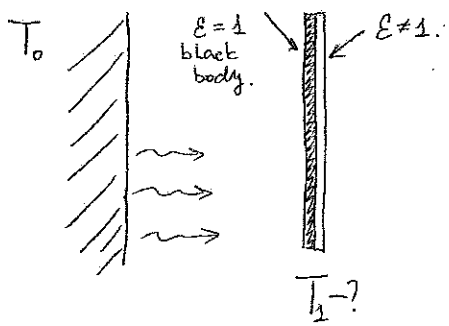
3760
2009 final

(80 points)
#1
Molecules of gas in a container have the Maxwell velocity distribution at temperature T. The mass of each molecule is m. Find the average speed ($v=\left(v_x^2+v_y^2+v_z^2\right)^{1/2}$) of a fraction of molecules moving in positive x-direction (in other words having positive component of v_x).

#2
A metallic plate is kept at constant temperature T and emits photons as a black body radiator. Parallel to this plate there is a second metallic plate. The surface of the second plate facing the first plate has properties of a black body radiator. The other surface has emissivity ε. The radiation coming form the first plate is the only source of heat for the second plate. What is the steady-state temperature of the second plate?



#3
The first quantum exited state of rotational motion of a CO molecule has energy E and orbital momentum L. The state is degenerate, that is quantum states with the projection of momentum L_Z= - 1, 0, +1 all have the same energy E. CO molecules are in equilibrium at temperature T. What is the ratio of a probability to find a CO molecule in the first exited state to a probability to find a molecule in the ground state?

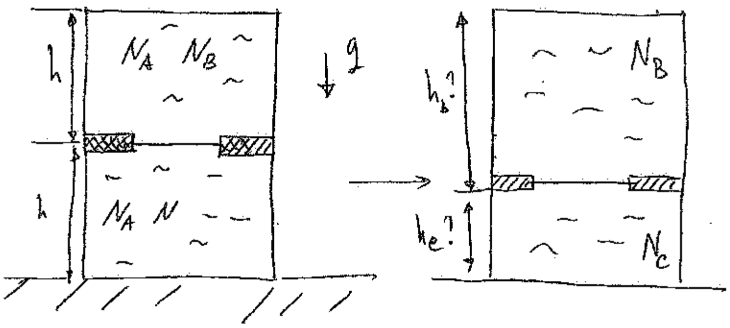
#4
A system of eight spins (s=1/2) is in zero magnetic field and is in equilibrium at temperature T. What is the probability to find this system in the quantum state with spins one, two, three and four pointing up and with spins five, six, seven and eight pointing down?

↑ ↑ ↑ ↑ ↓ ↓ ↓ ↓

1 2 3 4 5 6 7 8

#5

A container is *initially* separated into two equal compartments by a movable piston of mass m. A cross sectional area of each compartment is S, the high is h, all system is kept at constant temperature T_0 and in the gravitational field of Earth. The upper compartment has N_A molecules of water and N_B molecules of solvent B. The lower compartment has the same number N_A of molecules of water and N_C molecules of solvent C. A part of the piston is a membrane that can be penetrated by water molecules but can not be penetrated by molecules of B and C solvents. The piston is released. What will be an equilibrium position of a piston? (Consider water incompressible, hydrostatic pressure created by water weight is much smaller that the pressure created by a piston and can be neglected)



Final (solutions)

1.) $$P(v_x, v_y, v_z) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} dv_x dv_y dv_z.$$

The distribution in spherical coordinates is:

$$P(v, \theta, \varphi) = \left(\frac{m}{2\pi kT}\right)^{3/2} \cdot e^{-\frac{mv^2}{2kT}} \cdot \sin\theta \, v^2 \, dv \, d\theta \, d\varphi.$$

Formally, the mean speed of molecules moving in the positive x direction can be computed as

$$\langle v^x \rangle = \frac{\int_0^\infty dv \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \cdot v \cdot \left(\frac{m}{2\pi kT}\right)^{3/2} \cdot \exp\left(-\frac{mv^2}{2kT}\right) \sin\theta \, v^2}{1/2}.$$

1/2 accounts for the fact that only half of molecules moves in the positive x-direction.

$$\langle v^x \rangle = 4\pi \times 1 \times \int_0^\infty v^3 \cdot \left(\frac{m}{2\pi kT}\right)^{3/2} \cdot \exp\left(-\frac{mv^2}{2kT}\right) dv =$$

$$= 4\pi \times 1 \times \left(\frac{m}{2\pi kT}\right)^{3/2} \cdot \frac{1}{2} \cdot \left(\frac{2kT}{m}\right)^2 = \boxed{\sqrt{\frac{kT}{m}} \cdot \sqrt{\frac{8}{\pi}}}$$

However even without computation you may conclude that average speed of molecules moving to the right is the same as average speed of molecules moving to the left, and therefore is the same as just the average speed of molecules. It appears in lecture notes and in Laws and equations sheet.

Final 2009 3760

#2.

Power emitted by the first plate is.
$W_0 = \sigma A T_0^4$. It is completely absorbed by the second plate. The power emitted by the second plate is.

$$W_1 = \underbrace{\sigma A T_1^4}_{\text{black body side}} + \underbrace{\varepsilon \sigma A T_1^4}_{\text{side with emissivity } \varepsilon} = \sigma A T_1^4 (1+\varepsilon)$$

$W_1 = W_0$ in steady state.

$$\sigma A_1 T_0^4 = (1+\varepsilon) \sigma A T_1^4$$

$$T_1 = \frac{T_0}{(1+\varepsilon)^{1/4}}.$$

#3.

