PROOF, RULES AND MEANING: THE CASE OF NECESSITY AND IDENTITY

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My aim in this talk is to explore the upshot of my ongoing research project on *Proof, Rules and Meaning* to our understanding of the concepts of *necessity* and *identity*, and especially, their interaction. Other papers that are a part of this project can be downloaded from my website, at https://consequently.org/writing/. For background see, especially, [34–38].

1 MY APPROACH

The bulk of my work in philosophical logic has come in three phases. The first, from my PhD onward, focussed on non-classical logics, the semantic paradoxes, and so-called substructural logics [30]. The second, from the late 1990s, explored *logical pluralism* [3, 4, 31], the idea that there can be more than just "one true logic". The third, from the 2010s, has centred around the interaction between *proof*, *rules* and *meaning* [32–38].

NORMATIVE PRAGMATISM: Setting this project in its wider context, this project can be understood as broadly *normative pragmatist*, [9,10,21], in the sense that fundamental semantic features of a language are understood as *rules* for *use*. To take utterances or inscriptions to have meaning is to interpret them, not only as causal indicators of features their environment, but to understand them as governed by semantic rules.

SEMANTIC ANTI-REALISM: The traditional logical vocabulary, of concepts like 'and', 'if', 'not', 'all', and so on, are distinctive because we can understand rules that govern them not (at first, at least) by way of reference to some pre-existing independent reality, but by understanding their function in some other way. If the semantic *realist* understands the meaning of an expression in terms of some underlying reference relation (or any kind of language – world relation), then on this approach (at least at first) the semantics of these logical expressions is semantically *anti-realist* [7, 8, 17]. This does not mean that words like "and" or "not" have an expressivist semantics in the sense that their use expresses some inner attitude (in the way that the expressivist concerning moral vocabulary, or epistemic modals, thinks that "good" or "might" do [2]), but it does mean that the semantics is *not* given, in the first instance, by way of taking a piece of logical vocabulary as having as its semantic value a truth function, or some such abstract object.[1]

DIALOGICAL: Finally, the approach is *dialogical* and *communicative* [18]. Language is, first and foremost, a social and communicative phenomenon, and any account of linguistic meaning must start here. The social practice is filled with different kinds of acts: asserting, denying, supposing, conjecturing, questioning, answering, proposing, withdrawing, suggesting, asking, commanding, and much, much more. For the philosopher and for the logician it is unsurprising that *assertion* and *inference* are seen to be the heart of our linguistic practice, in the same way that belief is thought to be central to cognition, but assertion is by no means all there is to our communicative practice [5].[2]

Since my "Multiple Conclusions" [32], I have argued that we get a useful perspective on the semantics of our logical vocabulary (and much more) if we keep 'score' in dialogue in the following way. Instead of focussing only on what has been explicity stated by a participant of a dialogue, we should keep track of what has been *ruled in* and what has been *ruled out*. I use the notation '$[X : Y]$' for a *position* in which X is a collection of claims that have been *asserted* and Y is a collection of claims that have been denied.[3] Key constraints on such positions are that a position like $[X, A : A, Y]$, where the one and the same claim A has been asserted and has been denied, is self-undercutting, or *out of bounds*, no matter what else is in the set X (no matter what else has been asserted) and what is in Y (what else has been denied).[4] Furthermore, I have argued for the following principle: if $[X : A, Y]$ is out of bounds and $[X, A : Y]$ is out of bounds, then so is $[X : Y]$, since if we have taken up an available position $[X : Y]$ and it would be *out of bounds* to assert A, then the polar question $\ulcorner A?\urcorner$ is implicitly answered in the negative, so adding it as a denial would merely make explicit what was implicit in our commitments.[5] We use the notation $X \succ Y$ to make the claim that the position $[X : Y]$ is out of bounds. It is a *sequent*, in the logicians' sense. There is also a connection between sequents and inference, which can further be explored. If we have $X \succ A, Y$, then there is an important sense in which we can *infer* the conclusion A from a position $[X : Y]$. We have a way to answer *yes* to the question $\ulcorner A?\urcorner$ from our prior commitments.

The notion of the *bounds* for a language is quite general, and it makes no commitment at all to the kind of claims that are being made in the underlying vocabulary, or what it takes for those claims to be *true* or *false*. Regardless, we have enough at hand to understand what it might take to add new vocabulary to a language. The key idea of a *defining rule* for an item of vocabulary is that it *interprets* speech acts using the new vocabulary in terms of the bounds given in the original vocabulary. It shows how one is to treat claims involving the new vocabulary, if one already has the means to interpret the original vocabulary, to which the new

[1] Though, this is not to say that there is no connection between logical connectives and truth functions. However, from the semantic anti-realist perspective, any such connection is something to be *discovered*, and not the starting point for your theory of meaning [34].

[2] Brandom, for example, puts it this way: "Inferential practices of producing and consuming *reasons* are *downtown* in the region of linguistic practice. Suburban linguistic practices utilize and depend on the conceptual contents forged in the game of giving and asking for reasons, are parasitic on it" [10, p. 14]. The emphasis on assertion, denial and inference in my work should not be understood as endorsing Brandom's account of the importance of reasons and inference, though it is consistent with that view of the lay of the land.

[3] It is important to clarify the senses of 'assert' and 'deny' in play here [15,19,39]. If the assertion of p can be expressed equally by a 'yes' answer to the polar question $\ulcorner p?\urcorner$ then the salient sense of denial is that expressed by the '*no*' answer to that same question.

