Lecture slides on Calculus 1

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Main References

1. James Stewart: *Calculus*, 8th edition, Cengage Learning (2016) [Textbook]
2. George B. Thomas, Maurice D. Weir, Joel R. Hass, Frank R. Giordano: *Thomas' Calculus*, 11th edition, Addison-Wesley (2004)
3. G. James: *Modern Engineering Mathematics*, 3rd edition, Pearson Education (2004)

Several images in these slides are sourced from the Internet and are used solely for teaching purposes.

The lecture slides are updated periodically to enhance readability.

1. Integration
   - Indefinite integration
   - Definite integration
   - Fundamental Theorem of Calculus
   - Approximate Integration
   - Improper Integrals

Definition

If $F'(x) = f(x)$ for all x in an interval I, then $F(x)$ is called an **antiderivative** of $f(x)$ on I.

Example: The antiderivative of $f(x) = x$ is $F(x) = \frac{x^2}{2}$, because

$$\left(\frac{x^2}{2}\right)' = x.$$

Theorem

If F is an antiderivative of f, then every antiderivative of f has the form $F(x) + C$, where C is a real constant.

Why? Because

$$(F(x) + C)' = F'(x) = f(x).$$

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Definition (Indefinite Integral)

If a function f has an antiderivative, then its **indefinite integral** is the collection of all antiderivatives of f, denoted by

$$\int f(x)\,dx.$$

If F is an antiderivative of f, that is, $F'(x) = f(x)$, then

$$\boxed{\int f(x)\,dx = F(x) + C,}$$

where C is an arbitrary real constant.

Example:

$$\int x^2\,dx = \frac{x^3}{3} + C,$$

since

$$\left(\frac{x^3}{3}\right)' = x^2.$$

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Basic Integration Rules (Part 1)

Power Rule (for $\alpha \neq -1$)

$$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

Constant Rule

$$\int a \, dx = ax + C$$

where a is a constant.

Exponential Functions

$$\int e^x \, dx = e^x + C \quad \text{and} \quad \int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0, \; a \neq 1)$$

Basic Integration Rules (Part 2)

Trigonometric Functions

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

Note: $\sec x = \dfrac{1}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$

Logarithmic Function

$$\int \frac{1}{x} \, dx = \ln|x| + C \quad (x \neq 0)$$

Integration Method: Substitution

This method is inspired by the Chain Rule:

$$\frac{d}{dx}\left(\underbrace{u(v(x))}_{F(x)}\right) = \underbrace{u'(v(x))\,v'(x)}_{f(x)}.$$

$$\Rightarrow \int \underbrace{u'(v(x))\,v'(x)}_{f(x)}\,dx = \int u'(v)dv = \underbrace{u(v(x))}_{F(x)} + C.$$

Example.

$$\int x\cos(x^2)\,dx = \frac{1}{2}\int \cos(x^2)\,d(x^2) = \frac{1}{2}\sin(x^2) + C.$$

Integration Method: By Parts

This method is inspired by the product rule:

$$(\underbrace{uv}_{F})' = \underbrace{u'v + uv'}_{f}.$$

$$\Rightarrow \int \underbrace{(u'v + uv')}_{f}\,dx = \underbrace{uv}_{F} + C \quad \Longrightarrow \quad \int u\,dv = uv - \int v\,du.$$

Example:

$$\int xe^x\,dx$$

Choose parts:

$$\begin{cases} u = x & \Rightarrow \quad du = dx, \\ dv = e^x\,dx & \Rightarrow \quad v = e^x. \end{cases}$$

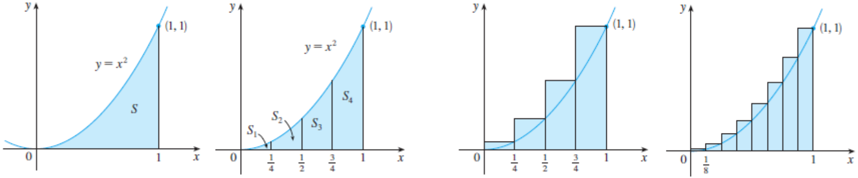
$$\Rightarrow \int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C.$$

Area Between Curves

To motivate the idea of a definite integral, consider the following area problem.

