$$

2. Differentiate with respect to t:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) = m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right)$$

3. Compute $\frac{\partial L}{\partial \theta_2}$:

$$\frac{\partial L}{\partial \theta_2} = -m_2 g l_2 \sin(\theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

4. Substitute into the Euler-Lagrange equation:

$$m_2 \left(l_2^2 \ddot{\theta}_2 + l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right)$$

$$+ m_2 g l_2 \sin(\theta_2) = 0$$

Hamiltonian Formulation of the Double Pendulum

The Hamiltonian formulation is an alternative framework for analyzing the dynamics of the double pendulum. I never actually took anything beyond introductory mechanics in College, so I've only ever encountered the Hamiltonian in a Quantum environment. Nice to see it popping up in regular old classical mechanics too. The Hamiltonian actually comes from the Irishman Sir William Rowan Hamilton of Dublin (win).

The Hamiltonian method offers another way to study how a double pendulum moves. While the Lagrangian method uses angles (θ_1, θ_2) and their speeds $(\dot{\theta}_1, \dot{\theta}_2)$ to describe the system, the Hamiltonian method focuses on the angles and their corresponding momenta.

This is especially helpful for understanding complex behaviors like chaos and for identifying conserved properties, such as total energy. The Hamiltonian, H, represents the system's total energy and serves as a key tool for analyzing dynamics.

$$H = T + V$$

and is derived from the Lagrangian, L, using soemthing called the **Legendre transformation**. The transformation relates the generalized velocities to the conjugate momenta via:

$$p_{\theta_i} = \frac{\partial L}{\partial \dot{\theta}_i}$$

for each coordinate θ_i.

Legendre Transformation

The Legendre transformation rewrites the Lagrangian in terms of the conjugate momenta:

$$H(\theta_1, \theta_2, p_{\theta_1}, p_{\theta_2}) = \sum_{i=1}^{2} p_{\theta_i}\dot{\theta}_i - L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

where:

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1}, \quad p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2}$$

The Hamiltonian H then becomes a function of the coordinates and momenta.

Hamilton's Equations of Motion

The Hamiltonian formulation provides the equations of motion via Hamilton's equations:

$$\dot{\theta}_i = \frac{\partial H}{\partial p_{\theta_i}}, \quad \dot{p}_{\theta_i} = -\frac{\partial H}{\partial \theta_i}$$

for $i = 1, 2$.

Hamiltonian for the Double Pendulum

From the Lagrangian L, we compute the conjugate momenta:

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1}, \quad p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2}$$

and use the Legendre transformation to express the Hamiltonian:

$$H = \sum_{i=1}^{2} p_{\theta_i}\dot{\theta}_i - L$$

The total kinetic and potential energies derived earlier are:

$$T = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right),$$

$$V = m_1 g l_1 \cos(\theta_1) + m_2 g \left(l_1 \cos(\theta_1) + l_2 \cos(\theta_2) \right),$$

so:

$$H = T + V$$

Equations of Motion

1. Compute $\dot{\theta}_1$ and $\dot{\theta}_2$ from $\frac{\partial H}{\partial p_{\theta_1}}$ and $\frac{\partial H}{\partial p_{\theta_2}}$:

$$\dot{\theta}_1 = \frac{\partial H}{\partial p_{\theta_1}}, \quad \dot{\theta}_2 = \frac{\partial H}{\partial p_{\theta_2}}$$

2. Compute $\dot{p}_{\theta_1}$ and $\dot{p}_{\theta_2}$ from $-\frac{\partial H}{\partial \theta_1}$ and $-\frac{\partial H}{\partial \theta_2}$:

$$\dot{p}_{\theta_1} = -\frac{\partial H}{\partial \theta_1}, \quad \dot{p}_{\theta_2} = -\frac{\partial H}{\partial \theta_2}$$

After explicitly substituting the expressions for H, the final equations of motion are derived as coupled differential equations in terms of θ_1, θ_2, and their time derivatives.

