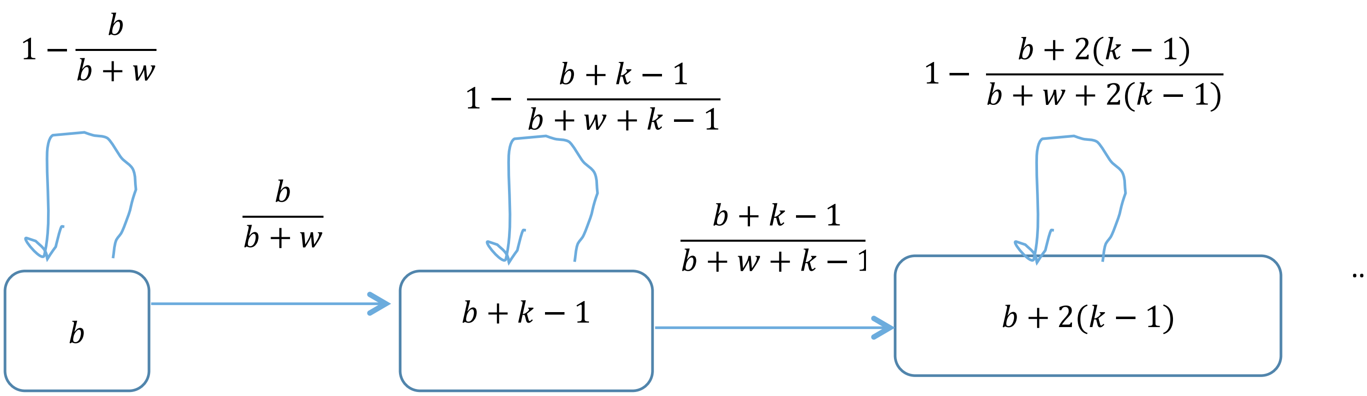
Blake Conrad
HW 1
IMSE 866 Applied Stochastic Processes
10/17/2018

Problem 1

We have been asked to provide a transition probability matrix to $\{X_n, n \geq 0\}$ or disprove that one exists; disproving time-homogeneity for the DTMC.

The problem states we start with parameters w, b, and k. At any given time step, X_n represents the number of black balls in the urn at that time. We also know that *we can only obtain more* black balls, never less, so we define our state-space for X_n to be $= \{b, b+k-1, b+2(k-1), \ldots\} = \{b, b+1, b+2, \ldots\}\ \forall k \geq 1$. We begin by stepping through some transitions if we only selected black balls from the urn, to get a feel for this DTMC:



Clearly, if we select a white ball from the urn at any of the states above, we immediately change our transition probability matrix, because $p_{i,\square}$ on time n+1 just got less likely to pull a black ball because of time n. Hence, depending on the selection at a given time n, we cannot know our probability for n+1. In other words, we are *time dependent* in this problem. We can, however, construct a non-time homogenous DTMC:

$$X_{n+1} = \begin{cases} X_n + k - 1 \ \boldsymbol{w.p.} \ \dfrac{X_n + n(k-1)}{b + w + n(k-1)} \\ X_n \ \boldsymbol{w.p.} \ 1 - \left(\dfrac{X_n + n(k-1)}{b + w + n(k-1)}\right) \end{cases}$$

$\therefore$ By *Definition 2.1*, $\{X_n, n \geq 0\}$ is a DTMC, however because it is not time-homogeneous p_{ij} cannot be constructed for this DTMC.

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Remarks:
We could construct a DTMC that accounts for history for any given time. Consider the following:
Let $W_n = \{0,1\}$ if we selected a white ball at time n. Then,

$$X_{n+1} = \begin{cases} X_n + k - 1 \ \boldsymbol{w.p.} \ \dfrac{X_n + n(k-1)}{b + w + n(k-1) + \left(\sum_{r=0}^{n} W_n\right)(k-1)} \\ X_n \ \boldsymbol{w.p.} \ 1 - \left(\dfrac{X_n + n(k-1)}{b + w + n(k-1) + \left(\sum_{r=0}^{n} W_n\right)(k-1)}\right) \end{cases}$$

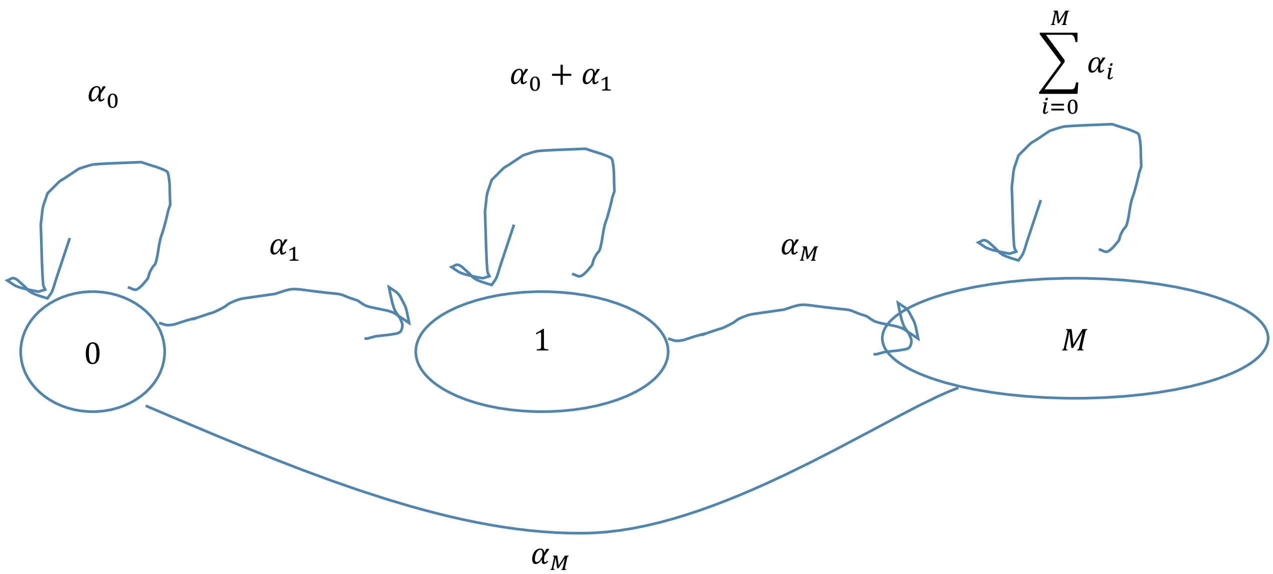
allows for a representation of X_n $\forall n$ with an account of the number of white ball replacements $(k-1)\sum_{r=0}^{n} W_n$ at every time stamp, diluting our probability of getting a black ball for each white ball that has been selected in our history.