[4] This is called the *identity* rule in the sequent calculus: it has the form $X, A \succ A, Y$.

[5] This is called the *Cut* rule in the sequent calculus. *Cut* is defensible, but not universally agreed-upon [40, 41], but I will not relitigate that argument here.

concept is added. The defining rule for *conjunction* has this form:

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\,{\wedge}Df$$

This is to be understood as a two-way rule. It says that a position in which the conjunction $A \wedge B$ is asserted is (in the *new* vocabulary) out of bounds if and only if the position involving assertions of A and of B are out of bounds. If A and B themselves don't also contain the introduced symbol '$\wedge$', this is a question settled by the *old* vocabulary. It is a way to *define* the significance of claims involving the logicians conjunction '$\wedge$', saying that making a *conjunctive* assertion has the same force as asserting both conjuncts.

This defining rule is not a 'definition-by-paraphrase' that asks you to replace $A \wedge B$ everywhere by the two claims A, B, since we have also said that declarative sentences may be denied as well as asserted. One can *deny* $A \wedge B$, and this does not have the same force as denying A or denying B, but has a new kind of significance, one that may be new to the language. This is the *value* of having a concept like conjunction that we can express: with it we can *do* things that we cannot do without it. What does our rule say about the significance of denying a conjunction? Using the rules of *Identity* and *Cut* we can see that the expected behaviour of '$\wedge$' emerges from the defining rule:

$$\dfrac{X \succ A, Y \qquad \dfrac{X \succ B, Y \qquad \dfrac{A \wedge B \succ A \wedge B}{A, B \succ A \wedge B}\,{\wedge}Df}{X, A \succ A \wedge B, Y}\,Cut}{X \succ A \wedge B, Y}\,Cut$$

In this little derivation, we see first that $[A, B : A \wedge B]$ is out of bounds—so denying $A \wedge B$ clashes with asserting A and asserting B, as you would expect—and *if* A and B both follow from some position $[X : Y]$, so does $A \wedge B$, as you would also expect. The usual 'introduction' and 'elimination' rules follow from this one defining rule.[6] What goes for conjunction goes for other logical vocabulary, such as negation:

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\,{\neg}Df$$

according to which the assertion of a negation has the force of the denial of the thing negated; and the material conditional:

$$\frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y}\,{\rightarrow}Df$$

for which denying $A \rightarrow B$ has the same force as asserting A and denying B, or equivalently, to *prove* $A \rightarrow B$ you suppose A (add it, temporarily, to the stock of things granted in your position) and prove B.

If the original language is stratified into *predicates* and *singular terms* then we can add rules for *quantifiers*, too. A simple defining rule for a universal quantifier goes like this:

$$\frac{X \succ Fn, Y}{X \succ \forall x Fx, Y}\,{\forall}Df$$

where in the conclusion of the rule, the singular term n does not appear. The key idea here is that the universal quantifier expresses *generality*. To prove that everything is F you prove that n is F, making no assumptions at all about what n is. If you *deny* that everything is F, this is out of bounds if and only if denying Fn is out of bounds, where n is a fresh singular term. (Consider the moves we make in dialogue: I deny that *everything* has some feature. You're well within your rights to say: well, suppose that n doesn't have that feature ... and if you can reduce that position to absurdity, you've undercut my claim.) The distinctive behaviour of the universal quantifier comes from the fact that if I have proved $\forall x Fx$ using this rule, then I have a proof for Fn where the name n is arbitrary, and so, that proof could apply to this object, or that object, or anything to which I would like to apply the claim $\forall x Fx$.[7] This kind of defining rule is slighgtly less general than the rules for the propositional connectives, like $\wedge$, $\neg$ and $\rightarrow$: for it to apply to a language, that language must be regimented into predicates and singular terms.

In each case, with these defining rules we can see that the defined concept 'makes explicit' in our vocabulary something that was implicit (joint assertion, the duality between assertion and denial, or the inference of a conclusion from a premise, generality, etc.) in our practice before the addition of the concept.

It is appropriate to call these concepts *defined* by way of these rules because, like definitions by paraphrase, their addition is *conservative* over the old vocabulary (no position in the old vocabulary is rendered out of bounds if it was not out of bounds already—the original vocabulary is undisturbed), and the addition is *uniquely defined* (if we add *two* concepts rather than one, using a rule of the same form, say two conjunctions: $\wedge$ and $\&$, then these are strongly equivalent in the sense that we have immediate derivations from $A \wedge B$ to $A \;\&\; B$ and back). Rules like these do just what we expect from definitions by paraphrase, while extending the language's expressive power.

Furthermore, each of these rules for our sharply defined concepts, like $\wedge$, $\neg$, $\rightarrow$, $\forall$, involve the kinds of moves in dialogue that we *do* for our everyday concepts such as 'and', 'not', 'if' and 'all',[8] and the defining rules themselves are simple, well-behaved, useful and tractable. One way to understand the concepts given by these rules is that they are Carnapian *explications* of our everyday concepts, and the logicians' introduction of rules like these is a kind of conceptual engineering. Here, though, the conceptual engineering is not achieved by shifting the extension of some predicate or the denotation of a singular term from a worse candidate to a better one (connectives and quantifiers aren't predicates or singular terms, after all), but by making explicit inference rules in this controlled and well-defined way. When is having such sharply delineated concepts valuable? It's especially valuable in those places where the *boundaries* are most important.[9] Whenever we want to understand what *can* be stated and what *cannot*, or what can be proved and what is beyond our grasp to confirm, the kind of precision and explicitness available in defining rules like these is most welcome.

