Fractional Dynamics in Physical Science

Conrad Ho

Presented for Partial Fulfilment of Bachelor's of Science

Objectives

- History of Fractional Calculus (FC)
- Preliminaries
- Memory Effect
- Application in Physics
- Future of FC and its application to Physics

A Brief History of Fractional Calculus

From Leibniz To Us

Leibniz's Letter in 1695

- Informal letter to L'Hoptial

Question:

What if the differentiation/integration order is non-integral arbitrary number?

What is half derivatives!? 3/2 integrals!? π order derivative!?

Timeline (not to scale)

- **1695** — Leibniz's Letter
- **1832** — Riemann–Liouville
- **1868** — Grünwald–Letnikov
- **1967** — Caputo
- **2002** — 'Fractional Schrodinger equation' (L. Naskin)
- **2008** — Fractional vector calculus and fractional Maxwell's equations (V. Tarasov)
- **2011** — 'Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media' (V. Tarasov)
- **2015** — Caputo–Fabrizio
- **2023** — 'Experimental realisations of the fractional Schrödinger equation in the temporal domain' (S. Liu et al.)

Fractional Calculus 101

A crash course in basics of FC

Euler's Notation

Defining a new notation (aka Euler's Notation:

$$D_x^n f(x) \equiv J_x^{-n} f(x) \equiv f^{(n)}(x) \equiv \frac{d^n}{dx^n} f(x)$$

Example:

$$D_x^2 f(x) \equiv f''(x) \equiv \frac{d^2}{dx^2} f(x)$$

$$D_x^{-1} f(x) \equiv J_x^1 f(x) \equiv f^{(-1)}(x) \equiv \int f(x)\, dx$$

(N.B often subscript is dropped when it is apparent what we are applying the operator wrt.)

The Curious Question

$$D_x^n f(x) \equiv J_x^{-n} f(x) \equiv f^{(n)}(x) \equiv \frac{d^n}{dx^n} f(x)$$

'What if n is not an integer ($n \notin \mathbb{Q}$)?'

The starting point

Intuition:

Derivatives are more “well behaved” → Start by generalising derivatives

Reality…?

(often cruel…)

No :D

The starting point

Cauchy formula for repeated integration:

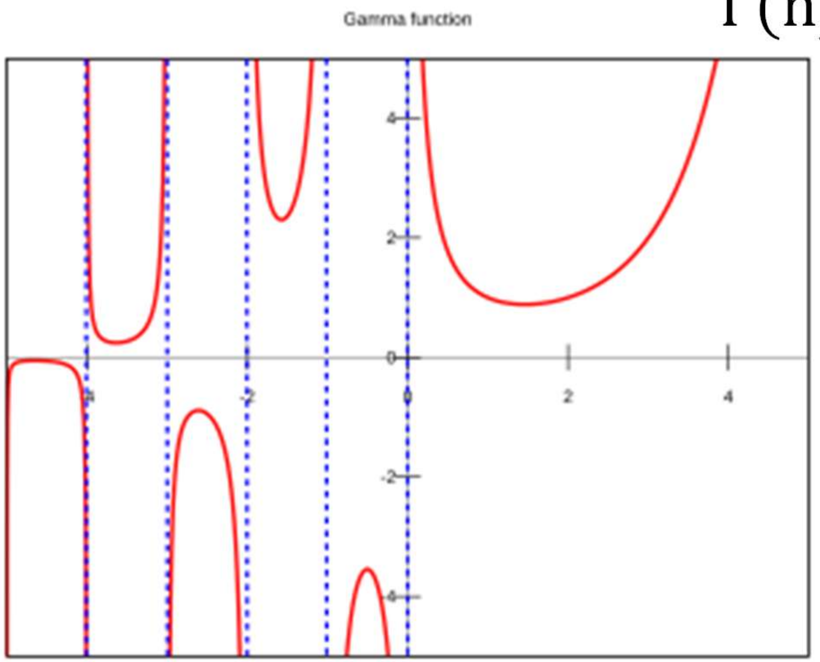
$$(J^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\, dt$$

→ Only works for $n \in \mathbb{N}$ b.c. factorial is undefined otherwise!

A neat 'trick'

Gamma function:

$$\Gamma(n) = (n-1)!$$



Generally, well defined for n>0

Back to the drawing board...

We generalise the formula from:

$$(J^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\, dt$$

To:

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

Where α is now the order of integral

(can be fraction!)

