1. Given the following knapsack constraint.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x∈B10

This problem has a lot of covers. Find an extended cover inequality that is facet defining. (prove that it is a facet)

Examine the cover {2,7,10} The extended cover is {1,2,7,10} and the associated inequality is

x1+ x2+x7+x10≤2.

The point (24/35,0,0,0,0,0,1,0,0,1) is a feasible linear relaxation point that is eliminated by this inequality.

The dimension of Pch is 10 as proved in class.

This is a valid inequality as it is an extended cover inequality.

(0,0,0,0,0,0,0,0,0,0) ϵP and 0<2 so dim(F)<10.

The following points are clearly affinely independent and satisfy the inequality at equality. Thus, this inequality is facet defining.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 7 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 10 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

c. Take the cover {6,7,8,9} and sequentially lift it to make it facet-defining. Also prove that it is facet defining.

I am going to lift backwards so I will lift x10, x5,...,x1

Initial inequality

x6+x7+x8+x9≤3

The solution to

Maximize x6+x7+x8+x9.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x10=1, x∈B10

is z=3, so α10=3-3=0.

The new inequality is

x6+x7+x8+x9≤3

The solution to

Maximize x6+x7+x8+x9.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x5=1, x∈B10

is z=2, so α5=3-2=1.

The new inequality is

x6+x7+x8+x9≤3

The solution to

Maximize x6+x7+x8+x9.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x5=1, x∈B10

is z=2, so α5=3-2=1.

The new inequality is

x5+ x6+x7+x8+x9≤3

The solution to

Maximize x5+ x6+x7+x8+x9.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x4=1, x∈B10

is z=2, so α4=3-2=1.

The new inequality is

x4+ x5+ x6+x7+x8+x9≤3

The solution to

Maximize x4+ x5+x6+x7+x8+x9.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x3=1, x∈B10

is z=1, so α3=3-2=1.

The new inequality is

x3 +x4+ x5++x7+x8+x9≤3

The solution to

Maximize x3 +x4+ x5+ x6+x7+x8+x9

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x2=1, x∈B10

is z=1, so α2=3-1=1.

The new inequality is

2x2+ x3 +x4+ x5+ x6+x7+x8+x9≤3

The solution to

Maximize 2x2+ x3 +x4+ x5+ x6+x7+x8+x9

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x1=1, x∈B10

is z=0, so α2=3-0=3.

The final inequality is

3x1 +2x2+ x3 +x4+ x5+ x6+x7+x8+x9≤3

It is valid due to the lifting procedure. It doesn’t define the whole space since 0 is feasible and 0<3. The following 10 pts are affinely independent and satify the inequality at equality. Thus it is a facet defining inequality.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 6 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| 9 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 10 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

The point (1,0,0,0,0,0,0,0,.5,0) is a feasible linear relaxation point that is eliminated by this inequality.

d. Take the cover from c and simultaneously lift {3,4,5} and then simultaneously lift {1,2}. (Visit www.opl.com) for software to aid in solving integer programs. What is the dimension of the inequalities new face?

Initial inequality

x6+x7+x8+x9≤3

I am going to simultaneously lift 3,4,5. Since I could feasible set at most 2 of these variables to 1, I will start my α at 3/2= rhs/feasible solution.

So the inequality is 1.5\*( x3 +x4+ x5)+x6+x7+x8+x9≤3

The solution to

Maximize 1.5\*( x3 +x4+ x5)+x6+x7+x8+x9

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x∈B10

is z=3.5 which is more than 3 with the solution of x4= x7=x8=1 and all others = 0.

Putting this into the general formula you get α(0+1+0)+0+1+1+0=3. Thus α=1.

So the new inequality is 1( x3 +x4+ x5)+x6+x7+x8+x9≤3

The solution to

Maximize x3 +x4+ x5+x6+x7+x8+x9

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x∈B10

is z=3.0 and so the inequality is valid.

Now to add {1,2}, observe that at most one of them can be 1, so α=3/1

So the new inequality is 3(x1+x2)+ x3 +x4+ x5+x6+x7+x8+x9≤3

The solution to

Maximizes 3(x1+x2)+ x3 +x4+ x5+x6+x7+x8+x9

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x∈B10

is z=4.0 which is more than 3 with the solution of x2= x9=1 and all others = 0.