2 THE QUESTIONS

Now at last I can come to the questions I want to address. I am interested in shedding some light on two concepts: necessity, and identity, and the interaction between them. Both are much stud-

[6] In general terms, defining rules of this shape produce introduction and elimination rules that are in 'harmony', unlike the pair of rules for Prior's infamous connective 'tonk' [26].

[7] There is much more to be said here about how generality is to be understood, and about whether this rule is appropriate in a language with 'defective' singular terms. For that discussion, see "Generality and Existence 1" [36].

[8] The match is nowhere near perfect in the case of 'if', of course [28,30], but this does not mean that the material conditional is not a useful concept to have.

[9] The analogy of the precision of the rules in backyard cricket compared to international Test cricket comes to mind.

ied by logicians, of course, and the concept of *identity* is most often taken to be a logical concept fundamental to first-order predicate logic. However, as straightforward and simple as the logic of identity might be, its addition to the *modal* logic of possibility and necessity raises a number of significant questions.

HOW DO WE GRASP THE CONCEPT OF NECESSITY? As philosophers, we not only regularly *make use* of the concepts of possibility and necessity—some of us have views about the meanings of those terms. To make things concrete, I will focus on the concepts of possibility and necessity as applied in locutions of the form "it is possible that p" ($\Diamond p$) and "it is necessary that p" ($\Box p$). Interest in modal *logic* in its current form exploded in the mid-20th Century, when the 'new' techniques in logic—brewing in the latter part of the 19th Century and gaining currency in the first part of the 20th—began to be applied to the concepts of possibility and necessity, by pioneers such as work of C. I. Lewis [22], Georg von Wright [43], Arthur Prior [25] and many others. The idea of 'possible worlds' has a long and interesting history [11, 12], and since the mid-1960s, the tools of *possible worlds semantics* became ubiquitous in the study of modality. The core idea of a possible worlds *model* for the concepts of possibility and necessity is simple and straightforward. Instead of sentences being true or false *simpliciter*, we model them as being true or false at a range of *worlds*. A sentence $\Box A$ is true at a particular world if and only if A is true at *every* world (in that model). A sentence $\Diamond A$ is true at a particular world if and only if A is true at *some* world (in that model).[10]

Models like these have proved invaluable for working with the concepts of possibility and necessity. They are extremely useful for exploring the logical relationships between modal claims, and for training our understanding. In recent years, possible worlds semantics and modal logic in general has proved useful not only for our understanding of what philosophers call *metaphysical possibility* (where possible worlds are understood as different *ways things could have gone*), but also *epistemic possibility* (where possible worlds are understood as different *ways things could be—for all we know*).[11] This distinction is marked in the grammar of English. Consider the difference between the subjunctive conditional 'if Oswald *hadn't* shot Kennedy' (we consider different ways things could have turned out, had Oswald not shot Kennedy—maybe Kennedy would have lived to a ripe old age), and the indicative conditional 'if Oswald *didn't* shoot Kennedy' (I suppose, then, that somene else shot Kennedy). The modal notions of possibility and necessity have application in both these 'metaphysical' and 'epistemic' interpretations.[12]

The concepts of possibility and necessity raise *many* questions. One that I *won't* broach is the question of the metaphysics of possible worlds. *One* way to take these models seriously as models of reality is to take there to really *be* some collection of possible worlds 'out there' [23]. My question, though, is different. The concepts of necessity and possibility are important to us, and we seem to have access to them in some way or other. How do we grasp those concepts? How *could* we grasp those concepts? Do we grasp those concepts by having some prior grasp of the concept of possible worlds, and the concept of truth in a possible world? According to Arthur Prior, the answer to that last question is 'no'.[13]

> . . . to say that something is true in a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case . . . We understand 'truth in states of affairs' because we understand 'necessarily'; not *vice versa*. [27]

For Prior, we understand what it is for something to be true in a possible world by first grasping the concepts of possibility and necessity. I agree. But this more questions. How *do* we grasp those concepts? And why do the concepts that we have grasped have the structure that makes possible worlds *models* such a good model for them? That can't be a coincidence, can it?

HOW DO WE GRASP THE CONCEPT OF IDENTITY? Compared to the concepts of possibility and necessity (in both their metaphysical and epistemic form), the concept of *identity*, concieved of as a binary relation, seems simple. As David Lewis said:

> Identity is utterly simple and unproblematic. Everything is identical to itself; nothing is ever identical to anything else except itself . . . There might be a problem about how to define identity to someone sufficiently lacking in conceptual resources—we note that it won't suffice to teach him certain rules of inference—but since such unfortunates are rare, even among philosophers, we needn't worry much if their condition is incurable. [23, pp. 192, 193]

Yes, in one sense identity is simple and unproblematic. However, the slogan that Lewis uses: "Everything is identical to itself; nothing is ever identical to anything else except itself" to help pin down the notion does not do a very good job at it. If we grant that "except x" is a paraphrase for "not identical to x" then the slogan says: "Everything bears relation X to itself; nothing ever bears relation X to anything to which it bears relation X". This does not say much about relation X beyond the claim that it is reflexive. (The universal relation, which relates anything to anything at all, satisfies this slogan, as does any other reflexive relation.) So, if as Lewis says, we cannot define the relation of identity using inference rules, then this raises a curious question: how do we grasp the concept of identity? How can we be sure that we *aren't* one of the unfortunates that lacks the concept of identity and has grasped the *wrong thing*? Is it just innate to a well-functioning mind, or is there more we can say?

