

Exercise Sheet: Relations

1. Let \mathbb{Z} be the set of all integers.
Describe the set $\mathbb{Z} \times \mathbb{Z}$.

2. Let $|A| = m$ and $|B| = n$.
How many relations exist from A to B ?

3. Is

$$f(x) = \frac{x}{x+1}$$

invertible?

4. For which real numbers a is the function

$$f(x) = ax, \quad x \in \mathbb{R}$$

invertible? Determine the inverse function.

5. Let R_m be a relation of the integers \mathbb{Z} defined by:

$$(x, y) \in R_m \Leftrightarrow m|(x - y)$$

Interpretation: $(x, y) \in R_m$ if and only if x and y have the same remainder when divided by m .

We denote this by:

$$x \equiv y \pmod{m}$$

i.e. x is congruent with y modulo m .

Example: We have $1 \equiv 11 \pmod{10}$, since both 1 and 11 have the remainder 1 when divided by 10.

Your task: Investigate if R_5 is reflexive, symmetric, antisymmetric and transitive!

6. Is the equality relation, i.e. $=$, an equivalence relation?
7. Is the relation $>$ an equivalence relation?
8. Is the relation \geq an equivalence relation?
9. Let

$$A = \{ \text{all positive integers divisible by 2 and not exceeding 30} \}.$$

(a) Find $|A|$

(b) Let

$B = \{ \text{all positive integers divisible by 6 and not exceeding 30} \}.$

$C = \{ \text{all positive integers divisible by 8 and not exceeding 30} \}.$

Find $B \cup C$, $B \cap C$, and binary string representations for B , C , $B \cup C$, $B \cap C$, \bar{B} .

10. Let $M_{n \times n}$ be the set of matrices of size $n \times n$. Let

$$R = \{(A, B) : A, B \in M_{n \times n} \text{ and there exists an invertible matrix } P \text{ such that } A = P.B.P^{-1}\}.$$

Show that R is an equivalence relation.

11. Given A the set of all webpages. Let $R = \{(a, b) \in A \times A : \text{there is at least 1 common link on } a \text{ and } b\}$. Investigate following properties of R

(a) reflexive

(b) symmetric

(c) transitive

(d) antisymmetric

(e) equivalent

(f) partial order

12. Given $A = \{1, 2\}$. Let

$$R = \{(B, C) \in 2^A \times 2^A : B \subseteq C\}.$$

(a) Find matrix representation M_R for R and $|R|$.

(b) Investigate reflexive (symmetric, antisymmetric, transitive) properties of R .

13. Given a set $A = \{a, b, c\}$ and a relation R with matrix representation

$$M_R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) Find R^{-1} and its matrix representation

(b) Find \bar{R} and its matrix representation

(c) Find R^2 and its matrix representation

(d) Find $R \cup S$, $R \circ S$ and $S \circ R$ where

$$S = \{(a, b), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

14. Given matrix representations of relations

$$M_{R_1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_{R_2} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Determine whether R_1 and R_2 are equivalence relations or not.

15. Let R be the relation on the set of all cities in the world such that (a, b) in R if there is a direct non-stop airline flight from a to b . When is (a, b) in

- (a) R^2
- (b) R^3
- (c) R^{-1}

16. Let R be the relation $\{(a, b) : a \neq b\}$ on the set of integers. What is the reflexive closure of R ?

17. Let R be the relation $\{(a, b) : a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R ?

18. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When (a, b) in

- (a) R^2
- (b) R^3
- (c) R^*

19. Given the matrix representation of the relation R on $\{a, b, c, d\}$ as following

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Find the transitive closure of R using naive and Warshall's algorithms.

20. Given the relation $R = \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$ on $\{1, 2, 3, 4\}$. Find

- (a) Reflexive closure of R
- (b) Symmetric closure of R
- (c) Transitive closure of R using naive algorithm and Warshall's algorithm

- (d) Reflexive transitive closure of R
 - (e) Equivalence closure of R .
21. Given the relation $R = \{(1, 2), (1, 4), (3, 3), (4, 1)\}$ on $\{1, 2, 3, 4\}$. Find
- (a) reflexive and transitive closure of R
 - (b) symmetric and transitive closure of R
 - (c) equivalence closure of R .
22. We say that two matrices A and B of size $n \times n$ are *similar* if there is an invertible matrix P such that $A = P^{-1}.B.P$. Show that the similar relation, which is defined by pairs (A, B) such that A is similar to B , is an equivalence relation.
23. Given the relation R on the set of all bit strings such that $(s, t) \in R$ if and only if s and t contain the same number of 1s.
- (a) Prove that R is an equivalence class.
 - (b) List all bit strings of length 4 equivalent to 01001.
 - (c) How many bit strings of length n with exactly 2 occurrences of 1s are there?
24. Let R be the relation on the set ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.
25. Given matrix representation for relations on $A = \{a, b, c\}$ below. Determine if the relations are partial ordering relation or not?
- (a) $M_{R_1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
 - (b) $M_{R_2} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
 - (c) $M_{R_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
26. Let R be a partial ordering relation on A . Show that the inverse relation R^{-1} is a partial ordering relation on A .
27. Ordering the following strings in the lexicographic order:

0, 01, 11, 001, 010, 011, 0001, 0101.

28. Draw Hasse diagram for the following posets

- (a) $A = \{2, 4, 5, 10, 12, 20, 25\}$ with the divisibility relation.
- (b) $B = 2^{\{a,b,c\}}$ with the containment relation.
- (c) $C = \{0, 01, 11, 001, 010, 011, 0001, 0101\}$ with the lexicographic ordering relation.