Answer keys: Linear Equation Systems

1. (a) No
   (b) No
   (c) Yes
   (d) Yes

2. (a) Coefficient matrix: $\begin{pmatrix} 1 & -7 & 2 & -5 & 8 \\ 0 & 1 & -3 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$

   Augmented matrix: $\begin{pmatrix} 1 & -7 & 2 & -5 & 8 & 10 \\ 0 & 1 & -3 & 3 & 1 & -5 \\ 0 & 0 & 0 & 1 & -1 & 4 \end{pmatrix}$

   Free variables: x_3, x_5; Dependent variables: x_1, x_2, x_4

   Solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -89 \\ -17 \\ 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 19 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -31 \\ -4 \\ 0 \\ 1 \\ 1 \end{pmatrix} x_5$, where x_3 and x_5 are arbitrary.

   (b) Coefficient matrix: $\begin{pmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   Augmented matrix: $\begin{pmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix}$

   Free variables: x_2; Dependent variables: x_1, x_3, x_4

   Solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -17 \\ 0 \\ -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2$, where x_2 is arbitrary.

3. (a) Yes/Inconsistent
   (b) Yes/$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} -\frac{10}{3} \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, where y, t are arbitrary.
   (c) No/$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$/Inconsistent
   (d) Yes/Inconsistent.

4. (a) $x = 5, y = -2, z = 4$
   (b) $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ -3 \\ 1 \end{pmatrix} t$, where t is arbitrary.
   (c) $\begin{pmatrix} x \\ y \\ z \\ t \\ v \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} -4 \\ 0 \\ -3 \\ -5 \\ 1 \end{pmatrix} v$, where y, v are arbitrary.

(d) $\begin{pmatrix} x \\ y \\ z \\ t \\ v \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} y + \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} t$, where y, t are arbitrary.

5. (a) $x = -17, y = 11, z = 3$.

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} \\ \frac{1}{7} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{7} \\ -\frac{4}{7} \\ 1 \end{pmatrix} z$, where z is arbitrary.

(c) $\begin{pmatrix} x \\ y \\ z \\ t \\ w \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{11}{3} \\ \frac{7}{3} \\ -7 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ 0 \\ 0 \\ 1 \end{pmatrix} w$, where t, w are arbitrary.

(d)

(e) $I_1 = -1; I_2 = 0; I_3 = 1; I_4 = 2$.

(f) $u = 1, v = 2, w = 3, z = 4$.

6.
- The system has exactly one solution if and only if $a \neq \pm 4$.
- The system has no solution if and only if $a = -4$.
- The system has infinitely many solutions if and only if $a = 4$.