Blake Conrad

Problem 2

The problem states that $X_n = \max\{Y_1, Y_2, \dots Y_n\}$. And each Y_k has a probability $\alpha_k$of being in a state k. Immediately we see our state-space for $Y_k \in \{0..M\}$. Therefore, our state-space for $X_n \in \{0..M\}$ because the max at any time step cannot exceed M. We also know $X_0 = 0$. By sketching a few transitions, we gain an intuition for the problem:

Clearly, if we select any number from k $\in \{0..M\}$, then next if we select a number less than or



equal to k, the probability of selecting k is now the union of all of the previous events including itself, hence the sum of all states prior up to and including itself. The only special case is state 0 with its own unique probability α_0. Let's illustrate with a quick example.

We know there is a completely equal probability of α_k, so everyone is accessible from state 0 with their own custom probability defined from the problem. Clearly X_1 will at first be 0 with probability α_0 then we transfer to X_2 with the probability α_2, now we transfer to X_3 with the value as the max of all previous occurrences. This is 2. So in other words, the probability of seeing a 2 at this point is the probability of seeing all other events less than or equal to 2, I.e., $\alpha_0 + \alpha_1 + \alpha_2$. So $X_3 = 2$.

∴ Therefore, we can construct a closed-form transition probability matrix to the DTMC $\{X_n, n \geq 0\}$:

$$P_{ij} = \begin{cases} p_{ij} = \alpha_j, j > i \in \{0..M\} \\ p_{ii} = \sum_{k=0}^{i} \alpha_k, i \in \{0..M\} \\ p_{ij} = 0, Otherwise \end{cases}$$

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Problem 3

Given:

- $n = Stage\ in\ the\ clinical\ trial.$
- Let $S_{1n} = The\ success\ of\ treatment\ 1\ on\ stage\ n\ \in \{0,1\}$
- Let $S_{2n} = The\ success\ of\ treatment\ 2\ on\ stage\ n\ \in \{0,1\}$
- Let $X_n = \sum_{r=0}^{n} S_{1n} - S_{2n} \in \{-k, -k+1, \ldots -1, 0, 1, \ldots, k-1, k\}$
- $p_1 > p_2$
- Successive trails are independent
- $\mathrm{P}[S_{1n} = 1] = p_1,$
- $\mathrm{P}[S_{2n} = 1] = p_2,$
- $\mathrm{P}[S_{1n} = 0] = 1 - p_1$, and
- $\mathrm{P}[S_{2n} = 0] = 1 - p_2$.

Assume:

- $X_0 = 0$
- S_{1n} and S_{2n} are independent
- *"The correct decision"* of a clinical trial is when it reaches the most probable conclusion→ p_1 → the state k determines our expected decision.

Notice:

$\mathrm{P}[X_1 = 1 | X_0 = 0] = \mathrm{P}[S_{11} = 1, S_{21} = 0\ ] = p_1 * (1 - p_2)$ //success of treatment 1 failure of treatment 2

$\mathrm{P}[X_1 = -1 | X_0 = 0] = \mathrm{P}[S_{11} = 0, S_{21} = 1\ ] = (1 - p_1) * p_2$ //failure of treatment 1 success of treatment 1

$\mathrm{P}[X_1 = 0 | X_0 = 0] = \mathrm{P}[\{S_{11} = 1, S_{21} = 1\} \cup \{S_{11} = 0, S_{21} = 1\}\ ] = (1 - p_1) * (1 - p_2) + p_1 * p_2$ //failure of both treatments, success of both treatments

Recall:

$$
\begin{aligned}
X_n &= \sum_{r=0}^{n} S_{1n} - S_{2n} \\
&= \left(\sum_{r=0}^{n-1} S_{1n} - S_{2n}\right) + S_{1n} - S_{2n} \\
&= X_{n-1} + S_{1n} - S_{2n}
\end{aligned}
$$

Considering the success of a treatment at any stage n, we are able to construct a closed form representation of the transition probability matrix:

$$
P_{ij} = \begin{cases}
p_{i,i+1} = p_1 * (1 - p_2) \\
p_{i,i-1} = (1 - p_1) * p_2 \\
p_{i,i} = (1 - p_1) * (1 - p_2) + p_1 * p_2, i \neq -k, k \\
p_{i,i} = 1, i = -k, k
\end{cases}
$$

Claim *(With additional assumptions)*

$$f(p_1, p_2, k, n) = \binom{n-1}{\frac{n-k}{2}} (p_{01})^{(k)} (p_{0-1})^{\left(\frac{n-k}{2}\right)} (p_{01})^{\left(\frac{n-k}{2}\right)}, \forall n \geq k : \exists r \in \mathbb{Z}_+ \rightarrow (n = 2r \; and \; k = 2r) \; or \; (n = 2r + 1 \; and \; k = 2r + 1).$$

Basically, for n at least k, n and k are either both odd or both even, the generalization holds.