After a lengthy derivation which I'm going to avoid (but can be found anywhere online if you're a dork), we are left with:

Equations of Motion for the Double Pendulum

The equations of motion for the double pendulum are given by:

$$\dot{\theta}_1 = \frac{l_2 p_1 - l_1 p_2 \cos(\theta_1 - \theta_2)}{l_1^2 l_2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]}$$

$$\dot{\theta}_2 = \frac{l_1 \left(m_1 + m_2\right) p_2 - l_2 m_2 p_1 \cos(\theta_1 - \theta_2)}{l_1 l_2^2 m_2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]}$$

$$\dot{p}_1 = -(m_1 + m_2) g l_1 \sin(\theta_1) - A + B$$

$$\dot{p}_2 = -m_2 g l_2 \sin(\theta_2) + A - B$$

where the terms A and B are given by:

$$A \equiv \frac{p_1 p_2 \sin(\theta_1 - \theta_2)}{l_1 l_2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]}$$

$$B \equiv \frac{l_2^2 m_2 p_1^2 + l_1^2 (m_1 + m_2) p_2^2 - l_1 l_2 m_2 p_1 p_2 \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]^2} \sin\left[2(\theta_1 - \theta_2)\right]$$

2 Python Code

This section analyzes a Python code implementation for simulating the motion of a double pendulum. The simulation incorporates physics-based modeling using Runge-Kutta Fourth Order methods for the above (semi-derived ODE's) and provides an interactive visualization interface using the Streamlit library.

Core Functionality

Pendulum Class

The `Pendulum` class encapsulates the physics of a double pendulum system:

Initialization (`__init__`)

The `__init__` constructor initializes the pendulum's parameters:

- Initial angles (`theta1`, `theta2`) in Radians.
- Lengths (`l1`, `l2`) and masses (`m1`, `m2`).
- Momenta (`p1`, `p2`) and time step (`dt`).

```
def __init__(self, theta1, theta2, dt, l1=1, l2=1, p1=0, p2=0, m1=1,
    m2=1):
    self.theta1 = theta1
    self.theta2 = theta2
    self.p1 = p1
    self.p2 = p2
    self.l1 = l1
    self.l2 = l2
    self.m1 = m1
    self.m2 = m2
    self.dt = dt
    self.g = 9.81
    self.trajectory = [self.polar_to_cartesian()]
    self.time = 0.0
```

Listing 1: Pendulum Class Constructor

Physics Calculations

The class defines several methods to compute the dynamics and update the pendulum's state.

Polar to Cartesian Conversion (`polar_to_cartesian`) This method converts the polar coordinates of the pendulum bobs into Cartesian coordinates for visualization.

```
def polar_to_cartesian(self):
    x1 = self.l1 * np.sin(self.theta1)
    y1 = -self.l1 * np.cos(self.theta1)
    x2 = x1 + self.l2 * np.sin(self.theta2)
    y2 = y1 - self.l2 * np.cos(self.theta2)
    return np.array([[0.0, 0.0], [x1, y1], [x2, y2]])
```

Listing 2: Polar to Cartesian Conversion

Derivatives (`derivatives`) This method computes the time derivatives of the state variables using the equations of motion derived from Lagrangian mechanics in the first section.

```python
def derivatives(self, state):
    theta1, theta2, p1, p2 = state
    sin_diff = np.sin(theta1 - theta2)
    cos_diff = np.cos(theta1 - theta2)
    denom = self.l1 * self.l2 * (self.m1 + self.m2 * sin_diff**2)
    dtheta1 = (self.l2 * p1 - self.l1 * p2 * cos_diff) / (self.l1**2 * denom)
    dtheta2 = (self.l1 * (self.m1 + self.m2) * p2 - self.l2 * self.m2 * p1 * cos_diff) / (self.l2**2 * self.m2 * denom)
    dp1 = -(self.m1 + self.m2) * self.g * self.l1 * np.sin(theta1) - ((p1 * p2 * sin_diff) / denom) + \
          (((self.l2**2 * self.m2 * p1**2 + self.l1**2 * (self.m1 + self.m2) * p2**2 - self.l1 * self.l2 * self.m2 * p1 * p2 * cos_diff) * np.sin(2 * (theta1 - theta2))) / (2 * denom**2))
    dp2 = -self.m2 * self.g * self.l2 * np.sin(theta2) + ((p1 * p2 * sin_diff) / denom) - \
          (((self.l2**2 * self.m2 * p1**2 + self.l1**2 * (self.m1 + self.m2) * p2**2 - self.l1 * self.l2 * self.m2 * p1 * p2 * cos_diff) * np.sin(2 * (theta1 - theta2))) / (2 * denom**2))
    return np.array([dtheta1, dtheta2, dp1, dp2])
```

