CHAPTER 18

Applications of Integral Calculus to Business

18.1 INTRODUCTION

The process of evaluating an integral of a function is called integration. Integration is also the inverse process of differentiation.

If the differential coefficient of a function $f(x)$ is $g(x)$, i.e., if $(d/dx)\,(f(x)) = g(x)$

Then $f(x)$ is said to be an integral or an anti-derivative or a primitive of $g(x)$ The same can be expressed in the form

$$\int g(x)dx = f(x)$$

where $\int$ is the symbol for integration, $g(x)$ is the integrand and dx integration with respect to x: $f(x)$ - outcome.

Integrand

The function to be integrated is referred to as integrand. The element dx is known as the element of integration and it specifies that variable with respect to which the given function is to be integrated.

Also,

$$(d/dx)\,(\int f(x)dx = f(x)$$

The study of integral calculus is related to developing techniques for the evaluation of integral of a given function. It has a wide application in geometry, social sciences and business decision making.

18.2 CONSTANT OF INTEGRATION

If $\int f(x)dx - F(x)$ then,

$\int f(x)dx = F(x) + C = \phi(x)$ where C is an arbitrary constant.

Clearly,

$$F'(x) = \phi'(x) = f(x)$$

This implies that

$F(x) - \phi(x) = \text{constant}$

Based on this, one may conclude that the integral of a function is not unique. That is, if $F(x)$ be any one integral of $f(x)$, then $F(x) + C$ is referred to as its general integral. The constant C is called a constant of integration.

Note: The constant of integration will generally be omitted.

18.3 LIST OF STANDARD INTEGRALS

1.
2. $\int x^{-1} dx = \log x$
3. $\int e^x dx = e^x$

4.
5. $\int \sin x dx = -\cos x$
6. $\int \cos x dx = \sin x$
7. $\int \sec x^2\, dx = \tan x$
8. $\int \text{cosec} x^2 dx = -\cot x$
9. $\int \sec x \tan x dx = \sec x$
10. $\int \text{cosec}\, x \cot x dx = -\text{cosec}\, x$

11.

18.4 STANDARD RESULTS

1. $\int a\, f(x) dx = a \int f(x) dx$

   where ‘a’ is a constant.

2. $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$
3. $\int [/(\text{x}) - g(x)] dx = \int f(x) dx - \int g(x) dx$

4.

where a and b are constants.

EXAMPLES

1. Evaluate

$$\int (x^3 + x^2 + 5)dx$$

Let

$$I = \int (x^3 + x^2 + 5)dx$$

2. Evaluate $\int (2^x + 4/x - (a/x^{1/3}))dx$; Let $I = \int (2^x + 4/x - (a/x^{1/3}))dx$

3. Evaluate

Let

$$= \int (x^2 - 1)dx + \tan^{-1}(x)$$

$$= \int (x^2\, dx - \int dx + \tan^{-1}(x)$$

4. Evaluate

Let

Simplification of the integrals leads to

Using Eq. (2) in Eq. (1),

18.5 DEFINITE INTEGRAL

In the application of integral calculus, it is highly essential to evaluate the difference in the values of an integral of a function $f(x)$ for two specific values assigned to the independent variable x, say a, b.

If $\int f(x)dx = F(x)$, the difference we mean $[(F(x)_{x=b}) - (F(x)_{x=a})] = (F(b) - F(a))$ is called the definite integral of $f(x)$ over the interval $[a,b]$. The same can be expressed mathematically as

where a is the lower limit and b is the upper limit of the integration.

Notes:

1. The value of the definite integral is unique.

   The arbitrary constant C is eliminated in the process.

2. Evaluation of integration without limits is referred to as indefinite integral.

18.5.1 Properties of Definite Integral

1.

2. where $a < c < b$

EXAMPLES

1. Evaluate

   Let

2. Evaluate

   Let

$$= 12 - 2\log 5$$

18.6 INTEGRATION BY SUBSTITUTION

It refers to the method of evaluating the value of the given integral $\int f(x)\,dx$ by transforming $\int f(x)\,dx$ into its equivalent valued integral $\int f(\phi(t))\,d(\phi(t))$ by using the relation $x = \phi(t)$.

That is,

$$\int f(x)dx = \int f(\phi(t))\,d(\phi(t))$$
$$= \int f(\phi(t))\,(\phi'(t)dt)$$

Note: Whenever the integral is definite in nature, the corresponding change of limits must be evaluated properly.

$$[x = \phi(t); \text{When } x = a, a = \phi(t)\ t = \phi^{-1}(a)\ \text{When } x = b, t = \phi^{-1}(b)]$$

