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Dual index production and subcontracting policies for assemble-to-order systems

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ABSTRACT

We analyze tradeoffs related to production and subcontracting decisions in an assemble-to-order system with capacity constraints and stochastic lead times. We assume that component replenishment is carried out by orders to a subcontractor and component stock levels at the manufacturer are determined by dual index-based policies. Furthermore, customer demands for the final product are immediately satisfied if all of the required components are in stock; otherwise, they are back-ordered. In order to maintain high service levels, the manufacturer reserves the option to produce components internally. Using queuing models, we analyze the tradeoffs related to internal manufacturing versus subcontracting under different types of dual index policies. We use Matrix-Geometric methods to conduct an exact analysis for an assemble-to-order system with two components and develop a decomposition-based algorithm to analyze the performance of systems with more than two products. Numerical studies provide useful insights on the performance of the various dual index policies under study.

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Subcontracting; assemble to order; dual index based policy; matrix geometric methods; queuing models

1. Introduction

In order to cut costs and reduce lead times, many manufacturers design their products and processes so that the final product can be quickly assembled from its components. In the literature, these systems are commonly referred to as assemble-to-order (ATO) systems. ATO systems combine the benefits of make-to-order (MTO) systems and make-to-stock (MTS) systems to provide custom products at short lead times. The strategy initially found popularity in the computer industry and since then the concept has gained acceptance in several other industries. Our research is motivated by collaborations with a large manufacturer of custom drilling motors for industrial applications. These drilling motors vary significantly in terms of their power requirement, motor speed, motor size, and shape. A typical drilling motor is assembled from various components such as rotor, stator, shaft, and connection box that have different specifications and ratings. Since the manufacture of some of these components takes a considerable amount of time and machining resources, the manufacturer often builds the critical components to stock. On receipt of an order, the drilling motor is assembled to the required specification from the components in stock. Therefore, the availability of components is critical to guarantee high service levels and short lead times.

In such a setting the manufacturer could either subcontract some components to a local vendor or produce them in-house to minimize component stockouts. Consequently, the manufacturer needs to balance the tradeoffs (in-house production costs, subcontracting costs, on-hand inventory costs, and back-ordering costs) and determine when and how much quantity of components need to be made in-house versus at the subcontractor. For example, if the manufacturer can produce the components at shorter lead times, the benefits of short lead times might outweigh the higher in-house production costs. In contrast, if the manufacturer is constrained by capacity, it might subcontract manufacturing of components to a local subcontractor and incur the subcontracting costs. Understandably, the manufacturer needs to analyze these tradeoffs while making production and subcontracting decisions.

In this article, we analyze an ATO system that uses a combination of stocking policies and subcontracting strategies to improve component availability. The ATO system assembles a single end-product from N components that are build to stock. Production and stocking decisions are made based on one of three dual index policies, namely, the dual base stock policy (*DB* policy), the on-hand inventory-based policy (*OH* policy), and the lead time–based policy (*LT* policy), respectively. The stock for the components is replenished either from a local subcontractor or by the in-house manufacturing facility. Both facilities have finite production capacity and stochastic lead times. We use the Matrix-Geometric approach (Neuts, 1981) and exploit the structure in the sparse transition matrix to provide an exact solution to estimate system performance in moderately sized systems with two components ($N = 2$).

For larger systems with more than two components ($N > 2$), state space explosion prevents an exact analysis. We overcome this challenge through a novel approximation method that uses decomposition of the Markov chain to efficiently evaluate the system performance. The approximation method has several advantages. First, the approach scales well with an increase in the number of components (N). Second, the approach can be easily adapted to analyze system performance under various dual index policies. Third, the approach yields reasonably accurate estimates of performance to guide managerial decisions.

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Using numerical studies we illustrate the performance of our approach for a system operating under three dual index policies (namely, *DB*, *OH*, and *LT*). We point out an operational ambiguity that arises when a system operates under the *DB* policy and then show that the *OH* policy and *LT* policy provide insights into how this ambiguity can be resolved in the *DB* policy to realize its benefits in practice.

The rest of the article is organized as follows. Section 2 describes the system model and assumptions. Section 3 presents the Markov chain formulation of the proposed system. Using these formulations, we develop an exact solution methodology for solving ATO systems and analyze its computational challenge for large-scale problems. Section 4 presents an approximation method to solve large systems for dual index polices. We also extend the approximation method for an ATO system with multiple components. Section 5 summarizes numerical studies for the proposed policies. Finally, Section 6 summarizes model insights and conclusions.

2. Literature review

Since our research falls at the interface of ATO systems and production systems with subcontracting, we review the literature related to both these topics.

Many studies on ATO systems build on the classic results reported by Rosling (1989) and Clark and Scarf (1960). These studies analyze a multi-stage assembly system and show that base stock policy is optimal when the system does not have any capacity constraints. Song and Yao (2002) modeled a single-product ATO system with random replenishment lead times for components and evaluated its performance in terms of average backorders delays. Gallien and Wein (2001) studied ATO systems with independent and identically distributed replenishment lead times and derived an approximate solution for the timing of component procurement. Lu *et al.* (2003) extended the analysis in Song and Yao (2002) to a multi-product ATO system with batch arrivals of customer orders. They derived an expression for the joint distribution of the outstanding customer orders to optimize service levels of the customer and identified the impact of variability on system performance. Lu and Song (2005) considered the correlation between component stocks and determined the optimal base stock level in a multi-product setting. Glasserman and Wang (1998) established tradeoffs between delivery performance and the average on-hand inventory of components in ATO systems with a continuous-review base stock policy. Later, Ko *et al.* (2011) modeled a single-product ATO system and derived a closed-form expression using linear bounds to approximate the lead time distribution of component replenishment. Karaarslan *et al.* (2013) derived the optimality condition for an ATO system under two variations of the pure base stock policy. All of these studies share two similarities: (i) the component stock replenishment is done using a base stock policy and (ii) the lead time distribution for component replenishment is known. Our research also assumes that the component stock replenishment is done using a base stock policy but, in contrast, it assumes that the replenishment can be done either in-house or at a subcontractor facility based on the stock level. This additional flexibility, in turn, influences the distribution of the lead time for component replenishment orders.

Studies on subcontracting have often focused on the value it provides to manufacturing firms to improve service levels to the customers (Li and Kouvelis, 1999; Jiang *et al.*, 2006; Yao *et al.*, 2010). However, subcontracting strategies often have lead time and cost implications. Lee and Zipkin (1989) analyzed a capacitated system with a zero replenishment lead time to provide the optimal make or buy quantities using a dynamic programming algorithm. Platts *et al.* (2002) assumed constant lead times for procurement and determined the quantity to be produced in-house and purchased from external suppliers. Atamturk *et al.* (2001) developed a deterministic linear capacity acquisition and subcontracting model for non-stationary demand. They also provided insights on the tradeoffs between capacity acquisition, production, subcontracting, and inventory decisions. Sethi *et al.* (2003) analyzed manufacturing systems with subcontractors that differ in delivery rates and costs and determined optimal (s, S) policies for these settings.

Bradley (2005) analyzed an in-house production and subcontracting model with exponential processing times for orders and showed that the *DB* policy for component replenishment is optimal. He considered the setting where unit production costs at the subcontractor are less than the in-house manufacturing cost and derived a closed-form expression for the optimal threshold in the *DB* policy.

Our research focuses on make and buy decisions (as opposed to make versus buy); i.e., the manufacturing facility primarily procures components but reserves the option to make parts in-house to meet service-level obligations. Our focus on an ATO system makes our analysis more complex than that in the study of Bradley (2005). Veeraraghavan and Schellar-Wolf (2008) determined optimal order quantities for both the subcontractor as well as the in-house manufacturer under the assumption of deterministic lead times. They proposed a dual index policy that was near-optimal. Our work builds on the dual base stock and dual index policies discussed in the literature but extends their application to a broad class of ATO systems with multiple components. In particular, depending on the sourcing and replenishment decisions, the replenishment lead times of these components could vary with the workload at both in-house manufacturing and local subcontracting facilities. ATO systems that operate under dual index policies recognize the sensitivity of the lead time to workloads at the in-house manufacturing facility and local subcontractor to suitably adapt their production and subcontracting decisions to improve system performance. In the subsequent sections, we present the relevant analytical models.