Problem. Find the area of the region S enclosed by the curve $y = x^2$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

A natural way to approximate this area is to divide the interval $[0, 1]$ into small strips and approximate each strip by a rectangle. The more subintervals we choose, the better the approximation becomes.



Example

Find the area of the region S enclosed by the curve $y = x^2$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

Step 1. Divide the interval $[0, 1]$ into N subintervals of equal width:

$$\Delta x = \frac{1 - 0}{N} = \frac{1}{N}$$

Then the endpoints are:

$$x_0 = 0,\ x_1 = \frac{1}{N},\ x_2 = \frac{2}{N},\ \ldots,\ x_i = \frac{i}{N},\ \ldots,\ x_N = 1$$

Step 2. Use the right endpoints $x_i^* = x_i = \frac{i}{N}$, where $i = 1, 2, \ldots, N$.

The function value at each point:

$$f(x_i^*) = \left(\frac{i}{N}\right)^2$$

So the area is approximated by:

$$\text{Area} \approx \sum_{i=1}^{N} f(x_i)\,\Delta x = \sum_{i=1}^{N} \left(\frac{i}{N}\right)^2 \cdot \frac{1}{N} = \frac{1}{N^3} \sum_{i=1}^{N} i^2$$

Recall:

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

Now compute the full expression:

$$\text{Area} \approx \sum_{i=1}^{N} f(x_i^*)\Delta x = \frac{1}{N^3} \cdot \frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6N^2}.$$

Approximation for different values of N:

- $N = 2 : \quad I_2 = \frac{3 \cdot 5}{6 \cdot 4} = \frac{15}{24} = 0.625$
- $N = 4 : \quad I_4 = \frac{5 \cdot 9}{6 \cdot 16} = \frac{45}{96} = 0.46875$
- $N = 6 : \quad I_6 = \frac{7 \cdot 13}{6 \cdot 36} = \frac{91}{216} \approx 0.42130$
- $N = 8 : \quad I_8 = \frac{9 \cdot 17}{6 \cdot 64} = \frac{153}{384} = 0.39844$

Step 3. Take the limit as $N \to \infty$:

$$\text{Area} = \lim_{N\to\infty} \frac{(N+1)(2N+1)}{6N^2} = \frac{1}{3}.$$

Therefore:

$$\text{Area} = \boxed{\frac{1}{3}}.$$

Observation: As $N \to \infty$, the partition becomes finer:

$$\Delta x \to 0 \quad \Rightarrow \quad x_i - x_{i-1} \to 0$$

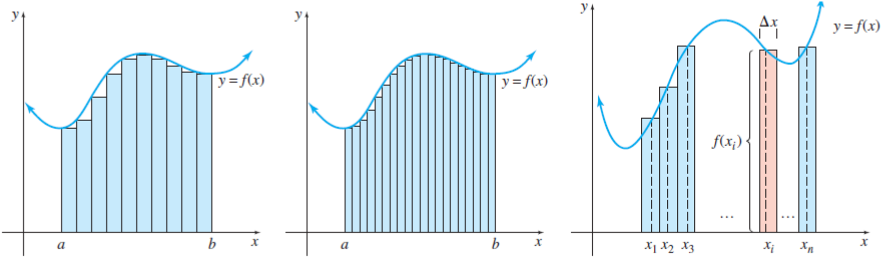
so the function values satisfy

$$f(x_i) \approx f(x_{i-1}).$$

This illustrates how Riemann sums approximate the area under the curve.

Area Between Curves

Generally, consider the region S bounded by $y = f(x) \geq 0$, the x-axis, and the vertical lines $x = a$ and $x = b$, where f is continuous on $[a, b]$.



Ten rectangles of equal width — Twenty rectangles of equal width — n rectangles of equal width

As the number of partitions increases, the rectangular approximations better capture the shape of the region.

Area Between Curves and Riemann Sums

Let $[a, b]$ be divided into N subintervals of width

$$\Delta x = \frac{b-a}{N}.$$

Choose any sample point x_i^* in each subinterval. The area of the i-th approximating rectangle is $f(x_i^*)\,\Delta x$.

