Sensitivity Analysis and Duality HW sols

Ch 6 review problems

5a.
The dual is

$$\begin{aligned} &\min w = 6y_1 + 4y_2 + 3y_3 \\ &\text{s.t.}\quad 3y_1 + 2y_2 + y_3 \geq 4 \\ &\qquad y_1 + y_2 + y_3 \geq 1 \\ &\qquad y_1 \leq 0\ \ y_2 \leq 0,\ y_3 \text{ urs} \end{aligned}$$

Primal Dual

Since x1=3, then e1=0.
Since e1=3, y1=0;
Since e2=3, then y2=0;
Now I don't know the values of e2 or y3, so I am solving
y3=4 and
y3-e2=1.

So y3 = 4, e2=3 and all others equal to 0.
w=12

(Note, observe the top row of the tableau to also get these values.)

8a. \$274 + (-3) (\$2.60) = \$266.20

8b. Obj. function coefficient for P2 is now 39.50(.33) = 13.035, a decrease of .165. Thus current basis remains optimal and current values of decision variables remain optimal but new z- value = (old z-value) + (-.165) (20) = \$270.70

8c. \$10 + (Row 4 Dual Price) = \$12.60

8d. Row 3 Dual Price = 20¢

8e. Obj. Func. Coefficient for P3 = .8(24) = \$19.2. New column for P3 is

$$\begin{bmatrix} -1 \\ 7 \\ 0 \\ 0 \end{bmatrix}$$

Product 3 prices out to -1 (-12.6) + 7(.2)-19.2 = -5.2. Thus P3 should be produced!

9a. min $w = 50y_1 + 15y_2 + 10y_3$

s.t. $y_1 + 2y_2 + y_3 \geq 3$

$y_1 - y_2 + y_3 \geq 4$

$y_1 + y_2 \geq 1$

$y_1 \geq 0$, $y_2 \leq 0$, y_3 u.r.s.

Primal Dual

From the tableau, x2=10, so e2=0,
x3=40, so e3=0,
e2=15, so y2=0;
Now I am left solving
y1+y3-e1=3
y1+y3=4
y1=1;

So the optimal solution is

The optimal dual solution is w = 80 $y_1 = 1$, $y_2 = 0$, $y_3 = 3$, e1=1, e2=0, e3=0.

22a. Total Cont. to Profit = $\{25-(5(1) + (6)(2) + 5)\}x_1$
$+ \{22-(5(2) + 6(1) + 4)\}x_2 = 3x_1 + 2x_2$

s.t. Laborer 1 Hour Used≤40 ($x_1 + 2x_2 \leq 40$)

Laborer 2 Hours Used≤50 ($2x_1 + x_2 \leq 50$)

All variables non-negative

22b. Let objective function coefficient of x_1 be Δ.. Then

$$\bar{\mathbf{C}}\mathbf{s_1} = [\Delta\ ,\ 2]\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = -1/3\ \Delta + 4/3 \geq 0, \text{ so } \Delta \leq 4.$$

$$\bar{\mathbf{C}}\mathbf{s_2} = [\Delta\ ,\ 2]\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 2/3\ \Delta - 2/3 \geq 0, \text{ so } \Delta \geq 1.$$

Thus if Δ≤4 and Δ ≥1 the current basis remains optimal. Thus current basis remains optimal if \$23
2≤ Radio 1 Price ≤ \$26.

22c. Let Δ be coefficient of x_2 in the objective function. Then

$$\bar{C}_{s_1} = [3, \Delta] \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = -1 + 2/3\Delta \geq 0, \text{ so } \Delta \geq 3/2.$$

$$\bar{C}_{s_2} = [3, \Delta] \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = 2 - 1/3\,\Delta \geq 0, \text{ so } \Delta \leq 6.$$

Thus current basis remains optimal if $21.50 ≤ Price of Radio 2≤ $26.

22d. The RHS of the optimal tableau is now

$$\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 30 \\ 50 \end{bmatrix} = \begin{bmatrix} 70/3 \\ 10/3 \end{bmatrix}.$$

Since both constraints still have nonnegative rhs, the current basis remains optimal. No need for the dual simplex.

