Proofs and Models in Philosophical Logic

propositional logic, modal logic, and identity

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Soundness and Completeness

Propositional Logic: the case of classical proof

Modal Logic: the significance of worlds

Identity: *necessary* or *contingent*?

SOUNDNESS AND COMPLETENESS

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Can we harness *both* proofs and models
to do interesting philosophical *work*?

PROPOSITIONAL LOGIC: THE CASE OF CLASSICAL PROOF

Models for classical logic

p	q	$p \wedge q$	$p \vee q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

$$A \not\models B \text{ iff } (\exists v)(v(A) = 1 \text{ and } v(B) = 0)$$

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$$\ldots \text{ iff } (\exists v)(v^-(A^-) = 0 \text{ and } v^-(B^-) = 1) \text{ iff } B^- \not\models A^-.$$

Standard Proof Systems are *not* self-dual: Hilbert Proofs

1.	$(A \to (A \to A)) \to ((A \to ((A \to A) \to A)) \to (A \to A))$	(DIST)
2.	$A \to (A \to A)$	(WEAK)
3.	$(A \to ((A \to A) \to A)) \to (A \to A)$	1,2 (MP)
4.	$A \to ((A \to A) \to A)$	(WEAK)
5.	$A \to A$	3,4 (MP)

Standard Proof Systems are *not* self-dual: Natural Deduction

$$\dfrac{\dfrac{p}{p \vee q}\vee I \qquad p}{(p \vee q) \wedge p}\wedge I$$

$$\dfrac{p \vee (p \wedge q) \qquad [p]^1 \qquad \dfrac{[p \wedge q]^2}{p}\wedge E}{p}\vee E^{1,2}$$

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$$p \vdash (p \vee q) \wedge p \qquad\qquad p \vee (q \wedge p) \vdash p$$

The Sequent Calculus *is* self-dual

$$
\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\,\lor R \qquad p \succ p}{p, p \succ (p \lor q) \land p}\,\land R
\qquad\qquad
\dfrac{p \succ p \qquad \dfrac{p \succ p}{p \land q \succ p}\,\land L}{p \lor (p \land q) \succ p, p}\,\lor L
$$

From $X \succ A$ to $X \succ Y$.

Cut (two forms)

$$\frac{X \succ A, Y \qquad X', A \succ Y'}{X, X' \succ Y, Y'}\,{}_{mCut}$$

$$\frac{X \succ A, Y \qquad X, A \succ Y}{X \succ Y}\,{}_{aCut}$$

The Completeness Proof is *beautiful*

If there is no derivation of $X \succ Y$, then there is a valuation assigning 1 to each member of X and 0 to each member of Y.

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. . . and take this process *to the limit*.

Connective Rules

$$\frac{X, A, B \succ Y}{X, A \land B \succ Y}\land Df \qquad \frac{X \succ A, B, Y}{X \succ A \lor B, Y}\lor Df$$

$$\frac{X, A \succ Y}{X \succ \neg A, Y}\neg Df \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}\to Df$$

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$$X, A \succ A, Y \qquad \frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\,\mathit{aCut}$$

Me, in 2005

MULTIPLE CONCLUSIONS

"Multiple Conclusions," in *Logic, Methodology and Philosophy of Science: Proceedings of the Twelfth International Congress*, edited by Petr Hajek, Luis Valdes-Villanueva and Dag Westerstahl, Kings' College Publications, 2005, 189–205.

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I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with multiple premises and multiple conclusions. Gentzen's multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for classical logic as it does for intuitionistic logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us understand and adjudicate debates between proponents of classical and non-classical logics.

This paper has now been reprinted in *Analysis and Metaphysics*, 6, 2007, 14-34.

`https://consequently.org/writing/multipleconclusions/`

Invertible Rules look like *definitions*

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\,{\wedge}Df \qquad \frac{X \succ A, B, Y}{X \succ A \vee B, Y}\,{\vee}Df$$

$$\frac{X, A \succ Y}{X \succ \neg A, Y}\,{\neg}Df \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}\,{\to}Df$$

Are sequent derivations *proofs*?