The Riemann sum is

$$S_N = \sum_{i=1}^{N} f(x_i^*)\,\Delta x.$$

As $N \to \infty$, the sum approaches the exact area:

$$\boxed{\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)\,\Delta x}$$

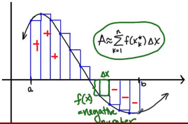
Definite Integral as a Limit of Riemann Sums

We now extend the definition of the definite integral to functions that may take negative values.

- If $f(x) \geq 0$, the integral $\int_a^b f(x)\,dx$ represents the usual geometric area under the curve.
- If f takes both positive and negative values, $\int_a^b f(x)\,dx$ gives the **signed area**: regions above the x-axis contribute positively, regions below contribute negatively.

Hence, the definite integral represents the **net area** between the graph of f and the x-axis over $[a, b]$:

$$\boxed{\text{Net Area} = (\text{area above the } x\text{-axis}) - (\text{area below the } x\text{-axis})}$$



Integrability of Continuous Functions

Theorem

Let $a, b \in \mathbb{R}$*. If* f *is continuous on the closed interval* $[a, b]$*, then* f *is integrable on* $[a, b]$*.*

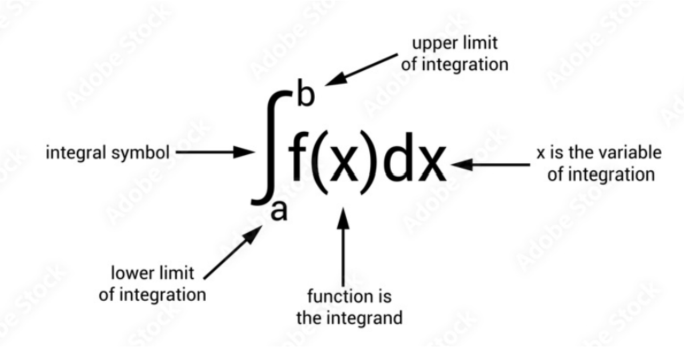
Definition

The **definite integral** of an integrable function f over $[a, b]$ is defined as the limit of its Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(c_i)\,\Delta x,$$

where $\Delta x = \frac{b-a}{N}$ and c_i is any sample point in the i-th subinterval.

Integral Notation



Properties of the Definite Integral

Let f and g be integrable on an interval containing a and b, and α, β be constants.

1. $$\int_a^a f(x)\,dx = 0$$

   Example: $\int_1^1 \sin x\,dx = 0$

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

   Example: $\int_1^0 x^2\,dx = -\int_0^1 x^2\,dx = -1/3$

3. $$\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

   Example: $\int_0^1 (x^2+3)\,dx = \int_0^1 x^2\,dx + 3\int_0^1 1\,dx = 1/3 + 3 = 10/3$

Properties of the Definite Integral

4. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

   Example: $\int_0^{1/2} x^2\,dx + \int_{1/2}^1 x^2\,dx = \int_0^1 x^2\,dx = 1/3$

5. If $f(x) \leq g(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

   Example: $f(x) = x^2, g(x) = 1$ on $[0, 1]$:
   $\int_0^1 x^2\,dx = 1/3 \leq \int_0^1 1\,dx = 1$

6. $\displaystyle\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

   Example:

$$\left|\int_0^{\pi} \sin x\,dx\right| = 2 \leq \int_0^{\pi} |\sin x|\,dx = \int_0^{\pi} 1\,dx = \pi.$$

Properties of the Definite Integral

7. If f is *odd*, meaning $f(-x) = -f(x)$ for all $x \in [-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

Example: $f(x) = x^3 \Rightarrow \int_{-2}^{2} x^3\,dx = 0$

8. If f is *even*, meaning $f(-x) = f(x)$ for all $x \in [-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Example: $f(x) = x^2 \Rightarrow \int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx = \frac{2}{3}$

Fundamental Theorem of Calculus

Computing a Riemann sum directly can often be challenging. However, the Fundamental Theorem of Calculus provides a far more efficient method.

Theorem (Fundamental Theorem of Calculus (Newton–Leibniz))

Let f be continuous on an interval I and let $a \in I$.

