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**26 April 2013**

# Approaches to Operations Research Guide

**TRADOC Analysis Center**
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The Department of Defense analytic community commonly uses a small set of Operations Research (OR) and statistics techniques. More techniques may be available that would improve decision support analyses. Therefore, the Methods and Research Office developed a reference guide that collects and presents the breadth of OR techniques available within the analytic community in order to improve TRAC studies and analyses. This guide has two main purposes: 1) Provide TRAC leadership with a quick reference to enable effective use of OR techniques for decisions analysis support, and 2) Provide TRAC analysts with a reference to enhance their analytic skills and provide exemplar approaches to applications for various OR techniques.

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# Approaches to Operations Research Guide (AORG)

Operations research (OR) was first formally applied to warfare applications with the formation of the Anti-Aircraft Command Research Group (AACRG) within the British Ministry of Supply and its tasking to improve the British air defense system. This initial OR team was established in September 1940 and led by Sir Patrick Blackett; the team came to be known as "Blackett's Circus."[1] Blackett's Circus rapidly began to apply formal mathematical techniques to a wide range of problems, including air defense, tank lethality, information flow, etc.

In 1942, an offshoot of the AACRG was established to directly oversee the work of all British Army OR analysts; this was the Army Operational Research Group (AORG).[2] Personnel from AORG landed at Normandy in 1944 and followed British forces as they advanced across Europe, analyzing a wide range of equipment-related topics (e.g., artillery effectiveness, aerial bombing, anti-tank effectiveness) and providing analysis to operational commanders.

In keeping with the historical foundation of OR and to recognize the contributions of these pioneers in our profession, we have chosen to use the same acronym for our document.

## How to Use this Guide

This guide has two main purposes:

- Provide TRAC leadership with a quick reference to enable effective use of OR techniques for decision analysis support.
- Provide TRAC analysts with a reference to enhance their analytic skills and provide exemplar approaches to applications for various OR techniques.

This guide does not provide step-by-step instructions on how to perform or use individual techniques. Instead, it provides general technique outlines and past, and potential future, relevant applications. Each technique outline provides the necessary assumptions that must be made to apply the method, limitations of the method, other considerations from applications lessons learned, resources and tools that can be used to execute, and references containing more detailed descriptions of the technique. Also, if a word is blue and underlined, that means it can be found in the terms of reference and glossary.

The following two pages highlight various applications to which these techniques may be applied. In some cases, the application may be a situation in which you find yourself (e.g., small data set). The exemplar guide shows techniques you may want to consider. The techniques may not fit each application exactly, but they give an idea of where to look when conducting certain applications.

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[1] The principal members of "Blackett's Circus" were: P. M. S. Blackett, physicist (1948 Nobel Laureate in Physics); B. Schonland, physicist; D. K. Hill, physiologist; A. F. Huxley, physiologist (1963 Nobel Laureate in Physiology/Medicine); L. E. Bayliss, physiologist; A. Porter, mathematical physicist; F. R. N. Nabarro, mathematical physicist; H. E. Butler, astrophysicist; I. Evans, physicist; G. W. Raybould, surveyor; A. J. Skinner, mathematician; and M. Keast, mathematician.

[2] A second OR organization with the same name and acronym was established in 1948, incorporating elements from the original World War II organization. AORG, in 1962, was renamed the Army Operational Research Establishment and was incorporated into the Defence Evaluation Research Agency (DERA) in 1995. DERA was sliced into the Defence Science and Technology Laboratory and Qinetiq in 2001.

# Exemplar Applications

| Application Category | Exemplar Applications (Not exhaustive list) |
|---|---|
| **Modeling Queues** | Node-arc systems (e.g., supply chains, etc.); queue arrivals (e.g., units crossing a bridge). |
| **Using Binary Data/Systems** | Binary state systems (e.g., missile hit or miss). |
| **Survey Data (with either a small or large set of data available)** | SME estimates (e.g., current OE state (based on quantitative and qualitative info)). |
| **Estimating Distributions** | Processing times and when point estimates are unknown (e.g., MI section intelligence processing time, SME scalar inputs on time to assess a situation, etc.). |
| **Risk Assessment** | COA impact assessments. |
| **Cost** | Cost estimates, cost trade assessments, cost benefit analysis. |
| **Comparing Populations** | Assessing differences between two data populations (e.g., different alternatives in multiple simulation runs, two SME population assessments). |
| **Analyzing Model Outputs** | Assessing differences for multiple alternatives through model outputs. |
| **Finding Relationships in Data** | Cause and effect relationships (e.g., miles driven vs number of IEDs encountered). |
| **Developing/ Examining Ranked Data** | Assessing similarity between multiple SME COAs/options/capability comparative utility rankings. Dealing with 1-N rankings. |
| **Optimization** | Calculating an optimal value for a multi-variate system (e.g., optimizing number of reconnaissance missions flown with various constraints). |
| **Comparing Multiple Attributes** | Finding “best” alternative comparing multiple attributes. |
| **Limited Data/Small Sample Size** | Limited data or small sample sizes available. |

# Exemplar Applications Guide

| OR Technique | Application Category | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Modeling Queues | Using Binary Data / Systems | Survey Data<br>Small Data / Sample Set | Survey Data<br>Large Data / Sample Set | Estimating Distributions | Risk Assessment | Cost | Comparing Populations | Analyzing Model Outputs | Finding Relationships in Data | Developing / Examining Ranked Data | Optimization | Comparing Multiple Attributes | Limited Data/Small Sample Size |
| Binomial Distribution | | x | | | | | | | | | | | | |
| Bootstrapping | | | | | x | x | | | | | | | | x |
| Borda Majority Count | | | x | x | | | | | | | x | | | |
| Exponential Distribution | x | | | | | | | | | | | | | |
| Intensity of Preference Adjusted Borda Count | | | x | x | | | | | | | x | | | |
| FORCES Cost Model | | | | | | | x | | | | | | | |
| Kendall's Tau | | | x | | | | | x | x | x | x | | | x |
| Math Programming | | | | | | | | | | | | x | | |
| Multi-Attribute Decision Making | | | | | | | | | | | | | x | |
| Normal Distribution | | | | x | | | | | | | | | | |
| One-Way ANOVA | | | | | | | | x | x | | | | | |
| Poisson Distribution | x | | | | | | | | | | | | | x |
| Simple Linear Regression | | | | | | | | | x | x | | | | |
| Triangular Distribution | | | | | x | x | | | | | | | | |

## Terms of Reference and Glossary

**Assumption** – A statement related to the study that is taken as true in the absence of facts, often to accommodate a limitation. In this guide, any assumption listed for a particular method must be assumed in order to conduct that method. If an assumption is false in your analysis, that method cannot be used.

