Homework 6 solutions

1. a. Is $f(x,y) = x^2+y^3-3xy$ convex?

The gradient of f is $\begin{vmatrix} 2x-3y \\ 3y^2-3x \end{vmatrix}$

So the hessian is $\begin{vmatrix} 2 & -3 \\ -3 & 6y \end{vmatrix}$

The determinant of the hessian is 12y-9. If y=0, then the determinant is negative and so the function is not convex.

2b. Is $f(x,y,z) = x^2+y^2 +z^2+6x+6y+6z+2xyz$ convex?

The gradient of f is $\begin{vmatrix} 2x +6+yz \\ 2y+6+xz \\ 2z+6+xy \end{vmatrix}$

So the hessian is $\begin{vmatrix} 2 & 2z & 2y \\ 2z & 2 & 2x \\ 2y & 2x & 2 \end{vmatrix}$

The determinant of the first square principle diagonal matrix is 2.

The determinant of the second square principle diagonal matrix is $4 – 4z^2$, which is negative if z>1, and so the function is not convex.

2. a Starting with x_l=-5 and x_r=11, perform the bisection method for 4 iterations on $f(x)=x^4+x^2$

$f'(x)=4x^3+2x$

x_{left}	x_{mid}	x_{right}	$f'(x_{mid})$
-5	3	11	114
-5	-1	3	-6
-1	1	3	6
-1	0	1	0

Since $f'(x_{mid})$ =0, a local optimal solution occurs at 0. If not I would have reported a local optimal exists between –1 and 1 and my best guess is to let x=0.

2. b Starting with x_l=-5 and x_r=11, perform the bisection method for 4 iterations on $f(x)=4x^4-x^2+5$
$f'(x)=16x^3-2x$

x_{left}	x_{mid}	x_{right}	$f'(x_{mid})$
-5	3	11	426
-5	-1	3	-14
-1	1	3	14
-1	0	1	0

Since $f'(x_{mid}) = 0$, a local optimal solution occurs at 0. If not I would have reported a local optimal exists between –1 and 1 and my best guess is to let x=0.

3a. Starting with x_l=-5 and x_r=11, perform the golden search method for 4 iterations for $f(x)=x^4+x^2$.

xleft	xmid1	xmid2	xright	f(xleft)	f(xmid1)	f(xmid2)	f(xright)
-5	1.112	4.888	11	650	2.765585	594.7462	14762
-5	-1.22278	1.110784	4.888	650	3.730826	2.756205	594.7462
-1.22278	1.111535	2.553681	4.888	3.730826	2.761999	49.04843	594.7462
-1.22278	0.217273	1.106943	2.547	3.730826	0.049436	2.726735	48.57109

I know a local optimal solution exists between –1.22 and 1.1. My best guess is x=.217 with a objective value of .05.

3b Starting with x_l=-5 and x_r=11, perform the golden search method for 4 iterations on $f(x)=4x^4-x^2+5$

xleft	xmid1	xmid2	xright	f(xleft)	f(xmid1)	f(xmid2)	f(xright)
-5	1.112	4.888	11	2480	9.87962	2264.522	58448
-5	-1.22278	1.110784	4.888	2480	12.4473	9.855614	2264.522
-1.22278	1.111535	2.553681	4.888	12.4473	9.87044	168.5873	2264.522
-1.22278	0.217273	1.106943	2.547	12.4473	4.961707	9.780332	166.8483

I know a local optimal solution exists between –1.22 and 1.1. My best guess is x=.217 with a objective value of 4.96.

4a. From a starting points of 0 and 1, perform the intial search algorithm to determine an optimal region for $x^2-10x+25$.

xleft	xmid	xright	f(xleft)	f(xmid1)	f(xright)
0	0.5	1	25	20.25	16
0	1	2	25	16	9
0	2	4	25	9	1
0	4	8	25	1	9

I know that there exists a local optimal between 0 and 8. If I wanted to find it I would use x_{left}=0 and x_{right} =8 in either bisection or the golden search methods.

4a. From a starting points of 0 and 1, perform the intial search algorithm to determine an optimal region for x^2-.2x+.04.

xleft	xmid	xright	f(xleft)	f(xmid1)	f(xright)
0	0.5	1	0.04	0.19	0.84
0	0.25	0.5	0.04	0.0525	0.19
0	0.125	0.25	0.04	0.030625	0.0525

I know that there exists a local optimal between 0 and .25. If I wanted to find it I would use x_{left}=0 and x_{right} =.25 in either bisection or the golden search methods.

5a. Perform two iterations of the gradient search method on $f(x,y)= x^2+4xy+2y^2+2x+2y$. Use (0,0) as a starting point.

The gradient of f is $\begin{vmatrix} 2x+4y+2 \\ 4x+4y+2 \end{vmatrix}$

d^1= the negative of the gradient of f at the point (0,0) = $-\begin{vmatrix} 2 \\ 2 \end{vmatrix}$

So
$x^1=(0,0) +\lambda(-2,-2) = (-2\lambda,-2\lambda)$.

Plugging this into f(x,y) results in
$f(\lambda)=4\lambda^2+16\lambda^2+8\lambda^2-4\lambda-4\lambda=28\lambda^2-8\lambda$
Taking the deriviative and setting it equal to 0 to find the min results in
$f'(\lambda)=56\lambda-8=0$
$\lambda=1/7$

So $x^1=(-2/7,-2/7)$.

d^2= the negative of the gradient of f at the point (-2/7,-2/7) = $-\begin{vmatrix} 2/7 \\ -2/7 \end{vmatrix}$

So
$x^1=(-2/7,-2/7) +\lambda(-2/7,2/7) = (-2/7-2\lambda/7, -2/7+2\lambda/7)$.

Plugging this into f(x,y) results in the following simplified expression.

$f(\lambda)=-4/49\lambda^2 -8/49\lambda-28/49$

Taking the deriviative and setting it equal to 0 to find the min results in
$f'(\lambda)= -8/49\lambda -8/49 =0$
$\lambda=-1$

So $x^2=(0,-4/7)$.

6. Find all KKT points of

$$\begin{array}{ll} \text{Minimize} & -x^3-xy \\ \text{Subject to} & x+y=4 \\ & x\leq 2 \end{array}$$

So the KKT conditions are

$$\begin{vmatrix} -3x^2-y \\ -x \end{vmatrix} + u \begin{vmatrix} 1 \\ 0 \end{vmatrix} + v \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad \begin{array}{l} \text{(i)} \\ \text{(ii)} \end{array}$$

$u(x-2)=0$ (iii)
$u\geq 0$ (iv)
$x\leq 2$ (v)
$x+y=4$ (vi)

Case 1: $u>0$
Thus $x=2$, from the (iii). By (vi) $y=2$. Using this in (ii) results in $v=2$. So plugginf all of this into (i) results in $-12-2+u+2=0$ and so $u=12$.

There is a KKT point at $x=2$, $y=2$, $u=12$, $v=2$. It's objective value is -12.

Case 2: $u=0$.
So (i) becomes $-3x^2-y \ +v = 0$. From (ii) we get $x=v$, and substituting this into the new (i), we get $-3x^2-y \ +x = 0$. From (vi) we get $y=4-x$. Substituting this into the most recent version of (i), $-3x^2-4+x \ +x = 0$. So $-3x^2+2x-4 = 0$. Using the quadratic formula results in$(-2 \pm (4-48)^{1/2})/6$. Since this is an imaginary number, there are no KKT points with $u=0$.

The only KKT point is at $x=2$, $y=2$, $u=12$, $v=2$. It's objective value is -12.