Proof

The derivation proof consists of finding all combinations for k=2, n=1,2,3,4,5, then again for k=3, n=1,2,3,4,5. Having these two data points we can generalize to the above expression.

The following situation generalizes for both even and odd n, for a given congruent k:

k	n	Probability
3	1	0
3	2	0
3	3	$(p_{01})^{(3)}$
3	4	0
3	5	$\binom{4}{1}(p_{01})^{(3)} \cdot p_{0-1}p_{01}$
3	6	0
3	7	$\binom{6}{2}(p_{01})^{(3)} \cdot p_{0-1}p_{01}$

And the pattern goes on for as large of an n as specified.

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Claim *(Without additional assumptions)*

The probability that the clinical trial reaches the correct decision, as a function of $p_1, p_2, k, and\ n$ is:

Where,

$$p_{0,-1} = (1 - p_1)p_2$$
$$p_{0,0} = p_1 p_2 + (1 - p_1)(1 - p_2)$$
$$p_{0,1} = p_1(1 - p_2)$$

$$f(p_1, p_2, k, n) = 1 - \left(\begin{array}{c} \left(p_{0,-1} \left(p_{0,-1} + p_{0,1}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)} + p_{0,1} \cdot \sum_{i=0}^{n-(k-1)} {p_{0,1}}^{i} \left(\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)} \right) \right) \right) + \\ \left(p_{0,0}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{n}\right) + \\ p_{0,1} \left(p_{0,-1}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1} + p_{0,0} \cdot \sum_{i=0}^{n-(k-1)} {p_{0,0}}^{i} \left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1} \right) \end{array} \right),$$

$$\forall n \geq k$$

$\forall n < k \; we \; know \; that \; P(reaching \; state \; k) = 0.$ Hence we define $f: p_1, p_2, k, n \rightarrow \mathbb{R} \in [0,1]$

Plan:
Using basic recursive properties, we will utilize the fact that our complementary cdf $\nu_k = 0$ and $\nu_{-k} = 1$. By doing such, we will show the general form of all recursions in terms of these two recursive branches, manipulating the $\nu_k = 0$ and $\nu_{-k} = 1$ to achieve a general solution without any loss of generality. Then by summing these two branches we will obtain the full probability for n>k or n=k time steps that we will never reach state k. By subtracting this value from 1 we acquire the probability that we do achieve a state k given we start from a state 0.

Proof:
Let $T_k = \min\{n \geq 0 : X_n = k\}$ be the first passage time we enter a state k.
Let u_0= P$[T_k < \infty \mid X_0 = 0]$ be the probability we reach a state k given we start from a state 0.

We know $u_0 = 1 - \nu_0$ is one minus the complementary cdf, which represents the probability that we will never reach a state k from 0. By solving this, we find the probability we reach a state k given we start from a state 0.

We know $\nu_0 = \sum_{j \neq k}^{\infty} p_{0j}\nu_j$. This is the same as stating the probability we never reach a state k from 0 in all possible circumstances. We know this is true only in three ways from state 0, if we go left, if we stay, or if we move right ($p_{0,-1}, p_{0,0}, and\ p_{0,1}$ respectively),

$$\nu_0 = p_{0,-1}\nu_{-1} + p_{0,0}\nu_0 + p_{0,1}\nu_1 \qquad \text{(Equation 1)}$$

We immediately notice that if we step to the left, our probabilities are exactly the same,

$$\begin{aligned} \nu_{-1} &= p_{-1,-2}\nu_{-2} + p_{-1,-1}\nu_{-1} + p_{-1,0}\nu_0 \\ &= \quad p_{0,-1}\nu_{-1} + p_{0,0}\nu_0 + p_{0,1}\nu_1 \end{aligned}$$

The same holds for $\nu_{-2}, \nu_{-3}, \ldots \nu_{-k+2}$. The same reasoning holds for $\nu_1, \nu_2, \ldots \nu_{k-2}$. We will consider our edge cases $\nu_{-k+1}\ and\ \nu_{k-1}$ separately, and define the recursion completely in terms of them. We can rewrite Equation 1 for all cases except ν_{-k+1} and ν_{k-1} (which we will account for next),

$$\nu_0 = p_{0,-1}\nu_0 + p_{0,0}\nu_0 + p_{0,1}\nu_0$$

After some algebra, we can factor out ν_0 without any loss of generality in our recursion (for the first k-1 steps),

$$\nu_0 = \nu_0(p_{0,-1} + p_{0,0} + p_{0,1})$$

After k-1 iterations in this recursion, we can quickly see that,

$$\nu_0 = \nu_{-k+1}(p_{0,-1} + p_{0,0} + p_{0,1})^{k-1} + \nu_0(p_{0,-1} + p_{0,0} + p_{0,1})^{n} + \nu_{k-1}(p_{0,-1} + p_{0,0} + p_{0,1})^{k-1}$$