The mistake in this position, however, resides in the idea that any formal game incorporating what appear to be inference rules will confer meanings on its logical symbols. Adherence to inferentialism importantly constrains one's choice of proof-theoretic frameworks and thus requires one to reject Carnap's amoralism about logic: the inferentialist must remain faithful to our ordinary inferential practice. Only those deductive systems that answer to the use we put our logical vocabulary to fit the bill. After all, it is the practice represented, not the formalism as such, that confers meanings. Therefore, the formalism is of meaning-theoretic significance and hence of interest to the inferentialist only if it succeeds in capturing (in a perhaps idealised form) the relevant meaning-constituting features of our practice. It is in this sense, then, that the inferentialist position imposes strict demands on the form deductive systems may take. For future reference, let us refer to these demands as the

> *Principle of answerability* only such deductive systems are permissible as can be seen to be suitably connected to our ordinary deductive inferential practices.

Florian Steinberger, "Why Conclusions Should Remain Single"
JPL (2011) 40:333–355 `https://dx.doi.org/10.1007/s10992-010-9153-3`

This is not just *conservatism*

What is a proof of A?

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(Not every way to meet a justification request is a *proof*, but proofs meet justification requests in a *very* stringent way.)

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- If it's other assertions (*assumptions*), the natural *shape* is $X \succ A$.
- If it's *bilateralist*, the natural shape is $X \succ A; Y$.

What is justified *in* that context?

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- If it's assertions only, the shape is $X \succ A; Y$.
- If it's assertions *and denials*, we have $X \succ A; Y$, and $X; A \succ Y$, and duality is restored.
- In the first case, we have the $\lambda\mu$ calculus, and in the second, fully *bilateralist* natural deduction.

MODAL LOGIC: THE SIGNIFICANCE OF WORLDS

Necessity and Possibility

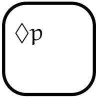
It is necessary that p It is possible that p

Necessity and Possibility

It is necessary that p | It is possible that p

$\Box p \qquad \Diamond p$

Possible Worlds Models



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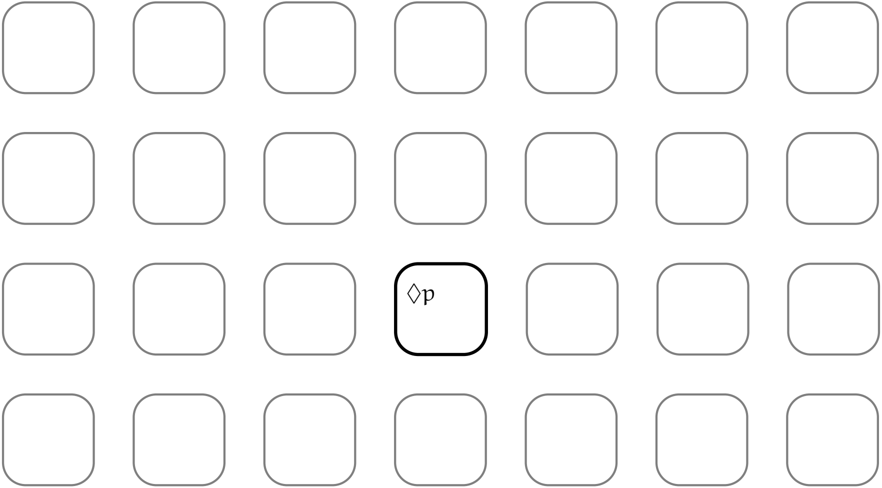

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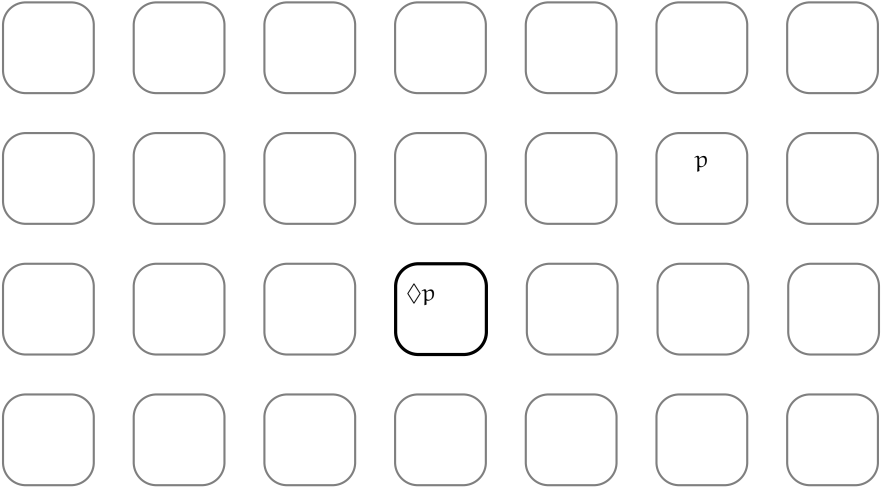