Back to the drawing board...

For α =

Very useful property that we will use to verify results later on!

$f(t)\,dt$

i.e the

Toolbox

From that fundamental, we can derive:

$$
\begin{aligned}
{}_aD_t^\alpha f(t) &= \frac{d^n}{dt^n}\, {}_aD_t^{-(n-\alpha)} f(t) \\
&= \frac{d^n}{dt^n}\, {}_aI_t^{n-\alpha} f(t) \\
{}_tD_b^\alpha f(t) &= \frac{d^n}{dt^n}\, {}_tD_b^{-(n-\alpha)} f(t) \\
&= \frac{d^n}{dt^n}\, {}_tI_b^{n-\alpha} f(t)
\end{aligned}
$$

RL Derivative
(‘left hand’ and ‘right hand’)

$$
{}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau.
$$

Caputo Derivative

$$
{}^{\mathrm{CF}}_{\ a}D_t^\alpha f(t) = \frac{1}{1-\alpha} \int_a^t f'(\tau)\, e^{\left(-\alpha \frac{t-\tau}{1-\alpha}\right)}\, d\tau
$$

Caputo–Fabrizio Derivative

$$
\mathcal{F}\left\{ \frac{\partial^\alpha u}{\partial |x|^\alpha} \right\}(k) = -|k|^\alpha \mathcal{F}\{u\}(k),
$$

Riesz derivative

Standard Results:

From that fundamental, we can derive:

$$D^\alpha e^{kx} = k^\alpha e^{kx} \tag{14}$$

$$D^\alpha \sin(kx) = k^\alpha \sin(kx + \alpha\frac{\pi}{2}) \tag{15}$$

$$D^\alpha \cos(kx) = k^\alpha \cos(kx + \alpha\frac{\pi}{2}) \tag{16}$$

$$D^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \tag{17}$$

$$D^\alpha f(x)g(x) = \sum_{k=0}^{\lceil\alpha\rceil} \binom{\lceil\alpha\rceil}{k} f^{(k)}(x)g^{(\alpha)}(x) \tag{18}$$

The Physics Bits

SHO, Memory Effect and Schrodinger

Tautochrone

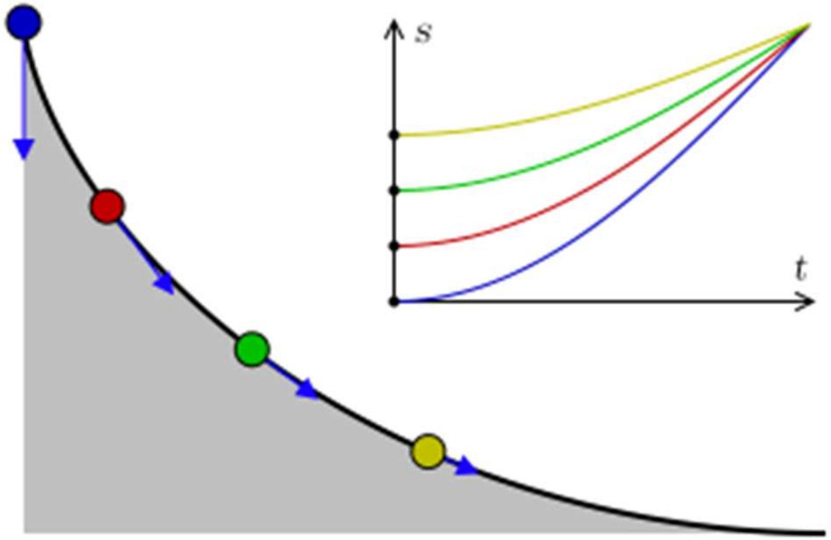
Aim:

Curve such that time falling from a height to the lowest point is independent of starting position

Time travelled is given by:

$$T(y_0 \;\ldots\; \int_{y}^{y_0} \;\ldots\; dy$$

Looks familiar?