Since our recursive terms $\nu_{-1}, \nu_0, and\ \nu_1$ are completely identical until our edge cases, we can include the k-1 recursive steps and step into the k-1th recursive step for our left tail.

$$\nu_{-k+1} = \left(p_{0,-1}\nu_{-k} + p_{0,0}\nu_{-k+1} + p_{0,1}\nu_{-k+2}\right)$$

Clearly, if we are in the state ν_{-k}, the probability of never reaching a state k is 100%, because it is an absorbing state.

$$\nu_{-k+1} = p_{0,-1} * (1) + p_{0,0}\nu_{-k+1} + p_{0,1}\nu_{-k+2}$$

We can also notice that in this edge case, ν_{-k+1} will behave similar to ν_0 with the addition of a terminating character each iteration. While ν_{-k+2} behaves identical to ν_0. We know

$$\nu_{-k+1} = \left(p_{0,-1} + \left(p_{0,0}\nu_{-k+1} + p_{0,1}\nu_{-k+2}\right)\right)$$

$$\nu_{-k+1} = \left(p_{0,-1} + p_{0,0}\nu_{-k+1} + p_{0,1}\nu_0\right)$$

We now see our left tail generalization,

$$\nu_{-k+1} = \left(p_{0,-1} + p_{0,1}\left(\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)} + p_{0,1} \sum_{i=0}^{n-(k-1)} {p_{0,1}}^{i}\left(\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)}\right)\right)\right)$$

We now examine the right tail,

$$\nu_{k-1} = p_{0,-1}\nu_{k-2} + p_{0,0}\nu_{k-1} + p_{0,1}\nu_k$$

Clearly, ν_k, the probability that we never reach a state k from a state k is impossible, lending itself to 0% chance or a probability of 0. So the cascading effect from the left tail holds itself identical to the right tail with the exception of terminating recursions with 0 instead of 1. Our k-1th step rewritten now becomes,

$$\nu_{k-1} = p_{0,-1}\nu_{k-2} + p_{0,0}\nu_{k-1} + 0$$

As before, our ν_{k-2} behaves as ν_0 whereas ν_{k-1} behaves as a trimmed version appending a subset,

$$\nu_{k-1} = p_{0,-1}(\nu_0) + p_{0,0}\nu_{k-1}$$

$$\nu_{k-1} = p_{0,-1}\left(\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{n-(k-1)} + p_{0,0}\nu_{k-1}\right.$$

We now see our right tail generalization,

$$\nu_{k-1} = p_{0,-1}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1} + p_{0,0} \sum_{i=0}^{n-(k-1)} p_{0,0}{}^{i}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1}$$

Bringing both tails together, we rewrite our original expression,

$$\nu_0 = p_{0,-1}\nu_{-1} + p_{0,0}\nu_0 + p_{0,1}\nu_1$$

where,

$$\nu_{-k+1} = \left(p_{0,-1} + p_{0,1}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)} + p_{0,1} \textstyle\sum_{i=0}^{n-(k-1)} p_{0,1}{}^{i}\left(\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)}\right),\right.$$

$$\nu_0 = \left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{n}$$

$$\nu_{k-1} = \left(p_{0,-1}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1} + p_{0,0} \textstyle\sum_{i=0}^{n-(k-1)} p_{0,0}{}^{i}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1}\right),$$

$\therefore$

$$\nu_0 = p_{0,-1}\left(p_{0,-1} + p_{0,1}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)} + p_{0,1} \textstyle\sum_{i=0}^{n-(k-1)} p_{0,1}{}^{i}\left(\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{(k-1)}\right) +\right.$$
$$p_{0,0}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{n} +$$
$$p_{0,1}\left(p_{0,-1}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1} + p_{0,0} \textstyle\sum_{i=0}^{n-(k-1)} p_{0,0}{}^{i}\left(p_{0,-1} + p_{0,0} + p_{0,1}\right)^{k-1}\right),$$

$\therefore$ 1-ν_0 becomes our probability we reach a state k as a function of p1, p2, and k upper bounded by n.

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Problem 4a

The occupancy time of being in state 1 after two years is the expected amount of time we go from a state I to a state 1 with n=2.

$$M_{ij}{}^{(2)} = \sum_{r=0}^{2} P^r = I + P + P^2$$

$$P^2 = \begin{pmatrix} 0.54 & 0.05 & 0.13 & 0.28 \\ 0.05 & 0.34 & 0.3 & 0.31 \\ 0.32 & 0.2 & 0.32 & 0.16 \\ 0.28 & 0.11 & 0.04 & 0.57 \end{pmatrix}$$

$$M_{ij}{}^{(2)} = \begin{pmatrix} 1.74 & 0.15 & 0.13 & 0.98 \\ 0.15 & 1.64 & 0.9 & 0.31 \\ 0.32 & 0.6 & 1.52 & 0.56 \\ 0.98 & 0.11 & 0.14 & 1.77 \end{pmatrix}$$

Occupancy Time of being in 1 after two years = $\bar{a}^T M_{i1}{}^{(2)}$

$$\bar{a}^T M_{i1}{}^{(2)} = \langle 0.25 \quad 0.25 \quad 0.25 \quad 0.25 \rangle \cdot \langle 0.15 \quad 1.64 \quad 0.6 \quad 0.11 \rangle^T = 0.625$$

∴ The expected number of visits to a state 1 after two years is 0.625.

