Markov Homework Solution

1. Suppose that today's weather is dependent upon the last two days. If it was sunny yesterday, then it will be sunny tomorrow with probablility .7. If it was rainy for the past two days, then there is a 50% chance of rain tomorrow. If it has only rained for only 1 day, then there is a 40% chance of rain tomorrow.

The state space is the last two days (day_i, day_{i+1}). So the possibilities are SS, SR, RS and RR where S is for sunny and R is for rainy.

P=

	SS	SR	RS	RR
SS	.7	.3	0	0
SR	0	0	.4	.6
RS	.7	.3	0	0
RR	0	0	.5	.5

If the Wed and Thursday are both rainy, what is the probability that Saturday and Sunday will both be sunny?

P would take Wed and Thursday to Thursday and Friday.
P^2 would take Wed. and Thursday to Friday and Saturday.
P^3 would take Wed. and Thursday to Saturday and Sunday.

P^3=

	SS	SR	RS	RR
SS	0.427	0.183	0.174	0.216
SR	0.406	0.174	0.198	0.222
RS	0.427	0.183	0.174	0.216
RR	0.42	0.18	0.185	0.215

So the probability that Saturday and Sunday will be sunny is RR to SS, .42.

What percent of weekends have both sunny or both rainy days.

Need to find the long run probablities, so solve $\pi P=\pi$ and $\Sigma_{\text{all } i}\, \pi_i=1$.

$.7\pi_{SS} +.7\pi_{RS}=\pi_{SS}$
$.3\pi_{SS} +.3\pi_{RS}=\pi_{SR}$
$.4\pi_{SR} +.5\pi_{RR}=\pi_{RS}$
$.6\pi_{SR} +.5\pi_{RR} =\pi_{RR}$
$\pi_{SS}+\pi_{SR} +\pi_{RS}+\pi_{RR}=1$

Solving the first and 4th constraints lead to
$\pi_{SS}=7/3\pi_{SR}$ and $\pi_{RR}=6/5\pi_{SR}$.
Substituting this into the second equation gives
$.7\pi_{SR}+.3\pi_{RS}=\pi_{SR}$, which is $\pi_{SR}=\pi_{RS}$.
Substituting these values into the last equation gives
$7/3\pi_{SR}+\pi_{SR}+\pi_{SR}+6/5\pi_{SR}=1$.

So $\pi_{SR}=15/83$ and
$\pi_{RS}=15/83$
$\pi_{SS}=35/83$
$\pi_{RR}=18/83$

So the porportion of weekends with both sunny days is 35/83.

Assume that you are hosting an outdoor volleyball tournament. If both days are rainy, you will lose $1000. If both days are sunny, you will gain $2000, if either day is sunny, you will gain $200. What is your expected payoff for the volleyball tournament.

The expected gain for the tournament is

-1000*18/83+2000*35/83+200*(15/83+15/83) = $698.8

2. A machine produces great, good or defective set of parts. If a defective set of parts is produced, then the process is stopped for the time it takes to create one set of parts and the machine is readjusted during this time. When the machine is back up and running the next set will be good and great with equal probability. If the machine produced a great set of parts, then the machine is equally likely to produce either a good set or a great set next. If the machine produces a good set of parts, then it will produce a good set 70% and a bad set 30% of the time.

The state is the most recent set of parts that was produced (GR, GO, DE, RE). The RE stage represents that the machine is being readjusted.

	GR	GO	DE	RE
GR	.5	.5	0	0
GO	0	.7	.3	0
DE	0	0	0	1
RE	.5	.5	0	0

What porportion of time is the machine getting readjusted?

Need to find the long run probablities, so solve $\pi P=\pi$ and $\Sigma_{\text{all } i}\, \pi_i=1$.

