Simpsons One-Third Rule for Integration

Conor Sheehan

April 2024

1 Introduction

The Simpson method, also known as Simpson's rule, is a numerical technique used to approximate the value of a definite integral. It is named after the mathematician Thomas Simpson and is one of the most widely used methods for numerical integration.

Simpson's rule falls under the category of Newton-Cotes formulas, which approximate the area under a curve by subdividing the interval into smaller segments and approximating the curve within each segment using simpler geometric shapes.

The Simpson method provides a more accurate approximation compared to simpler methods like the trapezoidal rule by using quadratic interpolations to approximate the curve within each segment.

2 Objective

The primary objective of the Simpson method is to provide an accurate estimate of a definite integral. Definite integrals arise in many areas of science and engineering, representing quantities such as areas, volumes, and accumulated changes over time.

Simpson's rule offers a practical way to approximate definite integrals, particularly when analytical methods are not feasible or efficient. Its accuracy and efficiency make it a valuable tool for numerical integration in various scientific and engineering applications.

3 Mathematics

Simpson's rule approximates the integral of a function $f(x)$ over an interval [a,b] by dividing the interval into n subintervals of equal width $h = \frac{b-a}{n}$. It is important to note that n must be even.

It then fits a second-degree polynomial (a quadratic) to each pair of adjacent sub-intervals, effectively approximating the curve within each segment.

The formula for Simpsons rule can be expressed as follows:

$$\int_a^b f(x) \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1}^{\frac{N}{2}} f(x_{i-1}) + 2\sum_{i=1}^{\frac{N}{2}} f(x_i) + f(x_N)\right] \tag{1}$$

Where $h = \frac{b-a}{n}$

4 Algorithm

The Simpson method follows a systematic algorithm for approximating definite integrals:

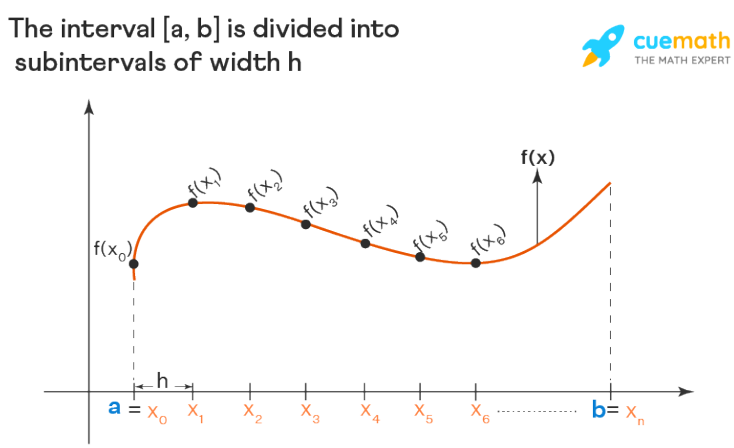

Figure 1: Simpsons Rule

- Divide the interval into n different subintervals of width h.
- Evaluate the function at each endpoint $f(x_N)$ of each subinterval.
- Use Simpsons Rule in Equation (1) above to calculate the approximate integral.

5 Conditions for Convergence

Simpson's method converges under certain conditions related to the smoothness and behavior of the integrand function f(x) over the interval [a,b].

The function should be sufficiently smooth, meaning that it should have continuous derivatives up to a certain order within the interval.

Rapid oscillations, discontinuities, or singularities in the function may affect convergence and accuracy.

6 Accuracy

Simpson's rule provides a higher degree of accuracy compared to simpler methods like the trapezoidal rule, especially for functions with smooth or moderately oscillatory behavior.

The error in Simpson's rule decreases rapidly as the step size h decreases. Specifically, the error is proportional to h^4, leading to quadratic convergence.

This means that Simpson's method can provide accurate results with relatively few function evaluations, making it an efficient and reliable technique for numerical integration in many practical applications.