Two-Way ANOVA
Part 4: T-Tests and Contrasts

STAT 705: Regression and Analysis of Variance

Introduction

- In the last lesson, we saw how to use F tests to determine whether or not the *effects* are significant
- We now turn our attention to the *means*
- We can test and get confidence intervals for
  - one mean (treatment or marginal)
  - difference of two means
  - contrasts of means
    (recall that we discussed contrasts in part 6 of the ANOVA module)

Inference for Means

- Tests and confidence intervals require three things
  1) a point estimate, and
  2) a standard error of the estimate, and
  3) the reference distribution (for the critical value)
- For means, we have already calculated estimates for
  - treatment (cell) means
  - marginal means
- We now get the standard errors of the estimated means

Standard Error of a Sample Mean

- When the main effects are significant, we usually want to follow up with tests of specific means
- The standard error of a sample mean (either a treatment mean or a marginal mean) measures how accurately the sample mean estimates the corresponding population mean
- Notation
  - Let n be the number of observations that go into computing the sample mean for which we want to find the standard error
  - Let S_p be the square root of MSE.
- The formula for the standard error of the mean is $SE = \frac{S_p}{\sqrt{n}}$

Standard Errors for the Fabric Data

	Untreated mean (n)	$CaCO_3$ mean (n)	$CaCl_2$ mean (n)	Marginal mean (n)
Level 1	838.33 (3)	727.00 (3)	723.67 (3)	763.00 (9)
Level 2	915.00 (3)	778.67 (3)	783.67 (3)	825.78 (9)
Marg. mean (n)	876.67 (6)	752.84 (6)	753.67 (6)	

- From the ANOVA table, MSE = 519.89 ⇨ S_p = 22.8
- Cell means: n = 3 ⇨ SE = 22.8 / root(3) = 13.2
- Marginal means

  For Level: n = 9 ⇨ SE = 22.8 / root(9) = 7.6

  For Salt: n = 6 ⇨ SE = 22.8 / root(6) = 9.3

Inference on Sample Means

- We can perform hypothesis tests and construct confidence intervals for any of the sample means
- For individual means
  - We are usually interested in confidence intervals
  - Formal hypothesis tests are (often) less informative
- For combinations of means
  - Such as the difference of two means or a contrast involving several means
  - Usually interested in testing whether the means are equal (e.g. whether the difference is 0)
  - Confidence intervals can also be constructed

Confidence Intervals for Means

- Basic form of a confidence interval is

  (sample mean) $\pm$ (critical value)$\times$SE

- Critical value
  - From the t distribution with degrees of freedom dfE
  - For the fabric data, dfE = 12
  - Critical value (two-sided, for α = .05) is 2.179
- Confidence interval for the 'Untreated' marginal mean
  - $876.67 \pm 2.179 \times 9.3$, or (856.4, 896.9) degrees
  - 95% of the time, we expect untreated fabric to ignite when the temperature is between 856.4 and 896.9 degrees

Difference of Means

- Suppose two sample means do not involve any of the same observations. This would be the case with
  - any of the treatment means
  - marginal means for factor A
  - marginal means for factor B
- Let SE_1 and SE_2 represent the standard errors of the two sample means
- The standard error of the difference of the means is given by

$$\text{SE diff} = \sqrt{SE_1^2 + SE_2^2}$$

Hypothesis Test for Difference

- Hypotheses are $H_0: \mu_1 = \mu_2$ vs. $H_a: \mu_1 \neq \mu_2$
  - Can also be written $H_0: \mu_1 - \mu_2 = 0$ vs. $H_a: \mu_1 - \mu_2 \neq 0$
  - population means: μ_1 and μ_2 (both unknown)
- Test statistic: $t = \dfrac{\left|\hat{\mu}_1 - \hat{\mu}_2\right|}{\text{SE diff}}$
- Critical value
  - Same as with a single mean
  - From the t distribution with degrees of freedom dfE

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Hypothesis Test: Example

- Test for differences between the marginal means of $CaCO_3$ and $CaCl_2$
  - Use subscript 1 for $CaCO_3$ and subscript 2 for $CaCl_2$
  - Sample means: $\hat{\mu}_1 = 752.84$ and $\hat{\mu}_2 = 753.67$
  - Standard errors of the means
    - » $SE_1 = 9.3$ and $SE_2 = 9.3$
    - » $SE\,\text{diff} = \sqrt{9.3^2 + 9.3^2} = 13.15$
- Critical value is 2.179
  - From t distribution with α=.05 and df = dfE = 12

Hypothesis Test: Conclusion

- Test statistic = $t = \dfrac{|752.84 - 753.67|}{13.15} = 0.063$
- Do not reject H_0
  - Test statistic is not greater than critical value
  - 0.063 is not greater than 2.179
- At significance level 0.05, there is not enough evidence to conclude that the mean temperature at which fabric ignites is different for fabric treated with $CaCO_3$ or $CaCl_2$.

