What do fund flows reveal about asset pricing models and investor sophistication?

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Recent papers use the relative strength of the relation between fund flows and alphas with respect to various multifactor models to draw inferences about the best asset pricing model and about investor sophistication. This paper analytically shows that such inferences are tenable only under a number of additional assumptions. The results of our simulations and empirical tests indicate that such comparisons are not reliable tests of asset pricing models. We also find that parsimonious factor models better predict future alphas than models with additional factors, and discuss its implications for evaluating investor sophistication.

An extensive literature documents that net fund flows into mutual funds are driven by their past performance. For example, Patel, Zeckhauser, and Hendricks (1991) document that equity mutual funds with bigger returns attract more cash inflows and they offer various behavioral explanations for this phenomenon. Other papers that document a positive relation between fund flows and past performance include Ippolito (1992), Chevalier and Ellison (1997), and Sirri and Tufano (1998).

Some papers in the early literature also examine whether abnormal performance (or alphas) measured with respect to some benchmarks better predict fund flows than others. For example, Gruber (1996) and Del Guercio and Tkac (2002) compare the mutual fund flow-performance relation for alphas measured with respect to one- and four-factor models. Fung et. al. (2008) makes similar comparisons with a different set of factor models for a sample of hedge funds.

While flow-alpha comparison was not the primary focus of the earlier papers, recent papers in this area have shown a renewed interest in such comparison using a broader range of asset pricing models. The primary driving force for this resurgence is their arguments that these comparisons can help us answer important economic questions that extend beyond a descriptive analysis of mutual fund flows. For example, Barber, Huang and Odean (2016) (hereafter "BHO") compare the relation between fund flows and alphas measured with respect to various models to evaluate the level of mutual fund investors' sophistication. They argue that sophisticated investors should use all common factors to evaluate fund performance regardless of the underlying true asset pricing model. BHO also find that fund flows are more highly correlated with CAPM alphas than with other alphas. Because investors do not seem to be using alphas with respect to a model that includes all factors, BHO conclude that investors in aggregate are not sophisticated in how they use past returns to assess fund performance.

Berk and van Binsbergen (2016) (hereafter "BvB") argue that such comparisons serve as a new and fundamentally different test of asset pricing models and that the results can determine which asset pricing model is the closest to the true asset pricing model in the economy. Because of the asset pricing model implications, they include several versions of equilibrium consumption-CAPM as well as recent multifactor asset pricing models in their comparisons.

Agarwal, Green and Ren (2017) and Blocher and Molyboga (2017) carry out similar tests with samples of hedge funds.

BvB find that fund flows are most highly correlated with alphas computed with respect to CAPM in their tests as well. They conclude that therefore CAPM "is still the best method to use to compute the cost of capital of an investment opportunity." Berk and van Binsbergen (2017) prescribe that practitioner should use CAPM to make capital budgeting decisions. The true asset pricing model has been a holy grail of the finance literature and BvB's conclusions potentially have broad implications that go well beyond just the mutual fund literature.

The far reaching inferences drawn in the recent literature based on comparisons of flow-alpha relations stand in contrast with the much more limited inferences drawn in the early literature. A natural question that arises is, under what assumptions can one draw reliable inferences about asset pricing models or about investor sophistication based on these results? Are the inferences about asset pricing models and investor sophistication in the recent literature tenable?

We need answers to these fundamental questions to assess whether we can potentially gain new economic insights that go beyond a more limited description of the relation between alphas and flows. We address these questions in this paper. We analytically show that one cannot draw any reliable inferences about the true asset pricing model based on flow-alpha relations without a number of additional assumptions. For example, it is possible that in some situations CAPM may not be true and none of the common factors including the market factor may be priced but investors may still optimally use the market model to estimate alphas. Also, in some other situations, it is possible that a factor other than the market factor may be priced but investors may optimally use the market model to estimate alphas. There are also situations where investors may optimally use the market model to estimate alpha when CAPM is true, which would justify inferences about asset pricing model. Therefore, one cannot identify the true asset pricing model solely based on flow-alpha comparison without further tests to determine which of these multiple possibilities are true in the data.

We find similar issues with drawing inferences about investor sophistication as well. We analytically show that investors may optimally not use some or all common factors in the return generating process to estimate alphas. We show that investors should optimally consider the following elements to decide whether a factor should be used to estimate alphas: the underlying asset pricing model, the incremental explanatory power of the factor and the potential error in estimating betas.[1] Here again, one cannot reliably conclude that investors are not sophisticated solely because the factor model with respect to which alphas are computed that yields the biggest correlation with fund flows is not the model that includes all factors.

We carry out two set of tests to empirically assess whether inferences about asset pricing models and investor sophistication based on alpha-flow relations are tenable. The first set of tests estimate the relevant parameters from the data and run simulation experiments under various "true" asset pricing models. The second set of tests examines the relation between past alphas computed based on various models and future alphas. These tests also help us determine the answer to the following question: which model should one use to best predict future alphas? The answer to this question would serve as the benchmark to assess investor sophistication and it will also help investors with their model selection decisions.

Our findings have important implications for the performance evaluation literature as well. Since there is no consensus on the true asset pricing model, this literature evaluates performance under multiple models and alphas are estimated with the same set of factors that are assumed to be in the asset pricing models. However, we show that the multifactor model that most precisely estimates true alphas is in general not the "true" asset pricing model. For example, even if one believes that CAPM is the true asset pricing model, the four-factor model provides a more precise estimate of CAPM alpha for a one-year holding period.

1. Fund flows and alphas: Foundation for empirical tests and inferences

This section presents a model to examine the relation between aggregate flows and alphas under different asset pricing models. This model forms the basis for our analysis of the implications of flow-alpha relations for asset pricing models and tests of investor sophistication.

1.1 Return generating process and Asset Pricing model

The following K-factor model is the true asset pricing model:

[1] BHO discuss the possibility that investors may not use non-market factors in their decisions because of measurement errors in betas, but rule out the possibility. They do not consider or evaluate how the other two factors we discuss here affect model choice.

$$E[r_i] = r_f + \sum_{k=1}^{K} \beta_{k,i}\gamma_k \tag{1}$$

where $E[r_i]$ is the expected return on asset *i*, r_f is the risk-free rate, $\beta_{k,i}$ is the beta of asset *i* with respect to factor *k*, and γ_k is the premium for a unit of factor risk. If K=0 then all assets have same expected returns and $E[r_i] = E[r_m]$, where r_m is the market return. We refer to the model with K=0 as the "no-beta risk" model. For the CAPM, K=1 and for Fama-French three-factor model, which we refer to as FF3, K=3.

Asset returns follow the J-factor model below:

$$r_{i,t} = E[r_i] + \sum_{k=1}^{J} \beta_{k,i}\, f_{k,t} + \xi_{i,t} \tag{2}$$

where $f_{k,t}$ is the realization of the common factor *k*, and $\xi_{i,t}$ asset specific return at time *t*. The factor realization $f_{k,t}$ is the innovation or the unexpected component of factor *k*. For instance, if $F_{k,t}$ is the total factor realization of the kth factor then $f_{k,t} = F_{k,t} - E[F_{k,t}]$ and $E[f_{k,t}] = 0$. If this factor is traded then $E[F_{k,t}] = \gamma_k$. The return generating process has J common factors and in general $J \geq K$, where K factors are priced and J-K factors are unpriced.

Suppose there are P mutual funds in this economy, and suppose the manager of mutual fund *p* has stock selection skills that on average earn a risk adjusted return of α_p^K. We use the superscript K to emphasize that investors use the true K-factor asset pricing model to adjust for risk and compute alphas. The expected return from investing in fund *p* is:

$$E[r_p] = \alpha_p^K + r_f + \sum_{k=1}^{K} \beta_{k,p}\gamma_k \tag{3}$$

Investors seek to invest in a fund or a set of funds that they believe have the biggest alphas. How do investors form expectations about alphas? Under the rational expectations hypothesis, investors in aggregate have rational expectations about the true asset pricing model in the economy. Investors form their expectations with this knowledge and with all available information to most precisely estimate α_p^K.

The implication of the rational expectation hypothesis in the context of mutual fund flows can potentially form a basis for tests of asset pricing models. For example, since investors in aggregate know the true asset pricing model, BvB note that aggregate flows reveal their preferences and hence can be used to test the validity of various asset pricing models. However, even when investors know the true K-factor model, it is possible that investors may use additional factors that are not among the K priced factors to estimate α_p^K more precisely than what they can with just the K factors.

BHO also conduct similar tests but their stated goal is to evaluate the level of investor sophistication. Since sophisticated investors would attempt to estimate alphas as precisely as possible using all available information, BHO assume that they should use all J factors in the return generating process to obtain the most precise estimate of α_p^K. However, as we show below it is possible that some of the factors among the J factors may not increase the precision of estimates of α_p^K and hence sophisticated investors may rationally ignore them.

To judge whether we can draw reliable inferences about asset pricing models and investor sophistication based on correlations between alphas with respect to various asset pricing models and fund flows we need to determine which particular factor model would yield the most precise estimates of α_p^K.

1.2 Alphas and Fund flows

Let $\hat{\alpha}_{p,\eta,t}$ be an unbiased estimate of alpha with respect to an η-factor model for fund p as of month t. To compare alphas-flow relations across models, consider the following univariate cross-sectional regression:

$$\Gamma_{p,t} = a_\eta + b_\eta \times \hat{\alpha}_{p,\eta,t} + \omega_{p,\eta,t}, \tag{4}$$

where $\Gamma_{p,t}$ is the flow into fund p during month t. Let $\hat{b}_\eta$ be the estimate of the slope coefficient across all months.

The estimate $\hat{b}_\eta$ measures the strength of the alpha-flow relation of each η-factor model. Suppose we run horse race among multifactor models with $\hat{b}_\eta$. What can we learn about the true asset pricing model or about investor sophistication based on this horse race?

The answer to this question depends on how investors arrive at their investment decisions. We assume that investors are rational agents and we model a simple rational expectations economy to characterize investors' decisions. We then analyze what we can learn from the horse race in this economy.

Suppose each agent in the economy receives a noisy signal about α_p^K, the true alpha of fund *p*, and they make investment decisions based on this signal. Since each investor's decision depends on a number of personal factors such as their liquidity needs, other investment opportunities and the precision of their individual signals, we do not specify a functional relation between fund flows and signals. However, since the signals contain information about α_p^K, their investment decisions is positively correlated with α_p^K. As a result, the aggregate flow into this fund into fund *p*, which we denote as Γ_p is also positively correlated with α_p^K.

The estimate of alpha, $\hat{\alpha}_{p,\eta}$ measures α_p^K, the true alpha with respect to the K-factor model, with error. Let:[2]

$$\hat{\alpha}_{p,\eta} = \alpha_p^K + \left(\alpha_{p,\eta} - \alpha_p^K\right) + \varepsilon_{p,\eta} \tag{5}$$

where the term $\varepsilon_{p,\eta}$ is the statistical estimation error. For example, if one were to estimate $\hat{\alpha}_{p,\eta}$ using OLS regression with a η-factor model, then $\hat{\alpha}_{p,\eta}$ contains statistical estimation error. In addition, the estimate also contains a model-misspecification error because it is estimated using a η-factor model, while the true model is a K-factor model. The standard model misspecification error is the term $\left(\alpha_{p,\eta} - \alpha_p^K\right) \equiv \theta_{p,\eta}$. For ease of exposition, we assume for now that aggregate flows are uncorrelated with the terms $\left(\alpha_{p,\eta} - \alpha_p^K\right)$ and $\varepsilon_{p,\eta}$ in Equation (5). We later show that all our results hold in rational expectations economy without this assumption.

The large sample estimate of the slope coefficient in Regression (4) with alphas estimated using an η-factor model is:

$$plim\ b_\eta = \frac{Cov\left(\Gamma_p, \hat{\alpha}_{p,\eta}\right)}{Var\left(\hat{\alpha}_{p,\eta}\right)} = Cov\left(\Gamma_p, \alpha_p^K\right) \left(\frac{1}{\sigma_{\alpha^K}^2 + \sigma_{\theta_\eta}^2 + \sigma_{\varepsilon_\eta}^2 + 2cov\left(\theta_{p,\eta}, \varepsilon_{p,\eta}\right)}\right), \tag{6}$$

[2] For ease of exposition, we omit the time subscript *t* when there is no ambiguity.

where σ^2 is the variance of the subscripted variable and $cov(\theta_{p,\eta}, \varepsilon_{p,\eta})$ is the covariance between model misspecification error and statistical measurement error. $\sigma^2_{\theta_\eta}$ is the variance of the model misspecification across the funds in the cross-section. The slope coefficient is a decreasing function of $\sigma^2_{\theta_\eta}$ and $\sigma^2_{\varepsilon_\eta}$.

To intuitively understand equation (6), we can rewrite it as:

$$plim\ b_\eta = \underbrace{\frac{Cov(\Gamma_p, \alpha_p^K)}{\sigma^2_{\alpha^K}}}_{\text{"true" slope}} \times \underbrace{\left(\frac{\sigma^2_{\alpha^K}}{\sigma^2_{\alpha^K} + \sigma^2_{\theta_\eta} + \sigma^2_{\varepsilon_\eta} + 2cov(\theta_{p,\eta}, \varepsilon_{p,\eta})}\right)}_{EIV\ Bias} \tag{7}$$

The first term on the right-hand side is the slope coefficient we would get if we use true alphas in linear regression (4) instead of estimates of alphas, which we label as "true" slope.[3] When we use estimates of alphas, the slope coefficient is smaller because of an errors-in-variables bias and the magnitude of bias is inversely related to the variance of measurement errors, given by $\sigma^2_{\theta_\eta} + \sigma^2_{\varepsilon_\eta} + 2cov(\theta_{p,\eta}, \varepsilon_{p,\eta})$.