3. ATO system with production and subcontracting

Figure 1 illustrates an ATO system that assembles a single product with two components. Component k, $k = 1, 2$ can be manufactured by the the in-house manufacturing facility M_k and the local subcontractor S_k. The components are stored at inventory location L_k and are assembled at station A to satisfy the demand for the final product. We assume that the customer orders for the final product arrive according to a Poisson process $N(t)$, $t \geq 0$ with rate λ and are satisfied on a first come–first serve (FCFS)

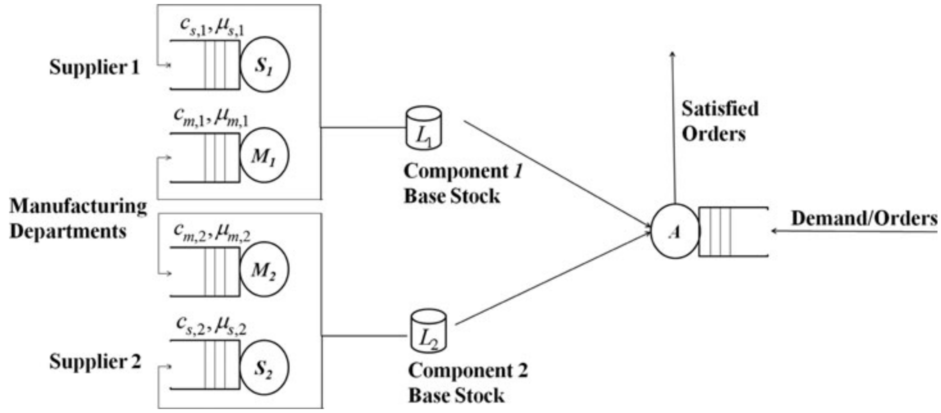

Figure 1. Supply chain model for an ATO system.

basis at assembly station A. Assembly operations of this station are instantaneous; i.e., if both components are available at the demand arrival epoch, then the demand for the final product is immediately satisfied. If one or more component(s) is unavailable, then the demand for the final product is backordered and the customer's order stays in the queue at station A. We model the local suncontractor S_k and the in-house manufacturing facility M_k, $k = 1, 2$ as a single-server queue with exponentially distributed service time with mean $\mu_{s,k}^{-1}$ and $\mu_{m,k}^{-1}$, respectively. This allows us to model the effect of workload on lead times at these facilities. The production cost per unit is $c_{s,k}$ and $c_{m,k}$ for component k at S_k and M_k, respectively. Without loss of generality, we assume that $\mu_{s,k} < \mu_{m,k}$ and $c_{s,k} < c_{m,k}$, $k = 1, 2$.

We assume that the system maintains a base stock level z_k for each component k; i.e., we ensure that the net inventory position is z_k through orders for replenishing inventory placed at demand arrival epochs. Let $O_{m,k}(t)$, $O_{s,k}(t)$, $I_k(t)$, $B_k(t)$ denote the in-house manufacturer's on-orders, subcontractor's on-orders, on-hand inventory quantity, and backorders for component k respectively at decision epoch t. Then, since the system maintains a base stock policy for each component, the following equation holds:

$$z_k = O_{m,k}(t) + O_{s,k}(t) + I_k(t) - B_k(t),\ k = 1, 2, \forall t. \qquad (1)$$

Note that at anytime t, $I_k(t)B_k(t) = 0$. For this system, we analyze system performance under three ordering policies: the *DB* policy, the *OH* policy, and the *LT* policy.

The DB policy: Under the *DB* policy, if at any instant t corresponds to a demand arrival, $I_k(t) < e_k$ (where e_k is a predefined inventory threshold limit), then the manufacturer uses all available capacity at its internal manufacturing facility, M_k, and the local subcontractor, S_k, to replenish the inventory for component k. If instead at the demand arrival epoch t, $z_k > I_k(t) \geq e_k$, the manufacturer places an order to replenish inventory for component k only to its local subcontractor S_k. If $I_k(t) = z_k$ at the demand arrival epoch, no replenishment order is placed for component k. Note that when $I_k(t) < e_k$, the *DB* policy does not specify who should get the order (M_k or S_k) as long as the order ensures that both the internal manufacturing facility M_k and local subcontractor S_k are busy. However, from an order fulfillment point of view, it is important to determine how the workload of replenishing inventory must be distributed between M_k and S_k. Therefore, we will consider two variations of the *DB* policy, namely, the *OH* policy and the *LT* policy, that specify who gets the order when $I_k(t) < e_k$.

1. *The OH policy*: Under the *OH* policy, if at a demand arrival epoch t, $I_k(t) < j_k$ (where j_k is a predefined inventory threshold limit), then the manufacturer places the order for component k to its internal manufacturing facility M_k. If at the demand arrival epoch t, $z_k > I_k(t) \geq j_k$, the manufacturer places the order for component k to the local subcontractor S_k.
2. *The LT policy*: Under the *LT* policy, the manufacturer first determines the estimates of the lead time $\hat{L}_{m,k}(t) = O_{m,k}(t)/\mu_{m,k}$ for the in-house manufacturing facility and $\hat{L}_{s,k}(t) = O_{s,k}(t)/\mu_{s,k}$ for the local subcontractor of component k, respectively. Then, if at the demand arrival epoch t, $\hat{L}_{m,k}(t) < l_k\hat{L}_{s,k}(t)$ (where l_k is a predefined lead time threshold limit), the manufacturer places the order for component k to its internal manufacturing facility M_k, and to the local subcontractor S_k when $\hat{L}_{m,k}(t) \geq l_k\hat{L}_{s,k}(t)$.

Note that we intentionally use separate notations e_k, j_k, and l_k to denote the thresholds corresponding to the *DB*, *OH*, and *LT* policies, as these thresholds could be different from each other. In the next section, we present an exact analysis of this system under these three ordering policies. We shall show through numerical studies in Section 6 that despite the operational ambiguities in the *DB* policy, the policy actually yields better performance. The performance of the *OH* and *LT* policies then provides an intuitive explanation for this superior performance and also resolves this operational dilemma of who should get specific orders in the *DB* policy.

4. Exact analysis of the ATO system

This section presents an exact approach to determine the steady-state probabilities Π for an ATO system with subcontracting flexibility.

4.1. *Exact analysis under the DB policy*

Under the *DB* policy, let each state in the state space Σ^{DB} be defined as $\sigma^{DB} = (I_1, I_2)$, where I_k is the inventory position of component k, $k = 1, 2$. Then, the system evolution can be modeled as a Markov chain. Let Π^{DB} denote the steady-state probability vector and $\pi^{DB}(I_1, I_2)$ denote the steady-state probability of state (I_1, I_2). Let N_k^{DB} denote the possible values of I_k under the DB policy and $B_{\max}$ denote the finite maximum limit for backorders of any component k, $k = 1, 2$. Then, the total number of states in Σ^{DB} is $N_1^{DB} N_2^{DB}$ and under the DB policy, $N_k^{DB} = (z_k + B_{\max})$, $k = 1, 2$. Note that our assumption that $B_{\max}$ is finite is not restrictive and the analysis in the sections below can be extended to the case where $B_{\max} = \infty$ with minimal modifications. However, setting $B_{\max}$ to be finite allows us to see the impact of various policies on the structure of the transition probability matrices and limits computations to finite matrices.

The state space Σ^{DB} can be re-written as $\Sigma^{DB} = \mathbb{A}_1^{DB} \times \mathbb{A}_2^{DB}$ where $\mathbb{A}_k^{DB}$, $k = 1, 2$ is the set of all possible values of the (m_k, s_k) pair. To construct the transition matrix $\mathbb{Q}^{DB}$, we exploit the similarities in the transition probabilities for states belonging to a particular set within each $\mathbb{A}_k^{DB}$. Each set $\mathbb{A}_k^{DB}$, $k = 1, 2$ can be further partitioned into five mutually exclusive subsets, $\mathbb{A}_{k,i}^{DB} \subset \mathbb{A}_k^{DB}$, $i = 1, 2, \ldots, 5$ where $\cup_i \mathbb{A}_{k,i}^{DB} = \mathbb{A}_k^{DB}$ and

$$
\begin{aligned}
A_{k,1}^{DB} &= \{I_k : I_k = -B_{max}\},\\
A_{k,2}^{DB} &= \{I_k : 1 - B_{max} \le I_k \le e_k - 1\},\\
A_{k,3}^{DB} &= \{I_k : I_k = e_k\},\\
A_{k,4}^{DB} &= \{I_k : e_k + 1 \le I_k \le z_k - 1\},\\
A_{k,5}^{DB} &= \{I_k : I_k = z_k\}.
\end{aligned}
$$

Note that if $B_{\max} = \infty$, $A_{k,1}^{DB}$ and $A_{k,2}^{DB}$ merge into one subset. For notational simplicity, we drop the superscript DB in the rest of this section. Using these subsets $\mathbb{A}_{k,i} \subset \mathbb{A}_k$, $i = 1, 2, \ldots, 5$, Chapman–Kolmogorov (C–K) equations can be written. For instance, for $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_2$ the C-K equations are written as follows:

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,1}$:

$$
\begin{aligned}
(\mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(I_1, I_2) = \lambda\pi(I_1 + 1, I_2 + 1)&\\
+(\mu_{m,1} + \mu_{s,1})\pi&\\
\times(I_1 - 1, I_2).& \qquad (2)
\end{aligned}
$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,2}$:

$$
\begin{aligned}
&(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(I_1, I_2)\\
&\quad = \lambda\pi(I_1 + 1, I_2 + 1) + (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2)\\
&\qquad +(\mu_{m,2} + \mu_{s,2})\pi(I_1, I_2 - 1). \qquad (3)
\end{aligned}
$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,3}$:

$$
\begin{aligned}
&(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(I_1, I_2)\\
&\quad = \lambda\pi(I_1 + 1, I_2 + 1) + (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2)\\
&\qquad +(\mu_{m,2} + \mu_{s,2})\pi(I_1, I_2 - 1). \qquad (4)
\end{aligned}
$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,4}$:

$$
\begin{aligned}
&(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(I_1, I_2)\\
&\quad = \lambda\pi(I_1 + 1, I_2 + 1) + (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2)\\
&\qquad +\mu_{s,2}\pi(I_1, I_2 - 1). \qquad (5)
\end{aligned}
$$

For $I_1 \in \mathbb{A}_{1,2}$ and $I_2 \in \mathbb{A}_{2,5}$:

$$
\begin{aligned}
&(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(I_1, I_2)\\
&\quad = (\mu_{m,1} + \mu_{s,1})\pi(I_1 - 1, I_2) + \mu_{s,2}\pi(I_1, I_2 - 1). \qquad (6)
\end{aligned}
$$

The C–K equations for other pairs $I_1 \in \mathbb{A}$, $i = 1, \ldots, 5$ and $I_2 \in \mathbb{A}_{k,j}$ can be written in a similar way. Unfortunately, these C–K equations yield a large and sparse transition matrix $\mathbb{Q}$. However, we can exploit the structural properties of the transition matrix using the Matrix-Geometric representation. We discuss the details below.

