# Black-Scholes-Merton model

- The nice thing about setting up external code bases in `.py` files is that you can easily import the code in a new notebook
- We'll now reload the `options.py` file containing the *Option* class definition into a new notebook/project

In [1]:
```
## Some preliminaries
%load_ext autoreload
%autoreload 2
import pandas as pd
import numpy as np
import os
os.chdir('g:/My Drive/teaching/1_boot camp python');
```

```
In [25]: import options as opts ## imports the g:/My Drive/teaching/1_boot camp/options.py file
         sigma,rf,TT,S0,KK = 0.15,0.02,1,30,35
opt = opts.Option(sigma,rf,TT,S0,KK) ## the class definition is inside opts
print(opt)
```

```
sigma:0.15  rf:0.02  TT (mat):1  S0:30  KK (strike):35
```

# The underlying stochastic process

- It can be shown that the limiting process modeled by the binmial tree is the geometric Brownian motion

$$\Delta S(t+h) = \mu S(t)h + \sigma S(t)\epsilon(t+h)$$

  where $\epsilon(t+h) \sim N(0,h)$ (the variance is $h$)
- A common way of writing the above discrete-time process as $h \to 0$ is

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

- There is a lot of cool stuff that you can do with this representation once you learn stochatic calculus
- For example you can derive an exact option pricing formula (you'll see how soon)

## Black-Scholes-Merton

- The Black-Scholes-Merton model gives a closed form solution for the price of a European option

$$C = N(d_1)S_0 - N(d_2)Ke^{-rt}$$

  where $N(\cdot)$ is the standard normal cdf
- The put price is obtained using put-call parity, a no-arbitrage relationship between the stock price, the call and put prices, and the short-term interest rate

$$P + S = C + Ke^{-rt}$$

  where $P$ and $C$ are the put and call prices of the same-strike, $t$-maturity options
- The formula assumes no early exercise and no dividends
- We'll examine put-call parity in a few minutes

In [3]:
```
import scipy.stats as stats
def price_european_option_bsm(sigma,rf,TT,S0,KK,kind):
    '''
    sigma, rr, TT, S0, KK -- the annual volatility, risk-free rate, maturity, stock price, strike
    kind -- 'call' or 'put'
    '''
    assert kind in ['call','put']

    d1 = (np.log(S0/KK) + (rf+sigma**2/2)*TT)/(sigma*np.sqrt(TT))
    d2 = d1 - sigma * np.sqrt(TT)
    call = stats.norm.cdf(d1)*S0 - stats.norm.cdf(d2)*KK*np.exp(-rf*TT)

    if kind == 'call':
        return call
    else:
        return call + KK*np.exp(-rf*TT) - S0
```

In [4]:
```
bspr = price_european_option_bsm(sigma,rf,TT,S0,KK,'call')
ss,pp = opt.price_european_option_with_tree(50,'call')
print(bspr,pp[0],bspr-pp[0][0])
```

```
0.4874900952556471 [0.4903088783432341] -0.0028187830875869846
```

Conveniently, you can redefine classes on the fly in Python

- Let's add the closed form method to our option class
- Pay attention to the role of the *self* argument $\leftarrow$ What is this?

```
In [5]: ## methods in classes take the argument 'self' to refer to the object in question
        def price_european_option_bsm_wrapper(self,kind):
    return price_european_option_bsm(self.sigma,self.rf,self.TT,self.S0,self.KK,kind)

## once we add the new method to the class, all existing instances of the class have access to the method
opts.Option.price_european_option_bsm = price_european_option_bsm_wrapper
print(bspr,opt.price_european_option_bsm('call'))
```

```
0.4874900952556471 0.4874900952556471
```

- This is okay for teaching or for playing around with a class, but
- It is better to put the definition into the `g:/My Drive/teaching/1_boot camp python/options.py` file

# Put-call parity

- Recall that we claimed there is a no-arbitrage relationship called put-call parity