Our analysis so far assumes that fund flow Γ_p is uncorrelated with $(\alpha_{p,\eta} - \alpha_p^K) + \varepsilon_{p,\eta}$. Let $v_{p,\eta} \equiv (\alpha_{p,\eta} - \alpha_p^K) + \varepsilon_{p,\eta}$. It is possible that Γ_p and $v_{p,\eta}$ are correlated in practice. For example, investors may use the same η-factor model as the econometrician to estimate alphas and hence measurement error in alphas would be correlated with the aggregate investors' signal and aggregate flow. Allowing for such correlation, we get:

$$\begin{aligned} plim\ b_\eta = {} & \frac{Cov(\Gamma_p, \alpha_p^K)}{\sigma^2_{\alpha^K}} \times \left(\frac{\sigma^2_{\alpha^K}}{\sigma^2_{\alpha^K} + \sigma^2_{\theta_\eta} + \sigma^2_{\varepsilon_\eta} + 2cov(\theta_{p,\eta}, \varepsilon_{p,\eta})}\right) \\ & + Cov(\Gamma_p, v_{p,\eta}) \times \left(\frac{1}{\sigma^2_{\alpha^K} + \sigma^2_{\theta_\eta} + \sigma^2_{\varepsilon_\eta} + 2cov(\theta_{p,\eta}, \varepsilon_{p,\eta})}\right). \end{aligned} \tag{8}$$

[3] We should emphasize that we do not assume a linear or any particular functional relation between flow and alpha. Equation (7) presents the large sample slope coefficient of a linear regression regardless of the underlying functional relation.

In a rational expectations economy, investors use the most precise estimator for true alpha to form their expectations. Therefore, the η-factor model that investors use to form their expectations would have the smallest $\sigma^2_{v_\eta} \equiv \sigma^2_{\theta_\eta} + \sigma^2_{\varepsilon_\eta} + 2cov(\theta_{p,\eta}, \varepsilon_{p,\eta})$, which in turn would have the biggest $Cov(\Gamma_p, v_{p,\eta})$. As we show earlier, the particular η-factor model that most precisely estimates true alpha would win the horse race when we assume $Cov(\Gamma_p, v_{p,\eta})$=0. In a rational expectations economy, removing this assumption would further strengthen the win for this particular η-factor model.

One component of $\sigma^2_{v_\eta}$ is $\sigma^2_{\theta_\eta}$, which is due to model misspecification. If this were the only component then the model with the biggest slope coefficient would be the one closest to the true asset pricing model. However, $\sigma^2_{\varepsilon_\eta}$ also contributes to the variance of measurement error in alpha. A model that is not the closest to the true asset pricing could have the biggest slope coefficient if $\sigma^2_{\varepsilon_\eta}$with respect to that model is sufficiently small. Therefore, we can reliably use the horse race with b_η to determine the model that is the closest to the true asset pricing model only if model misspecification is the empirically dominant source of measurement error.

If the winner of the horse race is not the particular η-factor model that most precisely estimates true alphas then one could conclude that investors are not sophisticated. However, the most precise model to estimate alpha need not be the model that includes all factors. For example, if $\sigma^2_{\theta_\eta}$is the dominant source of measurement error, then the K-factor model, which likely excludes some of the common factors in the return generating process, would yield the most precise estimates. Also, as we show below, even if $\sigma^2_{\theta_\eta}$were not the dominant source the most precise model may optimally exclude some of the factors in the return generating process.

1.3 Precision of alpha estimator across factor models: An illustrative example

This subsection discusses the factors that affect the precision of alphas using a two-step alpha estimator where the first step estimates factor betas and the second step estimates one-month ahead alphas. Suppose we estimate betas with respect to an η-factor model using the following time-series regression for each fund in the first step:

$$\left(r_{p,t} - r_{f,t}\right) = \alpha_{p,\eta} + \sum_{k=1}^{\eta} \beta_{k,p}\, F_{k,t} \, + \, e_{p,\eta,t} \tag{9}$$

The alpha estimates for month *t+1* are:

$$\hat{\alpha}_{p,\eta,t+1} = r_{p,t+1} - \left(r_{f,t+1} + \sum_{k=1}^{\eta} \hat{\beta}_{k,p} F_{k,t+1} \right) = e_{p,\eta,t+1} - \sum_{k=1}^{\eta} \left(\hat{\beta}_{k,p} - \beta_{k,p}\right) F_{k,t+1} \tag{10}$$

where $\hat{\beta}_{k,p}$ are estimated betas from regression (9).

Suppose a factor *j* belongs to the η-factor model but is not in the true K-factor model. Equation (10) considers the expected portion of the factor j, i.e. $\hat{\beta}_{j,p} \times E\left[F_{j,t+1}\right]$, as a part of fund p's expected return while it is not under the true model. Suppose a factor κ belongs to the true K-factor model, but is not included in the η-factor model. Then Equation (10) would mistakenly leave out the term $\beta_{\kappa,p} \times E\left[F_{k,t+1}\right]$ when it computes alpha. The net effect of the omitted factors and the non-priced factors is the model misspecification error $\theta_{p,\eta}$.

1.3.1 No measurement error in betas

What are the factors that determine $\sigma^2_{\varepsilon_\eta}$? As a starting point, suppose betas are estimated without error. In that case, the estimation error in $\hat{\alpha}_{p,\eta}$ in Equation (10) equals fund-specific return $e_{p,\eta}$ from Equation (9). Therefore:[4]

$$\sigma^2_{e_{\eta,p}} = \sigma^2_{r_p}\left(1 - R^2_{adj,p}\right) \tag{11}$$

where $\sigma^2_{r_p}$ is the variance of fund returns and $R^2_{adj,p}$ is the adjusted-R^2 for regression (9). For ease of exposition, also suppose R^2_{adj} is the same for all funds and suppose $e_{p,\eta}$ is uncorrelated across funds.[5] Then, $\sigma^2_{\varepsilon_\eta}$ is the cross-sectional average of $\sigma^2_{e_{p,\eta}}$ under these assumptions. Therefore,

$$\sigma^2_{\varepsilon_\eta} = \overline{\sigma^2_{e_{\eta,p}}} = \left(1 - R^2_{adj}\right) \overline{\sigma^2_{r_p}} \tag{12}$$

[4] Recall that $\varepsilon_{\eta,p}$ is the alpha estimation error in Equation (5) and $e_{\eta,p}$ is the fund specific return in Equation (9).
[5] In our simulations, $e_{p,\eta}$'s are correlated across funds and we allow R^2_{adj} to vary across funds.

where the overbar indicates average across funds.

1.3.2 Measurement error in betas

Measurement errors in betas would also affect the choice of variables that one would include in the time-series regression (9). For instance, a factor that may marginally increase R^2_{adj} of the regression may still not be desirable if the measurement error in beta with respect to that factor increases the alpha estimation error. This issue is particularly important if that factor is correlated with other factors in the regression because the addition of that factor would increase the measurement errors of other factor betas as well.

There are two potential sources of measurement error in betas. Even if true betas were constant, beta estimates using regression (9) would contain statistical estimation errors. Additionally, fund betas would vary over time because individual stock betas may be time-varying and active funds typically revise their portfolios over time. Therefore, the difference between the true betas at time $t+1$ and the average beta during the estimation period would add to the measurement error in betas. For these reasons, investors could optimally omit some factors from regression (9).

1.4 CAPM vs. No-beta risk model

This subsection considers an example that illustrates the contribution of $\sigma^2_{\varepsilon_\eta}$ and $\sigma^2_{\theta_\eta}$ to precision of the alpha estimates. Suppose asset returns are generated by the following single factor model:

$$r_{p,t} = E[r_p] + \beta_p \times f_t + \xi_{p,t}. \tag{13}$$

Expected returns are determined by one of the following two models:

i. No-beta risk model: The expected returns on all stocks are equal, i.e.

$$E[r_p] = E[r_m] \;\; \forall\, p \tag{14}$$

where $E[r_m]$ is the expected return on the market portfolio.

ii. CAPM:

$$E[r_p] = r_f + \beta_p(E[r_m] - r_f) \tag{15}$$

Consider the following two Estimators of alpha:

- Market adjustment (No-beta risk model):

$$\hat{\alpha}_{p,0} = r_{p,t} - r_{m,t} \tag{16}$$

- Market model adjustment (CAPM):

$$\hat{\alpha}_{p,1} = r_{p,t} - [r_f + \hat{\beta}_p(r_{m,t} - r_f)] \tag{17}$$

where $\hat{\beta}_p$ is computed using market model regression.

The variance of measurement errors of $\hat{\alpha}_{p,0}$ and $\hat{\alpha}_{p,1}$, which include both model misspecification error and statistical estimation error are tabulated below (Appendix 1 presents the derivations):

	Alpha Estimator	
True model:	a. Estimated with Mkt Adj. (Eq. 16)	b. CAPM (Eq. 17)
i. No-beta risk	$\sigma_u^2 \| r_{m,t} = \sigma_\beta^2 (r_{m,t} - E[r_m])^2 + \sigma_\xi^2 \| r_{m,t}$	$\sigma_u^2 \| r_{m,t} = \sigma_\beta^2 E(r_m)^2 + \sigma_{\hat{\beta}-\beta}^2 r_{m,t}^2 + \sigma_\xi^2 \| r_{m,t}$
ii. CAPM	$\sigma_u^2 \| r_{m,t} = \sigma_\beta^2\, r_{m,t}^2 + \sigma_\xi^2 \| r_{m,t}$	$\sigma_u^2 \| r_{m,t} = \sigma_{\hat{\beta}-\beta}^2 r_{m,t}^2 + \sigma_\xi^2 \| r_{m,t}$

The variables in the table above are:

Variables	Definition
$\sigma_u^2 \| r_{m,t}$	Variance of total measurement error conditional on the realization of market return i.e. $\sigma_u^2 \| r_{m,t} = \sigma_{\hat{\alpha}-\alpha}^2 \| r_{m,t}$
σ_β^2	Variance of true beta across funds.
$\sigma_{\hat{\beta}-\beta}^2$	Variance of measurement error across funds both due to the standard error of regression estimates and also due to time-variation in beta.
$\sigma_{\hat{\beta}}^2$	Variance of $\hat{\beta}_p$ across funds = $\left(\sigma_\beta^2 + \sigma_{\hat{\beta}-\beta}^2\right)$

$\sigma_{\xi}^{2} \vert r_{m,t}$	Variance of fund specific returns (assumed to be the same for all funds for expositional convenience) conditional on the realization of market return

The results in the above table illustrate the factors that contribute to total measurement error and the inherent trade-offs. For example, the term in cell (i)(b) can be grouped as:

$$\sigma_{u}^{2}|r_{m,t} = \underbrace{\sigma_{\beta}^{2}E[r_{m}]^{2}}_{Model\ Misspecification\ Error} + \underbrace{\sigma_{\hat{\beta}-\beta}^{2}r_{m,t}^{2} + \sigma_{\xi}^{2}|r_{m,t}}_{Estimation\ Error} \tag{18}$$

The first term in this expression is variance of model misspecification error, which arises because of using market model adjustment in equation (17) when ‘no-beta risk’ model is true. The last two terms are due to statistical estimation error.

To consider the trade-offs between model misspecification error and estimation error, consider the last row where CAPM is true. The variances of estimation errors in alpha using Equations (16) and (17) are given in the last row of the table, and they both contain the term σ_{ξ}^{2}. The variance of alpha estimated with Equation (16) contains the additional term $\sigma_{\beta}^{2}r_{m,t}^{2}$, which is the cross-sectional variation of true fund beta, and that with Equation (17) contains the term $\sigma_{\hat{\beta}-\beta}^{2}$, which is the variance of measurement error in beta. If the beta estimates are sufficiently noisy (i.e. big $\sigma_{\hat{\beta}-\beta}^{2}$) or if differences in betas across funds are small, then the variance of measurement error with Eq. (16) could be smaller than with estimator (17). In this case, we can infer from equation (7) that the slope coefficient b_{η} in equation (4) would be bigger for the market adjusted $\hat{\alpha}$ from estimator (16) compared to the market model $\hat{\alpha}$ from estimator (17). In other words, estimate of alpha using Eq. (16) would win out in a horse race of slope coefficients against alpha estimated with Eq. (17) even when CAPM is true (a counterexample to the underlying assumption in BvB), and even if investors were truly sophisticated (a counterexample to the underlying assumption in BHO because investors optimally do not use all factors in the return generating process). Of course, this is only an illustrative example, and we should empirically examine the true parameters to understand what we can learn from the horse races.

2. A comparison with prior literature

Our analysis in the last section assumes a linear relation between investors' signals and their fund flows. Suppose the aggregate investor receives an unbiased signal $\propto_p^{aggregate}$ about the skill of fund p. Let:

$$\propto_p^{aggregate} = \alpha_p^K + u_p.$$

Suppose the aggregate flow to fund p, Γ_p, is monotonically related to $\propto_p^{aggregate}$, but we would like to keep the functional relation as not necessarily linear. For example, it is possible that the relation between alpha and flow is piecewise linear with a steeper slope for positive alpha and a flatter slope for negative alpha. To allow for a more general relation, one may prefer a test specification based on the signs of alpha and flow as implemented by BvB than a linear regression. BvB transform flows and alpha estimates to binary variables and run the regression with these transformed variables. Specifically, the transformed binary variables are defined as follows:

$$Q_x = \begin{cases} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{cases} \tag{19}$$

where x is any random variable. BvB run the following OLS regression:

$$Q_{\Gamma_p} = A_\eta + B_\eta \times Q_{\hat{\alpha}_{p,\eta}} + o_{p,\eta}, \tag{20}$$

and compare $\hat{B}_\eta$. To relate our analysis based on Regression (4) to that based on Regression (20), we first establish the following propositions:

Proposition 1: Let $\hat{\alpha}_{p,\eta_1}$ and $\hat{\alpha}_{p,\eta_2}$ be the alphas computed with respect to η_1- and η_2-factor models using Equation (10). $\hat{b}_{\eta_1}$ and $\hat{b}_{\eta_2}$ are the corresponding Regression (4) slope coefficients and $\hat{B}_{\eta_1}$ and $\hat{B}_{\eta_2}$ are the corresponding Regression (20) slope coefficients. Assume the following:

a) α_p^K, $\hat{\alpha}_{p,\eta_1}$ and $\hat{\alpha}_{p,\eta_2}$ are normally distributed with mean equal to zero. [6]
b) $Cov(\Gamma_p, v_{p,\eta})$=0.

[6] We assume normality distributions for analytic convenience, but we can show that this proposition holds if we assume symmetric distributions.

If $\hat{b}_{\eta_1} > \hat{b}_{\eta_2}$ then $\hat{B}_{\eta_1} > \hat{B}_{\eta_2}$, when the number of funds in the sample is sufficiently large.

Proof: See Appendix 2.

Corollary: Under the assumptions of Proposition 1, the ordering of multifactor models based on the slope coefficients of Regression (4) and Regression (20) are identical.