HOW DO IDENTITY AND NECESSITY INTERACT? Understanding the concept of identity a little more clearly would help when it comes time to use the concept of identity and the concept of necessity in the same breath. For combining them raises interesting and challenging questions all of their own, as Ruth Barcan Marcus showed [24]. Barcan Marcus systematically explored the interaction between the *modal logic* (developed in the first part of the 20th Century in the work of C. I. Lewis and others), with the emerging treatment of the *quantifiers* and *identity* in what was coming to be known as first-order predicate logic *with identity*. She was the first to clearly specify the way that modal predicate logic raises important questions such as the necessity or contingency of identity statements.

[10] There are various bells and whistles to be added for a notion of *relative* possibility, where some worlds are possible relative to others, but this detail is not so important for my purposes in this talk.

[11] Philosophers have invariably focused on 'metaphysical' possibility [23, 42]. On the other hand, there is a thriving industry in *epistemic logic* that uses modal logic for just this purpose [16].

[12] This distinction is parallel to the distiction between objective and subjective probabilities. A subjective probability is some kind of degree of belief, while an objective probability is some kind of measure of 'how things are' in some empirical setup.

[13] Here, Prior uses 'possible state of affairs' instead of 'possible world', which has become more common.

- If x is identical to y, is it *necessary* that x is identical to y, or could they have been distinct? (In formal notation: do we have $x = y \to \Box(x = y)$?)
- If it is *possible* that there is something with feature F, does it follow that there is something that *possibly* has that feature? (Do we have $\Diamond\exists xFx \to \exists x\Diamond Fx$?)
- Do these questions have the same answer for metaphysical and epistemic senses of the modal operators?
- On what *basis* could we answer these questions? Where should we look for answers?

More questions like these arise, but these will be enough to be going on with for now.

3 DEFINING IDENTITY

Since identity is the simpler of the concepts, let's start there. Lewis was too pessimistic in his claim that there was no way to define identity using inference rules.[14] Stephen Read has shown us how to do so in his 2004 paper "Identity and Harmony" [29]. The rules Read presents in that paper are not quite defining rules in my sense, but they are close enough to massage into my preferred form:

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y}\ {=}Df$$

Here, the predicate F in the premises of the rule is required to be *arbitrary*, in just the same way that the name n was arbitrary in the rule $\forall Df$. The idea is simple. To *prove* $a = b$, we must prove that any feature a has, b has too, and *vice versa*.[15] To deny that $a = b$ means that we take there to be *some* feature (possibly not expressible in my vocabulary, but some feature nevertheless) which holds of one and not the other. Identity is, therefore, a kind of logical indistinguishability, but this does not mean that one has to grasp the concept of "indistinguishability" in order to grasp the concept of identity defined in this way. Rather, you need to take distinguishing a from b (by means of some feature or other) as a conclusive reason to deny $a = b$, and to take a to be identical to b when you have proved that any feature a has, b has, and vice versa. No more, no less.[16]

The concept defined by $=Df$ is easily shown to be reflexive, symmetric and transitive, and furthermore, if the predicate F is truly *general* (in the sense that logical rule that holds of F holds of any other predicate, too) then if $a = b$ and $A(a)$, we have $A(b)$ for any sentence $A(x)$ with free variable x.[17] This rule seems to fit well with the precise notion of a binary *identity* predicate in Lewis's sense, and the defining rule is more informative than Lewis' slogan.

Furthermore, this defining rule can be shown to be *uniquely defining* under certain natural conditions. If *you* have a predicate $=$ given by the defining rule $=Df$, and *I* have a predicate $\approx$ given by a defining rule of the same shape, then if we attempt to communicate, we see that our predicates are equivalent. If your '$=$' is now a predicate in my language, and *vice versa*, we can reason as follows:[18]

$$\dfrac{\dfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a \approx a}\ {\approx}Df \qquad \dfrac{a = b \succ a = b}{a \approx a, a = b \succ a \approx b}\ {=}Df}{a = b \succ a \approx b}\ Cut$$

$$\dfrac{\dfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a = a}\ {=}Df \qquad \dfrac{a \approx b \succ a \approx b}{a = a, a \approx b \succ a = b}\ {\approx}Df}{a \approx b \succ a = b}\ Cut$$

4 DEFINING NECESSITY

Let's turn to the concept(s) of necessity. We can start with the fact that in any discourse with modal vocabulary, we are likely to find different kinds of *supposition*, introducing kinds of *context shift*. We say things like:

Oswald shot Kennedy. But suppose he *hadn't* ...

In the one and the same breath, we assert that Oswald shot Kennedy, and then we put the denial (or the negation) on the table for discussion, too. The two claims are separated by a context shift, which I will mark by a vertical bar.

Oswald shot Kennedy | Oswald didn't shoot Kennedy

and these are both on the table for discussion, and exploration. In each of the two different contexts (or 'zones') we might have certain claims ruled in and certain claims ruled out, so a position can take this structure:

$$[X : Y] \mid [X' : Y']$$

where some things have been ruled in and other things have been ruled out, on each alternative. There is absolutely no inconsistency in this position:

$$[p :] \mid [: p]$$

where p is ruled *in* on the left and ruled *out* on the right. In this dialogue, we are simply considering alternative possibilities, one in which p holds and the other in which p fails. Once we shift between contexts like this, it is natural to flag, in some way, connections between contexts. At the very least, we would expect *this* position to be out of bounds:

$$[\Box p :] \mid [: p]$$

[14]There is a technical sense in which Lewis is correct. There is no theory in the language of first-order logic *without* identity that suffices to ensure that a binary relation R in the language is forced to have as its extension the identity relation. However, that falls very far short of the claim that there is no way to define an identity predicate with *rules of inference* in any more general sense.