1. *Define*
$$F(x) = \int_a^x f(t)\,dt, \qquad x \in I.$$
Then F is differentiable on I and
$$F'(x) = f(x).$$

2. *(Newton–Leibniz formula) If G is any antiderivative of f on I, then for any $b \in I$,*
$$\int_a^b f(x)\,dx = G(b) - G(a) = G(x)\big|_a^b.$$

Example: Evaluate

$$I = \int_0^x t^2\,dt.$$

Solution:

1. **Find an antiderivative:**

$$F(t) = \frac{t^3}{3}.$$

2. **Apply the Fundamental Theorem of Calculus:**

$$I = F(x) - F(0) = \frac{x^3}{3}.$$

Hence, for $x = 1$:

$$\int_0^1 t^2\,dt = \frac{1^3}{3} = \frac{1}{3}.$$

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}.$$

Example: Find the derivative of

$$A(x) = \int_2^x e^{-t^2}\, dt.$$

Solution:

1. Express the integral using an antiderivative $G(t)$:

$$A(x) = \int_2^x e^{-t^2}\, dt = G(t = x) - G(t = 2), \quad \frac{dG}{dt} = e^{-t^2}.$$

2. Differentiate using the chain rule:

$$\begin{aligned} \frac{d}{dx}A(x) &= \frac{d}{dx}\left[G(t = x) - G(t = 2)\right] \\ &= \frac{dG}{dt}\bigg|_{t=x} \times \frac{dt}{dx} - 0 \\ &= e^{-x^2} \times 1 = e^{-x^2} \implies \boxed{A'(x) = e^{-x^2}} \end{aligned}$$

Example: Find the derivative of

$$A(x) = \int_{x^3}^{2} e^{-t^2}\,dt.$$

Solution:

1. Express the integral using an antiderivative $G(t)$:

$$A(x) = \int_{x^3}^{2} e^{-t^2}\,dt = G(t=2) - G(t=x^3), \quad \frac{dG}{dt} = e^{-t^2}.$$

2. Differentiate using the chain rule:

$$\frac{d}{dx}A(x) = \frac{d}{dx}\left[G(t=2) - G(t=x^3)\right]$$

$$= 0 - \left.\frac{dG}{dt}\right|_{t=x^3} \times \frac{dt}{dx}$$

$$= -e^{-(x^3)^2} \times 3x^2 = -3x^2 e^{-x^6} \implies \boxed{A'(x) = -3x^2 e^{-x^6}}$$

Approximate Integration

Some functions, e.g., $\sin(x^2)$, $1/\ln(x)$, $\sqrt{1+x^4}$, do not have elementary antiderivatives.

When an antiderivative is not available, we can approximate the integral by:

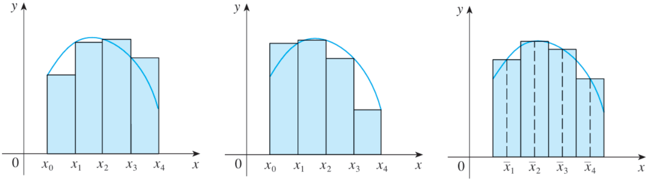
- Partitioning the interval of integration
- Approximating the function with simple polynomials on each subinterval
- Summing the areas of these approximations

This procedure is called **numerical integration**.

Midpoint Rule

To approximate $\int_a^b f(x)\,dx$ with n subintervals of equal length $\Delta x = \frac{b-a}{n}$, denote $x_i = a + i\Delta x, i = 0, 1, ..., n$.

$$\boxed{\int_a^b f(x)\,dx \approx M_n = \Delta x \sum_{i=1}^{n} f(x_i^*), \quad x_i^* = \frac{x_{i-1} + x_i}{2}}$$



Left endpoint approximation Right endpoint approximation Midpoint approximation

Example: Midpoint Rule

Approximate $\int_1^2 \frac{dx}{x}$ using $n = 5$ subintervals.

1. **Partition and endpoints:**

$$a = 1, \quad b = 2, \quad n = 5 \quad \Longrightarrow \quad \Delta x = \frac{2-1}{5} = 0.2$$

$$x_0 = 1,\ x_1 = 1.2,\ x_2 = 1.4,\ x_3 = 1.6,\ x_4 = 1.8,\ x_5 = 2$$

2. **Midpoints:**

$$x_1^* = 1.1, \quad x_2^* = 1.3, \quad x_3^* = 1.5, \quad x_4^* = 1.7, \quad x_5^* = 1.9$$

3. **Midpoint Riemann sum:**

$$\int_1^2 \frac{dx}{x} \approx \Delta x \left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right]$$

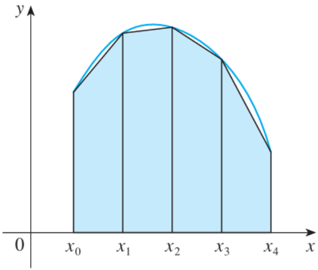
$$= 0.2 \left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right) \approx 0.69191$$

Trapezoidal Rule

The Trapezoidal Rule for n subintervals:

$$\boxed{\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]}$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$.