Let $\mathbb{C} = \text{diag}(0, \lambda, \lambda, \lambda)$. Define $\mathbb{I}_1$ as an identity matrix of size $N_1 \times N_1$, $\mathbb{B}_{2,1} = \mathbb{B}_{1,1} + \mathbb{C} - (\mu_{m,2} + \mu_{s,2})\mathbb{I}_1$, $\mathbb{B}_{3,1} = \mathbb{B}_{1,1} - (\mu_{m,2} + \mu_{s,2})\mathbb{I}_1$, and $\mathbb{B}_{4,1} = \mathbb{B}_{1,1} - (\mu_{s,2})\mathbb{I}_1$. The corresponding matrices ($\mathbb{B}_{2,2}$, $\mathbb{B}_{3,2}$, and $\mathbb{B}_{4,2}$) for component 2 are defined in a similar way. The matrices $\mathbb{B}_{1,k}$, $k = 1, 2$ and $\mathbb{D}$ are defined as follows:

$$
\mathbb{D} = \begin{vmatrix} 0 & 0 & 0 & 0\\ \lambda & 0 & 0 & 0\\ 0 & \lambda & 0 & 0\\ 0 & 0 & \lambda & 0 \end{vmatrix},
$$

$$
\mathbb{B}_{1,k} = \begin{vmatrix} -(\mu_{m,k} + \mu_{s,k}) & (\mu_{m,k} + \mu_{s,k}) & 0 & 0\\ 0 & -(\lambda + \mu_{m,k} + \mu_{s,k}) & (\mu_{m,k} + \mu_{s,k}) & 0\\ 0 & 0 & -(\lambda + \mu_{m,k} + \mu_{s,k}) & \mu_{s,k}\\ 0 & 0 & 0 & -(\lambda + \mu_{m,k} + \mu_{s,k}) \end{vmatrix},
$$

$$
\mathbb{Q} = \begin{array}{c|cccc}
 & \overrightarrow{(0,*)}^{\top} & \overrightarrow{(1,*)}^{\top} & \overrightarrow{2,*)}^{\top} & \overrightarrow{(3,*)}^{\top}\\
\hline
\overrightarrow{(0,*)} & \mathbb{B}_{2,2} & (\mu_{m,1} + \mu_{s,1})\mathbb{I}_2 & & \\
\overrightarrow{(1,*)} & \mathbb{D} & \mathbb{B}_{3,2} & (\mu_{m,1} + \mu_{s,1})\mathbb{I}_2 & \\
\overrightarrow{(2,*)} & & \mathbb{D} & \mathbb{B}_{4,2} & \mu_{s,1}\mathbb{I}_2\\
\overrightarrow{(0,*)} & & & \mathbb{D} & \mathbb{B}_{1,2}
\end{array}. \qquad (7)
$$

Then the transition matrix $\mathbb{Q}$ can be constructed using the above-mentioned matrices as shown in Equation (7) and the steady-state probabilities can be calculated using the system of Equations (8) and (9) and the Matrix-Geometric technique described in Neuts (1981).

$$
\Pi\mathbb{Q} = \mathbf{0}, \qquad (8)
$$

$$
\Pi\mathbf{e} = 1. \qquad (9)
$$

Here, $\mathbf{e} = [1, \ldots, 1]$ of size $1 \times (N_1 \times N_2)$ and $\overrightarrow{(0,*)}$ represents a vector with all states having $I_1 = 0$ in the DB policy. From the solutions to Equations (8) and (9), the expected on-hand inventory levels $E[I_1]$ and expected backorders $E[B_1]$ for component 1 can be calculated using Equations (10) and (11). The performance measures for component 2 can be calculated in a similar way. The in-house throughput, $TH_{m,1}$, and the supplier's throughput, $TH_{s,1}$, for component 1 are also computed using Equations (12) and (13). Similarly, we can define the performance measures for component 2. Note that in these equations $\pi(*, *)$ denotes the steady-state probability at the

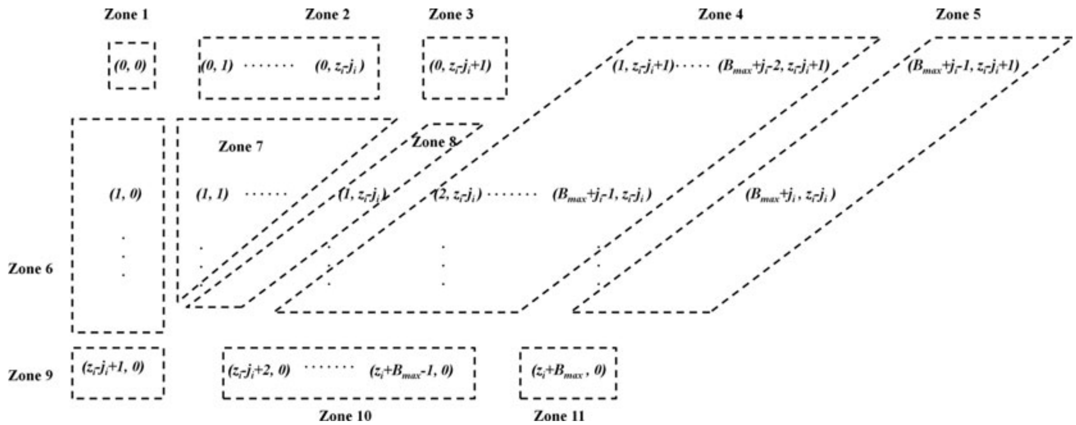

Figure 2. Subset classification of the states for component k under *OH* policy.

particular state:

$$E[B_1] = \sum_{I_1} \max(-I_1, 0)\pi(I_1, *), \tag{10}$$

$$E[I_1] = \sum_{I_1} \max(I_1, 0)\pi(I_1, *), \tag{11}$$

$$TH_{m,1} = \sum_{I_1<e_1} \mu_{m,1}\pi(I_1, *), \tag{12}$$

$$TH_{s,1} = \sum_{I_1<z_k} \mu_{s,1}\pi(I_1, *). \tag{13}$$

4.2. *Exact analysis under the OH and LT policies*

When a system operates under policy $\mathcal{P}$, where $\mathcal{P} \in \{OH, LT\}$, the state of the system is completely defined only if $O_{m,k}(t)$ and $O_{s,k}(t)$ are known for each component k, $k = 1, 2$ at any time t. Thus, we define a four-dimensional state variable to describe the system state under these policies. Let $\sigma^{\mathcal{P}}$ denote a state in the state space $\Sigma^{\mathcal{P}}$, where $\sigma^{\mathcal{P}} = (m_1, s_1, m_2, s_2)$ and m_k (or s_k) represents the on-order quantity of component k at in-house manufacturer M_k (or local subcontractor S_k). Then, the system evolution can be modeled as a Markov chain. Let $\Pi^{\mathcal{P}}$ denote the steady-state probability vector and $\pi^{\mathcal{P}}(m_1, s_1, m_2, s_2)$ denote the steady-state probability of state (m_1, s_1, m_2, s_2). Let $N_k^{\mathcal{P}}$ denote the possible values in the tuple (m_k, s_k). Then the total number of states in $\Sigma^{\mathcal{P}}$ is $N_1^{\mathcal{P}} N_2^{\mathcal{P}}$. Under the *OH* policy, $N_k^{OH} = (z_k - j_k + 2)(B_{\max} + (z_k + j_k + 1)/2)$ and under the *LT* policy, $N_k^{LT} = (z_k + B_{\max})(z_k + B_{\max} + 1)/2$ for $k = 1, 2$.

The state space $\Sigma^{\mathcal{P}}$ can be re-written as $\Sigma^{\mathcal{P}} = \mathbb{A}_1^{\mathcal{P}} \times \mathbb{A}_2^{\mathcal{P}}$, where $\mathbb{A}_k^{\mathcal{P}}$, $k = 1, 2$ is the set of all possible values of the (m_k, s_k) pair. To construct the transition matrix $\mathbb{Q}^{\mathcal{P}}$ we exploit the similarities in the transition probabilities for states belonging to a particular set within each $\mathbb{A}_k^{\mathcal{P}}$. For instance, each set $\mathbb{A}_k^{OH}$, $k = 1, 2$ can be further partitioned into 11 mutually exclusive subsets, $\mathbb{A}_{k,i}^{OH} \subset \mathbb{A}_k^{OH}$, $i = 1, 2, \ldots, 11$ where $\cup_i \mathbb{A}_{k,i}^{OH} = \mathbb{A}_k^{OH}$ and

$$\begin{aligned}
A_{k,1}^{OH} &= \{(m_k, s_k) : m_k = 0,\ s_k = 0\},\\
A_{k,2}^{OH} &= \{(m_k, s_k) : m_k = 0,\ 1 \le s_k \le z_k - j_k\},\\
A_{k,3}^{OH} &= \{(m_k, s_k) : m_k = 0,\ s_k = z_k - j_k + 1\},\\
A_{k,4}^{OH} &= \{(m_k, s_k) : q_k \le m_k \le B_{max} + j_k + q_k - 3,\ s_k\\
&= z_k - j_k - q_k + 2,\ q_k\\
&= \{1, \ldots, z_k - j_k + 1\}\},\\
A_{k,5}^{OH} &= \{(m_k, s_k) : m_k = B_{max} + j_k + q_k - 2,\ s_k\\
&= z_k - j_k - q_k + 2,\ q_k = \{1, \ldots, z_k - j_k + 1\}\},\\
A_{k,6}^{OH} &= \{(m_k, s_k) : 1 \le m_k \le z_k - j_k,\ s_k = 0\},\\
A_{k,7}^{OH} &= \{(m_k, s_k) : 1 \le m_k \le z_k - j_k,\ 1 \le s_k \le z_k - j_k - q_k\\
&\quad +1,\ q_k = \{2, \ldots, z_k - j_k + 1\}\},\\
A_{k,8}^{OH} &= \{(m_k, s_k) : 1 \le m_k \le z_k - j_k,\ s_k\\
&= z_k - j_k - q_k + 2,\ q_k\\
&= \{2, \ldots, z_k - j_k + 1\}\},\\
A_{k,9}^{OH} &= \{(m_k, s_k) : m_k = z_k - j_k + 1,\ s_k = 0\},\\
A_{k,10}^{OH} &= \{(m_k, s_k) : z_k - j_k + 2 \le m_k \le z_k + B_{max} - 1,\ s_k = 0\},\\
A_{k,11}^{OH} &= \{(m_k, s_k) : m_k = z_k + B_{max},\ s_k = 0\}.
\end{aligned}$$