[15]In Read's paper, only one premise is used, rather than two, and this suffices define a symmetric relation of identity if a classical negation connective is present in the vocabulary. For it to be truly symmetric, even in the absence of other connectives, the more general form given here is more appropriate.

[16]What, then, of Lewis's "technical" sense in which identity cannot be defined? We know that there are models in which the extension of the identity predicate "$=$" is not identity, and in which all the theses of the logic of identity are satisfied: there are distinct objects in the domain where the identity predicate relates those two distinct objects. The only condition required is that absolutely every feature that holds of one of these holds of the other. That is fair enough. This model, on its own, is not a violation of any logical principle concerning identity. However, once we add a new predicate to the language and interpret it such that it holds of one of these objects and not the other, the model that results *is* in violation of the defining rule $=Df$. If we rule out any such extensions of the original model, requiring that any predicate that holds of the one domain object holds of the other too, then these models are merely a non-standard model of the same logic of identity where two objects are present in the model to stand in for the one and the same object. The presence of such models is no bar to taking the defining rule to truly define the concept of *identity*.

[17]It is easiest to show this in the presence of an operator of λ-abstraction that transforms any sentence $A(x)$ into a one-place predicate $\lambda x.A(x)$, expressing the property of being an x such that $A(x)$ holds.

[18]Here, to keep the proof short, I have been treating $a = x$ and $a \approx x$ as predicates. To make more technically correct, you can form the explicitly unary predicate $\lambda x.(a = x)$ and use λ abstraction to move from $\lambda x.(a = x)b$ to $a = b$ and back.

If I assert $\Box p$ in one context and deny p in the other, this does clash. Necessity expresses a kind of preservation across contexts in a discourse. As with traditional single-context positions, we use *sequent* notation for positions that are out of bounds. So, the fact above is expressed in the *hypersequent* [33],

$$\Box p \succ \mid \succ p$$

which can be interpreted as ruling out asserting $\Box p$ in one context and denying p in another, or that granting $\Box p$ in one position is grounds to assert p in another. The *defining* rule that underwrites this claim goes like this:

$$\frac{\succ A \mid X \succ Y \mid \mathcal{S}}{X \succ \Box A, Y \mid \mathcal{S}}\ \Box Df$$

Denying $\Box A$ in one context is out of bounds if and only if denying A icontext with no other commitments is out of bounds. (Here '$\mathcal{S}$' stands for some number of ancillary contexts). If you wish to follow the details for how rules like these suffice to underwrite the traditional modal logic s5, see [33], and to see why this setting gives rise to exactly the logical principles that correspond to the usual models for modal logic, by 'reifying' contexts into worlds, see [34].

We have seen, though, that there is reason to posit be more than one kind of context shift in discourse. We can *subjunctively* shift context (suppose things had gone otherwise) or *indicatively* shift (suppose, instead, that you are right). In "A Cut-Free Sequent System for Two-Dimensional Modal Logic" [35], I show how using these two shifts *in concert* gives a normative pragmatist interpretation of a two-dimensional modal logic of the usual kind, with two notions of necessity [14], starting from dialogical principles rather than the usual worlds-based semantics.

There is reason to believe that the two kinds of context shift, marked in our grammar of indicative and subjunctive supposition, are not merely surface features of this or that language, but reflect different kinds of communicative practice that are significant and important for any creature like us—any creatures that plan, and act on the basis of shared views. Since we have different perspectives on the world and it is important at times to pool our information to learn from each other. Unreliable creatures that we are, we cannot always take everyone else's views for granted, so instead of simply *absorbing* another's commitments, we should expect to need to shift perspectives to merely temporarily take on another's position, for the sake of the argument, only agreeing with what is claimed on occasion. Indicative supposition plays this kind of role in dialogue. Any creature that plans future action in an uncertain world will need to take future possibilities into account. I suppose it might rain, and decide to take my umbrella when I go out in case that eventuality occurs. If I forget to do take my umbrella and it does rain, I might think: supposing I *had* taken my umbrella, I wouldn't be getting wet right now.[19] The role of the two different kinds of supposition in the cognitive and communicative life of creatures like us is discussed at length in Mark Lance and Heath White's "Stereoscopic Vision" [20]. Instead of pursuing this further, I will end this section with the conclusion that there is good reason to think that these two kinds of supposition are important to us, and it behooves us to explore the kinds of conceptual and communicative possibilities they facilitate, including their associated concepts of necessity and possibility.[20]

[19]Notice that this thought involving a subjunctive supposition is very different from the thought: suppose I in fact *do* have my umbrella right now . . . which would raise a very different question of why I cannot find it despite having it in my possession.

5 THEIR INTERACTION

Barcan Marcus' questions, such as the necessity of true identity statements, arises once we combine modality and identity. A simple natural deduction proof suffices to make the point:

$$\dfrac{a = b \qquad \dfrac{\dfrac{}{a = a}\ {=}Df}{\Box(a = a)}\ \Box Df}{\Box(a = b)}\ {=}Df$$

We can prove $a = a$ using $=$ *Df* from no premises at all, in any context at all. So, we can prove $\Box(a = a)$ in any context. Given the assumption $a = b$, we can conclude $\Box(a = b)$ from the premise $\Box(a = a)$, if we think of $\lambda x.\Box(a = x)$ (being necessarily identical to a) as the feature had by a that must also be possessed by b. This simple proof shows that identities are necessary — or so it seems.

