Solution to HW5

8.19 (a) For $n = 64$, $\sigma_{\bar{X}} = 5.6/8 = 0.7$, whereas for $n = 196$, $\sigma_{\bar{X}} = 5.6/14 = 0.4$. Therefore, the variance of the sample mean is reduced from 0.49 to 0.16 when the sample size is increased from 64 to 196.

(b) For $n = 784$, $\sigma_{\bar{X}} = 5.6/28 = 0.2$, whereas for $n = 49$, $\sigma_{\bar{X}} = 5.6/7 = 0.8$. Therefore, the variance of the sample mean is increased from 0.04 to 0.64 when the sample size is decreased from 784 to 49.

8.33 (a) When the population equals the limit, the probability of a sample mean exceeding the limit would be 1/2 due the symmetry of the approximated normal distribution.

(b) $P(\bar{X} \geq 7960 \mid \mu = 7950) = P(Z \geq (7960 - 7950)/(100/\sqrt{25})) = P(Z \geq 0.5) = 0.3085$. No, this is not very strong evidence that the population mean of the process exceeds the government limit.

8.34 (a) $\sigma_{\bar{X}_A - \bar{X}_B} = \sqrt{\frac{5^2}{30} + \frac{5^2}{30}} = 1.29$ and $z = \frac{4-0}{1.29} = 3.10$. So,

$$P(\bar{X}_A - \bar{X}_B > 4 \mid \mu_A = \mu_B) = P(Z > 3.10) = 0.0010.$$

Such a small probability means that the difference of 4 is not likely if the two population means are equal.

(b) Yes, the data strongly support alloy A.

8.44 (a) 2.145.

(b) -1.372.

(c) -3.499.

8.45 (a) $P(T < 2.365) = 1 - 0.025 = 0.975$.

(b) $P(T > 1.318) = 0.10$.

(c) $P(T < 2.179) = 1 - 0.025 = 0.975$, $P(T < -1.356) = P(T > 1.356) = 0.10$. Therefore, $P(-1.356 < T < 2.179) = 0.975 - 0.010 = 0.875$.

(d) $P(T > -2.567) = 1 - P(T > 2.567) = 1 - 0.01 = 0.99$.

8.48 From Table A.4 we find $t_{0.025} = 2.131$ for $v = 15$ degrees of freedom. Since the value

$$t = \frac{27.5 - 30}{5/4} = -2.00$$

falls between -2.131 and 2.131, the claim is valid.

Homework problems from the textbook:

8.19, 8.33, 8.34, 8.44, 8.45, 8.48

Extra problems:

Table of values of $\bar{X}$

$x_1 \backslash x_2$	0	1	2	3
0	0	1/2	1	3/2
1	1/2	1	3/2	2
2	1	3/2	2	5/2
3	3/2	2	5/2	3

Problem 1

Random variable X follows the distribution below:

X	0	1	2	3
$p(x)$	0.1	0.2	0.4	0.3

With a random sample of size $n = 2$, for the sample average $\bar{X} := \bar{X}_2$,

a) Calculate $\Pr(\bar{X} = 1), \Pr(\bar{X} = 2)$.

Let x_1 and x_2 be the two samples. Then

$\bar{x} = 1 \Rightarrow \{ x_1 = 0, x_2 = 2$, or $x_1 = 2, x_2 = 0$, or $x_1 = 1, x_2 = 1$

Therefore $p(\bar{x} = 1) = p(x_1 = 0, x_2 = 2) + P(x_1 = 2, x_2 = 0) + P(x_1 = 1, x_2 = 1) = 0.12$

Similarly $P(\bar{x} = 2) = 0.28$

b) What is the distribution of $\bar{X}$? (*Hint: complete the distribution table.*)

$\bar{x}$	0	1/2	1	1 1/2	2	2 1/2	3
$p(x)$	0.01	0.04	0.12	0.22	0.28	0.24	0.09

c) Calculate $E(\bar{X})$ and $SD(\bar{X})$.

