Uniqueness of Friend Cluster in a Social Network on Non-amenable Graphs

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Abstract

We study a social network based on a connected infinite graph G, where people travel through the edges and socialize at the vertices. For each vertex x in G, there are initially $N(x)$ people at x, where $N(x)$'s are i.i.d., mean-λ Poisson random variables. Each person performs a lazy simple random walk (SRW), independently of others. When two people meet at a vertex, they befriend each other and each other's friends. We answer the following question for non-amenable G: Will every two people eventually be friends a.s.?

The Social Network Model

1. The underlying graph G is connected, infinite, regular and non-amenable.
2. For each vertex x in G, there are initially $N(x)$ people at x, where $N(x)$'s are i.i.d., mean-λ Poisson random variables.
3. Each person performs a lazy simple random walk on G, independently from others.
4. When two people meet at a vertex, they befriend each other and each other's friends (*coalescent of friend clusters*).

Remark: Most results are extendable to the case of a non-regular graph with bounded degree.

Uniqueness of Friend Cluster

Will any two people be friends a.s. at time ∞?

Related Work & Motivation

- For a finite graph G of size n and $\lambda = 1$, [2] shows that the first time that all the people are in the same cluster is w.h.p. $O(\log^3 n)$.
- For infinite graphs it makes more sense to ask what happens as time $\to \infty$.

The case $G = \mathbb{Z}^d$ is well understood.

- *Infection model*: a particle model in which healthy particles are infected by contagious ones when they coincide. [3] studies this model on $\mathbb{Z}^d$ with Poisson(λ) starting configuration. Their results lead to uniqueness of friend cluster in the social network model on $\mathbb{Z}^d$ for all $\lambda > 0$.
- *Frog model*: a particle system where inactive particles are "awakened" by active ones and turned active. [1] studies this model on $\mathbb{Z}^d$ and shows all the particles are eventually active a.s. for one-particle-per-site starting condition.

More generally, we proved:

Proposition

If G is a vertex-transitive and amenable, then every two people eventually become friends a.s. for all $\lambda > 0$.

- Intuitively, the more G "spreads out", the more likely some friend clusters never merge.
- The uniqueness of infinite open cluster in Bernoulli percolation depends on amenability.
- Thus *amenability* is perhaps the deciding factor.

Main Results for Non-amenable G

- There exists $\lambda_c > 0$, such that for all $\lambda \in (\lambda_c, \infty)$, every two people eventually become friends a.s.; and for all $\lambda \in (0, \lambda_c)$, there are two people who never become friends with positive probability.
- If G is regular with degree d and spectral radius ρ, the following bounds on λ_c hold:

Lazy SRWs	Lower Bound	Upper Bound
holding prob. $\frac{1}{d+1}$	$O(1/\rho)$	$O(d/(1-\rho))$
holding prob. $\frac{1}{2}$	$O(\log(d))$	$O(\log(d)/(1-\rho))$

- For $G = d$-regular tree and holding probability $= 1/(d+1)$, λ_c is of order $O(\sqrt{d})$ for all large d.

Important Remark

The sign of the critical value λ_c provides a dichotomy between amenable and non-amenable graphs !

Theorem (Lower Bound)

Suppose that holding probability $= 1/(d+1)$. If

$$\lambda < \frac{1}{2\rho} - 1, \tag{1}$$

then there exist two people who never become friends with positive probability.

Elements of the proof

- Stochastically dominate a friend cluster by a lazy branching random walk (LBRW).
- The LBRW is transient for small λ, by a heat kernel estimate.

Remark: The proof of the $O(\log(d))$ lower bound uses similar ideas and is more involved.

Theorem (Upper Bound)

Suppose holding probability $= 1/(d+1)$. There exists constant C such that there is exactly one friend cluster at time ∞ a.s. if

$$\lambda > \frac{Cd}{1-\rho}. \tag{2}$$

Elements of the proof

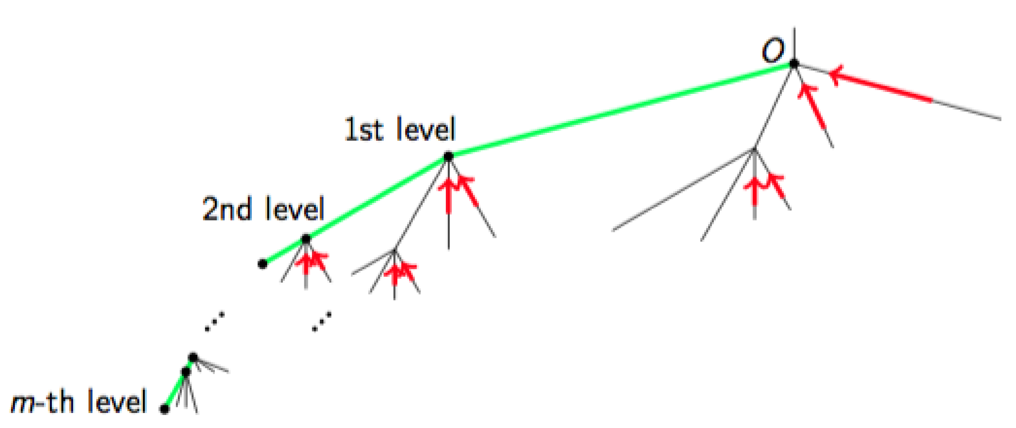
- Construct two random paths that eventually collide a.s., implying coalescence of two friend clusters.
- Exponential decay of heat kernel guarantees independence between the path increments.

Theorem (d-Regular Trees)

Suppose that G is a d-regular tree, $d \geq 5$, and holding probability $= 1/(d+1)$. If

$$\left\lfloor \frac{d-1}{2} \right\rfloor \left[1 - \exp\left(-\frac{\lambda}{6d} \right) \right]^2 > 1, \tag{3}$$

there is exactly one friend cluster at time ∞ a.s.



Idea behind the proof

Stochastically dominate a supercritical Bernoulli bond percolation by the trajectories of the walkers from a friend cluster.

Future Work

Does there exist $t_\lambda < \infty$ such that there is a.s. an infinite friend cluster at any time $t > t_\lambda$?

References

[1] O.S.M. Alves, F.P. Machado, and S.Yu. Popov, The shape theorem for the frog model, Annals of Applied Probability [2002].

[2] I. Benjamini, G. Kozma, and J. Hermon, Unpublished preprint [2013].

[3] H. Kesten and V. Sidoravicius, The spread of a rumor or infection in a moving population, Annal of Probability [2005].