EXAMPLES

1. Evaluate

Let

put

$$x^6 = t \qquad (2)$$

Differentiate Eq. (2) with respect to x,

Using Eqs (2) and (3) in Eq. (1),

2. Evaluate

Let

Put

$$e^x + e^{-x} = t \qquad (2)$$

Differentiate Eq. (2) with respect to x,

$$\Rightarrow (e^x - e^{-x})dx = dt \qquad (3)$$

Using Eqs (2) and (3) in Eq. (1),

Hence,

3. Show that

Let

Put

$$1 + x^2 = t \qquad (2)$$

Differentiate Eq. (2) with respect to x,

$$2xdx = dt \quad \Rightarrow \quad xdx = \quad dt \qquad (3)$$

Since the given integral is a definite integral, the corresponding change over limits values for t must be evaluated using Eq. (2),

When $x = 0; 1 + 0 = t \Rightarrow t = 1$; when $x = 1$; $1 + 1^2 = t$; $t = 2 \qquad (4)$

Using Eqs (2), (3) and (4) in Eq. (1),

Hence,

4. Evaluate

Let

put

2.

$$x^2 = t \qquad (2)$$

Differentiate Eq. (2) with respect to x,

$$\Rightarrow 2xdx = dt$$

$$\Rightarrow xdx = \quad dt \qquad (3)$$

Since the given integral is definite integral, find the equivalent value for t using Eq. (2).

When $x = 1$; $t = 1^2 = 1$ and when $x = 2$; $t = 2^2 = 4$

Using Eqs (2), (3) and (4) in Eq. (1),

18.6.1 Some Special Types

1.

2.

EXAMPLES

1. Evaluate

Let

$$d(x^2 + 9) = d(x^2) + d(9) = 2xdx + 0 = 2xdx$$

i.e.,

$$d(x^2+9) = 2xdx \qquad (2)$$

Using Eq. (2) in Eq. (1),

Hence,

2. Evaluate

Let

$$d(x^3 + 5) = d(x^3) + d(5) = 3x^2dx + 0 = 3x^2dx$$

i.e.,

$$d(x^3 + 5) = 3x^2\, dx$$

Hence,

18.7 METHOD OF PARTIAL FRACTIONS

Rules for resolving [$f(x)/g(x)$] into partial fractions

EXAMPLES

1. If the degree of $f(x)$ is less than the degree of $g(x)$ implies that it is a proper fraction. Factorize the denominator $g(x)$ and get the factors of the smallest degree possible. For each factor, there is a corresponding partial fraction.

Type of the Factor	Corresponding Partial Fraction

Type of the Factor	Corresponding Partial Fraction
…………	…………
…………	…………

The constants A, B, C and D, A_1, A_2 ,..., A_n, B_1, B_2 ,..., B_n must be evaluated.

When the degree of the denominator is less than or equal to the degree of numerator, it implies that the given fraction is improper.

First divide $f(x)$ by $g(x)$ completely. Write down the quotient and the remainder.

Clearly,

Consider the part ; if necessary, resolve it into partial fraction.

Note: Number of partial fractions is equal to the number of factors $g(x)$ contains.

Case (i): $g(x)$ contains linear factors

2. Evaluate

Compare Eq. (1) with the standard integral,

$f(x) = 1 \Rightarrow$ degree of $f(x)$: 0

$g(x) = (x + 1)(x + 2) \Rightarrow$ degree of $g(x)$: 2

since the degree of $f(x) <$ degree of $g(x)$. It implies that is a proper fraction.

The given fraction can be split into a sum of two partial fractions.

where A and B are arbitrary constants. (Number of partial fractions equal to the number of factor $g(x)$ contains). To find the value of A and B, multiply both the sides of eq. (2) by the factor $(x+1)(x+2)$.

$$\Rightarrow 1 = A(x+2) + B(x+1) \qquad (3)$$

Evaluation of A and B using Eq. (3) can be done in two ways.

Method 1

$$1 = A(x+2) + B(x+1) = Ax + 2A + Bx + B = x(A+B) + (2A+B)$$

Equating the like powers of x on both sides, implies

$$A + B = 0 \qquad (4)$$

and

$$2A + B = 1 \qquad (5)$$

Eq. (4) $\Rightarrow A = -B$

Using the value of A in Eq. (5), $-2B + B = 1$

$\Rightarrow -B = 1 \Rightarrow B = -1 \Rightarrow A = -B = -(-1) = 1$

Hence, $A = 1$ and $B = -1$

Method 2

$$1 = A(x + 2) + B(x + 1)$$

Substitute some specific values for x in such a way that the factor vanishes.

Put $x = -1$ Clearly $(x + 1)$ vanishes.

$\Rightarrow A(-1 + 2) = 1 \Rightarrow A = 1$

Put $x = -2$ Clearly $(x + 2)$ vanishes

$\Rightarrow A(-2 + 2) + B(-2 + 1) = \Rightarrow -B = 1 \Rightarrow B = -1$

Hence, $A = 1$ and $B = -1$

Note: In both ways, we should get the same value.

Then Eq. (2) becomes

Integrating Eq. (6) on either side with respect to x,

Hence,

3. Evaluate

Let

Here, is not a proper fraction

The degree of numerator is 2 and the degree of denominator is also 2 since both are equal, first divide $f(x)$ by $g(x)$ completely.