**Constraint** – A restriction imposed by the study sponsor that limits the study team's options in conducting the study. For the purpose of this guide, a constraint is defined as a condition that a solution to an optimization problem is required by the problem itself to satisfy.

**Cumulative Distribution Function (CDF)** – A function that gives the probability that a random variable is less than or equal to the independent variable of the function.

**Dependent** – When one variable's value is determined by the value assumed by the independent variable.

**DOTMLPF** – Doctrine, organization, training, materiel, leadership and education, personnel, and facilities.

**Exogenously** – Obtained from subject matter expert(s).

**Independence of Irrelevant Alternatives** – The social preferences between alternatives x and y depend only on the individual preferences between x and y.

**Independent** – One event, trial, observation, occurrence, or variable does not affect any other.

**Inference** – The theory, methods, and practice of forming judgments about the parameters of a population.

**Limitation** – An inability of the study team to fully meet the study objectives or fully investigate the study issues. In this guide, a limitation is a disadvantage and weaknesses of a particular method.

**Monotonic** – Having the property either of never increasing or of never decreasing as the values of the independent variable or the subscripts of the terms increase.

**Monte Carlo Simulations** – A broad class of computational algorithms that rely on repeated, random sampling to obtain numerical results.

**Nominal** – A set of information organized by category or name.

**Non-parametric Statistics** – Techniques that do not rely on data belonging to any particular distribution; techniques that do not assume that the structure of a model is fixed.

**Null Hypothesis** – A statistical hypothesis to be tested and accepted or rejected in favor of an alternative.

**Ordinal** – A categorical variable in which variables are clearly ordered.

**Pairwise Preference Matrix** – The rows and columns of the pairwise preference matrix are labeled with the alternatives being voted on. The entry in the row labeled X and the column labeled Y is the number of voters who prefer alternative X to Y in the election being considered.

**Parametric Statistics** – A branch of statistics that assumes that the data has come from a type of probability distribution and makes inferences about the parameters of the distribution.

**Probability Density Function (PDF)** – A function of a continuous random variable whose integral over an interval gives the probability that its value will fall within the interval.

**Residual** – The difference between the observed value and the estimated function value.

**Scalar** – A quantity that is completely specified by its magnitude and has no direction.

**Standard Deviation** – How much variation or dispersion exists from the average (mean) or expected value.

**(Statistical) Power** – The probability that a statistical test will reject the null hypothesis when the null hypothesis is false; the probability of not committing a Type II error.

**Type I Error** – The rejection of the null hypothesis ($H_o$) when the null hypothesis is true.

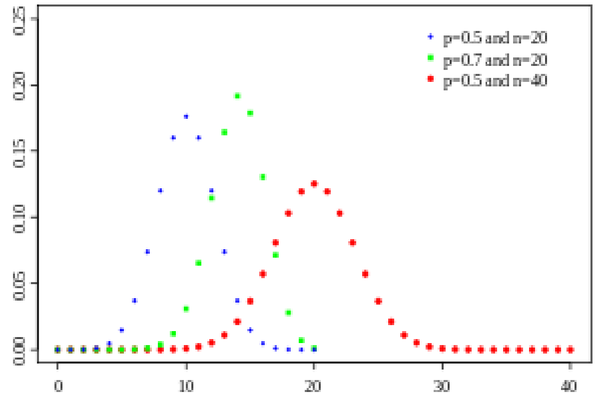
**Type II Error** – The failure to reject a false null hypothesis.

**Variance** – A numerical value used to indicate how widely individuals in a group vary (i.e., if individual observations are far away from the group mean, the variance is large).

**With Replacement** – When sampling, one can choose a member of the population more than once (i.e., the sample is put back into the population before selecting another sample).

**Without Replacement** – When sampling, one cannot choose any member of the population more than once (i.e., the sample is not put back in the population before selecting another sample).

# Binomial Distribution




**Description:** The Binomial Distribution is a discrete probability distribution that models the number of successes, *k*, in multiple identical and independent success/failure experiments, *n*. In these experiments, also called Bernoulli trials, only two outcomes are possible in each experiment, and the probabilities of success and failure are constant from experiment to experiment. When we look at the Binomial Distribution, we look at the descriptions involved with successes, like the mean number of successes. The corresponding statistics about failures are similar.

**Applications:** The Binomial Distribution is often used to model and analyze the number of successes (*k*) in a sample drawn (with replacement) from a population (*n*) when the observations represent one of two outcomes (e.g., "success" and "failure"). Possible use cases include (but are not limited to): single-shot, hit-or-miss probabilities for weapons-target pairings; selection of one type of ordnance for a fire mission out of two possible selections; the number of dud submunitions within a cluster munition; and the number of vehicles that last longer than time *T* without engine failure. We start with the probability of a success or failure on any trial, and we observe the outcomes in each trial. The Binomial Distribution tells us the number of successes (or whatever the specific binomial option may be) we can expect over the set of trials.

**Assumptions:**

- Trials are independent (i.e., success on any trial does not affect any other trial).
- The number of observations (n) is fixed.
- Observations represent one of two outcomes (e.g., "success" or "failure").
- The probability of success is fixed (e.g., no learning effect or fatigue) for each outcome.
- The sequence of experiments are independent from one another.

**Limitations:**

- Success probabilities must be constant.
- The probability of success may need to be obtained exogenously.
- If the sampling is carried out without replacement, the outcomes are not independent. Thus, the resulting distribution is not binomial, but rather a Hypergeometric Distribution. However, for $n >> k$, the Binomial Distribution is a good approximation.

**Other Considerations:**

- True independence of events is rare.
- The Binomial Distribution is driven solely by the probability of success and the number of trials. Look to these parameters for sensitivity analysis.
- A Geometric Distribution is used to determine the probability to achieve the first success after *n* identical, independent experiments.

