# Simple Linear Regression
## Part 2: Model Assumptions

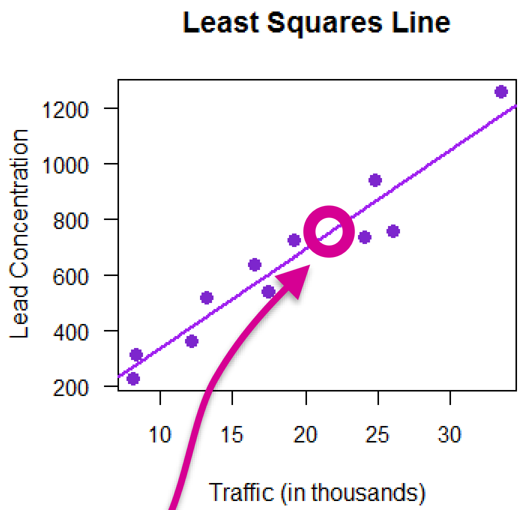
STAT 705: Regression and Analysis of Variance

# Recall

- For the Lead vs. Traffic example
  - estimated intercept = -21
  - estimated slope = 35.7
- You should know how to calculate these values
- $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X \quad \Leftrightarrow \quad \text{Lead} = -21 + 35.7 \times \text{Traffic}$

# Estimated Least Squares Line

- Lead = -21 + 35.7*Traffic
- Suppose there is another site that has traffic 22 (thousand vehicles)
- How much lead would we expect to see in the tree bark at this site?
  - Lead = -21 + 35.7∗22 = 764.4 micrograms of lead per gram of bark
  - This is the point on the line at $X$ = 22.



# A Trick Question

- Could we estimate the lead concentration for a site that has 50,000 vehicles?
- We can calculate it
  - Lead = -21 + 35.7$*$50 = 1764 micrograms per gram of bark
- But does it make sense?
- Short answer: It does <u>not</u> make sense.
  - The traffic values in the data set range from 8.1 to 33.6 (thousand)
  - 50 thousand vehicles is MUCH larger than the largest value in the data
  - We cannot assume the relationship between Traffic and Lead remains the same when Traffic is much larger than the observed values
  - This is EXTRAPOLATION, and it must be avoided

# Model vs. Estimated Line

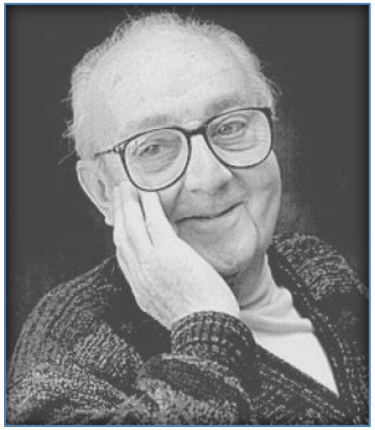
- Regression model: $Y = \beta_0 + \beta_1 X + \varepsilon$
  - Applies to the entire population (e.g., all the sites that could possibly be selected along the highway)
  - $\beta_0$ and $\beta_1$ are population parameters that describe the "true" relationship between $X$ and $Y$ for ALL items in the population
- Estimated regression line: $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$
  - This is our estimate of the "true" relationship
  - To assess whether or not our estimated line is close to the 'true' line, we can compare $\hat{\beta}_0$ to $\beta_0$, $\hat{\beta}_1$ to $\beta_1$, and $\hat{Y}$ to $Y$

# Are There Other Lines?

- Other criteria can be used to construct lines that fit the data
- Least Squares Lines have many desirable statistical properties
- Unless there is a specific reason to use a different method, Least Squares is preferred

# Statistical Model

- Is a conceptualization of a real process
- Is a simplification of a much more complex phenomenon
- The data provide clues about the process
- For some data sets, there might be many “good” models
- For other data sets, finding even one good model can be difficult



*“All models are wrong, but some are useful.”*

George E. P. Box
(1921 – 2013)

# Model Assumptions

- Model: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$
- Anything that is random has a probability distribution
  - $\beta_0$, $\beta_1$ and $X_i$ are fixed (not random)
  - $\varepsilon_i$ and $Y_i$ are random

**ASSUME: $\varepsilon_i \sim$ NIID(0, $\sigma^2$)**

- NIID is shorthand for Normally, Identically, and Independently Distributed
- We need to check that this assumption is reasonably satisfied … or at least not grossly violated!

