1 Gaussian elimination and matrices

1.1 Solving linear systems using elimination method

Recall: Common method to solve linear systems using Gaussian/Gauss-Jordan elimination:

1) Construct coefficient matrix and augmented matrix for the system
2) Using *row elementary operations* on matrices to reduce the matrix to a (the) (reduced) row-echelon form.
3) Using back-substitution or Gauss-Jordan elimination to solve.

*E*xercises:

1. Which of the following are linear equations in x_1, x_2 and x_3?

   (a) $x_1 + 5x_2 - \sqrt{x}_3 = 1$

   (b) $x_1^{-2} + x_2 + 8x_3 = 5$

   (c) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{1/3}$

   (d) $x_1 - 3x_2 + \frac{\sqrt{4-\sqrt{32}}}{\sqrt{5}}x_3 = 2$

2. Determine coefficient matrix and augmented matrix for the following linear equation systems. Circle their pivot positions and determine their free variables. Using back-substitution to solve them.

   (a)

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\ x_2 - 3x_3 + 3x_4 + \quad x_5 &= -5 \\ x_4 - \quad x_5 &= 4 \end{aligned}$$

   (b)

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ x_3 + x_4 &= 3 \\ x_4 &= 5 \end{aligned}$$

3. Given augmented matrices of linear equation systems. Determine which matrices are in row-echelon form. If it is not, please transform it to row-echelon form and use back-substitution method for solving the corresponding linear equation systems.

   (a) $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 & -1 & 3 & 0 & 4 \\ 0 & 0 & 3 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 5 & 6 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 0 & 0 & 0 & 7 \\ 1 & 2 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$

(d)
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 11 \\ 0 & 2 & 0 & 2 & 2 & 12 \\ 0 & 0 & 0 & 3 & 3 & 13 \\ 0 & 0 & 0 & 0 & 4 & 14 \end{bmatrix}$$

4. Rewriting the matrix-forms into systems of equations. Then solving the systems

(a)
$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5. Solving the following linear systems using Gaussian or Gauss-Jordan (back-substitution) elimination. Write clearly all elementary row operations you used.

(a)
$$\begin{aligned} x + y + 2z &= 0 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0. \end{aligned}$$

(b)
$$\begin{aligned} 2x + 2y + 2z &= 0 \\ -2x + 5y + 2z &= 1 \\ 8x + y + 4z &= -1. \end{aligned}$$

(c)
$$\begin{aligned} x + 2y - t + w &= 1 \\ 3y + z - w &= 2 \\ z + 7t &= 1. \end{aligned}$$

(d)
$$\begin{aligned} x - 2y + z - 4t &= 1 \\ x + 3y + 7z + 2t &= 2 \\ x - 12y - 11z - 16t &= 5. \end{aligned}$$

(e)
$$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - \quad 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10. \end{aligned}$$

(f)
$$\begin{aligned} 2u - v &= 0 \\ -u + 2v - w &= 0 \\ -v + 2w - z &= 0 \\ -w + 2z &= 5. \end{aligned}$$

6. For which values of a will the following system have no solutions? Exactly one solution? Infinitely

many solutions?

$$
\begin{aligned}
x + 2y - 3z &= 4 \\
3x - y + 5z &= 2 \\
4x + y + (a^2 - 14)z &= a + 2.
\end{aligned}
$$

1.2 Operations on matrices

Recall:

1) Essential matrices: Identity, zero matrix, symmetric matrix...
2) Operations: addition, scalar multiply, transposition, multiplication, inverse.
3) General properties: Associative, commutative, distributive

Exercises:

7. Suppose that A, B, C, D, E are matrices of the following sizes: $A : 4 \times 5$; $B : (4 \times 5)$; $C : 5 \times 2$; $D : 4 \times 2$; $E : 5 \times 5$. Determine (if there exists) the size of the following matrices:

   (a) BA;
   (b) $AC + D$;
   (c) $AE + B$;
   (d) $AB + B$;
   (e) $2E(A + B)$;
   (f) $E(AC)$;
   (g) $E^t A$;
   (h) $(A^t + E)D$.