Using Eq. (2) in Eq. (1),

Clearly is a proper fraction. According to the formula given in table.

Evaluate the arbitrary constants A and B.

Multiply by $(x - 2)^2$ on both the sides of Eq. (4),

$$4x - 4 = A(x - 2) + B$$

$$4x - 4 = Ax + (B - 2A) \qquad (5)$$

Equating the like powers of x on both the sides of Eq. (5),

$$A = 4 \quad \text{and} \quad B - 2A = -4$$

$$\Rightarrow B - 8 = -4 \Rightarrow B = 4$$

Using the values of A and B in Eq. (3),

Using Eq. (6) in Eq. (3),

Put

$$x - 2 = t \qquad (8)$$

Differentiate Eq. (8) with respect to x

Using Eq. (8) and Eq. (9) in Eq. (7),

$$= x + 4\log(t) - 4(t^{-1})$$

Using Eq. (8),

Hence,

4. Evaluate

Consider the integral,

. It is a proper fraction.

Resolve it into equivalent partial fractions.

Multiply both sides of Eq. (2) by $(x+1)^2(x^2+1)$,

$$1 = A(x+1)(x^2+1) + B(x^2+1) + (cx + D)(x+1)^2 \quad (3)$$

Evaluate the constants A, B, C and D.

When $x = -1$, $B((-1)^2+1) = 1$; $2B = 1$; $B = 1/2$

Expanding the right side of Eq. (3) and grouping the like powers of x,

$$1 = A[x^3 + x + x^2 + 1] + B[x^2 + 1] + (Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D)$$

$$1 = x^3[A + C] + x^2[A + B + 2C + D] + x[A + C + 2D] + [A + B + D]$$

Comparing the like powers of x on both the sides, $A + C = 0$ (4)

Eq. (4) implies that $A = -C$

$$A + B + 2C + D = 0 \quad (5)$$

Using Eq. (4) in Eq. (5); $B + C + D = 0$ (6)

$$A + C + 2D = 0 \quad (7)$$

$$A + B + D = 1 \quad (8)$$

String Eq. (6) and (8);

$$A - C = 1 \qquad (9)$$

Using Eqs. (4) and (9);

$$A = \quad ;$$

$$C = - \quad ;$$

$$B = \quad ;$$

$$D = 0;$$

Hence Eq. (2) ⇒

Using Eq. (10) in Eq. (1),

Let

Then Eq. (11) becomes

$$I_2 = \int (x+1)^{-2}\, dx$$

put

$$x+1=t$$

Differentiate with respect to x

$$dx=dt$$

put

$$x^2+1=t$$

Differentiate with respect to x

Hence,

18.8 METHOD OF INTEGRATION BY PARTS

Let u and v be functions of x.

Then

$$\int udv = uv - \int vdu \qquad (1)$$

Equation (1) is called the integration by parts formula.

Note: Selection of u and v plays *a* vital role in the application of this model.

(That is selecting *a* function as u, such a fashion that repeated differention of u leads to zero quickly).

Examples

1. Evaluate $\int \log xdx$

   Let

$$I = \int(\log x)\, dx \qquad (1)$$

   Let

$$u = \log x \qquad (2)$$

$$dv = dx \qquad (3)$$

   We know that

$$\int udv = uv - \int vdu \qquad (4)$$

   Differentiate Eq. (2) with respect to x on both the sides,

   Integrating on both the sides of Eq. (3),

$$\int dv = \int du \text{ ; } v = x \qquad (6)$$

Using Eqs (2), (5) and (6) in Eq. (4),

$$= x \log x - \int dx; = x \log x - x$$

$$\int (\log x) dx = x \log x - x$$

2. Evaluate $\int x^2 e^x \, dx$

   Let

$$I = \int x^2 e^x \, dx \qquad (1)$$

   Let

$$u = x^2 \qquad (2)$$

   Let

$$dv = e^x \, dx = d(e^x)$$

$$\Rightarrow v = e^x \qquad (3)$$

$$du = 2xdx \qquad (4)$$

   We know that

$$\int udv = uv - \int vdu \qquad (5)$$

   Using Eqs (2), (3) and (4) in Eq. (5),

$$I = x^2 e^x - \int e^x (2xdx)$$

$$= x^2 e^x - 2\int e^x x dx$$

$$= x^2e^x - 2\int x\,d(e^x)$$

$$= x^2 e^x - 2 [xe^x - \int e^x dx]$$

$$= x^2 e^x - 2 [xe^x - e^x]$$

$$= x^2 e^x - 2xe^x + 2e^x$$

$$=e^x [x^2 - 2x + 2]$$

$$\Rightarrow \int x^2e^x dx = e^x [x^2 - 2x + 2]$$

3. Find the value of

   Let

   Take

$$u = x \qquad (2)$$

$$dv = e^{-5x}dx \qquad (3)$$

   Differentiating Eq. (2) with respect to x, $du = dx$ (4)

   Integrating on both the sides of Eq. (3) with respect to x,

$$\int dv = \int e^{-5x}dx$$

   We know that

$$\int udv = uv - \int vdu \qquad (6)$$

Using Eqs (2), (3) and (5) in Eq. (6),

Hence,

18.9 APPLICATION OF INTEGRAL CALCULUS TO BUSINESS

Integral calculus has its applications in the following areas.