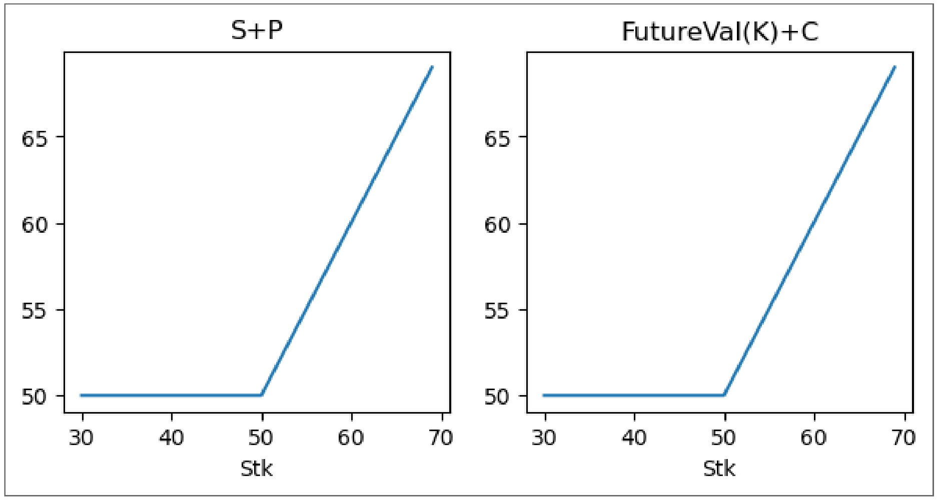
$$P_0 + S_0 = C_0 + Ke^{-rt} \tag{†}$$

- Let's verify this programmatically using Pandas and *lambda* functions
  - Lambda functions are a convenient tool to define simple function on the fly
  - We'll be using some features of Pandas you may not be familiar with
  - It's okay -- look over the code after class to understand what's going on
- We want to check that the *future* values of the left and right-hand sides of (†) are always the same
  - This ensures that the relationship in (†) -- which is in terms of *today's* prices -- holds via no arbitrage

```
In [6]: rf, TT, KK, S0 = 0.05, 3, 50, 60
        C0 = KK*np.exp(-rf*TT)  ## starting cash alongside call
put_price = lambda stk: max(KK-stk,0)   ## i.e., buy at 'stk' and sell at 'KK'
call_price = lambda stk: max(stk-KK,0)  ## i.e., buy at 'KK' and sell at 'stk'
future_cash_value = lambda cash: cash * np.exp(rf * TT)

vals = []
for stk in range(30,70):
    vals.append({'Stk':stk,
                 'Put':put_price(stk),
                 'Call':call_price(stk),
                 'Cash':future_cash_value(C0)})
vals = pd.DataFrame(vals) ## like a Series, but a DataFrame can store multiple columns
vals.set_index('Stk',inplace=True)  ## this tells the DataFrame to make the 'Stk' into its index
print(vals.iloc[19:23])  ## this is how you reference a DataFrame by numerical index
print(vals.loc[49:52])   ## this references by the actual index value (note it is *inclusive*)
```

```
     Put  Call  Cash
Stk
49     1     0  50.0
50     0     0  50.0
51     0     1  50.0
52     0     2  50.0
     Put  Call  Cash
Stk
49     1     0  50.0
50     0     0  50.0
51     0     1  50.0
52     0     2  50.0
```

```
In [7]: ## Now let's check put call parity
        import matplotlib.pyplot as plt
fig, axs = plt.subplots(1,2,figsize=(7,3))
## Pandas automatically labels x-axis with the index values
(vals.index + vals.Put).plot(ax=axs[0],title='S+P')  ## stock price in index
(vals.Cash + vals.Call).plot(ax=axs[1],title='FutureVal(K)+C');
```



## Comparing binomial trees to the BSM model

- We have not yet explored the sensitivity of option prices to the parameters (vols, stock price, etc.)
    - We'll talk more about this soon
- For the moment, let's focus on one particular parameter of the model: the number of steps in the tree
- What happens to the option price as we increase the number of steps and therefore decrease the step size, which equals $T/N$
- Take a look at the paper: Leisen and Reimer, 1996, "Binomial models for option valuation -- examining and improving convergence" (thanks to Kaiwen Hou for reference)