8. Let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}.$$

   Compute the following (where possible)

   (a) $2B - C$;
   (b) $3D - 2E^t$;
   (c) $3D^t - 2E)$ and $Tr(3D - 2E^t)$;
   (d) AB and BA;
   (e) AC and CA;
   (f) $(C^t B)A^t$ and $Tr((C^t B)A^t)$;
   (g) $tr(DD^t)$.
   (h) $D^t E^t - (ED)^t$.

9. Write down the 2 by 2 matrices A and B that have entries $a_{ij} = i + j$ and $b_{ij} = (-1)^{i+j}$. Multiply them to find AB and BA. Is the product of A and B commutative?

10. True or false? Give a specific counterexample when false.

    (a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB.
    (b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB.
    (c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of AB.
    (d) $(AB)^2 = A^2 B^2$.

11. Which of the following matrices are guaranteed to equal $(A + B)^2$

(a) $A^2 + 2AB + B^2$

(b) $A(A+B) + B(A+B)$

(c) $(A+B)(B+A)$

(d) $A^2 + AB + BA + B^2$

12. By trial and error find examples of 2 by 2 matrices such that
    (a) $A^2 = -I$, A having only real entries.
    (b) $B^2 = 0$, although $B \neq 0$;
    (c) $CD = -DC$, not allowing the case $CD = 0$.
    (d) $EF = 0$, although no entries of E or F are zero.

13. Three companies A, B and C are producing a new kind of moose food. In the beginning, the market shares are as follows: A has 40 percent, B has 20 percent and C has 40 percent. During the first year, A keeps 85% of its customers, loses 5% to B an 10% to C. B keeps 75% of its customers, loses 15% to A and 10% to C. C keeps 90% and loses 5% to A and 5% to B.

    Determine a matrix showing the changes of the market shares during the first year and calculate the market shares at the end of the first year.

14. Determine a, b, c such that
$$\begin{pmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 0 \end{pmatrix}$$
    is symmetric.

15. Suppose A commutes with every 2 by 2 matrix (that is $AB = BA$), and in particular
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
    Show that $a = d$ and $b = c = 0$. Consequently, prove that if $AB = BA$ for all matrices B, then A is a multiple of the identity.

16. In each part find matrices A, X, B which express the given system of linear equations as a single matrix equation $AX = B$. Solve those equations.

    (a)
$$\begin{aligned} x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 \quad + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

    (b)
$$\begin{aligned} x_1 \qquad - 3x_3 + x_4 \quad &= 7 \\ 5x_1 + x_2 \qquad - 8x_4 \quad &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 \quad &= 0 \\ 3x_2 \quad - x_3 + 7x_4 &= 2 \end{aligned}$$

17. Find the powers A^2, A^3, B^2, B^3, C^2, C^3. What are A^k, B^k and C^k for a given k?

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } C = AB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

1.3 Inverse matrix

Recall: For finding the inverse of a matrix, we can either

- use elementary row operations to bring $[A|I]$ into $[I|A^{-1}]$, or
- use determinants and calculate adjoint matrices.

Exercise:

18. Show that if A and B are invertible matrices then
    (a) A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$
    (b) AB are invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

19. Use the Gauss-Jordan method to invert the following matrices then solve the equations $Ax = b$ for $b = (-1, 2, 7)$.

    (a)
    $$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

    (c)
    $$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

    (e)
    $$A_5 = \begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

    (b)
    $$A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

    (d)
    $$A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

    (f)
    $$A_6 = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{10} \end{bmatrix}$$

20. True or false (with a counterexample if false and a reason if true):
    (a) A 4 by 4 matrix with a row of zeros is not invertible.
    (b) If A is invertible then A^{-1} is invertible
    (c) If A^t is invertible, then A is invertible.

21. If a matrix A has row 1 + row 2 = row 3, show that A is not invertible:
    (a) Explain why $Ax = (1, 0, 0)$ cannot have a solution.
    (b) Which right-hand sides (b_1, b_2, b_3) might allow a solution to $Ax = b$?
    (c) What happens to row 3 in elimination?