Source: https://en.wikipedia.org/wiki/Tautochrone_curve

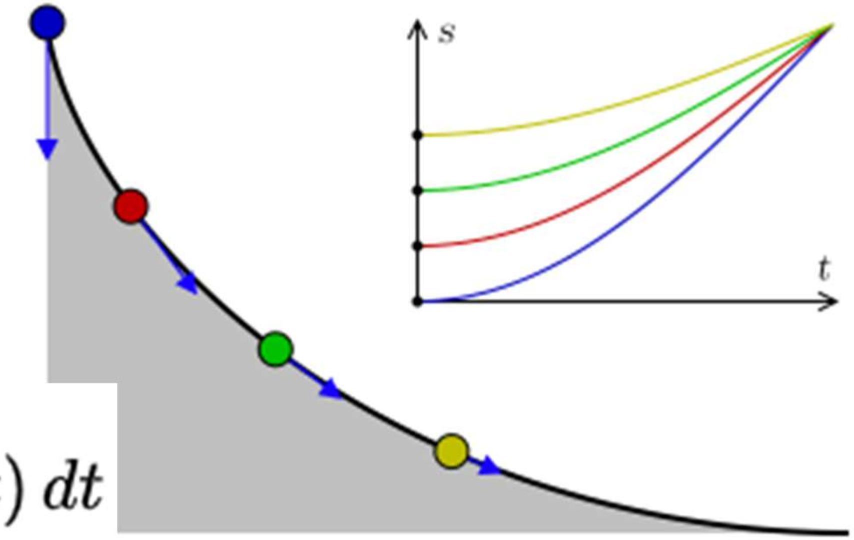
Applying FC...

Time travelled is given by:

$$T(y_0) = \int_{y}^{y_0} \frac{F(y)}{\sqrt{y_0 - y}} dy$$

Recall:

$$\left(J^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$



Tautochrone using FC

RL Integral Definition:

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

Therefore, we can write:

$$T(y_0) = \int_y^{y_0} \frac{\phi(y)}{\sqrt{y_0 - y}} dy$$

as:

$$T(y_0) = \sqrt{\pi}\, \boldsymbol{J}^{\frac{1}{2}}\{\phi(y_0)\}$$

and

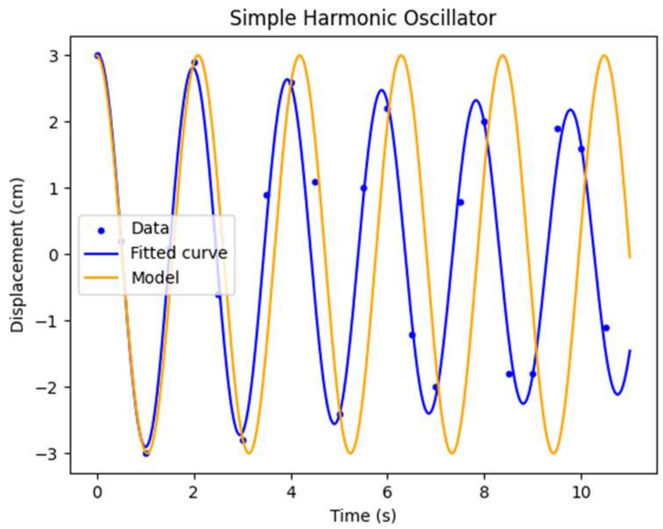
$$\phi(y_0) = \frac{1}{\pi} \boldsymbol{D}^{\frac{1}{2}}\{T(y_0)\}$$

SHO:

Recall:

$$\dot{x} + \omega^2 x = 0$$
$$x(t) = A\cos(\omega t + \varphi)$$

Plot:



(from an actual experimental result)

SHO:

$$\dot{x} + \omega^2 x = 0$$

What if it is to a fractional order?

$$D^{\alpha} x + \omega^2 x = 0$$

Side Note:

$$D^{\alpha}x + \omega^2 x = 0$$

It often hard to solve symbolically...