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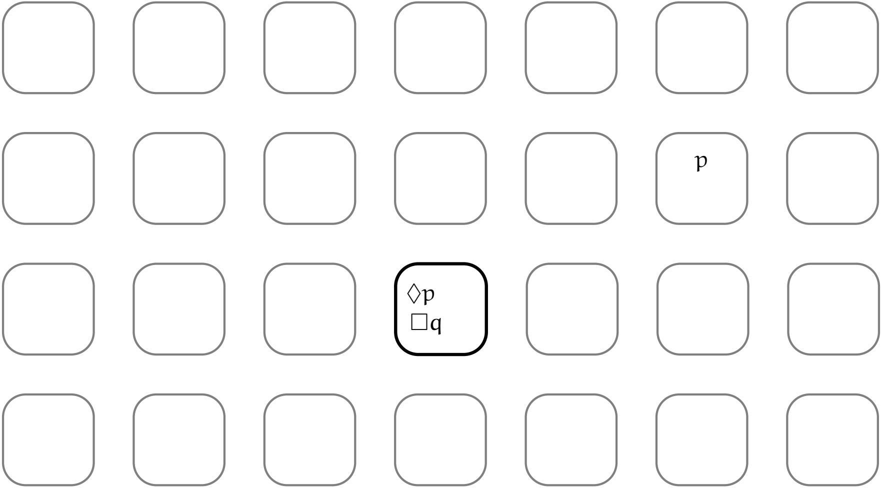

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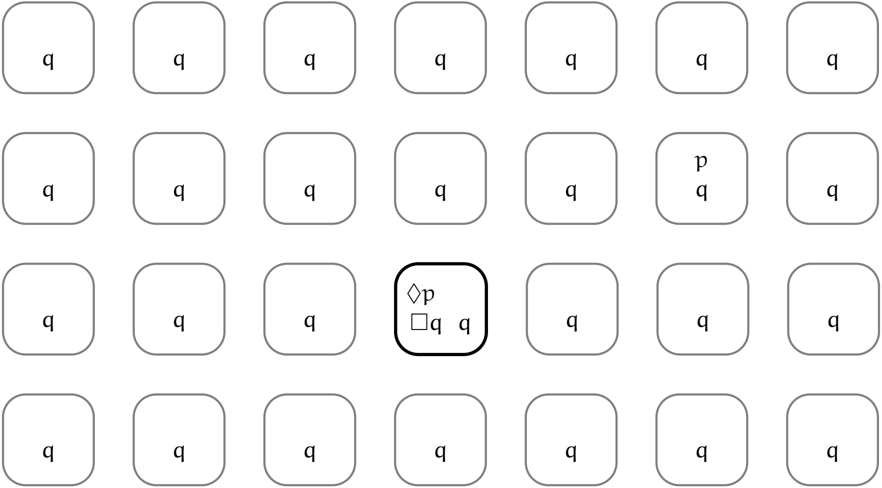

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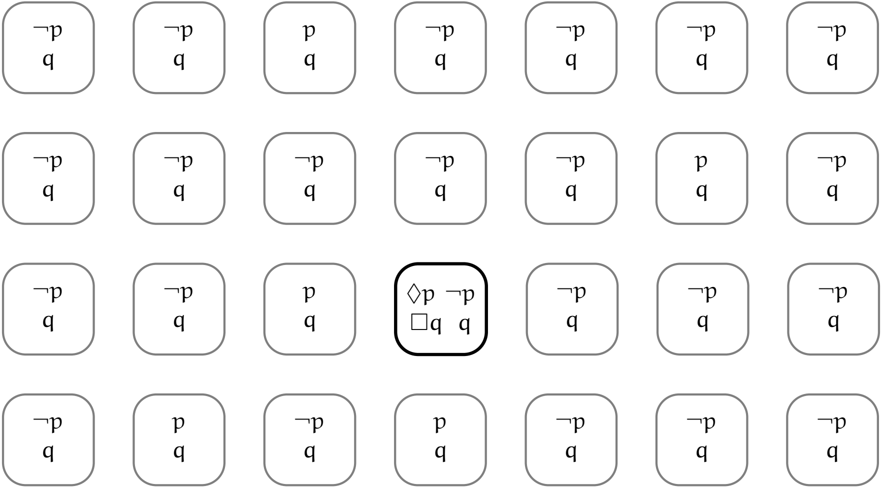

How seriously should we take these models?

