Geometric Models for Relevant Logics

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May 14, 2020

ABSTRACT

Alasdair Urquhart's work on models for relevant logics is distinctive in a number of different ways. One key theme, present in both his undecidability proof for the relevant logic R, and his proof of the failure of interpolation in R, is the use of techniques from geometry. In this paper, inspired by Urquhart's work, I explore ways to generate natural models of R^+ from geometries, and different constraints that an accessibility relation in such a model might satisfy. I end by showing that a set of natural conditions on an accessibility relation, motivated by geometric considerations, is jointly unsatisfiable.

1. MODELS FOR RELEVANT LOGICS

If a conditional is to be *relevant*—if $A \to B$ is to be true only when there is a genuine *connection* between the antecedent A and the consequent B—any 'worlds' semantics for that conditional must look rather unlike the well-known modal semantics for strict conditionals, counterfactuals and other non-classical conditional connectives. If I wish to evaluate the conditional $A \to B$ at some 'world' x, it will never suffice to find some class of worlds, related to x (whether that choice depends on A, or on B, or on anything else) and then check of those worlds where A is true, whether B is true at those selected worlds, too. For then, the identity conditional $A \to A$ (in which the consequent is identical to the antecedent) is guaranteed to be true at absolutely any world whatsoever. You may not think that this is a problem, since the conditional $A \to A$, seems to satisfy the canons of relevance as well as any conditional does, and more than most,[1] but

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[1]Of course the identity conditional $A \to A$ might be seen to be problematic for other reasons, [4, 5], but this to explore non-circular logics, in which true conditionals do not beg the question, would take us altogether too far away from our current topic.

there is a problem for relevance nonetheless: For consider the status of the conditional $p \to (q \to q)$ at the world x, where p and q are atoms, chosen to have nothing in common at all. Select worlds, related to x, for checking in whatever manner your semantics dictates. If we want to know whether all of those worlds where p is true, also have $q \to q$ true, we know the answer already: It's a 'yes', since $q \to q$ is true *everywhere*. But there is no requirement that $q \to q$ have anything to do with p. To require this is to enforce as blatant a violation of relevance as we should ever expect to see.

So, a worlds semantics for a relevant conditional must differ from the standard worlds semantics for strict or counterfactual conditionals. One of Alasdair Urquhart's key insights was the appropriate modification is not that difficult. We separate the world at which we check the conditional's *antecedent* from the worlds where we check its *consequent*. For what has been come to be called Urquhart's *semilattice* semantics, a conditional $A \to B$ holds at a world x if and only if for every world y at which A is true, the consequent B is true at the world $x \sqcup y$. What *is* this world $x \sqcup y$? It is found by *combining* x and y, with a semilattice (that is, commutative, associative and idempotent) operation $\sqcup$. This interpretation of a conditional makes some intuitive sense if we think of a conditional $A \to B$ as saying that we have the means to use whatever verifies A to produce something that verifies B. The x is the resource that we have, the y is the input (whatever verifies A) and the means of production is the *combination* (with $\sqcup$) of our initial resource x with the input y.

The points in this kind of model look rather less like *worlds* than the points of a Kripke model for a modal logic do, so Urquhart uses a different name for them. In his 1972 paper [12], he calls them *pieces of information*. It is easy to see how pieces of information could plausibly be combined to give us new pieces of information, in a mode of combination that turns out to be associative, commutative and idempotent. Rather than follow Urquhart in calling our points 'pieces of information', I will use the more neutral 'point' in what follows, not only because the name is shorter, but also because our target models will be geometries, in which the points are, well, *points*.

When we utilise a semilattice combination operation on points to interpret our conditional, we have all the tools necessary to allow for identity conditionals $A \to A$ to fail at some of our points. All we need is are points x and y where A holds at y but fails at $x \sqcup y$. If this occurs, we have a counterexample to $A \to A$ at x. If our information combination is not *cumulative* (if the result of combining y with x does not preserve all the information given by x alone) then the point x gives us the means to convert A-points into non-A-points, because the application of x to y transforms something that verifies A into something that no longer does so.[2]

Of course, the fact that identity conditionals $A \to A$ can fail *somewhere* does not mean that we want them to fail *everywhere*. Urquhart's semilattice semantics has a facility for this. A semilattice model has a *zero* point 0 where $0 \sqcup x = x$, for each point x. Then

[2] There are many different things we could say about how we might interpret 'combination', and much ink has been spilled on this very issue [1, 3, 6, 7, 11], both concerning this semilattice semantics and its generalisation, the ternary relational semantics of Routley and Meyer [9, 10]. It is not my place in this short paper to address those issues. Instead, we will look at how geometries provide a rich playground for developing models for these logics.

it is straightforward to verify that at the zero point, $A \to A$ is true. The logical truths of the logic $\mathsf{R}_{\to}$ (the implicational fragment of R) are those formulas which hold at the zero point in each model, not those formulas that hold everywhere.

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Urquhart's elegant semilattice semantics faces difficulties when we extend it with other connectives, specifically *disjunction*. Once disjunction is present in the language, we are faced with a dilemma. A natural thought is to take a disjunction to hold at a point if and only if at least one of the disjuncts holds at that point. But this sits uneasily with the modelling clause for a conditional. Take a point supporting $p \to q \vee r$ and *apply* it (using our combination operator) to some point supporting p. By the interaction between the application operation and the conditional, the resulting point will satisfy $q \vee r$. For classical logic, this would be no problem, since $p \to q \vee r$ entails $(p \to q) \vee (p \to r)$ if the conditional is interpreted classically. But this entailment fails Urquhart's target logic R^+.[3] The dilemma is this: either complicate the semantics for disjunction, so a disjunction can hold at a point when neither disjunct holds at that point,[4] or bite the bullet and build a semantics for a logic other than R. That is the basic choice for a semantics in which the conditional is interpreted with an operation like $\sqcup$ on points.