	GR	GO	DE	RE
GR	.5	.5	0	0
GO	0	.7	.3	0
DE	0	0	0	1
RE	.5	.5	0	0

$.5\pi_{GR}+.5\pi_{RE}=\pi_{GR}$
$.5\pi_{GR}+.7\pi_{GO}+.5\pi_{RE}=\pi_{GO}$
$.3\pi_{GO}=\pi_{DE}$
$\pi_{DE}=\pi_{RE}$
$\pi_{GR}+\pi_{GO}+\pi_{DE}+\pi_{RE}=1$

So from the first, third and fourth equations one gets $\pi_{GR}=\pi_{RE}$, $.3\pi_{GO}=\pi_{DE}$, $\pi_{DE}=\pi_{RE}$. Substituting into the last equation yields

$.3\pi_{GO}+\pi_{GO}+.3\pi_{GO}+.3\pi_{GO}=1.$

So

$\pi_{GR}=3/19$
$\pi_{GO}=10/19$
$\pi_{DE}=3/19$
$\pi_{RE}=3/19$

The machine is getting reset 3/19 of the time.

If a great set of parts is worth \$300, and a good set is worth \$250. A defective set is worth \$100 with probability .3 and \$50 with probability .7. Resetting the machine costs \$300. What is the expected payoff per time unit?

The expected profit is
300*3/19+250*10/19+.3*100*3/19+.7*50*3/19-300*3/19= 47.37

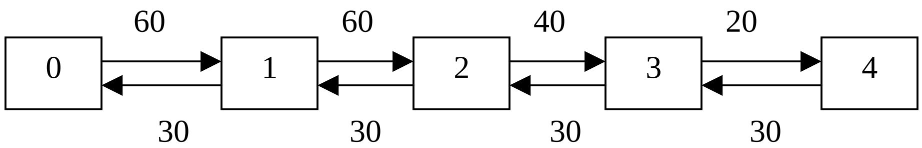
3. A fastfood place in the mall has its service times expo(30) distributed per hour. The interarrival times are exponetially distributed at a rate of 60 per hour. Many arrivals balk according to the number in line as follows.

Number in line	**% that balk.**
0	**0**
1	**33.333%**
2	**66.6666%**
3	**100%.**

If the value of a balk is worth \$1, do you think that it is worth adding a second set of workers?

SYSTEM 1

The state space is the number of people in the system, {0,1,2,3,4}. The following rate transition diagram describes the system. Not that when the state is in 3, that there are 2 people in line and one person ordering/paying/etc.. The arrrival rate is 60, but 66.666% balk so only 33.3333% stay and so the transistion rate from 4 to 5 is 1/3*60=20.



Solving for the steady state distribution leads to the following equations
Rate in = Rate out
$30P_1=60P_0$
$30P_2+60P_0=60P_1+30P_1$
$30P_3+60P_1=40P_2+30P_2$
$30P_4+40P_2=20P_3+30P_3$
$20P_3=30P_4$
$P_0 +P_1+ P_2+ P_3+ P_4=1$

Crossing out terms yields
$30P_1=60P_0$
$30P_2=60P_1$
$30P_3=40P_2$
$30P_4=20P_3$

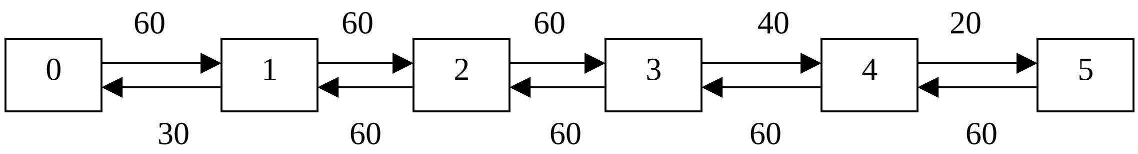
So $P_1=2P_0$, $P_2=2P_1$, $P_3=4/3P_2$ and $P_4=2/3P_3$.
Thus, $P_2=4P_0$, $P_3=16/3\ P_0$, $P_4=32/9\ P_0$, substituting into the last equation gives

$P_0 +2P_0+4P_0+16/3P_0+32/9P_0=1$

So $P_0 =9/143$
$P_1 =18/143$
$P_2 =36/143$
$P_3 =48/143$
$P_4 =32/143$

SYSTEM 2

If there are two servers in the system and both servers are working then their rate is the sum of the individual rates. Thus, if there are 2 or more people in the system, then the service rate is 60, which leads to the following rate transition diagram.