Proposition 2: Let $\hat{\alpha}_{p,\eta_{max}}$, the estimate of alphas with respect to η_{max}-factor model, be the most precise estimate of true alphas. Assume the following:

a) α_p^K and $\hat{\alpha}_{p,\eta}\ \forall\, \eta$ are normally distributed with mean equal to zero.
b) Investors are rational

The biggest slope coefficients in Regressions (4) and Regression (20) are $\hat{b}_{\eta_{max}}$ and $\hat{B}_{\eta_{max}}$, respectively, when the number of funds in the sample is sufficiently large.

Proof: See Appendix 2.

As discussed earlier, in rational expectations economy, the covariance between flow and measurement error is the biggest for the most precise estimator of alphas. Therefore, we can show that the same estimator would win horse race based on regressions (4) and (20).[7]

BvB show that the winner of the $\hat{B}_\eta$ horse race is the closest to the true asset pricing model under the assumption that "if a true risk model exists, any false risk model cannot have additional explanatory power." BvB note this assumption "rules out the possibility that ε_{it}^c contains information about managerial ability that is not also contained in ε_{it}." In the context of our model and the notations that we use, this assumption is equivalent to an assumption that alpha estimated with the true asset pricing model (i.e. $\hat{\alpha}_{p,K}$ estimated using the K-factor model) is a more precise estimate of true alpha than alphas estimated with respect to any other multifactor model $\left(\text{i.e. } \hat{\alpha}_{p,\eta}\ \forall\, \eta \neq K\right)$. Intuitively, a more precise estimate of a quantity, true $\alpha_{p,K}$ in this context, is more informative than a less precise estimate of the same quantity.

Equation (5) shows that model misspecification error is only one source of estimation error but the precision of alpha estimates is determined by the cumulative effect of model

[7] We cannot determine the ordering of $Cov\left(\Gamma_p, v_{p,\eta}\right)$ across other η-factor models by appealing to the rational expectations hypothesis and hence the ordering of $\hat{b}_\eta$ and $\hat{B}_\eta$ need not be identical.

misspecification error and statistical estimation error. It is possible that the cumulative effect may be smaller for alpha with respect to some other η-factor model than that with respect to the true model if smaller estimation error offsets the adverse effects of model misspecification error. The particular model that yields the most precise estimates of alpha depends on the return generating process and magnitudes of risk premiums, which should be empirically determined and is not a matter of assumption. Comparison of fund flow-alpha relations can yield reliable inferences about the true asset pricing model only if alphas estimated with the true model are the most precise estimate of true alphas.[8]

The analysis in the last section also helps us set a benchmark to evaluate investor sophistication. BHO argue that sophisticated investors should consider all priced and unpriced common factors to obtain the most precise estimates of alpha. However, Equation (5) shows that inclusion of some of the factors may decrease precision of alpha estimates. For example, some factors may actually increase alpha measurement error because of their small incremental contribution to R^2_{adj} relative to the contribution of measurement errors in betas with respect to them. Therefore, it is important to evaluate how these sources of measurement error affect the overall precision of alpha estimates to set benchmarks.

Overall, this subsection shows that under some assumptions we can draw reliable inferences about asset pricing models based on comparisons of flow-alpha relation. These are assumptions about the empirical distribution of returns which may not hold in the data. Unless those assumptions are empirically validated, any inferences about asset pricing models based on flow-alpha comparisons are not tenable. Also, the results show that the benchmark model for assessing investor sophistication may not be the J-factor model. Regardless of the underlying model, model error and estimation error would always be present in practice and their relative importance can only be empirically assessed.

[8] BvB argue that their inferences about asset pricing models based on fund flow-alpha comparisons apply to anyone who could potentially invest in mutual funds, which likely includes all investors and the entire economy. They present several justifications for this argument. However, BHO argue that the asset pricing model inferences based on such tests apply only to mutual fund investors and they "do not inform us about the beliefs of non-mutual fund investors." Our concern is entirely different. We show that unless BvB's foundational assumption holds in the data, their findings are not informative even about mutual fund investors' beliefs about the true asset pricing model.

3. Simulation Experiment

The hypotheses that comparison of flow-alpha relation is a test of asset pricing models and that investors are not sophisticated if they do not use a J-factor model for alpha estimation can be stated as:

Suppose the true asset pricing model is a K-factor asset pricing model and returns are generated by a J-factor model. Among all η-factor model regressions, where $0 \leq \eta \leq J$, let the biggest slope coefficient in regression (4) obtain when η=M i.e. when $\hat{\alpha}_{p,M}$ is computed with respect to an M-factor model.

A1. Asset Pricing test hypothesis: The model that yields the biggest correlation is the true asset pricing model, i.e. M=K.

A2. Investor Sophistication hypothesis (BHO): The most accurate model is the J-factor model that generates asset returns, i.e. M=J.

We can test them against the null that these hypotheses are not true.

3.1 Experimental design

The analysis in Section 1 shows that the flow-alpha relation is a monotonic function of the accuracy of alpha estimates, which in turn broadly depends on the following factors: (i) extent to which various factor models explain fund returns (i.e. model R^2_{adj}), (ii) beta estimation error $\left(\sigma^2_{\hat{\beta}-\beta}\right)$, (iii) variation of betas across funds $\left(\sigma^2_{\beta}\right)$ and (iv) the "true" asset pricing model. We can estimate the first three items from the data but we do not know the "true" asset pricing model. Therefore, in our first set of tests we estimate the first three items from the data and use these parameters generate simulated returns under each asset pricing model.

We generate fund returns using the following seven-factor model:

$$r_{p,t} = \alpha_p^{true} + E^{model}(r_p) + \beta_{p,m} \times \left(\widetilde{mkt - r_f}\right)_t + \beta_{p,smb} \times \widetilde{SMB}_t + \beta_{p,hml} \times \widetilde{HML}_t + \beta_{p,umd} \times \widetilde{UMD}_t + \beta_{p,ind1} \times \widetilde{IND1}_t + \beta_{p,ind2} \times \widetilde{IND2}_t + \beta_{p,ind3} \times \widetilde{IND3}_t + \epsilon_{p,t} \quad (21)$$

where $(\widetilde{mkt - rf})_t$ is the market factor, which equals market excess return in month *t* minus the sample mean of market excess return during the January 1990 to June 2017 sample period. The other factors are analogously defined based on Fama-French SMB and HML factors and Carhart (1996) UMD factor. When K=4, we get the Fama-French-Carhart four-factor model, which we refer to as FFC4. We also consider multifactor models which add three industry factors to FFC4. Following BHO, we construct the three industry factors as the first three principal components of residuals from regressing Fama-French 17 equal weighted industry portfolios on FFC4 factors.

These demeaned factors from the data are the time-series realization of these factors, and we keep these factor realizations the same for all simulations. The demeaned factor realizations are random draws in the data and we retain them in the simulation. Therefore, we ensure that the common factors have identical distribution properties and correlation structure as that in the data.

We generate the other components for returns as follows:

a. **Betas**: We randomly generate the seven factor betas for each fund from a normal distribution with means and standard deviations equal to the parameters tabulated in Table 4. Each factor beta is drawn independently and is constant over the entire sample period.
b. **Fund specific return**: We generate monthly fund specific return $\epsilon_{p,t}$ for each fund from a normal distribution with mean zero and standard deviation equal to 1.8%.
c. **Alpha**: The parameter α_p^{true} is the manager's skill which we keep constant over the entire sample period. We randomly draw α_p^{true} for each fund from a normal distribution with mean 0 and standard deviation of 0.2% per month.[9]
d. **Asset pricing model and expected returns**: Steps (a) through (c) describe the return generating process for the funds and this process does not vary with the asset pricing model. However, different common factors that are priced vary across asset pricing models and hence different asset pricing models imply different expected return for each fund. The term $E^{model}(r_p - r_f)$ is the "true" expected excess return and it depends on the model. We conduct simulations under three asset pricing models and expected excess returns under each model are computed as follows:

[9] The monthly cross-sectional variance of $\hat{\alpha}$s in the real data is the variance of true alphas plus the measurement error of alphas. The measurement error variance in $\hat{\alpha}$s is the squared OLS standard errors from the time-series regressions used to estimate alphas. The average standard deviation of the difference across models is roughly 0.2% per month.

- No-beta risk model: $E^{NR}(r_p - r_f) = 0.699\%$,
- CAPM: $E^{CAPM}(r_p - r_f) = \beta_{p,m} \times (\overline{mkt - r_f})$, (22)
- Fama-French three factor model (FF3): $E^{FF3}(r_p - r_f) = -0.016\% + \beta_{p,m} \times (\overline{mkt - r_f}) + \beta_{p,smb} \times (\overline{SMB}) + \beta_{p,hml} \times (\overline{HML})$.

The overbars above common factor returns indicate sample means. The constant in the equation for each model is chosen so that the average fund returns equal sample average of market excess returns. Although we use UMD and industry factors to generate returns, we do not use them as part of asset pricing models in the simulation.

e. **Fund return**: For each fund, we generate return for month *t* using Equation (21)

f. **Fund flow:** For each month, we compute flows into fund P as:

$$flow_{p,t} = a + b \times \alpha_p^{true} + \psi_{p,t} \tag{23}$$

where a = -0.003 and b = 2.5. We draw $\psi_{p,t}$ from a normal distribution with mean zero and standard deviation of 0.09 (9%). All these parameters match the corresponding parameters in the actual data.

We start the simulation with the same number of funds equal to that on January 1990 in the data and we match the entry and exit of funds in the simulation to that in the actual data. When a fund exits the sample in the data, we select a fund from the simulated sample that started on the same date as the exiting fund and remove it from simulated data. Therefore, the number of funds in the simulated sample exactly matches the real sample each month. We repeat the simulations 100 times.

3.2 Data

We estimate the parameters that we use in the simulation with the sample of funds in the CRSP survivor-bias free mutual fund database. Our sample includes all actively managed domestic equity funds in the January 1990 to June 2017 sample period. Our sample is comprised of all actively managed domestic equity funds. CRSP identifies these funds with objective codes ‘EDC’ and ‘EDY.’ When a fund has multiple share classes, we add assets in all share classes to compute its TNA and we compute fund level return as the weighted average of returns of individual share classes. The sample for month t includes all funds with at least \$10 million assets under management as of the end of month *t-1*. We follow BHO and exclude funds that had flows smaller than -90% or greater than 1000% in any month from the sample to avoid the effect of outliers. The sample for month *t* includes only funds that have returns data in all months from *t*-61 to *t*-1 to estimate betas.[10]

Table 1 presents the summary statistics for the funds in our sample. The sample is comprised of 1224 funds per month on average. The average monthly fund flow into a fund is 0.25% of its TNA the previous month. Around half of the funds in the sample have either an entry or exit load.

3.3 R^2_{adj} and beta measurement error: A first look

Before we proceed with the simulation, we take a first look at some of the determinants of the accuracy of alpha estimates. One important determinant is R^2_{adj} for Regression (9). For each model, we fit Regression (9) each month *t* using data for each fund from months t-60 to t-1. We then compute average R^2_{adj} for each model, and Table 2 reports the time-series averages. For the market model, we compute R^2_{adj} as $1-\left(\sum\left(r_{p,t}-r_{mkt}\right)^2/\sum\left(r_{p,t}-\bar{r}_p\right)^2\right)$.

Market-adjusted returns have the lowest R^2_{adj} of .774. The R^2_{adj} for the single factor market model is bigger at .820. R^2_{adj} jumps to .892 for the Fama-French three-factor model, but the increase is fairly gradual as we go from the Fama-French three-factor model to the Fama-French-Carhart model with three industry factors.

For the consumption CAPM, we use the following approximate beta-linear representation for expected return from Cochrane (2001) [chapter 1, equation (1.16)]:

$$E\left(r_p\right)=r_f+\beta_{p,\Delta c}\times\lambda_{\Delta c} \tag{24}$$

where $\lambda_{\Delta c}=\gamma\times var(\Delta c)$ and $\beta_{p,\Delta c}=\frac{cov(R_p,\Delta c)}{var(\Delta c)}$, Δc is the consumption growth and γ is the risk-aversion coefficient. We set γ=10 as in BvB. We then run the following time-series regression to get R^2_{adj}:

[10] This sample selection criterion excludes funds from the sample during the first 60 months of their existence. Therefore, our sample is not exposed to potential incubation bias that Evans (2010) and Elton, Gruber and Blake (2001) document.

$$\left(r_{p,t} - r_{f,t}\right) = \alpha_p + \beta_p \times (\Delta c_t - E(\Delta c_t) + \lambda_{\Delta c}) + \epsilon_{p,t} \tag{25}$$

where $E(\Delta c_t)$ is the unconditional sample mean of consumption growth. R^2_{adj} for CCAPM is .027, which is smaller than that for market-adjusted returns. Because R^2_{adj} is fairly small, even if CCAPM is true, estimation error in α estimated using Regression (25) may offset any gain in precision due to potential elimination of model misspecification error.

Another important component in the measurement error of $\hat{\alpha}$ is the variance of measurement error in betas across funds $\left(\sigma^2_{\hat{\beta}-\beta}\right)$. The term $\sigma^2_{\hat{\beta}-\beta}$ would differ from the time series variance of OLS estimation error in regression (4) for two reasons. First, if the fund-specific returns are correlated across funds, then the average variance of time-series errors will not equal $\sigma^2_{\hat{\beta}-\beta}$. Secondly, as we discussed earlier the OLS estimates are unbiased estimates of mean betas during the estimation periods and any difference between this average and the realized beta in month *t+1* is an additional source of measurement error.

To estimate the magnitude of this error we first estimate the following regressions for each fund for each month:

$$\begin{aligned} \left(r_{p,\tau} - r_{f,\tau}\right) &= \alpha^{past}_{p,k,t} + \beta^{past}_{p,k,t} F_{k,\tau} + e_{p,k,\tau} \qquad && \tau = t - 60 \text{ to } t - 1, \\ \left(r_{p,\tau} - r_{f,\tau}\right) &= \alpha^{future}_{p,k,t} + \beta^{future}_{p,k,t} F_{k,\tau} + e_{p,k,\tau} \qquad && \tau = t \text{ to } t + 11 \end{aligned} \tag{26}$$

where $F_{k,\tau}$ is the factor with respect to which betas are estimated. Suppose betas for a particular fund are constant over time.

$$\begin{aligned} \hat{\beta}^{past}_{p,k,t} &= \beta_{p,k} + u^{past}_{p,k,t}, \text{ and} \\ \hat{\beta}^{future}_{p,k,t} &= \beta_{p,k} + u^{future}_{p,k,t} \end{aligned} \tag{27}$$

where $\beta_{p,k}$is fund *p*'s true beta with respect to factor *k*.