- What sort of thing *is* a world?
- Given a world, what is *true* at that world?
- Given any such world, how can we *know* what holds there?

… possible worlds, in the sense of possible states of affairs are not *really* individuals (just as numbers are not *really* individuals).

To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case …

We understand 'truth in states of affairs' because we understand 'necessarily'; not *vice versa*.
— "Worlds, Times and Selves" (1969)

How *do* we understand '*necessarily*' and '*possibly*'?

How *do* we understand ‘*necessarily*’ and ‘*possibly*’?

… and what’s more, why does their logic have possible worlds models *as models*?

Modal reasoning trades on *supposition* and *context shift*

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But let's suppose that *you're* right and he *didn't*.

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The New York Times.

KENNEDY IS KILLED BY SNIPER AS HE RIDES IN CAR IN DALLAS; JOHNSON SWORN IN ON PLANE

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Why America Weeps

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Two notions of necessity

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+ Possible worlds semantics is useful in modelling *both* modal notions.
+ Let's consider both these options in the development of a *proof system*.

Hypersequents

$$\Box A \succ \;\; \mid \;\; \succ A$$

A Defining Rule for Necessity

$$\frac{\succ A \mid X \succ Y \mid \mathcal{S}}{X \succ \Box A, Y \mid \mathcal{S}} \, \Box Df$$

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(In the rule, the line between premise and conclusion is a double line.)

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The *form* of the rule is the same in either case, though the *content* differs.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box p \succ \Box p}{\succ p \mid \Box p \succ}\,\Box Df{\uparrow}}{\succ \Box p \mid \Box p \succ}\,\Box Df{\downarrow} \qquad \dfrac{\neg\Box p \succ \neg\Box p}{\Box p, \neg\Box p \succ}\,\neg Df{\uparrow}}{\neg\Box p \succ \mid \Box p \succ}\,mCut}{\neg\Box p \succ \mid \succ \neg\Box p}\,\neg Df{\downarrow}}{\neg\Box p \succ \Box\neg\Box p}\,\Box Df{\downarrow}$$

Cut (multiplicative and additive)

$$\frac{X \succ A, Y \mid \mathcal{S} \qquad X', A \succ Y' \mid \mathcal{S}'}{X, X' \succ Y, Y' \mid \mathcal{S} \mid \mathcal{S}'}\,mCut$$

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$$\frac{X \succ A, Y \mid \mathcal{S} \qquad X, A \succ Y \mid \mathcal{S}}{X \succ Y \mid \mathcal{S}}\,aCut$$

Expanding Positions

$$\frac{X \succ A, Y \mid \mathcal{S} \qquad X, A \succ Y \mid \mathcal{S}}{X \succ Y \mid \mathcal{S}}\,aCut \qquad\qquad \frac{\succ A \mid X \succ \Box A, Y \mid \mathcal{S}}{X \succ \Box A, Y \mid \mathcal{S}}\,\Box R'$$

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- $\Box A$ is false in one iff A is false in one.
- $\Box A$ is true in one iff A is true in all.

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- We *understand* modal concepts by governing our assertions and denials (and inferences) by way of these rules, connecting the modal operators with zone shifts.
- Since they have *this* logical structure, possible worlds models get their behaviour *right*.
- But those models are only ever *models* of the contexts arising out of our *use*.

This is explanatorily *rich*

- The models work equally well for *epistemic* and *metaphysical* modals.

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- The models work equally well for *epistemic* and *metaphysical* modals.
- The simple (S5) hypersequent structure can be generalised (see Poggiolesi on *Tree Hypersequents*).
- The result is two different perspectives on modal vocabularly:
  (1) *representational* (models), and (2) *normative pragmatic* (proofs).

IDENTITY: NECESSARY OR CONTINGENT?

