Two-Way ANOVA
Part 2: Hypotheses

STAT 705: Regression and Analysis of Variance

Introduction

- In the last lesson, we considered additive and non-additive (interaction) models for two-way ANOVA.
- In this lesson, we define important hypotheses that are tested in two-way ANOVA.
- Our hypotheses are about main effects and interaction effects.
- In subsequent lessons we will do statistical tests of these hypotheses to determine which effects are significant and which are not. Now we are interested in defining the hypotheses and understanding what they mean.

The Main Effect Hypotheses

- We define the hypotheses for main effects and interaction effects in terms of the interaction model

  $Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$

- Main effect hypotheses for factor A

  H_0: $\alpha_i = 0$ for all levels of A

  vs. H_a: not all α_i's are 0

  If H_0 is true, we say there are no A main effects

  If H_a is true, we say there are A main effects

- Main effect hypotheses for factor B are similar, but use β_j's instead of α_i's

Main Effects and Marginal Means

- The 'main effect' hypotheses for a factor may be defined in terms of marginal means of the factor. Here are the main effect hypotheses for A:
  - H_0: marginal means for A are all the same
    (this is equivalent to H_0: $\alpha_i = 0$ for all levels of A)
  - H_a: marginal means for A are not all the same
    (this is equivalent to H_a: not all α_i's are 0)
- To see why this works, suppose $\alpha_i = 0$ for all levels of A. By definition, $\alpha_i = \mu_{Ai} - \mu$, so if these terms are 0, we have $\mu_{Ai} = \mu$ for all levels of factor A, so all the marginal means are the same.

Example 1

- Consider the fictitious population means in the table below
- The marginal means for factor A are both 12, so there are no A main effects.
- The marginal means for factor B are 10, 16 and 10. Because they are not all the same, there are B main effects.

B / A	B1	B2	B3	Marginal Means
A1	9	18	9	12
A2	11	14	11	12
Marginal Means	10	16	10	12

Interaction Hypotheses

- We are still working from the interaction model

  $Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$

- The interaction hypotheses are

  H_0: $(\alpha\beta)_{ij} = 0$ for all treatments

  vs. H_a: not all $(\alpha\beta)_{ij}$ are 0

  If H_0 is true we say there is no interaction

  If H_a is true we say there is interaction

- If there is no interaction, then we have the additive model: $Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk}$

Interaction Contrasts

- Interactions can be written as contrasts of treatment means
- Consider the population means formed by combining levels A1 and A2 with levels B1 and B2.

Factor Levels	B1	B2	B3	Marginal Means
A1	μ_{11}	μ_{12}	μ_{13}	μ_{A1}
A2	μ_{21}	μ_{22}	μ_{23}	μ_{A2}
Marginal Means	μ_{B1}	μ_{B2}	μ_{B3}	μ

- The contrast $(\mu_{11} - \mu_{12}) - (\mu_{21} - \mu_{22})$ is called an interaction contrast
- With this contrast, we are comparing $(\mu_{11} - \mu_{12})$ to $(\mu_{21} - \mu_{22})$ to see if the differences are equal or not

Example 2

- Consider the hypothetical population means in the table
- The contrast involving A1 and A2 with B1 and B2 is (9 – 18) – (11 – 14) = –6

Factor Levels	B1	B2	B3	Marginal Means
A1	9	18	9	12
A2	11	14	11	12
Marginal Means	10	16	10	12

- We may compute this interaction contrast in the other direction and get the same result: (9 – 11) – (18 – 14) = –6
- The contrast involving A1 and A2 with B2 and B3 is (18 – 9) – (14 – 11) = 6. This tells us that in going from B2 to B3, the change in means is 6 units more under A1 than it is under A2.

Interaction Hypotheses and Contrasts

- The interaction hypotheses can be expressed in terms of contrasts
  - H_0: all interaction contrasts are 0

    (this is equivalent to H_0: $(\alpha\beta)_{ij}$ = 0 for all treatments)
  - H_a: at least one interaction contrast is not 0

    (this is equivalent to H_a: not all $(\alpha\beta)_{ij}$ are 0)

To see why this works, consider the contrast involving A1 and A2 with B1 and B2, i.e. $(\mu_{11} - \mu_{12}) - (\mu_{21} - \mu_{22})$. With no interaction, the effects are additive so that

$$\mu_{11} = \mu + \alpha_1 + \beta_1 \text{ and } \mu_{12} = \mu + \alpha_1 + \beta_2$$

When we take the difference, μ and α_1 cancel out, so $\mu_{11} - \mu_{12} = \beta_1 - \beta_2$. Similarly, $\mu_{21} - \mu_{22} = \beta_1 - \beta_2$, so

$$(\mu_{11} - \mu_{12}) - (\mu_{21} - \mu_{22}) = (\beta_1 - \beta_2) - (\beta_1 - \beta_2) = 0.$$

In other words, the contrast is 0.

Example 3

A hypothetical agronomy study…

We will consider two cases of population means to illustrate how to use marginal means and contrasts to determine whether or not there are main effects and interactions.

