CHAPTER 16

Application of Matrix Concept

16.1 INPUT−OUTPUT ANALYSIS

The mathematical model *Input−Output Analysis* was initially proposed by Wassily W. Leotief in 1930's. This model was built based on the concept 'economic interdependence' which implies that every industrial sector of economy is closely related to each other type of sector. That is, the different types of sectors are all inter-dependent and highly inter-related. Due to this close relatedness, if there is any change in one sector (like strike etc.) it will affect all other industries to a varying degree.

16.1.1 Assumptions of a Model

1. An economy is segregated into 'n' different sectors (industries), and each of these produces only one type of product. Each sector uses input that is nothing but the output of the other sectors.

   Let x_j (j = 1, 2, …, n) be the gross production (output) of the jth sector.

2. Let a_{ij} stands for a rupee value of the output from sector i which sector j must consume to produce one rupee worth of its own product. The same can be evaluated as

   The technical input−output coefficient Matrix A can be given as

   Matrix A remains unchanged so long as the structure of the economy remains unchanged.

3. The gross product of each sector is sufficient to satisfy the final demand and along with the demands of other sectors

Let d_j (j = 1, 2, …, n) be the final demand for product produced by each of n-sectors. The unitization is in terms of rupee value.

If the economy is assumed to be in a state of equilibrium, we have

Output = Input

= Need of each sector + Final demand

i.e.,

System (1) is linear it can be represented in a matrix form

$$X = AX + D \qquad (2)$$

We know that using Eq. (2) in Eq. (3),

$$IX = X \qquad (3)$$
$$IX = AX + D$$
$$IX - AX = D$$
$$(I - A)X = D \qquad (4)$$

Assume that $(I - A)^{-1}$ exists.

Premultiplying by $(I - A)^{-1}$ on both the sides of Eq. (4),

$$(I - A)^{-1} (I - A)X = (I - A)^{-1} D$$
$$IX = (I - A)^{-1} D$$
$$X = (I - A)^{-1} D$$

Each sector should produce *X* amount in order to satisfy the final demand as well as the demand of all sectors themselves.

EXAMPLES

1. Given the following input–output table, calculate the gross output so as to meet the final demand of 200 units of agriculture and 800 units of industry.

   Find the total demand using the relation.

Total demand = Agriculture + Industry + Final demand

Construct the technological matrix A,

where

Likewise,

Hence,

Given

Find

$= 0.28 - 0.12$

$= 0.16$

$| I - A | \neq 0$ implies $(I - A)^{-1}$ exits.

Hence, the gross output of agriculture and industry must be 2000 units and 4000 units, respectively.

2. Let the input−output coefficient Matrix for the two Industries be given by

and final demand vector be . It is required to know how much each industry should produce in order to meet the final demand.

Given the technological matrix

and

Find

The gross-output $X = (I - A)^{-1} D$

3. From the following inter-industry transaction demand table, evaluate the output when the final demand vector changes to

The technological matrix A can be given as

and

Find $(I - A)$

Find $| I - A |$

$$| I - A | = 105/896$$

$$X = (I - A)^{-1} D$$

Hence, the required gross output for the given demand is 128, 168 and 112 for the industries A, B and C, respectively.

16.2 MARKOV ANALYSIS

Markov analysis is a process of analysing the current movement of some variable in an effort to forecast its future movement. It is mainly used as a management tool in the marketing area to examine and predict the behaviour of customers from the stand point of the loyalty to one brand and their switching patterns to other brands.

16.2.1 Stochastic Process

A collection of random variables that are indicated by a parameter such as time and space are known as stochastic process. It is also called random process or chance process. Consider that there are *r* cells and an infinitely large number of identical balls and that one throw equally likely to get into any of the cells.

Suppose X_n is the number of occupied cells after *n* throws,

Then $\{X_n; n \geq 0\}$ constitute a stochastic process.