We can avoid the disjunction dilemma entirely if we generalise the semilattice semantics just a little. Instead of thinking that the application of a point x to a point y is *deterministic*, resulting in one and only one point $x \sqcup y$, we could take it to be *non-deterministic*. We have a ternary relation R, according to which $Rxyz$ if and only if z is a possible result of applying x to y. The result is Routley and Meyer's ternary relational semantics [9, 10]. This semantics can be thought to be a generalisation of Urquhart's semilattice semantics (though they were developed independently), where we move from the specific case of a ternary relation defined by a binary operation—$x \sqcup y = z$—to the more general case of a ternary relation. In ternary relational models, we can evaluate disjunctions in using the straightforward evaluation clause without risk of overshooting the logic R^+. In general, a ternary relational model has this structure: A FRAME is a 4-tuple $\langle P, R, \sqsubseteq, N\rangle$, where P is a non-empty set of points, R is a ternary relation on P, $\sqsubseteq$ is a binary relation on P, and N is a subset of P, where the following conditions are satisfied.

- $\sqsubseteq$ is a partial order on P.
- R is $\sqsubseteq$-downward preserved in the first two positions, and $\sqsubseteq$-upward preserved in the third. That is, if $Rxyz$ and $x^- \sqsubseteq x$, $y^- \sqsubseteq y$ and $z \sqsubseteq z^+$ then $Rx^-y^-z^+$.
- $y \sqsubseteq z$ if and only if there is some x where Nx and $Rxyz$.

Here, N generalises the *zero* point 0 of Urquhart's models, and $\sqsubseteq$ is an inclusion relation on the information carried by points. In some frames, this relation is the *identity*

[3]This failure need not be blamed on any particular property of *relevance*. The entailment from $p \to q \vee r$ to $(p \to q) \vee (p \to r)$ fails in intuitionistic logic, too.

[4]This is Lloyd Humberstone's approach, in his "Operational Semantics for Positive R" [2].

relation, in which case the preservation condition on R is vacuously satisfied. In other frames, a non-vacuous inclusion relation plays an important role.

Then, given a frame we can evaluate formulas at points in the usual way. First, we fix truth for *atoms* at points, imposing the constraint that for any atom, p, if $x \sqsubseteq x^+$ then when $x \Vdash p$ we have $x^+ \Vdash p$ too. The evaluation relation $\Vdash$ generalises to the whole language with the following inductive truth conditions:

- $x \Vdash A \to B$ iff for each y, z where $Rxyz$, if $y \Vdash A$ then $z \Vdash B$
- $x \Vdash A \land B$ iff $x \Vdash A$ and $x \Vdash B$
- $x \Vdash A \lor B$ iff $x \Vdash A$ or $x \Vdash B$

The frames provide the means to interpret two more logical concepts, $\circ$ and t, as follows:

- $x \Vdash A \circ B$ iff for some y, z where $Ryzx$, $y \Vdash A$ and $z \Vdash B$
- $x \Vdash t$ iff $x \in N$.

We say that A *entails* B in a model iff whenever $x \Vdash A$, then $x \Vdash B$ too, and that A and B are *equivalent* (on the model) if A entails B and vice versa. It is a good exercise to show that in any model of this kind, $A \circ B$ entails C if and only if A entails $B \to C$, and $t \circ A$ is equivalent to A.

In the general class of ternary relational models, we impose no constraints analogous to the requirement that $\sqcup$ be a semilattice operator. This means that ternary relational frames can provide models for logics much weaker than R^+. To model R^+, we can impose conditions on the ternary relation analogous to associativity, commutativity and idempotence for $\sqcup$.

- If $(\exists u)(Rxyu \land Ruzw)$ then $(\exists v)(Ryzv \land Rxvw)$
- If $Rxyz$ then $Ryxz$
- $Rxxx$

It must be said that while the ternary relational semantics gains marks for generality, it loses some with regards to elegance and simplicity. The model theory is cumbersome, in that the semilattice operator $\sqcup$ (with identity 0) is replaced by a triple $R, \sqsubseteq, N$ of a three-place relation, a two-place relation and a set, with interconnecting conditions between all three items. Despite this increase in complexity, the gain in *generality* allows us to consider interesting models, and in particular, models which allow us to engage our *geometric* intuitions.

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One reason that geometry is such a natural partner for ternary relational semantics is the ternary relation R at the heart of our models. Ternary relations are, naturally, harder to reason with than binary relations or sets. *Geometry* is a natural domain providing *genuine* ternary relations. One of the simplest naturally occurring ternary relations in a geometric setting is the *collinearity* relation: x, y and z are collinear ($Cxyz$) iff there is

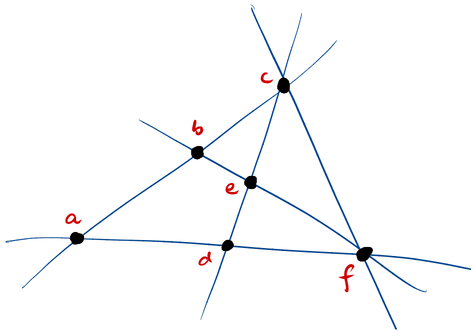

Figure 1: Collinearity and Betweenness

some line on which they all occur. Consider the diagram in Figure 1. Here, $Cabc$ and $Ccde$, hold (for example) but none of $Cabd$, $Cabe$ or $Cabf$ or $Cabg$ hold. Furthermore, if $Cxyz$ holds, then so does $Cyxz$, $Cyzx$, and any other statement we can find by permuting the terms in the predication.

Another natural relation on some spaces (for example, each real euclidean space $\mathbb{R}^n$) is the *betweenness* relation B where $Bxyz$ holds iff $Cxyz$ and y is *between* x and z on the line xz. If we think of the diagram above as depicting the real plane $\mathbb{R}^2$, we have $Babc$ but we do *not* have $Bbac$, since a does not occur between b and c.

Collinearity and betweenness are two examples of naturally occurring ternary relations, so we would hope that they might provide us a way to model relevant logics like R^+. However, as Urquhart noticed, the specifics of ternary relational models make this connection *close* but the fit is not exact. Consider first the order relation $\sqsubseteq$ in ternary relational models. We have whenever $x \sqsubseteq x'$ then if $Rxyz$ then $Rx'yz$. If R is collinearity or betweenness, then this order $\sqsubseteq$ collapses into identity, since given any point x, the set of pairs $\langle y, z\rangle$ such that x is collinear with y and z (or between y and z) uniquely determines the point x. There is no *other* point also satisfying those conditions. So, in spaces like these, the order relation must be identity.

