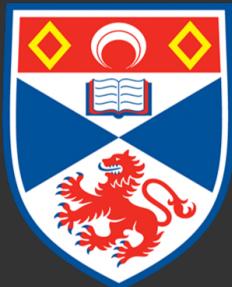


TYPE THEORY & THEMES IN PHILOSOPHICAL LOGIC

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You can download these slides
<https://consequently.org/p/2025/tt-tpl>

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1 INTRODUCTION

2 MODAL & SUBSTRUCTURAL LOGICS

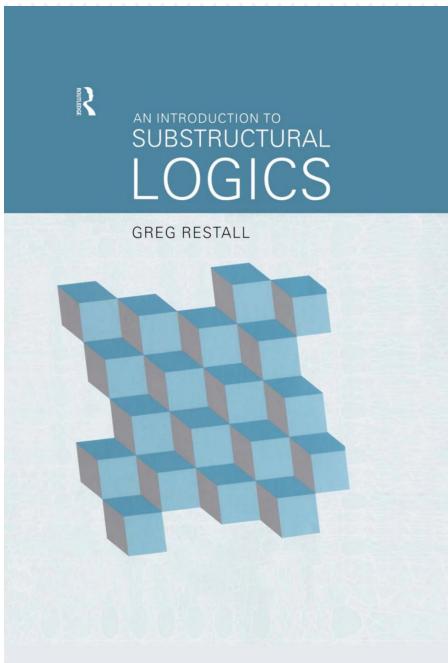
3 INTENSIONALITY & IDENTITY

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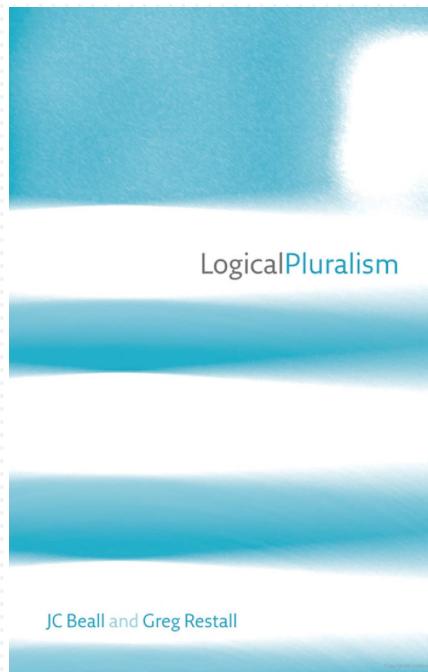
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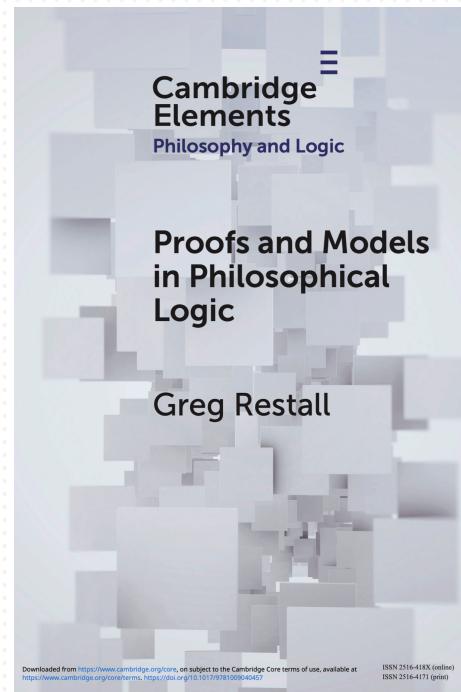
I work in PHILOSOPHICAL LOGIC



2000



2006



2022

I work to understand the connections between different techniques, traditions & approaches in logic & philosophy

TYPE THEORY is an exciting world I am beginning to explore.

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MODAL LOGICS — possibility & necessity ; reasoning over times, --

- Massive industry
 - "metaphysical necessity"
 - Epistemic logics
 - Montague-style type theory in linguistics

Central tool : Kripke - style "possible worlds semantics."