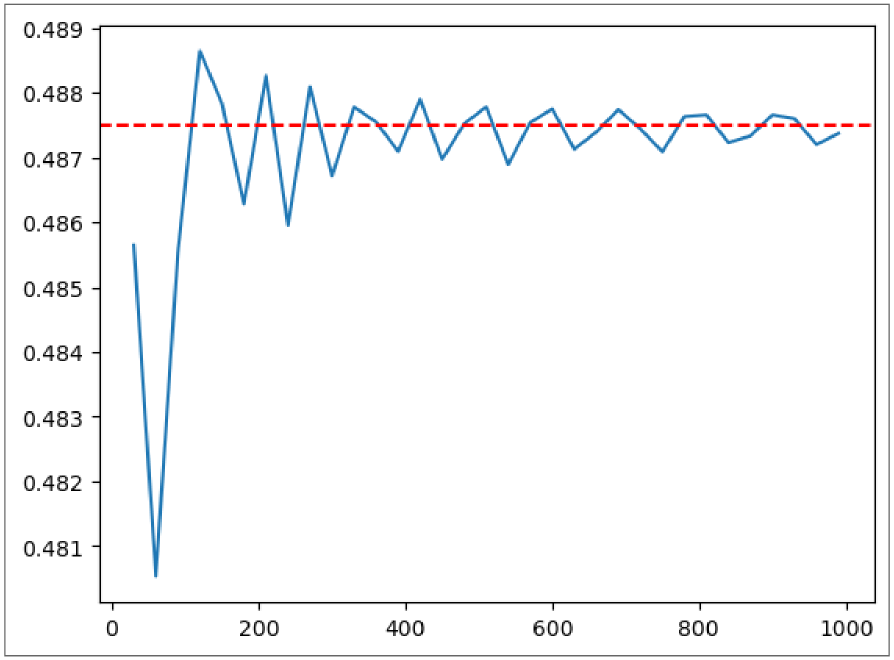
In [8]:
```
callpr = {}
print()
for NN in range(30,1010,30):
    print(NN,end=' ')
    ss,pp=opt.price_european_option_with_tree(NN,'call')
    ## NOTE that pp is a dictionary of lists, with the first list having one element: the option price at time
    ## take the option prices from step 0 in the tree, and the 1st element of that array
    callpr[NN] = pp[0][0]
print()
```

```
30 60 90 120 150 180 210 240 270 300 330 360 390 420 450 480 510 540 570 600 630 660 690 720
750 780 810 840 870 900 930 960 990
```

In [9]:
```
## convert to a Pandas series
        callpr = pd.Series(callpr) ## convert to Pandas series for ease of plotting
ax=callpr.plot()
ax.axhline(bspr,color='red',linestyle='--')
```

Out[9]:

```
<matplotlib.lines.Line2D at 0x13d65975c40>
```



Notice this code block in the *prices_and_rnprobs_at_step* method in `options.py`:

```
try:
    QQs = [comb(nn,ctr)*(QQ**(nn-ctr))*((1-QQ)**ctr) for ctr in range(0,nn+1)]
except:
    print('oops')
    QQs = [np.nan for ctr in range(0,nn+1)]
```

Why is the `try...except` code needed?

- Consider our risk-neutral probability calculation:

$$\binom{N}{K} Q^{N-K}(1-Q)^K$$

- What happens when $N$ and $K$ are very large?

```
In [10]: from math import comb
         QQ = 0.501
for NN in [1010,1030]:
    kk = int(NN/2)
    ans = comb(NN,kk)*QQ**(NN-kk)*(1-QQ)**kk
    print(ans)
```

```
0.025049242628127386

---------------------------------------------------------------------------
OverflowError                             Traceback (most recent call last)
Cell In[10], line 5
      3 for NN in [1010,1030]:
      4     kk = int(NN/2)
----> 5     ans = comb(NN,kk)*QQ**(NN-kk)*(1-QQ)**kk
      6     print(ans)

OverflowError: int too large to convert to float
```

- Note that we don't even use the risk-neutral probability vector in the binomial pricing.
- We only needed it as a robustness check and to plot the stock distribution.
- We either need fancier analytics that recognize that the probabilities that multiply the n choose k term drive the combination to be very low...
- Or we can tell the code to ignore errors.

In [11]:
```
from math import comb
QQ = 0.501
for NN in [1010,1030]:
    kk = int(NN/2)
    try:
        ans = comb(NN,kk)*QQ**(NN-kk)*(1-QQ)**kk
    except:
        ans = np.nan
    print(ans)
```

```
0.025049242628127386
nan
```

## Summarizing

- Binomial trees are intuitively easier to grasp
- The get arbitrarily close to the "correct" Black-Scholes-Merton price, but the rate of convergence is slow (and weird)
- Numerous methods have been proposed to accelerate the rate of convergence (e.g., **https://link.springer.com/article/10.1007/s007800200094**)
- There are other schemes, like *trinomial trees* (**https://warwick.ac.uk/fac/sci/maths/people/staff/oleg_zaboronski/fm/trinomial**
- When working with numerical schemes (e.g., dynamic programming, binomial trees, simulations, etc.) you are never going to be the expert
- Find resources that discuss how to implement efficient numerical schemes (this is something applied mathematicians do for a living)
- It is very good practice to benchmark against a known solution (like our BSM benchmark) -- then the calibration that works well for this will (hopefully) for in cases where the exact solution is not known