This, then, puts pressure on the presence of normal points, those in the set N, in our frame if the relation R is anything like collinearity or betweenness for points. Whenever $x \in N$ we must have $Rxyz$ only when $y \sqsubseteq z$, which in our case means $y = z$. In a projective space or an affine space, lines have at least *three* points. There is no point x where for every y and z, x, y and z are collinear only when $y = z$. For any point x we can find *some* distinct points y and z where x, y and z are collinear, or indeed, where x is between y and z. So, no point in this space will count as *normal*.

So, although betweenness and collinearity provide elegant and straightforward ways to depict natural and mathematically rich ternary relations, these spaces do not provide ternary relational models for relevant logics like R^+, in and of themselves. They must undergo some corrective surgery, in order to satisfy the conditions for being a ternary relational model, where in this case, the required operation involves the addition of an extra point (or more) to provide us with some normal points of our model, and the expansion of the relations $\sqsubseteq$ and R to incorporate this new point (or points). This can be done in a general and uniform way, independently of the detail of the ternary relation involved in the underlying frame. To see how we can do this most easily, however, it will help to change our perspective on ternary relational frames just a little, and so we turn to this in our next section. It will turn out, too, that when we take this wider perspective, we will see how we could, in fact, do without those additions, and these geometric spaces will provide models for relevant logics all by themselves, though the logics that are so modelled prove somewhat different from R and its familiar cousins.

2. COLLECTION FRAMES

In a ternary frame $\langle P, R, \sqsubseteq, N\rangle$, we have a non-empty set P of points, a ternary relation R, a binary relation $\sqsubseteq$, and a set (or a unary 'relation' or property) N. *Three*, *two*, *one*. The key insight involved in understanding collection frames is taking these three pieces of data, the ternary, binary and unary R, $\sqsubseteq$ and N as facets of one underlying *multi*-ary relation, which relates *collections* of points to points. A *collection* relation. The R of a ternary frame is given by the case of the collection involving *two* items, the $\sqsubseteq$ is the case of the collection involving *one* item, and N is given in the case where the collection is empty. The coherence conditions, connecting N, $\sqsubseteq$ and R then converge into a single condition governing the one underlying collection relation. Different logics can then be modelled not only by imposing different conditions on the collection relation, but also by different choices for what kinds of collections our relation relates. For example, it is one thing to think of our points as collected together in some kind of order (say, a list), so the question of whether $\langle a, b\rangle Rc$ or not may have a different answer to the question of whether $\langle b, a\rangle Rc$ or not. If we move from lists to *multisets* (which keep track of the multiplicity of membership, but not order) so $[a, b]$ is the same multiset as $[b, a]$ but it differs from $[a, a, b]$, then the fact that $[a, b]Rc$ holds if and only if $[b, a]Rc$ holds is not so much a special constraint holding of the collection relation R, but rather an inevitability, given that R here relates multisets.

But for this paper we will go further, and consider relations on *sets*. Here, not only does $\{a, b\}Rc$ hold iff $\{b, a\}Rc$, but $\{a, a\}Rc$ holds iff $\{a\}Rc$. The contraction rule holds in a very strong form in set frames. Multiplicities of membership are ignored completely. We are in the realm of models for R^+, and this will be a natural place to explore geometric models, because from this perspective, we can see our frames in a new light.

A reflexive *set* frame is a pair $\langle P, R\rangle$ consisting of a non-empty set P of points, and a relation R on $\mathcal{P}^{\text{fin}}(P) \times P$, relating finite subsets of P to elements of P. A possible intuitive understanding of XRy is that the points collected together in X can be represented by the individual point y.[5] For a reflexive set frame, relation R must satisfy the following two conditions:

1. $(\forall x \in P)(\{x\}Rx)$.
2. $(\forall X, Y \in \mathcal{P}^{\text{fin}}(P))(\forall z \in P)((X \cup Y)Rz \leftrightarrow (\exists y \in P)(YRy \wedge (X \cup \{y\})Rz)$,

[5] How 'represented' may be understood can, of course, vary from application to application, or model to model.

Condition 1 we call *REFLEXIVITY*, for obvious reasons, and Condition 2 is *COMPOSITIONALITY*. From right to left it tells us that if Y is R-related to some point y, which then, when bundled together with X, is R-related to z, then R also vouchsafes the relation between X together with Y to z.[6] From left to right, it tells us that this process can be reversed. If a set is a union of X and Y then whatever it R-relates to (say z) can be found as the target of an R relation of X together with some R-representative of Y.

This may be unfamiliar to you if you have not considered set relations before, but it turns out that relations satisfying the compositionality condition are widespread, and rather natural. Consider these set relations on ω, the natural numbers:

[MAXIMUM] Here, XRy if and only if y is the largest member of X, and is 0 if X is empty. Call this value, $\max(X)$. R, so defined, is clearly reflexive, since $\max\{x\} = x$. For compositionality, it suffices to notice that for any sets X and Y (whether empty or not) $\max(X \cup Y) = \max(X \cup \{\max(Y)\})$.

[SPECTRUM] $\{x_1, \ldots, x_m\}Ry$ iff for some naturals $n_1, \ldots, n_m$, we have $y = 0 + n_1x_1 + \cdots + n_mx_m$, where again, $\{\,\}Ry$ if and only if $y = 0$. So, XRy iff y is some sum (of any multiplicity) members of X. So, $\{1\}Rx$ for *every* x, $\{2\}Rx$ for *even* x, and in general, $\{n\}$ is related to the multiples of n (including 0), while $\{2, 3\}$ is related to every number other than 1 (including 0). This relation is reflexive by design, and proving compositionality is a straightforward case of spelling out the definition. If $(X \cup Y)Rz$ then $z = 0 + \Sigma n_ix_i + \Sigma m_jy_j$ for some choices of $x_i \in X$, $y_j \in Y$ and naturals n_i and m_j. Choose $0 + \Sigma m_jy_j$ for the number related to Y and we have $(X \cup \{0 + \Sigma m_jy_j\})Rz$ straightforwardly. Conversely, if YRy (since $y = 0 + \Sigma m_jy_j$ for some appropriate choices of values for m_j and y_j) and $(X \cup \{y\})Rz$, that is, we have $z = 0 + \Sigma n_ix_i + n\Sigma m_jy_j$, then clearly $(X \cup Y)Rz$ too.

