Solution to HW3

3.5 (a) $c = 1/30$ since $1 = \sum_{x=0}^{3} c(x^2+4) = 30c$.

(b) $c = 1/10$ since

$$1 = \sum_{x=0}^{2} c\binom{2}{x}\binom{3}{3-x} = c\left[\binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1}\right] = 10c.$$

3.25 Let T be the total value of the three coins. Let D and N stand for a dime and nickel, respectively. Since we are selecting without replacement, the sample space containing elements for which $t = 20, 25$, and 30 cents corresponding to the selecting of 2 nickels and 1 dime, 1 nickel and 2 dimes, and 3 dimes. Therefore, $P(T=20) = \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5}$,

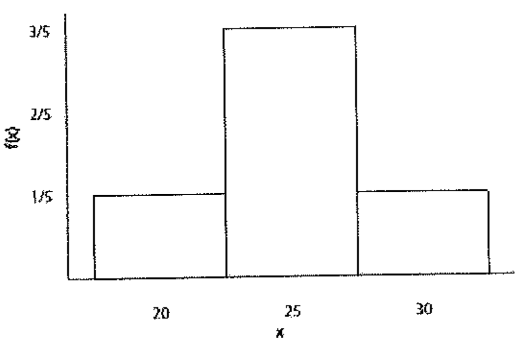
$P(T=25) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5}$,

$P(T=30) = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}$,

and the probability distribution in tabular form is

t	20	25	30
$P(T=t)$	1/5	3/5	1/5

As a probability histogram



3.30 (a) $1 = k\int_{-1}^{1}(3-x^2)\,dx = k\left(3x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{16}{3}k$. So, $k = \frac{3}{16}$.

(b) For $-1 \le x < 1$, $F(x) = \frac{3}{16}\int_{-1}^{x}(3-t^2)\,dt = \left(3t - \frac{1}{3}t^3\right)\Big|_{-1}^{x} = \frac{1}{2} + \frac{9}{16}x - \frac{x^3}{16}$.
So, $P\left(X < \frac{1}{2}\right) = \frac{1}{2} - \left(\frac{9}{16}\right)\left(\frac{1}{2}\right) - \frac{1}{16}\left(\frac{1}{2}\right)^3 = \frac{99}{128}$.

(c) $P(|X| < 0.8) = P(X < -0.8) + P(X > 0.8) = F(-0.8) + 1 - F(0.8)$
$= 1 + \left(\frac{1}{2} - \frac{9}{16}0.8 + \frac{1}{16}0.8^3\right) - \left(\frac{1}{2} + \frac{9}{16}0.8 - \frac{1}{16}0.8^3\right) = 0.164$.

4.3 $\mu = E(X) = (20)(1/5) + (25)(3/5) + (30)(1/5) = 25$ cents.

4.7 Expected gain $= E(X) = (4000)(0.3) + (-1000)(0.7) = \$500$.

4.12 $E(X) = \int_0^1 2x(1-x)\,dx = 1/3$. So, $(1/3)(\$5,000) = \$1,667.67$.

4.17 The probability density function is,

x	-3	6	9
$f(x)$	$1/6$	$1/2$	$1/3$
$g(x)$	25	169	361

$\mu_{g(X)} = E[(2X+1)^2] = (25)(1/6) + (169)(1/2) + (361)(1/3) = 209.$

4.34 $\mu = (-2)(0.3) + (3)(0.2) + (5)(0.5) = 2.5$ and
$E(X^2) = (-2)^2(0.3) + (3)^2(0.2) + (5)^2(0.5) = 15.5.$
So, $\sigma^2 = E(X^2) - \mu^2 = 9.25$ and $\sigma = 3.041.$

4.49 $E(X) = (0)(0.41) + (1)(0.37) + (2)(0.16) + (3)(0.05) + (4)(0.01) = 0.88$
and $E(X^2) = (0)^2(0.41) + (1)^2(0.37) + (2)^2(0.16) + (3)^2(0.05) + (4)^2(0.01) = 1.62.$
So, $Var(X) = 1.62 - 0.88^2 = 0.8456$ and $\sigma = \sqrt{0.8456} = 0.9196.$

4.50 $E(X) = 2\int_0^1 x(1-x)\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$ and
$E(X^2) = 2\int_0^1 x^2(1-x)\,dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{6}.$ Hence,
$Var(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$, and $\sigma = \sqrt{1/18} = 0.2357.$

4.57 $E(X) = (-3)(1/6) + (6)(1/2) + (9)(1/3) = 11/2,$
$E(X^2) = (-3)^2(1/6) + (6)^2(1/2) + (9)^2(1/3) = 93/2.$ So,
$E[(2X+1)^2] = 4E(X^2) + 4E(X) + 1 = (4)(93/2) + (4)(11/2) + 1 = 209.$

Homework problems from the textbook:

3.5, 3.25, 3.30, 4.3, 4.7, 4.12, 4.17, 4.34, 4.49, 4.50, 4.57

Extra problems:

Problem 1

Suppose the probability model for the random variable X is given as

x	100	200	300	400	500
$\Pr(X = x)$	0.1	0.2	0.3	0.2	?