Problem 4b

$$T = min\{n \geq 0 : X_n = 1\}$$
$$v_2 = P(T > 3 | X_0 = 2)$$
$$v(3) = B^3 e,$$

$$\boldsymbol{B^3 e} = \begin{pmatrix} 0.302 & 0.042 & 0.455 \\ 0.168 & 0.032 & 0.26 \\ 0.455 & 0.065 & 0.326 \end{pmatrix} \cdot \langle 1 \quad 1 \quad 1 \rangle^T = \langle 0.799 \quad 0.46 \quad 0.846 \rangle^T$$

$v_2 = 0.46$

 $v_2 = 0.46$, telling us that the probability of reaching state 1 starting from state 2 after 3 steps.

Problem 4c

$$T = min\{n \geq 0 : X_n = 1\}$$
$$m_2 = E(T | X_0 = 2)$$
$$m(1) = e + Bm(1)$$
$$m = (I - B)^{-1} e$$

$$m = \begin{pmatrix} 6.81818182 & 0.79545455 & 6.36363636 \\ 3.18181818 & 1.70454545 & 3.63636364 \\ 6.36363636 & 0.90909091 & 7.27272727 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13.97 \\ 8.52 \\ 14.54 \end{pmatrix}$$

$m_2 = 8.52$

$\therefore$ This is interpreted as the expected amount of time (provided in units the same as n from X_n) it takes to go from a state 2 to a state 1.

Problem 4d

$$T = min\{n \geq 0: X_n = 1\}$$
$$m(2)_2 = Var(T|X_0 = 2) = m(2)_2 + m(1)_2 - {m(1)_2}^2 = 206.74 + 8.52 - 8.52^2 = 142.62$$

Where,

$$m(1) = (I - B)^{-1}e = \begin{pmatrix} 13.97 \\ 8.52 \\ 14.54 \end{pmatrix}$$

$$m(2) = 2Bm(1) + Bm(2)$$

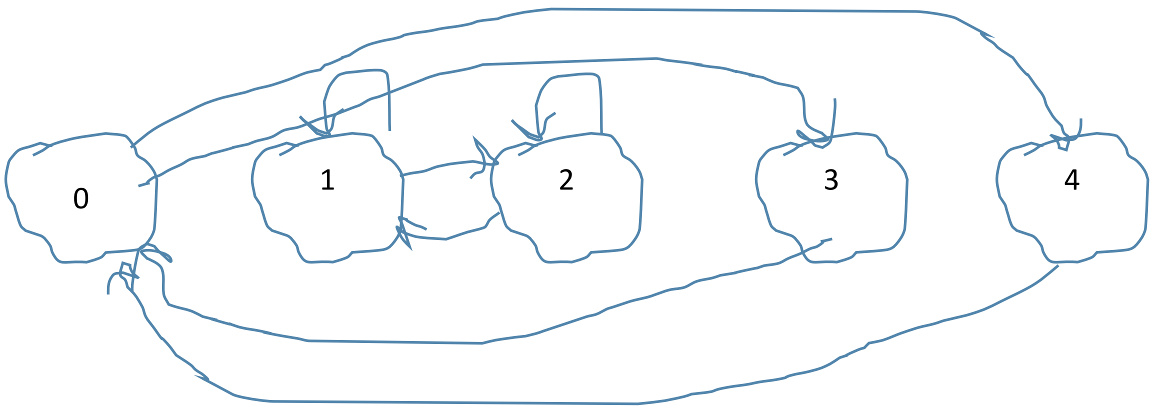
$$m(2) = (I - B)^{-1}2Bm(1)$$
$$= 2 * \begin{pmatrix} 6.81818182 & 0.79545455 & 6.36363636 \\ 3.18181818 & 1.70454545 & 3.63636364 \\ 6.36363636 & 0.90909091 & 7.27272727 \end{pmatrix} \cdot \begin{pmatrix} 0.2 & 0.0 & 0.7 \\ 0.0 & 0.2 & 0.4 \\ 0.7 & 0.1 & 0.2 \end{pmatrix} \begin{pmatrix} 13.97 \\ 8.52 \\ 14.54 \end{pmatrix}$$
$$= \begin{pmatrix} 361.32 \\ 206.74 \\ 375.86 \end{pmatrix}$$

** Theorem 4.8 shows the period of a communicating class will be the same for all states within that communicating class. **

Problem 5a

This is a reducible DTMC with the following state classifications:

0. Recurrent
1. Recurrent
2. Recurrent
3. Recurrent
4. Recurrent

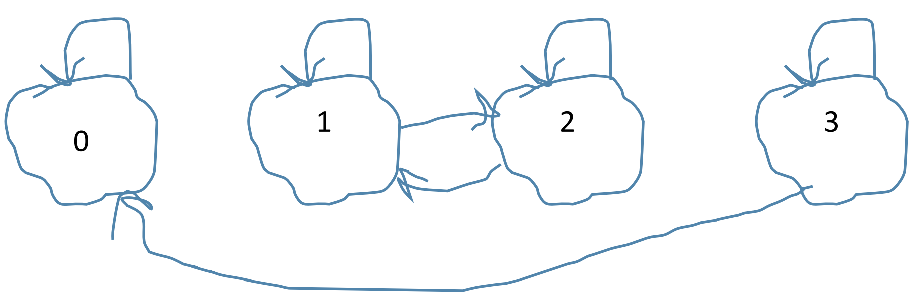


- Communicating Classes: $C_1 = \{1,2\}, C_2 = \{0,3,4\} \subset S = \{0,1,2,3,4\}$.
- $Period(C_1)$=2
- $Period(C_2)$=2
- Clearly $\forall\, i \in S\ \exists\, j \in S :\ i \not\Leftrightarrow j \rightarrow reducible$.