These two set relations are very different. The relation MAXIMUM is *functional*. For each X there is a unique y where XRy. It follows that the binary relation $\sqsubseteq$ induced by R on ω, given by setting $x \sqsubseteq y$ iff $\{x\}Ry$, is the identity relation. On the other hand, the SPECTRUM relation is anything but functional. The ordering $\sqsubseteq$ induced by this relation relates n to each number $n \times x$.

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We have already introduced some of the notation that connects collection relations with the machinery underlying ternary frames. For any *set frame* $\langle P, R\rangle$ there is an underlying binary relation $\sqsubseteq$ on P, given by setting $x \sqsubseteq y$ iff $\{x\}Ry$. Since R is reflexive, so is $\sqsubseteq$. Since R is compositional, $\sqsubseteq$ is transitive. For if $\{x\}Ry$ and $\{y\}Rz$ then $(\{\,\} \cup \{y\})Rz$ and by compositionality, $(\{\,\} \cup \{x\})Rz$, and hence $\{x\}Rz$. So, we have a partial order on our frame, and the notation $\sqsubseteq$ for this binary relation is deserved.

What holds for the binary relation $\sqsubseteq$ on Routley–Meyer frames also holds for the ternary relation R, and the set N of normal points. We have the following fact:

[6] This may be reminiscent to a *Cut* rule, taking us from and $\Gamma, A \succ B$ and $\Delta \succ A$ to $\Gamma, \Delta \succ B$.

FACT 1 [FROM SET FRAMES TO ROUTLEY–MEYER FRAMES] *If* $\langle P, R\rangle$ *is a set frame, then* $\langle P, R, \sqsubseteq, N\rangle$*, where we define:*

- $Rxyz$ *iff* $\{x, y\}Rz$
- $x \sqsubseteq y$ *iff* $\{x\}Ry$
- Nx *iff* $\{\,\}Rx$

is a Routley–Meyer frame for R^{+}*, satisfying the standard frame conditions connecting* N*,* $\sqsubseteq$ *and* R*, and the usual frame conditions on* R *of associativity, commutativity and idempotence.*

Establishing this fact is an enjoyable matter of applying the reflexivity and compositionality conditions on R in specific cases. We have already proved that $\sqsubseteq$ is a partial order. To show that N is closed upward under $\sqsubseteq$ and that the induced ternary R is downward preserved in the first two positions and upward preserved in the third, we notice that these two facts:

If XRy and $y \sqsubseteq z$ then XRz
If $x \sqsubseteq y$ and $(X \cup \{y\})Rz$ then $(X \cup \{x\})Rz$

are all we need to show those conditions on N and on R, and they are both instances of the left-to-right parts of compositionality.

For the structural conditions on R for idempotence, symmetry and associativity, we reason as follows: We have $Rxxx$ since $\{x, x\}Rx$ is a restatement of $\{x\}Rx$, which is given by reflexivity. For symmetry, if $Rxyz$ then $\{x, y\}Rz$ and hence $Ryxz$. For associativity, if $Rxyu$ and $Ruzw$ then $\{x, y, z\}Rw$, and hence, there is some v where $\{y, z\}Rv$ and $\{x, v\}Rw$, i.e., $Ryzv$ and $Rxvw$.

With set frames in view, let's reconsider what it is for a model to interpret a *relevant* conditional. In Urquhart's semilattice semantics, we allowed application to fail to be cumulative: a formula A might be true at y but no longer hold true at $x \sqcup y$, and in that case, $A \to A$ would fail to hold at x. If we restate this in the context of set frames, it means that we allow $\{x, y\}Ry$ to fail, on occasions. By compositionality, this generalises. Sometimes we have XRy, but there is a subset X' of X, where $X'Ry$ fails. This is the fundamental requirement for a relation on a *relevant* set frame. The principle of *weakening* is satisfied if whenever XRy and $X' \subseteq X$ then $X'Ry$ too. In models for relevant logics, weakening can fail.

Set frames are a simple way to view the three distinct moving parts of a ternary frame (the normal worlds, the binary ordering relation and the ternary accessibility relation) as three distinct facets of one underlying *set* relation. They provide a natural way to understand frame models for relevant logics.[7]

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[7] For more details concerning set frames, and their cousins, multiset frames, list frames and the like, the reader is referred to the paper "Collection Frames" [8], written with Shawn Standefer.

With set frames in mind, let's return to the relations of collinearity and betweenness. How does revisiting these in the light of set frames change our perspective? We have already seen that when we consider these ternary relations on the set $\mathbb{R}^2$, we almost have a ternary frame, but there is nothing corresponding to the set of normal points that we would hope to find. This phenomenon occurs repeatedly with collection frames in general, and set frames in particular. Recall the two concrete examples of compositional set relations we have considered (*MAX* and *SPECTRUM*). Why did I not consider *MIN*, the natural dual of *MAX*? Isn't it the case that for every X and Y, $\min(X \cup Y) = \min(X \cup \{\min(Y)\})$? Well, *yes*, but only when Y is non-empty. In the case where Y is empty, we are left with no value to choose that allows compositionality to be satisfied. If we stipulated that $\min\{\} = n$, then we would have for any m, $m = \min\{m\} = \min(\{m\} \cup \{\})$ which, by compositionality would be $\min(\{m, n\})$ which would differ from m when $m > n$. So, compositionality fails of *MIN*, because there is no value to choose to relate to the *empty* set, in just the same way that there is no point you can choose on $\mathbb{R}^2$ with collinearity (or betweenness) that could count as *normal* in the Routley–Meyer sense.