Consider the following cross-sectional regression for month *t*:

$$\hat{\beta}^{future}_{p,k,t} = a_t + b_t \times \hat{\beta}^{past}_{p,k,t} + e_{p,t} \tag{28}$$

Since we use non-overlapping sample periods to estimate $\beta_{p,k,t}^{past}$ and $\beta_{p,k,t}^{future}$, $u_{p,k,t}^{past}$ and $u_{p,k,t}^{future}$ are uncorrelated. With a sufficiently large number of funds, the probability limit of the slope coefficient is:

$$\text{plim}\, b_t = \frac{var(\beta_{p,k})}{var(\beta_{p,k}) + var\left(u_{p,k,t}^{past}\right)} \tag{29}$$

Therefore, the slope coefficient of regression (28) is the ratio of the cross-sectional variance of the factor betas divided by the sum of this variance plus the variance of the measurement error. If this slope coefficient is smaller than 0.5 then the variance of true beta is smaller than the variance of measurement error.

We fit regression (28) each month for each of the betas. All betas are estimated using univariate regressions as per equation (26). Table 3 reports the time-series averages of the slope coefficients for each beta. The slope coefficients are all greater than .75 for SMB betas with respect to the three Fama-French factors, but they are less than .5 for UMD and industry factors. Therefore, the variance of measurement error is bigger than the variance of true betas for the latter set of factors.

3.4 Simulation: Tests and results

We first examine the relation between alphas and fund flows under various models. For each η-factor model, we compute betas for each month *t* using the following time series regression:[11]

$$r_{p,\tau} = a_{p,\eta,t} + \sum_{k=1}^{\eta} \beta_{k,p,t} F_{k,\tau} + e_{p,\eta,\tau}, \qquad \tau = t - 60 \text{ to } t - 1 \tag{30}$$

We then compute alpha for month *t* as follows:

$$\hat{\alpha}_{p,\eta,t} = r_{p,t} - \sum_{k=1}^{\eta} \hat{\beta}_{k,p,t} F_{k,t} \tag{31}$$

[11] Since we generate excess returns in simulations, Eq. (30) does not use risk-free rate.

To facilitate comparison with BHO, we examine the relation between month *t* fund flows and alphas computed as an exponentially weighted average of past alphas from Equation (31) over the prior 18 months. Specifically, we compute weighted alpha for month *t* as:

$$\hat{\alpha}_{p,\eta,t}^{exp} = \frac{1}{10.843} \sum_{\tau=1}^{18} e^{-0.0666*(\tau-1)} \times \hat{\alpha}_{p,\eta,t-\tau} \tag{32}$$

We then fit the following cross-sectional regression for each model:

$$flow_{p,t} = a_{\eta,t} + b_{\eta,t} \times \hat{\alpha}_{p,\eta,t}^{exp} + e_{p,\eta,t} \tag{33}$$

Although the illustrative example we discuss in the last section uses one-month alpha estimates using Equation (10), the general results we derive apply with any unbiased estimator of alpha. In untabulated tests, we find qualitatively similar results when we use one-month alphas or their 12-month backward moving averages in Regression (33) instead of exponentially weighted alphas.

Table 5 reports the time-series averages of the slope coefficients and the standard errors computed across 100 simulations. Panel A reports the results when the true model is the no-beta risk model where the expected returns for all stocks are equal. The slope coefficient with market adjusted alpha is .193. In comparison, the slope coefficient with CAPM is slightly bigger at .196 although the true model is the no-beta risk model. The slope coefficients in Table 5 monotonically increase as we add more factors in the first stage regression. Column (7) in the table presents the slope coefficients in a multivariate version of regression (33) with all alphas on the right hand side. Here again the model with the most number of factors wins the horse race.

CAPM and Fama-French three-factor model are the true asset pricing models in Panels B and C respectively. We find similar results in both panels to those in Panel A. The slope coefficient on CAPM alpha is .196 in Panel A, where CAPM is not the true model and it is .202 in Panel B, where CAPM is the true model. These results indicate that the relation between alpha and flow is not particularly sensitive to the true asset pricing model.

To assess the economic significance of the differences across models, we examine the differences in skill level among the top and bottom deciles of the funds identified by each model. In each month t we rank funds based on their $\hat{\alpha}_{p,\eta,t}^{exp}$'s for each η-factor model and form equal-weighted portfolios of the funds in the extreme deciles. Panel A of Table 6 presents the average true monthly alphas in various top decile portfolios. We find that the seven-factor model wins this horse race as well, regardless of the true asset pricing model. For example, when CAPM is the true model, the average skill of funds identified as top decile funds using CAPM alpha is .105% compared that of .127% using seven-factor alpha.

We also conduct BvB tests with simulated data. For these tests, we compute betas for each fund over the entire life of the fund. We compute alphas for rolling windows of lengths ranging from three months to four years using full sample betas. We then estimate Regression (20) with alphas computed over various windows as explanatory variables and flows within the windows as dependent variables.

Table 7 reports the results for BvB tests. The table reports 100×(1 + slope coefficient)/2. The transformed estimate will equal 100, 0 or 50 when the correlation is +1, -1 or 0, respectively. Irrespective of the true asset pricing model, the seven-factor model wins at all horizons, indicating that the effect of bigger R_{adj}^2 more than offsets other sources of measurement errors.

The main take away from this section is that correlation between alphas with respect to various models and fund flows cannot be used to reliably assess which model is the true asset pricing model. In fact, the true asset pricing model has a minimal impact on which model wins the race and the true model is never the winner of the horse race. The results indicate that alpha with respect to the multi factor models that includes all factors, even ones that do not command a risk premium, almost always comes out the winner regardless of the true asset pricing model.

The results support BHO's argument that investors should use all factors including ones that do not command a risk premium to get the most efficient alpha estimate. However, this result should be treated with caution. Our simulation assumes constant betas for all funds but in reality fund betas vary over time. Any time-variation in betas would adversely affect the precision of alpha estimates and some of the factors with marginal incremental explanatory power may not be helpful in such situations. We are currently conducting simulations with time-varying betas to examine this issue.

4. Predicting Alpha

Our tests so far examine the relation between past alphas and fund flows. However, fund flows reflect investor expectations about future alphas and investors are interested in a fund's past alpha only to the extent that it is useful in predicting future alphas. How well do past alphas computed relative to various multifactor models predict future alphas? Does the ability of these alphas to predict future alphas depend on the underlying true asset pricing model? This section addresses these questions.

The focus of our tests of alpha predictability differs from that in a number of papers that examine performance persistence of funds. For example, Hendricks, Patel, and Zeckhauser (1993), Carhart (1996), Busse, Goyal and Wahal (2010) and others examine whether funds that earn abnormal returns in one period continue to earn abnormal returns in the same direction in the future. The primary focus of their tests is on evaluating whether superior fund performance is due to skill or luck. In contrast, our tests examine which factor model performs the best in identifying skill and whether the results are sensitive to assumptions about the underlying asset pricing models.

We examine the predictability of alphas over horizons of one, six and 12 months. Our first set of tests runs the following regressions of future alphas on past alphas:

$$\hat{\alpha}_{p,\eta,t}^{future}(T) = a_{\eta,t} + b_{\eta,t} \times \frac{\hat{\alpha}_{p,\eta,t}^{past}}{\sigma_{\hat{\alpha}(\eta,t)}} + e_{p,\eta,t}, \tag{34}$$

where $\hat{\alpha}_{p,\eta,t}^{future}(T)$ is the future alpha of fund p over a holding period T, computed using an η-factor model. To compute $\hat{\alpha}_{p,\eta,t}^{future}(T)$, we first compound monthly fund and factor returns over the holding period. We then compute T-period alphas starting from month t as:

$$\hat{\alpha}_{p,\eta,t}^{future}(T) = \left(\left(\prod_{\tau=t}^{t+T-1}(1+r_{p,\tau})\right)-1\right) - \left(\left(\prod_{\tau=t}^{t+T-1}(1+r_{f,\tau})\right)-1\right) - \sum_{k=1}^{n}\left(\hat{\beta}_{p,k,t}^{future}\left(\left(\prod_{\tau=t}^{t+T-1}(1+F_{k,\tau})\right)-1\right)\right) \tag{35}$$

We estimate $\hat{\beta}_{p,k,t}^{future}$ in Equation (35) for each fund p using corresponding multifactor time-series regressions with data from month $t+12$ through $t+47$. If fund p drops out of the sample at any point during months t through $t+47$, we replace the missing returns with value-weighted index returns to avoid a look-ahead bias. Effectively, we assume that the proceeds from investing in the fund are invested in the market if the fund drops out during this period.[12]

For this set of tests, we also compute alphas with respect to funds' self-declared benchmarks. We compute benchmark-adjusted alphas as the difference between *T*-period fund returns and contemporaneous benchmark index returns.[13] We compute market-adjusted alphas similarly, using value-weighted index as benchmark for all funds.

Past alphas in Regression (34) are 18-moth exponentially weighted alphas computed using equation (32). Equation (32) uses betas estimated over the pre-holding period to compute past alphas. Therefore, the sample period used to estimate betas used in computing past and future alphas do not overlap. We use these non-overlapping sample periods for estimation to avoid spurious correlation due to common measurement errors in betas. We scale $\hat{\alpha}_{p,\eta,t}^{past}$ by $\sigma_{\hat{\alpha}(\eta,t)}$, the standard deviation of $\hat{\alpha}_{p,\eta,t}^{past}$ across funds in month *t*, to standardize the independent variables across models and facilitate direct comparisons of the slope coefficients across models. We do not standardize $\hat{\alpha}_{p,\eta,t}^{future}(T)$ because this variable does not change across models.

Table 8 presents the slope coefficients in Regression (34) estimated using the Fama-MacBeth procedure and standard errors computed using the Newey-West procedure with lag lengths equal to *T*-1. For T=1, market-adjusted alpha has the biggest slope coefficients under all asset pricing models, except the CAPM. The market model alpha has the biggest coefficient when future alphas are computed under CAPM.

For T=12, market-adjusted past alpha has the biggest slope coefficients under market-adjusted, FF3 models while Fama-French-Carhart four-factor model ("FFC4") has the biggest slope coefficients under the CAPM and under benchmark adjusted model. In fact, the slope coefficient on market-adjusted alpha is not statistically significant when future alphas are measured with respect to the benchmark adjusted model or CAPM. In contrast, the FFC4 slope coefficient is significant for future alphas computed with respect to all models.

The standard errors of the slope coefficients with FFC4 alphas are smaller than that with market-adjusted alphas in all four columns. Therefore, even when the market-adjusted model has bigger slope coefficients, using FFC4 alphas improves the precision of the estimates. The

[12] We find qualitatively similar results if we exclude funds that drop for any month during month *t* through *t*+47 from the sample.
[13] We thank Martijn Cremers for providing us with benchmark returns data.

standard errors for alphas with FFC4 model augmented with industry factors are also smaller than that with the market-adjusted model. However, the addition of industry factors does not always result in a decline in standard errors. For example, for T=12 under FF3 model, the standard error with FFC4 is .236, compared with that of .253 when FFC4 is augmented with the three industry factors.

To assess the economic significance of alpha predictability, we examine the future performance of portfolios of mutual funds formed based on the same set of past alphas. Table 9 reports the alphas for the top- and bottom decile funds and for the top minus bottom decile funds. For T=1, market adjusted alphas produce the biggest top minus bottom decile spread under no beta risk, CAPM, and FF3 models. Benchmark-adjusted alpha wins out in the other case.

For T=12, we find the biggest slope coefficient with past market-adjusted alphas in two instances but the coefficients are not statistically significant in both cases. The slope coefficients are significant for future alphas under all asset pricing models only for FFC4 past alphas. The standard errors of the slope coefficient with FFC4 alphas are also smaller than that with market-adjusted alphas. For example, the standard errors under the no-beta risk model are 2.20 and .978 for market adjusted and FFC4 past alphas.[14]

Overall, the results in this section do not support the idea that the multifactor model best suited to predict future alphas are the same as what may be the true asset pricing model. For example, even if investors are truly concerned about CAPM alphas in the future, they would optimally use past alphas based on FFC4 for their prediction and hence flows should be more highly correlated with FFC4 alphas.

Also, the results here indicate that FFC4 alphas predict future performance better than FFC4 plus one or three industry factors. The precision of the slope coefficients in Table 8 are also about equal when we use alphas computed with the FFC4 model plus industry factors. Therefore, sophisticated investors may also not necessarily use all priced and unpriced factors to compute alphas for fund selection.

[14] The standard errors for the top decile and bottom decile portfolios are also smaller when funds are sorted based on past FFC4 alphas than on market-adjusted alphas. For example, the ratio of standard errors when sorted based on FFC4 alphas to that sorted based on market-adjusted alphas is 84% for the top decile compared with 43% for the top minus bottom decile. Sorting based on FFC4 alphas results in more closely matched factor betas for top and bottom decile funds and therefore as a result the top minus bottom decile funds have smaller systematic risk with this sorting.

5. **Conclusion**

Investors reveal their preferences for mutual funds through investments in or withdrawals from them. Since non-satiated investors prefer more abnormal returns than less, investors' fund flows reveal their views on abnormal returns that they can earn from their investments. Because flows reveal investors' perceptions, the recent literature has proposed that a comparison of relations between fund flows and alphas measured with respect to a number of models can be used to identify the best asset pricing model.

We show analytically that we cannot draw reliable inferences about asset pricing models based on such comparisons without critically important additional assumptions. In particular, we show that for such comparisons to be valid asset pricing tests we should assume that model misspecification error dominates statistical estimation error when alphas are measured with respect to various models. To examine whether this is a tenable assumption, we estimate the relevant parameters from the data and run simulation experiments under various "true" asset pricing models. Our simulation evidence indicates that the alphas measured with respect to the "true" asset pricing model are not the most precise estimates. For example, even if CAPM is the true asset pricing model, alphas measured with respect to CAPM are less precise than alphas measured with respect to other models. Therefore, comparisons of alpha flow relations cannot be a reliable test of asset pricing models.