## The greeks

- The main parameters that go into a European option price and the option price sensitivity to them:

| Parameter | Greek |
|---|---|
| Stock price | Delta |
| 2nd stock deriv | Gamma |
| Sigma | Vega |
| $r_f$ | Rho |
| $T$ | Theta |

- Let's write a generic function to calculate

$$\frac{dP}{d\xi}$$

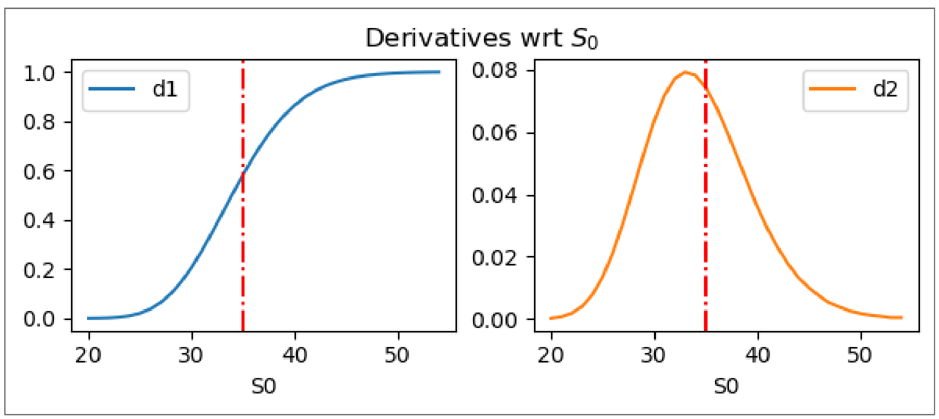
  where $P$ is the option price and $\xi$ is the parameter of interest

- As a mini-hw exercise, check what happens to the code below when `eps = 1e-8`
    - Numerical analysis is an *art* and a science

In [24]:
```
def calc_opt_deriv(self,param,kind):
        '''
    param -- the name of the parameter who sensitivity we want
    '''

    p1 = self.price_european_option_bsm(kind=kind)
    orig = getattr(self,param)

    ## perturb
    eps = 1e-5  ## experiment with this
    setattr(self,param,getattr(self,param)+eps)
    p2 = self.price_european_option_bsm(kind=kind)

    ## and again
    setattr(self,param,getattr(self,param)+eps)
    p3 = self.price_european_option_bsm(kind=kind)

    ## restore original value
    setattr(self,param,orig)

    ## return first and second derivs
    return {'d1':(p2-p1)/eps, 'd2':(p3-2*p2+p1)/eps**2}

## update class
opts.Option.calc_opt_deriv = calc_opt_deriv
```

## Interpret the option delta and gamma

```
In [26]: S0_save = opt.S0  ## make sure to restore the object state
         S0s = range(20,55,1)
derivs = []
for S0 in S0s:
    opt.S0 = S0
    res = opt.calc_opt_deriv('S0','call')
    res['S0'] = S0
    derivs.append(res)
derivs = pd.DataFrame(derivs)
derivs.set_index('S0',inplace=True)
opt.S0 = S0_save  ## restore the orignal value
axs = derivs.plot(figsize=(7,2.25),title='Derivatives wrt $S_0$',subplots=True,layout=(1,2))
for ii in range(2): axs[0,ii].axvline(opt.KK,color='red',linestyle='-.');
```



