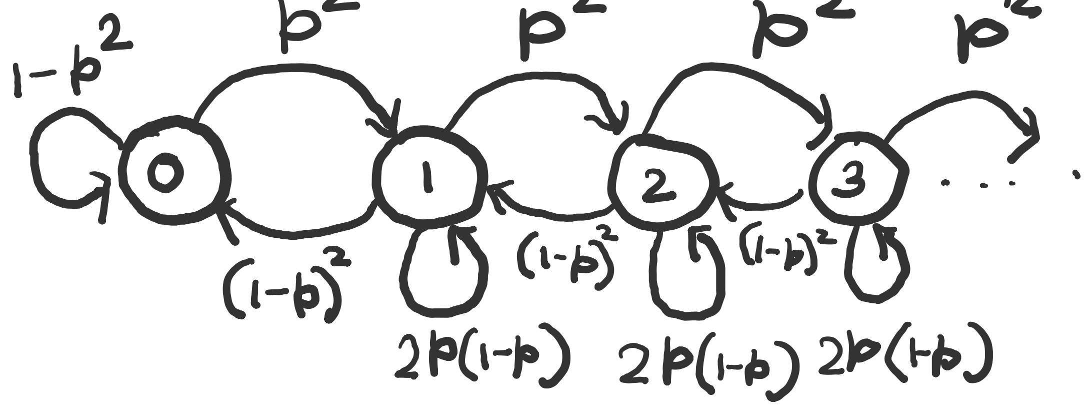
HW 1 Problem 6

X_n - number of items in the storage at the begining

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p^2 \\ X_n & \text{with prob } 2p(1-p) \\ \max(0, X_n - 1) & \text{with prob } (1-p) \end{cases}$$

Clearly, state at $(n+1)$ time period depends on the state at n time period.
So, it is a DTMC.

Limiting Distribution



$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & \cdots \\ \hline 0 & 1-p^2 & p^2 & 0 & \cdots \\ 1 & (1-p)^2 & 2p-2p^2 & p^2 & \cdots \\ 2 & 0 & (1-p)^2 & 2p-2p^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

$$I-P = \begin{array}{c|cccc} & 0 & 1 & 2 & \cdots \\ \hline 0 & p^2 & -p^2 & 0 & \cdots \\ 1 & -(1-p)^2 & 1-2p+2p^2 & -p^2 & \cdots \\ 2 & 0 & -(1-p)^2 & 1-2p+2p^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

For limiting distribution

$$\pi(I-P) = 0$$

$$\therefore \quad \pi_0 p^2 = \pi_1 (1-p)^2$$

$$\Rightarrow \boxed{\pi_1 = \frac{p^2}{(1-p)^2} \pi_0}$$

Now Lets do the same for π_2 and so on

$$\Rightarrow \pi_0 p^2 + \pi_2 (1-p)^2 = \pi_1 (1-2p+2p^2)$$

Now, substituting the value of π_1

$$\Rightarrow \pi_0 p^2 + \pi_2 (1-p)^2 = \frac{p^2(1-2p+2p^2)}{(1-p)^2}\pi_0$$

$$\Rightarrow \pi_2 (1-p)^2 = \left[\frac{p^2[(1-p)^2+p^2]}{(1-p)^2} - p^2\right]\pi_0$$

$$\Rightarrow \pi_2 (1-p)^2 = \left[\frac{\cancel{(1-p)^2}+p^2-\cancel{(1-p)^2}}{(1-p)^2}\right]p^2\pi_0$$

$$\Rightarrow \boxed{\pi_2 = \frac{p^4}{(1-p)^4}\pi_0}$$

In general $\pi_K = \dfrac{p^{2K}}{(1-p)^{2K}}\pi_0$

We know

$$\pi_0 + \pi_1 + \pi_2 + \cdots = 1$$

$$\Rightarrow \pi_0 \left[1 + \frac{p^2}{(1-p)^2} + \frac{p^4}{(1-p)^4} + \cdots \right] = 1 \quad \text{—— (1)}$$

Its a geometric series that converges

when $\frac{p^2}{(1-p)^2} < 1$

$$\Rightarrow p^2 < 1 - 2p + p^2$$

$$\Rightarrow \boxed{p < 1/2}$$

That also ensures unique solution and makes the DTMC positive recurrent

From eq (1)

$$\pi_0 \left[\frac{1}{1 - \frac{p^2}{(1-p)^2}} \right] = 1$$

$$\Rightarrow \boxed{\pi_0 = 1 - \frac{p^2}{(1-p)^2}}$$

So, $$\pi_K = \frac{p^{2K}}{(1-p)^{2K}} \left[1 - \frac{p^2}{(1-p)^2} \right]$$