



Two-Way ANOVA

Part 3: ANOVA Table and F-Tests

STAT 705: Regression and Analysis of Variance

Introduction

- We continue to work with two-way data, concentrating on main effects and interaction effects
- So far, we have used *population means* to define main effects and interactions effects
- Now we move to *sample data*
 - Develop formal hypothesis tests to determine if effects are statistically significant
 - Define the test statistics
 - Display the test results

Hypotheses We Want to Test

- These are based on the interaction model

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}, \text{ i.e. } Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

- A main effects hypothesis: $H_0: \alpha_i = 0$ for all levels of A
vs. H_a : not all α_i 's are 0
- B main effects hypothesis: $H_0: \beta_j = 0$ for all levels of B
vs. H_a : not all β_j 's are 0
- Interaction hypothesis: $H_0: (\alpha\beta)_{ij} = 0$ for all treatments
vs. H_a : not all $(\alpha\beta)_{ij}$ are 0

Test Statistics for the Hypotheses

- The test statistics depend on
 - point estimates for the parameters, and
 - separating the total variability in the response into components attributed to
 - » A main effects
 - » B main effects
 - » interaction effects

Least Squares Estimates for Means

- Grand mean

$\hat{\mu}$ = average of all observations

- Treatment (i.e. cell) means

$\hat{\mu}_{ij}$ = average of all observations in treatment (A_i, B_j)

- Marginal mean for i^{th} level of factor A

$\hat{\mu}_{A_i}$ = average of all the cell means involving factor A_i

- Marginal mean for j^{th} level of factor B

$\hat{\mu}_{B_j}$ = average of all the cell means involving factor B_j

Technical note: These definitions are based on Type 3 sums of squares, which lead to Type 3 estimates and hypothesis tests. There are also Type 1, Type 2 and Type 4 definitions, but we will not consider those in this course. Differences between the four types arise only when the data are not balanced.

Least Squares Estimates for Effects

- To estimate the main effects and interaction effects, use
 - a) the definitions for these effects, and
 - b) the least squares estimates for the means (on the previous slide)
- The least squares estimates are
 - Effect of the i^{th} level of Factor A: $\hat{\alpha}_i = \hat{\mu}_{Ai} - \hat{\mu}$
 - Effect of the j^{th} level of Factor B: $\hat{\beta}_j = \hat{\mu}_{Bj} - \hat{\mu}$
 - Interaction effect: $\hat{\mu}_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j$
- Note that if we substitute the estimates for α_i and β_j , the interaction effect can be written $\hat{\mu}_{ij} - \hat{\mu}_{Ai} - \hat{\mu}_{Bj} + \hat{\mu}$

Partitioning the Total Variation

- Test statistics for the hypotheses are based on the previous estimates and separating (partitioning) the total variation in the response
- This is the same approach that we used in regression
- The total variation in the response ($SSTot$) is split into
 - 1) the variation that is explained by the model ($SSModel$), and
 - 2) the variation that is not explained (SSE)

$$\begin{aligned} SSTot &= \sum (Y_{ijk} - \bar{Y})^2 = \sum (Y_{ijk} - \hat{\mu})^2 \\ &= \sum (Y_{ijk} - \hat{Y}_{ijk} + \hat{Y}_{ijk} - \hat{\mu})^2 \\ &= \underbrace{\sum (Y_{ijk} - \hat{Y}_{ijk})^2}_{\text{SSE}} + \underbrace{\sum (\hat{Y}_{ijk} - \hat{\mu})^2}_{\text{SSModel}} \end{aligned}$$

Partitioning the Model SS

We continue partitioning by splitting the model sum of squares into components for the effects

$$\hat{Y}_{ijk} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + (\hat{\alpha}\hat{\beta})_{ij}$$

$$\hat{Y}_{ijk} - \hat{\mu} = \hat{\alpha}_i + \hat{\beta}_j + (\hat{\alpha}\hat{\beta})_{ij}$$

$$\hat{Y}_{ijk} - \hat{\mu} = (\hat{\mu}_{Ai} - \hat{\mu}) + (\hat{\mu}_{Bj} - \hat{\mu}) + (\hat{\mu}_{ij} - \hat{\mu}_{Ai} - \hat{\mu}_{Bj} + \hat{\mu})$$