Listing 3: Derivatives Method

State Evolution (`evolve`) The `evolve` method updates the pendulum's state using the Runge-Kutta 4th order integration.

```python
def evolve(self):
    state = np.array([self.theta1, self.theta2, self.p1, self.p2])
    k1 = self.derivatives(state)
    k2 = self.derivatives(state + self.dt * k1 / 2)
    k3 = self.derivatives(state + self.dt * k2 / 2)
    k4 = self.derivatives(state + self.dt * k3)
    new_state = state + self.dt * (k1 + 2 * k2 + 2 * k3 + k4) / 6
    self.theta1, self.theta2, self.p1, self.p2 = new_state
    self.time += self.dt
    self.trajectory.append(self.polar_to_cartesian())
    return self.trajectory[-1]
```

Listing 4: Evolve Method

Streamlit Integration

Streamlit is used to create an interactive simulation interface:

- A sidebar allows users to adjust parameters such as initial angles, lengths, masses, and time steps.
- Real-time visualization is achieved using `matplotlib`.

- A table displays key metrics: time, velocities, and the position of the second bob.

```
with st.sidebar:
    theta1 = st.slider("Initial Angle 1 (radians)", 0.0, 2 * np.pi,
        np.pi)
    theta2 = st.slider("Initial Angle 2 (radians)", 0.0, 2 * np.pi,
        np.pi - 0.01)
    l1 = st.slider("Length of Rod 1", 0.5, 2.0, 1.0)
    l2 = st.slider("Length of Rod 2", 0.5, 2.0, 1.0)
    m1 = st.slider("Mass of Bob 1", 0.5, 5.0, 1.0)
    m2 = st.slider("Mass of Bob 2", 0.5, 5.0, 1.0)
    dt = st.slider("Time Step (s)", 0.001, 0.1, 0.01)
    num_steps = st.slider("Number of Steps", 100, 2000, 500)
    trace_switch = st.checkbox("Show Bob 2 Trace", value=True)
    start_button = st.button("Start/Restart Simulation")
```

Listing 5: Streamlit Sidebar

3 Example Trajectories

Here in this section, I'm just gonna show some example trajectories. The Streamlit link is live, so if you're bored you can play around with it yourself too. Note in these only the trajectory of m2 is traced, given that m1 can only rotate in a circle.