22e. RHS of optimal tableau is now

$$\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 40 \\ 89 \end{bmatrix} = \begin{bmatrix} 138/3 \\ -9/3 \end{bmatrix} = \begin{bmatrix} 46 \\ -3 \end{bmatrix} \text{ z is now } [3,2] \begin{bmatrix} 46 \\ -3 \end{bmatrix} = 132$$

Thus current basis is infeasible so we use the dual simplex method to obtain a new solution.

z	x1	x2	s1	s2	RHS
1	0	0	1/3	4/3	132
0	1	0	-1/3	2/3	46
0	0	1	2/3	-1/3	-3.

So the pivot is on the bottom row and the s2 column.

The new tableau is

z	x1	x2	s1	s2	RHS
1	0	4	7/3	0	120
0	1	2	1	0	38
0	0	1	-2	1	9.

So the new solution is x1=38, x2=0, and z=120.

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22f. Increasing Laborer 1 time by 1 hour increases profits by \$1/3. This includes the \$5 cost of paying for the extra labor hour. Thus we could pay up to 5 + 1/3 = \$5.33 for an extra hour of work by Laborer 1 and still be better off.

22g. $\Delta b_2 = -2$. If current basis remains optimal, (it does!) then new profit = 80 - 2(4/3) = \$77 1/3. To check this we determine the new optimal solution via

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 40 \\ 48 \end{bmatrix} = \begin{bmatrix} 56/3 \\ 32/3 \end{bmatrix}$$

and Profit = 3(56/3) + 2(32/3) = 232/3 = \$77 1/3

22h. Price out the new activity:

$\mathbf{c}_{BV}B^{-1}\mathbf{a}_3 - c_3 = [1/3\ \ 4/3] \begin{bmatrix} 2 \\ 2 \end{bmatrix} - (30 - 10 - 12 - 3) = -5/3$ Thus the current basis is no longer optimal and a pivot will need to be performed. The new column is

$$\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}.$$ So the new tableau is

z	x1	x2	x3	s1	s2	RHS
1	0	0	-5/3	1/3	4/3	80
0	1	0	2/3	-1/3	2/3	20
0	0	1	2/3	2/3	-1/3	10

Pivot on x3 column and last row leads to

z	x1	x2	x3	s1	s2	RHS
1	0	5/2	0	2	1/2	105
0	1	1	0	-1	1	10
0	0	3/2	1	1	-1/2	15.

So the optimal solution is to produce 15 of the new product.

23a. min $w = 40y_1 + 30y_2 + 40y_3$

s.t. $y_1 + y_2 + 2y_3 \geq 40$ (Ale Const.)

$2y_1 + y_2 + y_3 \geq 50$ (Beer Const.)

$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

Optimal dual solution is $w = 1200$, $y_1 = 20$, $y_2 = 0$, $y_3 = 10$.

23b. Now $\mathbf{c}_{BV} = [50\ 0\ 40 + \Delta]$. This changes $\mathbf{c}_{BV}B^{-1}$ to

$$\mathbf{c}_{BV}B^{-1} = [50\ 0\ 40+\Delta]\begin{bmatrix} 2/3 & 0 & -1/3 \\ -1/3 & 1 & -1/3 \\ -1/3 & 0 & 2/3 \end{bmatrix} = [20 - \Delta/3 \quad 0 \quad 10 + 2\Delta/3]$$

Thus Row 0 is now
$z + (20 - \Delta/3)s_1 + (10 + 2\Delta/3)s_3$.

The current basis remains optimal if $20 - \Delta/3) \geq 0$ and $10 + 2\Delta/3 \geq 0$ or $\Delta \leq 60$ and $\Delta \geq -15$. Thus current basis remains optimal if $25 = 40-15 \leq$ Ale Price $\leq 40 + 60 = 100$.

