g. The goal is to apply the disjunctive procedure on $\underline{\delta = 2}$ for

$C_1: \; x_1 + 8x_2 - 9x_3 + 15x_4 \leq 7$

$C_2: \; 4x_1 + 8x_2 - 9x_3 + 15x_4 \leq 15$

Notice x_1 as the target region. Apply γ to C_2, the larger.

$$C_2: \; x_1(4-\gamma) + 8x_2 - 9x_3 + 15x_4 + \gamma(x_1 - 3) \leq 15 - 3\gamma$$

$$C_1: \; x_1(1+\lambda) + 8x_2 - 9x_3 + 15x_4 - \lambda(x_1 - 2) \leq 7 + 2\lambda$$

set $4 - \gamma = 1 + \lambda$ and $15 - 3\gamma = 7 + 2\lambda$

$\lambda = 3 - \gamma \longrightarrow 15 - 3\gamma = 7 + 2(3 - \gamma)$

$8 - 3\gamma = 6 - 2\gamma$

$\lambda = 1 \longleftarrow 2 = \gamma$

Since $\lambda, \gamma > 0$ it is valid.

$$x_1(4-2) + 8x_2 - 9x_3 + 15x_4 + \ldots \leq 15 - 3(2)$$

$$x_1(1+1) + 8x_2 - 9x_3 + 15x_4 + \ldots \leq 7 + 2(1)$$

$$\Rightarrow \boxed{2x_1 + 8x_2 - 9x_3 + 15x_4 \leq 9}$$

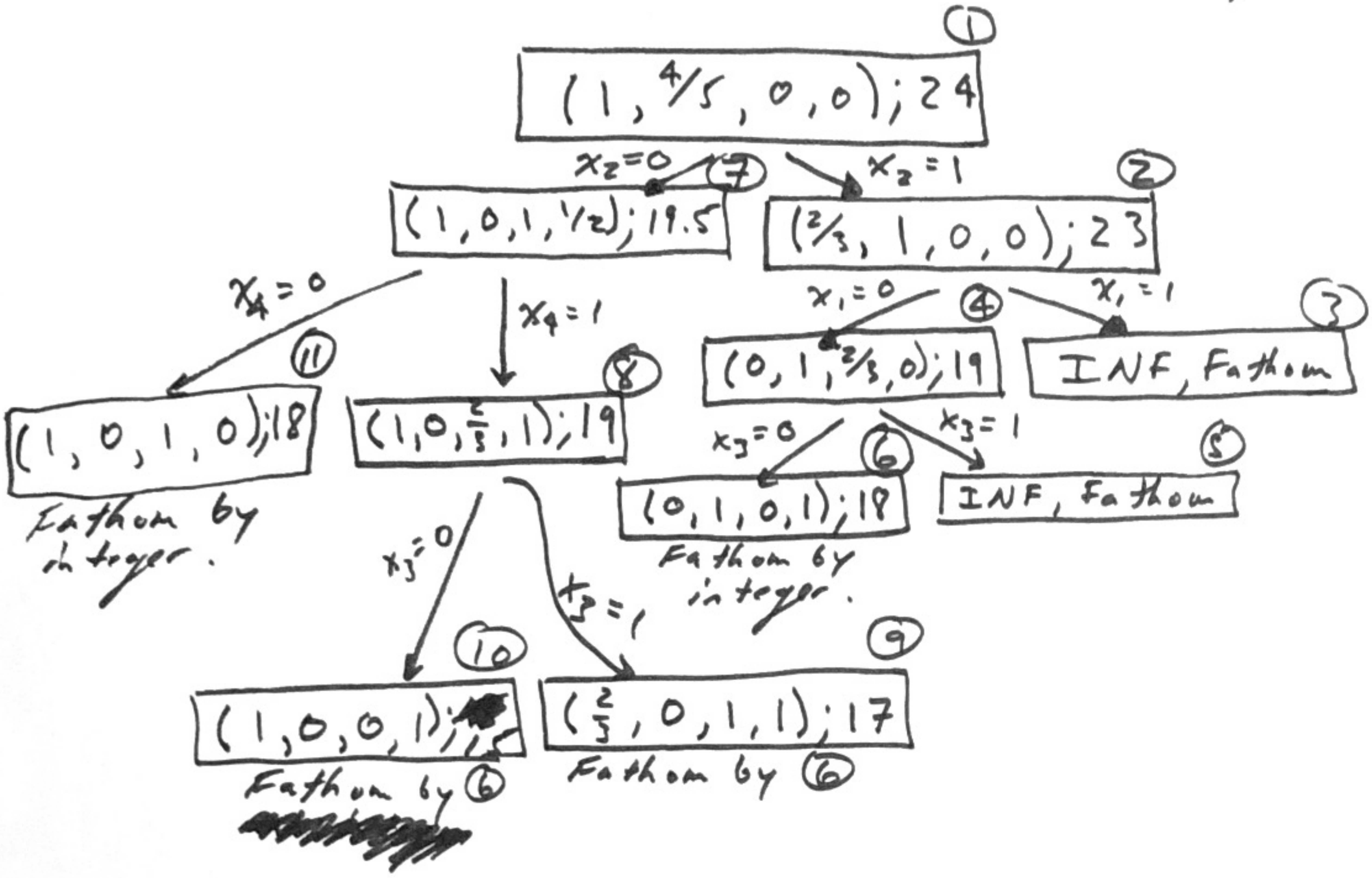
1h. Given a final row of the simplex tableau $9/8 x_1 + 13/2 x_2 - 3/4 x_3 + x_4 + 1/4 s_1 = 15/4$,
by applying a gomory cut we arrive at (after mod 1),
the following
$1/8 x_1 + 1/2 x_2 + 1/4 x_3 + 1/4 s_1 \geq 3/4$ as the cutting plane
to remove the LR BFS $x_{BV} = \{x_4\} : x_4 = 15/4$, because
$0 < 3/4$. Now if one gomory branch is
$1/8 x_1 + 1/2 x_2 + 1/4 x_3 + 1/4 s_1 = 3/4$, then the
other branch must capture the rest of the
half-space defined by the cut. So the other
branch is, $1/8 x_1 + 1/2 x_2 + 1/4 x_3 + 1/4 s_1 > 3/4$.