Putting this into the general formula you get α(0+1)+0+0+0+0+0+0+1=3. Thus α=2.

So the new inequality is 2(x1+x2)+ x3 +x4+ x5+x6+x7+x8+x9≤3

The solution to

Maximize 2(x1+x2)+ x3 +x4+ x5+x6+x7+x8+x9

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

x∈B10

is z=3.0 and so the inequality is valid.

The final inequality is then

2(x1+x2)+ x3 +x4+ x5+x6+x7+x8+x9≤3

This inequality has dimension 8 as the following 9 affinely independent points show. This is not facet defining.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 6 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 8 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 9 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

The point (0,1,0,0,0,0,0,.5,1,0) is a feasible linear relaxation point that is eliminated by this inequality.

c.

35x1+27x2 +23x3+19x4+15x5+15x6+12x7+8x8+6x9+3x10≤39.

With cover {6,7,8,9}

µ0=0

µ1=15

µ2=15+12=27

µ3=27+8 =35

µ4=35+6=41

λ=41-39=2

Ranges

0≤αi≤ 15-2 =13 implies αi =0

13<αi< 15 =13 implies αi ϵ[0,1]

15≤αi≤ 27-2 =25 implies αi =1

25<αi< 27 implies αi ϵ[1,2]

25≤αi≤ 33 implies αi =2

33<αi< 35 implies αi ϵ[2,3]

35≤αi≤ 39 implies αi =3

39<αi< 41 implies αi ϵ[3,4]

Thus,

3x1+2x2 +x3+x4+x5+x6+x7+x8+x9≤3 is a valid inequality. General comments, no items existed in the second class of ranges. If I had a knapsack inequality of this form,

34x1+34x2 +33x3+26x4+26x5+15x6+12x7+8x8+6x9+14x10≤39, then the following inequalities would all be valid. Notice that the cover didn’t change.

3x1+2x2 +2x3+x4+x5+x6+x7+x8+x9 +0x10 ≤3

2x1+3x2 +2x3+x4+x5+x6+x7+x8+x9 +0x10 ≤3

2x1+2x2 +2x3+2x4+x5+x6+x7+x8+x9 +0x10 ≤3

2x1+2x2 +2x3+x4+2x5+x6+x7+x8+x9 +0x10 ≤3

2x1+2x2 +2x3+x4+x5+x6+x7+x8+x9 +1x10 ≤3

The point (1,0,0,0,0,0,0,0,.5,0) is a feasible linear relaxation point that is eliminated by this inequality.

2. Find and prove a facet defining inequality to the following IP. Your facet defining inequality must be useful and cutoff a linear relaxation point.

36x1+27x2 +12x3+6x4+4x5+2x6 =39.

x∈{0,1}6

This is an equality and b is odd. Since a2 is the only odd, it must be 1. Therefore one is looking for all solutions to 36x1 +12x3+6x4+4x5+2x6 =12. Clearly x1=0. From here it is easy to see that there are only two feasible points in P

(0,1,1,0,0,0) and (0,1,0,1,1,1). Thus, dim(Pch)=1. Consequently, any facet will have dimension 0, which is a point. Thus, any inequality passing exactly one of these points would be facet defining as long as the inequality was valid and didn’t satisfy the other point.

To help understand,

x1≥0 is not facet defining as F=Pch

However

x3≥0 is facet defining, since it goes through the second point, but doesn’t meet the first point at equality.

Some other equivalent facet defining inequalities are

x4+x5+x6 ≤3

x5+x6 ≤2

x4+x6 ≤2

x4+x5 ≤2

x4 ≤1

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There are an infinite number of inequalities that could represent this facet. To see this examine the inequality

x4-λx3 ≤1

That is valid for any λ≥0 and none of these inequalities would be similar and yet they define the exact same facet. Hopefully this helps you understand facets in nonfull dimensions better.

A similar facet is x3+x5+x6 ≤2

36x1+27x2 +12x3+6x4+4x5+2x6 =39.

The point (23/36,0,1,0,1,1) is a feasible linear relaxation point that is eliminated by this inequality.