# Implications of the Assumptions

Model: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

Assume: (1) $\varepsilon_i \sim$ NIID(0, $\sigma^2$) and (2) $\beta_0$, $\beta_1$ and $X_i$ are constants

Recall

- The expected value (mean) of a constant is the constant
- The variance of a constant is 0

| Assumption | Implication |
|---|---|
| $E(\varepsilon_i) = 0$ | $E(Y_i) = \beta_0 + \beta_1 X_i$ |
| $Var(\varepsilon_i) = \sigma^2$ | $Var(Y_i) = Var(\varepsilon_i) = \sigma^2$ |
| $\varepsilon_i$ 's are independent | $Y_i$ 's are independent |
| $\varepsilon_i$ 's are normal | $Y_i$ 's are normal |

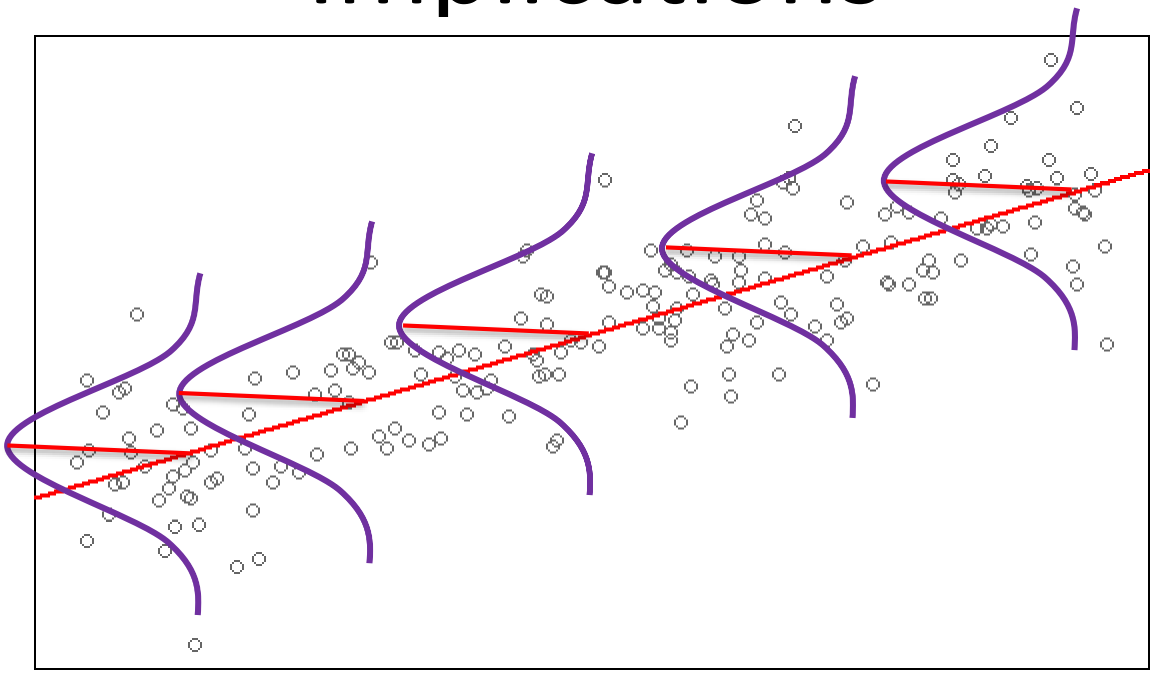
# Implications of Model Assumptions

- In shorthand,

  $\varepsilon_i \sim \text{NIID}(0, \sigma^2) \;\Rightarrow\; Y_i \sim \text{NID}(\beta_0 + \beta_1 X_i\,, \sigma^2)$

- Notes:
  - "NID" stands for a normal, independent, distribution (not identical)
  - $Y$ has a probability distribution
  - The mean of $Y$ depends on $X$
  - The variance of $Y$ does <u>not</u> depend on $X$

# Implications



Mean of $Y$ is $\beta_0 + \beta_1 X$ (depends on $X$)

(This is the Y value on the line for the specific X)

Variance of $Y$ is $\sigma^2$ (does NOT depend on $X$)

(Each normal curve has exactly the same width)

# More Implications

Gauss-Markov Theorem:

"Under the conditions of the linear regression model, the least squares estimators for $\beta_0$ and $\beta_1$ are unbiased and have minimum variance among all unbiased linear estimators."