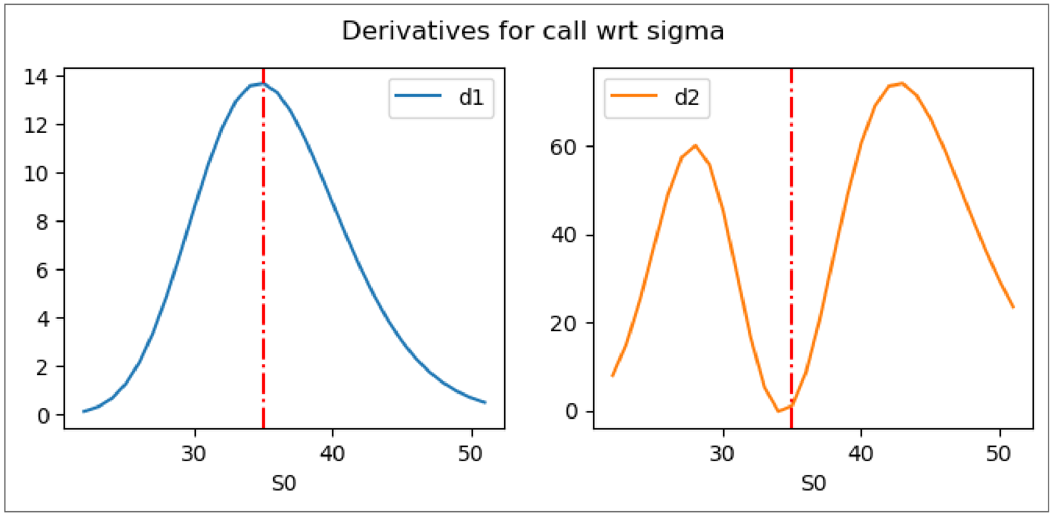

- How do you interpret the above gamma plot?

# Let's make the code above more general and calculate the other greeks

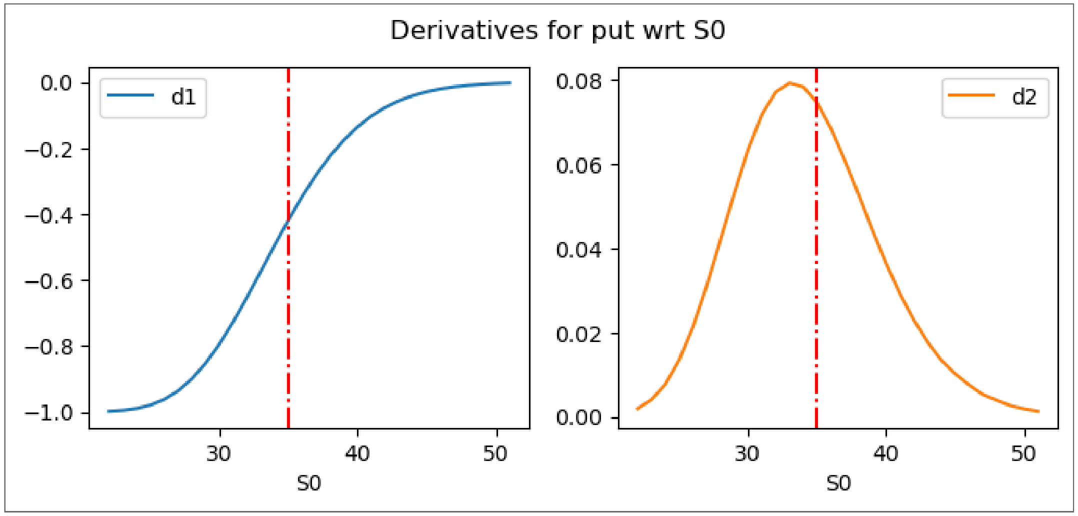
In [15]:
```
def show_greeks(self,param,kind):
    '''
    param -- which parameter to perturb (the x-axis is assumed to be S0, but this can
    be changed to reflect the variable being perturbed)
    kind -- 'call' or 'put'
    '''
    assert param in ['S0','KK','sigma','rf','TT']  ## sanity check
    S0_save = self.S0  ## make sure to restore the object state
    S0s = range(int(0.65*opt.KK),int(1.5*opt.KK),1) ## don't hard code range
    try: ## any time you modify object state, should be in a try call so it can be restored
        derivs = []
        for S0 in S0s:
            self.S0 = S0
            res = self.calc_opt_deriv(param,kind)
            res['S0'] = S0
            derivs.append(res)
        derivs = pd.DataFrame(derivs)
        derivs.set_index('S0',inplace=True)
    except:
        print('Oops')
    self.S0 = S0_save  ## restore the original value
    axs = derivs.plot(figsize=(8,3),title=f'Derivatives for {kind} wrt {param}',subplots=True,layout=(1,2))
    for ii in range(2):
        axs[0,ii].axvline(self.KK,color='red',linestyle='-.')

opts.Option.show_greeks = show_greeks ## extend class definition
```