The ground state of rotating molecule is molecule at rest. The energy of this state is $0 = \mathcal{E}_g$ and the degeneracy is 1.
The first excited state has energy $\mathcal{E}_1$ and degeneracy 3, because of three possible values of L_z.

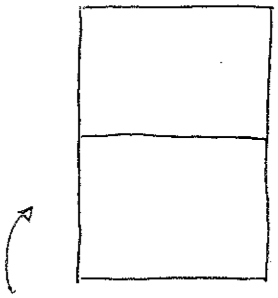
$$\frac{P(\mathcal{E}_1)}{P(\mathcal{E}_g)} = 3 \times \exp\left(-\frac{\mathcal{E}_1}{kT}\right)$$

4.

In zero magnetic field $P(\uparrow) = P(\downarrow) = \frac{1}{2}$ for a single spin. The description of the state of the system of 8 spins given in the statement of the problem is in fact exact description of the quantum state or the microstate of the system.
Its probability is $\left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^8$

\# 5

Let's start computing chemical potential of solvet (A) adding step by step complications to our system.



μ_A' – chemical potential of solvent in uppor compartment

μ_A'' – chemical potential of solvent in lower compartment

1) $V_{upper} = V_{Lower}$, $N_A' = N_A'' = N_A$, no solutes, piston is held and does not produce any pressure.

$$\mu_A' = \mu_0(T, P_0) \qquad \mu_A'' = \mu_0(T, P_0)$$

P_0 – some initial pressure, We neglect the hydrostatic pressure caused by water weight so $P_0' = P_0''$

V', P_0', N_A' – notations for upper compartment.

V'', P_0'', N_A'' – notations for lower compartment

2) $V' = V'' = V_0$, $N_A' = N_A''$, piston is held, but solutes are added.

$$\mu_A' = \mu_0(T, P_0) - \frac{N_B}{N_A} \cdot kT$$

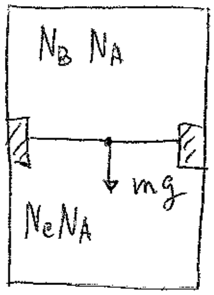
$$\mu_A'' = \mu_0(T, P_0) - \frac{N_C}{N_A} \cdot kT$$

3) Piston is released, but it is still in initial position.

$$\mu_A' = \mu_0(T, P_0) - \frac{N_B}{N_A} \cdot kT$$

$$\mu_A'' = \mu_0(T, P_0) - \frac{N_C}{N_A} \cdot kT + \left(\frac{\partial \mu_A}{\partial P}\right)_{T,N} \Big(\underbrace{P_0 + \frac{mg}{S}}_{\text{final pressure}} - \underbrace{P_0}_{\text{initial pressure}} \Big)$$

This is not equilibrium



$\Delta P = \frac{mg}{S}$, piston is released but it is still in initial position.

$\left(\frac{\partial \mu_A}{\partial P}\right)$ - ? for pure solvent.

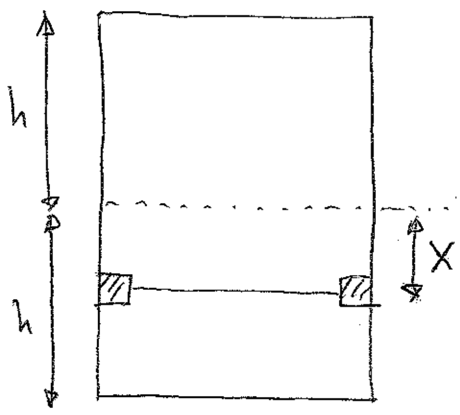
$G = \mu_A \cdot N \qquad dG = -SdT + VdP$

$\left(\frac{\partial \mu_A}{\partial P}\right) = \frac{1}{N_A} \cdot \left(\frac{\partial G}{\partial P}\right) = \frac{V}{N_a} = v_a$ - volume per molecule of water.

$v_a = \frac{h \cdot S}{N_A}$. Because we consider water incompessible. v_a does not change when piston slides. Pressure created by the piston also does not depend on position of the piston.

4) Piston slided to the position where diffusive equilibrium is established.

$N_A' \neq N_A''$



$\mu' = \mu_0 - \frac{N_B}{N_A'} \cdot kT$

$\mu'' = \mu_0 - \frac{N_C}{N_A''} \cdot kT + \frac{h \cdot S}{N_A} \cdot \frac{mg}{S}$

$N_A' = N_A \cdot \frac{h + x}{h}$

$N_A'' = N_A \cdot \frac{h - x}{h}$

$$\mu' = \mu''$$

$$-\frac{N_B\,kT}{A\cdot\frac{h+x}{h}} = -\frac{N_c\,kT}{N_A\cdot\frac{h-x}{h}} + \frac{h\cdot S}{N_A}\cdot\frac{mg}{S}$$

$$-\frac{N_B}{(h+x)} = -\frac{N_c}{(h-x)} + mg/kT$$

$$\frac{mg}{kT}\cdot x^2 - \frac{mg}{kT}h^2 - (h-x)N_B + (h+x)N_c = 0.$$

$$\frac{mg}{kT}x^2 + (N_B+N_c)x + (N_c-N_B)h - \frac{mg}{kT}h^2 = 0$$

$$x = \frac{-(N_B+N_c) \pm \sqrt{(N_c+N_B)^2 - 4\frac{mg}{kT}\left((N_c-N_b)h - \frac{mgh^2}{kT}\right)}}{2\frac{mg}{kT}}$$

If $N_c = N_B$ the piston slides down and x must be positive. That means the solution with "$-$" sign in the above equation is physically irrelevant.