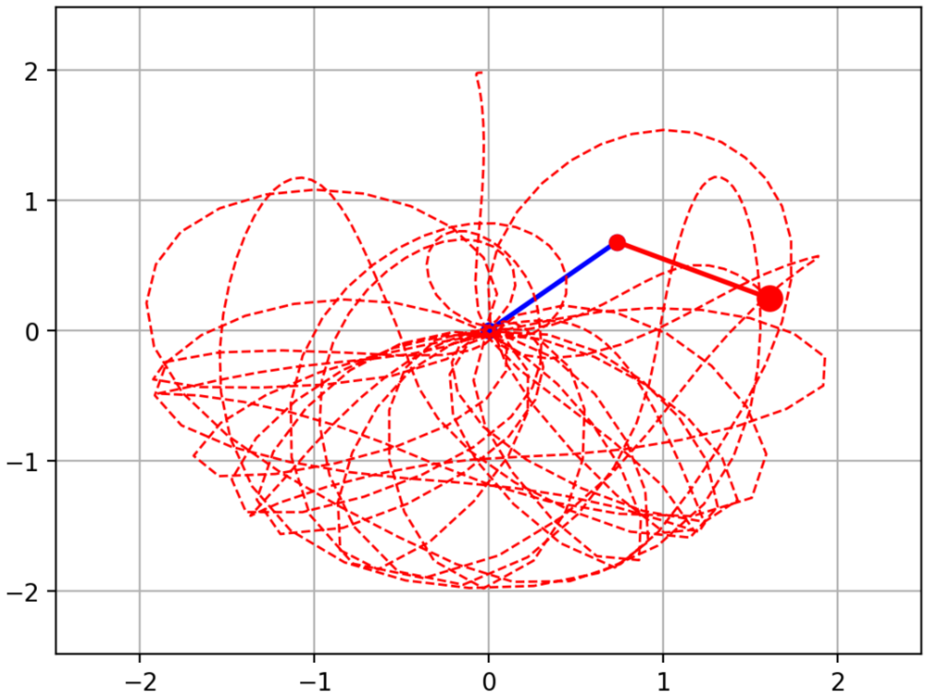


Figure 1: M1 = M2 = 1Kg, l1 = l2 = 1m

Here everything is set equal, and both masses are initially vertical, as if they were being held up. The chaotic nature of the movement of m2 can be seen here, with no symmetry about the x or y axis.

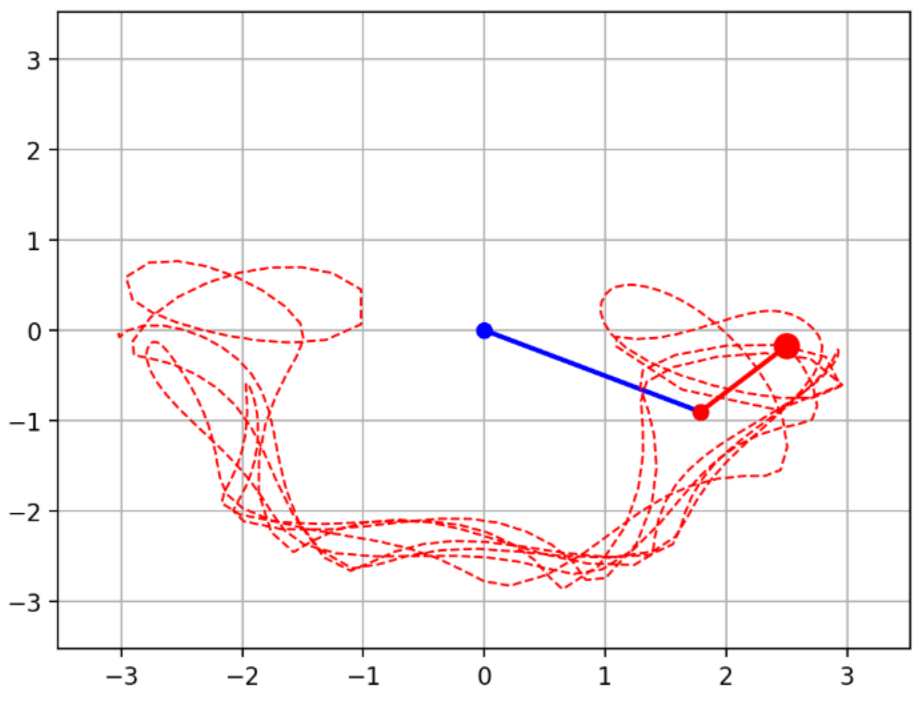


Figure 2: m2 = 5m1, l1 = 2l1

In this example, I aimed to see if by weighting the outer mass heavily and by making the inner length larger, I could simulate a pendulum-like motion. It can be seen in figure 2, with some level of self-persuasion, that this behaviour is seen.

In the example in figure 3, both lengths are equal, and the second mass is weighted