**Resources/Tools:** Excel, MATLAB, R, Minitab, Wolfram.

**Wiki Reference:** "Binomial Distribution." *Wikipedia.* http://en.wikipedia.org/wiki/Binomial_distribution.
**Academic and Professional References:** "Binomial Distribution." *NIST/SEMATECH e-Handbook of Statistical Methods*. http://www.itl.nist.gov/div898/handbook/eda/section3/eda366i.htm.

# Bootstrapping

**Description:** "The bootstrap is a data-based simulation method for statistical inference," (Efron and Tibshirani, 1993). Bootstrapping is primarily used to estimate the uncertainty associated with a point estimate. It is particularly valuable for complicated estimators and when distributional assumptions are not warranted - and it is usually simple to implement. Bootstrapping's great strength is its wide applicability to a breadth of problems in which no standard statistical method is applicable (e.g., when required assumptions for the latter are in doubt). Bootstrapping uses random resampling, with replacement, from an approximating distribution of the population (typically, the sample itself). As a nonparametric approach, no parametric assumptions are made. If distributional knowledge is available, the resampling can be done from a fitted parametric distribution. The central idea behind the bootstrap is that we assess the randomness inherent in our estimate by measuring it via simulation using the approximating distribution. From the approximating distribution, we generate a large number (many thousands) of bootstrap estimates based on random samples from the approximating distribution. We use the measured variability in the bootstrap samples to make inferences (e.g., interval estimates) about the variability in the original estimate.

**Applications:** Applications include the following:

- Developing input data: If an analyst lacks specific input data for a model, one method to obtain such data is from subject matter experts (SMEs). Using SME data from a small sample of experts, bootstrapping allows for the calculation of an estimated mean (for the hypothetical entire population of SMEs) for use as the model input data.
- Analyzing model output data: If (due to various resource constraints) a stochastic model could be run only a small number of times (i.e., $< 30$) to develop a sample set of values for a particular output variable, bootstrapping could be used to develop sample sets of values (from the original runs) to create an estimated mean for the output variable.
- TRAC analysts used bootstrapping in the Army of 2020 work to construct confidence intervals to overcome potential shortfalls of small sample sizes and to ensure that the ordering of the Brigade Combat Team (BCT) designs provided a valid representation of the BCT commanders' collective wisdom.

**Assumptions:**

- The original sample is random and independently drawn from the population.
- The original sample is a good approximation of the population.

**Limitations:** If the original sample is not representative of the population, bootstrapping may only exaggerate parameter estimate errors.

**Other Considerations:**

- Bootstrapping samples can be treated as if they were actual samples from the original population.
- Bootstrapping produces only estimates of the actual population's parameter(s).
- While there is a bootstrap equivalent to most conventional statistical tests, the reverse does not hold. If a standard statistical approach applies (e.g., the distribution generating the data is known, say normal), then the bootstrap approach will likely be less powerful.

**Resources/Tools:** Excel, MATLAB, Minitab, R, SAS.

**Wiki Reference:** "Bootstrapping (Statistics)." *Wikipedia*. http://en.wikipedia.org/wiki/Bootstrapping_(statistics).
**Academic and Professional References:** [1] Fricker, Ronald and Capt Christopher Goodhart. "Applying a Bootstrap Approach for Setting Reorder Points in Military Supply Systems." http://faculty.nps.edu/rdfricke/docs/Fricker4.pdf.
[2] Efron, B. and R.J. Tibishirani. *An Introduction to the Bootstrap*. New York: Chapman & Hall, 1993.

# Borda Majority Count

**<u>Description</u>:** The Borda Majority Count (BMC) is a technique from social choice theory used to choose one winner, among alternatives. Given a set of alternatives, BMC asks voters to give an evaluation of each alternative using a reasonable common language of grading (e.g., Excellent, Very Good, Good, Acceptable, Poor, Reject), rather than a large grading range (e.g., 0-99 point scale). These grades are converted into numerical scores. The alternative with the highest average score is the preferred alternative for the group.

**<u>Applications</u>:** For analyses where physical, real system data do not exist or are difficult to obtain (i.e., cost, time, or otherwise prohibitive), a common method is to obtain data estimates from subject matter experts. The BMC can be used to rank proposed alternatives by using the combined input from a range of such experts. The TRAC-led Force Design/Force Mix decision analysis support effort used BMC to rank a large number of force design updates. Interestingly, the Borda Count method is used to determine the Most Valuable Player in Major League Baseball.

**<u>Assumptions</u>:** None.

**<u>Limitations</u>:** BMC is vulnerable to voter manipulation (e.g., collusion, tactical voting), although its vulnerability is much less than is possible with a large grading range. By restricting the range of scores, the effects of manipulation in BMC are less dramatic. The study team should identify additional ways to control voter manipulation. For example, a workshop facilitator should become familiar with possible agendas that might be associated with voters in order to address outliers or inconsistent voting patterns.

**<u>Other Considerations</u>:**

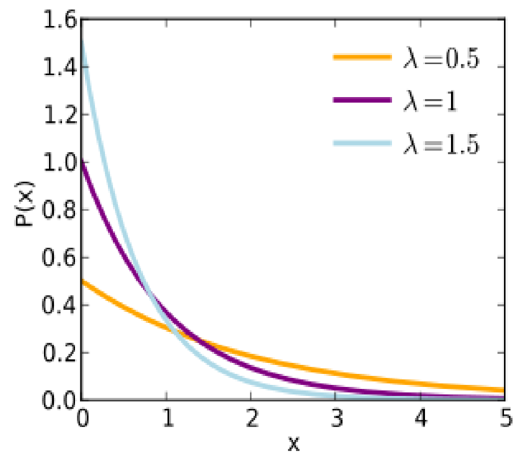
- Ties among alternatives are possible, although unlikely. If two or more alternatives have the same BMC, apply the BMC tie-breaking procedure (dropping grades from lower to higher until a winner is found).
- Other social choice methods to consider as an alternative to BMC are the Borda Count or the Intensity of Preference Adjusted Borda Count.
- Arrow's Impossibility Theorem mathematically establishes that any practical voting system will always violate at least one of five fairness axioms, meaning there can never be a fair voting system, including BMC.
- The study lead should select a voting tool that quickly and effectively collects data. For example, selecting an automated collection tool over a paper survey will mitigate vulnerability to human error and allow for quicker processing of data.

**<u>Resources/Tools</u>:** Excel, voting tool (e.g., Voting and Priority Ranking Tool, paper survey, SharePoint survey tool), Remark Office Optical Mark Recognition.