There is a straightforward response to this problem in each of these cases. We have already seen that different kinds of relations are found with different kinds of *collections* to be related. Multisets differ from sets, which differ from lists, or leaf-labelled trees. In the same way, *inhabited* (that is, non-empty) sets differ from *sets*, if only in one special case. It is a very small step from compositional *set* relations, on $\mathcal{P}^{\text{fin}}(P) \times P$, which we have seen, to compositional *inhabited-set* relations, on $\mathcal{P}^{\text{fin}*}(P) \times P$, where $\mathcal{P}^{\text{fin}*}(P)$ is the set of all *inhabited* finite subsets of the underlying point set P. We can say that a reflexive inhabited-set frame $\langle P, R\rangle$ is given by a non-empty point set P and a relation R on $\mathcal{P}^{\text{fin}*}(P) \times P$ satisfying the same two conditions of reflexivity and compositionality as before:

1. $(\forall x \in P)(\{x\}Rx)$.
2. $(\forall X, Y \in \mathcal{P}^{\text{fin}}(P))(\forall z \in P)((X \cup Y)Rz \leftrightarrow (\exists y \in P)(YRy \land (X \cup \{y\})Rz)$,

except that now we require that in each place that the relation R appears, the left hand sides ($\{x\}$, $X \cup Y$ and Y and $X \cup \{y\}$) are required to be inhabited. It is easy to see that this holds if and only if Y is inhabited, since $\{x\}$ and $X \cup \{y\}$ are inhabited in virtue of their form. (You may wonder why we allow X to be empty. When X is empty, compositionality reads as follows: $(\forall Y \in \mathcal{P}^{\text{fin}*}(P))(\forall z \in P)(YRz \leftrightarrow (\exists y \in P)(YRy \land \{y\}Rz))$. In the case where R is reflexive, the left-to-right part is trivially true—choose z as a witness for the existential quantifier. For the right-to-left part, it is a natural generalisation of the transitivity of $\sqsubseteq$, which is a natural constraint, even when R relates only inhabited sets.)

On any frame $\langle P, R\rangle$, whether R is a set relation or merely an inhabited-set relation, we can define an evaluation relation $\Vdash$ between points and formulas in a natural way. We define $\Vdash$ on atoms, with the one proviso, that $\Vdash$ be *hereditary* along the order $\sqsubseteq$. If $x \Vdash p$ and $\{x\}Ry$, then $y \Vdash p$ too. With that, the relation is extended to complex formulas as follows:

- $x \Vdash A \land B$ iff $x \Vdash A$ and $x \Vdash B$

- $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$
- $x \Vdash A \to B$ iff for each y where $y \Vdash A$, if $\{x, y\}Rz$ then $z \Vdash B$
- $x \Vdash A \circ B$ iff there are y, z where $\{y, z\}Rx$, $y \Vdash A$ and $z \Vdash B$.

If the relation R is a set relation, defined on the empty set, then we can also give truth conditions for the Ackermann constant t, using the empty set.

- $x \Vdash t$ iff $\{\,\}Rx$.

A frame equipped with an evaluation relation $\Vdash$ is said to be a *model*. On these models we can define a straightforward notion of entailment. We say that A entails B according to some model when for any point x if $x \Vdash A$ then $x \Vdash B$.

Set frames and inhabited-set frames provide a natural setting for models for relevant logics with or without normal points. Once we turn our attention to *inhabited*-set frames, we have more natural examples of compositional relations:

[MEMBERSHIP] For any non-empty point set P, if we define XRy as holding if and only if $y \in X$, then R is a reflexive, compositional inhabited-set relation. It is straightforward that R is reflexive, and the compositionality condition is verified immediately, since for any inhabited set Y we can choose an appropriate member.

[SUBSPACE] On any affine or projective geometry with point set P, we can define a compositional inhabited-set relation R on its point set, by setting XRy iff y is in the smallest subspace of P containing each member of X. So, $\{x\}Ry$ iff $y = x$ (a point is a subspace of dimension 0), if $x \neq y$, then $\{x, y\}Rz$ iff z is on the line xy, if x, y and z are all distinct, $\{x, y, z\}Rw$ iff w is on the plane xyz, and so on. The collinearity relation generalises to higher dimensions in a natural way. The relation R, so defined, is compositional.

[INSIDE] If our space, like $\mathbb{R}^n$, comes equipped with a notion of *betweenness*, then the three place relation of betweenness generalises appropriately. We have for any finite set X of points, XRy if and only if y is inside the shape inscribed by the points X (including its boundary). To show that this relation is compositional, we use an elementary geometric argument. If $(X \cup Y)Rz$ then z is in the shape bound by $X \cup Y$, then we can find something inside Y (say, y) where z is also in the shape bound by $X \cup \{y\}$. (For example, in the diagram in Figure 1, the point e is inside the shape inscribed by $\{a, c, f\} = \{a\} \cup \{c, f\}$. We can find something inside the shape inscribed by $\{a, f\}$ (namely the point d) where $\{c, d\}Re$.) Conversely, if $(X \cup \{y\})Rz$ and YRy it is clear that $(X \cup Y)Rz$.

So, we have three compositional relations on the point set $\mathcal{P}^{\text{fin}*}(P)$, two of which have natural geometric meanings. Here, the fit is perfect. If we are willing to forego the existence of normal points, geometries provide very natural examples of inhabited-set frames, and these frames can be used to model relevant logics. Set frames are models the relevant logic R^+. (This fact is an immediate consequence of what we have proved. Ternary relational frames satisfying idempotence, commutativity and associativity are models for R^+ and any set frame generates a ternary relational frame satisfying these conditions.) Inhabited-set frames, on the other hand, are not always models for R^+. The absence of normal points makes a difference. To show this, it suffices to note that $((p \to p) \to q) \to q$ is a theorem of R^+, or equivalently, that $(p \to p) \to q$ entails q on all R^+ models. (The reasoning is straightforward. If $x \Vdash (p \to p) \to q$ then since $(\{x\} \cup \{\,\})Rx$ there is some y where $\{\,\}Ry$ and $\{x, y\}Rx$. Since $\{\,\}Ry$, we have $y \Vdash p \to p$ since for every z where $z \Vdash p$, if $\{y, z\}Rw$, then since $\{\,\}Ry$, $\{z\}Rw$, and hence, $w \Vdash p$, too. So, since $y \Vdash p \to p$, $x \Vdash (p \to p) \to q$ and $\{x, y\}Rx$, we have $x \Vdash q$ as desired.)