Note that if $B_{\max} = \infty$, subsets $A_{k,4}^{OH}$ and $A_{k,5}^{OH}$ merge into one subset and subsets $A_{k,10}^{OH}$ and $A_{k,11}^{OH}$ merge into another subset. Figure 2 pictorially represents these subsets. For notational simplicity, we drop the superscript *OH* in the rest of this section. Using these subsets $\mathbb{A}_{k,i} \subset \mathbb{A}_k$, $i = 1, 2, \ldots 11$, C–K equations can be written. For instance, for $(m_1, s_1) \in \mathbb{A}_{1,4}$ and $(m_2, s_2) \in \mathbb{A}_2$ the C–K equations are written as follows:

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,1}$:

$$\begin{aligned}
&(\lambda + \mu_{s,1} + \mu_{m,1})\pi(m_1, s_1, m_2, s_2)\\
&\quad = \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\
&\qquad + \mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1).
\end{aligned} \tag{14}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,2}$:

$$\begin{aligned}
&(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2)\\
&\quad = \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1)\\
&\qquad + \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\
&\qquad + \mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2)\\
&\qquad + \mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1).
\end{aligned} \tag{15}$$

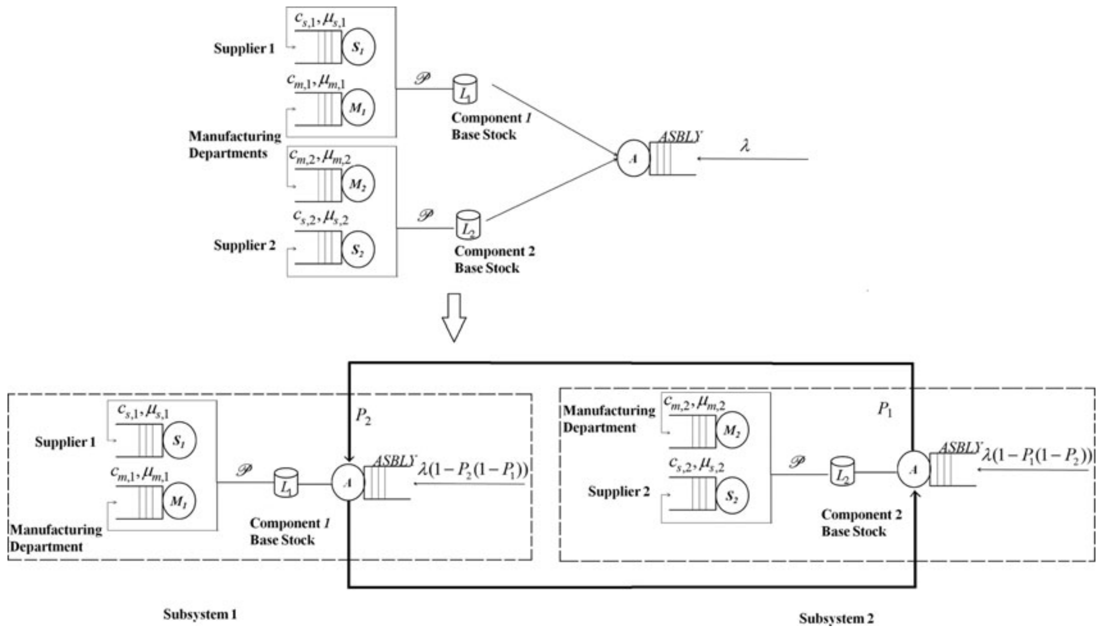

Figure 3. Illustration of the solution approach for $N = 2$.

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,3}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2)\\ = \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1)\\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2).\end{aligned} \tag{16}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,4}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2)\\ = \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2)\\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2).\end{aligned} \tag{17}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,5}$:

$$\begin{aligned}(\mu_{s,1} + \mu_{m,1} + \mu_{s,2} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2)\\ = \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2)\\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2).\end{aligned} \tag{18}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,6}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2)\\ = \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2)\\ +\mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1).\end{aligned} \tag{19}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,7}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2)\\ = \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1)\\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2)\\ +\mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1).\end{aligned} \tag{20}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,8}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2} + \mu_{s,2})\pi(m_1, s_1, m_2, s_2)\\ = \lambda\pi(m_1 - 1, s_1, m_2, s_2 - 1)\\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2).\end{aligned} \tag{21}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,9}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2)\\ = \mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2).\end{aligned} \tag{22}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,10}$:

$$\begin{aligned}(\lambda + \mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2)\\ = \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2)\\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2)\\ +\mu_{m,2}\pi(m_1, s_1, m_2 + 1, s_2)\\ +\mu_{s,2}\pi(m_1, s_1, m_2, s_2 + 1).\end{aligned} \tag{23}$$

For $(m_1, s_1) \in A_{1,4}$ and $(m_2, s_2) \in A_{2,11}$:

$$\begin{aligned}(\mu_{s,1} + \mu_{m,1} + \mu_{m,2})\pi(m_1, s_1, m_2, s_2)\\ = \lambda\pi(m_1 - 1, s_1, m_2 - 1, s_2)\\ +\mu_{m,1}\pi(m_1 + 1, s_1, m_2, s_2).\end{aligned} \tag{24}$$

The C–K equation for other states where $(m_1, s_1) \in \mathbb{A}_{k,i}$, $i = 1, \ldots, 11$ and $(m_2, s_2) \in \mathbb{A}_{k,j}$ can be written in a similar way. As in the case of the *DB* policy, we can exploit structural properties using the Matrix-Geometric representation of $\mathbb{Q}^{OH}$. Let $\mathbb{C} = \text{diag}(\lambda, \lambda, \lambda, 0, \lambda, \lambda, 0)$ and define $\mathbb{I}_k$ as an identity matrix of size $N_k^{OH} \times N_k^{OH}$. Also, define $\mathbb{B}_{3,k} = \mathbb{B}_{1,k} - \mu_{s,1}\mathbb{I}_k$, $\mathbb{B}_{4,k} = \mathbb{B}_{1,k} - (2\mu_{s,1}\mathbb{I}_k + \mu_{m,1}\mathbb{I}_k)$, $\mathbb{B}_{5,k} = \mathbb{B}_{1,k} - (\mathbb{C} + 2\mu_{s,1}\mathbb{I}_k + \mu_{m,1}\mathbb{I}_k)$, $\mathbb{B}_{9,k} = \mathbb{B}_{1,k} - \mu_{m,1}\mathbb{I}_k - \mu_{s,1}\mathbb{I}_k$, and $\mathbb{B}_{11,k} = \mathbb{B}_{1,k} - (\mathbb{C} + \mu_{m,1}\mathbb{I}_k) + \mu_{s,1}\mathbb{I}_k)$, where $\mathbb{B}_{1,k}$ for

$k = 1, 2$ and $\mathbb{D}$ are defined as follows:

$$\mathbb{D} = \begin{vmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\mathbb{B}_{1,k} = \begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_{s,k} & -(\lambda + \mu_{s,k}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_{m,k} & -(\lambda + \mu_{s,k} + \mu_{m,k}) & 0 & \mu_{s,k} & 0 & 0 \\ 0 & 0 & \mu_{m,k} & -(\mu_{s,k} + \mu_{m,k}) & 0 & \mu_{m,k} & 0 \\ \mu_{m,k} & 0 & 0 & 0 & -(\lambda + \mu_{m,k}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_{m,k} & -(\lambda + \mu_{m,k}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_{m,k} & -\mu_{m,k} \end{vmatrix}.$$

Then, we can construct the transition matrix $\mathbb{Q}^{OH}$ using the above-mentioned matrices as shown in Equation (25) and compute the steady-state probabilities by solving the system of equations (26) and (27) using the Matrix-Geometric technique described in Neuts (1981).

