Exercise Sheet: Linear Equation Systems

Recall: Common method to solve linear systems using Gaussian/Gauss-Jordan elimination:

i) Construct a matrix (called augmented matrix) corresponding to the system

ii) Using *elementary row operations* on matrices to reduced the matrix to the (reduced) row-echelon form.

iii) Using back-substitution elimination to solve.

1. Which of the following are linear equations in x_1, x_2 and x_3?

   (a) $x_1 + 5x_2 - \sqrt{x}_3 = 1$

   (b) $x_1^{-2} + x_2 + 8x_3 = 5$

   (c) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{1/3}$

   (d) $x_1 - 3x_2 + \frac{\sqrt{4-\sqrt{32}}}{\sqrt{5}}x_3 = 2$

2. Determine coefficient matrix and augmented matrix for the following linear equation systems. Circle their pivot positions and determine their free variables. Using back-substitution to solve them.

   (a)
$$\begin{aligned} x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\ x_2 - 3x_3 + 3x_4 + x_5 &= -5 \\ x_4 - x_5 &= 4 \end{aligned}$$

   (b)
$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ x_3 + x_4 &= 3 \\ x_4 &= 5 \end{aligned}$$

3. Given augmented matrices of linear equation systems. Which matrices has the corresponding coefficient matrix in a row-echelon form. If it is not, please transform it to row-echelon form and use back-substitution method for solving the corresponding linear equation systems.

   (a) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

4. Rewriting the matrix-forms into systems of equations. Then solving the systems

(a)

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 6 & 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5. Solving the following linear systems using Gaussian or Gauss-Jordan (back-substitution) elimination. Write clearly all elementary row operations you used.

(a)

$$\begin{aligned} x + y + 2z &= 0 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0. \end{aligned}$$

(b)

$$\begin{aligned} 2x + 2y + 2z &= 0 \\ -2x + 5y + 2z &= 1 \\ 8x + y + 4z &= -1. \end{aligned}$$

(c)

$$\begin{aligned} x + 2y - t + w &= 1 \\ 3y + z - w &= 2 \\ z + 7t &= 1. \end{aligned}$$

(d)

$$\begin{aligned} x - 2y + z - 4t &= 1 \\ x + 3y + 7z + 2t &= 2 \\ x - 12y - 11z - 16t &= 5. \end{aligned}$$

(e)

$$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10. \end{aligned}$$

(f)

$$\begin{aligned} 2u - v &= 0 \\ -u + 2v - w &= 0 \\ -v + 2w - z &= 0 \\ -w + 2z &= 5. \end{aligned}$$

6. For which values of a will the following system have no solutions? Exactly one solution? Infinitely many solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2. \end{aligned}$$