IMSE 884 Final
To avoid an incomplete, it is due Wednesday, May 13 at 11:59 pm. Please email your answers to me.

Name Blake Conrad Score /150

By signing below, I state that I have neither given nor received any help on this exam. Furthermore, I agree not to provide any form of assistance to any individual who has not yet taken this exam. Signature Blake C

Bonus question 2 pts. There were several people who were involved with finding facets for the knapsack polytope. Name two of them? Bryce Huschka, Julia Guiteriz

1. Short answer

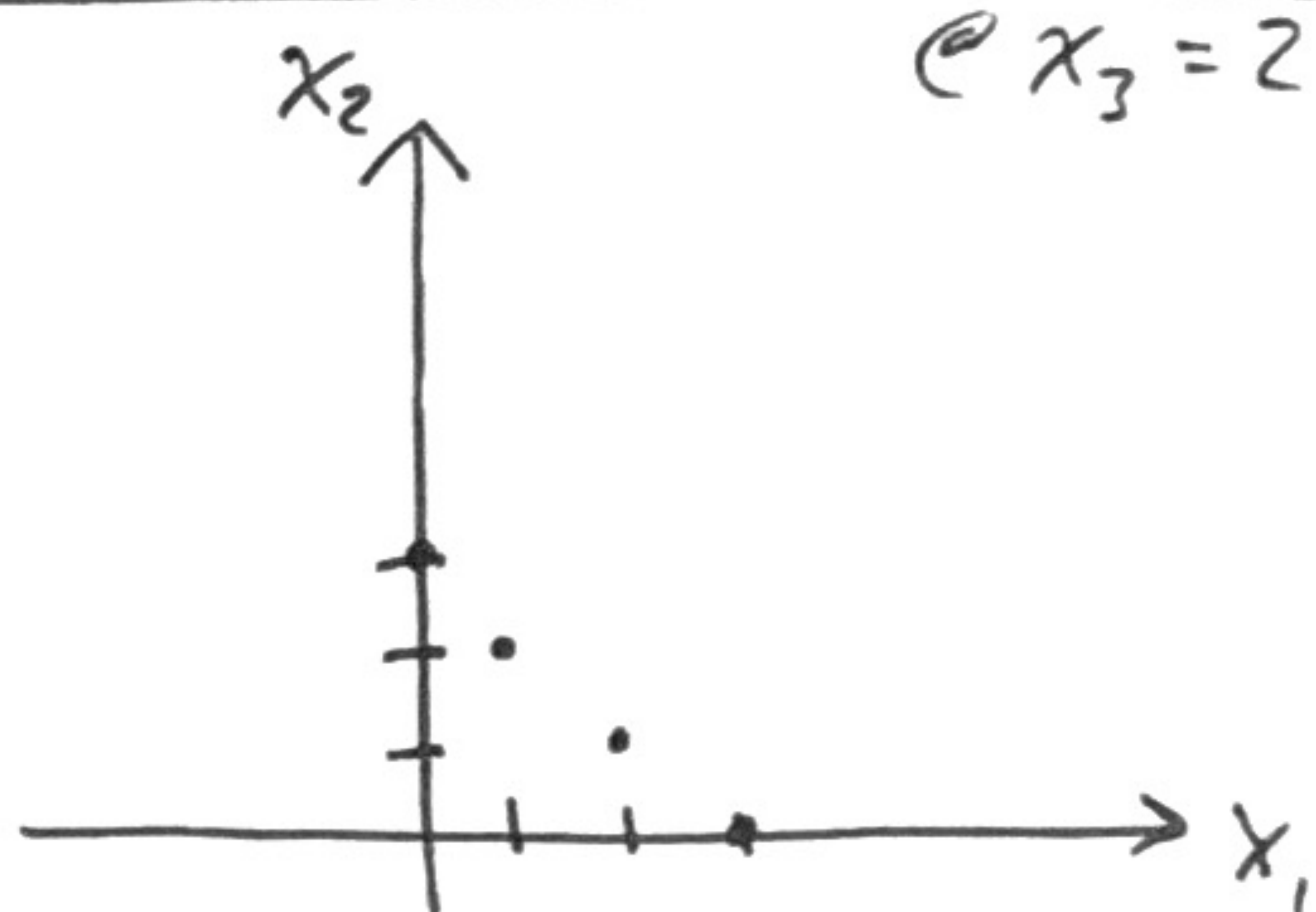
Classify any and all facets of P^{ch} for these two problems 5 points each.