22. Find the inverse (in any legal way) of

    (a)
    $$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix},$$

    (b)
    $$A_2 = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

(c)

$$A_3 = \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$$

23. For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

24. Give examples of matrices A and B such that

    (a) $A + B$ is not invertible although A and B are invertible.

    (b) $A + B$ is invertible although A and B are not invertible.

    (c) all of A, B, and $A + B$ are invertible.

    (d) In the last case use $A^{-1}(A + B)B^{-1} = B^{-1} + A^{-1}$ to show that $C = B^{-1} + A^{-1}$ is also invertible and find a formula for C

25. Show that $A^2 = 0$ is possible but $A^t A = 0$ is not possible (unless $A =$ zero matrix).

26. If the inverse of A^2 is B, show that the inverse of A is AB. Thus, A is invertible whenever A^2 invertible.

27. If $A = A^t$ and $B = B^t$, which of these matrices are certainly symmetric?

    (a) $A^2 - B^2$;

    (b) $(A + B)(A - B)$;

    (c) ABA;

    (d) $ABAB$.

1.4 Determinants

28. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Assume that $\det(A) = 7$, find

(a) $\det(3A)$;

(b) $\det(2A^{-1})$;

(c) $\det((2A)^{-1})$;

(d) $\det A^t$;

(e) $\det A^2$;

(f) $\left| \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix} \right|$.

29. Prove the identity without evaluating the determinants

(a) $\det \begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = 0$;

(b) $\begin{bmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$;

(c) $\begin{bmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{bmatrix} = -2 \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$;

30. Evaluate determinants by cofactor expansion along a row or column of your choice:

(a) $\begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$

(d) $\begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

31. Find the inverse of the following matrices by calculating its cofactors

(a) $\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$

32. Let $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$.

(a) Evaluate A^{-1} using its cofactors;

(b) Evaluate A^{-1} using the elimination method.

33. Solving equations using Cramer's Rule

(a) $\begin{cases} 7x_1 - 2x_2 & = 3 \\ 3x_1 + x_2 & = 5 \end{cases}$

(b) $\begin{cases} 4x + 5y & = 3 \\ 11x + y + 2z & = 3 \\ x + 5y + 2z & = 1 \end{cases}$

(c) $\begin{cases} 2x_1 + x_2 & = 3 \\ x_1 + 2x_2 + x_3 & = 70 \\ x_2 + 2x_3 & = 0 \end{cases}$

(d) $\begin{cases} x_1 - 3x_2 + x_3 & = 4 \\ 2x_1 - x_2 & = -2 \\ 4x_1 - 3x_3 & = 0 \end{cases}$

34. Let $v = (3, 2)$ and $w = (1, 4)$.

  (a) Find the area of the parallelogram with edges v and w;
  (b) Find the area of the triangle with sides v, w and $v + w$. Draw it.
  (c) Find the area of the triangle with sides v, w and $w - v$. Draw it.
  (d) The corners of a triangle are $(2, 1)$, $(3, 4)$ and $(0, 5)$. What is it area?

35. Knowing that the volume spanned by vectors $\{u, v, w\}$ in $\mathbb{R}^3$ is equal to 10. What is the volume spanned by vectors

  (a) $\{u, v, 2w\}$?
  (b) $\{u, v, 2w - 3u - v\}$?

2 Vector spaces

2.1 Vector spaces and subspaces

36. By checking the vector space axioms, determine which sets together with the two operations (scalar and addition) defined respectively are vector spaces:

    (a) The set of all triples of real numbers (x, y, z) with the following operations $(x, y, z)+(x', y', z') = (x + x', y + y', z + z')$ and $k \cdot (x, y, z) = (kx, y, z)$;

    (b) The set of all triples of real numbers (x, y, z) with the following operations $(x, y, z)+(x', y', z') = (x + x', y + y', z + z')$ and $k \cdot (x, y, z) = (0, 0, 0)$;

    (c) The set of all triples of real numbers (x, y, z) with the following operations $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1, z_1 + z_2 + 1)$ and $k \cdot (x, y, z) = (kx, ky, kz)$;