In inhabited-set frames, this reasoning does not apply, since we cannot appeal to the existence of some y where $\{\,\}Ry$. Not only does *that* reasoning not apply, we can find a counterexample to the entailment. For the counterexample, consider the inhabited-set frame on the real line $\mathbb{R}^1$ where XRy when y is in the interval bounded by X. On this frame, xy iff $\{x\}Ry$ iff $x = y$, so any set of points is a possible extension of an atomic formula, the heredity condition puts no constraints on where formulas can be true. So, let p be true at 0 and 1, but nowhere else. On this model, $p \to p$ is true *nowhere*. Take any x at all. We can find some y, where $y \Vdash p$ (either 0 or 1 will do) where there is some z between x and y where p fails. So, in this model, $p \to p$ is true nowhere, and it follows that $(p \to p) \to p$ is true at every point (vacuously). It follows from that that the point 2 (for example) provides a counterexample to the entailment from $(p \to p) \to p$ (which holds at 2) to p (which fails there). So, the argument from $(p \to p) \to p$ to p has a counterexample in this model, and so, the argument from $(p \to p) \to q$ to q also fails there. The logic of inhabited-set frames is weaker than the logic of set-frames. These frames are natural models of a substructural logic, but this logic is weaker than the relevant logic R in distinctive ways.

3. ADDING A NORMAL POINT

Of course, there reasons why, for certain applications, we may *not* like to forego the existence of normal points. In this section we will show that there are two distinct ways to add a single normal point to a non-empty set frame $\langle P, R\rangle$. The aim is to consider a new element $\infty \notin P$, and to define a new compositional relation between $\mathcal{P}^{\text{fin}}(P \cup \{\infty\})$ and $P \cup \{\infty\}$, extending the relation R, where in this new relation, the empty set $\{\,\}$ is related to the new element ∞, so this new element is a normal point in the extended frame.

If this newly defined relation is compositional, the result will be a R^+ frame, since all compositional set relations are R^+ frames. The key question to answer for any such extension R' is whether $XR'z$ holds or not, in cases where either $\infty \in X$ or $z = \infty$. In the case where $\infty \notin X$ and $z \neq \infty$, we will set R' to echo the verdict of R unchanged: $XR'z$ iff XRz.

We will see that there are two natural ways to extend R to $P \cup \{\infty\}$. We will call these R^+ and $R^\times$. Let's start with R^+. The choice we make for R^+ is straightforward:

$$XR^+z \quad \text{iff} \quad \text{either } (X \backslash \{\infty\})Rz \text{ or } z = \infty$$

That is, any set of points X is related to the new normal point ∞, and furthermore, the set of points X is related to one of the original points z iff the set X, with the point ∞

removed (if it was present at all), is related to that point z in the original frame. This is one choice for extending R to $\mathcal{P}^{\text{fin}}(\mathsf{P} \cup \{\infty\}) \times \mathsf{P} \cup \{\infty\}$. In this set frame, the added point ∞ is *above* each point in the ordering. We have $z \sqsubseteq^+ \infty$, since $\{z\}\mathsf{R}^+\infty$, for every z.

That is one natural way to extend R on non-empty finite sets to a new relation R^+ on all finite sets. This extension is simple, but it does significant shortcomings. For one, since the added point ∞ extends *every* point in the frame, there is no way for our frames to have divergent pairs of points. (Points x and y are divergent if there is no point z where $x \sqsubseteq z$ and $y \sqsubseteq z$). Since x and y converge in ∞, any apparently opposing positions x and y take on some claim is 'resolved' in ∞, which simply agrees with both x *and* y. This puts pressure on the interpretation of negation on our frames, and the idea that some points might be *worldlike*, in the sense of being comprehensive states of affairs. It is one thing for a logic to be paraconsistent (as the relevant logic R is), it is another to say that any and all circumstances can be subsumed into the one all-encompassing situation, ∞. The second issue with such an extension is that any compositional relation R^+ with such a point ∞ must satisfy the converse of the weakening condition: That is, we will have whenever $X\mathsf{R}^+y$ and $X \subseteq X'$ then $X'\mathsf{R}^+y$ too. It is straightforward to verify that if $X\mathsf{R}^+y$, then since $(X\cup\{\,\})\mathsf{R}^+y$ we have $(X\cup\{\infty\})\mathsf{R}^+y$ and since $X'\mathsf{R}^+\infty$, we conclude by compositionality $(X \cup X')\mathsf{R}^+y$. So, if $X \subseteq X'$ we have $X'\mathsf{R}^+y$, too. So, if R^+, so defined is compositional, it satisfies the converse of the weakening condition. This means that R^+ is compositional only when the underlying relation R already satisfied the converse of the weakening condition. We have seen some compositional inhabited-set frames that satisfy this condition (MEMBERSHIP, SUBSPACE and INSIDE all do so) while others do not (MAX and MIN notably do not), so R^+ will work as a technique for converting an inhabited-set frame into a set frame for only some of our frames.

It is just as well, then, that there is another natural way to extend R to a compositional relation on all finite sets, with the addition of ∞, and this relation does not have the constraints exhibited by R^+. The relation $\mathsf{R}^\times$ is defined differently.

$$X\mathsf{R}^\times z \text{ iff } \begin{cases} z = \infty, & X = \{\,\} \text{ or } X = \{\infty\} \\ (X\backslash\{\infty\})\mathsf{R}z, & X \neq \{\,\} \text{ and } X \neq \{\infty\} \end{cases}$$

Here, the relation takes the opposite policy to R^+, which makes *everything* related to the new normal point. In this case, a set X is related to ∞ only in the two cases where it absolutely *has* to be so related. In the case $\{\,\}\mathsf{R}^\times\infty$ (which was the design goal, that we have *some* point as the target of the empty set) and $\{\infty\}\mathsf{R}^\times\infty$, which is demanded by the reflexivity of $\mathsf{R}^\times$. In all other cases, we say that a set is *not* related to ∞, and we say that a set X is related no one of our original points z if and only if the set X *without* ∞, is related to z in the original frame.