a. $x_1 + x_2 + x_3 = 5$
$x_3 = 2$
$x_1, x_2, x_3 \geq 0$ and integer

b. $x_1 - x_2 = 2$
$x_1, x_2 \geq 0$ and integer

Blake Conrad

Final Exam



1a. $x_1 + x_2 + x_3 = 5$

$x_3 = 2$

$x_1, x_2, x_3 \geq 0$ and integer.

$\Rightarrow x_2 = -x_1 + 3$

$x_1, x_2, x_3 \geq 0$ and integer.

The integer line $x_2 = -x_1 + 3$ at $x_3 = 2$ is shown above. Clearly, there are only 4 points on the line and in p^{CH}. So $p^{CH} = \{(0,3),(1,2),(2,1),(3,0)\}$. Since $n=2$ and 2 AI points are $(0,3)$ and $(1,2)$, $\dim(p^{CH}) = 1$ (That is, a line). Any valid faces at this point must be points, that is $\dim(F) = 0$. So at this point only single dimensional facets can meet this expectation (that is 1 variable, or a point). Also, in order to be valid, we must show no integer point in p^{CH} is cut off by such a facet. Only 2 inequalities meet this expectation. 1) $x_1 \leq 3$ 2) $x_2 \leq 3$.

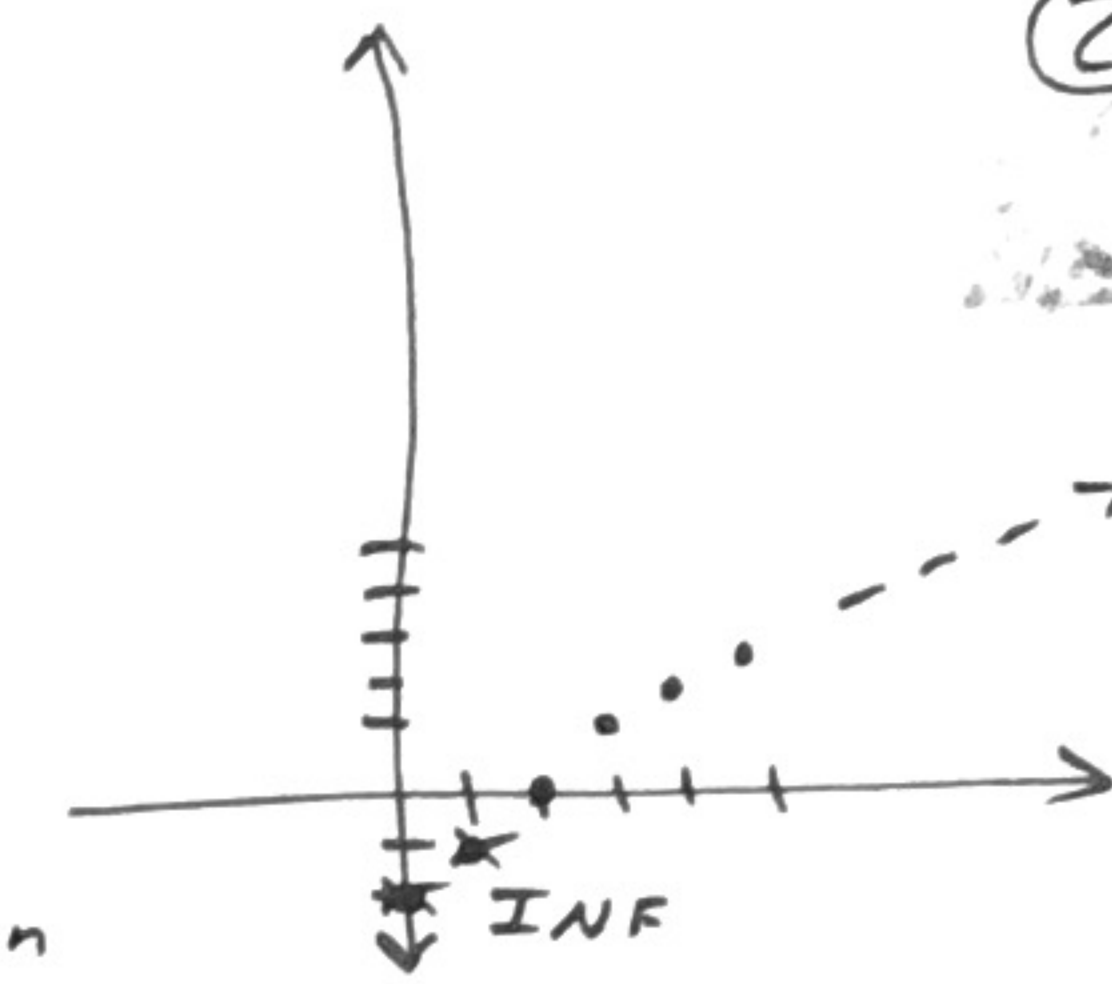
First verify a point where $x_1 = 3$, clearly $(3,0)$. Again, $x_2 = 3$, clearly $(0,3)$. Second, since [redacted] $(1,2)$ shows $x_1 = 1 < 3$ $\dim(F_1) \leq 0$. Likewise for $(2,1)$ $x_2 = 1 < 3$ so $\dim(F) \leq 0$. Lastly since each point exists $\in p^{CH}$ and on F_1 consider 1 AI point $x_1 = 3$, so $\dim(F_1) \geq 0$. Likewise for $x_2 = 3$ $\dim(F_2) \geq 0$. Hence, $\dim(F_1) = \dim(F_2) = 0$. These are both facet defining and any other point $\in p^{CH}$ would not be valid. □