This makes the gomory cut valid with 2
branches using the modular arithmetic.

2a. Solve the following with branch + bound DFS-Right. 10

$$\text{Max. } 12x_1 + 15x_2 + 6x_3 + 3x_4$$
$$\text{s.t. } 3x_1 + 5x_2 + 3x_3 + 2x_4 \le 7$$
$$x_i \in \{0,1\}$$

Notice $\frac{12}{3} > \frac{15}{5} > \frac{6}{3} > \frac{3}{2}$ so it is ordered in priority.

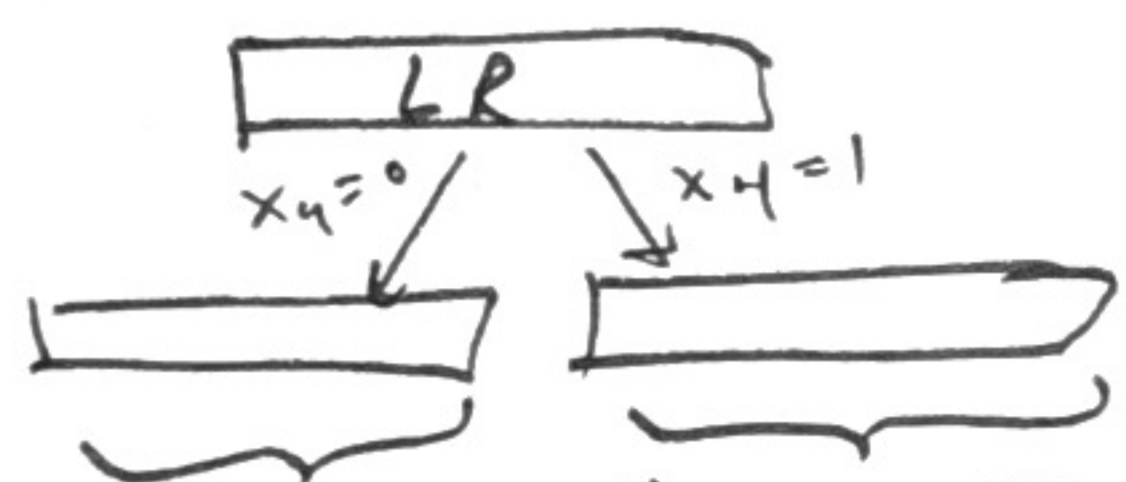


Therefore the solution to the above IP is $x^* = (0,1,0,1)$ $z^* = 18$.

- Explain the concept of branch and cut.