$$\mathbb{Q}^{OH} = \begin{array}{c|ccccccc} & \overrightarrow{(0,0,*,*)}^{\top} & \overrightarrow{(0,1,*,*)}^{\top} & \overrightarrow{(1,1,*,*)}^{\top} & \overrightarrow{(2,1,*,*)}^{\top} & \overrightarrow{(1,0,*,*)}^{\top} & \overrightarrow{(2,0,*,*)}^{\top} & \overrightarrow{(3,0,*,*)}^{\top} \\ \hline \overrightarrow{(0,0,*,*)} & \mathbb{B}_{1,2} & \mathbb{D} & & & & & \\ \overrightarrow{(0,1,*,*)} & \mu_{s,1}\mathbb{I}_2 & \mathbb{B}_{3,2} & \mathbb{D} & & & & \\ \overrightarrow{(1,1,*,*)} & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{4,2} & \mathbb{D} & \mu_{s,1}\mathbb{I}_2 & & \\ \overrightarrow{(2,1,*,*)} & & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{5,2} & & \mu_{s,1}\mathbb{I}_2 & \\ \overrightarrow{(1,0,*,*)} & \mu_{m,1}\mathbb{I}_2 & & & & \mathbb{B}_{9,2} & \mathbb{D} & \\ \overrightarrow{(2,0,*,*)} & & & & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{9,2} & \mathbb{D} \\ \overrightarrow{(3,0,*,*)} & & & & & & \mu_{m,1}\mathbb{I}_2 & \mathbb{B}_{11,2} \end{array} \quad (25)$$

$$\Pi \mathbb{Q}^{OH} = \mathbf{0}, \quad (26)$$

$$\Pi \mathbf{e} = 1. \quad (27)$$

Here, $\mathbf{e} = [1, \ldots, 1]$ is of size $1 \times (N_1^{OH} \times N_2^{OH})$ and $\overrightarrow{(0, 0, *, *)}$ denotes a vector with all states having $m_1 = s_1 = 0$ in the *OH* policy. The expected on-hand inventory levels $E[I_1]$ and expected backorders $E[B_1]$ for component 1 are also calculated using Equations (28) and (29). The performance measures for component 2 can be calculated in a similar way. Note that in these equations, $\pi(*, *, *, *)$ denote the steady-state probability of the particular state:

$$E[B_1] = \sum_{m_1+s_1>z_1} (m_1 + s_1 - z_1)\pi(m_1, s_1, *, *), \quad (28)$$

$$E[I_1] = \sum_{m_1+s_1\le z_1} (z_1 - m_1 - s_1)\pi(m_1, s_1, *, *), \quad (29)$$

$$TH_{m,1} = \sum_{m_1>0} \mu_{m,k}\pi(m_1, *, *, *), \quad (30)$$

$$TH_{s,1} = \sum_{s_1>0} \mu_{s,k}\pi(*, s_1, *, *). \quad (31)$$

Similarly, we can conduct the analysis for the *LT* policy. In Section 6, we conduct numerical analysis for these systems under the three polices (*DB*, *OH*, and *LT* policies).

5. Approximate analysis for ATO systems

The exact analysis of ATO systems described above becomes computationally challenging even with the Matrix-Geometric representation of the transition matrices. As we go from a two-component ATO system to a four-component ATO system, the total number of unique C–K equations increases from 25 to 625 for the *DB* policy and from 121 to 14 641 in the *OH* policy. This limits the use of the Matrix-Geometric approach to analyze large systems. To overcome this issue, we propose an approximation method that uses decomposition as shown in Figure 3. The key idea is to split the original Markov chain for a system with N components into N independent Markov chains, each corresponding to a subsystem that models the evolution of one of the components. However, for the decomposition technique to be accurate, in the Markov chain for component k, the effects of the other components need to be appropriately accounted for by using the effective demand arrival rate λ_k.

Note that in the original system described in Section 2, external orders are lost when backorders due to one or more components reach $B_{\max}$. However, in the decomposition analysis, demands for component k arrive at subsystem k and queue at station A as long as the backorders for that component is less than $B_{\max}$; i.e., the decomposition ignores the fact that orders could be lost due to the possibility that backorders for one or more of the other components, i $(i \neq k)$ reach $B_{\max}$. Therefore, the effective effective demand arrival rate λ_k for component k needs to be set to recognize this possibility. Let X_i denote the event that the backorders at subsystem i is equal to $B_{\max}$ and let P_i denote the probability of this event. Then assuming that X_i is independent of X_j for every pair i and j, $\prod_{i\neq k}(1 - P_i)$ is the probability that the backorders at all of the other subsystems i are less than $B_{\max}$. Then, $[1-\prod_{i\neq k}(1-P_i)][1-P_k]$ is the probability that the backorders at one or more of the other subsystems $i\neq k$ is equal to $B_{\max}$, with the backorders at subsystem k being less than $B_{\max}$. This implies that the effective demand arrival rate λ_k for component k in the analysis of subsystem for component k is given by

$$\lambda_k=\lambda\left(1-\left[1-\prod_{i\neq k}(1-P_i)\right][1-P_k]\right). \tag{32}$$

Clearly, the solution to subsystem k requires estimates of $P_i, \forall i \in 1,2,\ldots,N$ and $i\neq k$. In Sections 5.1 and 5.2 we characterize the subsystems under the *DB* and *OH* policies, and in Section 5.3 we present the approximate solution algorithm.

5.1. *Characterizing subsystems under the DB policy*

Let $\mathbb{Q}_k^{DB}$ denote the transition matrix for the subsystem corresponding to component k under the *DB* policy. Let λ_k denote the demand arrival rate corresponding to component k in the decomposition approach and let $\pi_k^{DB}(I_k)$ denote the corresponding steady-state probabilities. For notational simplicity, we drop the superscript *DB* in the rest of this section. Then, $\mathbb{Q}_k$ can be written using the C–K equations shown below:

For $I_k\in\mathbb{A}_{k,1}$:

$$(\mu_{s,k}+\mu_{m,k})\pi_k(I_k)=\lambda_k\pi_k(I_k+1). \tag{33}$$

For $I_k\in\mathbb{A}_{k,2}$:

$$\begin{aligned}(\lambda+\mu_{s,k}+\mu_{m,k})\pi_k(I_k)=\lambda_k\pi_k(I_k+1)&\\+(\mu_{m,k}+\mu_{s,k})\pi_k(I_k-1)&.\end{aligned} \tag{34}$$

For $I_k\in\mathbb{A}_{k,3}$:

$$\begin{aligned}(\lambda+\mu_{s,k})\pi_k(I_k)=\lambda_k\pi_k(I_k+1)&\\+(\mu_{m,k}+\mu_{s,k})\pi_k(I_k-1)&.\end{aligned} \tag{35}$$

For $I_k\in\mathbb{A}_{k,4}$:

$$(\lambda+\mu_{s,k})\pi_k(I_k)=\lambda_k\pi_k(I_k+1)+\mu_{s,k}\pi_k(I_k-1). \tag{36}$$

For $I_k\in\mathbb{A}_{k,5}$:

$$(\lambda+\mu_{s,k})\pi_k(I_k)=\mu_{s,k}\pi_k(I_k-1). \tag{37}$$

These equations are solved using the iterative algorithm described in Section 5.3.

5.2. *Characterizing subsystems under OH policy*

Let $\mathbb{Q}_k^{OH}$ denote the transition matrix for the subsystem corresponding to component k under the *OH* policy. Let λ_k denote the demand arrival rate corresponding to component k in the decomposition approach and let $\pi_k^{OH}(m_k,s_k)$ denote the corresponding steady-state probabilities. For notational simplicity, we drop the superscript *OH* in this rest of the section. Then, $\mathbb{Q}_k$ can be written using the C–K equations shown below:

For $(m_k,s_k)\in\mathbb{A}_{k,1}$:

$$\lambda_k\pi_k(m_k,s_k)=\mu_{s,k}\pi_k(m_k,s_k+1). \tag{38}$$

For $(m_k,s_k)\in\mathbb{A}_{k,2}$:

$$\begin{aligned}(\lambda_k+\mu_{s,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k,s_k-1)&\\+\mu_{m,k}\pi_k(m_k+1,s_k)&\\+\mu_{s,k}\pi_k(m_k,s_k+1)&.\end{aligned} \tag{39}$$

For $(m_k,s_k)\in\mathbb{A}_{k,3}$:

$$\begin{aligned}(\lambda_k+\mu_{s,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k,s_k-1)&\\+\mu_{m,k}\pi_k(m_k+1,s_k)&.\end{aligned} \tag{40}$$

For $(m_k,s_k)\in\mathbb{A}_{k,4}$:

$$\begin{aligned}(\lambda_k+\mu_{s,k}+\mu_{m,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k-1,s_k)&\\+\mu_{m,k}\pi_k(m_k+1,s_k)&.\end{aligned} \tag{41}$$

For $(m_k,s_k)\in\mathbb{A}_{k,5}$:

$$(\mu_{s,k}+\mu_{m,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k-1,s_k). \tag{42}$$

For $(m_k,s_k)\in\mathbb{A}_{k,6}$:

$$\begin{aligned}(\lambda_k+\mu_{m,k})\pi_k(m_k,s_k)=\mu_{m,k}\pi_k(m_k+1,s_k)&\\+\mu_{s,k}\pi_k(m_k,s_k+1)&.\end{aligned} \tag{43}$$

For $(m_k,s_k)\in\mathbb{A}_{k,7}$:

$$\begin{aligned}(\lambda_k+\mu_{m,k}+\mu_{s,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k,s_k-1)&\\+\mu_{m,k}\pi_k(m_k+1,s_k)&\\+\mu_{s,k}\pi_k(m_k,s_k+1)&.\end{aligned} \tag{44}$$

For $(m_k,s_k)\in\mathbb{A}_{k,8}$:

$$\begin{aligned}(\lambda_k+\mu_{m,k}+\mu_{s,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k,s_k-1)&\\+\mu_{m,k}\pi_k(m_k+1,s_k)&.\end{aligned} \tag{45}$$

For $(m_k,s_k)\in\mathbb{A}_{k,9}$:

$$(\lambda_k+\mu_{m,k})\pi_k(m_k,s_k)=\mu_{m,k}\pi_k(m_k+1,s_k). \tag{46}$$

For $(m_k,s_k)\in\mathbb{A}_{k,10}$:

$$\begin{aligned}(\lambda_k+\mu_{m,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k-1,s_k)&\\+\mu_{m,k}\pi_k(m_k+1,s_k)&\\+\mu_{s,k}\pi_k(m_k,s_k+1)&.\end{aligned} \tag{47}$$

For $(m_k,s_k)\in\mathbb{A}_{k,11}$:

$$(\lambda_k+\mu_{m,k})\pi_k(m_k,s_k)=\lambda_k\pi_k(m_k-1,s_k). \tag{48}$$

These equations are solved using the iterative algorithm described in Section 5.3.