This is not simply a feature of our proof rules. In the simplest *models* for quantified modal logic with identity, the same result holds. This is the crux of Williamson's argument for the necessity of true identity statements in *Modal Logic as Metaphysics*.

> For the necessitist or permantentist to mess with the modal or temporal logic of identity in order to avoid ontological inflation would be a lapse of methodological good taste, or good sense, for it means giving more weight to ontology than to the vastly better developed and more successful discipline of logic. More specifically, the classical modal or temporal logic is a strong, simple, and elegant theory. To weaken, complicate, and uglify it without overwhelming reason to do so merely in order to block the derivation of the necessity or permanence of identity would be as retrograde and wrongheaded a step in logic and metaphysics as natural scientists would consider a comparable sacrifice of those virtues in a physical theory. [42, pp. 26, 27].

Logic doesn't *compel* this conclusion—it is straightforward enough to build models for contingent identity—but for Williamson these models are weaker, more complicated, and more *ugly* than the "strong, simple, and elegant theory" he prefers.[21]

Now, as a matter of *taste*, one might take issue with Williamson's judgements concerning modal logics with contingent identity. The *Case-Intensional First Order Logic* of Belnap and Müller is nothing if not strong, simple and elegant, as well as

[20]These concepts are the logicians *idealisations* which simply take these two kinds of shifts and allow us to operate with shifts to an arbitrary extent, in the same way that the logician's universal quantifier is genuinely *universal*. The logician's $\Box$ and $\Diamond$ range over all shifts to all (subjunctive, on the one hand, or indicative on the other) contexts, and take no care to restrict attention to 'nearby' or 'local' shifts, where we shift from 'here' enough to accommodate the new supposition, but not so far as to throw away *everything* from the starting position except for what has been granted as *necessary*. Graded modalities, and other kinds of *counterfactual* semantics allow for more subtle kinds of modality and conditionality than the extreme forms that are our target here.

[21]See, too, footnote 17 on page 11 "As usual, we are not interested in epistemic readings of 'it is possible that'" [42]. This is unsurprising, but unfortunate, for to ignore epistemic possibility completely is to ignore a significant slice of the discipline of modal logic, and to ignore exactly that part of the discipline that makes understanding contingent identity statements a pressing issue, as well as a surprsingly tractable one.

expressively powerful [6]. The work, in particular, of Giovanna Corsi [13] and Maria Aloni [1] shows that models for modal logic with contingent identity have a wide range of uses in the analysis of significant epistemic puzzles. Given the depth and range of epistemic logics, to ignore them completely is to restrict one's attention needlessly to half the field at most. However, in my discussion of the necessity or contingency of identity, my focus is not on models, but on rules for *proof*. Since our proof rules involve positions with more than one *context* or *zone*, let's consider the issue in a concrete case. Hesperus is Phosphorous, since both 'Hesperus' and 'Phosphorous' are names for the one and the same planet, Venus. Consider the position involving asserting the identity $h = p$ in one zone of a dialogue while denying it in another, like this:

$$[h = p : \,] \mid [\, : h = p]$$

Is this coherent, or is this position out of bounds? To make things more concrete again, since in the second position here we are *denying* $h = p$, if identity is governed by the defining rule, we'll take there to be some feature had by h but not by p. We have:

$$[h = p : \,] \mid [Fh : Fp]$$

Is *this* coherent? Upon reflection, we can see that the answers differ depending on whether the shift is indicative or subjunctive. As far as indicative shifts are concerned, this is totally coherent. After all, we only learned that $h = p$ after some significant astronomical investigations. Until that discovery, it was plausible that the morning star was not the evening star, and that indeed, some features were had by the morning star that were not shared by the evening star. These were both live epistemic possibilties. There is nothing inconsistent or incoherent in granting this fact. Identities are among those claims that are open to us, and can only be closed (in some instances) by empirical investigation. A properly *epsitemic* sense of possibility and necessity must find some way to grant that some true identity statements are contingent, and not epistemically necessary.

The same cannot be said for *subjunctive* alternatives. If I am planning (prospectively) or analysing alternative courses of events (retrospectively) these do not seem to shift true identity statements like this. If $h = p$ and I am considering a vacation to h, there is no way that I can avoid p along the way, since $h = p$. Similarly, if the planet h were to have been hit by a large meterorite, then p would have 'also' borne the brunt of the impact, since $h = p$. That it is plausible that identities are preserved under *subjunctive* (metaphysical) alternatives does not mean that they are also preserved under *indicative* (epistemic) alternatives too. The two kinds of supposition seem to interact very differently with identity.[22]

