Exercise Sheet: Vector spaces

1. Given
$$W = \{(6a - b, a + b, -7a) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}.$$
   (a) Prove that W is a subspace of $\mathbb{R}^3$.
   (b) Find a spanning set for W.

2. Given vectors in $\mathbb{R}^3$:
$$v_1 = (1, -1, -2), \quad v_2 = (5, -4, -7), \quad , v_3 = (-3, 1, 0), \quad v = (-4, 3, h).$$
   For which value of h will $v \in span\{v_1, v_2, v_3\}$?

3. Which of the following descriptions are correct? The solutions x of
$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
   form
   (a) a plane;
   (b) a line;
   (c) a point;
   (d) a subspace;
   (e) the nullspace of A;
   (f) the column space of A.

4. For which vectors (b_1, b_2, b_3) is each system below consistent?
   (a) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$
   (c) $\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

5. Decide whether or not the following vectors are linearly independent, by solving the equation $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$, where
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then decide also if they span $\mathbb{R}^4$. If it not, find the largest number of independent vectors among them and find the dimension of the subspace spanned by these vectors.

6. Which of the followings are bases for $\mathbb{R}^3$?
   (a) $(1, 2, 0)$ and $(0, 1, -1)$?
   (b) $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$?
   (c) $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$?
   (d) $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$?

7. True or False
   (a) The vectors $(1, 3, 2), (2, 1, 3)$ and $(3, 2, 1)$ are independent;
   (b) The vectors $(1, -3, 2), (2, 1, -3), (-3, 2, 1)$ are independent;
   (c) The two vectors $(1, 1, -1)$ and $(-1, -1, 1)$ span $\mathbb{R}^3$;
   (d) The three vectors $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ span $\mathbb{R}^3$ and it forms a basis of $\mathbb{R}^3$;
   (e) $\mathbb{R}^3$ is spanned by the columns of a 3 by 5 echelon matrix with 2 pivots?

8. Given vectors $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$ in $\mathbb{R}^3$. Let $H = \{(s, s, 0) : s \in \mathbb{R}\}$.
   (a) Prove that H is a subspace of $span\{v_1, v_2\}$.
   (b) Is $\{v_1, v_2\}$ a basis for H?
   (c) Find a basis for H.

9. Given $v_1 = (0, 2, -1)$, $v_2 = (2, 2, 0)$, $v_3 = (6, 16, -5)$.
   (a) Is $\{v_1, v_2, v_3\}$ linearly independent?
   (b) Find a basis for $span\{v_1, v_2, v_3\}$.

10. Find a basis for
    (a) the set of points on the line $y = -3x$ in the space $\mathbb{R}^3$;
    (b) the space of s of $x - 3y + 2z = 0$ in the space $\mathbb{R}^3$.

11. Given a matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & -3 & 2 \\ 4 & -2 & 2 \end{pmatrix}$

(a) Show that the solution set to the system $Ax = 0$ forms a vector subspace of $\mathbb{R}^3$.

(b) Find a basis and the dimension for that space.

12. Let w_1, w_2, w_3 be linearly independent vectors. Are the following vectors independent or dependent? If they are dependent, find a their non-zero linear combination that gives zero. Does the claims hold if the given vectors w_1, w_2, w_3 are dependent.

    (a) $v_1 = w_1 - w_2, v_2 = w_2 - w_3, v_3 = w_3 - w_1$;

    (b) $v_1 = w_1 + w_2, v_2 = w_2 + w_3, v_3 = w_3 + w_1$.

13. Find a basis for each of these subspaces of $\mathbb{R}^4$:

    (a) All vectors whose components are equal;

    (b) All vectors whose components add to zero.

    (c) All vectors that are perpendicular to $(1, 1, 0, 1)$ and $(1, 0, 1, 1)$.

    (d) The column space (in $\mathbb{R}^2$) and null space (in $\mathbb{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.
    Find their dimensions.