**<u>Wiki Reference</u>:** None yet.

**<u>Academic and Professional References</u>:** [1] Dabkowski, Matthew and Others. "Force Design/Force Mix: Building the Best Army Possible with Reduced End-Strength." https://www.tkeportal.army.mil/sites/TRAC/Projects/AORG/Project%20Files/TRAC-FLVN%20FD%20FM%20Final%20Report%20(as%20of%2020110816).pdf.
[2] de Sward, Harrie and Manzoor Ahmad Zahid. "The Borda Majority Count." http://www2.eco.uva.es/presad/SSEAC/documents/BMC%20final.pdf.

# Exponential Distribution




**<u>Description</u>:** The Exponential Distribution is a continuous probability distribution that models intervals of time per an event happening. This distribution is derived from the Poisson process, meaning it is close to the Poisson Distribution. Here, however, we are observing continuous intervals of time between event occurrences vice events happening in a fixed interval. The Exponential Distribution has one parameter, $\lambda$, which is called the interarrival rate, the time between two consecutive events happening (i.e., time between arrivals). Service times (i.e., the time in a system) are also observable, given the distribution. The average events per unit of time are shown by $1/\lambda$.

**<u>Applications</u>:** An analyst most often uses an Exponential Distribution when modeling queues. The key points of interest in the application are the interarrival and service rates. From this information, an analyst can describe the queue system from the length of the line to the average wait time and various other insightful properties. The Exponential Distribution is the most common continuous distribution in queuing theory and will be the most helpful in examining queue dynamics, such as identifying bottlenecks within a queue. In the Aerial Reconnaissance and Surveillance Mix Study, the study team used this distribution while performing the processing, exploitation, and dissemination (PED) analysis to examine how intelligence goes from collection to important information. For example, consider a fleet of Unmanned Aircraft Systems collecting intelligence on various targets. This intelligence must go through the PED process to be transformed from data into actionable information. To study the PED process, an analyst's most valuable tool is queuing theory, especially when looking to identify bottlenecks in a queue.

**<u>Assumptions</u>:**

- Events are independent (i.e., one event does not affect any other events).
- Time periods between event occurrences (i.e., interarrival times/service times) are being observed.

**<u>Limitations</u>:**

- True independence of events is rare.
- The parameter $\lambda$ may need to be obtained through subject matter expertise.

**<u>Other Considerations</u>:**

- Be mindful of the parameter and how that value is obtained, because this distribution is solely determined by $\lambda$.
- All descriptors of interest are solely influenced by the parameter $\lambda$. The functions such as PDF and CDF involve Euler's constant, $e$. Therefore, the sensitivity of the parameter and this distribution can be easily seen by observing the behavior of these straightforward functions.

**<u>Resources/Tools</u>:** Excel, MATLAB, R, Minitab, Wolfram.


**<u>Wiki Reference</u>:** "Exponential Distribution." *Wikipedia.* http://en.wikipedia.org/wiki/Exponential_distribution.
**<u>Academic and Professional References</u>:** "Exponential Distribution." *NIST/SEMATECH e-Handbook of Statistical Methods*. http://www.itl.nist.gov/div898/handbook/eda/section3/eda3667.htm.

# FORCES Cost Model

**Description:** The Force and Organization Cost Estimating System (FORCES) Cost Model (FCM) provides realistic, current and supportable force cost estimates of Active and Reserve Component table of organization and equipment (TOE) units, by standard requirements code (SRC), for acquisition, activation, operations, movement, modification, and inactivation. The model is sensitive to operating tempo (OPTEMPO), authorized level of organization, geographic location, year, and component.

**Applications:** FCM is used to provide programmed cost estimates for Army formations. FCM is useful for studies looking at Army force structure, force design, operational energy, mix analyses, TOE changes, and any associated costs.

TRAC studies that have used FCM: Brigade Combat Team Mix Analysis (force structure), Multi-function and Functional Brigade Study (force structure).

**Assumptions:** None.

**Limitations:** The data supporting FCM are derived from official Army sources, but may not reflect the latest updates or changes of that data or the exact methodology of the various proponents. For example, the latest SRCs being considered in the most recent Total Army Analysis may not be reflected in the model.

**Other Considerations:**

- The FCM provides Department of the Army Military Operations – Training (DAMO-TR)'s current cost estimates for Army SRCs. However, care must be taken in how the outputs of the models are used. The outputs do not reflect the official Army position for the budget or program objective memorandum (POM). Users with budget/POM issues should contact DAMO-TR concerning OPTEMPO and Department of the Army's Assistant Chief of Staff for Installation Management – Resources Division concerning Installation Services.
- FCM is requirements-based – built on force unit authorizations rather than actual fielded units.
- Outputs include operation and maintenance costs, ammunitions initial issue cost, acquisition cost drivers, and estimated unit personnel by grade, for example. All data in the supporting databases are in constant dollars of the year corresponding to the version number of the model, with few exceptions. When an out year is selected, all cost factors except the equipment direct operations cost factors are inflated to the selected then year dollars.

**Resources/Tools:**

- Register for access to FCM at: https://www.osmisweb.army.mil/.
- Force Management System Web Site: https://fmsweb.army.mil/.
- If you have questions, contact the Deputy Assistant Secretary of the Army for Cost and Economics (DASA-CE) FORCES Program Manager at (703) 692-5259.

**Wiki Reference:** None.

**Academic and Professional References:** [1] "FORCES Information." *Army Financial Management: Assistant Secretary of the Army for Financial Management & Comptroller.* http://asafm.army.mil/offices/CE/Forces Info.aspx?OfficeCode=1400. [2] FCM Help Menu (within the model).

# Intensity of Preference Adjusted Borda Count

**Description:** Intensity of Preference Adjusted Borda Count (IPABC) is a technique from social choice theory used to rank alternatives. IPABC first asks voters to rank all alternatives from most to least preferred (i.e., Borda Count). IPABC then asks voters to express their intensity of preference (IP) between each alternative and the next lower-ranked alternative. An effective scale on which voters will express their IP is a finite list of values that are common language (e.g., Absolutely Prefer, Strongly Prefer, Moderately Prefer, Weakly Prefer, Indifferent). IP scores are converted into numeric values and used to construct pairwise preference matrices for each voter. The resulting intensity-adjusted pairwise scores for each alternative are averaged across all voters. The average for each alternative is the IPABC score and is used to rank the alternatives.