5.3. *Solution algorithm and performance measures*

Recall that the solution to subsystem k requires the estimates of $P_i, \forall i, i\neq k$ must be obtained from the solution to subsystem $\forall i, i\neq k$ and *vice versa*. Therefore, we use an iterative approach. The steps of the iterative approach are shown below.

Step 0: Initialize $P_k^{(0)} = 0$, $\epsilon = 10^{-5}$, $\delta_k = 1$, and calculate an estimate of $\lambda_k^{(0)}$ using Equation (32) for $k = 1, \ldots, N$. At each iteration n, $n = 1, \ldots,$ while $\delta_k \geq \epsilon$.

Step 1: Solve subsystem k, $k = 1, \ldots, N$ using $\lambda_k^{(n-1)}$, $i > 1$
- For *DB* policy: Solve Equations (33) to (37).
- For *OH* policy: Solve Equations (38) to (48).
- Compute estimates of $P_k^{(n)}$ from the steady-state probabilities.

Step 2: Calculate new estimate of $\lambda_k^{(n)}$ using $P_k^{(n)}$.

Step 3: Compute $\delta_k = |P_k^{(n)} - P_k^{(n-1)}|$.

If $\delta_k < \epsilon$, $\forall k$, stop. Else, repeat Steps 1 to 3.

Using a similar approach, we can also analyze the performance of an ATO system operating under *LT* policy.

The expected on-hand quantities $E[I_k]$, expected backorders $E[B_k]$, throughput $TH_{m,k}$ for component k at the manufacturing facility, and $TH_{s,k}$ at the subcontractor facility are calculated using Equations (49) to (52) for the *DB* policy:

$$E[B_k] = \sum_{I_k} \max(-I_k, 0)\pi_k(I_k), \tag{49}$$

$$E[I_k] = \sum_{I_k} \max(I_k, 0)\pi_k(I_k), \tag{50}$$

$$TH_{m,k} = \sum_{I_k < e_k} \mu_{m,k}\pi_k(I_k), \tag{51}$$

$$TH_{s,k} = \sum_{I_k < z_k} \mu_{s,k}\pi_k(I_k). \tag{52}$$

The corresponding equations for the *OH* policy are given by Equations (53) to (56):

$$E[B_k] = \sum_{m_k + s_k > z_k} (m_k + s_k - z_k)\pi_k(m_1, s_1), \tag{53}$$

$$E[I_k] = \sum_{m_k + s_k \leq z_k} (z_k - m_k - s_k)\pi_k(m_k, s_k), \tag{54}$$

$$TH_{m,k} = \sum_{m_k > 0} \mu_{m,k}\pi_k(m_k, *), \tag{55}$$

$$TH_{s,k} = \sum_{s_k > 0} \mu_{s,k}\pi_k(*, s_k). \tag{56}$$

Note that the performance measures obtained using the above equations use an approximation method. Alternatively, for smaller systems, these performance measures could be obtained from exact solutions using Equations (2) to (6) for the *DB* policy and Equations (14) to (24) for the *OH* policy, respectively. For larger systems, where exact solution of the Markov chain becomes computationally challenging, we derive results from detailed simulation models. We compare the accuracy of these estimates as part of our numerical studies in Section 6.

6. Numerical analysis

In this section, we discuss the numerical experiments conducted to compare the performance of policies under different scenarios. We define the total cost function $TC = \sum_k (c_{m,k}TH_{m,k} + c_{s,k}TH_{s,k} + b_kE[B_k] + h_kE[I_k])$ where b_k is the cost of backordering per unit of component k and h_k is the holding cost per unit for component k. We conducted three sets of experiments. The first two experiments compared the performance of *DB*, *OH*, and *LT* policies (see Sections 6.1 and 6.2), whereas the third experiment investigated the performance of the decomposition algorithm for ATO systems with $N \geq 2$ (see Section 6.3). In all of our experiments, we observed that the algorithm converges within 0.5 seconds on a personal computer with an Intel core i5 processor. Furthermore, we always obtained a unique solution to the system. Although we do not have a proof for the convergence, in our numerical computations we found that the time for convergence does not increase with increase in the number of components.

6.1. *Performance comparison of the DB, OH, and LT policies*

We present the properties and a comparison of the expected costs for a system operating under the *DB* policy, *OH* policy, and *LT* policy using the Matrix-Geometric approach.

We analyzed an ATO system with two components and use the Matrix-Geometric approach to solve and compare the costs in the *DB* policy, *LT* policy, and *OH* policy. We considered a symmetric case where all of the parameters for component 1 are equal to that of component 2 (see Table 1). We set $B_{\max}$ to be large enough so that the probability of lost sales is less than 10^{-3} in all cases. For $z_k = 10$, we varied the threshold limits e_k, j_k, and l_k from one to z_k and calculated in each case the total costs for each policy.

Figure 4 plots different costs (in-house throughput costs, subcontractor's throughput costs, on-hand inventory costs, and backordering costs) versus the threshold limit (e_k, j_k, and l_k) for the *DB*, *OH*, and *LT* policies. In Fig. 4(a) we observe that as the threshold (e_k, j_k, l_k) increases, the throughput cost at the manufacturing facility increases under all three policies (*DB*, *OH*, and *LT* policies). This is to be expected, as in these policies when the threshold is high, orders for component k are placed more with the manufacturing facility than the external subcontractor. This results in an increase in the throughput cost at the manufacturing facility. However, under the *LT* policy, the in-house throughput cost increases at low thresholds and becomes flat at high thresholds. Correspondingly, in Fig. 4(b) as the threshold (e_k, j_k, l_k) increases, we observe that the throughput cost at the external subcontractor decreases under all three policies.

In Fig. 4(c) as the threshold (e_k, j_k, l_k) increases, we observe that the on-hand inventory cost increases under the *DB*, *OH*, and *LT* policies. In these three policies, as the threshold increases, more orders for component k are placed with the manufacturing facility, which has a faster production rate and therefore replenishes the component inventory at a faster rate.

Table 1. System parameters and costs used in the numerical experiments.

System parameters		*Costs*	
λ	1.5	$c_{m,k}$, $k = 1, 2$	10
z_k, $k = 1, 2$	10	$c_{s,k}$, $k = 1, 2$	5
$\mu_{m,k}$, $k = 1, 2$	2	b_k, $k = 1, 2$	20
$\mu_{s,k}$, $k = 1, 2$	1	h_k, $k = 1, 2$	1