We also compare the predictability of future alphas based on past alphas computed with respect to various multifactor models. Here again we do not find that past alpha computed with respect to a particular asset pricing model is the best predictor of future alphas computed under the same model. These results also support our simulation results that even if, say CAPM is the true asset pricing model, investors need not use past CAPM alphas to identify funds in which they would invest.

Our simulation experiment finds that alphas estimated with respect to a seven-factor model that includes the market, SMB, HML and UMD factors augmented with three industry factors are the most precise estimates regardless of the "true" asset pricing model. However, our empirical tests find that alphas estimated with respect to a four-factor model with market, SMB, HML and UMD factors predict future alphas under all asset pricing models better than the seven-factor model.[15] Therefore, sophisticated investors would likely use only some of the factors in the return generating process to estimate alphas that guide their investment decisions.

[15] The difference between the simulation and empirical results is likely due to the fact that we assume fund betas are constant in our simulation, but they may change over time in real data because of fund turnover.

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Table 1: Summary statistics

This table presents the summary statistics for the sample of funds included in the sample. The number of fund-month observations is 404,042. The table first computes the respective statistics across funds each month and reports the averages over the entire sample period. The sample period is from January, 1990 to June, 2017.

	Mean	Std. Dev.	Median
Number of funds each month	1224		
Flow (%)	0.25	10.8	-0.42
TNA ($ mn)	1120.4	4507.4	223.6
Age (months)	376.8	306.6	299.2
Expense Ratio (%)	1.22	0.45	1.19
Load Dummy	0.49	0.50	0
Ret. Volatility (t-1,t-12)	4.7	2.3	4.2

Table 2: Factor model R^2

This table fits the following regression:

$$\left(r_{p,t} - r_{f,t}\right) = \alpha_{p,\eta} + \sum_{k=1}^{\eta} \beta_{k,p} F_{k,t} + e_{p,\eta,t}.$$

Where $r_{p,t}, r_{f,t}$ and $F_{k,t}$ are fund return, risk-free rate and realization of factor k in month t, respectively. For each month t, the regression is fitted from $t-60$ to $t-1$. The table reports the cross-sectional averages of time-series means of adjusted R^2 of the OLS regressions under each model. For market-adjusted and benchmark-adjusted returns we compute this metric as $1 - (\sum(r_{it} - r_{mkt})^2 / \sum(r_{it} - \bar{r}_i)^2)$, $1 - \left(\sum\left(r_{it} - r_{b/m}\right)^2 / \sum(r_{it} - \bar{r}_i)^2\right)$ using full sample of returns for each. Benchmark is the fund benchmark identified by Cremers and Petajisto (2009). For CCAPM, we use an approximate beta-linear as $E(R_i) = R_f + \beta_{i,\Delta c} \times \lambda_{\Delta c}$ where $\lambda_{\Delta c} = \gamma \times var(\Delta c)$ and $\beta_{i,\Delta c} = cov(R_i, \Delta c)/var(\Delta c)$ where Δc is the consumption growth. The sample period is January, 1990 to June, 2017.

Model	Adj. R^2
Market Adj. Return	0.774
Benchmark Adj. Return	0.870
CAPM	0.820
FF3	0.892
FFC4	0.901
FFC4 + 3 IND	0.910
CCAPM	0.027

Table 3: Measurement Errors in betas

This table reports the slope coefficients from the following cross-sectional regressions:

$$\hat{\beta}_{p,k,t}^{future} = a_t + b_t \times \hat{\beta}_{p,k,t}^{past} + e_{p,t},$$

where for each fund f, $\hat{\beta}_{p,k,t}^{future}$ and $\hat{\beta}_{p,k,t}^{past}$ are estimated using time-series regressions with data from t to t+11, and t-1 to t-60, respectively. All betas are estimated with univariate time-series regressions. The above regression is fitted each month for betas with respect to each factor and the table reports time-series averages of the slope coefficients. Standard errors from the second stage of Fama-MacBeth regressions are adjusted for serial correlation using Newey-West correction with lag length of 11 months. Sample period for these regressions is Jan-1990 to Jul-2016. ***, **, * indicate statistical significance at the 1%, 5%, and 10% levels respectively.

Betas	Average b_t	Std. Err.
Market	0.821***	0.07
SMB	0.876***	0.03
HML	0.765***	0.05
UMD	0.409***	0.08
IND1	0.356***	0.09
IND2	0.362***	0.09
IND3	0.090	0.10

Table 4: Simulation Parameters

This table shows the parameters used in generating simulated returns and flows during 1990-2017. We generate returns using the following seven-factor model:

$$r_{p,t} = \alpha_p + E^{model}(r_p) + \beta_{p,m} \times (\widetilde{mkt - rf})_t + \beta_{p,smb} \times \widetilde{SMB}_t + \beta_{p,hml} \times \widetilde{HML}_t + \quad \beta_{p,umd} \times \widetilde{UMD}_t + \beta_{p,ind1} \times \widetilde{IND1}_t * + \beta_{p,ind2} \times \widetilde{IND2}_t + \beta_{p,ind3} \times \widetilde{IND3}_t + \epsilon_{p,t}$$

where α_p if fund manager skill, the variables under $\sim$ are demeaned realizations of the following factors: market, SMB, HML, UMD, and three industry factors and betas are corresponding factor sensitivities. We draw α, βs and $\epsilon_{p,t}$ from normal distributions with means and standard deviations shown below, which roughly match their respective sample values. We generate monthly flows $flow_{p,t}$ using the following equation:

$$flow_{p,t} = -0.003 + 2.5 \times \alpha_p + \psi_{p,t}.$$

We generate ψ from a normal distribution with mean and standard deviation shown below.

Parameter	Mean	Standard Deviation
α	0	0.2%
β_{mkt}	1	0.17
β_{smb}	0.25	0.35
β_{hml}	0	0.30
β_{umd}	0	0.15
β_{IND1}	0	0.06
β_{IND2}	0	0.06
β_{IND3}	0	0.08
ϵ	0	1.8%
ψ	0	9%

Table 5: Flow-Performance relation in simulated sample

This table presents flow-performance regression results in the simulated sample. Panels A, B and C report the results with true expected returns generated under the market adjusted, CAPM, and FF3 models. The alphas are computed with respect to the models indicated in the first column, and the exponentially weighted alphas from t-18 through t-1 ($\hat{\alpha}_{p,t}^{exp}$) under each model are the independent variables. $\hat{\beta}$s for each month are estimated using the prior month returns. Monthly flow is simulated in the sample as $flow_{p.t} = -0.003 + 2.5 * \alpha_p^{model} + \psi_{p,t}$ which is the dependent variable. The table presents the average value of slope coefficients with flows as the dependent variable and alphas as independent variables across 100 simulated samples.

Panel A: True model is Market Adjusted Return							
Dependent Variable is:	Flow						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Alpha Estimated Using:							
Market Adjusted Ret	0.193***					0.066***	0.071***
	(0.003)					(0.006)	(0.006)
CAPM		0.196***				-0.028***	-0.033***
		(0.003)				(0.007)	(0.007)
FF3			0.262***			0.046***	0.055***
			(0.003)			(0.008)	(0.008)
FFC4				0.288***		0.213***	-0.007
				(0.003)		(0.008)	(0.010)
FFC4+3 IND					0.328***		0.257***
					(0.004)		(0.007)
Adj. R^2 (%)	0.024	0.024	0.032	0.035	0.041	0.037	0.043
Obs	233131	233131	233131	233131	233131	233131	233131

Panel B: True model is CAPM							
Dependent Variable is:	Flow						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Alpha Estimated Using:							
Market Adjusted Ret	0.189***					0.031***	0.029***
	(0.003)					(0.006)	(0.006)
CAPM		0.202***				0.003	0.004
		(0.003)				(0.007)	(0.007)
FF3			0.272***			0.051***	0.062***
			(0.003)			(0.008)	(0.008)
FFC4				0.300***		0.221***	-0.006
				(0.003)		(0.008)	(0.010)
FFC4+3 IND					0.343***		0.268***
					(0.003)		(0.007)
Adj. R^2 (%)	0.023	0.025	0.034	0.037	0.042	0.038	0.044
Obs	233131	233131	233131	233131	233131	233131	233131

Panel B: True model is FF3							
Dependent Variable is:	Flow						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Alpha Estimated Using:							
Market Adjusted Ret	0.188***					0.022***	0.019***
	(0.003)					(0.006)	(0.006)
CAPM		0.202***				0.007	0.009
		(0.003)				(0.007)	(0.007)
FF3			0.276***			0.056***	0.067***
			(0.003)			(0.008)	(0.008)
FFC4				0.304***		0.225***	-0.005
				(0.003)		(0.008)	(0.010)
FFC4+3 IND					0.349***		0.273***
					(0.004)		(0.007)
Adj. R^2 (%)	0.023	0.025	0.034	0.038	0.043	0.038	0.045
Obs	233131	233131	233131	233131	233131	233131	233131

Table 6: Economic significance

This table presents "true" alpha, averaged across 100 simulations, for portfolio of mutual funds formed on the basis of various past $\hat{\alpha}$s. In each month *t* we group funds into portfolios based on exponentially weighted alphas computed for months [t-18, t-1]. Panels A and B report the equal weighted average of true alphas of funds in the top and bottom deciles, respectively.

Panel A: Next Month True Alpha in (%) for Top Decile				
	True Model is:			
	Market Adj.	CAPM	FF3	FFC4+3 IND
Deciles formed on:				
Market Adj.	0.106	0.103	0.103	0.106
CAPM	0.105	0.107	0.107	0.105
FF3	0.116	0.118	0.119	0.116
FFC4	0.120	0.122	0.123	0.120
FFC4 + 3 IND	0.127	0.130	0.131	0.127

Panel B: Next Month True Alpha in (%) for Bottom Decile				
	True Model is:			
	Market Adj.	CAPM	FF3	FFC4+3 IND
Deciles formed on:				
Market Adj.	-0.106	-0.104	-0.104	-0.106
CAPM	-0.105	-0.107	-0.108	-0.105
FF3	-0.116	-0.119	-0.120	-0.116
FFC4	-0.120	-0.123	-0.124	-0.120
FFC4 + 3 IND	-0.127	-0.130	-0.131	-0.127

Table 7: Alphas and Flows: Sign regression

This table reports the results based on the following regression:

$$Q_{\Gamma_p} = A_\eta + B_\eta \times Q_{\hat{\alpha}_{p,\eta}} + \nu_{p,\eta},$$

where $Q_{\Gamma_p} = 1$ if $\Gamma_p \geq 0$, and 0 otherwise, Γ_p is the net flow into fund p. $Q_{\hat{\alpha}_{p,\eta}}$ is defined analogously, and $\hat{\alpha}_{p,\eta}$ is alpha estimated using an η-factor model. Expected returns are generated under the no-beta risk model, CAPM and Fama-French three-factor models in Panels A, B and C. For each model, we estimate $\hat{\beta}$s for each fund using the full sample of returns and use these betas to compute monthly alphas using the following six models: market adjusted, CAPM, FF3, FFC4, FFC4+1 IND, and FFC4+3IND. We then compound these alphas over rolling windows of length specified in the column header, and we compute fund flows within contemporaneous rolling windows. We fit regression with data generated in the simulation experiment described in the text. The table presents 100*(.5+$B_\eta/2$) averaged across 100 simulations.

Panel A: No-beta risk model						
	Horizon					
Model:	3 months	6 months	1 year	2 years	3 years	4 years
Return	50.24	50.50	50.99	51.06	51.21	52.23
Return-R_f	50.24	50.51	50.95	51.23	51.17	51.87
Return-Market	50.44	50.88	51.51	52.61	53.52	54.40
CAPM	50.45	50.88	51.50	52.68	53.48	54.40
FF3	50.53	51.03	51.81	53.03	54.02	54.88
FFC4	50.55	51.07	51.87	53.16	54.19	55.06
FFC4+3 IND	50.60	51.16	52.05	53.52	54.68	55.58
Order of coefficients:						
1st Best	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND
2nd Best	FFC4	FFC4	FFC4	FFC4	FFC4	FFC4
3rd Best	FF3	FF3	FF3	FF3	FF3	FF3

Panel B: CAPM						
	Horizon					
Model:	3 months	6 months	1 year	2 years	3 years	4 years
Return	50.24	50.51	50.97	51.08	51.21	52.34
Return-R_f	50.25	50.51	50.93	51.23	51.19	51.96
Return-Market	50.44	50.87	51.48	52.59	53.43	54.27
CAPM	50.45	50.87	51.55	52.70	53.59	54.51
FF3	50.53	51.05	51.86	53.11	54.15	55.13
FFC4	50.54	51.08	51.91	53.26	54.38	55.37
FFC4+3 IND	50.61	51.17	52.10	53.65	54.86	55.88
Order of coefficients:						
1st Best	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND
2nd Best	FFC4	FFC4	FFC4	FFC4	FFC4	FFC4
3rd Best	FF3	FF3	FF3	FF3	FF3	FF3

Panel C: FF3						
	Horizon					
Model:	3 months	6 months	1 year	2 years	3 years	4 years
Return	50.24	50.51	50.99	51.08	51.20	52.37
Return-R_f	50.25	50.52	50.95	51.21	51.20	51.97
Return-Market	50.44	50.87	51.46	52.55	53.43	54.30
CAPM	50.45	50.88	51.56	52.70	53.58	54.46
FF3	50.53	51.05	51.86	53.09	54.19	55.21
FFC4	50.55	51.08	51.92	53.29	54.42	55.44
FFC4+3 IND	50.61	51.18	52.13	53.67	54.95	56.03

Order of coefficients:						
1st Best	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND
2nd Best	FFC4	FFC4	FFC4	FFC4	FFC4	FFC4
3rd Best	FF3	FF3	FF3	FF3	FF3	FF3

Table 8: Alpha predictability

This table presents the slope coefficients from univariate Fama-MacBeth regressions of future alphas indicated at the top of the column on past alphas shown in the rows. We consider three horizons for future alphas: 1 month (panel A), 6 months (panel B), 12 months (panel C). For each month *t* we use returns in [t-60, t-1] to get backward looking betas and then compute monthly alphas as (return – expected return from the factor model). We then compute the exponentially weighted 18 month backward looking alphas and standardize them by the monthly cross-sectional standard deviation to get the independent variables. We compute forward looking alphas over various horizons as the difference of buy-and-hold returns over the horizon and the expected return from the factor model with factors compounded over the same horizon. We use betas estimated in [t+12, t+47] to compute these forward looking alphas. If a fund drops out of the sample in [t, t+47], we replace its return with value weighted index return to avoid look-ahead bias. The last three rows of each panel present the predictors with the three biggest coefficients in predicting the column variable. Newey-West standard errors with lag length equal to horizon length minus one are reported in parentheses. The sample period is from January, 1990 to June, 2013.