**Applications:** For analyses where physical, real system data do not exist or are difficult (i.e., cost, time, or otherwise prohibitive) to obtain, a common method is to obtain data estimates from subject matter experts. The IPABC can be used to rank proposed alternatives by using the combined input from a range of such experts. TRAC studies that have used IPABC include Force Design/Force Mix, Army of 2020, TRAC-FLVN Quality of Life Survey, TRADOC's Total Army Analysis Comprehensive Force Design Update Assessment, and the Reconnaissance and Surveillance Brigade Study.

**Assumptions:**

- Voters' IP scores are monotonic.
- IP scores are complementary (i.e., if a voter prefers $x_i$ to $x_j$ with an IP of 0.6, then that voter prefers $x_j$ to $x_i$ with an IP of 0.4).

**Limitations:**

- As with any voting system, there is risk of voter manipulation. The team should identify ways to mitigate that risk. For example, voting should be conducted independently and simultaneously (without pressure or undue influence) with a diverse voting body.
- Due to its relative scale, IPABC does not provide any absolute information on the quality of the proposed solutions.

**Other Considerations:**

- The study team may have voters bin alternatives into absolute preference bins before ranking the alternatives. While ranking may be done within each bin, it is recommended that IP scoring be done collectively, outside of bins. Binning may provide absolute information on the quality of alternatives, which may be used to rank alternatives using the Borda Majority Count (BMC) technique.
- It is recommended that IPABC be performed in conjunction with BMC (whether binning with IPABC or asked independently). If alternatives are dropped or added during the course of a study, BMC will assure independence of irrelevant alternatives (IIA).
- IPABC can rank alternative across multiple objectives (e.g., metrics, scenarios).
- Arrow's Impossibility Theorem mathematically establishes that any practical voting system will always violate at least one of five fairness axioms, meaning there can never be a fair voting system, including IPABC (which fails to meet the IIA).
- The study lead should select a voting tool that quickly and effectively collects data. For example, selecting an automated collection tool over a paper survey will mitigate vulnerability to human error and allow for quicker processing of data.

**Resources/Tools:** Excel, Access database, Java, voting tool (e.g., Voting and Priority Ranking Tool, Word or Access form).

**Wiki Reference:** None yet.

**Academic and Professional References:** [1] Dabkowski, Matthew and Others. "Force Design/Force Mix: Building the Best Army Possible with Reduced End-Strength." https://www.tkeportal.army.mil/sites/TRAC/Projects/AORG/Project%20Files /TRAC-FLVN%20FD%20FM%20Final%20Report%20(as%20of%2020110816).pdf. [2] García-Lapresta, Jose Luis, and Miguel Martínez-Paneros. "Borda count versus approval voting: a fuzzy approach." https://www.tkeportal.army.mil/sites/TRAC/Projects/AORG/Project%20Files/Borda%20Count%20versus%20Approval%20Voting%20-%20A%20Fuzzy%20Approach.pdf.

# Kendall's Tau

**Description:** Kendall's Tau (τ) is a non-parametric test of statistical dependence between two ordered variables in a data set. It is a measure of concordance, which is defined as the tendency of larger values in one variable to be associated with larger values in the other variable, and vice versa. This assessment is made for every pair of measurements, which in essence is the same as measuring the correlation (either +1 or –1 in most cases) by fitting a straight line that connects them. Kendall's Tau combines these pairwise concordance assessments into a single measure of association (Conover, 1999). The inputs include the number of concordant pairs ($N_c$), the number of discordant pairs ($N_d$), and the number of observations ($n$). If the data have ties, as often occurs when at least one of the measured variables is discrete or ordinal, tied observations are treated as being "half concordant and half discordant" in calculating the required count statistics.

**Applications:** Kendall's Tau is used to find dependent relationships between two variables. It can be used with data that are numerical or ordinal, and it is robust to outliers. Tau is completely determined from the ranks of the two variables. General examples include analysis of simulation output (e.g., COMBATXXI, AWARS, and OneSAF outputs), judgment analysis (i.e., subject matter expert elicitation), and survey analysis (i.e., Kendall's Tau may be used to associate responses to survey questions that are presented on Likert scales). Recently Kendall's Tau was used to analyze trends of relationships between African countries and the United States.

**Assumptions:**

- The observations are independent (i.e., the occurrence of any single observation does not affect any other observations).
- The variables are either scalar or ordinal.
- Any dependent relationship that may be present is monotonic (either increasing or decreasing).

**Limitations:** Because Kendall's τ does not assume a distribution, there is some loss of power relative to the best parametric test for the same situation. The power of a statistical test describes how well the test can discern between null and alternative hypotheses. Statistically, it is 1 minus the type II error or false negative. If you have a 10 percent chance of committing a false negative, the power is 90 percent.

**Other Considerations:**

- If the above assumptions are not met, consider using the binomial test (i.e., two variables and they are nominal) or the chi-squared test (i.e., more than two variables and they are nominal).
- This technique is particularly useful with a small sample size.
- Kendall's τ assumes values between −1 and 1.
  - If the larger values of $X$ tend to pair with the larger values of $Y$, Kendall's τ tends to be positive (i.e., value is increasing).
  - If the larger values of $X$ tend to pair with the smaller values of $Y$ Kendall's τ tends to be negative.
  - If the values of $X$ seem to be randomly paired to those of $Y$, Kendall's τ should be fairly close to zero.

**Resources/Tools:** Excel, R, Statplus, Minitab, MATLAB, Octave, Mathematica, JMP.

**Wiki Reference:** "Kendall Tau Rank Correlation Coefficient." *Wikipedia*. http://en.wikipedia.org/wiki/Kendall_tau_rank_correlation_coefficient.

**Academic and Professional References:** Conover, W. J.. *Practical nonparametric statistics*. 3rd ed. Upper Saddle River, NJ: John Wiley and Sons, Inc., 1999.

# Math Programming

**Description:** Mathematical programming (also called optimization) helps a decision maker determine the best solution, or decision, from a set of feasible solutions. Mathematical constraints impose restrictions on possible decisions and define the set of feasible solutions. An objective function maps each solution to an objective value that reflects the solution's quality. The optimal solution yields the best possible objective value while satisfying all constraints.