From lecture, the idea behind brand and cut is to select a LR cut for some IP problem, this initial branch is selected in such a way to create a cut on one whole section of a tree (local) or the whole tree (global). The best example of this is to see a branch for some variable that then automatically implies a cover inequality. Thereafter the cover inequality will dominate the branching in that section of the space. This is shown in an example below,

$$20x_1 + 19x_2 + 17x_3 + 15x_4 \leq 48$$



Now $x_1+x_2+x_3 \leq 2$ cover inequality applies to all children.

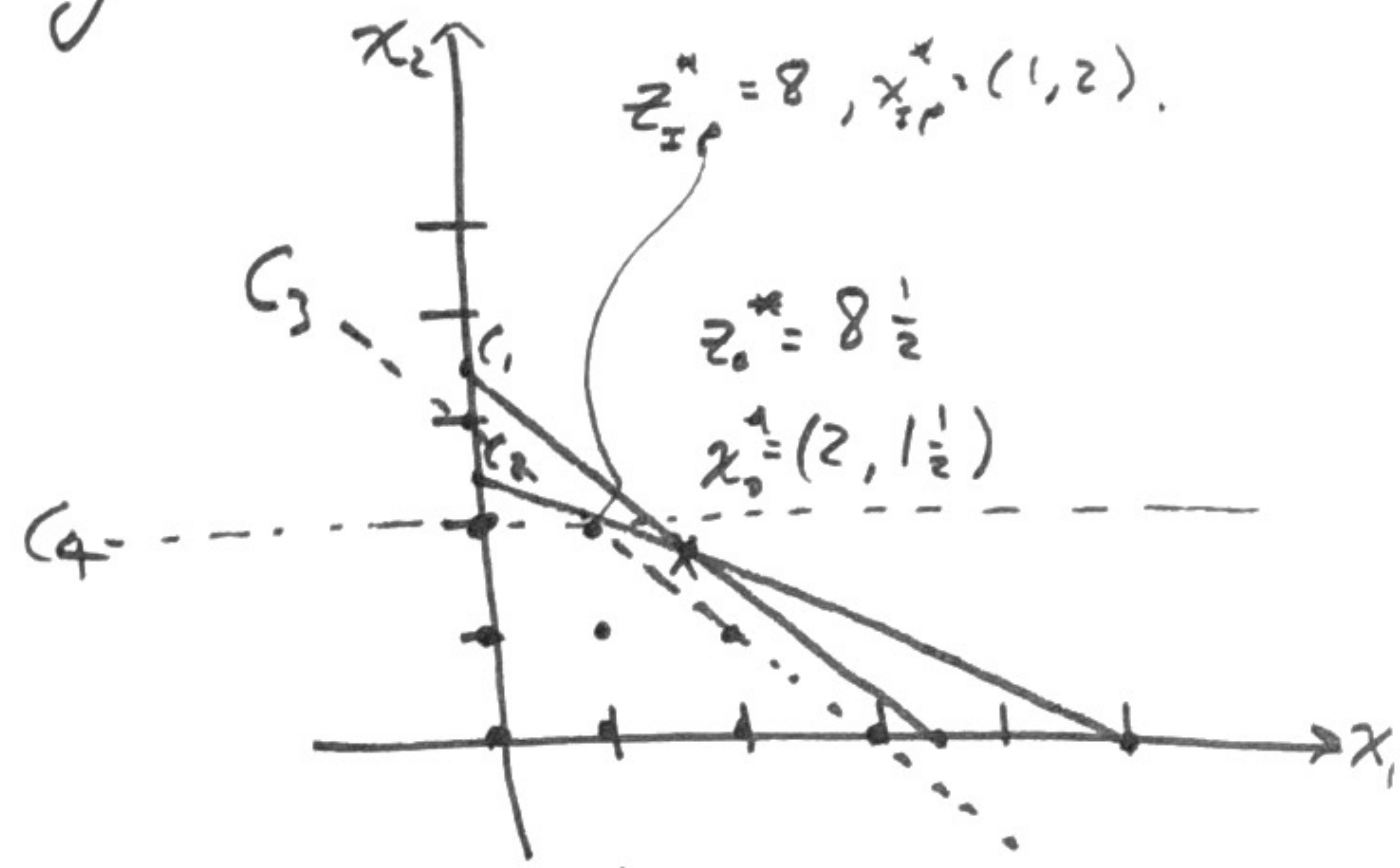
Now $x_1+x_2+x_3 \leq 1$ cover inequality applies to all children.

This reduces the search space with a more powerful cut than just a branch + bound procedure.