1. Cost Function
2. Revenue Function
3. Customer's Surplus and Producer's Surplus
4. Compound Interest and Rate of Growth
5. Capital Formation
6. Amount of Annuity

18.9.1 Cost Function

Whenever the total function $C(x)$ is given, it is possible for us to find the marginal cost function MC(x) by differentiating $C(x)$ with respect to x.

That is, $\mathrm{MC}(x) = (d/dx)(C(x))$

In turn, if the marginal cost function is given directly and to find the cost function we seek the help of integration. Integration is the reversal process of the differentiation.

That is,

$$C(x) = \int MC(x)dx + C_1$$

When $x = 0, C(0) = C_1 \neq 0$. It implies that C_1 be the fixed cost of production. This is because fixed cost does not respond to change in output.

EXAMPLES

1. The marginal cost function for producing x units is $y = 23 + 16x - 3x^2$ and the total cost for producing one unit is 40. Obtain the total cost function and the average cost function.

   Given

   The marginal cost function $y = 23 + 16x - 3x^2$; $C_1 = 40$ when $x = 1$ *(1)*

   We know that

   Cost function = $\int$(Marginal cost function) dx

$$\Rightarrow C = \int y\, dx$$

$$= \int(23 + 16x - 3x^2)\, dx + C_1; \text{ (where } C_1 \text{ is a constant)}$$

$$= \int 23\, dx + \int 16x\, dx - \int 3x^2\, dx$$

$$C = 23x + 8x^2 - x^3 + C_1 \qquad (2)$$

Using the given condition in Eq. (1),

$$40 = 23(1) + 8(1)^2 - (1)^3 + C_1$$

$$40 = 23 + 8 - 1 + C_1 \Rightarrow C_1 = 10$$

$$\Rightarrow C = 23x + 8x^2 - x^3 + 10$$

$$\text{Average cost} = C/x$$

$$= (23x + 8x^2 - x^3 + 10)/x$$

$$= 23 + 8x - x^2 + (10/x)$$

Hence, the required cost function is $C = 23x + 8x^2 - x^3 + 10$ and the average cost function is

$$\text{AC} = 23 + 8x - x^2 + (10/x)$$

2. The marginal cost for a certain product is $5 + x^2$. Find the total cost and the average cost function if the fixed cost is 50.

   Given

$$\text{Marginal cost} = MC = 5 + x^2$$

$$\text{Fixed cost} = 50$$

That is, $C = 50$ When $x = 0$ (1)

$$C = \int(MC)\,dx = \int(5 + x^2)\,dx = \int 5\,dx + \int x^2\,dx$$

Using the given condition in Eq. (1), Eq. (2) reduced to $5 = C_1$

Hence, the required cost function is $C = 5x + (x^3/3) + 50$ and the average cost function is

$$AC = 5 + (x^2/3) + (50/x)$$

3. If the marginal cost function of x units of output is , and if the cost of output is zero, find the total cost as a function of x.

   Given

and $C = 0$ when $x = 0$ (1)

$$C = \int (\mathrm{MC})dx$$

Put $ax+b=t$ (3)

Differentiate Eq. (3) with respect to x,

Using Eqs (3) and (4) in Eq. (2),

$$c = 0, \text{ when } x = 0$$

Hence, the cost function is

18.9.2 Revenue Function

Whenever the total revenue function $R(x)$ is given, it is possible for us to find the marginal revenue function $\mathrm{MR}(x)$ by differentiating $R(x)$ with respect to x.

That is, $\mathrm{MR}(x) = (d/dx)(R(x))$

In turn, if the marginal cost function is given directly and to find the cost function we seek the help of Integration.

That is,

$$\mathrm{R}(x) = \int \mathrm{MR}(x)dx + C_1$$

Moreover, when $x = 0$ certainly $\mathrm{R}(x) = 0$. Hence $\mathrm{R}(x) = \int \mathrm{MR}(x)dx + C_1$

EXAMPLES

1. If the marginal revenue function for output x is given by Find the total revenue function and the demand function.

   Given

   We know that

$$R(x) = \int MR(x)dx$$

Consider

Use the method of substitution,

Put $x + 2 = t$ and (4)

Differentiate Eq. (4) with respect to x

Using Eqs (4) and (5) in Eq. (3),

Using Eq. (6) in Eq. (2),

When $x = 0$, $R(x) = 0$, it implies that

We know that

$R(x) = x \times p(x)$ where $p(x)$ is the demand function.

Hence, the required revenue function is

and the demand function is

Note: . The demand function and the average revenue function are one and the same.

2. The *XYZ* company limited has approximated the marginal revenue function for one of its products by MR = $20x - 2x^2$. The marginal cost function is approximated by MC = $81 - 16x + x^2$. Cost is 40 when $x = 1$. Determine the profit function.