Identity and harmony

Stephen Read

1. Harmony

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960–61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if 'tonk' has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of 'p tonk q' is given by Prior's rule:

$$\frac{p}{p \text{ tonk } q}\ \text{tonk-I}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{p \text{ tonk } q \qquad \begin{matrix}(p)\\ r\end{matrix}}{r}\ \text{tonk-E}$$

A Defining Rule for Identity

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y} {=}Df$$

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Or equivalently, to prove that $a = b$, prove Fb from the assumption Fa (and *vice versa*), where the predicate F is *arbitrary*.

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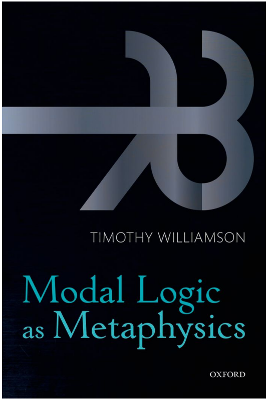
Or equivalently, to prove that $a = b$, prove Fb from the assumption Fa (and *vice versa*), where the predicate F is *arbitrary*.

Identity is a kind of *indistinguishability*.

Doesn't identity become necessary, *automatically*?

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\succ a = a}\,\mathit{Refl}}{\succ \Box(a = a)}\,\Box I}{\succ \lambda x.\Box(a = x)a}\,\lambda\mathit{Df}{\downarrow} \qquad \dfrac{}{a = b, \lambda x.\Box(a = x)a \succ \lambda x.\Box(a = x)b}\,{=}\mathit{L.ax}_1}{\dfrac{a = b \succ \lambda x.\Box(a = x)b}{a = b \succ \Box(a = b)}\,\lambda\mathit{Df}{\uparrow}}\,\mathit{Cut}
$$

Willamson *defends* the necessity of identity



Williamson on the necessity of identity

…to mess with the modal or temporal logic of identity in order to avoid ontological inflation would be a lapse of methodological good taste, or good sense, for it means giving more weight to ontology than to the vastly better developed and more successful discipline of logic.

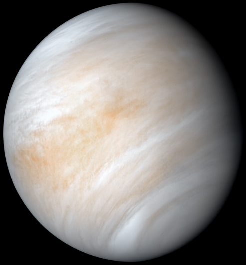
More specifically, the classical modal or temporal logic is a strong, simple, and elegant theory. To weaken, complicate, and uglify it without overwhelming reason to do so merely in order to block the derivation of the necessity or permanence of identity would be as retrograde and wrong-headed a step in logic and metaphysics as natural scientists would consider a comparable sacrifice of those virtues in a physical theory.

— Timothy Williamson, *Modal Logic as Metaphysics*, pp. 26, 27.

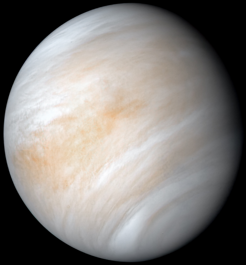
Modal Logic with Contingent Identity

Giovanna Corsi "Counterpart Semantics" 2002

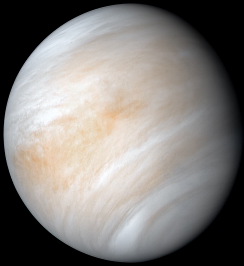
Maria Aloni "Individual Concepts in Modal Predicate Logic" 2005



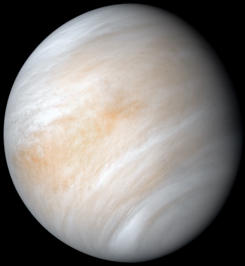
Hesperus



Phosphorous



Hesperus



Phosphorous

$$h = p \mid h \neq p$$

Identity and Context Shift

$$a = b \succ \quad | \; Fa \succ Fb$$

Is it consistent to grant that $a = b$ in one context, while distinguishing a from b in another?

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Subjunctive shifts (*planning*): This is much less plausible.

($h = p$. Were I to travel to h, then of necessity, I am going to p.)

More from Williamson

> … we are not interested in epistemic readings of 'it is possible that'.
>
> — Timothy Williamson, *Modal Logic as Metaphysics*, p. 11, fn. 17.