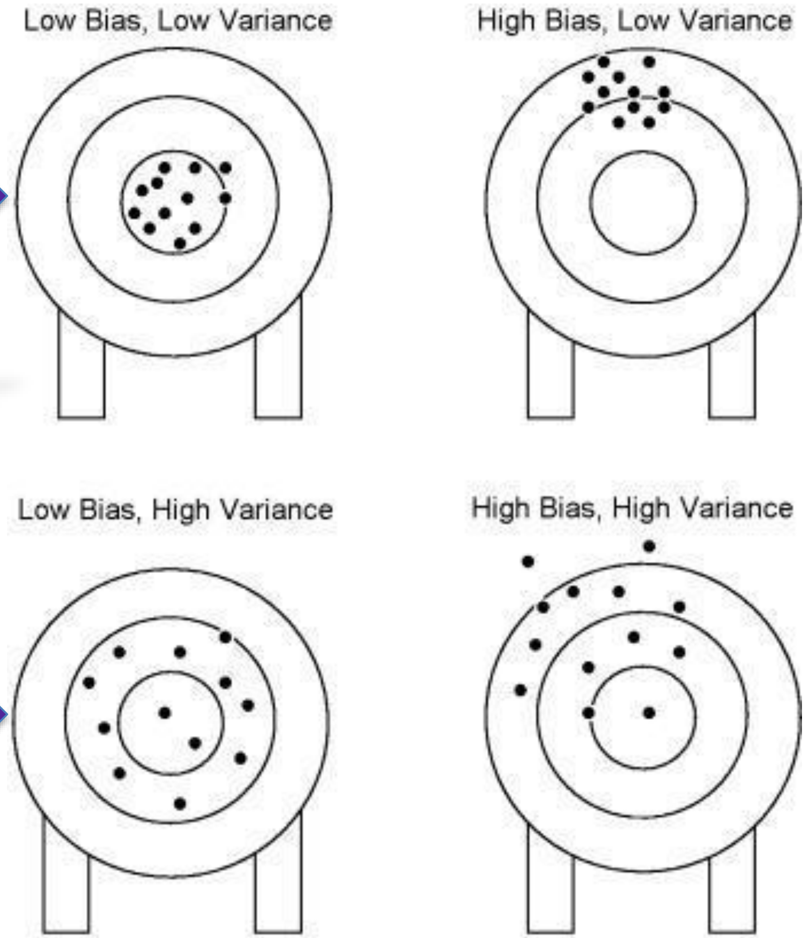
This is the statistical 'gold standard' for estimators.

# Unbiased, Minimum Variance

Reliable and accurate

BEST

Reliable, but not accurate




Accurate (on average), but not reliable

Not reliable, not accurate

Image source: http://www.amstat.org/publications/jse/v11n2/martin.html

# Reasons for Performing Regression

- Get point estimates for parameters
  - Intercept and slope ($\beta_0$ and $\beta_1$)
  - Error variance ($\sigma^2$)
- Inference on parameters
  - Confidence intervals (i.e., plausible values)
  - Hypothesis tests
- Estimate the mean of $Y$ for a given $X$
- Predict a new value of $Y$ for a given $X$

# Point Estimates

- Intercept: The point estimate for $\beta_0$ is $\hat{\beta}_0$
  - This is the estimate for $Y$ when $X$ = 0
  - This may be nonsensical. Example: For a location that has zero traffic, do we really expect -21 micrograms of lead?
- Slope: The point estimate for $\beta_1$ is $\hat{\beta}_1$
  - Average change in $Y$ for each one-unit increase in $X$
  - Example: If the traffic increases by one (thousand vehicles) then, on average, we expect the lead to increase by 35.7 micrograms per gram of bark.

# Point Estimate for $\sigma^2$

- $\varepsilon_i$ ~ NIID(0, $\sigma^2$)
- $\sigma^2$ is the variability in the Y values that is NOT explained by the regression equation
- It is based on the sum of squared errors (residuals)
- Total variability around the line = $\text{SSE} = \sum_{i=1}^{n} r_i^2$
- Average variability around the line = $\text{MSE} = \dfrac{\text{SSE}}{n-2}$
- Point estimate for is $\sigma^2$ is $\hat{\sigma}^2 = \text{MSE}$
- For Lead vs. Traffic example, MSE = 7867.2 (see next slide)

# Calculate MSE for Lead Example

| Site (i) | Traffic (X) | Lead (Y) | Est'd Lead | Residual | Resid^2 |
|---|---|---|---|---|---|
| 1 | 8.1 | 227 | 268.17 | -41.17 | 1695.0 |
| 2 | 8.3 | 312 | 275.31 | 36.69 | 1346.2 |
| 3 | 12.1 | 362 | 410.97 | -48.97 | 2398.1 |
| 4 | 13.2 | 521 | 450.24 | 70.76 | 5007.0 |
| 5 | 16.5 | 640 | 568.05 | 71.95 | 5176.8 |
| 6 | 17.5 | 539 | 603.75 | -64.75 | 4192.6 |
| 7 | 19.2 | 728 | 664.44 | 63.56 | 4039.9 |
| 8 | 24.8 | 945 | 864.36 | 80.64 | 6502.8 |
| 9 | 24.1 | 738 | 839.37 | -101.37 | 10275.9 |
| 10 | 26.1 | 759 | 910.77 | -151.77 | 23034.1 |
| 11 | 33.6 | 1263 | 1178.52 | 84.48 | 7136.9 |

Est'd Lead =
-21 + 35.7*Traffic

Residual =
Lead – Est'd Lead

Sum this column
SSE = 70,805.1

MSE = 70,805.1 / 9
MSE = 7867.2

# Things You Should Know

- The difference between the regression model and the estimated equation
- The assumptions of the linear regression model
- Understand the process for calculating and know how to interpret
  - a point estimate for the slope
  - a point estimate for the intercept
  - the mean of Y for a given X
  - the predicted value of Y for a given X
  - a point estimate for the residual variance