**Applications:** Math programming is useful for finding a feasible, optimal decision from a large set of solutions. Common applications include determining optimal routings, schedules, assignments, configurations, and procurements. For example, the Aerial Reconnaissance and Surveillance study built a mixed integer program to determine the optimal force mix for aerial intelligence, surveillance, and reconnaissance platforms and sensors given a set of perceived mission demands and operating constraints over the next 12 years.

**Assumptions:** A math programmer's goal is to identify those assumptions that result in an efficient formulation. The formulation requires a number of problem-specific assumptions in order to identify:

- A set of decision variables that reflect real-world decisions being made.
- Those constraints needed to appropriately restrict the values of the decision variables.
- An objective function that calculates the value or cost of a proposed solution.

**Limitations:**

- A very simple model may not provide meaningful solutions. A more complex model may better reflect the real problem but can be difficult (or impossible) to solve. The analyst must find the right balance.
- For certain problems, commercial software is not guaranteed to find an optimal solution and may find only a feasible solution. A trained analyst can identify such problems and may be able to mitigate them through reformulation or specialized solution techniques.

**Other Considerations:**

- The nature of the objective function and constraints dictates the solution method and tractability of a math program. There are three basic types of math programs: linear programs (LPs), nonlinear programs (NLPs), and integer linear programs (ILPs). LPs are the most tractable. Large LPs can be solved to optimality very efficiently. NLPs can be solved efficiently, but commercial software is not always guaranteed to return an optimal solution. Standard algorithms for ILPs are theoretically guaranteed to find an optimal solution, if one exists. Solution time for such algorithms may not be practical for more complex models. Frequently ILPs are solved close enough to optimal, for some definition of "close enough," to reduce solution time.
- Math programming is a prescriptive tool, meaning that it is used to decide upon a course of action. The robustness of math programming solutions to uncertainty in the input data is often determined through sensitivity analysis or simulation.
- Care must be taken when modeling factors that are difficult to quantify, such as social or political considerations.

**Resources/Tools:** Excel, MATLAB, GAMS, LPSolve, ILOG OPL, CPLEX.

**Wiki Reference:** "Mathematical Optimization." *Wikipedia*.http://en.wikipedia.org/wiki/Mathematical_optimization.
**Academic and Professional References:** [1] Hillier, Frederick and Gerald Lieberman. *Introduction to Operations Research*. McGraw-Hill, 2010. [2] Rardin, Ronald L. *Optimization in Operations Research.* Prentice Hall, 1998.
[3] Brown, Gerald G. and Richard E. Rosenthal. "Optimization Tradecraft: Hard-Won Insights from Real-World Decision Support." *Interfaces* (INFORMS) 38, no. 5 (September–October 2008): 356-366.

# Multi-Attribute Decision Making

**Description:** In today's complex environment, DOTMLPF decisions are impacted by a wide range of criteria. Evaluating all of these criteria is necessary to develop a comprehensive understanding for decision analysis support. The Multi-Attribute Decision Making (MADM) technique allows one to consider multiple attributes, quantitative as well as qualitative, to deliver a broader analytical perspective of the alternatives and how they compare with one another. Various models fall under this technique.

**Applications:** This technique is a quick way to enhance most analyses. For example, when conducting an analysis of alternatives (AoA), an analyst could use this technique to compare the alternatives by looking at multiple decision parameters: protection, mobility, reliability, etc. Such a comparison is well-suited for MADM application. For the Joint Light Tactical Vehicle (JLTV) AoA, the study team obtained the objective and threshold key performance parameter values from the JLTV Capability Development Document (CDD) and compared those with how the alternatives actually performed in physical system tests. All of the values were then grouped into three main categories: protection, performance, and payload. Each category was assigned a comparative weight, using inputs from subject matter experts (SMEs), and a final score was calculated for each alternative. The scores from this method allowed the team to determine an operationally preferred alternative that aligned with the rest of the analysis done throughout the AoA.

**Assumptions:** Depending on which model is used for MADM, there are varying conditions that must be met. These are discussed in TRAC's MADM Code of Best Practices.

**Limitations:**

- Depending on the number of attributes, required SME inputs for comparative weights may be substantial and time consuming.
- Even with the same SME input, different models may produce different conclusions.

**Other Considerations:**

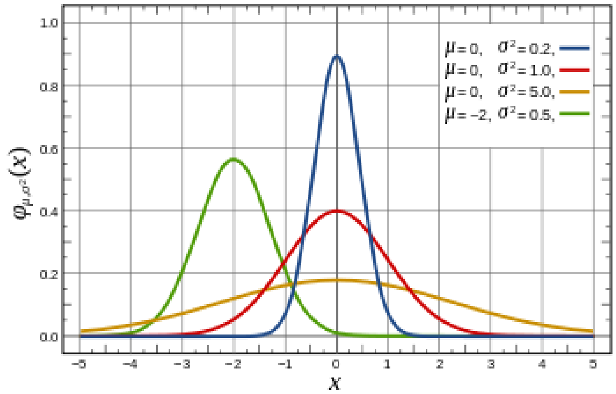
- Rank reversals may occur in some models (i.e., A is preferred if B is the only other alternative, but B becomes preferred if another alternative, C, is introduced into the mix).
- Units of measure do not have to be the same across all attributes to use this technique.
- Besides the overall score, a "best" alternative may also be observed within various sub-categories.
- This technique easily allows for sensitivity analysis by changing the values of any associated weights.
- From the application above: Sensitivity analysis could also be conducted by modifying requirements from the CDD to see what effect that has on the alternatives. This information can help inform a cost trades assessment.
- Plotting effectiveness scores against cost can offer an effective visual.
- The Office of the Secretary of Defense (OSD) leadership prefers a larger sample of SME input when determining weights. If only a small set of SMEs is available to determine the weights, obtain concurrence from additional SMEs before going to OSD.

**Resources/Tools:** Excel.

**Wiki Reference:** "Multi-Criteria Decision Analysis." *Wikipedia*. http://en.wikipedia.org/wiki/Multi-criteria_decision_analysis.

**Academic and Professional References:** Anderson, Michael R., PhD and Eric E. Johnson. "Multi-Attribute Decision Making (MADM) Code of Best Practices (COBP)." https://www.tkeportal.army.mil/sites/TRAC/HQ/Code%20of%20Best%20Practice%20Guidelines/Forms/AllItems.aspx.