In [16]:
```
print(opt)
opt.show_greeks('sigma','call')
```

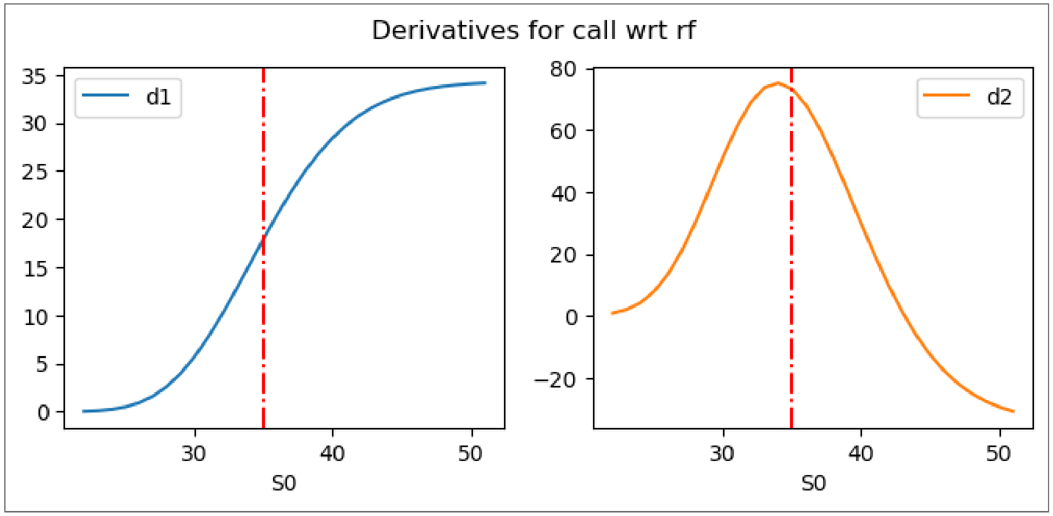
```
sigma:0.15  rf:0.02  TT (mat):1  S0:30  KK (strike):35
```



```
In [17]: opt.show_greeks('S0','put')
```



```
In [18]: opt.show_greeks('rf','call')
```



# Simulations and delta hedging

- Our assumed stock price process is

$$\Delta S(t+h) = \mu S(t)h + \sigma S(t)\epsilon(t+h)$$

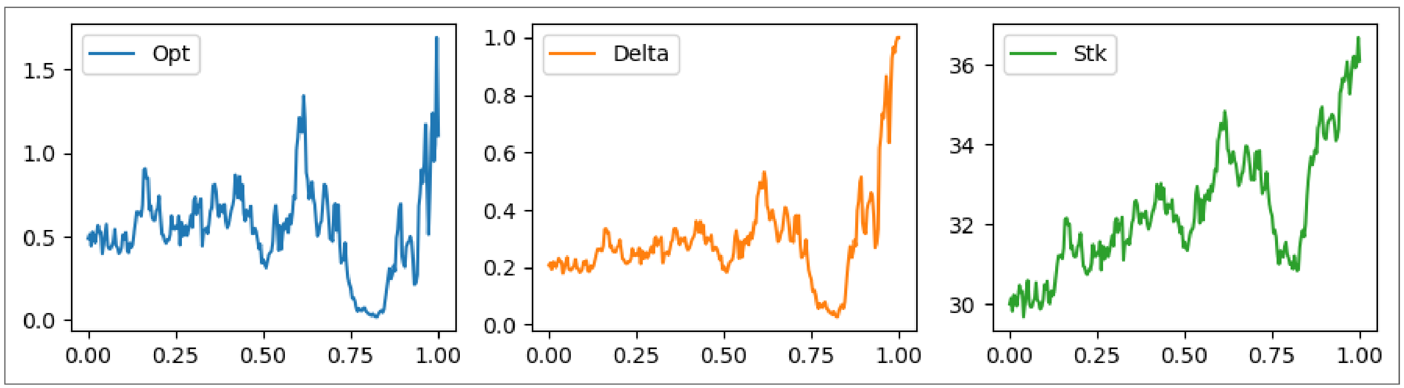
- Let's take this for a spin, along with our option pricing machinery
- Let's simulate one path of the above process every day for one year (so $h$ = one day) and see what the option price looks like
- Notice that thus far $\mu$ has not been used in any of our analysis!
- This is because when pricing under the risk-neutral measure, the stock's drift equals $r_f$
- But for running our simulations, we'll do them under the *physical measure* and use $\mu$
- But first, let's recall what the Option class we've constructed looks like

```
In [19]: help(opts.Option)
```

```
Help on class Option in module options:

class Option(builtins.object)
 |  Option(sigma, rf, TT, S0, KK)
 |
 |  Methods defined here:
 |
 |  __init__(self, sigma, rf, TT, S0, KK)
 |      This is the constructor and self refers to the object instance. It must be the first
argument
 |      passed into the constructor. The constructor is called with the information to creat
e an Option
 |      object, and then the class provides operations to perform on this object.
 |
 |  __repr__(self)
 |      Return repr(self).
 |
 |  calc_opt_deriv(self, param, kind)
 |      param -- the name of the parameter who sensitivity we want
 |
 |  get_Q(self, tau)
 |
 |  plot_rn_price_dist_at_step(self, nn, NN)
 |
 |  price_european_option_bsm = price_european_option_bsm_wrapper(self, kind)
 |      ## methods in classes take the argument 'self' to refer to the object in question
 |
 |  price_european_option_with_tree(self, NN, kind)
 |      NN -- number of steps in the tree
 |      kind -- the type of option ('call' or 'put')
 |
 |  prices_and_rnprobs_at_step(self, nn, NN)
 |      NN -- steps in the tree
 |      nn -- which step
```

```
 |
 |  show_greeks(self, param, kind)
 |      param -- which parameter to perturb (the x-axis is assumed to be S0, but this can
 |      be changed to reflect the variable being perturbed)
 |      kind -- 'call' or 'put'
 |
 |  ----------------------------------------------------------------------
 |  Data descriptors defined here:
 |
 |  __dict__
 |      dictionary for instance variables (if defined)
 |
 |  __weakref__
 |      list of weak references to the object (if defined)
```