Problem 5b

This is a reducible DTMC with the following state classifications:

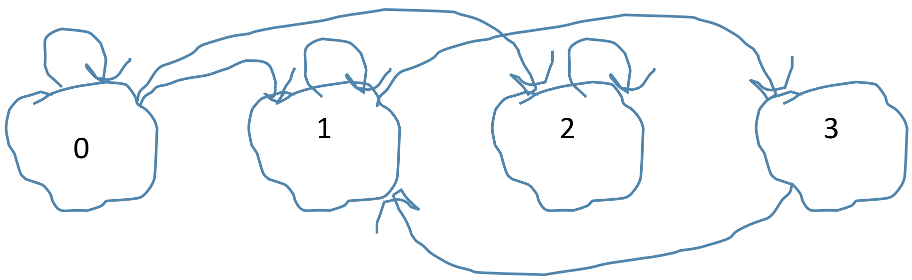
0. Positive Recurrent
1. Recurrent
2. Recurrent
3. Transient



- Communicating Classes: $C_1 = \{0\}, C_2 = \{1,2\}, C_3 = \{3\} \subset S = \{0,1,2,3,4\}$.
- $Period(C_1) = 1 \rightarrow Aperiodic$
- $Period(C_2) = 2$

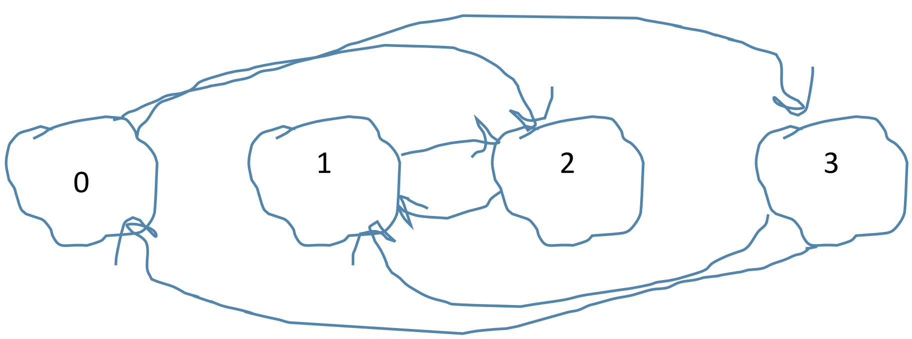
- $Period(C_3) = 1 \rightarrow Aperiodic$
- Notice $2 \nLeftrightarrow 0$, showing that this DTMC is not a closed communicating class. Hence it is not irreducible.

Problem 5c
This is a reducible DTMC with the following state classifications:
0. Transient
1. Recurrent
2. Positive Recurrent
3. Recurrent



- Communicating Classes: $C_1 = \{0\}, C_2 = \{1,3\}, C_3 = \{2\} \subset S = \{0,1,2,3,4\}$.
- $Period(C_1) = 1 \rightarrow Aperiodic$
- $Period(C_2) = 2$
- $Period(C_3) = 1 \rightarrow Aperiodic$
- Notice $2 \nLeftrightarrow 0$, showing that this DTMC is not a closed communicating class. Hence it is not irreducible.

Problem 5d
This is a reducible DTMC with the following state classifications:
0. Transient
1. Positive Recurrent
2. Positive Recurrent
3. Transient



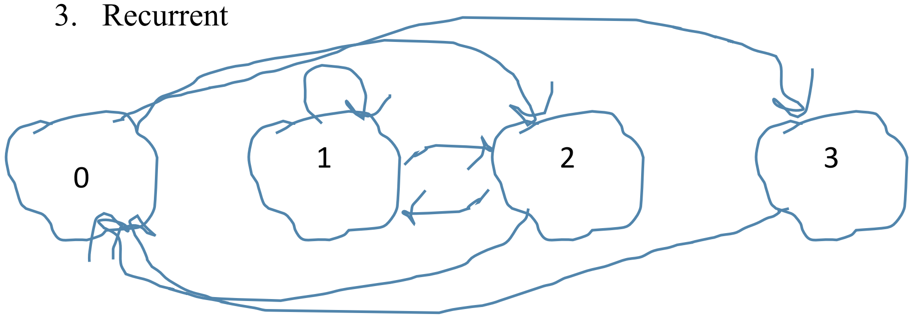
- Communicating Classes: $C_1 = \{0,3\}, C_2 = \{1,2\} \subset S = \{0,1,2,3,4\}$.
- $Period(C_1) = 2$
- $Period(C_2) = 2$

- Notice $1 \nLeftrightarrow 0$, showing that this DTMC is not a closed communicating class. Hence it is not irreducible.

Problem 5e

This is an irreducible DTMC with the following state classifications:

0. Recurrent
1. Recurrent
2. Recurrent
3. Recurrent



- Communicating Classes: $C_1 = \{0,1,2,3\} \subset S = \{0,1,2,3,4\}$.
- $Period(C_1) = 2$
- Notice that all states communicate with one another. This is an irreducible DTMC.