An Agronomist studied how an insecticide and a herbicide (bug and weed killers) affected the growth of plants.

The agronomist applied combinations of two levels of insecticide (0, 4) and three levels of herbicide (0, 1, 2) to containers of plants and measure their dry weights after the growing period.

Example 3, Case 1

- For the population means in this case we will show that there are main effects for both insecticide and herbicide, but there is no interaction.

Herb. / Insect.	H=0	H=1	H=2	Marginal Means
I=0	100	70	70	80
I=4	90	60	60	70
Marginal Means	95	65	65	75

- There are insecticide main effects because the two marginal means for insecticide are not the same. There are herbicide main effects because the marginal means for herbicide are not all the same.
- These main effects may or may not be statistically significant. We will perform formal hypothesis tests later.

Example 3, Case 1 continued

- There is no interaction because any interaction contrast involving these means is 0.

Herb. / Insect.	H=0	H=1	H=2	Marginal Means
I=0	100	70	70	80
I=4	90	60	60	70
Marginal Means	95	65	65	75

- A simple way to show this is to note that the differences between I=0 and I=4 is the same for all levels of H:

  100 – 90 = 70 – 60 = 70 – 60 = 10

  So any contrast involving these differences will be 0.

Example 3, Case 2

For the population means in this case, we will show

» there is a main effect for insecticide

» there is a main effect for herbicide

» there is an interaction

Insect. \ Herb.	H=0	H=1	H=2	Marginal Means
I=0	100	90	80	90
I=4	90	70	50	70
Marginal Means	95	80	65	80

- There are main effects for both factors because the marginal means for each factor are not the same.
- To show that there is an interaction, we have to find just one interaction contrast that is not 0.
- Consider the contrast involving I=0 and I=4 with H=0 and H=1.
  (100 – 90) – (90 – 70) = -10 , which is not 0
  So there is interaction in this case.

Example 3, Case 2 continued

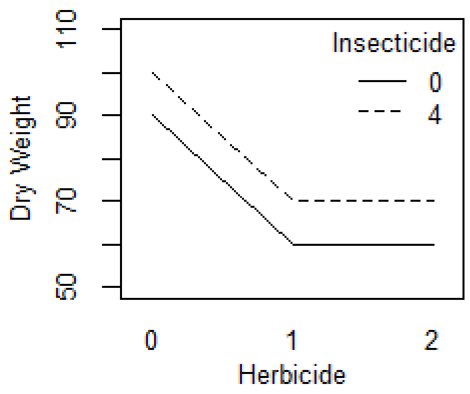
- We should always try to interpret interaction in the context of the study being done.
- When I = 0, dry weight decreases 10 units for each increase in level of herbicide.
  - Dry weight goes from 100 to 90 to 80.
- When I = 4, dry weight decreases 20 units for each increase in level of herbicide.
  - Dry weight goes from 90 to 70 to 50.
- Interaction occurs because the insecticide ‘accelerates’ the effect of herbicide.

Interaction Plots

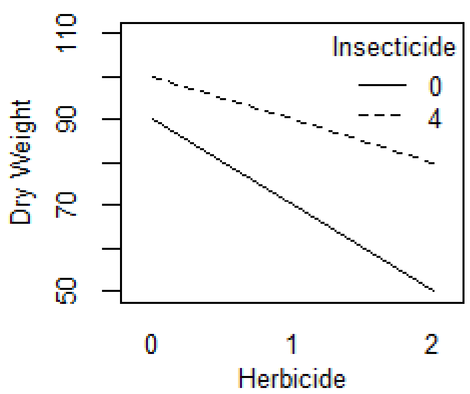
- An interaction plot is also called a mean profile plot
- Treatment means are plotted so that levels of one factor are on the x axis, and levels of the other factor are joined by lines
- If the lines of the plot are parallel, there is no interaction.
- If the lines are not parallel, there is interaction. In a later lesson, we will use formal hypothesis tests to determine if the interaction is significant.

Interaction Plots for Example 3

- For Case 1, there is no interaction so the interaction plot shows parallel lines.



- For Case 2, there is interaction, so the interaction plot shows non-parallel lines.



Interaction and Non-additivity

- In the previous lesson, we considered additive and non-additive models.
- In the current lesson, we have considered models with and without interaction
- These model are related
  - No interaction ⟺ Additive model
  - Interaction ⟺ Non-additive model
- If there is no interaction, the factors affect the response in a simple additive way. Otherwise, the factors affect the response in a more complicated way which needs to be investigated.

What You Should Know

- How to write the hypotheses for main effects and interactions in terms of model parameters
- How to calculate main effects and interactions from a provided table of means
- Identify whether or not interaction is present by looking at the mean profile plot (a.k.a., the interaction plot)
- Interpret main effects and interactions

Remember:
A main effect or an interaction can be present, but it may not be statistically significant. We have not yet conducted any formal hypothesis tests for main effects or interactions.