Two-Way ANOVA
Part 3: ANOVA Table and F-Tests

STAT 705: Regression and Analysis of Variance

Introduction

- We continue to work with two-way data, concentrating on main effects and interaction effects
- So far, we have used *population means* to define main effects and interactions effects
- Now we move to *sample data*
  - Develop formal hypothesis tests to determine if effects are statistically significant
  - Define the test statistics
  - Display the test results

Hypotheses We Want to Test

- These are based on the interaction model

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}\,, \quad \text{i.e.} \quad Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

- A main effects hypothesis: H_0: $\alpha_i = 0$ for all levels of A
  vs. H_a: not all α_i's are 0
- B main effects hypothesis: H_0: $\beta_j = 0$ for all levels of B
  vs. H_a: not all β_j's are 0
- Interaction hypothesis: H_0: $(\alpha\beta)_{ij} = 0$ for all treatments
  vs. H_a: not all $(\alpha\beta)_{ij}$ are 0

Test Statistics for the Hypotheses

- The test statistics depend on
  - point estimates for the parameters, and
  - separating the total variability in the response into components attributed to
    - A main effects
    - B main effects
    - interaction effects

Least Squares Estimates for Means

- Grand mean

  $\hat{\mu}$ = average of all observations

- Treatment (i.e. cell) means

  $\hat{\mu}_{ij}$ = average of all observations in treatment (Ai, Bj)

- Marginal mean for i^{th} level of factor A

  $\hat{\mu}_{Ai}$ = average of all the cell means involving factor Ai

- Marginal mean for j^{th} level of factor B

  $\hat{\mu}_{Bj}$ = average of all the cell means involving factor Bj

Technical note: These definitions are based on Type 3 sums of squares, which lead to Type 3 estimates and hypothesis tests. There are also Type 1, Type 2 and Type 4 definitions, but we will not consider those in this course. Differences between the four types arise only when the data are not balanced.

Least Squares Estimates for Effects

- To estimate the main effects and interaction effects, use
  a) the definitions for these effects, and
  b) the least squares estimates for the means (on the previous slide)
- The least squares estimates are
  - Effect of the i^{th} level of Factor A: $\hat{\alpha}_i = \hat{\mu}_{Ai} - \hat{\mu}$
  - Effect of the j^{th} level of Factor B: $\hat{\beta}_j = \hat{\mu}_{Bj} - \hat{\mu}$
  - Interaction effect: $\hat{\mu}_{ij} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j$
- Note that if we substitute the estimates for α_i and β_j, the interaction effect can be written $\hat{\mu}_{ij} - \hat{\mu}_{Ai} - \hat{\mu}_{Bj} + \hat{\mu}$

Partitioning the Total Variation

- Test statistics for the hypotheses are based on the previous estimates and separating (partitioning) the total variation in the response
- This is the same approach that we used in regression
- The total variation in the response (SSTot) is split into
  1) the variation that is explained by the model (SSModel), and
  2) the variation that is not explained (SSE)

$$\begin{aligned} SSTot &= \sum\left(Y_{ijk} - \bar{Y}\right)^2 = \sum\left(Y_{ijk} - \hat{\mu}\right)^2 \\ &= \sum\left(Y_{ijk} - \hat{Y}_{ijk} + \hat{Y}_{ijk} - \hat{\mu}\right)^2 \\ &= \underbrace{\sum\left(Y_{ijk} - \hat{Y}_{ijk}\right)^2}_{\textbf{SSE}} + \underbrace{\sum\left(\hat{Y}_{ijk} - \hat{\mu}\right)^2}_{\textbf{SSModel}} \end{aligned}$$

Partitioning the Model SS

We continue partitioning by splitting the model sum of squares into components for the effects

$$\hat{Y}_{ijk} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + (\widehat{\alpha\beta})_{ij}$$

$$\hat{Y}_{ijk} - \hat{\mu} = \hat{\alpha}_i + \hat{\beta}_j + (\widehat{\alpha\beta})_{ij}$$

$$\hat{Y}_{ijk} - \hat{\mu} = \left(\hat{\mu}_{Ai} - \hat{\mu}\right) + \left(\hat{\mu}_{Bj} - \hat{\mu}\right) + \left(\hat{\mu}_{ij} - \hat{\mu}_{Ai} - \hat{\mu}_{Bj} + \hat{\mu}\right)$$

Square each term and sum over all observations

$$\sum\left(\hat{Y}_{ijk} - \hat{\mu}\right)^2 = b \cdot r\sum\left(\hat{\mu}_{Ai} - \hat{\mu}\right)^2 + a \cdot r\sum\left(\hat{\mu}_{Bj} - \hat{\mu}\right)^2 + r\sum\left(\hat{\mu}_{ij} - \hat{\mu}_{Ai} - \hat{\mu}_{Bj} + \hat{\mu}\right)^2$$