23c. Now $\mathbf{c}_{BV}B^{-1} = [50 + \Delta\ 0\ 40]$ and

$$\mathbf{c}_{BV}B^{-1} = [50 + \Delta\ 0\ 40]\begin{bmatrix} 2/3 & 0 & -1/3 \\ -1/3 & 1 & -1/3 \\ -1/3 & 0 & 2/3 \end{bmatrix} = [20 + 2\Delta/3\ 0\ 10 - \Delta/3]$$

Now Row 0 is $z + (20 + 2\Delta/3)s_1 + (10 - \Delta/3)s_3$. Thus current basis remains optimal iff

$20 + 2\Delta/3 \geq 0$ (or $\Delta \geq -30$)

and $10 - \Delta/3 \geq 0$ (or $\Delta \leq 30$)

Thus current basis remains optimal for $-30 \leq \Delta \leq 30$ or current basis remains optimal if $20 = 50 - 30 \leq$ Beer Price $\leq 50 + 30 = 80$.

23d. If $B^{-1}\begin{bmatrix} 40+\Delta \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ -1/3 & 1 & -1/3 \\ -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 40+\Delta \\ 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 40/3 & + & 2\Delta/3 \\ 10/3 & - & \Delta/3 \\ 40/3 & - & \Delta/3 \end{bmatrix}$ is $\geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The current basis will remain optimal. Thus current basis remains optimal if $\Delta \geq -20$, $\Delta \leq 10$ and $\Delta \leq 40$. Thus current basis remains optimal if $-20 \leq \Delta \leq 10$ or $20 = 40 - 20 \leq$ Corn Available $\leq 40 + 10 = 50$.

23e. Current basis remains optimal if

$$B^{-1}\begin{bmatrix} 40 \\ 30+\Delta \\ 40 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ -1/3 & 1 & -1/3 \\ -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 40 \\ 30+\Delta \\ 40 \end{bmatrix} = \begin{bmatrix} 40/3 \\ 10/3+\Delta \\ 40/3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus current basis remains optimal if $\Delta \geq -10/3$ or
$80/3 = 30 - 10/3 \leq$ Available Hops

23f. Current basis remains optimal if

$$B^{-1}\begin{bmatrix} 40 \\ 30 \\ 40+\Delta \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & -1/3 \\ -1/3 & 1 & -1/3 \\ -1/3 & 0 & 2/3 \end{bmatrix}\begin{bmatrix} 40 \\ 30 \\ 40+\Delta \end{bmatrix}\begin{bmatrix} 40/3-\Delta/3 \\ 10/3-\Delta/3 \\ 40/3+2\Delta/3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus current basis remains optimal if

$40/3 - \Delta/3 \geq 0$ (or $\Delta \leq 40$)
and $10/3 - \Delta/3 \geq 0$ (or $\Delta \leq 10$)
and $40/3 + 2\Delta/3 \geq 0$ (or $\Delta \geq -20$) Thus current basis remains optimal for $-20 \leq \Delta \leq 10$ or $20 = 40 - 20 \leq$ Available Malt $\leq 40 + 10 = 50$.

23g. Price/barrel of malt liquor = \$50
Cost of Ingredients in barrel of malt liquor = 20(.5) + 0(3) + 10(3) = \$40. Since 50>40, Beerco should manufacture malt liquor.

23h. max z = 40ALE + 50BEER

$$
\begin{aligned}
\text{s.t.}\quad & 16ALE + 32BEER \leq 640 \text{ (CORN)} \\
& 16ALE + 16BEER \leq 480 \text{ (HOPS)} \\
& 32ALE + 16BEER \leq 640 \text{ (MALT)} \\
& ALE \geq 0,\ BEER \geq 0
\end{aligned}
$$

The dual to this LP is

$$
\begin{aligned}
& \min w = 640y_1 + 480y_2 + 640y_3 \\
\text{s.t.}\quad & 16y_1 + 16y_2 + 32y_3 \geq 40 \\
& 32y_1 + 16y_2 + 16y_3 \geq 40 \\
& y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0
\end{aligned}
$$

23i. The optimal solution to the dual in problem (23i) is $y_1 = 20/16$, $y_2 = 0/16$, $y_3 = 10/16$. This solution is dual feasible and has w = 1,200(the optimal primal obj fun. value). Thus this solution is optimal for the new dual. From this answer we find that Shadow Price for ‘Ounce’ Constraint = 1/16 (Shadow Price for ‘Pound’ Constraint). This is reasonable because the value of an ounce of a good should be 1/16 the value of a pound of a good.