Exercise Sheet: Matrices

1. Compute
$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 3 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 & 1 \\ 2 & 5 \\ 1 & 6 \end{pmatrix}$$

2. Compute
$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 3 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$$

3. Compute
$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 5 \\ 1 & 6 \end{pmatrix}$$

4. Compute
$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix}$$

5. Compute
$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 3 \end{pmatrix}^T$$

6. Compute
$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

7. Let
$$A = \begin{pmatrix} 1 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \; D = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 1 & 3 \end{pmatrix}$$

Compute

(a) $A \cdot B$

(b) $B \cdot A$

(c) $C \cdot D$

(d) $D \cdot C^T$

(e) $D^T \cdot C^T$

8. Let A and B be quadratic matrices. Compute

   (a) $(A + B)^2$

   (b) $(A - B)^2$

   (c) $(A + B)(A - B)$

   (d) $(A - B)(A + B)$

9. Three companies A, B and C are producing a new kind of moose food. In the beginning, the market shares are as follows: A has 40 percent, B has 20 percent and C has 40 percent. During the first year, A keeps 85% of its customers, loses 5% to B an 10% to C. B keeps 75% of its customers, loses 15% to A and 10% to C. C keeps 90% and loses 5% to A and 5% to B.

   Determine a matrix showing the changes of the market shares during the first year and calculate the market shares at the end of the first year.

10. Determine a, b, c such that

$$\begin{pmatrix} 2 & a - 2b + 2c & 2a + b + c \\ 3 & 5 & a + c \\ 0 & -2 & 0 \end{pmatrix}$$

   is symmetric.

11. Compute $A^T - 5B + 3C^T$ for

$$A = \begin{pmatrix} 3 & 4 & 0 \\ -1 & 5 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -3 & 3 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

12. Compute $(AB)^T$ and $B^T A^T$ for

$$A = \begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 1 \\ -3 & 2 & 5 \end{pmatrix}, \ B = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 5 \\ 2 & 3 & 8 \end{pmatrix}$$

13. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -5 \end{pmatrix}, \ B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Show:

$$AB = BA \Leftrightarrow a - c = 7b$$

14. Show that if A and B are matrices of the same size then

    (a) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$;

    (b) AB are invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

15. True or false (with a counterexample if false and a reason if true)

    (a) A 4 by 4 matrix with a row of zeros is invertible.

    (b) If a matrix A is invertible then its inverse A^{-1} is invertible too

    (c) If the transposed matrix A^T is invertible, then A is invertible.

16. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Assume that $\det(A) = 7$, find

(a) $\det(3A)$;

(b) $\det(2A^{-1})$;

(c) $\det((2A)^{-1})$;

(d) $\det(A^T)$;

(e) $\det(A^2)$;

(f) $\begin{pmatrix} a & g & d \\ b & h & e \\ c & i & f \end{pmatrix}$

17. Prove the identity without evaluating the determinants

(a) $\det \begin{pmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{pmatrix} = 0;$

(b) $\det \begin{bmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{bmatrix} = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$

(c) $\det \begin{pmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2] \\ a_3+b_3 & a_3-b_3 & c_3 \end{pmatrix} = -2\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix};$

18. Compute the following determinants:

(a)
$$\begin{vmatrix} 0 & -1 & 3 \\ 2 & -6 & 4 \\ 3 & 1 & 2 \end{vmatrix}$$

(b)
$$\begin{vmatrix} 7 & 2 & 8 \\ 1 & -5 & 1 \\ 2 & 64 & 4 \end{vmatrix}$$

(c)
$$\begin{vmatrix} 2 & 4 & 0 & 8 \\ 0 & 5 & 2 & 8 \\ 3 & 1 & 1 & 4 \\ 1 & 6 & 1 & 4 \end{vmatrix}$$

(d)

$$\begin{vmatrix} 2 & -3 & 0 & 1 \\ 1 & -4 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 3 & -5 \end{vmatrix}$$

(e)

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

(f)

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 \end{vmatrix}$$

(g)

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$

19. For which $x, y, z \in \mathbb{R}$ is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & y & 1 \\ 1 & 1 & 1 & z \end{vmatrix} = 0?$$

20. Compute the inverse of the matrix A below then solve the

system $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix}.$$

21. Compute the area of the triangle, determined by $(1,2)$, $(2,3)$ and $(3,5)$.

22. Solve the equation system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & - & 4x_4 & = & 2 \\ x_1 & - & 2x_2 & + & x_3 & - & 2x_4 & = & -2 \\ 3x_1 & - & 2x_2 & + & 5x_3 & - & 8x_4 & = & -2 \\ 3x_1 & + & 2x_2 & + & 7x_3 & - & 10x_4 & = & 2 \end{array}$$

23. For which values of a is the equation system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & - & 2x_2 & + & x_3 & = & 2 \\ x_1 & & & + & 2x_3 & = & a \end{array}$$

solvable? Solve the system for these a.

24. Solve the equation system

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & a \\ x & + & y & + & 3z & = & a+2 \\ 2x & - & 2y & + & (a+1)z & = & a+1 \end{array}$$

for all real values of a.

25. Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

(a) Compute the inverse A^{-1}.

(b) Determine the solutions to $A\vec{x} = \vec{b}$ for

$$\vec{b} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

(c) Determine the solutions to $A\vec{x} = \vec{b}$ for

$$\vec{b} = \begin{pmatrix} 10 \\ 5 \\ -5 \\ -5 \end{pmatrix}$$

26. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 3 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

(a) Determine the inverse matrices A^{-1} and B^{-1}.

(b) Determine for each of the tasks below a matrix X with

i. $A + X = B$

ii. $AX = B$

iii. $XA = B$

iv. $AX + B = 2A$

v. $AXB = B$

vi. $AXB + A = B$

27. Are the following matrices regular? Determine the inverse matrix, it it exists.

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 32 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 1 & -1 & -3 \\ 2 & -2 & -6 \\ 2 & 1 & -3 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 2 & 2^2 & 2^3 \\ 0 & 1 & 2 & 2^2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$$

28. Solve the following linear equation systems with Cramer's rule:

(a)

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 4x_3 & = & 1 \\ & & & & x_3 & = & 3 \\ 3x_1 & - & 3x_2 & + & 5x_3 & = & 0 \end{array}$$

(b)

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 7 \\ 4x_1 & + & 3x_2 & + & 2x_3 & = & 0 \\ x_1 & + & 3x_2 & - & x_3 & = & -2 \end{array}$$

29. Solve the following linear equation systems with Gauss-Jordan elimination:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 2x_2 & + & 8x_3 & = & -4 \\ x_1 & - & 2x_2 & & & = & 2 \end{array}$$

30. (a) Find the area of the parallelogram spanned by $(1,2)$ and $(3,4)$.

(b) Knowing that the area of the parallelogram spanned by two vectors u and v in the plane is 10. Find the area of the parallelogram spanned by the vectors $3u+v$ and $u-2v$.

(c) Find the volume of the parallelepiped spanned by the vectors $(1,3,0)$, $(2,-1,-1)$, and $(4,1,0)$.