Given the role and function of these kinds of zone shifts in resolution of disagreements and in planning, it is not surprising that this distinction is present. We would expect to have to resolve disagreements about identities. Identities are the kinds of things we can learn by engagement with the world around us, and they are open to us until we learn them. On the other hand, once we have learned an identity statement, that fact should constrain our view of how things *could go*, or how things *could have gone*. It follows that an appropriate defining rule for an identity involving context shifts defines it in terms of property preservation across *subjunctive* shifts, but not *epistemic* shifts. To expand $=Df$ to contexts involving more than one zone, we have:

$$\frac{Fa \succ Fb \mid X \succ Y \mid \mathcal{S} \qquad Fb \succ Fa \mid X \succ Y \mid \mathcal{S}}{X \succ a = b, Y \mid \mathcal{S}} {=}Df$$

where '|' marks a subjunctive alternative shift. Denying $a = b$ in some zone amounts to distinguishing a from b in *some* subjunctive alternative zone (including a fresh zone, like this). This seems motivated and defensible, and with this rule, we can show that positions like $[a = b : \,] \mid [Fa : Fb]$ are indeed out of bounds, as expected. On the other hand, such a rule is *not* well motivated when the shift in question is *indicative*. Marking this with the double line '‖', we would expect a position like $[a = b : \,] \parallel [Fa : Fb]$ to *not* be out of bounds, and neither would $[a = b : \,] \parallel [\, : a = b]$ in which two *indicative* alternative positions take opposing views on the issue of $a = b$. Since we have an *epistemic* modality $\Box_E$ that jumps over epistemic alternatives, this means that we can see how the following position is coherent:

$$[a = b, \Box_E(a = a) : \Box_E(a = b)] \parallel [\, : a = b]$$

Here, in our first context, we have asserted $a = b$ and $\Box_E(a = a)$, and denied $\Box_E(a = b)$, since we have an epistemic alternative position in which $a = b$ is denied. This means that the *epistemic* modal operator $\Box_E$ creates, as expected, a context is not closed under identity statements. The "property" $\lambda x.\Box_E(a = x)$ holds of a, but it does not hold of b, despite the fact that $a = b$. This is unsurprising, since $\Box_E$ is, by design, *epistemic*, in a loose sense.

We must be careful, therefore, in how we interpret the defining rules $=Df$, since if we allow for $\lambda x.\Box_E(a = x)$ to be a predicate to which $=Df$ applies, we either have a counterexample to our rule, identities turn out to be epistemically necessary as well as metaphysically necessary. If we wish to avoid both those horns of this dilemma, the way out is straightforward. We recall that in the motivation for $=Df$ in the first place, we motivated it by saying that identity was indistinguishability: a is identical to b iff any *feature* had by a is had by b and vice versa. It being epistemically necessary that it is identical to a is not a *feature* had by some object alone, but also by the way that object is represented. One option is to assume that all primitive predicates in our language represent features, while some complex expressions involving epistemic modalities allow us to form non-feature predicates. I prefer a more genereous route, that allows for our language to have primitive predicates that fail to represent features in this sense (that is, they are not *extensional* predicates, preserved under identity). There is a reading of 'famous' in English that fails to be extensional, if Superman can be famous while Clark Kent fails to be famous.[23] If our language has primitive *extensional* (feature) predicates and primitive *non-extensional* predicates, then it suffices to restrict $=Df$ to feature predicates, and the result is a well-behaved account of identity, and metaphysical and epistemic modalities, according to which identity statements are *metaphysically* necessary but *epistemically* contingent.

[22] This is not the only difference between the two zone shifts. If we keep track of the "actual" zone, the zone in which we are recording commitments about how things actually are, then subjunctive shifts take us to zones *not* marked as actual, while indicative shifts take us to zones marked as actual, since they are different views on how things actually are [35]. On this view, we have $@p \to \Box_M @p$ (if p actually holds then it's metaphysically necessary that it actually holds), while $@p \to \Box_E @p$ fails (if p actually holds then it is by no means *epistemically* necessary that it actually holds).

[23] In another sense, of course, Clark Kent *is* famous, since Clark Kent = Superman, who is famous. In general, we can *extensionalise* any predicate F into an extensional predicate $F^=$, by setting $F^= a$ as shorthand for $\exists y(y = a \land Fy)$. This is, by design, extensional, since if $a = b$ and $F^= a$ we clearly have $F^= b$ too.

6 THE UPSHOT

- I have *shown how* you can incorporate insights about *necessary* identity and *contingent* identity in the one framework, appealing to the structure of our communicative practice.
- I have given an example of how attention to *inference rules* allows for a kind of *conceptual engineering* sympathetic to the *normative pragmatist* and *semantic anti-realist*.
- I have *undercut* David Lewis' claim that we are unable to define a notion of identity.
- I have *vindicated* Arthur Prior's claim that we can grasp the concept(s) of necessity without *first* having to grasp the concept of possible worlds.

REFERENCES

[1] MARIA ALONI. "Individual Concepts in Modal Predicate Logic". *Journal of Philosophical Logic*, 34(1):1–64, Feb 2005.

[2] DORIT BAR-ON AND JAMES SIAS. "Varieties of Expressivism". *Philosophy Compass*, 8(8):699–713, 2013.

[3] JC BEALL AND GREG RESTALL. "Logical Pluralism". *Australasian Journal of Philosophy*, 78:475–493, 2000.

[4] JC BEALL AND GREG RESTALL. *Logical Pluralism*. Oxford University Press, Oxford, 2006.

[5] NUEL BELNAP. "Declaratives are not enough". *Philosophical Studies*, 59(1):1–30, 1990.

[6] NUEL BELNAP AND THOMAS MÜLLER. "CIFOL: Case-Intensional First Order Logic". *Journal of Philosophical Logic*, 43(2):393–437, 2013.

[7] SIMON BLACKBURN. *Spreading the Word*. Clarendon Press, 1984.

[8] SIMON BLACKBURN. "Morals and Modals". In *Fact, Science & Morality: Essays on A. J. Ayer's Language, Truth and Logic*, pages 119–141. Blackwell, Oxford, 1986.