Given

$$\text{MR} = 20\,x - 2\,x^2 \qquad (1)$$

$$\text{MC} = 81 - 16\,x + x^2 \qquad (2)$$

and $C = 40$ when $x = 1$. First evaluate the revenue function and the cost function.

$$R(x) = \int \text{MR}\,dx + C_1;\ R(x) = \int (20x - 2x^2)\,dx + C_1$$

$$= 20\int x\,dx - 2\int x^2\,dx + C_1$$

When $x = 0$, $R(0) = 0$; $\Rightarrow C_1 = 0$

$$C(x) = \int (\text{MC})\,dX + C_2$$

$$= \int (81 - 16x + x^2)\,dx + C_2$$

$$= 81\int dx - 16\int x\,dx + \int x^2\,dx + C_2$$

$C = 40$ when $x = 1$

$$\Rightarrow 40 = 81 - 8 + (1/3) + C_2$$

$$C_2 = 40 + 8 - 81 - (1/3) = -100/3$$

$$C(x) = 81x - 8x^2 + (x^3/3) - (100/3) \qquad (4)$$

Profit function

$$P(x) = R(x) - C(x)$$

$$P(x) = (10x^2 - (2/3)x^3) - (81\,x - 8x^2 + (x^3/3) - 100/3)$$

$$= 10x^2 - (2/3)x^3 - 81x + 8x^2 - (x^3/3) + (100/3)$$

$$P(x) = -81x + 18x^2 - x^3 + 100/3$$

Hence, the profit function is $P(x) = -81x + 18x^2 - x^3 + (100/3)$

3. An electric manufacturing company makes small household switches. The company estimates the marginal revenue function for these switches to be MR = $(x^2e^x/(x + 2)^2)$ where x represents the number of units (in thousands). Find the revenue function.

   Given

Then,

$$R(x) = \int (MR)dx + C_1$$

$$= \int (x^2 e^x) d(-(x + 2)^{-1}) + C_1$$

$$= -x^2 e^x (x + 2)^{-1} - \int (-(x + 2)^{-1})[2xe^x + x^2 e^x]dx + C_1$$

When $x = 0$; $R(0) = 0 \Rightarrow C_1 = 1$

Hence, the required revenue function is

18.9.3 Consumer's Surplus and Producer's Surplus

Consumer's Surplus

This theory was developed by the great economist Marshal. The demand function *p(x)* reveals the relationship between the quantities that the people would buy at a given price. It can be expressed as

$$p = f(x) \qquad (1)$$

Let us assume that the demand of the product $x = x_0$ when the price is p_0. But there can be some consumer who is ready to pay q_0 which is more than p_0 for the same quantity x_0. Any consumer who is ready to pay the price more than p_0 gains from the fact that the price is only p_0. This gain is called the consumer's surplus.

It is represented in the following diagram.

Mathematically, the consumer's surplus (CS) can be defined as

CS = (Area under the demand curve from $x = 0$ to $x = x_0$) − (Area of the rectangle OABC)

EXAMPLES

1. If the demand function is $p = 20-2x$, p is the price and x is the amount demanded. Find the consumer's surplus when $p = 6$.

   Given

   $$p(x) = 20 - 2\,x$$

   When $p = 6$; $6 = 20-2\,x \Rightarrow 2\,x = 14 \Rightarrow x = 7$

   Consumer's surplus when $p = 6$ and $x = 7$ is

$$= ((20 \times 7 - (7)^2) - 0) - 42 = 140 - 49 - 42; CS = 49$$

2. The demand function of a commodity is $y = 36 - x^2$. Find the consumer's surplus for $y_0 = 11$.

   Given

$$y = 36 - x^2$$

   and

$$y_0=11$$

$$y_0 = 11 \Rightarrow 11 = 36 - x^2$$

$$x^2 = 36 - 11 = 25 \Rightarrow x = 5$$

Hence, the consumer’s surplus is (250/3).

Producer’s Surplus

A supply function $s(x)$ represents the quantity that can be supplied at a price p. Let p_0 be the market price for the corresponding supply x_0. But there can be some producers who are willing to supply the commodity below the market price gain from the fact that the price is p_0. This gain is called the producer’s surplus. It is represented in the following diagram.

Mathematically, producer’s surplus (PS) can be defined as

PS = (Area of the rectangle OABC) − (Area below the supply function from $x = 0$ to $x = x_0$)

EXAMPLES

1. The supply curve for a commodity is and the quantity sold is 7 units. Find the producer's surplus.

   Given supply curve (1)

   and $x = 7$; using $x = 7$ in (1) ⇒

$$= 28 - \quad [(16)^{3/2} - (9)^{3/2}]$$

$$= 28 - \quad [64 - 27]$$

$$= 28 - \quad [37]$$

$$= \quad - 3.33$$

Hence, the producer's surplus is (10/3).