Good tools at the level of types, not terms.

(Kripke models represent what follows from what — not why.)

Minority tradition ~ algebras & proof theory for modal logics.

RESIDUATION Galois Connection	$\frac{\Diamond^{\leftarrow} a \leq b}{a \leq \Box^{\rightarrow} b}$	\Diamond^{\leftarrow} sometime in the past \Box^{\rightarrow} all times in the future	Generalises more naturally to categories & so, to type theoretical interpretation.
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SUBSTRUCTURAL LOGICS — resources, relevance, paradox, syntax

$$\frac{a \otimes b \leq c}{a \leq b \rightarrow c} \quad \left. \begin{array}{l} \text{"Substructural" since the standard structural} \\ \text{rules of contraction, weakening, permutation,} \\ \text{\& even associativity may be absent.} \end{array} \right\}$$

Kripke models for modal logics generalise to the substructural setting.

$$\begin{array}{ll} \square^\rightarrow \diamond^\leftarrow & - \text{unary connective, binary relation} \\ \rightarrow \otimes & - \text{binary connective, ternary relation} \end{array} \quad \begin{array}{ll} \square^\rightarrow & \text{universal forward} \\ \diamond^\leftarrow & \text{existential backward} \\ \rightarrow & \text{universal forward} \\ \otimes & \text{existential backward} \end{array}$$

These models extend distributive lattices with \rightarrow, \otimes .

Algebras, Coherence Spaces & Phase space models give natural non-distributive structures & categories.

1 INTRODUCTION

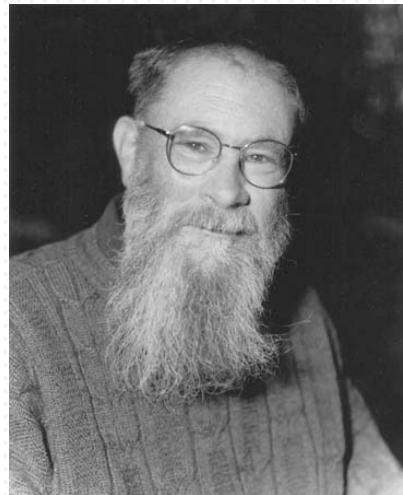
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Identity is utterly simple and unproblematic.
Everything is identical to itself ;
nothing is ever identical to anything else except itself.

— David LEWIS

On the Plurality of Worlds

- This is correct, but it is not the end of questions about identity

IDENTITY &

- NECESSITY
- PROOF
- CONSTRUCTION
- ISOMORPHISM

} And philosophers have worked on
these issues for a long time.

IDENTITY & NECESSITY

δ = the number of planets ✓

It is necessary that $\delta = \delta$ ✓

It is necessary that the number of planets = δ ✗

$\Box[(\text{The } n \text{ where } n = \#\text{planets}) \rightarrow n = \delta]$ ✗ — de dicto

(The n where $n = \#\text{planets}$) $\Box[n = \delta]$ ✓ — de re

Scope makes a difference

IDENTITY & KNOWLEDGE / PROOF

Clark Kent = Superman. ✓

Lois Lane knows that Clark Kent is Clark Kent. ✓

Lois Lane knows that Clark Kent is Superman. ?

$f(x) = y$ ✓

Lois Lane shows that $y = y$ ✓

Lois Lane shows that $f(y) = y$?

$s=t$'s' & 't' have the same referent (value)

they might not have the same sense,

ISOMORPHISM & IDENTITY

Mathematical Structures — When are G_1 & G_2 the same group?

What is the relationship between isomorphism & identity?

(this is a part of deciding what mathematical structure is.)

In philosophy of mathematics this is explored in
STRUCTURISM, which is congenial to category
theoretical (& HoTT, Cubical) presentation, but
these views are not identical.

(See, especially, Colin McLarty, Steve Awodey.)

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PROOF THEORY ❤ CONSTRUCTIVE LOGIC

Gentzen, Heyting, Dummett, PML, Prawitz, Girard

Understanding the Classical / Constructive boundary
is an active research area in many directions.