With these two definitions, we have the following fact:

FACT 2 *If* R *is a compositional inhabited-set relation on* P*, and* $\infty \notin \mathsf{P}$ *then* $\mathsf{R}^\times$*, defined above, is a compositional set-relation on* $\mathsf{P} \cup \{\infty\}$*. If* R *also satisfies the converse weakening* *condition, then* R^+*, also defined above, is also compositional. If* R *is reflexive, then so is* R^+ *and* $R^\times$.

The proof of this fact is a relatively straightforward set case analysis.

Proof: That R^+ and $R^\times$ are reflexive follows immediately from the reflexivity of R.

For compositionality, we will consider $R^\times$ first. Let's suppose that $(X \cup Y)R^\times z$, in order to find some y where $YR^\times y$ and $(X\cup\{y\})R^\times z$. By definition $(X\cup Y)R^\times z$ holds if and only if $z = \infty$ (if $X \cup Y$ is either empty, or $\{\infty\}$) or $((X \cup Y)\backslash\{\infty\})Rz$ (otherwise). Let's take these cases in turn. If $X \cup Y$ is either empty, or $\{\infty\}$ then clearly X and Y are both either empty or $\{\infty\}$, so in this case, both $YR^\times\infty$ and $(X \cup \{\infty\})R^\times\infty$, as desired. So, now consider the second case: we have $((X\cup Y)\backslash\{\infty\})Rz$ and $X\cup Y$ is neither empty nor $\{\infty\}$. We aim to find some y where $YR^\times y$ and $(X \cup \{y\})R^\times z$. If Y itself is empty or $\{\infty\}$, then we choose ∞ for y. We have, then, $YR^\times\infty$ and since $((X \cup Y)\backslash\{\infty\})Rz$, we have $(X\backslash\{\infty\})Rz$, so we have $(X\cup\{\infty\})R^\times z$ as desired. On the other hand, if Y has some element other than ∞, since $((X\cup Y)\backslash\{\infty\})Rz$, we have $((X\backslash\{\infty\})\cup(Y\backslash\{\infty\}))Rz$, and since R is compositional, there is some y where $(Y\backslash\{\infty\})Ry$ and $((X\backslash\{\infty\}) \cup \{y\})Rz$, which gives us $YR^\times y$ and $(X \cup \{y\})R^\times z$ as desired.

Now for the second half of the compositionality condition for $R^\times$, suppose that there is some y where $YR^\times y$ and $(X \cup \{y\})R^\times z$. We aim to show that $(X \cup Y)R^\times z$. If $YR^\times y$ then either $y = \infty$ and Y contains at most ∞, or otherwise $(Y\backslash\{\infty\})Ry$. In the first case, $(X \cup \{y\})R^\times z$ tells us that $(X \cup \{\infty\})R^\times z$, which means either that $(X\backslash\{\infty\})Rz$, or X also contains at most ∞ and then $z = \infty$. In the either of these cases, we have $(X \cup Y)R^\times z$, as desired. So, let's suppose $y \neq \infty$. In that case we have $(Y\backslash\{\infty\})Ry$, and then, since $(X \cup \{y\}))R^\times z$, we have $((X \cup \{y\})\backslash\{\infty\})Rz$, and by the compositionality of R, $((X \cup Y)\backslash\{\infty\})Rz$, which gives $(X \cup Y)R^\times z$, as desired.

Now consider R^+. Let's suppose that $(X \cup Y)R^+z$, in order to find some y where YR^+y and $(X \cup \{y\})R^+z$. By definition, $(X \cup Y)R^+z$ holds iff $((X \cup Y)\backslash\{\infty\})Rz$ or $z = \infty$. In the second case, we then have $(X \cup \{\infty\})R^+\infty$ and $YR^+\infty$ and we are done. In the first case, since $((X \cup Y)\backslash\{\infty\})Rz$ we have $((X\backslash\{\infty\}) \cup (Y\backslash\{\infty\}))Rz$, and so, by compositionality, there is some $y \in P$ where $(Y\backslash\{\infty\})Ry$ and $((X\backslash\{\infty\}) \cup \{y\})Rz$. It follows immediately that YR^+y and $(X \cup \{y\})R^+z$ as desired.

For the second half of the compositionality condition for R^+, suppose that there is some y where YR^+y and $(X \cup \{y\})R^+z$. We aim to show that $(X \cup Y)R^+z$. If $y = \infty$ then $(X\cup\{y\})R^+z$ ensures that XRz, and then we can appeal to the converse weakening condition, to get $(X \cup Y)Rz$, as desired. If $y \neq \infty$, then we have $(Y\backslash\{\infty\})Ry$. From $(X\cup\{y\})R^+z$ we have either $z = \infty$ (in which case $(X\cup Y)R^+z$ immediately), or $z \neq \infty$ and $((X\cup\{y\})\backslash\{\infty\})Rz$. In that case, $((X\backslash\{\infty\})\cup\{y\})Rz$, and by the compositionality of R, we have $((X\backslash\{\infty\}) \cup (Y\backslash\{\infty\}))Rz$, which gives us $(X \cup Y)R^+z$, as desired. ∎

So, we have two different ways to 'upgrade' inhabited-set frames to set frames. One technique (R^+) applies only to frames satisfying the converse weakening condition, while the other ($R^\times$) is applicable more widely. In fact, $R^\times$ is *essentially* relevant, in that the move from R to $R^\times$ fails to preserve weakening, if it is present. Where z is one of the original points in our frame, we have, by reflexivity, $\{z\}R^\times z$ in our new frame. However, the definition of $R^\times$ rules out $\{\}R^\times z$, so weakening fails. The addition of a normal point by the move from R to $R^\times$ is one which introduces a modicum of relevance, whether it was there in the original model, or not.