Example: Trapezoidal Rule

Approximate $\int_1^2 x^2\,dx$ using $n = 4$ subintervals.

1. **Partition and endpoints:**

$$a = 1, \quad b = 2, \quad n = 4 \quad \Longrightarrow \quad \Delta x = \frac{2-1}{4} = 0.25$$

$$x_0 = 1,\ x_1 = 1.25,\ x_2 = 1.5,\ x_3 = 1.75,\ x_4 = 2$$

2. **Function values:**

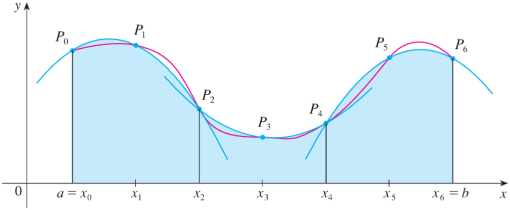
$$f(x) = x^2,\ f(1) = 1,\ \ f(1.25) = \frac{25}{16},\ \ f(1.5) = \frac{9}{4},\ \ f(1.75) = \frac{49}{16},\ \ f(2) = 4$$

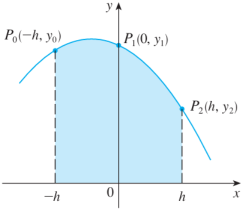
3. **Trapezoidal sum:**

$$T_4 = \frac{\Delta x}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3)) + f(x_4)\right]$$

$$= \frac{0.25}{2}\left[1 + 2\left(\frac{25}{16} + \frac{9}{4} + \frac{49}{16}\right) + 4\right] = \frac{75}{32} = 2.34375$$

Simpson's Rule





The area under the parabola $P_2(x) = Ax^2 + Bx + C$ from $-h$ to h is

$$\int_{-h}^{h} P_2(x)\,dx = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx$$

$$= 2\int_{0}^{h} (Ax^2 + C)\,dx = 2\left[\frac{Ax^3}{3} + Cx\right]_0^h$$

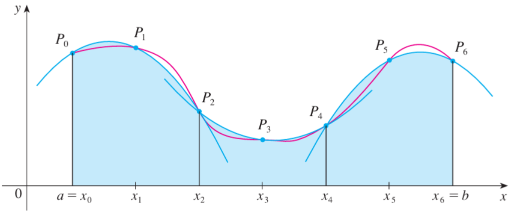
$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

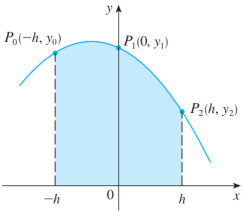
Simpson's Rule

For an even number n of subintervals:

$$\boxed{\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]}$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$.





Example: Simpson's Rule

Approximate $\int_0^2 5x^2\,dx$ using $n = 4$ subintervals.

1. **Partition and endpoints:**

$$a = 0, \quad b = 2, \quad n = 4 \quad \Longrightarrow \quad \Delta x = \frac{2-0}{4} = 0.5$$

$$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2$$

2. **Function values:**

$$f(x) = 5x^2, \qquad f(0) = 0,\ f(0.5) = \frac{5}{4},\ f(1) = 5,\ f(1.5) = \frac{45}{4},\ f(2) = 20$$

3. **Simpson's Rule:**

$$S_4 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{0.5}{3}\left[0 + 4\left(\frac{5}{4}\right) + 2(5) + 4\left(\frac{45}{4}\right) + 20\right] = \frac{40}{3}$$

Computing Definite Integrals

In summary, there are three main approaches to compute a definite integral:

- **By Definition:**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

where

$$x_0 = a, \quad x_n = b, \quad \Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x, \quad x_i^* \in [x_{i-1}, x_i].$$

- **By Theorem:**

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{(Newton–Leibniz formula)}$$

- **By Approximation:** When neither the definition nor an antiderivative is convenient, numerical methods can be applied:
  - Rectangular Approximation Method (RAM)
  - Trapezoidal Rule, or Simpson's Rule, etc.