Contrasts

- A contrast can be used to compare any linear combination of means, provided the coefficients sum to 0
- To illustrate, we will use the fabric data to test if the average temperature for salts $CaCO_3$ and $CaCl_2$ at Level 1 is equal to the average temperature of $CaCO_3$ and $CaCl_2$ at Level 2

(Before continuing, you might want to review contrasts in Part 6 of the ANOVA module)

Standard Error of the Contrast

Level	Treatment Means		
	Untreated	$CaCO_3$	$CaCl_2$
1	838.33 (μ_{11})	727.00 (μ_{12})	723.67 (μ_{13})
2	915.00 (μ_{21})	778.67 (μ_{22})	783.67 (μ_{23})

- Average of $CaCO_3$ and $CaCl_2$ at Level 1 = $(\mu_{12} + \mu_{13})/2$
- Average of $CaCO_3$ and $CaCl_2$ at Level 2 = $(\mu_{22} + \mu_{23})/2$
- Contrast = $(\mu_{12} + \mu_{13})/2 - (\mu_{22} + \mu_{23})/2$

  $= \frac{1}{2}\mu_{12} + \frac{1}{2}\mu_{13} - \frac{1}{2}\mu_{22} - \frac{1}{2}\mu_{23}$
- From the ANOVA table, MSE = 519.89 and dfE = 12
- SE of contrast = $\sqrt{MSE} \cdot \sqrt{\frac{(1/2)^2}{3} + \frac{(1/2)^2}{3} + \frac{(-1/2)^2}{3} + \frac{(-1/2)^2}{3}} = 13.164$

Test for “Contrast = 0”

- Hypotheses H_0: $\frac{1}{2}\mu_{12} + \frac{1}{2}\mu_{13} - \frac{1}{2}\mu_{22} - \frac{1}{2}\mu_{23} = 0$

  vs. H_a: $\frac{1}{2}\mu_{12} + \frac{1}{2}\mu_{13} - \frac{1}{2}\mu_{22} - \frac{1}{2}\mu_{23} \neq 0$
- Estimate of contrast

  $\frac{1}{2}(727 + 723.67) - \frac{1}{2}(778.67 + 783.67) = -55.835$
- Test statistic: t = (estimate) / (standard error)

  $t = -55.835 / 13.164 = -4.241$
- Critical value = 2.179 (from t distribution with df = dfE = 12)
- Decision: Reject H_0 because $|-4.241| > 2.179$
- At significance level 0.05, the average temperature for $CaCO_3$ and $CaCl_2$ is different between levels 1 and 2.

Pause and Reflect

Before continuing, you should re-examine two of the hypothesis tests we have conducted on the fabric data. At a glance, these two tests appear to be giving contradictory results. Look at the treatment means involved in each test, and make sure you understand why these two tests are not the same.

Test #1: Is there a difference between the marginal means of $CaCO_3$ and $CaCl_2$?

Our conclusion: There is NO difference

Test #2: Is the average temperature for $CaCO_3$ and $CaCl_2$ at Level 1 equal to the average temperature of $CaCO_3$ and $CaCl_2$ at Level 2?

Our conclusion: There IS a difference

General Guidelines

Use these guidelines when analyzing a multi-factor experiment

1. The experimental design dictates the model.
   Do NOT use the results from the data to derive the model.
2. Include the interaction term in the model, unless there is a specific reason to exclude it.
3. Examine the results of the interaction <u>before</u> examining the main effects.
   a) If the interaction is <u>not</u> significant, we can examine the main effects.
   b) If the interaction IS significant, we should not examine the main effects. Instead, we compare the means of interest individually (e.g. by using contrasts).

What You Should Know

- Know the difference between treatment means and marginal means
- Be able to calculate the standard errors for both treatment means and marginal means, as well as difference of means and contrasts of means
- For all of these means and combinations of means
  - Be able to conduct the t tests
  - Be able to construct confidence intervals

You do NOT have to construct the ANOVA table by hand.
We will use software for this.