Euler's Method

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1 Introduction

Euler's method is a numerical technique used to approximate the solutions of Ordinary Differential Equations (ODE's) numerically. It's names after the Swiss mathematician Leonhard Euler, who introduced the method in the 18^{th} century. ODE's arise in various scientific and engineering applications to model phenomena involving rates of change.

2 Objective

The objective of Euler's method is to approximate the solution to an initial value problem for an ordinary differential equation when an analytical solution is not feasible or computationally expensive to obtain. By iteratively stepping through small intervals along the independent variable, Euler's method provides an approximate solution that can be useful for understanding the behaviour of the system over time.

3 Mathematics

Euler's method is based on the concept of linear approximation. Given an initial value problem:

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

with an initial condition $y(x_0) = y_0$ where $f(x, y)$ represents the derivative function, Euler's method approximates the solution by iteritively stepping through small intervals of with h along the x-axis. At each step, the slope of the tangent line to the solution curve at the current point is approximated by the derivative function $f(x, y)$, and the change in y over the interval is estimated as $\Delta y = h.f(x, y)$. This approximation is used to update the value of y for the next step.

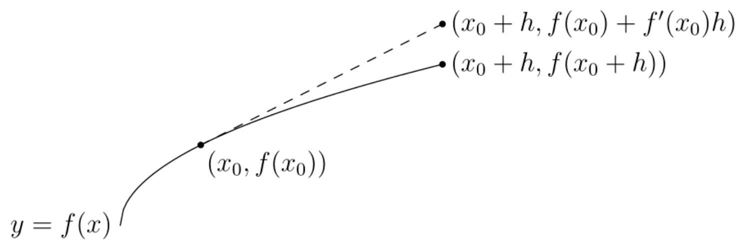

Figure 1: Euler's method

4 Algorithm

The algorithm for Euler's method can be summarized as follows:

- Given the initial condition (x_0, y_0) and the step size h, set $x = x_0$ and $y = y_0$.
- Repeat the following steps until the desired endpoint is reached:
  - Compute the derivaative $f(x, y)$ at the current point (x, y).
  - Estimate the change in y over the interval $[x, x+h]$ as $\Delta y = h.f(x, y)$.
  - Update the values of x and y for the next iteration: $x = x + h$ and $y = y + \Delta y$.

5 Convergence

Euler's method is a first-order numerical method, which means its global error decreases linearly with the step size h. While Euler's method is relatively simple and easy to implement, it may not always provide highly accurate results, especially for stiff ODEs or when using large step sizes. Other more sophisticated numerical methods, such as the Runge-Kutta methods, offer higher-order convergence and improved accuracy.

6 Conditions for Convergence:

Euler's method is guaranteed to converge under certain conditions, such as when the derivative function $f(x, y)$ is continuous and satisfies Lipschitz continuity with respect to y. Additionally, the step size h should be sufficiently small to ensure that the linear approximation remains valid over the interval $[x, x + h]$

6.1 Lipschitz Continuity

A function $f(x)$ is Lipschitz continuous on an interval $[a, b]$ if there exists a constant L such that for all x_1, x_2 in the interval $[a, b]$ the following inequality holds:

$$|f(x_2) - f(x_1)| \leq L.|x_2 - x_1| \tag{2}$$

Intuitively, Lipschitz continuity captures the idea that the function's rate of change (i.e., the slope of the tangent line) is bounded by a constant L over the entire interval $[a, b]$. This implies that the function cannot have arbitrarily steep slopes or sudden changes in behavior, which makes Lipschitz continuous functions well-behaved and predictable.

7 Accuracy

The accuracy of Euler's method depends on the step size h chosen for the approximation. Smaller step sizes generally result in higher accuracy, as they reduce the error introduced by the linear approximation. However, excessively small step sizes may lead to computational inefficiency. The choice of step size often involves a trade-off between accuracy and computational cost, and it may require experimentation or analysis to determine an appropriate value for a given problem.