3. Solve the following with cutting planes.
Show one is facet defining.

$$\max \; 2x_1 + 3x_2$$
$$\text{s.t. } C_1: 2x_1 + 2x_2 \le 7$$
$$C_2: x_1 + 2x_2 \le 5$$
$$x_1, x_2 \in \mathbb{Z}^+$$

$P^{CH} = \{(0,2), (0,1), (0,0), (1,0), (1,1), (1,2), (2,0), (2,1), (3,0)\}$



Consider the cut formed between the points $(1,2)$ and $(3,0)$. $m: \frac{0-2}{3-1} = -1 \rightarrow x_2 \le -x_1 + 3$. Call this cut C_3. If we resolve with this inequality as a new constraint we see 3 candidate corner points: $\underbrace{C_3 = C_2}_{z_1}$, $\underbrace{C_2 = (x_2 \text{ axis})}_{z_2}$, $\underbrace{C_3 = (x_1 \text{ axis})}_{z_3}$

These z values are now computed,

z_1:
$$\begin{array}{r} x_1 + x_2 \le 3 \\ -x_1 - 2x_2 \le -5 \\ \hline -x_2 = -2 \\ x_2 = 2 \\ x_1 = 1 \end{array}$$
$= 7$

$z_2: x_1 = 0, x_2 = 2.5$
$= 7.5$

$z_3: x_1 = 3, x_2 = 0$
$= 6$

Clearly z_2 with points $(0, 2.5)$ will be selected. So add 1 more cut $C_4: x_2 \le 2$. Now let's re-evaluate the LP with $C_1 - C_4$. Clearly all corner points are integer now $(0,2), (1,2), (3,0)$. Notice $z_2 = 8$ @ $(1,2)^*$ solves this problem optimally.

Next we show that one of the above cuts C_3 or C_4 is facet defining.

Consider $c_3 : x_2 \leq -x_1 + 3$.

First, $\dim(P^{(CH)}) \leq n = 2$ and notice 3 AI points $(0,0), (0,1), (1,0) \in P^{CH}$, so $\dim(P^{(CH)}) \geq 2$. Conclude that $\dim(P^{(CH)}) = 2$.

Second, notice that no integers in P^{CH} are cut off by c_3 and z_0^* the initial LR is cut off, so it is both valid and useful. See graph.

Third, if $F := x_2 = -x_1 + 3$, notice $x_2 < -x_1 + 3$ that then does exist a point $(0,0)$ that is valid and satisfying. So $\dim(F) \leq \dim(P^{CH}) - 1 \leq 1$

Also notice 2 AI points $\in F$, that is $(1,2), (3,0)$ clearly AI and $3 = 3$ in both instances. This means $\dim(F) \geq 2 - 1 = 1$. So $\dim(F) = 1$.

Therefore $c_3 : x_2 \leq -x_1 + 3$ is a valid Facet Defining Inequality. $\square$

4. The Node Packing IP is below,

$$\max. \sum_{i \in V} x_i$$
$$\text{s.t.} \quad x_i + x_j \le 1 \quad \forall \{i,j\} \in E$$
$$x_i \in \{0,1\} \quad \forall i \in V$$

Given a 2-cliqued odd hole where $p = 7$, $q = 5$ we seek to find a valid inequality. Start with just 1 clique-hole. Since p is odd we know that only $\lfloor \frac{7}{2} \rfloor$ can be selected. Start there: $\sum_{i \in P} x_i \le 3$. Next, sequentially lift all other nodes in the 2-cliqued odd hole.

$$\max. \sum_{i \in P} x_i \qquad \Rightarrow z^* = 0 \Rightarrow \alpha_j = 3 .$$
$$\text{s.t} \quad x_j = 1 \quad \text{some } j \in Q$$

We see the same situation for all nodes in the Q side, if selected, no $i \in P$ can be so $z_j^* = 0 \ \forall j$, and $\alpha_j = 3 \ \forall j$. Hence, we arrive at the (7,5) inequality: $3 \sum_{j \in Q} x_j + \sum_{i \in P} x_i \le 3$. In general, this is $\lfloor \frac{p}{2} \rfloor \sum_{j \in Q} x_j + \sum_{i \in P} x_i \le \lfloor \frac{p}{2} \rfloor$. In the next page we address validity and facet defining proof with conditions outside this induced sub-graph

1. $\dim(P^{CH}) \leq n = p+q$

Also we can find $p+q+1$ AI points in P^{CH}. Consider $e_i = 1$ else 0 $\forall i \in V$. This is an identity matrix

$$\begin{array}{c} 1 \\ 2 \\ \vdots \\ p \\ p+1 \\ \vdots \\ p+q \end{array} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

, Also $\bar{0} \in P^{CH}$.