# Normal Distribution




**Description:** The Normal Distribution refers to a family of continuous probability distributions useful across a wide range of applications. Many statistical tests assume normal distributions (most work even if the data are only approximately normal) and the normal distribution itself is easy to work with. Many physical and social science phenomena tend to be (at least approximately) normally distributed. The Central Limit Theorem states that the probability of a sufficiently large number of independent random variables, each with a well-defined mean and well-defined variance (defined as $\sigma^2$), will be approximately normally distributed. Two parameters govern the Normal Distribution - the mean ($\mu$) and the standard deviation ($\sigma$), which allow easy conversion of raw data to percentiles, and vice versa.

**Applications:** The Normal Distribution can be used as a generating distribution to model some physical and social science phenomena and for a range of statistical tests. An example implementing both applications is the analysis of survey data (with > 30 participants; see Limitations below). For example, an analyst reviewing the scores of the Armed Services Vocational Aptitude Battery Test, with and without various treatments to increase scores, can expect the test results data to fit a Normal Distribution. The analyst can use the properties of the Normal Distribution to perform a range of tests to determine the effectiveness of the various treatments.

**Assumptions:**

- Events are independent (i.e., one event does not affect any other event(s)).
- A sufficiently large number of data points exist to establish a Normal Distribution.

**Limitations:** To make the normality assumption and establish a Normal Distribution, a sufficient number of data points are typically required. The number of required data points is typically greater than 30, but the actual normality of the data about the sample mean determines the number required. Thirty is chosen for a typical sample that is expected to be normally distributed with approximately 5 percent of the data elements falling outside the bounds of the Normal Distribution.

**Other Considerations:**

- True independence of events is rare.
- It is always useful to assess the data (visually or by some other method) first to determine whether the data reasonably (a qualitative measure) fits a Normal Distribution.
- The standard deviation affects the shape of the curve. The mean shifts the distribution.
- The Normal Distribution can be transformed to have a mean of $\mu=0$ and a standard deviation of $\sigma=1$ to facilitate analysis.

**Resources/Tools:** Excel, MATLAB, R, Minitab, Wolfram.

**Wiki Reference:** “Normal Distribution.” *Wikipedia*. http://en.wikipedia.org/wiki/Normal_distribution.
**Academic and Professional References:** “Normal Distribution.” *NIST/SEMATECH e-Handbook of Statistical Methods*. http://www.itl.nist.gov/div898/handbook/eda/section3/eda3661.htm.

# One-Way Analysis of Variance

**Description:** One-way analysis of variance (ANOVA) is used to test hypotheses concerning whether the means of two or more populations/groups are equal. The null hypothesis for this test is that all of the population means ($\mu_i$) are equal (i.e., $H_o$: $\mu_1 = \mu_2 = \mu_3 = \ldots = \mu_n$). The alternative hypothesis is that at least two of the population means are not equal. In other words, if the null is rejected when performing this test, at least two of the population/group means are significantly different and more tests are necessary to assess these differences. If the null hypothesis is accepted, no significant difference exists between the means of the groups.

**Applications:** This simple technique can enhance most analyses. For example, if combat modeling was used to support a study, one-way Analysis of Variance (ANOVA) could be conducted on the output data to tell whether there is a significant difference between any of the alternatives/cases. Metrics, such as the number of blue vehicles killed, the number of threat killed, the amount of time until stopping conditions are met, etc., could be tested for all alternatives to see whether there are differences in the areas of survivability, lethality, etc.

**Assumptions:**

- Each population from which a sample is taken is normally distributed.
- The variances are equal across all populations.
- Each sample is randomly selected and independent.

**Limitations:**

- In real-world applications, one rarely has normally distributed data and equal variances. If the data do not meet these assumptions, ANOVA cannot be used.
- ANOVA indicates only that differences exist between population means. It will not, however, identify which population means are different.

**Other Considerations:**

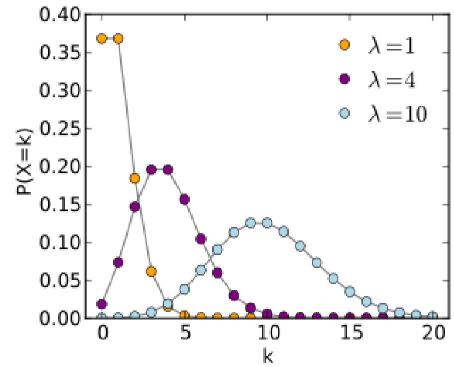
- A value for α, the probability of a Type I Error, must be selected before beginning this test.
- If the null hypothesis is rejected, a multiple comparison test (e.g., Fisher's LSD, Bonferroni, Tukey, Scheffé) can be used to determine which means differ. Each method has advantages and disadvantages and may be more appropriate than the others in certain circumstances. These tests are discussed elsewhere in this guide.

**Resources/Tools:** Excel, R, SAS, Minitab, MATLAB.

**Wiki Reference:** "One-Way Analysis of Variance." *Wikipedia*. http://en.wikipedia.org/wiki/One-way_analysis_of_variance.

**Academic and Professional References:** [1] "1-Way ANOVA Calculations." *NIST/SEMATECH e-Handbook of Statistical Methods*. http://www.itl.nist.gov/div898/handbook/prc/section4/prc434.htm. [2] Ingram, Marlynne. "One-way Fixed Effects ANOVA." https://wiki.uiowa.edu/display/bstat/One-way+fixed+effects+ANOVA.

# Poisson Distribution




**Description:** The Poisson Distribution is a discrete probability distribution that is often used as a model for the number of discrete event occurrences during a continuous interval. We usually see intervals of time; however, we can also look at occurrences over an interval of distance and other measurements. The only parameter for the Poisson Distribution is the mean, λ; it denotes the average number of events to occur during the continuous interval unit. λ is also the variance. If you add two Poison Distributions together, you get another Poisson Distribution with the mean equal to the sum of the means.

**Applications:** An analyst will encounter a Poisson Distribution when observing repeated events over a specified period of time, such as enemy forces crossing a border. Queuing applications and models are often represented with a Poisson Distribution. For example, counting the number of hostile forces crossing a border during an hour could be modeled with the number of arrivals and a Poisson Distribution. This information could then be used to answer questions about the length of the queue or the time in the system, and is helpful in examining queue dynamics, such as identifying bottlenecks within the queue. The Poisson Distribution could also provide the expected number of crossings over a given time period and the probabilities associated with a set number of occurrences happening in a certain interval. Another example would be using the Poisson Distribution to provide the expected number of indirect fire attacks on a combat outpost per month.

