Exercise Sheet: Linear Maps

1. Determine which of the following maps L are linear.

   (a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x, y, z) = (3x - 2y + z, 3x - 2y)$.

   (b) $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(x, y, z) = (x, y - 1, z)$.

   (c) $L : \mathbb{R}^2 \to \mathbb{R}$ defined by $L(x, y) = xy$.

   (d) $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x, y) = (x, -y)$ (the reflection by x-axis).

   (e) $L : \mathbb{R}^2 \to \mathbb{R}$ defined by $L(x, y) = \max\{x, y\}$.

   (f) $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x, y) = (0, 1)$.

   (g) $L : M_{m\times n} \to M_{n\times m}$ defined by $L(A) = A^T$.

2. Suppose a linear T transforms $(1, 1)$ to $(2, 2)$ and $(2, 0)$ to $(0, 0)$.

   (a) Find $T(2, 2)$.

   (b) Find $T(3, 1)$.

   (c) Find the matrix representation for T with respect to the standard basis of $\mathbb{R}^2$.

   (d) $v = (a, b)$, for $a, b \in \mathbb{R}$.

   (e) Find $T^2(2, 2)$.

3. Find the matrix representation with respect to standard bases for the following linear maps

   (a) the projection from the space $\mathbb{R}^3$ onto the $x - y$ plane.

   (b) the reflection from the space $\mathbb{R}^3$ through the $x - y$ plane.

   (c) $L : \mathbb{R}^2 \to \mathbb{R}^3$ such that $L(x, y) = (2x - y, 3x + 4y, x - 5y)$.

   (d) the ϕ-rotation from $\mathbb{R}^2$ to itself.

4. Find the range and kernel of T:

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(v_1, v_2) = (v_2, v_1)$;

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(v_1, v_2, v_3) = (v_1, v_2)$ (the projection);

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(v_1, v_2) = (0, 0)$;

   (d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(v_1, v_2) = (v_1, v_1)$;

   (e) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (2x - y, 3x + 4y, x - 5y)$.

5. Let $L : M_{n\times n} \to M_{n\times n}$ be the map such that $L(A) = \frac{A-A^T}{2}$.
   (a) Show that L is linear.
   (b) Describe the kernel of L, and determine its dimension.
   (c) What is the image of L?
6. For the following transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. Find $T(T(v))$
   (a) $T(v) = -v$;
   (b) $T(v) = v + (1, 1)$;
   (c) $T(v) = 90^o$ rotation $= (-v_2, v_1)$;
7. Let $L : V \to W$ be a linear map such that $\ker(L) = \{0\}$. Show that if $v_1, v_2, \ldots, v_n$ are linearly independent of V, then $L(v_1), L(v_2), \ldots, L(v_n)$ are linearly independent elements of W.
8. Let $L : V \to W$ be a linear map on finite dimensional spaces V and W. Then,
$$\dim V = \dim \operatorname{Ker}(L) + \dim \operatorname{Im}(L).$$