    (d) The set of all pairs of real numbers (x, y) with the following operations $(x, y) + (x', y') = (x + x', y + y')$ and $k \cdot (x, y) = (2kx, 2ky)$;

    (e) The set of all pairs of real numbers of the form (x, y) where $x \geq 0$, with the standard operations on $\mathbb{R}^2$;

    (f) All 2 by 2 matrices with the matrix addition and scalar multiplication;

    (g) The set of singular 2 by 2 matrices with the matrix addition and scalar multiplication;

    (h) The set of non-singular 2 by 2 matrices with the matrix addition and scalar multiplication;

    (i) All non-singular matrices 2 by 2 with the matrix scalar multiplication and the addition is the matrix multiplication.

    (j) The set of all 2 by 2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with the matrix addition and scalar multiplications;

    (k) The set of all 2 by 2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with the matrix addition and scalar multiplications;

    (l) All one variable polynomials of degree 2 with the scalar multiplication and addition are the scalar multiplication and additions in polynomials.

37. Which of the following subsets of R^3 are actually subspaces?

    (a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

    (b) The plane of vectors b with $b_1 = 1$.

    (c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).

    (d) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.

    (e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

38. Which of the followings are subspaces of $\mathbb{R}^\infty$?

    (a) All sequences like $(1, 0, 1, 0, \dots)$ that include infinitely many zeros?

(b) All sequences $x_1, x_2, \dots$ with $x_j = 0$ from some point onward?

(c) All decreasing sequences $x_{j+1} \leq x_j$ for each j?

(d) All convergent sequences?

(e) All arithmetic progression $x_{j+1} - x_j$ is the same for all j?

(f) All geometric progression $(x_1, kx_1, k^2x_1 \dots,)$ allowing all k and x_1?

39. Given

$$W = \{(6a - b, a + b, -7a) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}.$$

(a) Prove that W is a subspace of $\mathbb{R}^3$.

(b) Find a spanning set for W.

40. Given vectors in $\mathbb{R}^3$:

$$v_1 = (1, -1, -2), \quad v_2 = (5, -4, -7), \quad , v_3 = (-3, 1, 0), \quad v = (-4, 3, h).$$

For which value of h will $v \in span\{v_1, v_2, v_3\}$?

41. Which of the following descriptions are correct? The solutions x of

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

form

(a) a plane;

(b) a line;

(c) a point;

(d) a subspace;

(e) the nullspace of A;

(f) the column space of A.

42. Describe the smallest subspace of the 2 by 2 matrix space M that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

43. Describe the null space of the matrices

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}, \quad E = \begin{bmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{bmatrix}.$$

44. For which vectors (b_1, b_2, b_3) is each system below consistent?

(a) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$

(c) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

2.2 Solving $Ax = 0$ and $Ax = b$

45. Using Gauss or Gauss-Jordan elimination to find the rank of each given matrix. In each case, specify which variables are free? Describe the nullspaces. Solve the complete solution of the system $Ax = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

    (a) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$;

    (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

    (c) $\begin{bmatrix} 1 & 3 & -2 & 2 \\ -1 & -2 & -1 & -1 \\ -1 & -5 & 8 & -3 \end{bmatrix}$;

    (d) $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 4 & 0 \\ -1 & -1 & -4 & 1 \end{bmatrix}$.

46. What conditions on b_1, b_2, b_3, b_4 make each system solvable? Solve for x.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

47. If $Ax = b$ has two distinct solutions x_1 and x_2, find two solutions to $Ax = 0$. Then find another solution to $Ax = b$.

48. Suppose that you know x_p (free variables of equation $Ax = 0$) and all special solutions for $Ax = b$. Find x_p and all special solutions for these systems:

$$Ax = 2b, \quad \begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = b, \quad \begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}.$$

49. Write all known relations among r, m and n if $Ax = b$ has

    (a) no solution for some b;

    (b) infinite many solutions for every b;

    (c) exactly one solution for some b, no solution for other b.

    (d) exactly one solution for every b.