2. Find the producer's surplus defined by the supply curve $s(x) = 4x + 8$ and the quantity sold is 5 units.

   Given

$$s(x) = 4x + 8 \qquad (1)$$

   and $x = 5$; $p = s(x = 5) = 4(5) + 8 = 28$

   By definition

$$= 140 - [(2(5)^2 + 8(5)) - 0]$$

$$= 140 - [50 + 40]$$

$$= 140 - 90 = 50$$

Hence, the producer's surplus is 50.

3. The demand and supply functions of a commodity are

$p_d = 18 - 2x - x^2$ and $p_s = 2x - 3$. Find the consumer's surplus and producer's surplus at equilibrium prices.

$$p_d = 18 - 2x - x^2 \quad (1)$$

$$p_s = 2x - 3 \quad (2)$$

To get the value of x, use the concept of equilibrium price. We know that at equilibrium price, $P_d = P_s$.

$$\Rightarrow 18 - 2x - x^2 = 2x - 3 \Rightarrow x^2 + 4x - 21 = 0$$

$$x^2 + 7x - 3x - 21 = 0 \Rightarrow x(x + 7) - 3(x + 7) = 0$$

$$(x + 7)(x - 3) = 0 \Rightarrow x + 7 = 0 \text{ or } x - 3 = 0$$

$$x = -7 \text{ or } 3$$

The value of x cannot be −ve, $\Rightarrow x = 3$

When $x = 3$, the price $p = 18 - 2(3) - (3)^2 = 3$.

$$54 - 9 - 9 - 9$$

$$CS = 27$$

$$= 9 - [3^2 - 3(3)]$$

$$= 9 - [9 - 9] = 9$$

Hence at equilibrium price,

1. the consumer’s surplus is 27
2. the producer’s surplus is 9

4. When the price of the pen averaged 400, *ABC* Company Limited sold 20 every month. When the price dropped to an average of 100, 120 were sold every month. When the

price was 400, 200 pens were available per week for sale. When the price reached 100 only 50 remained. Determine consumer's surplus and producer's surplus.

Given

Cost of a pen (*p*)	**Units demanded (*x*)**
400	20
100	120

Clearly (400, 20) and (100, 120) are the two co-ordinate points on the demand equation. Hence, the equation can be given as

$$p - 400 = -3x + 60;\ p = -3x + 460 \qquad (1)$$

Also given

Supply *s*( )	**Availability *x***
400	200
100	50

Similarly proceeding,

$$s - 400 = 2x - 400 \Rightarrow s = 2x \qquad (2)$$

At equilibrium, $p = s$

$$\Rightarrow -3x + 460 = 2x \Rightarrow 5x = 460 \Rightarrow x = 92$$

Using the value of x in Eq. (1),

$$\Rightarrow p = -3(92) + 460 = 184$$

That is, when $x = 92$, the price $p =$ 184.

Consumer’s surplus,

$$= [-12696 + 42320] - 16928$$

$$= 29624 - 16928 = 12696$$

Producer's surplus,

$$= 16928 - [(92)^2 - 0]$$

$$= 16928 - (92)^2$$

$$= 16928 - 8464 = 8464$$

Hence, the consumer's surplus is 12,696 and the producer's surplus is 8464.

18.10 LEARNING CURVES

Almost all the organizations learn and improve over time. As organizations and the workers perform a task over and over, they learn how to perform more efficiently. This means that task times and costs decrease. Learning curves are drawn based on the principle that workers and organizations become better at their jobs because they are repeatedly doing the same. A learning curve graph displays labour hours per unit versus the number of units produced. From there, we see that the time needed to produce a unit decreases, usually following a negative exponential curve, as the person or a company produces more units. In other words, it takes less time to complete each additional unit a firm produces. The above figure explains the time *savings* in completing each subsequent unit *decreases*. These are the major attributes of the learning curve.

18.10.1 Arithmetic Approach

On the basis of *doubling* of production only the learning curve is erected. It means that when the production doubles, automatically the production time will come down this in turn affects the rate of the learning curve.

The learning curve principle can be given as

Time required for the *n*th unit to be produced = $T \times L^n$

where T stands for the unit cost/unit time for the first unit; L refers to the learning curve rate; and n stands for the number of times T is doubled.

EXAMPLES

If the first unit of a particular product took 8 hours/unit, and if a 80% learning curve is present, find the hours the fourth unit will take require doubling twice.

Given $T = 8$ hrs/unit $L = 80/100 = 0.8$ $n = 2$;   due to doubling it becomes 4.

Time required for the *n*th unit to be produced = $T \times L^n = 8 \times 0.8^2 = 5.12$ hrs

18.10.2 Logarithmic Approach

This approach permits us to evaluate the labour for any number of unit.

The structural formula can be given as $T_n = T_1 \times n^b$.

where T_n refers to the time for the nth unit; T_1 refers to the time required to produce the first unit and b refers to the slope of the learning curve and it can be defined as

$b = \{(\text{log of the learning rate})/(\log 2)\}$.