Table 1: Probability Model for random variable X.

a) In the table above, there is a missing value for $\Pr(X = 500)$. So what is it? 0.2

b) What is $\Pr(100 < X < 300)$? $= \Pr(X=200) = 0.2$

see below table

f) What is $E(X)$? $= 320$

g) What is $SD(X)$? $= \sqrt{Var(X)} = \sqrt{15600} = 124.9$

h) What is $Var(X)$? $= E[X^2] - (E[X])^2 = 118000 - 320^2 = 15{,}600$

i) What is $E(-2X+3), SD(-2X+3)$ and $Var(-2X+3)$?

$$E(-2X+3) = -2\,E(X) + 3 = -2(320) + 3 = -637$$

$$SD(-2X+3) = |-2|\,SD(X) = 249.8$$

$$Var(-2X+3) = (-2)^2\,Var(X) = 62{,}400$$

x	100	200	300	400	500	Sum
$p(x)$	.1	.2	.3	.2	.2	1.0
$x\,p(x)$	10	40	90	80	100	320
$x^2\,p(x)$	1,000	8,000	27,000	32,000	50,000	118,000

Problem 2

You draw a card from a standard deck of 52 cards. If you get a red card, you win nothing. If you got a spade, you win \$5. For any club, you win \$10 plus an extra \$20 for the ace of clubs.

a) Create a probability model for the amount you win at this game.

b) Find the expected amount you'll win in 1 game. And also the standard deviation.

c) Find the expected amount you'll win in 100 independent games. And also the standard deviation.

Card type	win(\$)	# Cards
Red	0	26
Spade	5	13
other club	10	12
Ace of club	30	1
		52

(a)

w	$p(w)$
0	$1/2 = .50$
5	$1/4 = .25$
10	$3/13 = .23077$
30	$1/52 = .01923$
	1

(b) $\mathbb{E}[W] = 0 + \frac{5}{4} + \frac{30}{13} + \frac{30}{52} = \frac{215}{52} = 4.13$

$\mathbb{E}[W^2] = 0 + \frac{25}{4} + \frac{300}{13} + \frac{900}{52} = \frac{2425}{52} = 46.6346$

$Var(W) = 46.6346 - 4.13^2 = 29.5777$

$SD(W) = \sqrt{29.5777} = 5.44$

(c) $\mathbb{E}[W_1 + W_2 + \cdots + W_{100}] = 100\,\mathbb{E}[W] = 413.46$

$SD(W_1 + W_2 + \cdots + W_{100}) = \sqrt{100}\; SD(W) = 54.39$

Problem 3

The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a) Calculate λ.

b) What is the probability that a computer will function between 50 and 150 hours before breaking down?

c) What is the probability that a computer will function less than 100 hours before breaking down?

a) $$1 = \int_0^\infty \lambda e^{-x/100} dx = 100\lambda \int_0^\infty e^{-y} dy = 100\lambda$$

$$\Rightarrow \lambda = \frac{1}{100} = 0.01$$

b) $$P_r(50 < X < 150) = \int_{50}^{150} 0.01\, e^{-0.01x} dx$$

$$= \int_{0.5}^{1.5} e^{-y} dy = -e^{-y}\Big|_{0.5}^{1.5}$$

$$= -e^{-1.5} + e^{-0.5} = \begin{matrix} -.223130 \\ +.606530 \end{matrix} = .3834$$

c) $$P_r(X < 100) = \int_0^{100} 0.01\, e^{-0.01x} dx$$

$$= \int_0^1 e^{-y} dy = 1 - e^{-1} = \begin{matrix} 1.0 \\ -0.3678794 \end{matrix}$$

$$= 0.6321206$$

Problem 4

The density function of a continuous random variable X is given by

$$f(x) = \begin{cases} a + bx^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

1. If $E(X) = 3/5$, find a and b.

Solve $\begin{cases} 1 = \int_0^1 (a+bx^2)\,dx = a + \frac{b}{3} \\ \frac{3}{5} = \int_0^1 x(a+bx^2)\,dx = \frac{a}{2} + \frac{b}{4} \end{cases}$

$\Rightarrow \frac{6}{5} = a + \frac{b}{2} \Rightarrow \frac{1}{5} = \frac{b}{6} \Rightarrow b = \frac{6}{5}$

and $a = \frac{3}{5}$

2. Let $g(x) = \frac{1}{1+2x^2}$. Calculate $E[g(X)]$.

$$E[g(X)] = \int_0^1 \frac{\frac{3}{5} + \frac{6}{5}x^2}{1 + 2x^2}\,dx = \frac{3}{5}\int_0^1 dx = \frac{3}{5}$$

3. What is the probability X is greater than 0.5, i.e. $\Pr(X > 0.5)$.

$$\Pr(X > 0.5) = \int_{0.5}^1 \left(\frac{3}{5} + \frac{6}{5}x^2\right) dx$$

$$= \frac{3}{5} \cdot \frac{1}{2} + \frac{6}{5} \cdot \frac{1^3 - 0.5^3}{3}$$

$$= \frac{3}{10} + \frac{2}{5} \cdot \frac{7}{8} = \frac{13}{20} = 0.65$$