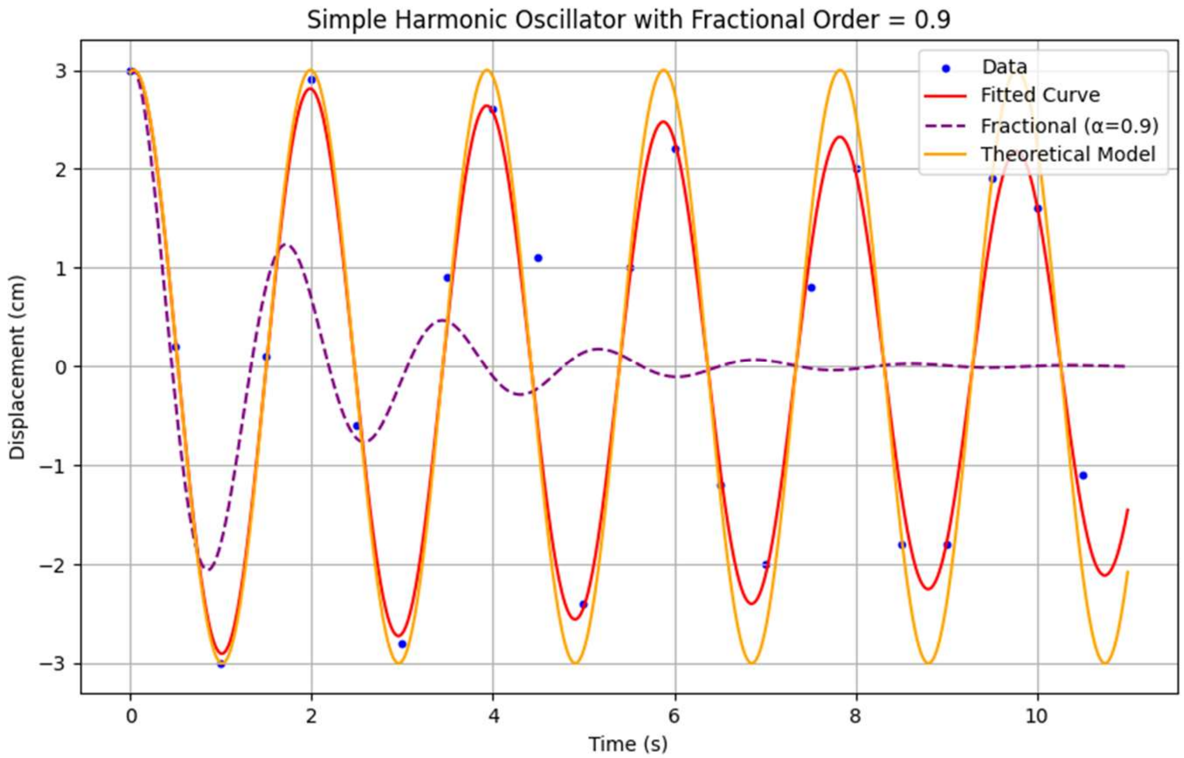
To save your life – Use Python:

- Differint – Computing DifferIntegrals (hence the name)
- Fodeint – Differential Equation Solver

Results in the following pages use the above 2 libraries!

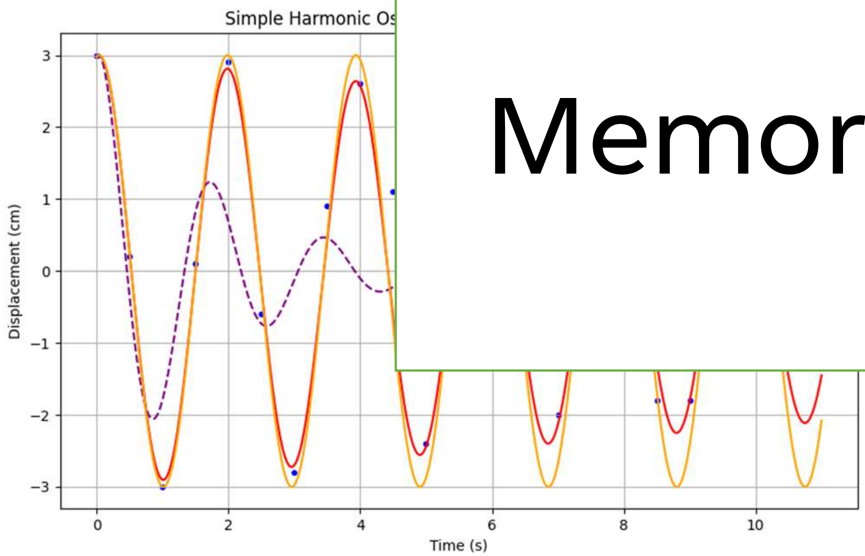
SHO Fractional:

Plot: (purple line)



Damping!?