	Panel A: Dependent Variable is 1-month future alpha			
	Future alpha based on:			
	Mkt Adj.	Benchmark Adj.	CAPM	FF3
Past alphas based on:				
Mkt Adj.	0.198***	0.145***	0.195***	0.170***
	(0.072)	(0.041)	(0.068)	(0.040)
Benchmark Adj.	0.119**	0.141***	0.114**	0.121***
	(0.055)	(0.054)	(0.050)	(0.033)
CAPM	0.182***	0.133***	0.211***	0.140***
	(0.068)	(0.039)	(0.066)	(0.038)
FF3	0.097*	0.106***	0.109**	0.117***
	(0.052)	(0.038)	(0.050)	(0.035)
FFC4	0.121***	0.114***	0.138***	0.131***
	(0.036)	(0.029)	(0.035)	(0.026)
FFC4+3 IND	0.112***	0.106***	0.125***	0.124***
	(0.034)	(0.028)	(0.033)	(0.024)
Max-Min	0.101**	0.039*	0.101**	0.053**
Max-CAPM	0.016	0.012	0	0.030***
Max-Market Adj.	0	0	0.015	0
Max Coefficient	Mkt Adj.	Mkt Adj.	CAPM	Mkt Adj.
2nd Biggest	CAPM	Benchmark Adj.	Mkt Adj.	CAPM
3rd Biggest	FFC4	CAPM	FFC4	FFC4

	Panel B: Dependent Variable is 6-month future alpha			
	Future alpha based on:			
	Mkt Adj.	Benchmark Adj.	CAPM	FF3
Past alphas based on:				
Mkt Adj.	0.713**	0.456**	0.705**	0.699***
	(0.336)	(0.200)	(0.321)	(0.206)
Benchmark Adj.	0.402	0.512**	0.383	0.495***
	(0.291)	(0.246)	(0.261)	(0.184)
CAPM	0.638**	0.378**	0.782***	0.497***
	(0.276)	(0.167)	(0.288)	(0.171)
FF3	0.359	0.389**	0.443*	0.475***
	(0.288)	(0.194)	(0.266)	(0.180)
FFC4	0.539***	0.498***	0.675***	0.626***
	(0.199)	(0.148)	(0.191)	(0.131)
FFC4+3 IND	0.517***	0.478***	0.614***	0.611***
	(0.199)	(0.147)	(0.193)	(0.134)
Max-Min	0.354	0.134	0.399**	0.224*
Max-CAPM	0.075	0.134	0	0.202***
Max-Market Adj.	0	0.056	0.077	0
Max Coefficient	Mkt Adj.	Benchmark Adj.	CAPM	Mkt Adj.
2nd Biggest	CAPM	FFC4	Mkt Adj.	FFC4
3rd Biggest	FFC4	FFC4+3 IND	FFC4	FFC4+3 IND

	Panel C: Dependent Variable is 12-month future alpha			
	Future alpha based on:			
	Mkt Adj.	Benchmark Adj.	CAPM	FF3
Past alphas based on:				
Mkt Adj.	0.920*	0.491	0.863	0.988***
	(0.531)	(0.314)	(0.542)	(0.305)
Benchmark Adj.	0.522	0.663	0.474	0.681**
	(0.474)	(0.432)	(0.421)	(0.304)
CAPM	0.765*	0.333	0.983*	0.590***
	(0.458)	(0.257)	(0.503)	(0.226)
FF3	0.499	0.452	0.664	0.708**
	(0.574)	(0.387)	(0.535)	(0.344)
FFC4	0.815**	0.687***	1.109***	0.966***
	(0.353)	(0.264)	(0.359)	(0.236)
FFC4+3 IND	0.760**	0.641**	0.963**	0.939***
	(0.364)	(0.265)	(0.376)	(0.253)
Max-Min	0.422	0.354*	0.635**	0.398***
Max-CAPM	0.156	0.354*	0.126	0.398***
Max-Market Adj.	0	0.196	0.246	0
Max Coefficient	Mkt Adj.	FFC4	FFC4	Mkt Adj.
2nd Biggest	FFC4	Benchmark Adj.	CAPM	FFC4
3rd Biggest	CAPM	FFC4+3 IND	FFC4+3 IND	FFC4+3 IND

Table 9: Economic significance of predictability

Each month, we identify top, bottom deciles of funds based on 18-month exponentially weighted alphas computed under the models in the first column. We compute value-weighted buy-and-hold abnormal returns for the top decile (panel A), bottom decile (panel B), and top-bottom (panel C) over the next 1 month, 6 months, 12 months using TNA from previous month as the weight. If a fund drops out of the sample in [t, t+47], we replace its return with value weighted index return to avoid look-ahead bias. The last three rows of each panel present the models used in forming deciles which gives the three biggest abnormal returns (by magnitude) measured using the model shown in the column. We report the Newey-West standard errors with lag length equal to holding period minus one in the parentheses. The sample period is from January, 1990 to June, 2013.

Panel A: Top Decile

	1-month holding period				6-month holding period				12-month holding period			
	Model to compute forward alpha				Model to compute forward alpha				Model to compute forward alpha			
	Market Adj.	B/M Adj.	CAPM	FF3	Market Adj.	B/M Adj.	CAPM	FF3	Market Adj.	B/M Adj.	CAPM	FF3
Sorted On Alpha from:												
Market Adj. Ret.	0.202	0.169	0.149	0.192	0.690	0.685	0.290	0.714	0.788	0.771	0.153	0.840
	(0.164)	(0.121)	(0.159)	(0.119)	(0.804)	(0.585)	(0.817)	(0.681)	(1.512)	(1.069)	(1.683)	(1.394)
Benchmark Adj. Ret.	0.195	0.208	0.128	0.170	0.746	0.984	0.209	0.632	0.928	1.488	0.152	0.722
	(0.158)	(0.135)	(0.153)	(0.115)	(0.826)	(0.644)	(0.782)	(0.583)	(1.556)	(1.193)	(1.522)	(1.238)
CAPM	0.219	0.213*	0.161	0.193*	0.790	0.813	0.243	0.565	0.718	0.975	-0.185	-0.075
	(0.153)	(0.112)	(0.149)	(0.113)	(0.736)	(0.518)	(0.751)	(0.598)	(1.323)	(0.835)	(1.559)	(1.007)
FF3	0.088	0.132	0.016	0.088	0.309	0.597	-0.310	0.221	0.150	0.783	-1.080	-0.150
	(0.146)	(0.117)	(0.141)	(0.102)	(0.723)	(0.572)	(0.699)	(0.533)	(1.437)	(1.093)	(1.455)	(0.990)
FFC4	0.201	0.193*	0.151	0.200**	0.716	0.828	0.275	0.695	0.812	1.161	-0.102	0.461
	(0.133)	(0.111)	(0.128)	(0.097)	(0.616)	(0.518)	(0.588)	(0.488)	(1.187)	(0.952)	(1.144)	(0.933)
FFC4+3 IND	0.144	0.190*	0.089	0.133	0.437	0.780	0.046	0.390	0.250	0.960	-0.460	-0.098
	(0.128)	(0.110)	(0.123)	(0.088)	(0.618)	(0.517)	(0.587)	(0.490)	(1.156)	(0.946)	(1.064)	(0.913)
Max-Min	0.131	0.081	0.145	0.113	0.480	0.386	0.600	0.493	0.778	0.718	1.233	0.990
Max-CAPM	0	0	0	0.007	0	0.171	0.048	0.148	0.210	0.513	0.338	0.915

Max-Mkt Adj.	0.018	0.044	0.012	0.008	0.100	0.299	0	0	0.140	0.718	0	0
Max Coefficient	CAPM	CAPM	CAPM	FFC4	CAPM	B/M Adj.	Market Adj.	Market Adj.	B/M Adj.	B/M Adj.	Market Adj.	Market Adj.
2nd Biggest	Market Adj.	B/M Adj.	FFC4	CAPM	B/M Adj.	FFC4	FFC4	FFC4	FFC4	FFC4	B/M Adj.	B/M Adj.
3rd Biggest	FFC4	FFC4	Market Adj.	Market Adj.	FFC4	CAPM	CAPM	B/M Adj.	Market Adj.	CAPM	FFC4	FFC4

Panel B: Bottom Decile												
	1-month holding period				6-month holding period				12-month holding period			
	Model to compute forward alpha				Model to compute forward alpha				Model to compute forward alpha			
	Market Adj.	B/M Adj.	CAPM	FF3	Market Adj.	B/M Adj.	CAPM	FF3	Market Adj.	B/M Adj.	CAPM	FF3
Sorted On Alpha from:												
Market Adj. Ret.	-0.340***	-0.181**	-0.391***	-0.414***	-1.193*	-0.541	-1.401**	-1.695***	-1.494	-0.255	-1.379	-2.898**
	(0.124)	(0.083)	(0.126)	(0.126)	(0.623)	(0.479)	(0.675)	(0.614)	(1.456)	(1.049)	(1.923)	(1.446)
Benchmark Adj. Ret.	-0.276**	-0.273***	-0.363***	-0.377***	-0.921*	-1.027*	-1.361**	-1.519***	-0.787	-1.207	-1.250	-2.283*
	(0.109)	(0.104)	(0.104)	(0.104)	(0.557)	(0.587)	(0.566)	(0.557)	(1.218)	(1.164)	(1.536)	(1.231)
CAPM	-0.290**	-0.224**	-0.360***	-0.389***	-0.919	-0.654	-1.174*	-1.401**	-0.953	-0.559	-0.906	-2.163
	(0.120)	(0.089)	(0.118)	(0.117)	(0.559)	(0.438)	(0.637)	(0.546)	(1.415)	(1.007)	(1.908)	(1.361)
FF3	-0.098	-0.167*	-0.140	-0.200**	-0.237	-0.512	-0.405	-0.811	-0.326	-0.548	-0.172	-1.871
	(0.099)	(0.088)	(0.100)	(0.096)	(0.562)	(0.481)	(0.654)	(0.513)	(1.313)	(1.056)	(1.706)	(1.140)
FFC4	-0.147	-0.186**	-0.219**	-0.236***	-0.732	-0.809*	-1.243***	-1.231***	-0.923	-0.943	-1.948*	-2.473**
	(0.094)	(0.084)	(0.089)	(0.078)	(0.516)	(0.472)	(0.443)	(0.418)	(1.266)	(1.012)	(1.116)	(1.016)
FFC4+3 IND	-0.088	-0.159*	-0.153	-0.177**	-0.462	-0.577	-0.949**	-0.933**	-0.532	-0.633	-1.522	-1.988*
	(0.098)	(0.085)	(0.094)	(0.085)	(0.514)	(0.493)	(0.444)	(0.456)	(1.267)	(1.048)	(1.137)	(1.172)

Max-Min	0.253	0.113	0.251	0.237	0.956	0.514	0.996	0.883	1.168	0.953	1.776	1.027
Min-CAPM	-0.050	-0.048	-0.031	-0.025	-0.274	-0.373	-0.227	-0.293	-0.541	-0.648	-1.042	-0.735
Min-Mkt Adj.	0.000	-0.092	0.000	0.000	-0.000	-0.486	0.000	-0.000	0.000	0	-0.569	0.000
Min Coefficient	Market Adj.	B/M Adj.	Market Adj.	Market Adj.	Market Adj.	B/M Adj.	Market Adj.	Market Adj.	Market Adj.	B/M Adj.	FFC4	Market Adj.
2nd Smallest	CAPM	CAPM	B/M Adj.	CAPM	B/M Adj.	FFC4	B/M Adj.	B/M Adj.	CAPM	FFC4	FFC4+3 IND	FFC4
3rd Smallest	B/M Adj.	FFC4	CAPM	B/M Adj.	CAPM	CAPM	FFC4	CAPM	FFC4	FFC4+1 IND	Market Adj.	B/M Adj.

Panel C: Top-Bottom Decile

	1-month holding period				6-month holding period				12-month holding period			
	Model to compute forward alpha				Model to compute forward alpha				Model to compute forward alpha			
	Market Adj.	B/M Adj.	CAPM	FF3	Market Adj.	B/M Adj.	CAPM	FF3	Market Adj.	B/M Adj.	CAPM	FF3
Sorted On Alpha from:												
Market Adj. Ret.	0.542**	0.350**	0.540**	0.606***	1.883	1.225*	1.691	2.408**	2.282	1.025	1.533	3.738
	(0.245)	(0.158)	(0.245)	(0.221)	(1.144)	(0.720)	(1.259)	(1.094)	(2.195)	(1.313)	(3.104)	(2.271)
Benchmark Adj. Ret.	0.472**	0.481**	0.490**	0.547***	1.667	2.010**	1.570	2.151**	1.715	2.696	1.401	3.005*
	(0.201)	(0.206)	(0.198)	(0.182)	(1.019)	(0.991)	(1.044)	(0.933)	(1.703)	(1.693)	(2.350)	(1.715)
CAPM	0.509**	0.437***	0.521**	0.582***	1.709*	1.467**	1.417	1.967**	1.670	1.535	0.722	2.088
	(0.226)	(0.153)	(0.226)	(0.204)	(0.932)	(0.630)	(1.103)	(0.899)	(1.827)	(1.054)	(2.900)	(1.651)
FF3	0.187	0.299**	0.156	0.287*	0.546	1.110	0.095	1.032	0.476	1.331	-0.907	1.721
	(0.180)	(0.151)	(0.182)	(0.156)	(0.926)	(0.685)	(1.087)	(0.752)	(1.924)	(1.313)	(2.618)	(1.280)
FFC4	0.348***	0.379***	0.370***	0.436***	1.448**	1.638***	1.517***	1.926***	1.736*	2.104**	1.846*	2.934***
	(0.130)	(0.119)	(0.128)	(0.124)	(0.571)	(0.530)	(0.569)	(0.610)	(0.978)	(0.816)	(1.107)	(0.998)
FFC4+1 IND	0.275**	0.381***	0.282**	0.349***	1.307**	1.588***	1.312**	1.700***	1.625	2.050**	1.611	2.711**
	(0.126)	(0.120)	(0.123)	(0.122)	(0.580)	(0.527)	(0.585)	(0.629)	(1.071)	(0.860)	(1.228)	(1.150)
FFC4+3 IND	0.231*	0.349***	0.242*	0.310**	0.899	1.357**	0.995*	1.323**	0.782	1.593*	1.061	1.890
	(0.127)	(0.121)	(0.126)	(0.121)	(0.597)	(0.549)	(0.583)	(0.627)	(1.089)	(0.864)	(1.185)	(1.230)

Max-Min	0.355	0.182	0.384	0.318	1.337	0.901	1.596	1.376	1.806	1.670	2.754	2.017
Max-CAPM	0.033	0.044	0.019	0.024	0.174	0.544	0.274	0.442	0.612	1.161	1.125	1.650
Max-Mkt Adj.	0	0.131	0	0	0	0.785	0	0	0	1.670	0.314	0
Max Coefficient	Market Adj.	B/M Adj.	Market Adj.	Market Adj.	Market Adj.	B/M Adj.	Market Adj.	Market Adj.	Market Adj.	B/M Adj.	FFC4	Market Adj.
2nd Biggest	CAPM	CAPM	CAPM	CAPM	CAPM	FFC4	B/M Adj.	B/M Adj.	FFC4	FFC4	Market Adj.	B/M Adj.
3rd Biggest	B/M Adj.	FFC4	B/M Adj.	B/M Adj.	B/M Adj.	CAPM	FFC4	CAPM	B/M Adj.	FFC4+3 IND	B/M Adj.	FFC4

Appendix 1:

This appendix derives the results presented in section 1.4 of the paper. For expositional convenience, we set the risk-free rate to zero.