Exercises: Approximate Integration

Approximate

$$\int_1^2 \sqrt{1+x^3}dx$$

using the following methods with $n = 4, 8, 16$:

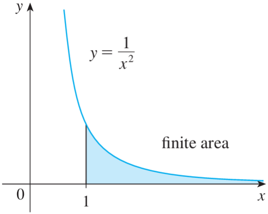
1. Midpoint Rule
2. Trapezoidal Rule
3. Simpson's Rule

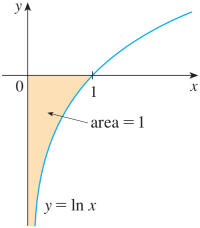
Improper Integrals

So far, definite integrals have required two conditions: a finite domain $[a, b]$ and a bounded integrand.

When one or both conditions fail, the integral is called an **improper integral**:

1. **Type I:** the interval of integration is infinite.
2. **Type II:** the integrand becomes unbounded at a point in the interval.

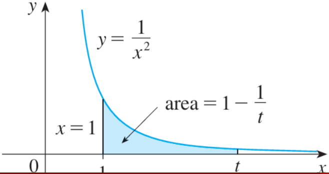




Example: Find the area under $y = \frac{1}{x^2}$ from $x = 1$ to $x = \infty$.

Solution:

1. Consider the area from $x = 1$ to a finite $t > 1$:
$$A(t) = \int_1^t \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}.$$
2. Define the total area as the limit as $t \to \infty$:
$$A = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1.$$
Hence, the area is
$$\int_1^\infty \frac{dx}{x^2} = 1.$$



Definition

An **improper integral of Type I** occurs when the interval of integration is infinite. For a continuous function f:

- On $[a, \infty)$:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

- On $(-\infty, b]$:

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

- On $(-\infty, \infty)$ (split at $c \in \mathbb{R}$):

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

An improper integral is **convergent** if the corresponding limit exists as a finite number, and **divergent** otherwise.

Example: Improper Integral of Type I

Evaluate

$$\int_0^{\infty} e^{-x/2}\,dx.$$

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1. **Express as a limit:**

$$\int_0^{\infty} e^{-x/2}\,dx = \lim_{t\to\infty}\int_0^{t} e^{-x/2}\,dx.$$

2. **Compute the integral over a finite interval $[0,t]$:**

$$I(t) = \int_0^{t} e^{-x/2}\,dx = \left[-2e^{-x/2}\right]_0^{t} = 2\big(1-e^{-t/2}\big).$$

3. **Take the limit as $t\to\infty$:**

$$\int_0^{\infty} e^{-x/2}\,dx = \lim_{t\to\infty} I(t) = 2.$$

Hence, the improper integral is convergent and equals 2.

Example: Improper Integral of Type I

Evaluate

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx.$$

---

1. **Express as a limit:**

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}}\,dx.$$

2. **Compute the integral over a finite interval $[1, t]$:**

$$I(t) = \int_1^t x^{-1/2}\,dx = \left[2\sqrt{x}\right]_1^t = 2\sqrt{t} - 2.$$

3. **Take the limit as $t \to \infty$:**

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\left(2\sqrt{t} - 2\right) = \infty.$$

Hence, the improper integral **diverges**.

Example: Evaluate

$$\int_0^\infty \frac{2x-1}{e^{3x}}\,dx$$

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1. **Express as a limit:**

$$\int_0^\infty \frac{2x-1}{e^{3x}}\,dx = \lim_{t\to\infty}\int_0^t \frac{2x-1}{e^{3x}}\,dx$$

2. **Compute the integral over $[0, t]$:**

$$\int \frac{2x-1}{e^{3x}}\,dx = \int (2x-1)e^{-3x}dx = \left[-\frac{2x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\right] + C$$

so

$$I(t) = \left[-\frac{2x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\right]_0^t = -\frac{2t}{3}e^{-3t} - \frac{1}{9}e^{-3t} + \frac{1}{9}.$$