2.3 Linear independence, dependence, spanning, bases and dimension

50. By using the definitions, show that v_1, v_2, v_3 are linearly independent but v_1, v_2, v_3, v_4 are dependent:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

51. Which of the followings are bases for $\mathbb{R}^3$?
    (a) $(1, 2, 0)$ and $(0, 1, -1)$?
    (b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$?
    (c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$?
    (d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$?

52. True or False
    (a) The vectors $(1, 3, 2), (2, 1, 3)$ and $(3, 2, 1)$ are independent;
    (b) The vectors $(1, -3, 2), (2, 1, -3), (-3, 2, 1)$ are independent;
    (c) The two vectors $(1, 1, -1)$ and $(-1, -1, 1)$ span $\mathbb{R}^3$; Describe the subspace spanned by two these vectors. Find its dimension.
    (d) The three vectors $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ span $\mathbb{R}^3$ and it forms a basis of $\mathbb{R}^3$;
    (e) $\mathbb{R}^3$ is spanned by the columns of a 3 by 5 echelon matrix with 2 pivots?
    (f) All vectors with positive components forms a spanning system for $\mathbb{R}^3$.

53. Given vectors $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$ in $\mathbb{R}^3$. Let $H = \{(s, s, 0) : s \in \mathbb{R}\}$.
    (a) Prove that H is a subspace of $span\{v_1, v_2\}$.
    (b) Is $\{v_1, v_2\}$ a basis for H?
    (c) Find a basis for H.

54. Given $v_1 = (0, 2, -1)$, $v_2 = (2, 2, 0)$, $v_3 = (6, 16, -5)$.
    (a) Is $\{v_1, v_2, v_3\}$ linearly independent?
    (b) Find a basis for $span\{v_1, v_2, v_3\}$.

55. Find a basis for
    (a) the set of points on the line $y = -3x$
    (b) the space of solutions of $x - 3y + 2z = 0$

56. Given vectors $S = \{1 + t, 1 - t, 2\}$ in $P_1(t)$.
    (a) Prove that S is linear dependent.
    (b) Find a basis for $span(S)$.

57. Let w_1, w_2, w_3 be linearly independent vectors. Are the following vectors independent or dependent? If they are dependent, find a their non-zero linear combination that gives zero. Does the claims hold if the given vectors w_1, w_2, w_3 are dependent.

   (a) $v_1 = w_1 - w_2, v_2 = w_2 - w_3, v_3 = w_3 - w_1$;

   (b) $v_1 = w_1 + w_2, v_2 = w_2 + w_3, v_3 = w_3 + w_1$.

58. Decide whether or not the following vectors are linearly independent, by solving the equation $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$, where

$$v_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \qquad v_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \qquad v_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \qquad v_4 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}.$$

   Then decide also if they span $\mathbb{R}^4$. If it not, find the largest number of independent vectors among them and find the dimension of the subspace spanned by these vectors.

59. To decide whether b is in the subspace spanned by $w_1, w_2, \ldots, w_m$, let A be matrix having column vectors w and try to solve the equation $Ax = b$. What is the result for

   (a) $w_1 = (1, 1, 0), w_2 = (2, 2, 1), w_3 = (0, 0, 2), b = (3, 4, 5)$;

   (b) $w_1 = (1, 2, 0), w_2 = (2, 5, 0), w_3 = (0, 0, 2), w_4 = (0, 0, 0)$ and any b?

60. For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix}1 & 2 & 5 & 0 & 5\\0 & 0 & c & 2 & 2\\0 & 0 & 0 & d & 2\end{bmatrix}, \qquad U = \begin{bmatrix}c & d\\d & c\end{bmatrix}.$$

61. Find a basis for each of these subspaces of $\mathbb{R}^4$:

   (a) All vectors whose components are equal;

   (b) All vectors whose components add to zero.

   (c) All vectors that are perpendicular to $(1, 1, 0, 1)$ and $(1, 0, 1, 1)$.

   (d) The column space (in $\mathbb{R}^2$) and null space (in $\mathbb{R}^5$) of $U = \begin{bmatrix}1 & 0 & 1 & 0 & 1\\0 & 1 & 0 & 1 & 0\end{bmatrix}$. Find their dimensions.