Let the returns be generated by a single factor model as shown in equation (13). The true model of expected returns is either a no-beta risk model in equation (14) or CAPM in equation (15). $\hat{\alpha}$ is estimated using either a market adjustment as shown in equation (16) or a market model adjustment as shown in equation (17).

In the cross-section of funds, the following hold true:

$$\begin{aligned} cov(\beta, \xi) &= 0 \\ cov(\hat{\beta}, \xi) &= 0 \\ cov(\hat{\beta} - \beta, \xi) &= 0, \end{aligned} \qquad (A.1.1)$$

where β represents true beta of a fund, $\hat{\beta}$ represents the estimated beta of the fund, $\hat{\beta} - \beta$ is the measurement error in estimated beta, and ξ represents the fund specific returns.

We also have, by definition:

$$cov(\hat{\beta} - \beta, \beta) = 0 \qquad (A.1.2)$$

From the two models of expected returns and two estimators, we have the following four cases.

Case 1: Market adjustment when the no-beta risk model is true

From equations (13), (14), (16):

$$\begin{aligned} \hat{\alpha}_{p,0} = r_{p,t} - r_{m,t} &= \alpha_p + E[r_m] + \beta_p \times f_t + \xi_{p,t} - r_{m,t} \\ &= \alpha_p + E[r_m] + \beta_p \times (r_{m,t} - E[r_m]) + \xi_{p,t} - r_{m,t} \\ &= \alpha_p + u_t \text{ where } u_t = (\beta_p - 1) \times (r_{m,t} - E[r_m]) + \xi_{p,t} \end{aligned}$$

Therefore, the cross-sectional variance of u_t after using the results in $(A.1.1)$ will be:

$$\begin{aligned} \sigma_u^2 \Big| r_{m,t} &= \left(r_{m,t} - E[r_m]\right)^2 \times var\left(\beta_p - 1 \middle| r_{m,t}\right) + \sigma_{\xi_{p,t}}^2 \Big| r_{m,t} \\ &= \left(r_{m,t} - E[r_m]\right)^2 \times \sigma_{\beta_p}^2 | r_{m,t} + \sigma_{\xi_{p,t}}^2 | r_{m,t} \end{aligned}$$

Since the true betas and the fund specific returns are drawn from identical distributions across funds, we can drop the subscript p to arrive at:

$$\sigma_u^2 | r_{m,t} = \left(r_{m,t} - E[r_m]\right)^2 \times \sigma_\beta^2 | r_{m,t} + \sigma_\xi^2 | r_{m,t} \qquad (A.1.3)$$

Case 2: Market model adjustment (i.e. CAPM) when the no-beta risk model is true

From equation (17):

$$
\begin{aligned}
\hat{\alpha}_{p,1} &= r_{p,t} - \hat{\beta}_p \times r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f \\
&= \alpha_p + E[r_m] + \beta_p \times f_t + \xi_{p,t} - \hat{\beta}_p r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f \text{ from equations (13), (14)} \\
&= \alpha_p + E[r_m] + \beta_p \times \left(r_{m,t} - E[r_m]\right) + \xi_{p,t} - \hat{\beta}_p r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f \\
&= \alpha_p + u_t \text{ where } u_t = \left(1 - \beta_p\right) \times E(r_m) - \left(\hat{\beta}_p - \beta_p\right) \times r_{m,t} - \left(1 - \hat{\beta}_p\right) \times r_f + \xi_{p,t}
\end{aligned}
$$

Using $(A.1.1), (A.1.2)$, and the following two results

$$
\begin{aligned}
Cov\left(1 - \beta_p, 1 - \hat{\beta}_p\right) &= var\left(\beta_p\right) \\
Cov\left(\hat{\beta}_p - \beta_p, 1 - \hat{\beta}_p\right) &= -var\left(\hat{\beta}_p - \beta_p\right),
\end{aligned}
$$

the cross-sectional variance of u_t will be:

$$
\begin{aligned}
\sigma_u^2 \middle| r_{m,t} = E(r_m) \times \left(E(r_m) - r_f\right) \times \sigma_{\hat{\beta}}^2 \middle| r_{m,t} + r_{m,t} \times (r_{m,t} - r_f) \\
\times \sigma_{\hat{\beta}-\beta}^2 \middle| r_{m,t} + r_f^2 \times \sigma_{\hat{\beta}}^2 \middle| r_{m,t} + \sigma_{\xi}^2 | r_{m,t}
\end{aligned} \quad (A.1.4)
$$

When the risk-free rate is set to zero:

$$
\sigma_u^2 | r_{m,t} = E(r_m)^2 \times \sigma_{\hat{\beta}}^2 | r_{m,t} + r_{m,t}^2 \times \sigma_{\hat{\beta}-\beta}^2 | r_{m,t} + \sigma_{\xi}^2 | r_{m,t} \quad (A.1.5)
$$

Case 3: Market adjustment when CAPM is true

From equations (13), (15), (16):

$$
\begin{aligned}
\hat{\alpha}_{p,0} &= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times f_t + \xi_{p,t} - r_{m,t} \\
&= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times \left(r_{m,t} - E[r_m]\right) + \xi_{p,t} - r_{m,t} \\
&= \alpha + u_t \text{ where } u_t = -\left(1 - \beta_p\right) \times (r_{m,t} - r_f) + \xi_{p,t}
\end{aligned}
$$

Using $(A.1.1)$, the cross-sectional variance of u_t is:

$$
\sigma_u^2 \middle| r_{m,t} = \left(r_{m,t} - r_f\right)^2 \times \sigma_{\beta_p}^2 \middle| r_{m,t} + \sigma_{\xi_{p,t}}^2 | r_{m,t} \quad (A.1.6)
$$

Dropping the subscript p since betas and fund specific returns are drawn from identical distributions across funds and with risk free rate set to zero, we get:

$$\sigma_u^2 \middle| r_{m,t} = r_{m,t}^2 \times \sigma_\beta^2 \middle| r_{m,t} + \sigma_\xi^2 | r_{m,t} \qquad (A.1.7)$$

Case 4: Market model adjustment (i.e. CAPM) when CAPM is true

From (13), (15), (17):

$$\begin{aligned}
\hat{\alpha}_{p,1} &= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times f_t + \xi_{p,t} - \left[r_f + \hat{\beta}_p\left(r_{m,t} - r_f\right)\right] \\
&= \alpha + r_f + \beta_p \times \left(E[r_m] - r_f\right) + \beta_p \times \left(r_{m,t} - E[r_m]\right) + \xi_{p,t} - \left[r_f + \hat{\beta}_p\left(r_{m,t} - r_f\right)\right] \\
&= \alpha + u_t \text{ where } u_t = -\left(\hat{\beta}_p - \beta_p\right) \times (r_{m,t} - r_f) + \xi_{p,t}
\end{aligned}$$

Using $(A.1.1)$, the cross-sectional variance of u_t is:

$$\sigma_u^2 \left| r_{m,t} = \left(r_{m,t} - r_f\right)^2 \times \sigma_{\hat{\beta}_p - \beta_p}^2 \right| r_{m,t} + \sigma_{\xi_{p,t}}^2 | r_{m,t} \qquad (A.1.8)$$

After dropping subscript p and setting risk free rate to zero, we get:

$$\sigma_u^2 \left| r_{m,t} = r_{m,t}^2 \times \sigma_{\hat{\beta} - \beta}^2 \right| r_{m,t} + \sigma_\xi^2 | r_{m,t} \qquad (A.1.9)$$

Appendix 2:

This appendix proves Propositions 1 and 2.

Proof of Proposition 1:

Denote

$$\hat{\alpha}_{p,\eta} = \alpha_p^K + \left(\alpha_{p,\eta} - \alpha_p^K\right) + \varepsilon_{p,\eta} \equiv \alpha_p^K + v_{p,\eta} \qquad (A.2.1)$$

Where the measurement error $v_{p,\eta}$ in $\hat{\alpha}_{p,\eta}$ is independent of flow Γ_p.

$\hat{\alpha}_{p,\eta 1}, \hat{\alpha}_{p,\eta 2}, \alpha_p^K$ are Normal with mean zero by assumption and are therefore symmetric around zero. By this, we get:

$$\begin{gathered} \Pr\left(Q_{p,\eta} = -1\right) = \Pr\left(Q_{p,\eta} = 1\right) = .5 \text{ for } \eta = \eta_1, \eta_2, \\ E\left(Q_{p,\eta_1}\right) = E\left(Q_{p,\eta_2}\right) = 0 \text{ , and} \\ Var\left(Q_{p,\eta_1}\right) = Var\left(Q_{p,\eta_2}\right) = 1. \end{gathered} \qquad (A.2.2)$$

It also follows from the definition in $(A.2.1)$ that:

$$E\left(v_{p,\eta}\right) = 0 \qquad (A.2.3)$$

Consider the following OLS regressions from (4) and (20):

$$\begin{gathered} \Gamma_p = a_\eta + b_\eta \hat{\alpha}_{p,\eta} + \omega_{p,\eta} \\ Q_{\Gamma_p} = A_\eta + B_\eta Q_{p,\eta} + o_{p,\eta} \end{gathered}$$

From Regression (4), after using $Cov\left(\Gamma_p, v_{p,\eta}\right) = 0$, we get:

$$b_\eta = \frac{cov(\Gamma_p, \hat{\alpha}_{p,\eta})}{var\left(\hat{\alpha}_{p,\eta}\right)} = \frac{cov(\Gamma_p, \alpha_p^K)}{var\left(\alpha_p^K\right) + var(v_{p,\eta})} \qquad (A.2.4)$$

Given that $\hat{b}_{\eta_1} > \hat{b}_{\eta_2}$. Therefore, from $(A.2.4)$ we get:

$$var\left(v_{p,\eta 1}\right) < var(v_{p,\eta 2}) \qquad (A.2.5)$$

From Regression (20), after using the result in $(A.2.2)$, we get:

$$B_\eta = \frac{Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right)}{Var\left(Q_{p,\eta}\right)} = Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) \quad (A.2.6)$$

To evaluate this covariance term, we use the law of total covariance which states:

$$cov(X,Y) = E\big(cov(X,Y|Z)\big) + cov(E(X|Z), E(Y|Z)) \quad (A.2.7)$$

Using $(A.2.7)$, we can write:

$$Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) = E\left(cov\left(Q_{\Gamma_p}, Q_{p,\eta}|Q_{p,K}\right)\right) + cov\left(E\left(Q_{\Gamma_p}|Q_{p,K}\right), E\left(Q_{p,\eta}|Q_{p,K}\right)\right) \quad (A.2.8)$$

Since Γ_p is independent of the measurement error part of $\hat{\alpha}_{p,\eta}$, the conditional covariance $cov\left(Q_{\Gamma_p}, Q_{p,\eta}|Q_{p,K}\right)$ will be zero on average. Hence the first term on the RHS of $(A.2.8)$ will be zero. Expanding the second term in $(A.2.8)$, we get:

$$\begin{aligned} Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) \\ &= E\left[E\left(Q_{\Gamma_p}|Q_{p,K}\right) \times E\left(Q_{p,\eta}|Q_{p,K}\right)\right] - E\left[E\left(Q_{\Gamma_p}|Q_{p,K}\right)\right] \\ &\times E\left[E\left(Q_{p,\eta}|Q_{p,K}\right)\right] \end{aligned} \quad (A.2.9)$$

The two terms in $(A.2.9)$ can further be expanded as:

$E\left[E\left(Q_{\Gamma_p}|Q_{p,K}\right) \times E\left(Q_{p,\eta}|Q_{p,K}\right)\right] = E\left(Q_{\Gamma_p}|Q_{p,K} = 1\right) \times E\left(Q_{p,\eta}|Q_{p,K} = 1\right) \times \Pr\left(Q_{p,K} = 1\right) + E\left(Q_{\Gamma_p}|Q_{p,K} = -1\right) \times E\left(Q_{p,\eta}|Q_{p,K} = -1\right) \times \Pr\left(Q_{p,K} = -1\right)$, and

$E\left[E\left(Q_{\Gamma_p}|Q_{p,K}\right)\right] \times E\left[E\left(Q_{p,\eta}|Q_{p,K}\right)\right] = \left\{E\left(Q_{\Gamma_p}|Q_{p,K} = 1\right) \times \Pr\left(Q_{p,K} = 1\right) + E\left(Q_{\Gamma_p}|Q_{p,K} = -1\right) \times \Pr\left(Q_{p,K} = -1\right)\right\} \times \left\{E\left(Q_{p,\eta}|Q_{p,K} = 1\right) \times \Pr\left(Q_{p,K} = 1\right) + E\left(Q_{p,\eta}|Q_{p,K} = -1\right) \times \Pr\left(Q_{p,K} = -1\right)\right\}$

Substituting these into $(A.2.9)$, using $1 - \Pr\left(Q_{p,K} = 1\right) = \Pr\left(Q_{p,K} = -1\right)$, and rearranging the terms yields:

$$
\begin{aligned}
&Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) \\
&\qquad = \left\{E\left(Q_{\Gamma_p}|Q_{p,K}=1\right) - E\left(Q_{\Gamma_p}|Q_{p,K}=-1\right)\right\} \\
&\qquad \times \left\{E\left(Q_{p,\eta}|Q_{p,K}=1\right) - E\left(Q_{p,\eta}|Q_{p,K}=-1\right)\right\} \times \Pr\left(Q_{p,K}=1\right) \\
&\qquad \times \Pr\left(Q_{p,K}=-1\right)
\end{aligned} \tag{A.2.10}
$$

From $(A.2.6)$ and $(A.2.10)$ we can see that comparing coefficients $B_{\eta 1}$, $B_{\eta 2}$ reduces to comparing $\left\{E\left(Q_{p,\eta}|Q_{p,K}=1\right) - E\left(Q_{p,\eta}|Q_{p,K}=-1\right)\right\}$ for $\eta = \eta_1$ & η_2, since true alpha α_p^K is same across the two models.