[9] ROBERT B. BRANDOM. *Making It Explicit*. Harvard University Press, 1994.

[10] ROBERT B. BRANDOM. *Articulating Reasons: an introduction to inferentialism*. Harvard University Press, 2000.

[11] B. JACK COPELAND. "The Genesis of Possible Worlds Semantics". *Journal of Philosophical Logic*, 31(2):99–137, 2002.

[12] B. JACK COPELAND. "Meredith, Prior, and the History of Possible Worlds Semantics". *Synthese*, 150(3):373–397, 2006.

[13] GIOVANNA CORSI. "Counterpart Semantics: A Foundational Study on Quantified Modal Logics". Technical Report ILLC 2002 PP-2002-20, ILLC/Department of Philosophy, University of Amsterdam, Amtsterdam, 2002.

[14] MARTIN DAVIES AND LLOYD HUMBERSTONE. "Two notions of necessity". *Philosophical Studies*, 38(1):1–30, 1980.

[15] IMOGEN DICKIE. "Negation, anti-realism, and the denial defence". *Philosophical Studies*, 150:161–185, 2010.

[16] HANS VAN DITMARSCH, WIEBE VAN DER HOEK, AND BARTELD KOOI. *Dynamic Epistemic Logic*. Synthese Library. Springer, 2007.

[17] MICHAEL DUMMETT. *The Logical Basis of Metaphysics*. Harvard University Press, 1991.

[18] CATARINA DUTILH NOVAES. *The Dialogical Roots of Deduction: Historical, Cognitive, and Philosophical Perspectives on Reasoning*. Cambridge University Press, 2020.

[19] LUCA INCURVATI AND JULIAN J. SCHLÖDER. "Weak Rejection". *Australasian Journal of Philosophy*, 95(4):741–760, 2017.

[20] MARK LANCE AND W. HEATH WHITE. "Stereoscopic Vision: Persons, Freedom, and Two Spaces of Material Inference". *Philosophers' Imprint*, 7(4):1–21, 2007.

[21] MARK NORRIS LANCE AND JOHN O'LEARY-HAWTHORNE. *The Grammar of Meaning*. Cambridge University Press, 1997.

[22] C. I. LEWIS AND C. H. LANGFORD. *Symbolic Logic*. The Century Co, New York and London, 1932.

[23] DAVID K. LEWIS. *On the Plurality of Worlds*. Blackwell, Oxford, 1986.

[24] RUTH BARCAN MARCUS. "Modalities and Intensional Languages". *Synthese*, 13:303–322, 1961.

[25] ARTHUR N. PRIOR. *Time and Modality*. Oxford University Press, 1957.

[26] ARTHUR N. PRIOR. "The Runabout Inference-Ticket". *Analysis*, 21(2):38–39, 1960.

[27] ARTHUR N. PRIOR AND KIT FINE. *Worlds, Times and Selves*. Duckworth, 1977.

[28] STEPHEN READ. *Relevant logic: a philosophical examination of inference*. Basil Blackwell, Oxford, 1988.

[29] STEPHEN READ. "Identity and Harmony". *Analysis*, 64(2):113–115, 2004.

[30] GREG RESTALL. *An Introduction to Substructural Logics*. Routledge, 2000.

[31] GREG RESTALL. "Carnap's Tolerance, Meaning and Logical Pluralism". *The Journal of Philosophy*, 99:426–443, 2002.

[32] GREG RESTALL. "Multiple Conclusions". In PETR HÁJEK, LUIS VALDÉS-VILLANUEVA, AND DAG WESTERSTÅHL, editors, *Logic, Methodology and Philosophy of Science: Proceedings of the Twelfth International Congress*, pages 189–205. KCL Publications, 2005.

[33] GREG RESTALL. "Proofnets for S5: sequents and circuits for modal logic". In COSTAS DIMITRACOPOULOS, LUDOMIR NEWELSKI, AND DAG NORMANN, editors, *Logic Colloquium 2005*, number 28 in Lecture Notes in Logic. Cambridge University Press, 2007.

[34] GREG RESTALL. "Truth Values and Proof Theory". *Studia Logica*, 92(2):241–264, 2009.

[35] GREG RESTALL. "A Cut-Free Sequent System for Two-Dimensional Modal Logic, and why it matters". *Annals of Pure and Applied Logic*, 163(11):1611–1623, 2012.

[36] GREG RESTALL. "Generality and Existence 1: Quantification and Free Logic". *Review of Symbolic Logic*, 12:1–29, 2019.

[37] GREG RESTALL. "Existence and Definedness: The Semantics of Possibility and Necessity". Article in progress, 2021.

[38] GREG RESTALL. "Speech Acts & the Quest for a Natural Account of Classcial Proof". Article in progress, 2021.

[39] DAVID RIPLEY. "Negation, Denial, and Rejection". *Philosophy Compass*, 6(9):622–629, 2011.

[40] DAVID RIPLEY. "Anything Goes". *Topoi*, 34(1):25–36, Apr 2015.

[41] DAVID RIPLEY. "Bilateralism, coherence, warrant". In FRIEDERIKE MOLTMANN AND MARK TEXTOR, editors, *Act-Based Conceptions of Propositional Content*, pages 307–324. Oxford University Press, 2017. Forthcoming.

[42] TIMOTHY WILLIAMSON. *Modal Logic as Metaphysics*. Oxford University Press, 2013.

[43] GEORG H. VON WRIGHT. *An Essay in Modal Logic*. North Holland Publishing Company, Amsterdam, 1951.