EXAMPLES

1. The learning rate of a specific operation is 75%, and the time for producing the first item is 150 hours, then find the time required to produce the 5th item.

   Given

   $T_1 = 150$ hrs

   Learning rate = 0.75

   $b = \{(\text{log of the learning rate})/(\log 2)\} = \log[0.75]/\log 2$

   Then to find $T_5 = ?$;

We know the structural formula to find the time for producing the nth item as:

$T_n = T \times n^b$.

$T_5 = 150 \times 5^{\{\log[075]/\log2\}}$.

2. ABC shipyard took 125,000 hrs to manufacture a first ship. You have a plan to purchase a ship from that company. The same company produced 2nd boat and the 3rd boat with the learning factor of 85%. The manufacturing cost becomes 2000/hr, what will the expected amount that you can think of to purchase the 4th boat?

Given

$T_1 = 125000$ hr

Production cost/hr = 2000

Learning rate = 0.85

$b = \{(\text{log of the learning rate})/(\log 2)\} = \log[0.85]/\log2$

Then to find $T_4 = ?$;

We know the structural formula to find the time for producing the nth item as $T_n = T_1 \times n^b$.

$T_4 = 125{,}000 \times 4^{\{\log[0.85]/\log2\}}$.

Expected cost = 2000 × T_4

EXERCISE

1. Evaluate $\int(x + 1/x)^2\, dx$
2. Integrate $(e^x - 1)$ $w.r.t.x$
3. Evaluate $\int x^{1/2}dx$
4. Evaluate $\int(x^3 + 4x^2 - 5x - 6)dx$
5. Evaluate $\int(x^2 - 2)^2\, dx$

Evaluate the following integrals

6.

7.

8.

9.

10.

11.

Evaluate the following with respect to x:

12. (Hint: put $x^2 + 9 = t$)

13.
14. $\int (5 + 3x)^2\, dx$

15.

16.
17. $\int x^3 e^{x^4}\, dx$ (Put $x^4 = t$)

Evaluate the following integrals:

18.

19.

20.

21.

22.

Evaluate the following:

Problem	Hint : for partial fraction
23.	
24.	
25.	First divide and then apply the partial fraction
26.	First divide and then apply the partial fraction
27.	
28.	

29. Evaluate $\int x \log x \, dx$

(Take $u = \log x$ and $dv = x dx$)

30. Evaluate $\int x e^{mx} \, dx$

(Take $u = x$ and $dv = e^{mx} \, dx$)

31. Find the value of

32. Evaluate $\int (\log x)^3 \, dx$

(Take $u = (\log x)^3$ and $dv = dx$)

33.

(Take $u = (\log x)^2$ and $dv = dx$)

34.

35. The marginal cost function for producing x units is given by $y' = 10 + 24x - 3x^2$; if the cost of producing one unit is 25, find the total cost function and the average cost function.

36. The marginal cost function of manufacturing x shoes is $6 + 10x - 6x^2$. The cost of producing a pair of shoes is 12. Find the total and average cost function. (Hint: Given $C = 12$ when $x = 1$)

37. The marginal cost function is $MC = 2 + 5e^x$

1. find C if $C(0) = 100$
2. Find AC
3. Calculate C, MC and AC for $x = 60$.

38. The marginal cost of production is found to be $MC = 1000 - 20x + x^2$ where x is the number of units produced. The fixed cost of production is 9000. Find the cost function.

39. A company has determined that the marginal cost function for a product of a particular commodity is given by $MC = 125 + 10x - (x^2/9)$. If the fixed cost is 250, what is the cost of producing 15 units? (Hint: Given $C = 250$ when $x = 0$. Find C by taking $x = 15$)

40. The marginal cost of production is found to be $MC = 2000 - 320x + 3x^2$ where x is the number of units produced. The fixed cost of production is 18000. Find the cost function. If the manufacturer fixed the price per unit at 6800.

1. Find the revenue function
2. Find the profit function
3. Find the sales volume that yields maximum profit.
4. What is the profit at this sales volume.

41. The marginal cost of production of a firm is given by $C'(x) = 5 + 0.13x$. Further marginal revenue $R'(x) = 18$. Also it is given that $C(0) =$ 120. Compute the profit function and the total profit. (Hint: To get the value of x, equate $C'(x) = R'(x)$)

42. If the marginal revenue function is $R'(x) = 15 - 9x - 3x^2$ find the Revenue function. (x − be the number of items sold)

43. The marginal revenue function of a firm is $((ab)/(x - \mathrm{b})^2) - C$. Find the total revenue function and hence prove AR = $((a/b) - x) - C$.

44. A business organization made an analysis of production which show that with present equipment and workers, the production is 10000 units per day. It is estimated that the rate of change of production P with respect to change in the number of additional

works x is What is the production (expressed in units per day) with 25 additional workers? (Hint: When $x = 0$; $P = 10000$)

45. A firm has the marginal revenue function given by MR = $(a/(x + b)^2) - C$, where x is the output and a, b, c are constants. Show that the demand function is given by $p = ((a/b)(x + b) - c$.