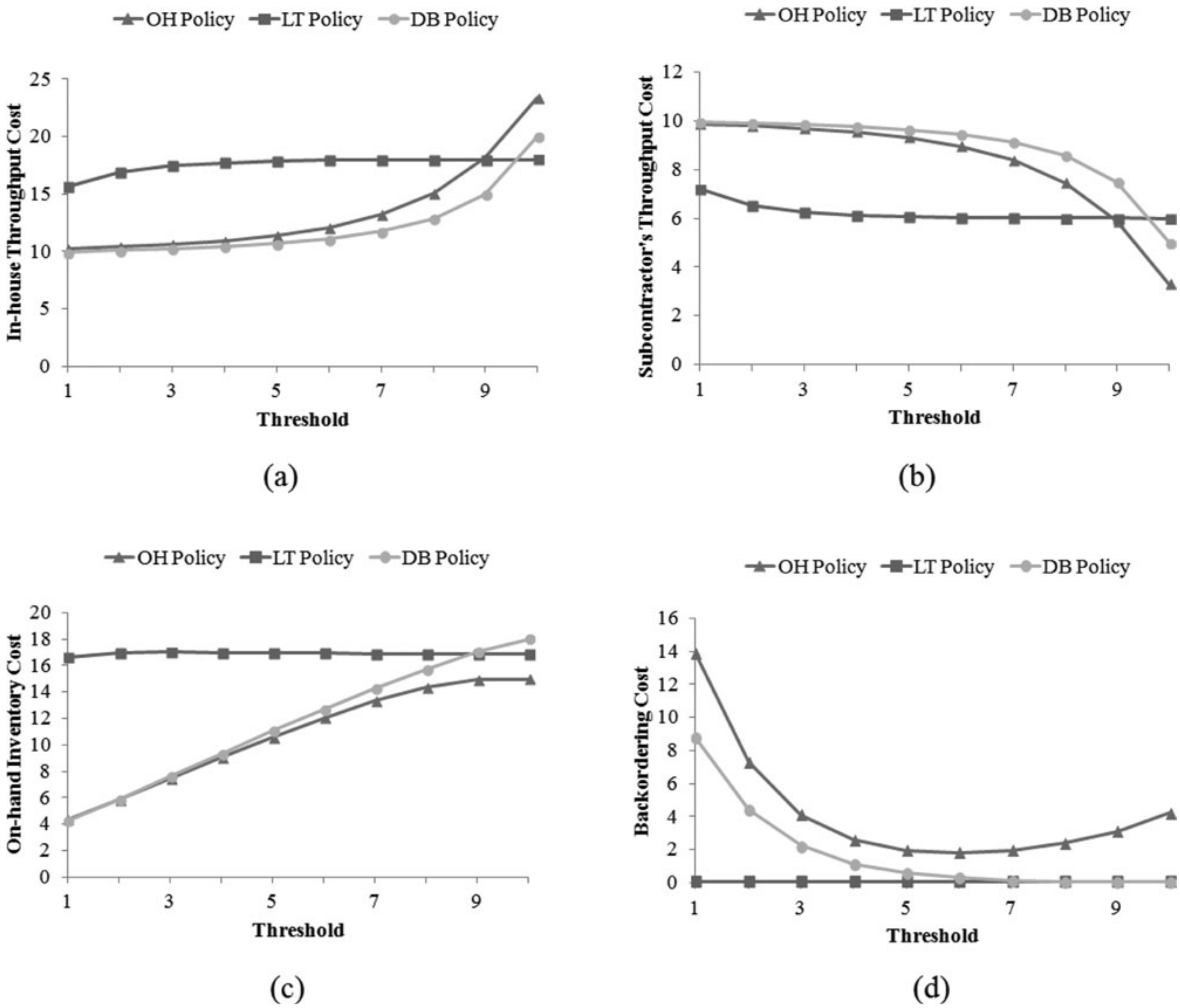

Figure 4. Estimated costs in policies at $z_k = 10$.

In Fig. 4(d), as the threshold increases, we observe that the backordering cost for the system operating under the *DB*, *OH*, and *LT* policies is convex. The backordering costs are convex due to the effects of increased queue length and lead times at the manufacturing facility at higher thresholds.

We can obtain the total cost *TC* by adding the above-mentioned costs. Our solution algorithm can then be used to numerically identify the optimal threshold e_k^*, j_k^*, and l_k^* that minimizes the total cost *TC*. We compared the total cost for these three policies and found that the *DB* policy is the best among the three policies.

6.2. Comparison of optimal costs under the DB, OH, and LT policies

We analyzed an ATO system with two components and used the decomposition approach to solve and compare the optimal costs in the *DB* policy, *LT* policy, and *OH* policy. The results are shown in Fig. 5. We considered the symmetric case shown in Table 1 where all of the parameters for component 1 are equal to that of component 2. In this experiment, we varied $z_k = 2$ to 25, and for each z_k we determined the corresponding optimal threshold limit (e_k^* for the *DB* policy, j_k^* for the *OH* policy, and l_k^* for the *LT* policy). Then, we calculated the optimal cost TC^* and compare this cost across the three policies. Therefore, determining each point in the corresponding plot in Fig. 5 itself requires a search procedure.

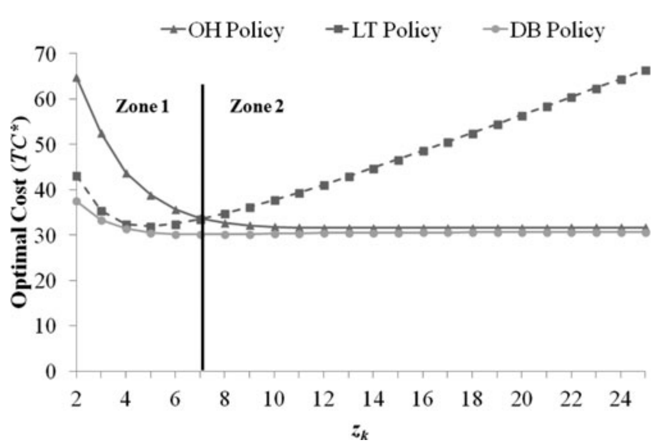

Figure 5. Optimal cost comparison for the considered policies.

In this subsection, we discuss insights related to the optimal solution for each policy using the same system parameters and costs listed in Table 1 and varying $z_k = 2$ to 25. Figure 5 shows the optimal costs for both the *OH* and *LT* policies with increasing z_k under different zones. In zone 1, the expected inventory cost is less than the expected backordering cost. Note that in the *LT* policy, orders for component k are placed based on the lead time estimates $(\hat{L}_{m,k}(t), \hat{L}_{s,k}(t))$. Therefore, the *LT* policy encourages reduction in the backordering cost as opposed to inventory costs. However, in the *OH* policy, orders for component k are placed based on the inventory levels $(I_k(t))$. Therefore, the *OH* policy encourages the reduction of the inventory cost as opposed to the backordering costs. In zone 1, the expected inventory cost is less significant than the expected backorder cost. Thus, the *LT* policy outperforms the *OH* policy in zone 1. Similarly, in zone 2, the expected inventory cost is more significant than the expected backorder cost. Thus, the *OH* policy outperforms the *LT* policy in zone 2. However, the *DB* policy always outperforms the *OH* policy and the *LT* policy, as it has the most flexibility. In the *DB* policy, in zone 1 if the inventory level is above a certain threshold, orders for component k are placed only to the external subcontractor. This results in the replenishment of the components at a slower rate $(\mu_{s,k})$, which in turn reduces the expected inventory cost. Similarly, in zone 2, if the inventory level is below a certain threshold, orders for component k are placed to both the in-house manufacturer and the external subcontractor. This results in the replenishment of the components at a faster rate $(\mu_{s,k} + \mu_{m.k})$, which in turn reduces the expected backorder cost.

Although the results for $z_k = 0$ and $z_k = M$ (M is significantly large, say $M = 1000$) are not discussed, we observed that when $z_k = 0$, the *LT* policy converged to the *DB* policy, and for large values of z_k, say $z_k = 1000$, the *OH* policy converged to the *DB* policy. This provides an intuitive solution to the operational ambiguity inherent in the *DB* policy, suggesting that the *LT* policy should be preferred at lower z_k and the *OH* policy should be preferred at higher z_k.

6.3. *Performance of the ATO system with N components*

In this subsection, we analyze the approximate method for an N-component ATO system and provide insights on its performance. We analyze the performance of the decomposition-based approximation for both the symmetric and asymmetric cases.

Under the *DB* policy, we analyzed an ATO system with N components ($N = 2, 4, 8, 16$) and compared the numerical accuracy of the decomposition-based approach with the exact solution for three choices of service time distributions, namely, exponential (E), shifted-exponential (S), and triangular (T) distributions. We used simulation models to obtain the exact solution for the ATO system with $N \geq 2$. We used five replications and a 99.99% confidence interval in our simulations and ensured that the half-width was less than 0.001% in all cases. For ease of readability, we do not report the half-width intervals in this article. The results from the decomposition were then compared with the exact results for several inputs. Table 2 presents the processing time parameters used for the exponential, shifted-exponential, and triangular distributions for the symmetric case. For case of comparison, we ensured that the mean processing time was the same for all three distributions.

Table 2. Distribution parameters used in the numerical experiments.

Distribution	*Processing time at M_k*	*Processing time at S_k*
Exponential (E)	$\exp(0.5)$	$\exp(1)$
Shifted-exponential (S)	$0.1 + \exp(0.4)$	$0.1 + \exp(0.9)$
Triangular (T)	$\text{triag}(0, 0.5, 1)$	$\text{triag}(0, 1, 2)$

We defined $\Delta[TH^E_{m,k}] = (TH^A_{m,k} - TH^E_{m,k})/TH^E_{m,k}$ as the error in the estimate of the throughput at the manufacturing facility M_k, and it was computed as the relative difference between the estimates from the approximation solution and the exact solution for the case that considered exponentially distributed processing times. Similarly, we defined the error measures for other performance measures and distributions. We analyzed the performance of the approximate method under two cases: (i) symmetry with respect to service rate for each component—i.e., $\mu_{s,i+1} = \mu_{s,i}$ and $\mu_{m,i+1} = \mu_{m,i}$, $i = 1, 2, \ldots, N-1$; and (ii) asymmetry with respect to service rate for each component with $\mu_{s,i+1} = 1.1\mu_{s,i}$ and $\mu_{m,i+1} = 1.1\mu_{m,i}$, $i = 1, 2, \ldots, N-1$. The rest of the parameters were the same as described in Table 1.

Table 3 shows the error in $TH_{s,1}$ obtained from the decomposition-based approach for 2-, 4-, 8-, and 16-component symmetric ATO systems under the *DB* policy. We observe that the estimates of the subcontractor's throughput, $TH_{s,1}$, and manufacturer's throughput, $TH_{m,1}$, are within 2% for the exponential (E) and shifted-exponential (S) case, and witin 4% for the triangular distribution (T) case.

Table 4 reports the error in $E[I_1]$ obtained from the decomposition-based approach for a symmetric ATO system with 2, 4, 8, and 16 components under the *DB* policy. We observe that the expected on-hand inventory $E[I_1]$ is within 3% for all cases, except for the low threshold case under the triangular distribution (T) where the error is less than 9%. A similar performance is also observed for expected backorders $E[B_1]$.

Table 5 reports the error in $TH_{s,1}$ in the approximation for 2-, 4-, 8-, and 16-component asymmetric ATO systems under *DB* policy. We observe that the estimates of the subcontractor's

Table 3. Error in $TH_{s,1}$ under the *DB* policy for the symmetric case.