Therefore, $\dim(P^{CH}) \geq p+q \Rightarrow \dim(P^{CH}) = p+q$.

2. Validity of $\lfloor \frac{p}{2} \rfloor \sum_{j \in Q} x_j + \sum_{i \in P} x_i \leq \lfloor \frac{p}{2} \rfloor$.

Assume it is not valid. Then $\exists x \in P^{CH}$: $\lfloor \frac{p}{2} \rfloor \sum_{j \in Q} x_j + \sum_{i \in P} x_i > \lfloor \frac{p}{2} \rfloor$. That is $\exists$ more than 1 x for $\lfloor \frac{p}{2} \rfloor \sum_{j \in Q} x_j$ OR more than $\lfloor \frac{p}{2} \rfloor$ x for $\sum_{i \in P} x_i$. Since $\sum_{i \in P} x_i$ is a node packing polyhedron this cannot happen, contradicting node packing condition. And since the structure is fully connected we can only select at most 1 item from Q in order to add any more from P. So P^{CH} has no points with more than 1 $x \in Q$. Since neither condition can be true the inequality must be valid

3. Show $F := \lfloor \frac{p}{2} \rfloor \sum_{j \in Q} x_j + \sum_{i \in P} x_i = \lfloor \frac{p}{2} \rfloor$ dimensionality.

- Clearly $0 < \lfloor \frac{p}{2} \rfloor \; \forall \; p \geq 2$, since p is odd assume $p > 1$ for all further analysis.

So, $\dim(F) \leq p+q-1$

- Next we find $p+q$ AI points $\in F$. Observe first that we can individually select each item $\in Q$ in isolation and be $\in F$. This is q points. Next observe the # of ways of selecting $\lfloor \frac{p}{2} \rfloor$ nodes from p selection. For $p=7$ this can be shown graphically. Notice $(a,c,e),(b,d,f),(c,e,g),(d,f,a),(e,g,b)$ (f,a,c), (g,b,d) then duplicates start occuring. So 7 points. Exactly what we want. So our points are below,

$$
\begin{array}{c}
q_1 \\ q_2 \\ \vdots \\ q_q \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \\ p_p
\end{array}
\begin{bmatrix}
1 & 0 & \cdots & 0 & & & \\
0 & 1 & \cdots & 0 & & 0 & \\
\vdots & \vdots & \ddots & \vdots & & & \\
0 & 0 & \cdots & 1 & & & \\
1 & 0 & 1 & & \cdots & & \\
0 & 1 & 0 & 1 & & \ddots & \\
1 & 0 & 1 & & & \ddots & \\
0 & 1 & 0 & 1 & & & \\
\vdots & \vdots & \vdots & & \cdots & & \\
0 & 1 & 0 & & \cdots & &
\end{bmatrix}
$$

The selection of the $p-1$ points will come from starting at $p_1 = 1, p_2 = 0$ and alternating until $\lfloor p/2 \rfloor$ are selected. Next $p_2 = 1, p_3 = 0$, and again alternating until $\lfloor p/2 \rfloor$ are selecting. This cyclic pattern will acquire p points that meet the inequality AT equality. Hence $p+q$ AI points $\in F$.

So $\dim(F) \geq p+q-1$.

And $\dim(F) = p+q-1$. So it is facet defining $\square$

The conditions needed to make this this facet defining for the entire graph are below:

1. At most 1 node can be connected to nodes $\in P$ and at most 1 node can be connected to nodes $\in Q$. This can disrupt the bipartite balance and cause for an invalid inequality.

2. Nodes connected to P or Q groups can not connect to more than 1 adjacent edge. Otherwise, this conflict between edge selection could disrupt the #AI points $\in F$ for it's dimensionality.

3. The max # of nodes allowed in connection to G (the 2-clique odd [redacted] hole) must be 2. If 3 are connected then could be a conflict in the counting and generalization of $\lfloor \frac{p}{2} \rfloor \sum_{j \in Q} x_j + \sum_{i \in P} x_i \leq \lfloor \frac{p}{2} \rfloor$ [redacted] and there might $\exists x \in P^{cm} : \lfloor \frac{p}{2} \rfloor \sum_{j \in Q} x_j + \sum_{i \in P} x_i > \lfloor \frac{p}{2} \rfloor$, making it invalid.