3. **Take the limit as $t \to \infty$ using L'Hospital's Rule:**

$$\begin{aligned}\lim_{t\to\infty}\left(-\frac{2t}{3}e^{-3t}\right) &= \lim_{t\to\infty}\frac{-2t/3}{e^{3t}} \quad \left(=\frac{\infty}{\infty}\right)\\ &= \lim_{t\to\infty}\frac{-2/3}{3e^{3t}}\\ &= 0\end{aligned}$$

Hence, the improper integral converges:

$$\int_0^\infty \frac{2x-1}{e^{3x}}\,dx = 0 + 0 - \frac{1}{9} = -\frac{1}{9}.$$

Hence, the improper integral is convergent and equals $-\frac{1}{9}$.

Exercises

Evaluate the following improper integrals:

1. $\displaystyle\int_0^{\infty} \frac{2\arctan(2x)}{1+4x^2}\,dx$ *(use substitution method)*
2. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,dx$ *(use integration by parts)*
3. $\displaystyle\int_0^{\infty} xe^{-x}\,dx$ *(use integration by parts)*

Improper Integrals of Type I: p-Integrals

$$\int_1^\infty \frac{dx}{x^p}.$$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b x^{-p}\, dx.$$

- If $p \neq 1$:

$$\int_1^b x^{-p} dx = \frac{b^{1-p} - 1}{1-p}.$$

  Taking the limit as $b \to \infty$:

$$\int_1^\infty x^{-p} dx = \begin{cases} \frac{1}{p-1}, & p > 1 \quad \text{(convergent)} \\ \infty, & p \leq 1 \quad \text{(divergent)} \end{cases}$$

- If $p = 1$,

$$\int_1^b \frac{dx}{x} = \ln b \to \infty \qquad \Rightarrow \text{divergent.}$$

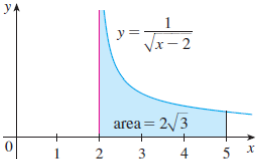
Improper Integrals of Type II: Vertical Asymptotes

If f is unbounded at a point in $[a, b]$, define:

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx \quad \text{or} \quad \lim_{c \to b^-} \int_a^c f(x)dx.$$

Example:

$$\int_2^5 \frac{dx}{\sqrt{x-2}} = \lim_{a \to 2^+} \int_a^5 \frac{dx}{\sqrt{x-2}} = 2\sqrt{3}.$$



Example: Determine convergence of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

Solution:

1. Rewrite using a limit:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-1/2}\,dx.$$

2. Evaluate the integral:

$$\int x^{-1/2}\,dx = 2x^{1/2}.$$

Hence,

$$\lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 = \lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right) = 2.$$

Conclusion:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ is convergent.}$$

Exercises

Evaluate the following improper integrals:

1. $\displaystyle\int_0^3 \frac{dx}{x-1}$
2. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$
3. $\displaystyle\int_0^1 \frac{dx}{(x-1)^{2/3}}$

Comparison Test for Improper Integrals

Theorem (Type I)

Let $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then:

1. *If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges.*
2. *If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ also diverges.*

This test also applies to Type II improper integrals.

Example

Determine convergence of $\int_0^{\infty} e^{-x^2}\, dx$.

Solution:

1. Split the integral:

$$\int_0^{\infty} e^{-x^2}\, dx = \int_0^{1} e^{-x^2}\, dx + \int_1^{\infty} e^{-x^2}\, dx$$

2. Estimate each part:
   - $\int_0^{1} e^{-x^2}\, dx \leq \int_0^{1} 1 dx = 1 \quad \text{(convergent)}$
   - $\int_1^{\infty} e^{-x^2}\, dx \leq \int_1^{\infty} e^{-x}\, dx = e^{-1} \quad \text{(convergent)}$

Conclusion: $\int_0^{\infty} e^{-x^2}\, dx$ is convergent.

Exercises

Determine whether each integral is convergent or divergent:

1. $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2+1}\,dx$ (use comparison test)
2. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ (use substitution method)
3. $\displaystyle\int_1^\infty \frac{x+1}{x^2+2x}\,dx$ (use comparison test)
4. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$ (use substitution method)
5. $\displaystyle\int_{-\infty}^\infty \frac{dx}{4x^2+4x+5}$

– The End of Topic –

Thank You!