**Assumptions:**

- Events are independent (i.e., one event does not affect any others).
- Discrete events over a continuous period of time (e.g., three hostiles crossing the border per day) are being observed.

**Limitations:**

- True independence of events is rare.
- The parameter λ may need to be obtained through subject matter expertise or data.

**Other Considerations:**

- The only input required is λ.
- Be mindful of the parameter λ, and how you obtained that value, because this distribution is completely influenced by the mean.
- With binary outcomes, the Binomial Distribution may be more useful. It can also be found in this guide.
- The Poisson Distribution is concerned primarily with the counts of event occurrences in an interval period as opposed to counting successes and failures or just a number of total occurrences.
- This distribution should be used when limited operational data are available; otherwise, consider more powerful distributions such as the Normal Distribution.

**Resources/Tools:** Excel, MATLAB, R, Minitab, Wolfram.

**Wiki Reference:** "Poisson Distribution." *Wikipedia*. http://en.wikipedia.org/wiki/Poisson_distribution.
**Academic and Professional References:** "Poisson Distribution." *NIST/SEMATECH e-Handbook of Statistical Methods*. http://www.itl.nist.gov/div898/handbook/eda/section3/eda366j.htm.

# Simple Linear Regression

**Description:** Simple linear regression is the process for finding the equation of a best-fit line, $y = \beta_0 + \beta_1 x + \varepsilon$, for a data set of independent ($x$) and dependent ($y$) variables, with y-intercept and slope $\beta_0$ and $\beta_1$, and residual (error term) ε. Residuals can be associated with a multiple causes: measurement errors, specific condition variations, and influencing variables that exist but are unknown. This method can be used to assess the contribution of an independent variable to a dependent variable, or to construct a model to predict the unknown value of a dependent variable for a known value of the independent variable. The first step is establishing a linear relationship between $x$ and $y$ (which may be apparent by viewing a scatterplot of $x$ vs $y$). A best-fit line is then calculated; a standard technique for doing this is the (ordinary) least squares method, which identifies the line that minimizes the sum of the squared residuals - the amounts by which the fitted values of $y$ differ from the actual values of $y$.

**Applications:** Any situation in which one variable is (primarily) linearly influenced by another variable is a possible candidate for this method. Examples of such variable relationships include: increased system weight (independent variable) vs number of systems that can be transported by the current aircraft fleet (dependent variable); number of Army brigades in theater (independent variable) vs number of non-combat fatalities (dependent variable); total number of miles traveled by ground vehicles (independent variable) vs number of IEDs encountered (dependent variable).

**Assumptions:**

- Linear relationship between independent and dependent variables exists.
- Constant variance of residuals exists.
- Residuals are independent of one another.
- Residuals are normally distributed about the regression line.

**Limitations:** As with all regression methods, this technique can establish only correlation between variables – not causation.

**Other Considerations:**

- Methods other than ordinary least squares may be used for linear regression; the selection of a method will depend on the analyst's objectives and the characteristics of the data.
- Predictions are for the average value of $y$ for a given value of $x$. Uncertainty should be included in predictions made using linear regression.

**Resources/Tools:** Excel, MATLAB, Mathematica.

**Wiki Reference:** "Simple Linear Regression." *Wikipedia*. http://en.wikipedia.org/wiki/Simple_linear_regression.
**Academic and Professional References:** [1] Carvalho, Carlos. "Introduction and Simple Linear Regression." http://www2.mccombs.utexas.edu/faculty/carlos.carvalho/teaching/Section1HMBA.pdf. [2] Nau, Robert. "Testing the Assumptions of Linear Regression." http://people.duke.edu/~rnau/testing.htm.

# Triangular Distribution

**Description:** The Triangular Distribution is a continuous distribution that can be used to model quantities for which the precise distribution is unknown, but where the minimum, maximum, and most likely values can be approximated. The probability density function for the Triangular Distribution with minimum $a$, maximum $b$, and most likely value (mode) $m$ is given by:

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(m-a)}, a \leq x \leq m \\ \dfrac{2(b-x)}{(b-a)(b-m)}, m \leq x \leq b \end{cases}$$

and where $f(x)=0$ elsewhere.

**Applications:** The Triangular Distribution is often used as a subjective description of a population, especially for Monte Carlo Simulations in military, business, and financial analyses. Triangular Distributions are also used in Program Evaluation Review Technique analysis to represent the time required to complete a task. Operational risk assessments can use Triangular Distributions for estimates of severity of unmet gaps. Instead of soliciting point estimates, subject matter experts (SMEs) can be asked to provide minimum, maximum, and most likely impact values. That is, for a given capability gap, SMEs indicate the highest and lowest level of impact that may result from leaving the gap unmet, as well as the most likely level of impact. Monte Carlo Simulations over the resulting distributions can then provide severity estimates along with likelihood estimates for operational risk calculations.

**Assumptions:** SMEs can make reasonably accurate estimates of distribution parameters ($a$, $b$, and $m$).

**Limitations:** Estimates for the minimum ($a$), maximum ($b$), and the most likely ($m$) values are based on experience and may not account for more extreme occurrences outside the experience of the SME(s).

**Other Considerations:**

- The Triangular Distribution need not be symmetrical about the most likely value.
- This method can provide a basis for performing sensitivity analysis.
- Some SMEs using this method in support of TRAC studies have indicated they were more comfortable being able to tailor their input by giving minimum/maximum/most likely values, rather than being restricted to providing point estimates.

**Resources/Tools:** Excel, MATLAB, Mathematica, R.

**Wiki Reference:** "Triangular Distribution." *Wikipedia*. http://en.wikipedia.org/wiki/Triangular_distribution.
**Academic and Professional References:** [1] McIlrath, Bonnie and Others. "Armored Multi-Purpose Vehicle Analysis of Alternatives Final Report." https://www.tkeportal.army.mil/sites/TRAC/Projects/AORG/Project%20Files/Forms/AllItems.aspx. [2] Wolfe, Michele. "Operational Risk Analysis: A New Approach." https://www.tkeportal.army.mil/sites/TRAC/Projects/AORG/Project%20Files/Forms/AllItems.aspx.