Subjunctive Context Shifts: *Necessary* Identity Rules

$$\dfrac{Fa \succ Fb \mid X \succ Y \mid \mathcal{S} \qquad Fb \succ Fa \mid X \succ Y \mid \mathcal{S}}{X \succ a = b, Y \mid \mathcal{S}} {=}Df$$

(the inference line is printed as a double line)

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To deny $a = b$ is to take a and b to be distinguish*able*.

$$
\dfrac{
\dfrac{
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\qquad
\dfrac{a = b \succ a = b}{a = b \succ \ \mid Fb \succ Fa}\,{\scriptstyle =Df\uparrow}
}{a = b \succ \ \mid \ \succ a = b}\,{\scriptstyle =Df\downarrow}
}{a = b \succ \Box a = b}\,{\scriptstyle \Box Df\downarrow}
$$

Epistemic Shifts: Contingent Identity

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  - To distingish a from b we want to find some *feature* a has that b lacks (or *vice versa*). Epistemic modalities do not always generate features.
  - After all, it's no argument against $h = p$ that it's epistemically necessary that $h = h$ and not epistemically necessary that $h = p$.

Two Kinds of Predicates

F: *feature predicates*   G: *general predicates*

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- Feature predicates are closed under the classical connectives and quantifiers and λ.
- Applying λ into $\Box_E$ contexts creates a *general* predicate.
- There are two kinds of *Spec* rule: one for feature predicates, and one for general predicates.

Contingent Identity Rules

$$\frac{X, Fa \succ Fb, Y \mid \mathcal{S} \qquad X, Fb \succ Fa, Y \mid \mathcal{S}}{a = b, X \succ Y \mid \mathcal{S}} {=}Df$$

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Two Dimensions — with *actuality*

$$
\begin{array}{ccccccccc}
X_1^1 \succ_{@} Y_1^1 & \mid & X_2^1 \succ Y_2^1 & \mid & \cdots & \mid & X_{m_1}^1 \succ Y_{m_1}^1 & \| \\
X_1^2 \succ_{@} Y_1^2 & \mid & X_2^2 \succ Y_2^2 & \mid & \cdots & \mid & X_{m_2}^2 \succ Y_{m_2}^2 & \| \\
\vdots & & \vdots & & & & \vdots & \\
X_1^n \succ_{@} Y_1^n & \mid & X_2^n \succ Y_2^n & \mid & \cdots & \mid & X_{m_n}^n \succ Y_{m_n}^n &
\end{array}
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The zones in correspond to *worlds*,
the formulas in the LHS are *true*
and those in the RHS are *false*.

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- Take the quotient of the terms by identity to form your 'domain'.
- The expected truth conditions hold for the logical operators.
- These are regular models for constant domain S5 with necessary identity.
- (The usual rules for quantifiers work, too.)

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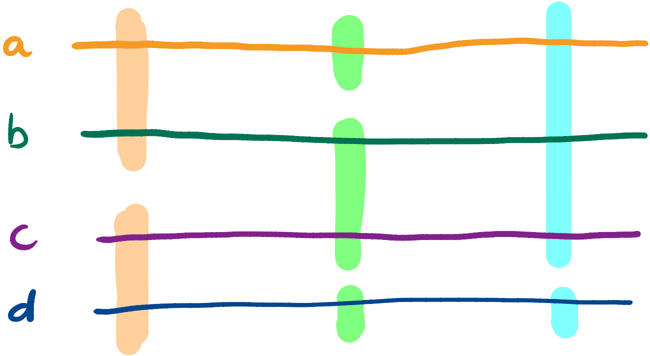

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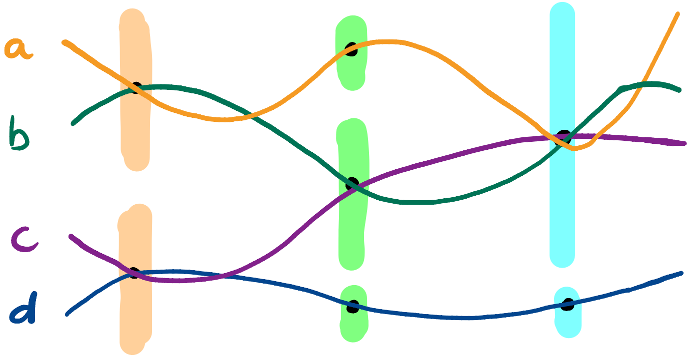
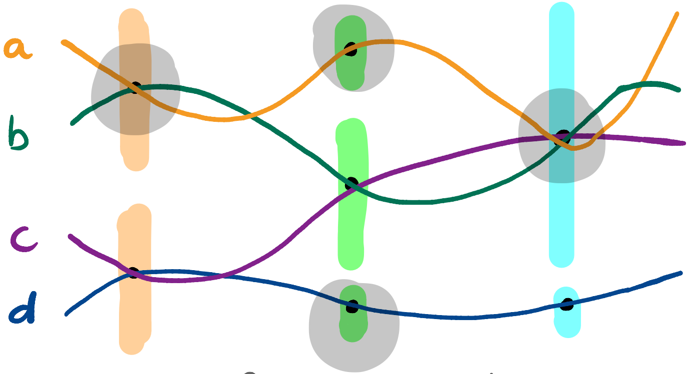


