Solve the following problem by adding cutting planes. Prove that at least one of your cutting planes is a facet-defining inequality.

max x1+x2

s.t. x1+4x2≤10

3x1+x2≤9

x1,x2∈Z+

See graph for picture.

LP relaxation z= 47/11 x1=26/11= x2=21/11

Add the cut 2x1+x2≤6 The new solution is z=6 and x1=2, x2 = 2 and so we are done.

Now prove that 2x1+x2≤6 is a facet-defining inequality

1.

The dimension of P = 2. There are 2 variables so dim(P)≤2.

The points (0,0), (1,0) and (0,1) are in P and affinely independent so the dim(P)≥2 and thus the dim(P)=2.

2. 2x1+x2≤6 is clearly a valid inequality, see graph.

3. (0,0) is feasible and 0<6 and so the dim({x∈P: 2x1+x2=6})≤ 2-1.

The points (3,0) and (2,2) are in P and satisfy 2x1+x2=6. Furthermore, these points are affinely independent. Thus dim({x∈P: 2x1+x2=6})≥2-1=1 and so 2x1+x2≤6 is a facet-defining inequality.

2. Create a two dimensional IP that has no facets

Consider the IP

Maximize 0x1+0x2

Subject to x1-x2=0

x1, x2 unrestricted.

The (-1,-1) and (0,0) are in Pch so dim(Pch)≥1. The point (0,5) is not in P, so dim(Pch)≤2-1, where the 2 comes from two variables. So dim(Pch ) =1.

There are only two valid inequalities that take the form x1-x2≥0 and x1-x2≤0. Any other inequality cannot be valid as it would eliminate some feasible points. The face of both of these valid inequalities is 1 and thus they are not facet defining.

3. Given the following knapsack constraint.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x∈B10

a. Draw the conflict graph for the above problem find a clique that is induces a facet. (prove that it is a facet)

5

6

7

2

1

8

9

4

3

10

There exists a maximal cliques of {1,2,3,4}, {1,2,5}, {1,2,6}, {1,7}, {1,8} and {1,9}.

I will show that K3={1,2,5} induces a facet defining inequality.

1a and b The problem has 10 variables, 0 and the ei for i = 1 to 10 are feasible and affinely independent so the polytope is full dimensional.

2 The inequality x1+x2+x5≤1 is valid, since 27+15>42.

3a. The 0 point does not satisfy this inequality at equality, thus the dim(x∈P: x1+x2+x5=1) ≤10 –1=9.

3 b the following points are affinely independent and satify the inequalaity at equality, which completes the proof.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

b. Find a hyperclique and show how large the dimension is. (edges must be of at least size 3).

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39

{1,2,3,4,10} is a K5,3. The edge {1,2,3} is in there because 35+27+23>39

Similarly the edges {1,2,4}, {1,2,10},{1,3,4},{1,3,10},{1,4,10}, {2,3,4},{2,3,10},{2,4,10} {3,4,10} are all there so it is a hyperclique. It is maximal since you can’t add on vertex 5 to this hyperclique.

The hyperclique inequality is x1+x2 +x3+x4+x10≤ 3-1 = 2 and defines a face of dimension 9 with the following points.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

There are many other hypercliques.

4. I like science fiction. So assume that the fourth dimension is time. Suppose I have a polyhedron that has length, width, depth and time as the dimensions. Furthermore, you may assume that the only constraints are bound constraints with each variable being restricted between 0 and 1. (The four dimensional hypercube.) Please describe in words at least two classes of facets for this polyhedron.

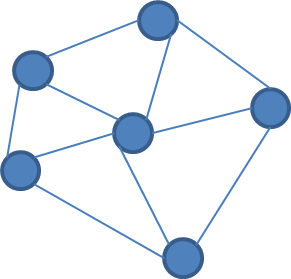
As described in class, the polyhedron can be represented by a cube starting at time 0 and going to time 1. Pick up a die, don’t blink and look at it for exactly one second. That is the polyhedron.

All facets will have dimension 3.

One facet is the cube at time 0, another is the cube at time 1. Think about taking a 3d picture at time 0 or 1. To make this picture work, the cube better be a glass so you can see all 6 sides at exactly that moment.

The other class of facets is any of the 6 squares faces of the die moving over time between 0 and 1. Thus, you would be a two dimensional plane over the entire second. You could achieve this by taking a 1 second video of exactly one side of the die. In this case, the die cannot be translucent. You must center the camera in such a fashion that you cannot see any of the other sides.

5. A wheel Wm, is a graph with m+1 nodes where the nodes 2, 3,..., m+1 form a hole and there exist edges {1,2}, {1,3}, ..., {1,m+1}. A picture of a W5 is below. Derive conditions where an induced Wm creates a facet defining inequality for the node packing polyhedron. Prove that the conditions you come up with do generate a facet-defining inequality. The more broad the condition, the more points given.



The wheel inequality for m≥5 and odd *is .*  This is clearly a valid inequality.

A wheel inequality is facet defining if every vertex is either in the wheel or the vertex is either not adjacent to the hub or adjacent to at most two of nodes in the whole.

Proof: Given an odd wheel Wm with m≥5 in a graph G=(V,E). Without loss of generality, assume the vertices of the wheel are v1, v2, …, vm such that v1 is the hub. If every uϵV\V(Wm) has that every {u,v} is not in E or that u is adjacent to at most two spoke vertices, then the wheel inequality is facet defining.

As shown in class dim(PNPch) = |V| as shown in class. Clearly, is a valid inequality. Furthermore, 0 is feasible and 0<, so the dim (F) ≤ |V|-1 where F= {xϵR|V|: xϵPNPch and

The points e1, for all j=1 to m. These m points are clearly in F and are affinely independent as the second set of points are just the cyclical permutation of an odd hole with alternating every other one. This was discussed in class.

Let uϵV\V(Wm) such that {u,v1} is not in E. Then the point eu+e1 is in F. Let uϵV\V(Wm) such that u is adjacent to at most two vertices in {v2, v3,…, vm}. Thus, there exists some j ϵ {1,…,m} such that eu+ is in F. Clearly, this is |V| affinely independent points in F. Thus, the wheel inequality is a facet defining inequality under these conditions.