Plot:



Memory Effect!!

ial

Memory Effect

Mathematically:

- Non-Local Operator (e.g. Fourier Transform)

Physically:

- Viscous material‘s behaviour
- Anomalous diffusion

Applying to QM

- Plane Wave
- Infinite Potential Wall

FC applied to QM

TISE:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x)+V(x)\psi(x)=E\psi(x)$$

Generalise to fractional order (FTISE):

$$-\frac{\hbar^2}{2m}\nabla^{2\alpha}\psi(x)+V(x)\psi(x)=E\psi(x)$$

where $0 < \alpha < 1$

Plane Wave

Time independent plane wave wavefunction:

$$\psi(t) = e^{ikx}$$

Applied to FTISE:

$$-\frac{\hbar^2}{2m}\nabla^{2\alpha}\psi(x)+V(x)\psi(x)=E\psi(x)$$

To simplify, we assume $V(x) \rightarrow\ 0$

Plane Wave

Recall Standard Results:

$$D^\alpha e^{kx} = k^\alpha e^{kx} \quad k \geq 0 \tag{14}$$

$$D^\alpha \sin(kx) = k^\alpha \sin(kx + \alpha\frac{\pi}{2}) \quad k \geq 0 \tag{15}$$

$$D^\alpha \cos(kx) = k^\alpha \cos(kx + \alpha\frac{\pi}{2}) \quad k \geq 0 \tag{16}$$

$$D^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{k-\alpha} \quad x \geq 0, k \neq -1, -2, -3, \ldots \tag{17}$$

$$D^\alpha f(x)g(x) = \sum_{k=0}^{\lceil\alpha\rceil} \binom{\lceil\alpha\rceil}{k} f^{(k)}(x)g^{(\alpha)}(x) \tag{18}$$

Plane Wave

Then:

$$\nabla^{2\alpha}\psi(x) = i^{2\alpha}k^{2\alpha}\,\psi(x)$$

$$D^\alpha e^{kx} = k^\alpha e^{kx} \quad k \geq 0 \tag{14}$$

$$D^\alpha \sin(kx) = k^\alpha \sin(kx + \alpha\frac{\pi}{2}) \quad k \geq 0 \tag{15}$$

$$D^\alpha \cos(kx) = k^\alpha \cos(kx + \alpha\frac{\pi}{2}) \quad k \geq 0 \tag{16}$$

$$D^\alpha x^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha} \quad x \geq 0, k \neq -1, -2, -3, \ldots \tag{17}$$

$$D^\alpha f(x)g(x) = \sum_{k=0}^{\lceil\alpha\rceil}\binom{\lceil\alpha\rceil}{k} f^{(k)}(x)g^{(\alpha)}(x) \tag{18}$$

Plane Wave

Putting back into FTISE and yield:

$$-\frac{\hbar^2}{2m} i^{2\alpha} k^{2\alpha} = E$$

How to simplify further?

Plane Wave

Euler's Identity:

$$e^{i\pi} = -1$$

Taking sqrt both side:

$$e^{i\pi/2} = i$$

Hence, $i^{2\alpha}$ from our previous equation becomes:

$$e^{i\pi\alpha} = i^{2\alpha}$$

Plane Wave

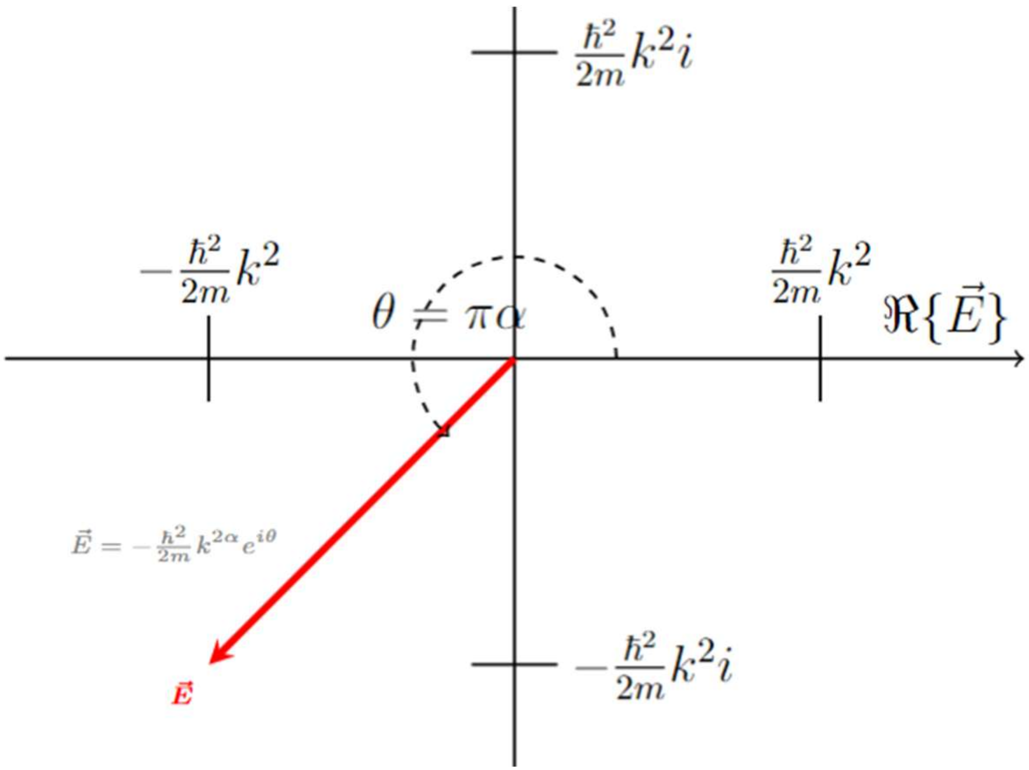
$$e^{i\pi\alpha} = i^{2\alpha}$$

Hence, our expression for the eigenvalues:

$$E = -\frac{\hbar^2}{2m} i^{2\alpha} k^{2\alpha}$$

$$E = -\frac{\hbar^2}{2m} k^{2\alpha} e^{i\pi\alpha}$$

Plane Wave

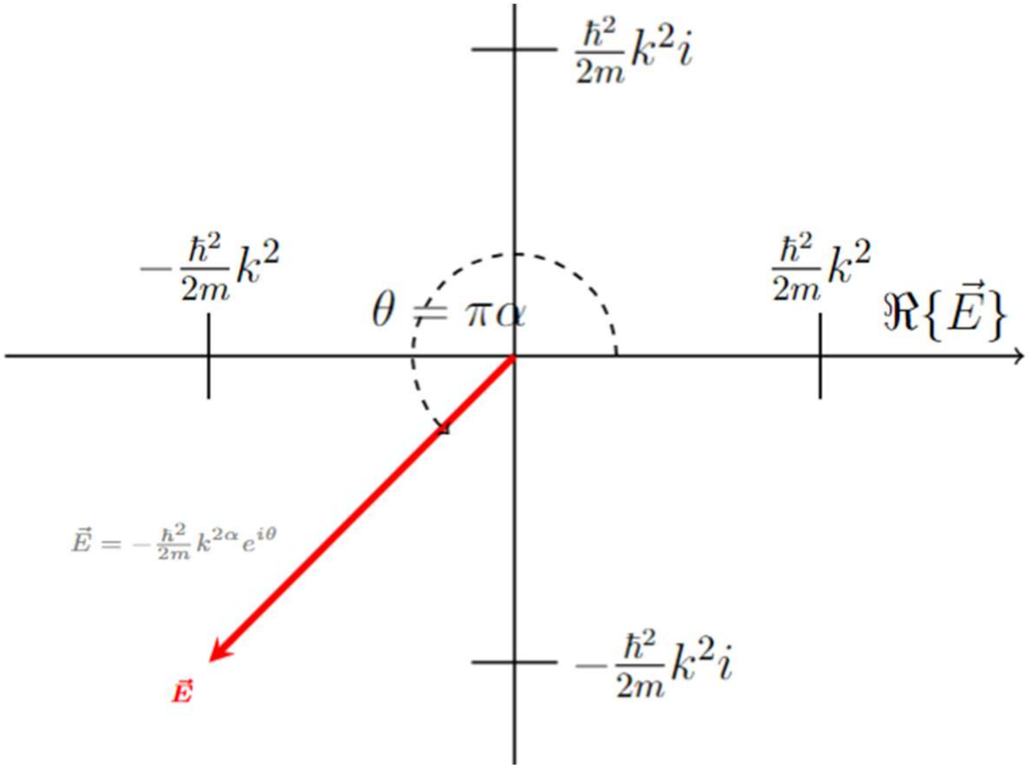


Plot this on an Argand Diagram

$$E = -\frac{\hbar^2}{2m}k^{2\alpha}e^{i\pi\alpha}$$

- Fractional order α rotates the Energy "vector" in the Re-Im Plane
- similar to a Phasor in PH1420
- Neat U(1) Symmetry

Plane Wave



Decomposed:

By Euler's equation:

$$E = -\frac{\hbar^2}{2m}k^{2\alpha}e^{i\pi\alpha}$$

$$E = -\frac{\hbar^2}{2m}k^{2\alpha}(\cos\pi\alpha - i\sin\pi\alpha)$$

- Real corresponds to observables
- Imaginary encodes some additional information?