Contingent Identity Models

Fa, Fb   Fc, Fd   Fa, Fd   Fb, Fc   Fa, Fb, Fc   Fd

$$a = b, c = d \succ b = c \mid b = c \succ a = d \mid a = b, b = c \succ a = d$$



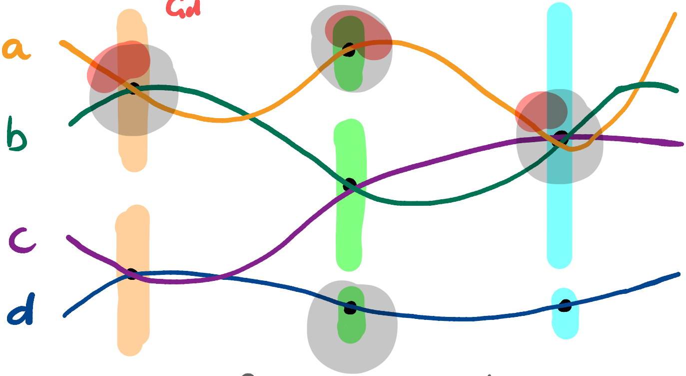
F: feature predicate

Contingent Identity Models

Fa, Fb Fc, Fd Fa, Fd Fb, Fc Fa, Fb, Fc Fd

$$a = b, c = d \succ b = c \mid b = c \succ a = d \mid a = b, b = c \succ a = d$$

Ga Gb, Gc, Gd Ga Gb, Gc, Gd Ga Gb, Gc, Gd



F: feature predicate

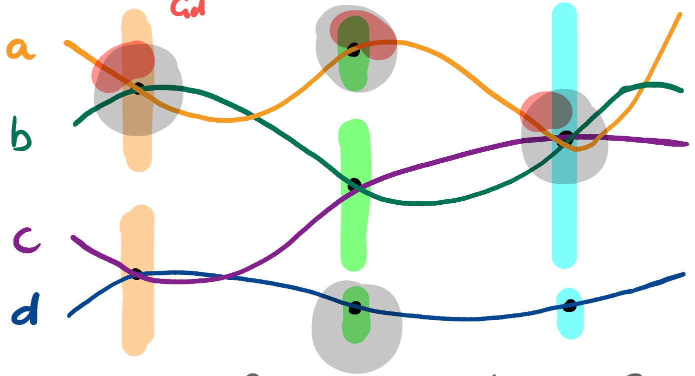
G: general predicate

Contingent Identity Models

Fa, Fb — Fc, Fd — Fa, Fd — Fb, Fc — Fa, Fb, Fc — Fd

$$a = b, c = d \succ b = c \mid b = c \succ a = d \mid a = b, b = c \succ a = d$$

Ga — Gb, Gc, Gd — Ga — Gb, Gc, Gd — Ga — Gb, Gc, Gd



F: feature predicate

G: general predicate

$Gx \equiv \mathsf{D}Fx$

BEYOND

Where to from here?

1. *Philosophically*:

   What speech acts and parts of speech play semantically significant roles?

   If we start from the normative pragmatics, what should we look for in a *model*?

2. *Formally*:

   There are many open questions around hypersequent calculi:

   Cut elimination procedures, expressive power, complexity,

   identity of proofs, and more . . .

Thank you!

SLIDES: `https://consequently.org/presentation/2021/pmpl-blc`

FEEDBACK: `@consequently` on *Twitter*,
or *email* at `greg@consequently.org`