4. *FUNCTIONAL* GEOMETRIC SET FRAMES

There is something appealing about the idea that a set relation be *functional*. This was the approach of the semilattice semantics, after all. At least at the level of pairs $\{x, y\}$ there is a unique point $x \sqcup y$ to which this pair is related. I will end this short paper exploring some of the constraints around functionality in the setting of geometric models, to end with another fact, showing that a collection of plausible of different constraints on set relations on geometries are jointly incompatible. We will start with some properties that a compositional inhabited-set relation might satisfy, starting with functionality:

- R is FUNCTIONAL iff whenever $\emptyset \neq X \subseteq \mathbb{R}$, there is some unique y where XRy. The inhabited-set relations MAX and MIN are functional. So is any *constant* relation given by setting XRy iff $y = c$ for a given constant c. (It is easy to verify that such a relation is also compositional.)
- R is INCLUSIVE iff whenever $\emptyset \neq X \subseteq \mathbb{R}$ and XRy, then $y \in b(X)$, where $b(X)$ is the region *bound* by the set X. In the case of subsets of the real line, $b(X) = \{z : (\exists x_1 \in X)(\exists x_2 \in X)(x_1 \leq z \leq x_2)\}$, so $b(\{x\}) = \{x\}$, $b(\{x, y\}) = [x, y]$ if $x \leq y$, and in general, $b(\{x_1, \ldots, x_n\}) = [\min(x_1, \ldots, x_n), \max(x_1, \ldots, x_n)]$. The relations MAX and MIN are inclusive, as is the *betweenness* relation, but the *collinearity* relation (which is the *universal* relation on $\mathbb{R}$, but is not trivial in $\mathbb{R}^n$ for $n > 2$) fails to be inclusive.
- R is PRESERVED UNDER TRANSLATION iff whenever $\emptyset \neq X \subseteq \mathbb{R}$, and $x \in \mathbb{R}$, if XRy then $(X + x)R(y + x)$. The examples MIN, MAX, as well as all of our geometric examples, are preserved under translation. (This notion generalises in a natural way on spaces like $\mathbb{R}^n$ where points and regions can be translated across space and preserved under rotations, and other length-preserving transformations in space.)
- R is REGIONAL iff whenever the sets X and Y bound the same region (that is $b(X) = b(Y)$) then for all z, XRz iff YRz. Again, our examples of MIN, MAX, collinearity and betweenness are all regional in this sense.

We will show that only two relations jointly satisfy this set of conditions on $\mathbb{R}$.

FACT 3 *The only two compositional inhabited-set relations on $\mathbb{R}$ that are functional, inclusive, preserved under translation and regional are* min *and* max.

Proof: Take a functional inclusive regional compositional inhabited-set relation R on $\mathbb{R}$, which is preserved under translation, but is not min or max. For simplicity, let's write $\{x, y\}Rz$ as $x * y = z$, since R is functional. We automatically have $x * y = y * x$ (since

R is a *set* relation) and by inclusivity, if $x \leq y$ then $x \leq x*y \leq y$, so we have $x*x = x$, for each x.

Since R is not min, there are values $x < y$ where $x < x*y$, and since R is inclusive, we must have $x < x*y \leq y$. Similarly, since R is not max, we have some $u < v$ where $u*v < v$, and since R is inclusive, we must have $u \leq u*v < v$.

Now, since R is regional, we have $z*(x*y) = x*y$ for any $z \in [x,y]$, since for any regional relation R, $\{x,y,z\}Ra$ iff $\{x,y\}Ra$, since $b(\{x,y,z\}) = b(\{x,y\})$. This means that $x*y$ is an input for which the $*$ function acts *locally* like a *maximum*: for the values z in that non-empty interval $[x, x*y]$, we have $z*(x*y) = x*y = \max(z, x*y)$.

For the same reason, $w*(u*v) = u*v$ for each $w \in [u,v]$. This means that $u*v$ is an input for which the $*$ function acts *locally* like a *minimum*: for the values w in the non-empty interval $[u*v, v]$, we have $w*(u*v) = u*v = \min(w, u*v)$.

Given this, consider the real value $\epsilon = \min(x*y - x, v - u*v) > 0$. It is the length of the shorter of the two intervals $[x, x*y]$ and $[u*v, v]$. If we consider the interval $[x, x*y]$, and translate this across $\mathbb{R}$ so the right end has moved to ϵ. The interval $[0,\epsilon]$ is no longer than $[x, x*y]$ and since R is preserved under translation, we have $0*\epsilon = \epsilon$ since for any $z \in [x, x*y]$, $z*(x*y) = x*y$. Similarly, we can translate the interval $[u*v, v]$ so the *left* end moves to 0. The interval $[0,\epsilon]$ is also no longer than $[u*v, v]$, and since R is preserved under translation, we have $0*\epsilon = 0$ since for any $z \in [u*v, v]$ we have $z*(u*v) = u*v$. Since $\epsilon \neq 0$ we have a contradiction from $0*\epsilon = \epsilon$ and $0*\epsilon = 0$. It follows that our relation R cannot fail to be both MIN and MAX. ■

There are meagre choices, then, on the menu of compositional, functional, inclusive, regional inhabited-set relations on $\mathbb{R}$ that are preserved under translation. There are *fewer* choices for such relations on $\mathbb{R}^n$ for $n \geq 2$. We will end with this fact:

FACT 4 *There are* no *compositional, functional, inclusive, regional inhabited-set relations on* $\mathbb{R}^n$ *that are preserved under translation, when* $n \geq 2$.

Proof: If R is such a relation on $\mathbb{R}^n$, then its restriction to any line in $\mathbb{R}^n$ must still be compositional, functional, inclusive, regional and preserved under translation on that subspace. It must, therefore, be either MIN or MAX on that line. However, a line can be translated onto the its mirror reflection, in $\mathbb{R}^2$ (and any higher dimensional space) by rotation. The relations MIN and MAX are not preserved under this translation, rather, they are swapped. So, no relation on $\mathbb{R}^n$ $(n \geq 2)$ jointly satisfies all these criteria. ■

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