IMSE866: Homework #1, Fall 2018

Due by 11:59 pm Wednesday Oct. 17 2018 (submit electronically)

Unless otherwise specified, please turn in your answers to the problems in a single file as either *Hw1F18.docx* or *Hw1F18.pdf* file with each problem clearly labeled. The homework has six questions. You will submit your work to the class Canvas site. This is an individual homework.

Problem 1: (15 points) An urn contains *w* white balls and *b* black balls initially. At each stage a ball is picked from the urn at random and is replaced by *k* balls of similar color (*k>=1*). Let X_n be the number of black balls in the urn after n stages. Is $\{X_n, n>=0\}$ a DTMC (prove it). If yes, give its transition probability matrix and transition diagram.

Problem 2: (15 points) Let $\{Y_n, n \geq 1\}$ be a sequence of iid random variables with common pmf

$$P(Y_n = k) = \alpha_k, k = 0,1,\dots,M$$

Define $X_0 = 0$ and

$$X_n = \max\{Y_1, Y_2, \dots, Y_n\}, n = 1,2,3,\dots$$

Show that $\{X_n, n \geq 0\}$ is a DTMC. Find its transition probability matrix.

Problem 3: (20 points) A clinical trial is designed to identify the better of two experiments. The trial consists of several stages. At each stage new patients are selected randomly from a common pool of patients and one is given treatment 1, and the other is given treatment 2. The stage is complete when the result of each treatment is known as a success or a failure. At the end of the nth stage we record X_n =the number of success under treatment 1 minus those on treatment 2 observed on all the stages so far. The trial stops as soon as X_n reaches $+k$ or $-k$, where k is a given positive integer. If the trial stops with $X_n = k$, treatment 1 is declared to be better than 2, else treatment 2 is declared better than 1. Now suppose the probability of success of the *i*th treatment is $p_i, i = 1,2$. Suppose $p_1 > p_2$. Assume that the result of the successive stages are independent. Compute the probability that the clinical trial reaches correct decision, as a function of p_1, p_2 and k.

Problem 4: (20 points) Let $\{X_n, n \geq 0\}$ be a DTMC on state space $S = \{0,1,2,3\}$ with transition probability matrix P

$$P = \begin{pmatrix} 0.2 & 0.1 & 0.0 & 0.7 \\ 0.1 & 0.3 & 0.6 & 0.0 \\ 0.0 & 0.4 & 0.2 & 0.4 \\ 0.7 & 0.0 & 0.1 & 0.2 \end{pmatrix}$$

Let $\mathrm{T} = \min\{n \geq 0 : X_n = 1\}$

Compute

(a) What is the occupancy time of being in state 1 after 2 years?
(b) $P(T \geq 4 | X_0 = 2)$
(c) $E(T | X_0 = 2)$
(d) $\mathrm{var}(T | X_0 = 2)$

Problem 5: (10 points) For each transition probability matrix below, identify and classify the states of the corresponding stochastic process. Also, indicate whether the process is irreducible.

a.

$$\begin{array}{cc} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & \times & \times \\ 0 & \times & \times & 0 & 0 \\ 0 & \times & \times & 0 & 0 \\ \times & 0 & 0 & 0 & 0 \\ \times & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

b.

$$\begin{array}{cc} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} \times & 0 & 0 & 0 \\ 0 & \times & \times & 0 \\ 0 & \times & \times & 0 \\ \times & 0 & 0 & \times \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

c.

$$\begin{array}{cc} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} \times & \times & \times & 0 \\ 0 & \times & 0 & \times \\ 0 & 0 & \times & 0 \\ 0 & \times & 0 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

d.

$$\begin{array}{cc} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & 0 \\ 0 & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

e.

$$\begin{array}{cc} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 0 & \times & \times \\ 0 & \times & \times & 0 \\ \times & \times & 0 & 0 \\ \times & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

Problem 6: (40 points) A machine produces two items per day. Each item is non-defective with probability p, the quality of the successive items being independent. Defective items are thrown away immediately, and the non-defective items are stored to satisfy demand of one item per day. Any demand that cannot be satisfied immediately is lost. Let X_n be the number of items in storage at the beginning of each day (before the production and demand for that day is taken into account). Show that $\{X_n, n \geq 0\}$ is DTMC. When is it positive recurrent? Compute its limiting distribution when it exists.