By definition, this term can be expressed as:

$$
\begin{aligned}
&E\left(Q_{p,\eta}|Q_{p,K}=1\right) - E\left(Q_{p,\eta}|Q_{p,K}=-1\right) \\
&\qquad = \Pr\left(\alpha_p^K + v_{p,\eta} \geq 0 | \alpha_p^K \geq 0\right) - \Pr\left(\alpha_p^K + v_{p,\eta} < 0 | \alpha_p^K \geq 0\right) \\
&\qquad - \Pr\left(\alpha_p^K + v_{p,\eta} \geq 0 | \alpha_p^K < 0\right) + \Pr\left(\alpha_p^K + v_{p,\eta} < 0 | \alpha_p^K < 0\right)
\end{aligned} \tag{A.2.11}
$$

Where the conditional probabilities are defined as:

$$
\Pr\left(\alpha_p^K + v_{p,\eta} \geq 0 | \alpha_p^K \geq 0\right) = \int_0^\infty \Pr\left(v_{p,\eta} \geq -\alpha_p^K | \alpha_p^K\right) \times f\left(\alpha_p^K \middle| \alpha_p^K \geq 0\right) \times d\alpha_p^K \tag{A.2.12}
$$

with $v_{p,\eta}|\alpha_p^K$ distributed as Normal with mean zero.

We get similar expressions for the remaining three terms on the RHS of equation $(A.2.11)$.

When $X \sim N(0, \sigma^2)$, the following definitions apply:

$$
\begin{aligned}
\Pr(X \leq a) &= F(a) = \frac{1}{2} \times \left[1 + \operatorname{erf}\left(\frac{a-\mu}{\sigma\sqrt{2}}\right)\right] = \frac{1}{2} \times \left[1 + \operatorname{erf}\left(\frac{a}{\sigma\sqrt{2}}\right)\right] \\
\Pr(X \geq a) &= 1 - F(a) = \frac{1}{2} \times \left[1 - \operatorname{erf}\left(\frac{a-\mu}{\sigma\sqrt{2}}\right)\right] = \frac{1}{2} \times \left[1 - \operatorname{erf}\left(\frac{a}{\sigma\sqrt{2}}\right)\right]
\end{aligned} \tag{A.2.13}
$$

Where $\operatorname{erf}(x)$ is the error function given by:

$$
\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \times \int_0^x e^{-t^2} dt
$$

This is an odd function with $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ and is monotonically increasing in its argument x. From these two properties and the definitions in $(A.2.13)$, we can infer the following:

$$\Pr(X \geq a) \text{ is } \begin{cases} \text{decreasing with } \sigma \text{ if } a < 0 \\ \text{increasing with } \sigma \text{ if } a > 0 \end{cases}$$
$$\Pr(X \leq a) \text{ is } \begin{cases} \text{increasing with } \sigma \text{ if } a < 0 \\ \text{decreasing with } \sigma \text{ if } a > 0 \end{cases} \qquad (A.2.14)$$

From $(A.2.5)$, we have $\sigma_{v_{p,\eta1}} < \sigma_{v_{p,\eta2}}$. Therefore, from $(A.2.12)$ and $(A.2.14)$, we can see that:

$$\begin{gathered} \Pr\left(\alpha_p^K + v_{p,\eta1} \geq 0 | \alpha_p^K \geq 0\right) > \Pr\left(\alpha_p^K + v_{p,\eta2} \geq 0 | \alpha_p^K \geq 0\right), \\ \Pr\left(\alpha_p^K + v_{p,\eta1} < 0 | \alpha_p^K < 0\right) > \Pr\left(\alpha_p^K + v_{p,\eta2} < 0 | \alpha_p^K < 0\right), \\ -\Pr\left(\alpha_p^K + v_{p,\eta1} < 0 | \alpha_p^K \geq 0\right) > -\Pr\left(\alpha_p^K + v_{p,\eta2} < 0 | \alpha_p^K \geq 0\right), \\ -\Pr\left(\alpha_p^K + v_{p,\eta1} \geq 0 | \alpha_p^K < 0\right) > -\Pr\left(\alpha_p^K + v_{p,\eta2} \geq 0 | \alpha_p^K < 0\right) \end{gathered} \qquad (A.2.15)$$

Substituting $(A.2.15)$ into $(A.2.11)$ gives:

$$E\left(Q_{p,\eta1} | Q_{p,K} = 1\right) - E\left(Q_{p,\eta1} | Q_{p,K} = -1\right) > E\left(Q_{p,\eta2} | Q_{p,K} = 1\right) - E\left(Q_{p,\eta2} | Q_{p,K} = -1\right) \qquad (A.2.16)$$

Finally, substituting this into $(A.2.10)$ and using the definition of B_η from $(A.2.6)$, we get $B_{\eta1} > B_{\eta2}$.

Proof of Proposition 2:

For analytic tractability, we assume $E(\Gamma_p) = 0$ in this proof.

From Regression (4), if $Cov\left(\Gamma_p, v_{p,\eta}\right) \neq 0$, we get:

$$b_\eta = \frac{cov(\Gamma_p, \hat{\alpha}_{p,\eta})}{var\left(\hat{\alpha}_{p,\eta}\right)} = \frac{cov(\Gamma_p, \alpha_p^K)}{var\left(\alpha_p^K\right) + var(v_{p,\eta})} + \frac{cov(\Gamma_p, v_{p,\eta})}{var\left(\alpha_p^K\right) + var(v_{p,\eta})} \qquad (A.2.17)$$

Since the η_{max}-factor model corresponds to the most precise estimate of true alphas, it follows that $var\left(v_{p,\eta_{max}}\right)$ is the minimum among all η-factor models. Therefore, in a rational expectations economy, $Cov(\Gamma_p, \hat{\alpha}_{p,\eta_{max}})$ is the biggest among all η-factor models, and hence $b_{\eta_{max}}$ is the biggest coefficient in Regression (4) among all η-factor models.

From Regression (4) it also follows that:

$$\sigma_{\Gamma_p}^2 = b_\eta^2 \times \sigma_{\hat{\alpha}_\eta}^2 + \sigma_{\omega_\eta}^2 = \frac{\left(cov(\Gamma_p, \hat{\alpha}_{p,\eta})\right)^2}{\sigma_{\hat{\alpha}_\eta}^2} + \sigma_{\omega_\eta}^2 \qquad (A.2.18)$$

Since $Cov(\Gamma_p, \hat{\alpha}_{p,\eta_{max}})$ is the biggest and $\sigma^2_{\hat{\alpha}_{\eta_{max}}}$ is the smallest among all η-factor models, and $\sigma^2_{\Gamma_p}$ is independent of the multifactor model used to estimate alphas, $\sigma^2_{\omega_{\eta_{max}}}$is the smallest among these models.

For Regression (20), following $(A.2.6)$ and $(A.2.2)$, we have:

$$B_\eta = \frac{Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right)}{Var\left(Q_{p,\eta}\right)} = Cov\left(Q_{\Gamma_p}, Q_{p,\eta}\right) = E(Q_{\Gamma_p} \times Q_{p,\eta}) \tag{A.2.19}$$

Using Law of Iterated Expectations,

$$\begin{aligned} E\left(Q_{\Gamma_p} \times Q_{p,\eta}\right) \\ &= E\left(Q_{\Gamma_p} \middle| Q_{p,\eta} = 1\right) \times \Pr\left(Q_{p,\eta} = 1\right) - E\left(Q_{\Gamma_p} \middle| Q_{p,\eta} = -1\right) \\ &\times \Pr(Q_{p,\eta} = -1) \end{aligned} \tag{A.2.20}$$

This can be simplified by writing the conditional expectations as:

$$E\left(Q_{\Gamma_p} \middle| Q_{p,\eta} = 1\right) = E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta} \geq 0\right) = \frac{\int_0^\infty E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta}}{\Pr\left(Q_{p,\eta} = 1\right)}$$

$$\Rightarrow E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta} \geq 0\right) \times \Pr\left(Q_{p,\eta} = 1\right) = \int_0^\infty E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta}$$

Similarly, $E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta} < 0\right) \times \Pr\left(Q_{p,\eta} = -1\right) = \int_{-\infty}^0 E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta}$

Substituting these in $(A.2.20)$,

$$\begin{aligned} E\left(Q_{\Gamma_p} \times Q_{p,\eta}\right) \\ &= \int_0^\infty E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} \\ &- \int_{-\infty}^0 E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} \end{aligned} \tag{A.2.21}$$

We have:

$$E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta}\right) = \Pr\left(\Gamma_p \geq 0 \middle| \hat{\alpha}_{p,\eta}\right) - \Pr\left(\Gamma_p < 0 \middle| \hat{\alpha}_{p,\eta}\right)$$

Using this in $(A.2.21)$,

$$E\left(Q_{\Gamma_p} \times Q_{p,\eta}\right) = \int_0^{\infty} \left[\Pr\left(\Gamma_p \geq 0 \middle| \hat{\alpha}_{p,\eta}\right) - \Pr\left(\Gamma_p < 0 \middle| \hat{\alpha}_{p,\eta}\right)\right] \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} - \int_{-\infty}^{0} \left[\Pr\left(\Gamma_p \geq 0 \middle| \hat{\alpha}_{p,\eta}\right) - \Pr\left(\Gamma_p < 0 \middle| \hat{\alpha}_{p,\eta}\right)\right] \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} \quad (A.2.22)$$

Since $E\left(\hat{\alpha}_{p,\eta}\right) = 0$, Regression (4) implies $\Gamma_p | \hat{\alpha}_\eta \sim N\left(E\left(\Gamma_p\right) + b_\eta \times \hat{\alpha}_\eta, \sigma_{\omega_\eta}^2\right)$

Using the definition of cumulative density of a Normal random variable from $(A.2.13)$ on $\Gamma_p | \hat{\alpha}_\eta$, the first term on the RHS of $(A.2.22)$ is:

$$\int_0^{\infty} \left[\Pr\left(\Gamma_p \geq 0 \middle| \hat{\alpha}_{p,\eta}\right) - \Pr\left(\Gamma_p < 0 \middle| \hat{\alpha}_{p,\eta}\right)\right] \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} = \int_0^{\infty} \operatorname{erf}\left(\frac{E\left(\Gamma_p\right) + b_\eta \hat{\alpha}_{p,\eta}}{\sigma_{\omega_\eta}\sqrt{2}}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} \quad (A.2.23)$$

We also use the property $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ in deriving $(A.2.23)$.

The second term on the RHS of $(A.2.22)$ can be expressed in a similar way:

$$\int_{-\infty}^{0} E\left(Q_{\Gamma_p} \middle| \hat{\alpha}_{p,\eta}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} = \int_{-\infty}^{0} \operatorname{erf}\left(\frac{E\left(\Gamma_p\right) + b_\eta \hat{\alpha}_{p,\eta}}{\sigma_{\omega_\eta}\sqrt{2}}\right) \times f\left(\hat{\alpha}_{p,\eta}\right) \times d\hat{\alpha}_{p,\eta} \quad (A.2.24)$$

Where $f\left(\hat{\alpha}_{p,\eta}\right) = \frac{1}{\sqrt{2\pi}\sigma_{\hat{\alpha}_\eta}} \times e^{-\frac{\hat{\alpha}_\eta^2}{2\sigma_{\hat{\alpha}_\eta}^2}}$.

The standard result for the integral of error function over a Normal density on $[0, \infty)$ is:

$$\int_0^{\infty} \operatorname{erf}(ax) \times e^{-b^2x^2} dx = \frac{1}{b\sqrt{\pi}} \tan^{-1}\left(\frac{a}{b}\right) \quad (A.2.25)$$

If we assume $E\left(\Gamma_p\right) \approx 0$, then we can use the result in $(A.2.25)$ to evaluate $(A.2.23)$ and $(A.2.24)$ and then substitute them in $(A.2.22)$ to get:

$$E\left(Q_{\Gamma_p} \times Q_{p,\eta}\right) = \frac{1}{\pi\sigma_{\hat{\alpha}_\eta}^2} \times \tan^{-1}\left(\frac{b_\eta \times \sigma_{\hat{\alpha}_\eta}}{\sigma_{\omega_\eta}}\right) = \frac{1}{\pi\sigma_{\hat{\alpha}_\eta}^2} \times \tan^{-1}\left(\frac{cov(\Gamma_p, \hat{\alpha}_{p,\eta})}{\sigma_{\hat{\alpha}_\eta} \times \sigma_{\omega_\eta}}\right) \qquad (A.2.26)$$

We already established that $\sigma_{\hat{\alpha}_{\eta_{max}}}^2$ is the smallest, $Cov(\Gamma_p, \hat{\alpha}_{p,\eta_{max}})$ is the biggest, and $\sigma_{\omega_{\eta_{max}}}^2$ is the smallest among all η-factor models. Since $\tan^{-1}(x)$ is an increasing function of x, these results and $(A.2.26)$ imply that $B_{\eta_{max}} = E\left(Q_{\Gamma_p} \times Q_{p,\eta_{max}}\right)$ is the biggest among all the η-factor model coefficients from Regression (20). Q.E.D.