Translations: Classical $\xrightarrow{\text{DN}}$ Constructive

Constructive $\xrightarrow{\text{S4}}$ Classical Modal
Topological

The Context of Deduction:

What is the difference between

$\Gamma \vdash A \ \notin$

$\Gamma \vdash A, \Delta ; \Gamma, A \vdash \Delta ?$

BILATERALISM: assertion & denial treated equally

A (MILDELY) BILATERAL PROOF OF PERLES (AN)

$$\frac{\frac{[(A \rightarrow B) \rightarrow A]^3}{\frac{A}{\frac{A}{\frac{[A]^1 [A]^2}{\frac{\#}{B}}}}}}{\frac{A \rightarrow B}{\frac{\rightarrow \bar{E}}{((A \rightarrow B) \rightarrow A) \rightarrow A}}} \quad \frac{\frac{[A]^2}{\frac{\#}{A}}}{\frac{A}{\frac{A}{\frac{\downarrow^2}{\frac{\rightarrow P^3}{((A \rightarrow B) \rightarrow A) \rightarrow A}}}}} \rightarrow P^1$$

Contexts contain positive & negative information. Judgments are positive or dead ends.

A (MILDELY) BILATERAL PROOF OF PERCET'S (AN)

(In M-Parigot's $\lambda\mu$ -calculus)

$$\frac{\frac{x[A] \quad \alpha[A]^L}{\alpha[x] \#} \downarrow \frac{\mu\beta.\alpha[x] \quad B}{\rightarrow P^1}}{y[(A \rightarrow B) \rightarrow A]^3 \quad \lambda x.\mu\beta.\alpha[x] \quad A \rightarrow B} \rightarrow \bar{E}$$

$$\frac{y(\lambda x.\mu\beta.\alpha[x]) \quad A}{\alpha[\alpha[A]^L] \uparrow} \quad \alpha[\alpha[\alpha[A]^L] \#]$$

$$\frac{\mu\alpha.\alpha[y(\lambda x.\mu\beta.\alpha[x])] \quad A}{\rightarrow P^3} \quad \downarrow^2$$

$$\lambda y.\mu\alpha.\alpha[y(\lambda x.\mu\beta.\alpha[x])] \quad ((A \rightarrow B) \rightarrow A) \rightarrow A$$

Contexts contain positive & negative information. Judgments are positive or dead ends.

A SYMMETRIC BILATERAL CALCULUS $\bar{\lambda}\mu\tilde{\nu}$ (CURRIEN & HERBEIN)

At the typing level, we obtain $LK_{\mu\tilde{\nu}}$ whose typing judgments are:

$c : (\Gamma \vdash \Delta)$	COMMANDS	CLASHES ASSERTIONS DENIALS
$\Gamma \vdash v : A \Delta$	TERMS	
$\Gamma e : A \vdash \Delta$	CONTEXTS	

and whose typing rules are:

$$\frac{}{\Gamma \vdash v : A | \Delta \quad \Gamma | e : A \vdash \Delta}$$

$$\frac{}{\langle v | e \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{}{\Gamma | \alpha : A \vdash \alpha : A, \Delta}$$

$$\frac{}{\Gamma, x : A \vdash x : A | \Delta}$$

$$\frac{c : (\Gamma \vdash \beta : B, \Delta)}{\Gamma \vdash \mu\beta.c : B | \Delta}$$

$$\frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma | \tilde{\mu}x.c : A \vdash \Delta}$$

$$\frac{\Gamma \vdash v : A | \Delta \quad \Gamma | e : B \vdash \Delta}{\Gamma | v \cdot e : A \rightarrow B \vdash \Delta}$$

$$\frac{}{\Gamma, x : A \vdash v : B | \Delta}$$

$$\frac{}{\Gamma \vdash \lambda x.v : A \rightarrow B | \Delta}$$

complex contexts / denials.

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FREGE'S BEGRIFFSCHRIFT

A — content, a thought, a proposition.