SSModel = SSA + SSB + SSAB

(r = # replications, a = # levels for A, b = # levels for B)

These sums of squares will be used to construct test statistics

Mean Squares

- Degrees of freedom
  - For the A main effect, with ‘a’ levels, dfA = a – 1
  - For the B main effect, with ‘b’ levels, dfB = b – 1
  - For the interaction, df AB = (a – 1)(b – 1)
- Mean square is the sum of squares divided by the degrees of freedom
  - MSA = SSA / dfA
  - MSB = SSB / dfB
  - MSAB = SSAB / dfAB

The Test Statistics

- For the A main effect: H_0 is $\alpha_i = 0$ for all levels of A
  - Test statistic is F = MSA / MSE
  - Use F distribution with degrees of freedom dfA and dfE
- For the B main effect: H_0 is $\beta_j = 0$ for all levels of B
  - Test statistic is F = MSB / MSE
  - Use F distribution with degrees of freedom dfB and dfE
- For the A*B interaction: H_0 is $(\alpha\beta)_{ij} = 0$ for all treatments
  - Test statistic is F = MSAB / MSE
  - Use F distribution with degrees of freedom dfAB and dfE

ANOVA Table for Two-Way Data

- The ANOVA table provides a concise way of recording the information we have been calculating
- N is the total number of observations (in all groups)
- We will rely on software to perform most of these calculations (especially the p-values)

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
A main effect	a – 1	SSA	SSA/dfA	MSA / MSE	
B main effect	b – 1	SSB	SSB / dfB	MSB / MSE	
A*B interaction	(a – 1)*(b – 1)	SSAB	SSAB / dfAB	MSAB / MSE	
Error	N – ab	SSE	SSE / dfE		
Total	N – 1	SSTot			

Fabric Data Re-visited

Level	Salt	Temperature
1	Untreated	812
1	Untreated	827
1	Untreated	876
2	Untreated	945
2	Untreated	881
2	Untreated	919
1	$CaCO_3$	733
1	$CaCO_3$	728
1	$CaCO_3$	720
2	$CaCO_3$	786
2	$CaCO_3$	771
2	$CaCO_3$	779
1	$CaCl_2$	725
1	$CaCl_2$	727
1	$CaCl_2$	719
2	$CaCl_2$	756
2	$CaCl_2$	781
2	$CaCl_2$	814

- Fabric was treated with one of three different inorganic salts, at one of two levels of concentration, for the purpose of measuring its effect on the flammability of the fabric
- Response variable is the temperature at which the fabric ignites

Fabric Data: ANOVA Table

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Level	1	17734.72	17734.72	34.11	<.0001
Salt	2	60928.78	30464.39	58.60	<.0001
Level*Salt	2	486.11	243.06	0.47	0.6375
Error	12	6238.67	519.89		
Corrected Total	17	85388.28			

Verify the degrees of freedom

- Factor A is ‘Level’; number of levels is 2; dfA = 1
- Factor B is ‘Salt’; number of levels is 3; dfB = 2
- Interaction: dfAB = 1*2 = 2
- Total number observations = N

  = (number of A’s)*(number of B’s)*(number of replications)

  = 2*3*3 = 18 ⇨dfTot = 17

Interpret the ANOVA Table

- **Examine the interaction first.**
  - It is not significant ($p = 0.6375$). This implies
    - » If Salt impacts Temperature, the effect is the same for both Level 1 and Level 2, and
    - » If Level impacts Temperature , the effect is the same for all three Salts
- Since the interaction is not significant, we can interpret the main effects.
  - Salt does affect Temperature ($F = 58.60$, $p < 0.0001$)
  - Level does affect Temperature ($F = 34.11$, $p < 0.0001$)

Relation to Nested Model F Test

- Each of the hypothesis tests for main effects and interactions can be viewed as a comparison-of-models F test
  - This is also called a 'nested model' F test
  - We discussed this in Multiple Regression, Part 4
- Consider the test for interaction. H_0: $(\alpha\beta)_{ij} = 0$ for all treatments
  - Full model is the interaction model: $Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$
  - Reduced model is the additive model: $Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$
- The test for interaction is comparing these two models.
  - If we reject H_0, then we need to keep the full (interaction) model.
  - If we do not reject H_0, then we can use the reduced (additive) model.

What You Should Know

- The concept behind partitioning the sums of squares
- How to interpret the ANOVA table
  - Look first at the interaction hypothesis
    - If there is no interaction, then consider the tests of main effects
    - If there is interaction, subsequent analysis gets more complicated (more about this later)
  - Know the hypotheses for each F test
  - Be able to hand-calculate the degrees of freedom