46. A firm has marginal revenue function given by MR = $(a/(x + b)) - c$ where x is the output and a, b, c are constants. Show that the demand function $p = (a/x) \log((x + b)/b) - c$.

47. If the marginal revenue and marginal cost of output of a commodity are given as MR = $5 - 4x + 3x^2$ and MC = $3 + 2x$ and if the fixed cost is zero find the profit function and the profit when the output is 4.

48. The marginal revenue and marginal cost of output of q is given by $dR/dq = 1/2q^{-1/2}$, where R stands for total revenue. What is the demand function?

49. The marginal revenue function of a monopolist is given as $R'(x) = 50 - 0.0002\, x^2$, where R' denotes marginal revenue and x denotes the quantity produced and sold. It is known that total revenue is zero, when $x = 0$. Find the market demand function for the commodity.

50. The demand and supply functions under pure competition are $p = 16 - x^2$ and $s = 2x^2 + 4$. Find the consumer's surplus and the producer's surplus. (Hint: equate the demand function with the supply function to get x).

51. The demand function of a commodity is $p = 100 - 5D - D^2$ (p represents price & D represents the quantity). Find the consumer's surplus when (i) $p = 16$, (ii) $p = 34$ (iii) $p = 50$.

52. Find the consumer's surplus and the producer's surplus defined by the demand curve $p(x) = 20 - 5x$ and the supply curve $s(x) = 4x + 8$.

53. Under pure competition for a commodity, the demand and supply laws are

$p_d = (8/(x + 1) - 2$, and $p_s = (x + 3)/2$ respectively

Determine the consumer's surplus and producer's surplus.

(Hint: to get x, equate p_d with p_s i.e., $p_d = p_s$)

54. The demand and supply function under perfect competition is $p_d = 16 - x^2$ and $p_s = 2x^2 + 4$ respectively. Find the market price and producer's surplus.

55. The demand function for a commodity is $p = e^{-x}$. Find the consumer's surplus when $p = 1/2$.

ANSWERS

1. $(x^3/3) + 2x - (1/x)$
2. $e^x - x$
3. $(2/3)\, x^{3/2}$
4. $(x^4/4) + (4/3)x^3 - (5/2)x^2 - 6x$
5. $(x^5/5) - 4(x^3/3) + 4x$
6. 4
7. log 2
8. $(2/3) + 2\tan^{-1} 2$

9.

10.

11.
12. (98/3)
13. 1/5

14.
15. (2/3)
16. $\tan^{-1}(x^2)$
17. $(1/4)\, e\, x^3$
18. $\log(1+x^3)$
19. $\log(1 + e^x)$
20. $\log(\log x)$
21. $\log(x^2 - 1)$

22.

23.

24.

25. $-\log(x+1) - 4\log(x+2) + (9/2)\log(x+3)$
26. $x^2 + 3x + 3\log(x-1)$
27. $2\log(x+3) - \log(5x^2 + 2x + 3)$

28.

29.

30.
31. $e^x \log x + c$
32. $x(\log x)^3 - 3x(\log x)^2 + 6x\log x - 6x$
33. $10(\log 10)^2 - 20(\log 10) + 18$
34. 1/2
35. $C = 10x + 12x^2 - x^3 + 4$

$\text{AC} = 10 + 12x - x^2 + (4/x)$

36. $C = 6x + 5x^2 - 2x^3 - 4$

$\text{AC} = 6 + 5x - 2x^2 - (4/x)$

37. $C = 2x + 5ex + 95$

$\text{AC} = 2 + 5(e^x/x) + (95/x)$

$(C)x = 60 = (3/5) + 5e^{60}$

$(\text{MC})x = 60 = 2 + 5e^{60}$

$(\text{AC})x = 60 = 3.6 + 0.08\, e^{60}$

38. $C = 9000 + 1000x - 10x^2 + (x^3/3)$
39. $C = 125x + 5x^2 - (x^3/27) + 250$;

$(C)_{15} = \quad 3125$

40.
    1. $R = 2000x - 160x^2 + x^3 + 18000$
    2. $P = 6800x - 2000x + 160x^2 - x^3 - 1800$
    3. when $x = 120$; the profit is maximum
    4. the maximum profit is 17,46,000
41. $P = 13 - 0.065x^2 - 120$

Profit function is maximum $x = 100$

maximum profit = 530

42. $R = 15x - (q/2)x^2 - x^3$

44. $P = 200x - 2x^{3/2} + 10000$;

Production = 14750

47. $P = 2x - 3x^2 + x^3$;

$P(x = 4) = 24$

48.

49.

50. $x = 12$, CS = (16/3); PS = (32/3)

51.

***P*( )**	**CS( )**
16	351.17
34	234
50	145.83

52. CS = (40/9); PS = (96/9)

53. CS = 8(log2) − 4; = 1/4

54. $x = 2$, p = 12,CS = (16/3), PS = (32/3)

55. $x = \log 2$; CS = (1/2) [1 − log2]