Plane Wave

Quick Check:

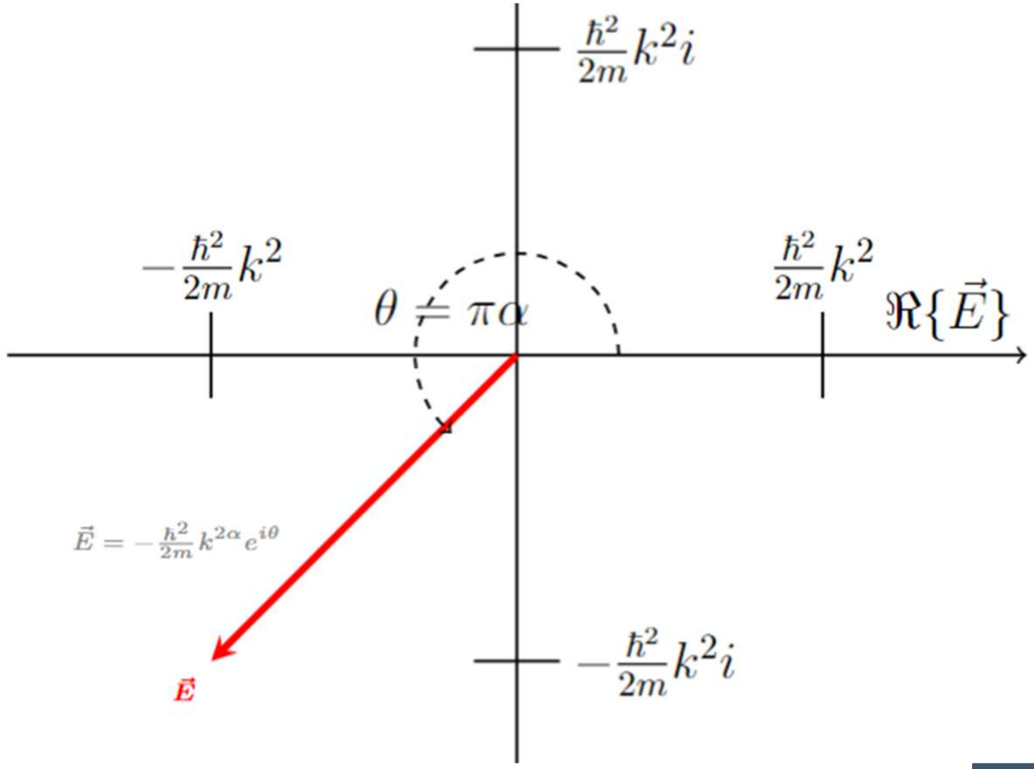
We know the property where $\alpha \to 1$, we recover the "typical" case

$$\Re(E) = \lim_{\alpha \to 1} -\frac{\hbar^2}{2m} k^{2\alpha} \cos \pi\alpha$$

$$\Re(E) = \frac{\hbar^2 k^2}{2m}$$

Let's see this on the diagram.

Plane Wave



$\frac{\hbar^2}{2m}k^2 i$
$-\frac{\hbar^2}{2m}k^2$
$\frac{\hbar^2}{2m}k^2$
$\theta = \pi\alpha$
$\Re\{\vec{E}\}$
$\vec{E} = -\frac{\hbar^2}{2m}k^{2\alpha}e^{i\theta}$
$-\frac{\hbar^2}{2m}k^2 i$
$\vec{E}$

Plane Wave

Does this make sense?

Sort Of...

Plane Wave

The Hamiltonian in the Fractional order does not satisfy self-adjoint -> Non-Hermitian

Weird Properties like these are sort of expected!

Particle in a Box

It's Weird!

Open Problems

Open Problems

- Physical Interpretation?
  - What does the fractional derivative of something physically mean?
  - Does it make sense physically?
- Applications to other fields
  - For example, in SR:
    - Can some order of derivative of a coefficient acts be in analogus to Lorentz Transformation?
    - Does this generalise to GR?

Summary

Summary

- A topic of relatively long history but advancements are fairly recent
- Definition
- Common Function
- Physics
  - Classical Mechanics: Tautochrone, SHO
  - QM: Plane Wave, Infinite Potential Well/Particle in a Box
- Memory Effect
  - Strange Behaviour!
- Open Problems
  - Physical Interpretation
  - Application to other fields

The End