16. $x_1 - x_2 = 2$

$x_1, x_2 \geq 0$ and integer

$\Rightarrow x_2 = x_1 - 2$

$x_1, x_2 \geq 0$ and integer



The integer line $x_1 - x_2 = 2$ is shown above. It clearly goes on forever. The line is 1-dim, and $P^{CH} = \{(2,0), (3,1), (4,2), \ldots (n, n-2)\}$), however n will never be reached. This is a line so $\dim(P^{CH}) = 1$. Any facet at this point must be a point, or a single variable facet. $\dim(P^{CH}) = 1$ can be varified by acknowledging $n=2$ and $(2,0), (3,1) \in P^{CH}$, and AI.

For any facet F, it will never be valid if $x_1 \leq \beta$ or $x_2 \leq \alpha$ because there always exists an integer point $x_1 > \beta$ and $x_2 > \alpha \in P^{CH}$, because the space goes on forever as $x_1 \to \infty$ and $x_2 \to \infty$. However, as $x_1 \to 2$ and $x_2 \to 0$ the space terminates. So in only 2 situations will a facet be valid. 1) $x_1 \geq 2$ 2) $x_2 \geq 0$. We quickly see $\nexists$ integer $x \in P^{CH} : x_1 < 2$ or $x_2 < 0$, so they are valid. Also $x_1 = 2$ and $x_2 = 0$ are $\in P^{CH}$ and F. We need find only 1 AI points $(2,0)$ to verify $\dim(F_1) \geq 0$ and $\dim(F_2) \geq 0$. [redacted]

Notice $\exists x \in P^{CH}$: $x_1 > 2$ and $x_2 > 0$
so $\dim(F_1) \leq 0$ and $\dim(F_2) \leq 0$.

Hence, $\dim(F_1) = \dim(F_2) = 0$.

