Primal Affine Scaling Algorithm

Consider the LP Problem (Standard Form)

$$\text{(P1)} \qquad \begin{array}{ll} \min & c^t x \\ \text{s.t.} & Ax = b, \qquad x \geq 0 \end{array}$$

Define the feasible region of P1:

$$P = \{x : Ax = b, x \geq 0\}$$

The relative interior of P is defined as:

$$P^0 = \{x : Ax = b, x > 0\}$$

Affine Scaling Transformation

Let $\quad x^k \in \Re^n$, and x>0 for $i = 1, 2, \ldots, n$.

$\Longrightarrow \quad x^k \in ri(\Re^{n+})$,

where $ri(\bullet)$ denotes the relative interior of $\Re^{n+}$ the non-negative octant of $\Re^n$

Define a $n \times n$ matrix

$$X_k = \begin{bmatrix} x_1^k & & & \\ & x_2^k & & \\ & & \ddots & \\ & & & x_n^k \end{bmatrix}$$

X_k is non-singular, hence X exists, which is also a diagonal matrix but with 1/ x being its i-th diagonal element for i = 1, 2, ..., n.

Reforming the AFFINE SCALING TRANSFORMATION as follows:

$\quad y = T_k(x) = X_x$

The following properties of $T_k(\bullet)$:

1. T_k is a well-defined mapping from $\Re^{n+}$ to $\Re^{n+}$;
2. $T_k(x^k) = c$;
3. $T_k(x) =$ is a vertex if x is a vertex;
4. $T_k(x)$ is a interior point if x is a interior point;
5. $T_k((x)$ is on the boundary of $\Re^{n+}$ if x is on the boundary of $\Re^{n+}$;
6. T_k is an one-to-one and onto mapping with an inverse transform T, such that $T, = X_k y$ for each $y \in \Re^{n+}$.

PRIMAL AFFINE SCALING ALGORITHM

Rewrite problem P1 using y:

$$\begin{array}{ll} \min & c_k^t y \\ \text{s.t.} & A_k y = b, y \geq 0, \\ \text{where} & \quad c_k = X_k c, \quad A_k = AX_k \end{array}$$

Basic Ideas:

a) Find an improving direction, d, such that

$$\begin{array}{l} c_k^t(y^k + \alpha^k d_y^k) < c_k^t y^k, \quad \forall \alpha^k > 0, \\ AX_k d_y^k = 0, \quad \Rightarrow d_y^k \text{ onto the null space of } A_k(= AX_k), \\ \Rightarrow A_k(y^k + \alpha^k d_y^k) = b \end{array}$$

b) Find a step-size, such that $y^k + \alpha^k d_y^k > 0$

c) Let $y^{k+1} = y^k + \alpha^k d_y^k$, and $x^{k+1} = X_k y^{k+1}$

a) Finding An Improving Direction (d):

Since we are minimizing $c_k^t y$, c_k is then an improving direction. To ensure d is also a feasible direction, we must make sure that $A_k d_y^k = AX_k d_y^k = 0$, we can project c_k onto the null space of A_k to define d.

Define the orthogonal projection matrix,

$$P_k = I - A_k^t(A_k A_k^t)^{-1}A_k = I - X_k A^t(AX_k^2 A^t)^{-1}AX_k$$

NOTE:

1. $P_k = P_k^2$
2. $P_k = P_k^t$ is symmetric
3. $(I - P_k)^t b P_k c = b^t(I - P_k)^t P_k c = b^t(P_k - P_k^2)^t c = 0$

   $(I - P_k)^t b$: the error vector.

The improving direction can be calculated as follows:

$$d_y^k = P_k(-c_k) = -[I - X_k A^t(AX_k^2 A^t)^{-1}AX_k](X_k c) = -P_k X_k c \ A_k y = b.$$

b) Finding An Appropriate Step Size (α^k):

If $\quad d \geq 0, \quad y^k + \alpha^k d_y^k > 0 \quad \forall \alpha^k > 0.$

If $\quad d_i < 0, \quad$ for some i, then $\quad \alpha^k < \dfrac{y_i^k}{-(d_y^k)_i} = \dfrac{1}{-(d_y^k)_i},$