16.2.2 Markov Process

If $\{X(t); t \in T\}$ is a stochastic process such that given the values of $X(s)$ the values of $X(t)$, $t > s$ does not depend on the values of $X(t)$, $t < s$. Then the process is said to be Markov process.

It helps to identify

1. a specific state of the system being studied.
2. the state−transition relationship

16.2.3 One Stage Transition Probability

The occurrence of an event at a specified point in time put the system in state S_n; if after the passage of one unit of time, another event occurs, that is the system moved from the state S_n to S_{n+1}. This movement is related to a probability distribution, there is a probability associated with each (move) transition from event S_n to S_{n+1}. This probability distribution is called one-stage transition probability.

EXAMPLE

ABC consultants have been hired to conduct a market response study regarding the introduction of a new product.

1st row: If the system is currently in state A, the probability that “now + 1” will find the system in state A is 0;

Movement from A to B is 0.8

Movement from A to C is 0.1

16.2.4 Markov Chain

It is an ordered series of states linked by an appropriate transition matrix, a rectangular array in which the elements are transition probabilities which are such that the probability of an event in time period (n + 1) depends only on the state of system in time period n.

16.2.5 State Transition Matrix

It is a rectangular array which summarizes the transition probabilities for a given Markov process. In such a matrix, the rows identify the current state of the system being studied and the columns identify the alternative states to which the system can move.

The general structure can be given as

Each element represents the probability that a customer will change his liking from one brand to the other in his next purchase. This is the reason for the calling then transition probabilities and

If we know the transition probabilities matrix at time $t = n$, then the successive transition probabilities can be evaluated for $t = n + 1, n + 2, \ldots$ using the recursive equation.

$$p(n + 1) = p(n) \times \mathrm{P};\ (n = 0, 1, 2, \ldots)$$

EXAMPLE

Given brand switching analysis of 7000 women from a society with three different brands.

Find the position in state 2.

Step 1: Find the transition probability matrix.

Step 2: Assume the initial state probabilities as

$$p_X(0) = [1\ 0\ 0]$$
$$p_Y(0) = [0\ 1\ 0]$$
$$p_Z(0) = [0\ 0\ 1]$$

Step 3: We know that

$$p_j(n + 1) = p_j(n) \times \mathrm{P};\ j = (x, y, z) \qquad (1)$$

Transition State 1: Put $n = 0$ in (1),

Transition State 2: Put $n = 2$

Similarly, we can proceed to get any transition state probabilities.

16.2.6 Steady State

For a problem, the position of the states at time (t) and ($t + 1$) are the same implies that the system attains its steady state. After that, it is in the same state at any time period. If P is a matrix of order $n \times n$ and π the steady state vector, then we have

16.2.7 Existence of the Steady State

If P is the transition probability matrix, find P^2 if all the elements in that matrix are +ve (>0) then it has a steady state.

EXAMPLES

1. The purchase patterns of two brands of toothpaste can be expressed as a Markov process with the following transition probabilities.

	Formula A	Formula B
Formula A	0.9	0.1
Formula B	0.05	0.95

2. Determine the market shares of each of the brand in equilibrium position.
3. Given

4.

5. Since P is of order 2 × 2, then the steady-state vector π can be defined as
6.
7. $\pi = [\pi_1 \; \pi_2]$ and $\pi_1 + \pi_2 = 1$ (1)
8. Steady-state condition,
9.
10. $\pi = \pi \times P$

11. $[\pi_1 \; \pi_2] = [\pi_1 \pi_2] \times$
12. $= [0.9\pi_1 + 0.05\pi_2 \;\; 0.1\pi_1 + 0.95\pi_2]$
13. $\Rightarrow \pi_1 = 0.9\pi_1 + 0.05\pi_2 \Rightarrow -0.1\pi_1 + 0.05\pi_2 = 0$ (2)
14. $\pi_2 = 0.1\pi_1 + 0.95\pi_2 \Rightarrow 0.1\pi_1 - 0.05\pi_2 = 0$ (3)
15. Consider Eqs (1) and (2)
16.
17. $(1) \Rightarrow \pi_1 + \pi_2 = 1$
18. $(2) \times 10 \Rightarrow -\pi_1 + 0.5\pi_2 = 0$
19. -------------------
20. Add:

21. $1.5\pi_2 = 1$

22. $\pi_2 = 2/3$

23. $\Rightarrow \pi_1 = 1 - \pi_2 = 1 - 2/3 = 1/3$

24. The steady state can be given as [1/3, 2/3]. Hence, the expected market shares in an equilibrium condition for formula A will be 0.33 and that of formula B will be 0.67.

25. There are three grocery stores (1, 2, 3) in a town. The stores conducted a study on the number of customers that each store retained, gained or lost from time A to B assuming that the number of total customer during the period remained the same. The following is the gain and loss results of the study:

Construct the state transition matrix.

Given

Elements (2, 1) = 50 and (3, 1) = 50 ⇒ out of 500 customers of store 1, 50 customers moved to store 2 and store 3, respectively.

Hence, the retention is 500 − (50 + 50) = 400

$$\Rightarrow p_{11} = 400/500 = 0.8$$

$$p_{12} = 50/500 = 0.1$$

$$p_{13} = 50/500 = 0.1$$

Element (1, 2) = 100 ⇒ Out of 400 customers of store 2, 100 customers moved to store 1. Hence the retention is

$$400 - 100 = 300$$

$$\Rightarrow p_{22} = 300/400 = 0.75$$

$$p_{21} = 100/400 = 0.25$$

$$p_{23} = 0/400 = 0$$

Similarly,

$$p_{33} = 475/500 = 0.95$$

$$p_{32} = 25/500 = 0.05$$

$$p_{31} = 0/500 = 0$$

Hence, the required transition probability matrix P = [p_{ij}]; i = 1, 2, 3, j =1, 2, 3

26. A customer owning a Maruti car right now has got the option to switch over to Maruti, Ambassador or Fiat next time with probability of (0.2 0.5 0.3). The following is the transition matrix.

Find the probabilities with his 4th purchase.

Given

We are given that the state 1 probabilities p_1 = [0.2 0.5 0.3].

EXERCISE

1. For the Markov Chain P

   Determine the steady-state probability distribution.

2. A marketing research firm has just completed a survey of consumer buying habits with respect to three brands of coffee. It estimates that at the present time, 40% of the customers buy brand A, 20% of the customers buy brand B and 40% of the customers buy brand C. Additionally, the marketing research firm has analysed its survey data and has determined that the following brand-switching matrix is appropriate for the three brands of coffee.

   What will be the expected distribution of customers two time periods later?

3. Consider the Matrix of transition probabilities of a product available in the market in two brands:

	Brand A	**Brand B**
Brand A	0.9	0.1
Brand B	0.3	0.7

4. Determine the market shares of each brand in equilibrium position.

5. For the following input−output table, calculate the technology matrix and write the balance equation for two sectors.

6. The following table gives the input−output coefficients for a two-sector economy consisting of agriculture and manufacturing industry.

Producing Sector	**User Sector**	
	A	**M**
A	0.10	1.50
M	0.20	0.25

7. The final demands for the two industries are 300 and 100 units, respectively. Find the gross output of two industries.

8. From the following transaction matrix, find the total output for the final demands of 180 and 440 units for Industry I and II, respectively.

9. In an economy, there are two industries A and B and the following table gives the supply and demand position of these in lakhs of rupees.

Determine the total output if the final demand changes to 12 for A and 18 for B.

10. Given the following transaction matrix, find gross output to meet the final demand of 300 units of agriculture and 900 units of industry.

ANSWERS

1. $\pi_1 = 1/2$ and $\pi_2 = 1/2$
2. [0.336, 0.416, 0.248]
3. $\pi_1 = 1/2$ and $\pi_2 = 1/2$

4. A

5. X

6.

7.

8.