$\vdash A$ — the assertion that A

If A then B. — this can be asserted, but the A & B are propositions
inside the conditional, but are not asserted.

Assertion is a speech act — there are others.

?A — polar question

?_nA(n) — find an n where A(n) question

!A — see to it that A is true } IMPERATIVE

{_BA — β promises to see to it that A } COMMITMENT

CONDITIONAL SPEECH ACTS

If A then is it the case that B?

If A then I promise to B.

If A then please do B.

Are these questions, promises & imperatives?

Certainly if the antecedent holds... maybe only then.

↳

If A is a restrictor of more than propositions.

Traditional formal grammars do not respect conditional speech acts — the grammar is independent of the semantics.

[A true]
⋮

A prop B prop

[A true]
⋮

A prop B promise

A > B prop

A > B promise

These are also entangled, but the dependence is in the other direction!

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TYPE THEORY CAN BE APPLIED IN DIFFERENT WAYS

$$\begin{array}{c}
 \frac{x:P \quad y:Q}{P \& Q} \& I \\
 \frac{}{Q > P \& Q} \rightarrow I y \\
 \frac{}{\rightarrow (Q > P \& Q) \rightarrow I x} \rightarrow I x
 \end{array}$$

$$\frac{x:\Gamma}{\Gamma, A} = \Pi^A = \Pi : A$$

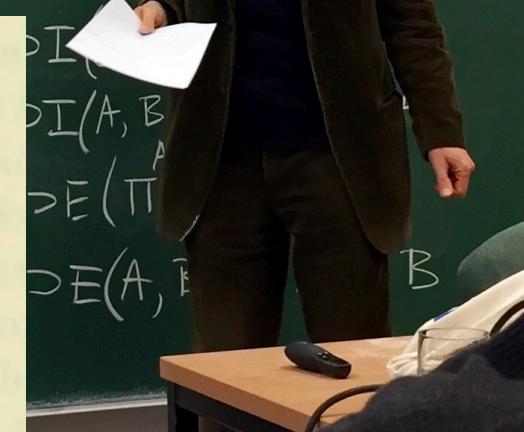


TABLE 1. KEY NOTIONS OF PROGRAMMING WITH MATHEMATICAL COUNTERPARTS

programming	mathematics
program, procedure, algorithm	function
input	argument
output, result	value
$x := e$	$x = e$
$S_1; S_2$	composition of functions
if B then S_1 else S_2	definition by cases
while B do S	definition by recursion
data structure	element, object
data type	set, type
value of a data type	element of a set, object of a type
$a:A$	$a \in A$
integer	Z
real	R
Boolean	{0, 1}
(c_1, \dots, c_n)	$\{c_1, \dots, c_n\}$
array [I] of T	$T^I, I \rightarrow T$
record $s_1:T_1; s_2:T_2$ end	$T_1 \times T_2$
record case $s:(c_1, c_2)$ of $c_1:(s_1:T_1); c_2:(S_2:T_2)$ end	$T_1 + T_2$
set of T	$\{0, 1\}^T, T \rightarrow \{0, 1\}$

PML - Constructive Mathematics & Computer programming (1984)

Computational type theory
sequents classify computational processes

Formal type theory
pure logic, backed only by the rules

$$\Gamma \vdash t : A$$

Conceptual type theory
sequents classify cognitive constructions
CONSTRUCTIVE LOGIC
INTUITIONISM - theories of judgement
PML, Dag Prawitz, Catarina Smulholm, ...

Dialogical type theory
sequents classify practices of
processes of reasoning & justification

NORMATIVE PRAGMATICS

Robert Brandom, Jaroslav Peregrin, ...

HYBRID TYPE THEORY?

Computational type theory
sequents classify computational processes

Formal type theory

$$\Gamma \vdash t : A$$

What about applications that encompass these domains? Justifications that include computation, computer aided reasoning, natural language program specification

Dialogical type theory

sequents classify practices of processes of reasoning & justification

Conceptual type theory

sequents classify cognitive constructions

It seems to me that many of these
intersections could be fruitful in
the years ahead.

Questions?