N	e_1	$TH^A_{s,1}$	$TH^E_{s,1}$	$\Delta[TH^E_{s,1}]$ (%)	$TH^S_{s,1}$	$\Delta[TH^S_{s,1}]$ (%)	$TH^T_{s,1}$	$\Delta[TH^T_{s,1}]$ (%)
2	2	0.990	0.989	0.07	0.993	−0.29	0.999	−0.90
	4	0.977	0.976	0.12	0.982	−0.52	0.995	−1.80
	6	0.945	0.942	0.31	0.952	−0.74	0.978	−3.32
	8	0.857	0.850	0.81	0.861	−0.45	0.892	−3.86
4	2	0.990	0.989	0.06	0.993	−0.29	0.999	−0.92
	4	0.977	0.976	0.13	0.982	−0.50	0.995	−1.81
	6	0.945	0.942	0.32	0.952	−0.72	0.978	−3.31
	8	0.857	0.850	0.84	0.861	−0.40	0.892	−3.86
8	2	0.989	0.989	0.01	0.992	−0.29	0.999	−0.97
	4	0.977	0.975	0.13	0.981	−0.46	0.995	−1.83
	6	0.945	0.941	0.39	0.952	−0.72	0.977	−3.31
	8	0.857	0.850	0.85	0.860	−0.38	0.891	−3.84
16	2	0.989	0.987	0.22	0.992	−0.25	0.999	−0.97
	4	0.977	0.973	0.34	0.981	−0.43	0.995	−1.83
	6	0.945	0.941	0.47	0.951	−0.60	0.977	−3.31
	8	0.857	0.848	1.04	0.860	−0.31	0.891	−3.83

Table 4. Error in $E[I_1]$ under the *DB* policy for the symmetric case.

N	e_1	$E[I_1^A]$	$E[I_1^E]$	$\Delta[E[I_1^E]]$ (%)	$E[I_1^S]$	$\Delta[E[I_1^S]]$ (%)	$E[I_1^T]$	$\Delta[E[I_1^T]]$ (%)
2	2	2.945	2.988	−1.45	2.939	0.20	2.707	8.76
	4	4.670	4.623	1.02	4.613	1.25	4.593	1.69
	6	6.353	6.225	2.05	6.258	1.51	6.452	−1.55
	8	7.860	7.635	2.96	7.728	1.71	8.083	−2.76
4	2	2.964	3.022	−1.91	2.982	−0.60	2.708	9.48
	4	4.675	4.674	0.01	4.618	1.23	4.607	1.47
	6	6.354	6.244	1.76	6.259	1.51	6.458	−1.62
	8	7.860	7.643	2.84	7.728	1.71	8.089	−2.83
8	2	3.001	3.050	−1.61	3.049	−1.57	2.760	8.72
	4	4.684	4.682	0.04	4.656	0.60	4.609	1.64
	6	6.356	6.246	1.75	6.273	1.31	6.460	−1.62
	8	7.861	7.673	2.45	7.731	1.68	8.090	−2.83
16	2	3.001	3.091	−2.89	3.064	−2.03	2.764	8.60
	4	4.684	4.815	−2.71	4.720	−0.75	4.609	1.63
	6	6.356	6.302	0.85	6.310	0.72	6.460	−1.62
	8	7.861	7.689	2.24	7.746	1.48	8.080	−2.72

Table 5. Error in $TH_{s,1}$ under the *DB* policy for the asymmetric case.

N	e_1	$TH_{s,1}^A$	$TH_{s,1}^E$	$\Delta[TH_{s,1}^E]$ (%)	$TH_{s,1}^S$	$\Delta[TH_{s,1}^S]$ (%)	$TH_{s,1}^T$	$\Delta[TH_{s,1}^T]$ (%)
2	2	0.990	0.989	0.06	0.993	−0.29	0.999	− 0.88
	4	0.977	0.976	0.10	0.982	−0.50	0.995	− 1.79
	6	0.945	0.943	0.23	0.952	−0.74	0.978	− 3.32
	8	0.857	0.850	0.81	0.861	−0.45	0.892	− 3.88
4	2	0.990	0.989	0.06	0.993	−0.30	0.999	− 0.89
	4	0.977	0.976	0.10	0.982	−0.50	0.995	− 1.79
	6	0.945	0.943	0.23	0.952	−0.74	0.978	− 3.30
	8	0.857	0.850	0.81	0.861	−0.45	0.892	− 3.88
8	2	0.990	0.989	0.06	0.993	−0.30	0.999	− 0.89
	4	0.977	0.976	0.10	0.982	−0.50	0.995	− 1.79
	6	0.945	0.943	0.23	0.952	−0.74	0.892	6.00
	8	0.857	0.850	0.81	0.861	−0.45	0.892	− 3.88
16	2	0.990	0.989	0.06	0.993	−0.30	0.999	− 0.89
	4	0.977	0.976	0.10	0.982	−0.50	0.995	− 1.79
	6	0.945	0.943	0.23	0.952	−0.74	0.978	− 3.30
	8	0.857	0.845	1.43	0.861	−0.45	0.892	− 3.88

throughput, $TH_{s,1}$, and manufacturer's throughput, $TH_{m,1}$, are within 4% of the exact values for most of the cases.

Table 6 reports the error in $E[I_1]$ obtained from the approximation for an asymmetric ATO system with 2, 4, 8, and 16 components under the *DB* policy. We observe that the expected on-hand inventory, $E[I_1]$ is within 3% of the exact values for all cases, except for the low threshold case under the triangular distribution (T) where the error is within 9%. A similar performance is also observed for the expected backorders $E[B_1]$.

Table 6. Error in $E[I_1]$ under the *DB* policy for the asymmetric case.

N	e_1	$E[I_1^A]$	$E[I_1^E]$	$\Delta[E[I_1^E]]$ (%)	$E[I_1^S]$	$\Delta[E[I_1^S]]$ (%)	$E[I_1^T]$	$\Delta[E[I_1^T]]$ (%)
2	2	2.939	3.027	− 2.89	2.905	1.17	2.695	9.06
	4	4.669	4.624	0.98	4.594	1.62	4.603	1.44
	6	6.352	6.220	2.13	6.255	1.56	6.452	− 1.54
	8	7.860	7.632	2.99	7.725	1.75	8.082	− 2.74
4	2	2.942	3.030	− 2.91	2.925	0.57	2.697	9.10
	4	4.669	4.624	0.97	4.602	1.47	4.603	1.45
	6	6.352	6.228	1.99	6.258	1.51	6.452	− 1.54
	8	7.860	7.633	2.97	7.726	1.73	8.083	− 2.76
8	2	2.942	3.033	− 2.99	2.930	0.43	2.698	9.07
	4	4.669	4.628	0.90	4.604	1.41	4.603	1.44
	6	6.352	6.229	1.98	6.263	1.42	6.460	− 1.67
	8	7.860	7.635	2.95	7.726	1.73	8.088	− 2.81
16	2	2.942	3.033	− 3.00	2.930	0.41	2.705	8.80
	4	4.669	4.629	0.88	4.607	1.36	4.603	1.44
	6	6.352	6.229	1.97	6.267	1.35	6.460	− 1.67
	8	7.860	7.636	2.93	7.727	1.73	8.088	− 2.82

For a given e_k as the number of components increases, we observe a decrease in the throughput at the external subcontractor and an increase in the expected inventory. This is because, as components increase, backorders increase and more orders for component k are placed with the internal manufacturing facility. This reduces throughput at the external subcontractor and increases the expected inventory. Again for any given N, as e_k increases, we observe a decrease in the throughput at the external subcontractor and an increase in the expected inventory. This is because, with an increase in the threshold, e_k, more orders for component k are placed with the internal manufacturing facility. This reduces the throughput at the external subcontractor and increases the expected inventory.

Our results suggest that the decomposition-based approach is fairly accurate for various choices of service time distributions. Furthermore, in terms of computational effort, it should be noted that all of these experiments were performed on a personal computer with a 1.6 GHz Intel Core i5 processor. We note that the runtime for the decomposition approach ranged from 0 to 20 seconds and did not increase significantly as we varied N from two to 16. This suggests that the approach can be used to analyze fairly large systems. In contrast, the runtime to obtain exact results using simulation ranged from 5 to 10 minutes for each case.

7. Conclusions

We considered a single-product ATO system where the product is assembled from multiple components. The components can be manufactured in-house or purchased from the local subcontractor with different system parameters and costs. We analyzed dual index policies (*DB*, *OH*, and *LT* policies) using the Matrix-Geometric approach for moderately sized systems and using a decomposition-based approximation for large systems. The *OH* policy uses thresholds limits on the inventory levels of both the components, whereas the *LT* policy uses thresholds limits on the on-order levels of both the components. The performance of these policies was compared to determine regions in the system design space where they each perform well. We observed that the *LT* policy works well at low base stock levels, whereas the *OH* policy works well at high base stock levels. The *DB* policy outperformed the other two policies, although it lacks the details necessary for implementation. However, we observed that in particular settings, the performance of the *OH* policy and *LT* policy closely resembles that of the *DB* policy and also provides the clarity needed for implementation. This suggests using a combination of the *LT* policy and the *OH* policy to overcome the operational ambiguity in the *DB* policy. For an ATO system with $N > 2$ components, we faced computational challenges with the exact approach used to analyze smaller systems. However, our proposed approximations exploited structural characteristics of the system to address this challenge. The approach not only provides reasonably accurate estimates of performance measures for large systems but also scales well in terms of computational effort. Developing similar decomposition-based approaches for ATO systems with both multiple components and end-products seems to be a promising area of future research.

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