Therefore we have found 2 0-dim FDI $\square$

6. Find a Gomory cut and show that it cuts off the current LP relaxation point.

The current $x^* = (0, 0, 0, 13/2, 0)$, $z^* = 78$. $x_{BV} = \{x_4\}$ $x_{NBV} = \{x_1, x_2, x_3, s_1\}$. The following Gomory cut process will eliminate the current BFS:

$$\text{mod1}\left(9/4\, x_1 + 13/4\, x_2 - 1/4\, x_3 + x_4 + 1/4\, s_1 \geq 13/2\right)$$

$$= 1/4\, x_1 + 1/4\, x_2 + 3/4\, x_3 + 0 x_4 + 1/4\, s_1 \geq 1/2$$

$$= \underline{1/4\, x_1 + 1/4\, x_2 + 3/4\, x_3 + 1/4\, s_1 \geq 1/2}$$

Clearly our BFS x^* implies $x_{NBV} = 0$, so $0 \geq 1/2$ forces the current LP relaxation point out of the basis, hence cutting off the point.

c. Lift x_1 into $2x_3 + x_4 + x_5 + x_6 \le 4$.

Lift x_1

$$\begin{aligned} \max.\ & 2x_3 + x_4 + x_5 + x_6 \\ \text{s.t.}\ & 2x_1 + 2x_2 + x_3 + x_4 \le 2 \\ & 2x_1 + x_2 + 2x_3 + x_4 + x_5 + x_6 \le 4 \\ & x_1 = 1 \end{aligned}$$

$z_1^* = 2$ so $\alpha_1 = 4 - 2 = 2$. New Eq^n: $2x_1 + 2x_3 + x_4 + x_5 + x_6 \le 4$

Lift x_2

$$\begin{aligned} \max.\ & 2x_1 + 2x_3 + x_4 + x_5 + x_6 \\ \text{s.t.}\ & 2x_1 + 2x_2 + x_3 + x_4 \le 2 \\ & 2x_1 + x_2 + 2x_3 + x_4 + x_5 + x_6 \le 4 \\ & x_2 = 1 \end{aligned}$$

$z_2^* = 2$ so $\alpha_2 = 4 - 2 = 2$. New Eq^n: $2x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6 \le 4$

Comments on FDI

Since $n = 6$ $\dim(P^{CH}) \le 6$. Since we can allocate 7 AI points in P^{CH}, that is $e_i\ \forall i$ in $\{1..6\}$ and $\bar{0}$, $\dim(P^{CH}) \ge 6$. So $\dim(P^{CH}) = 6$. In order for $2x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6 \le 4$ to be FDI it must be valid. We must show no integers $\in P^{CH}$ satisfy $2x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6 > 4$. Consider P^{CH}:

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, no points break validity. We also know $0 < 4 \in P^{CH}$ so $\dim(F) \le 5$. Finally, consider 6 AI $\in F$ (That is $2x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6 = 4$). However, this cannot be verified b/c only feasible solutions involving x_3, x_4 will allow a a sum to 4, and this is never on the face. <u>So, not FDI.</u>

D. Let $u_i = 1/3$

$$\frac{1}{3}\begin{pmatrix} x_1 + x_2 + x_3 \le 2 \\ x_1 + x_2 + x_4 \le 2 \\ x_1 + x_3 + x_4 \le 2 \\ x_2 + x_3 + x_4 \le 2 \end{pmatrix}$$

$+$

$$x_1 + x_2 + x_3 + x_4 \le \lfloor 2\tfrac{2}{3} \rfloor \le 2$$ Rank 1.

Notice $(2/3, 2/3, 2/3, 2/3)$ is LR feasible $6/3 \le 2$, but $8/3 > 2$ so not CG inequality feasible.

Observe the $\binom{5}{3}$ situation with 10 equation, the additional 6 are shown below, beside the original,

$$\underbrace{\underbrace{\begin{matrix} x_1 + x_2 + x_3 \le 2 \\ x_1 + x_2 + x_4 \le 2 \\ x_1 + x_2 + x_4 \le 2 \\ x_2 + x_3 + x_4 \le 2 \end{matrix}}_{\binom{4}{3}} \qquad \begin{matrix} x_1 + x_2 + x_5 \le 2 \\ x_1 + x_3 + x_5 \le 2 \\ x_1 + x_4 + x_5 \le 2 \\ x_2 + x_3 + x_5 \le 2 \\ x_3 + x_4 + x_5 \le 2 \\ x_2 + x_4 + x_5 \le 2 \end{matrix}}_{\binom{5}{3}}$$