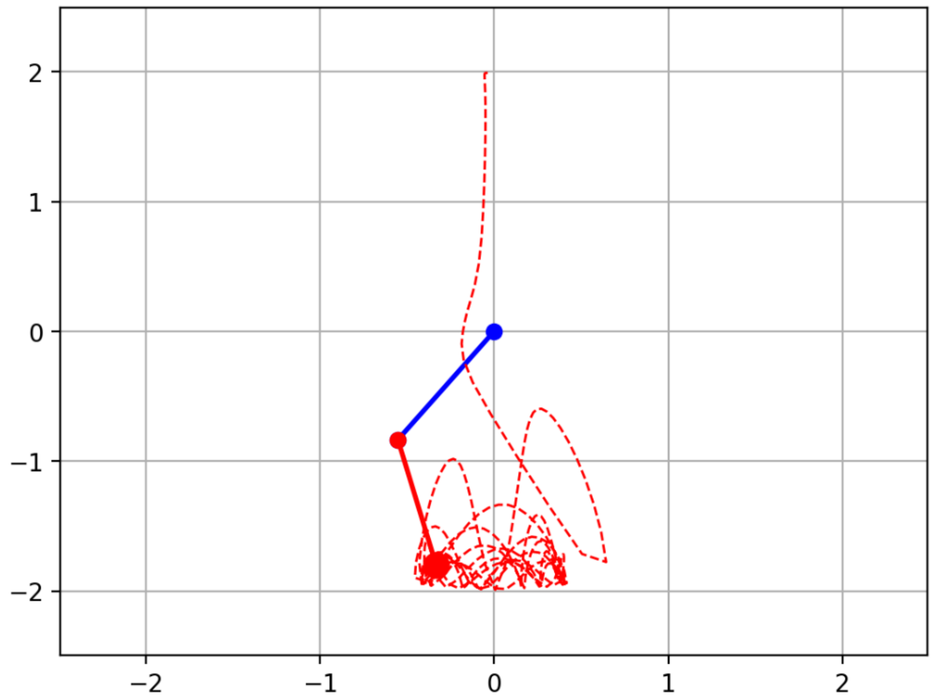


Figure 3: m2 = 5m1

heavily. The movement began with both lengths vertical, as in the first example. The second mass simply bobbed up and down, with little lateral locomotion. Given that this system is ideal, with no energy loss, I would think the fact that the initial movement had little to no x component most likely influenced this.

Finally, here in the example in figure 4, all I wanted to see was the effect of increasing the second length. Given that masses are equivalent here, this graph can almost be considered a simply more accentuated form of figure 1.

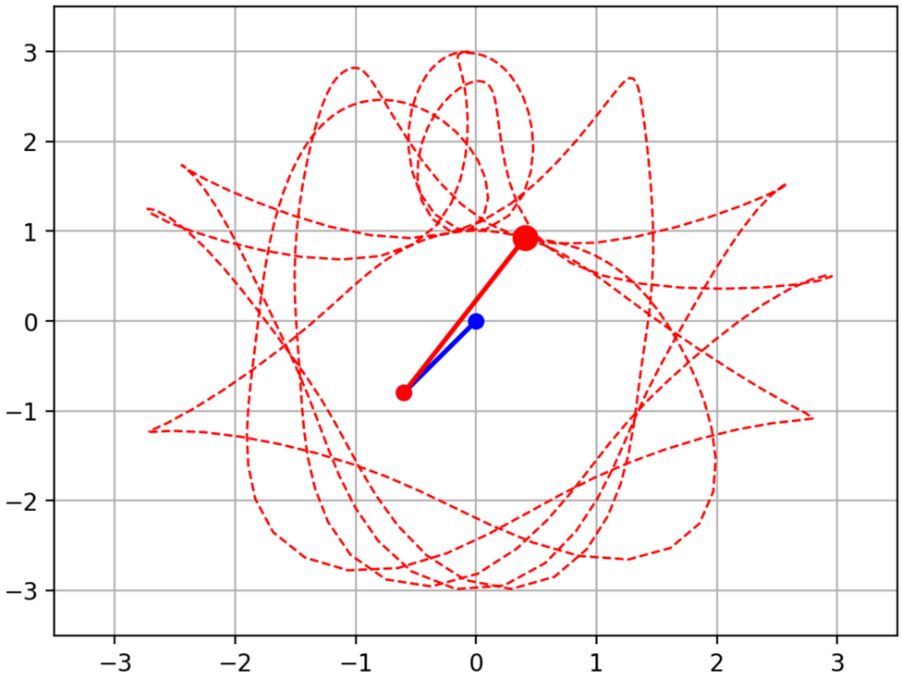


Figure 4: l2 = 2l2