From above $E(\bar{x}) = \sum x_i p(x_i) = 1.9$

$E(\bar{x}^2) = 4.055$

$\Rightarrow Var(\bar{x}) = 0.445$ & $SD(\bar{x}) = 0.6671$

d) Verify that $E(\bar{X}) = E(X), SD(\bar{X}) = SD(X)/\sqrt{n}$.

$E(x) = 0 + 0.2 + 0.8 + 0.9 = 1.9 = E(\bar{x})$

$E(x^2) = 4.5$

$Var(x) = 0.89 \Rightarrow SD(x) = 0.9434$

$\frac{SD(x)}{\sqrt{n}} = \frac{SD(x)}{\sqrt{2}} = 0.6671$

Problem 2

The scores of students on the ACT college entrance examination in a recent year had the Normal distribution with mean 18.6 and standard deviation 5.9.

1. What is the probability that a single student randomly chosen from all those taking the test scores 21 or higher?
2. Consider 20 randomly selected students who took the test. What is the distribution of the average score of 20 students? What are parameter values (mean and standard deviation) for this distribution?
3. What is the probability that the average score is 21 or higher?

1. Test score of any single student is normally distributed with mean 18.6 and sd 5.9

Therefor required probability is

$P[X > 21] = \text{normal cdf } (21, 9^9, \cancel{18}18.6, 5.9)$

$= 0.3421$

2. Average of i.i.d normal is normal and hence here $\bar{X}_{20} \sim \text{Normal}\left(\text{mean} = 18.6, \ sd = \frac{5.9}{\sqrt{20}}\right)$

3. $P[\bar{X}_{20} > 21] = \text{normal cdf}\left(21, 9^9, 18.6, \frac{5.9}{\sqrt{20}}\right)$

$= 0.6344$

Problem 3

Roll a fair six-sided die 50 times and let $\overline{X}$ denote the average of the rolls.

1. What is the probability that the average is between 3 and 4?
2. How many times should you roll the die to be 90% certain the average will fall between 3 and 4?

1. Let X_i be the value of the i^{th} die.
Then $E(X_i) = \frac{7}{2}$ and $Var(X_i) = \frac{35}{12}$
Since sample size 50 is large, we can approximate the distribution of average by using CLT. and

$$\overline{X}_{50} \approx \text{normal}\left(\frac{7}{2},\ sd = \sqrt{\frac{35}{12} \times \frac{1}{50}}\right)$$

$$\Rightarrow P[3 < \overline{X}_{50} < 4] = \text{normalcdf}\left(3, 4, \frac{7}{2}, \sqrt{\frac{35}{12} \times \frac{1}{50}}\right) = 0.9616$$

2. Let n be the number of times we need to roll the die such that $P[3 < \overline{X}_n < 4] = 0.90$

$$\equiv P\left[\frac{3-\frac{7}{2}}{\sqrt{\frac{35}{12} \times \frac{1}{n}}} < \frac{\overline{X}_n - \frac{7}{2}}{\sqrt{\frac{35}{12} \times \frac{1}{n}}} < \frac{4-\frac{7}{2}}{\sqrt{\frac{35}{12} \times \frac{1}{n}}}\right] = 0.90$$

$$\equiv P\left[\frac{-1/2}{\sqrt{\frac{35}{12} \times \frac{1}{n}}} < Z < \frac{1/2}{\sqrt{\frac{35}{12} \times \frac{1}{n}}}\right] = 0.90$$

That is we require the symmetric interval around zero for normal $(0,1)$ that contains 90% probability. That for $N(0,1)$ is the interval from 5^{th} to 95^{th} percentile

$$\Rightarrow \frac{\frac{1}{2}}{\sqrt{\frac{35}{12} \times \frac{1}{n}}} = \text{invnorm}(0.95, 0, 1) = 1.644854$$

$$\Rightarrow \sqrt{n} = 1.644634 \times \sqrt{\frac{35}{3}} \quad \Rightarrow \quad n \approx 32$$