Choose α^k based on the minimum ratio test:

$$\alpha^k = \min_i \{\frac{\gamma}{-(d_y^k)_i} : (d_y^k)_i < 0, i = 1, 2, ..., n\}, \qquad \text{where } 0 < \gamma < 1, \text{ e.g., } \gamma = 0.9.$$

c) Calculate New Solution (x^{k+1})

We can calculate the original vaule of x^k at the current iteration k,

$$\begin{aligned}
x^{k+1} &= T(y^{k+1}) = X_k y^{k+1} \\
&= X_k(y^k + \alpha^k d_y) \\
&= x^k + \alpha^k X_k d_y \\
&= x^k + \alpha^k X_k P_k X_k c \\
&= x^k - \alpha^k X_k [I - X_k A^t (A X_k^2 A^t)^{-1} A X_k (X_k c) \\
&= x^k - \alpha^k {X_k}^2 [c - A^t (A X_k^2 A^t)^{-1} A X_k^2 c] \\
&= x^k - \alpha^k X_k^2 [c - A^t w^k],
\end{aligned}$$

where w^k is the dual vector, such that

$$w^k = (A X_k^2 A^t)^{-1} A X_k^2 c$$

Hence, the improving direction at current point x^k denoted as d,

$$d_x^k = -X_k^2 [c - A^t w^k],$$

and the improving direction for y,

$$\begin{aligned}
d_y^k &= -X_k [c - A^t w^k] \\
&= X_k [A^t w^k - c], \quad \text{or}
\end{aligned}$$

$$(d_y^k)_j = x_j^k (z_j - c_j)$$

Conclusions:

1) $d_y^k = -P_k c_k = -[I - X_k A^t (A X_k^2 A^t)^{-1} A X_k](X_k c)$ and $d_x^k = X_k d_y^k = X_k^2 [A^t w^k - c]$

$$\begin{aligned}
c^t x^{k+1} &= c^t x^k + \alpha^k c^t X_k d_y^k \\
&= c^t x^k + \alpha^k (c_k^t) d_y^k \\
&= c^t x^k + \alpha^k (c_k^t) P_k^2 (-c_k) \\
&= c^t x^k + \alpha^k (P_k c_k)^t (-P_k c_k) \\
&= c^t x^k - \alpha^k (d_y^k)^t (d_y^k) \\
&= c^t x^k - \alpha^k \left\| d_y^k \right\|^2
\end{aligned}$$

2) If $x^k \in P^0$ and $d_y^k > 0$, P1 is unbounded.

3) If $x^k \in P^0$ and $d_y^k = 0, x^k$ is the optimum of P1.

4) If P1 has a finite optimal solution, then the sequence generated by the primal affine scaling algorithm, $\{c^t x^k | k = 1, 2, \ldots\}$ is well-defined and strictly decreasing.

5) Let the dual slack variables – reduced cost
$s^k = [c - A^t w^k]$

The duality gap

$c^t x^k - b^t w^k = \bar{e} X_k s^k$ if $\bar{e} X_k s^k \geq 0$, we have optimum at hand

Hence, $\quad d_y^k = -X_k s^k$.

PRIMAL AFFINE ALGORITHM

STEP 1 (Initialization) Set $k = 0$. Find $x^0 > 0$ and $Ax^0 = b$.

STEP 2 (Computation of the Dual Solutions)

$$\begin{aligned} w^k &= (AX_k^2A^t)^{-1}AX_k^2c \\ s^k &= c - A^tw^k \end{aligned}$$

STEP 3 (Computation of Improving Direction)

Let $d_y^k = -X_ks^k$

STEP 4 (Check for Optimality)

If $x^k \in P^0$ and $d_y^k > 0$, STOP, problem P1 is unbounded;

If $\bar{e}^td_y^k < \varepsilon$ and $s^k \geq 0$, STOP, we have the optimum of P1;

Otherwise, go to STEP 5.

STEP 5 (Computation of Step-Size)

Compute

$\alpha^k = \min_i\{\frac{\gamma}{-(d_y^k)_i} : (d_y^k)_i < 0, i = 1, 2, ..., n\}$, where $0 < \gamma < 1$.

STEP 6 (Computation of new solution)

$x^{k+1} \quad = x^k + \alpha^kX_kd$

$k \leftarrow k + 1$

go to STEP 2.