Problem 6

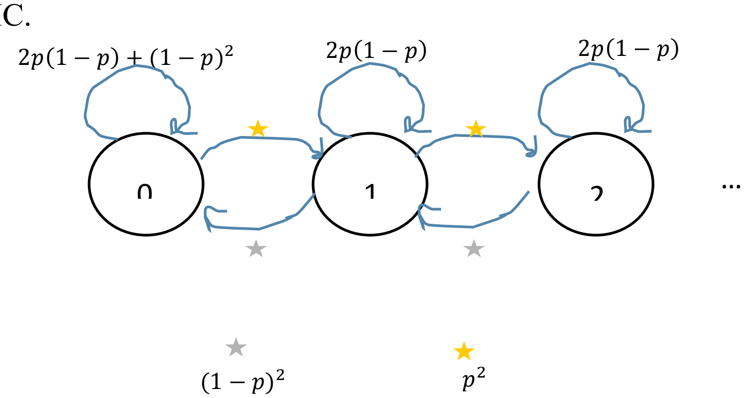
Given

- $P[S_{1n} = 1] = p$
- $P[S_{1n} = 0] = 1 - p$
- $P[S_{2n} = 1] = p$
- $P[S_{2n} = 1] = 1 - p$
- $P[S_{1n} = 1, S_{2n} = 1\ ] = p * p = p^2$
- $P[S_{1n} = 0, S_{2n} = 0\ ] = (1-p) * (1-p) = (1-p)^2$
- $P[\{S_{1n} = 1, S_{2n} = 0\ \} \cup \{S_{1n} = 0, S_{2n} = 1\ \}] = p * (1-p) + (1-p) * p = 2p(1-p)$

Assume

- Let S_{1n} be the success of machine 1 at time n $\in \{0,1\}$
- Let S_{2n} be the success of machine 2 at time n $\in \{0,1\}$
- Let $D_n \in \{1\}, \forall n$
- Let $X_n = \sum_{r=0}^{n} S_{1n} + S_{2n} - D_n = (\sum_{r=0}^{n-1} S_{1n} + S_{2n} - D_n) + (S_{1n} + S_{2n} - D_n)$
- $= X_{n-1} + (S_{1n} + S_{2n} - D_n)$
- Let $X_n = \begin{cases} X_{n-1} + 1\ (\boldsymbol{w.p})\ p^2 \\ X_{n-1}\ (\boldsymbol{w.p})\ 2p(1-p) \\ \max\{0, X_{n-1} - 1\}\ (\boldsymbol{w.p})\ (1-p)^2 \end{cases}$ on $S \in \{0,1,2,\dots\}$

We now define our transition probability matrix for the provided stochastic process, which is in fact a DTMC.



$$\boldsymbol{P_{ij}} = \begin{bmatrix} 2p(1-p) + (1-p)^2 & p^2 & 0 & \cdots & \cdots \\ (1-p)^2 & 2p(1-p) & p^2 & & \\ \vdots & & & \ddots & \vdots \\ \vdots & & & \cdots & \ddots \end{bmatrix}$$

$$\boldsymbol{P_{ij}} = \begin{cases} p_{0,0} = 2p(1-p) + (1-p)^2 \\ p_{i,i} = 2p(1-p), i \neq 0 \in S \\ p_{i,i+1} = p^2 \\ p_{i,i-1} = (1-p)^2 \end{cases}$$

Comment: The transition probability matrix regarding state 0 back to itself; the additional probability is gained because is our DTMC has two cases when we can approach state 0, if we break even on non-defective parts, or if we lose on non-defective parts in turn n-1. This produces a union, which due to the law of probability produces a sum of these two events.

Positive Recurrence of $\{X_n, n \geq 0\}$
Following theorem 4.6, by showing that a state in an irreducible DTMC is positive recurrent, we also show that all other states in that DTMC are also positive recurrent. Hence, we need only to prove that a single state in this DTMC is positive recurrent.

Proof
Let $T_0 = \min\{n \geq 0, X_n = 0\}$ be the first passage time the DTMC enters state 0.
Let $u_0 = P[T_0 < \infty \mid X_0 = 0]$ be the probability of return to state 0.
Let $v_0 = P[T_0 > \infty \mid X_0 = 0]$ be the probability we never return to state 0.
Let $m_0 = E[T_0 < \infty \mid X_0 = 0]$ be the probability of return to state 0 in finite time.
Therefore, by showing the following two conditions hold, we can say that the entire DTMC is positive recurrent:

- $u_0 = 1$
- $m_0 < \infty$

We proceed by finding when these are true.

We know that $u_0 = 1 - v_0$ will tell us the probability we return to state 0. Hence we decompose v_0. We know by definition that $v_0 = \sum_{i=1}^{\infty} p_{0,j} v_j$ is a recursively defined function, which tells us the probability we never reach a state 0 is completely determined the probability we leave from 0 into any other possible state, then by that new states probability of never reaching itself. Clearly for our DTMC, we only have one option if we never want to reach a state 0, we must progress to state 1. We now proceed to step through our recursive steps to determine v_0.