If we notice $\binom{5}{3}$ can be made up of $\binom{5}{4} = 4$, we can construct 4 rank 1 CG inequalities from $\binom{5}{3}$ that use combinations of $u_i = 1/3$, producing the following,

$$\begin{matrix} x_1 + x_2 + x_3 + x_4 \le 2 \\ x_1 + x_2 + x_3 + x_5 \le 2 \\ x_1 + x_2 + x_4 + x_5 \le 2 \\ x_2 + x_3 + x_4 + x_5 \le 2 \end{matrix} \implies$$

So let $u_i = 1/3$ again and obtain the rank 2 CG inequality,

$$x_1 + x_2 + x_3 + x_4 + x_5 \le \lfloor \tfrac{8}{3} \rfloor \le 2$$

This is rank 2.

5.5

The process continues for $\binom{6}{3}=20$ adding 10 new equations to $\binom{5}{3}=10$, an increase in 4 equations, from the previous difference [$\binom{4}{3}=4 \to \binom{5}{3}=10$ is 6, as is to $\binom{5}{3}=10$ $\binom{6}{3}=20$ is 10]. This pattern will continue in even and odd intervals because the number of edges is fixed, the pattern found is below,

$$\binom{4}{3} \quad \binom{5}{3} \quad \binom{6}{3} \quad \binom{7}{3} \quad \binom{8}{3} \quad \binom{9}{3} \quad \ldots\ldots$$

$$4 \quad 10 \quad 20 \quad 35 \quad 56 \quad 84 \quad \ldots\ldots$$

$$6 \quad 10 \quad 15 \quad 21 \quad 28 \quad \ldots\ldots$$

$$4 \quad \underbrace{5 \quad 6}_{rank_2} \quad \underbrace{7 \quad 8}_{rank_3} \quad \ldots\ldots$$

$rank_1$ (4), $rank_2$ (5, 6), $rank_3$ (7, 8), so the following is the estimate.

	4	5	6	7	8	9	10	11	12	13	14	15
rank	1	2	2	3	3	3	4	4	4	5	5	5

5. Consider partitions $Q_1 = \{1,2,8,9,5,10,11,12\}$ (6)
$Q_2 = \{4,7,6,3\}$.

The column-sums with this configuration are : $-1, 1, 1, 1, 1, 1, 0 \leq 1$ ✓

So the matrix is TUM, by the theorem in class, since $\bigcup_i Q_i = A$.

6. TUM is important because in LP if A is TUM and b is integer, then all extreme points of $Ax = b$ are integer.
In general this is <u>super</u> helpful b/c LP is polynomial solvable, but IP is NP-Complete.

F. Given $3.4x_1 + 7.6x_2 + 9.1x_3 \leq 15.2$

The strongest superadditive inequality is defined by $\alpha = 15.2 - \lfloor 15.2 \rfloor = 0.2$ in the following stepwise equation,

$$f_{0.2}(d) = \begin{cases} \lfloor d \rfloor & \text{if } 0 \leq d - \lfloor d \rfloor \leq 0.2 \\ \lfloor d \rfloor + \dfrac{d - \lfloor d \rfloor - 0.2}{1 - 0.2} & \text{else} \end{cases}$$

$$f_{0.2}(3.4) = \frac{0.4 - 0.2}{0.8} = \frac{0.2}{0.8} = 1/4 + 3 = 3\tfrac{1}{4}.$$

$$f_{0.2}(7.6) = \frac{0.6 - 0.2}{0.8} = \frac{0.4}{0.8} = 1/2 + 7 = 7\tfrac{1}{2}.$$

$$f_{0.2}(9.1) = 9.$$

$$f_{0.2}(15.2) = 15.$$

So the strongest equation is,

(1) $3\frac{1}{4}x_1 + 7\frac{1}{2}x_2 + 9x_3 \leq 15$.

Compare this to the common $f_{\lfloor\rfloor}(x) = \lfloor x \rfloor$,

(2) $3x_1 + 7x_2 + 9x_3 \leq 15$,

And we can see that $f_{0.2}(d)$ increases the RHS of the inequality slightly more than $f_{\lfloor\rfloor}(x)$ does. So (1) is the dominant inequality, because it cuts off more space than (2).