In [31]:
```python
np.random.seed(1234) ## to make this reproducible
sigma, rf, TT, S0, KK, mu, hh = 0.15, 0.02, 1.0, 30, 35, 0.08, 1/252
times = np.arange(0,TT+hh,hh)
SS = pd.Series(0,index=times,name='Stk')
opt_vals = pd.DataFrame(0,index=times,columns=['Opt','Delta'])
for ii in range(0,len(times)):
    ## sqrt(hh) because np.random.normal expects a std dev
    if ii == 0:SS.iloc[0] = S0
    else: SS.iloc[ii] = SS.iloc[ii-1] + \
        SS.iloc[ii-1] * (mu * hh + sigma * np.random.normal(0,np.sqrt(hh)))
    ## at each point calculate (the max is to trick it to price 0 maturity option)
    opt = opts.Option(sigma,rf,max(TT-times[ii],1e-5),SS.iloc[ii],KK)
    opt_vals.loc[times[ii],'Opt'] = opt.price_european_option_bsm('call')
    derivs = opt.calc_opt_deriv('S0','call')
    opt_vals.loc[times[ii],'Delta'] = derivs['d1']
## merge the data
opt_vals = opt_vals.merge(SS,left_index=True,right_index=True)
opt_vals.plot(subplots=True,layout=(1,3),figsize=(9,2.5)); plt.tight_layout()
```



Cool fact: The delta gives us a rule for option replication!

- Remember, this is how we priced our one period option in the binomial tree: We solved for the $\Delta$ and the cash position in the replicating portfolio, and said that by no-arbitrage the option price should equal

$$\text{Option price} = \Delta \times S + B$$

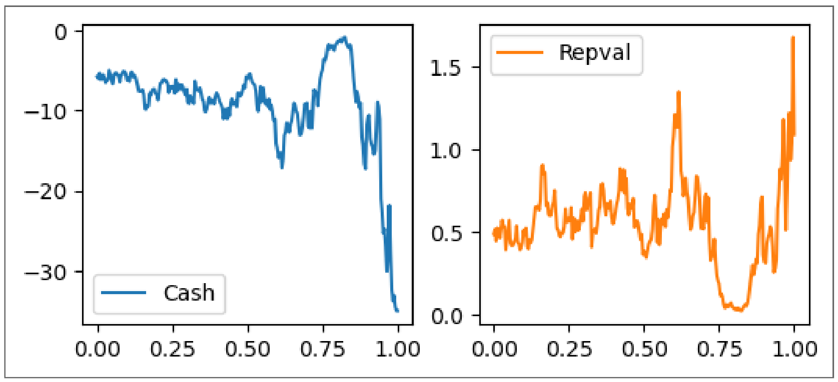
- Let's check whether how works in our simulation
- Pretend you're an option market maker and you sold the payout of this option to a client, who paid you an option price $C_0$ (plus a little bit of markup)
- Let us refer to the value of the replicating portfolio as $V_t$ and then $V_0 = C_0$
- The initial positions in stock and cash are $\Delta_0$ and $B_0 = C_0 - \Delta_0 \times S_0$ (why?)
- The value of the replicating portfolio next period is (note $\exp(r_f h) \approx 1 + r_f h$ for small $h$)