We have the following setup:

$$v_0 = p^2 v_1$$
$$v_1 = (1-p)^2 v_0 + p^2 v_2$$
$$v_2 = (1-p)^2 v_1 + p^2 v_3$$

So after just two iterations we can capture the terms that continue to be appended,

$$v_2 = (1-p)^2((1-p)^2(p^2 v_1) + p^2 v_2) + p^2 v_3$$

Which generalizes to,

$$p^2 \sum_{i=0}^{n} ((1-p)^2)^i \, p^2$$

So we look at how this term tends to infinity by taking its limit,

$$\lim_{n\to\infty} p^2 + p^2 \sum_{i=1}^{n} ((1-p)^2)^i$$

Since this is a convergent geometric series due to 1-p < 1, we obtain the following:

$$p^2 + p^2\left(\frac{1}{1-(1-p)}\right) = p^2 + \frac{p^2}{p} = p^2 + p$$

Hence, we seek to find when the following is true,

$$p^2 + p = 0$$
$$\therefore u_0 = 1 \; iff \; p = 0.$$

Next we show when $m_0 < \infty$,

$$m_0 = \lim_{n\to\infty} \sum_{m=0}^{n} \left(p_{0,0}\right)^n = \lim_{n\to\infty} \sum_{m=0}^{n} (2p(1-p) + (1-p)^2)^n$$

Since we know that our transition probability matrix must sum to 1, and this is only one of the two transitions, it therefore must be a real number between 0 and 1. This means we again have a convergent geometric series with the following limit:

$$m_0 = \lim_{n\to\infty} \frac{1}{1-(2p(1-p)+(1-p)^2)} < \infty$$

We see the limit is less than infinity only if,

$$-2p(1-p) - (1-p)^2 \; \neq 1$$

After some algebra of our expression, we can see that the following is our condition,

$$2p + 2p^2 - 1 + 2p - p^2 \; \neq 1$$
$$p^2 + 4p \; \neq 2$$
$$p(4+p) \; \neq 2$$

Clearly if p = 0, then our derived expression is not equal to 2. Hence our one qualifying condition for guaranteed absorption (shown above) is also a qualifying condition for return in finite time.

∴ Hence, we conclude for our DTMC, if p=0 we can guarantee positive recurrence for the entire chain. This also makes intuitive sense by simply plugging in p=0, causing an absorption in state 0; guaranteeing return and return in finite time.

$$u_0 = 1 \; and \; m_0 < \infty \; iff \;\; p = 0$$

■

Limiting Distribution of $\{X_n, n \geq 0\}$

The setup for showing this leverages the following formula:

$$\pi P = \pi$$

hence,

$$\pi(I - P) = 0$$

We begin by setting up each equation, then start looking for a pattern that we can generalize with:

<u>Equation 1.</u>

$$\pi_0(1 - 2p + 2p^2 - (1-p)^2) = \pi_1(1-p)^2 \rightarrow \pi_1 = \frac{p^2}{(1-p)^2} \cdot \pi_0$$

<u>Equation 2.</u>

$-p^2\pi_0 + \big(1 - 2p(1-p)\big)\pi_1 = (1-p)^2\pi_2$, which after extensive algebra → $\pi_2 = \frac{p^4}{(1-p)^4} \cdot \pi_0$

And since we know that from state 1 off to infinity we will have an identical pattern (because of the consistency of our transition matrix), we express our pattern in terms of our iteration k:

<u>Equation k:</u>

$$\pi_k = \frac{p^{2k}}{(1-p)^{2k}} \cdot \pi_0$$

Now that we have all k equations in terms of π_0, we can represent our final constraint $\sum_{i=0}^{\infty} \pi_i = 1$ as an infinite series.

$$S = \pi_0 + \frac{p^2}{(1-p)^2} \cdot \pi_0 + \frac{p^4}{(1-p)^4} \cdot \pi_0 + \cdots = \sum_{k=0}^{\infty} \frac{p^{2k}}{(1-p)^{2k}} \cdot \pi_0 = 1$$

where

$$S = \pi_0 \left(\sum_{k=0}^{\infty} \frac{p^{2k}}{(1-p)^{2k}} \right) = 1$$

Where we seek to find the convergence of the following:

$$S = \pi_0 \left(\sum_{k=0}^{\infty} \frac{p^{2k}}{(1-p)^{2k}} \right) = 1$$

We now rewrite our expression in a more familiar format

$$S = \pi_0 \left(\sum_{k=0}^{\infty} \left(\frac{p}{(1-p)} \right)^{2k} \right) = 1$$

Since we know p and 1-p are on a measure P which is a measurable function producing only real values between 0 and 1, we also know that their ratio must be less than 1 if it is to converge:

$$\frac{p}{1-p} < 1$$
$$p < 1 - p$$
$$2p < 1$$
$$p < \frac{1}{2}$$

Hence we can show that if p < 1/2, we gain the following representation of S:

$$S = \pi_0 \left(\frac{1}{(1 - (\frac{p}{1-p})} \right) = 1$$
$$S = \pi_0 = 1 - (\frac{p}{1-p})$$
$$S = \pi_0 = 1 - (\frac{p}{1-p})$$
$$\pi_0 = 1 - (\frac{p}{1-p})$$

Now that we have π_0, we can cascade through all of our equations, then generalize when the limiting distribution exists,

$$\pi_0 = 1 - (\frac{p}{1-p})$$

$$\pi_1 = \frac{p^2}{(1-p)^2}\left(1-(\frac{p}{1-p})\right) = \frac{p^2}{(1-p)^2} - \frac{p^3}{(1-p)^3}$$
$$\pi_2 = \frac{p^4}{(1-p)^4}\left(\frac{p^2}{(1-p)^2} - \frac{p^3}{(1-p)^3}\right) = \frac{p^6}{(1-p)^6} - \frac{p^7}{(1-p)^7}$$

$\therefore$ We now have enough to generalize,

$$\pi_j = \frac{p^{j^2+j}}{(1-p)^{j^2+j}} - \frac{p^{j^2+j+1}}{(1-p)^{j^2+j+1}} \forall p < \frac{1}{2}$$

■