2.4 Linear Transformations

Recall: Let T be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. T is linear transformation if T satisfies three properties:

i) $T(0) = 0$;

ii) $T(x + y) = T(x) + T(y)$ for any $x, y \in \mathbb{R}^n$;

iii) $T(cx) = cT(x)$.

Exercises:

62. Which of these transformations is not linear? The input is $v = (v_1, v_2)$. Find its matrix with respect to the standard basis $(1, 0)$ and $(0, 1)$?

    (a) $T(v) = (v_2, v_1)$,
    (b) $T(v) = (v_1, v_1)$,
    (c) $T(v) = (0, v_1)$,
    (d) $T(v) = (0, 1)$,
    (e) $T(v) = (2v_1 + v_2, v_1^2)$.

63. Suppose a linear T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$. Find $T(v)$ when

    (a) $v = (2, 2)$ (b) $v = (3, 1)$ (c) $v = (-1, 1)$ (d) $v = (a, b)$.

64. For these transformations of $V = \mathbb{R}^2$ to $W = \mathbb{R}^2$. Find $T(T(v))$

    (a) $T(v) = -v$;
    (b) $T(v) = v + (1, 1)$;
    (c) $T(v) = 90^o$ rotation $= (-v_2, v_1)$;

65. Find the range and kernel of T:

    (a) $T(v_1, v_2) = (v_2, v_1)$;
    (b) $T(v_1, v_2, v_3) = (v_1, v_2)$;
    (c) $T(v_1, v_2) = (0, 0)$;
    (d) $T(v_1, v_2) = (v_1, v_1)$.

66. From the cubics $\mathbb{P}_3$ to the fourth-degree polynomials $\mathbb{P}_4$, what matrix represents multiplication by $2 + 3t$? The columns of the 5 by 4 matrix A come from applying the transformation to $1, t, t^2, t^3$.

67. The space of all 2 by 2 matrices has the four basis "vectors":

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

    For the linear transformation of *"transposing"*, find its matrix A with respect to this matrix. Why is $A^2 = I$?

2.5 Reviews

Goal: Understand the equation $Ax = b$: the existence of solutions, the space of solutions (or nullspace) in case it has infinite solutions (find the dimension, the basis of the solution space); For which b the equation has a solution, the space of the vectors b such that $Ax = b$ (or the column space) has at least one solution. Find solutions of a linear equation, the inverse matrix, the dimension, the basis...

68. Given $A \in M_{4\times 7}$ with $r(A)$. Determine $\dim(null(A))$, $\dim(row(A))$, $r(A^T)$, $\dim(null(A^T))$.

69. Given $A \in M_{6\times 8}$ with 4 pivot columns. Find $r(A)$, $\dim(null(A))$.

70. Given $A = \begin{matrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{matrix}$ and $u = (3, -2, -1, 0)$ and $v = (3, -1, 3)$.

    - Determine if $u \in null(A)$? Could u be in $col(A)$?
    - Determine if $v \in col(A)$? Could v be in $null(A)$?

71. Find the basis and dimension for the following subspaces of $\mathbb{R}^4$:

    (a) The vectors for which $x_1 = 2x_4$;
    (b) The vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$;
    (c) The subspace spanned by $(1, 1, 1, 1), (1, 2, 3, 4), (2, 3, 4, 5)$;

72. What is the echelon form U of A? Find the dimensions of its four fundamental subspaces?

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & -2 \end{bmatrix}.$$

73. (a) $Ax = b$ has a solution under what conditions on b, for the following A and b:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

    (b) Find a basis for the null-space of A;
    (c) Find a basis for the column space of A;
    (d) Find the rank of A^T;
    (e) Find the general solution to $Ax = b$, when a solution exists;

74. Find the inverse of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. Then solve the equation $Ax = (0, 1, 2)$.

75. Write down the matrix representation of the following linear maps relative to the natural bases of $\mathbb{R}^n$:

    (a) $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by $T(x_1, x_2, x_3, x_4) = (x_2, x_3)$;
    (b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2 - x_3, 0, 3x_4)$;
    (c) $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = 3x$;