$$V_h = \Delta_0 \times S_h + B_0 \times (1 + r_f h)$$

- At this point, the delta is $\Delta_h$ and the cash position is $B_h = V_h - \Delta_h \times S_h$
- And we keep on going

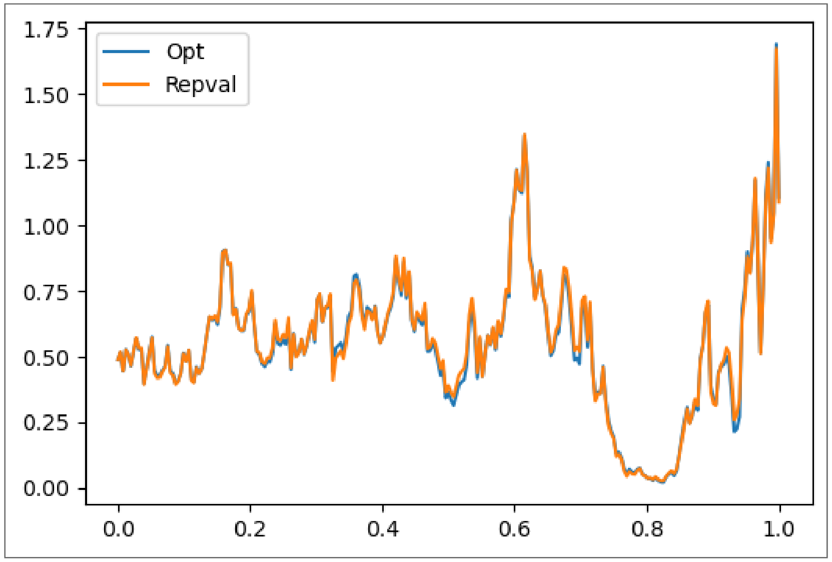
You will need to think about this offline to really get it

```
In [32]: ## Let's calculate the value of the replicating portfolio
         rep_vals = pd.DataFrame(0,index=times,columns=['Cash','Repval'])
for ii in range(0,len(times)):
    if ii == 0:
        rep_vals.loc[times[ii],'Repval'] = opt_vals.loc[times[ii],'Opt']
    else: ## note the delta and cash position are from the prior day
        rep_vals.loc[times[ii],'Repval'] = opt_vals.loc[times[ii-1],'Delta'] \
            * opt_vals.loc[times[ii],'Stk'] + \
            rep_vals.loc[times[ii-1],'Cash'] * (1+rf*hh)

    ## get the cash position
    rep_vals.loc[times[ii],'Cash'] = rep_vals.loc[times[ii],'Repval'] - \
        opt_vals.loc[times[ii],'Delta'] * opt_vals.loc[times[ii],'Stk']

rep_vals.plot(subplots=True,layout=(1,2),figsize=(6,2.5))
opt_vals = opt_vals.merge(rep_vals,left_index=True,right_index=True) ## merge the two DataFrames
```




- Does it work? Let's see how the value of replicating portfolio and the option evolve over time.

```
In [22]: opt_vals[['Opt','Repval']].plot(figsize=(6,4));
```




- As $h \to 0$ the replicating portfolio becomes exact

## Summary

- We saw in a one period setting, we can replicate an option payout using a position in the stock and cash (and the probabilility of up or down moves did not matter)
- This idea generalized to a binomial tree
- In an idealized, continuous-time setting, Black-Scholes-Merton showed how to derive a closed-form option pricing formula
- The tree answer converges (slowly) to the BSM one with number of steps
- Option greeks tell you the sensitivity of options to their input parameters, and the delta gives you a hedging strategy to replicate the option payout
- The Python tools and computing concepts we used in our analysis:
    - Functions and classes
    - Rich plotting functionality
    - Structuring a project and importing code into Jupyter
    - Dictionaries, lists, and Pandas
    - Numerical errors and convergence to the limit
    - Simulations and setting seeds for random generators