Solving for the steady state distribution leads to the following equations
Rate in = Rate out
$30P_1=60P_0$
$60P_2+60P_0=60P_1+30P_1$
$60P_3+60P_1=60P_2+60P_2$
$60P_4+60P_2=40P_3+60P_3$
$60P_5+40P_3=20P_4+60P_4$
$20P_4=60P_5$
$P_0 +P_1+ P_2+ P_3+ P_4+ P_5=1$

Crossing out terms yields
$30P_1=60P_0$
$60P_2=60P_1$
$60P_3=60P_2$
$60P_4=40P_3$
$60P_5=20P_4$

So $P_1=2P_0$, $P_2=P_1$, $P_3=P_2$, $P_4=2/3P_3$ and $P_5=1/3P_3$.
Thus, $P_2=2P_0$, $P_3=2\ P_0$, $P_4=4/3\ P_0$, and $P_5=4/9P_0$ and substituting into the last equation gives

$P_0 +2P_0+2P_0+2P_0+4/3P_0+4/9P_0=1$

So
$P_0 =9/79$
$P_1 =18/79$
$P_2 =18/79$
$P_3 =18/79$
$P_4 =12/79$
$P_5 =4/79$

Analysis

System 1: On average 60 people come in any hour. Balks only occur in states 2, 3 and 4. The rate of a balk is 20 in state 2, 40 in state 3 and 60 in state 4. Thus the expected number of balks in any hour is

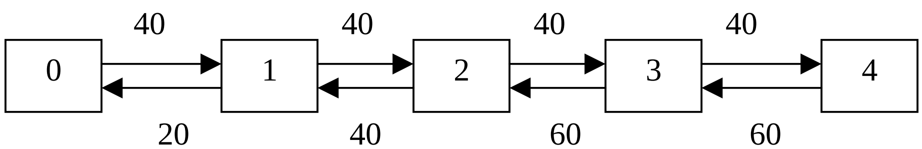
20* 36/143 +40*48/143 +60*32/143=31.8. The store is loosing $31 per hour in balks.

System 2: On average 60 people come in any hour. Balks only occur in states 3, 4 and 5. The rate of a balk is 20 in state 3, 40 in state 4 and 60 in state 5. Thus the expected number of balks in any hour is

20*18/79+40*12/79+60*4/79= 13.67. Thus, the company can make an additional $18 an hour to have a second system open. This would require at least one cashier and one cook, so I would recommend against it. However, it appears as though it is about an equal amount of money in either case.

4. A drive up bank has 3 tellers and location for 1 car to be waiting. If the service rates are expo(20) and the interarrival times are expo(40) (both in hours), then what is the average number in the system, and how many people balk in an 8 hour day?

The state space is the number of cars in the system, {0,1,2,3,4}. The following rate transition diagram describes the system.



Solving for the steady state distribution leads to the following equations
Rate in = Rate out
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$40P_2+40P_0=40P_1+20P_1$
$60P_3+40P_1=40P_2+40P_2$
$60P_4+40P_2=60P_3+40P_3$
$60P_4=40P_3$
$P_1+ P_2+ P_3+ P_4=1$

Crossing out terms yields
$20P_1=40P_0$
$40P_2=40P_1$
$60P_3=40P_2$
$60P_4=40P_3$

So $P_1=2P_0$, $P_2=P_1$, $P_3=2/3P_2$ and $P_4=2/3P_3$. Substituting into the last equation gives

$P_0 +2P_0+2P_0+4/3P_0+8/9P_0=1$

So $P_0 =9/65$
$P_1 =18/65$

P_2 =18/65
P_3 =12/65
P_4 =8/65

The average number in the system is
0*9/65+1*18/65 +2*18/65+3*12/65+4*8/65 = 1.88

The average number of balks per hour is
P_4*40/(40+60)= 16/325.