Square each term and sum over all observations

$$\sum (\hat{Y}_{ijk} - \hat{\mu})^2 = \sum (\hat{\mu}_{Ai} - \hat{\mu})^2 + \sum (\hat{\mu}_{Bj} - \hat{\mu})^2 + \sum (\hat{\mu}_{ij} - \hat{\mu}_{Ai} - \hat{\mu}_{Bj} + \hat{\mu})^2$$

$$\mathbf{SSModel} = \mathbf{SSA} + \mathbf{SSB} + \mathbf{SSAB}$$

These sums of squares will be used to construct test statistics

Mean Squares

- Degrees of freedom
 - For the A main effect, with 'a' levels, $df_A = a - 1$
 - For the B main effect, with 'b' levels, $df_B = b - 1$
 - For the interaction, $df_{AB} = (a - 1)(b - 1)$
- Mean square is the sum of squares divided by the degrees of freedom
 - $MSA = SSA / df_A$
 - $MSB = SSB / df_B$
 - $MSAB = SSAB / df_{AB}$

The Test Statistics

- For the A main effect: H_0 is $\alpha_i = 0$ for all levels of A
 - Test statistic is $F = MSA / MSE$
 - Use F distribution with degrees of freedom df_A and df_E
- For the B main effect: H_0 is $\beta_j = 0$ for all levels of B
 - Test statistic is $F = MSB / MSE$
 - Use F distribution with degrees of freedom df_B and df_E
- For the A*B interaction: H_0 is $(\alpha\beta)_{ij} = 0$ for all treatments
 - Test statistic is $F = MSAB / MSE$
 - Use F distribution with degrees of freedom df_{AB} and df_E

ANOVA Table for Two-Way Data

- The ANOVA table provides a concise way of recording the information we have been calculating
- N is the total number of observations (in all groups)
- We will rely on software to perform most of these calculations (especially the p-values)

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
A main effect	$a - 1$	SSA	SSA/df_A	MSA / MSE	
B main effect	$b - 1$	SSB	SSB / df_B	MSB / MSE	
A*B interaction	$(a - 1)*(b - 1)$	SSAB	$SSAB / df_{AB}$	$MSAB / MSE$	
Error	$N - ab$	SSE	SSE / df_E		
Total	$N - 1$	SSTot			

Fabric Data Re-visited

Level	Salt	Temperature
1	Untreated	812
1	Untreated	827
1	Untreated	876
2	Untreated	945
2	Untreated	881
2	Untreated	919
1	CaCO ₃	733
1	CaCO ₃	728
1	CaCO ₃	720
2	CaCO ₃	786
2	CaCO ₃	771
2	CaCO ₃	779
1	CaCl ₂	725
1	CaCl ₂	727
1	CaCl ₂	719
2	CaCl ₂	756
2	CaCl ₂	781
2	CaCl ₂	814

- Fabric was treated with one of three different inorganic salts, at one of two levels of concentration, for the purpose of measuring its effect on the flammability of the fabric
- Response variable is the temperature at which the fabric ignites

Fabric Data: ANOVA Table

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Level	1	17734.72	17734.72	34.11	<.0001
Salt	2	60928.78	30464.39	58.60	<.0001
Level*Salt	2	486.11	243.06	0.47	0.6375
Error	12	6238.67	519.89		
Corrected Total	17	85388.28			

Verify the degrees of freedom

- Factor A is 'Level'; number of levels is 2; $df_A = 1$
- Factor B is 'Salt'; number of levels is 3; $df_B = 2$
- Interaction: $df_{AB} = 1 * 2 = 2$
- Total number observations = N
= (number of A's)*(number of B's)*(number of replications)
= $2 * 3 * 3 = 18 \Rightarrow df_{Tot} = 17$

Interpret the ANOVA Table

- **Examine the interaction first.**
 - It is not significant ($p = 0.6375$). This implies
 - » If Salt impacts Temperature, the effect is the same for both Level 1 and Level 2, and
 - » If Level impacts Temperature, the effect is the same for all three Salts
- Since the interaction is not significant, we can interpret the main effects.
 - Salt does affect Temperature ($F = 58.60, p < 0.0001$)
 - Level does affect Temperature ($F = 34.11, p < 0.0001$)

Relation to Nested Model F Test

- Each of the hypothesis tests for main effects and interactions can be viewed as a comparison-of-models F test
 - This is also called a 'nested model' F test
 - We discussed this in Multiple Regression, Part 4
- Consider the test for interaction. $H_0: (\alpha\beta)_{ij} = 0$ for all treatments
 - Full model is the interaction model: $Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$
 - Reduced model is the additive model: $Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$
- The test for interaction is comparing these two models.
 - If we reject H_0 , then we need to keep the full (interaction) model.
 - If we do not reject H_0 , then we can use the reduced (additive) model.

What You Should Know

- The concept behind partitioning the sums of squares
- How to interpret the ANOVA table
 - Look first at the interaction hypothesis
 - If there is no interaction, then consider the tests of main effects
 - If there is interaction, subsequent analysis gets more complicated (more about this later)
 - Know the hypotheses for each F test
 - Be able to hand-calculate the degrees of freedom