Collection Frames

What, how & why?

Greg Restall

University of
St Andrews

November 2022

COLLECTION FRAMES

What, how & why?

This talk reports joint work with

GREG RESTALL

Shawn Standefer

University of St Andrews

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This talk reports joint work with

Shawn Standefer

but he is not responsible for <u>any</u> of the controversial things I say here

GREG RESTALL

University of St Andrews

NOVEMBER 2022

MY PLAN

1. WHAT ARE MODELS FOR?
2. THE COMPLEXITY OF ROUTLEY/MEYER FRAMES
3. THREE INTO ONE
4. BEYOND REFLEXIVITY
5. EXAMPLES
6. OPEN QUESTIONS

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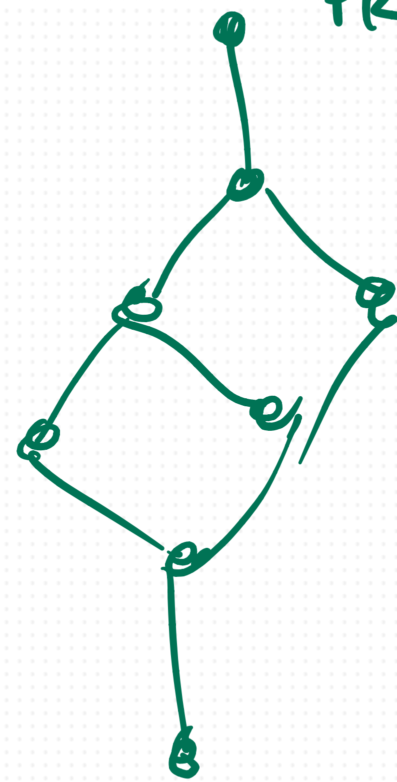
MODELS

LANGUAGE $\Longrightarrow$ THINGS

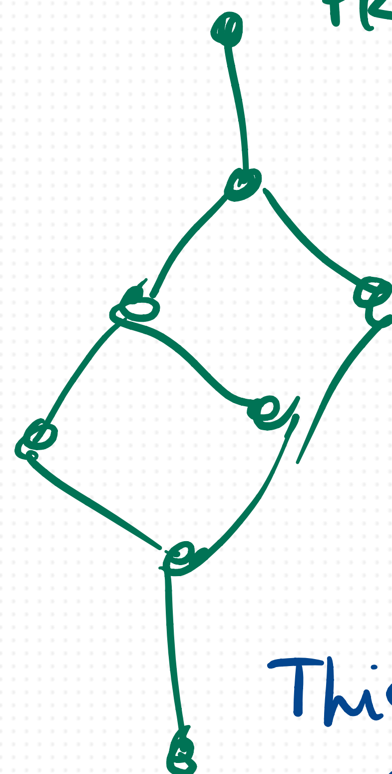
MODELS

LANGUAGE $\Longrightarrow$



MODELS

LANGUAGE $\Longrightarrow$



PROPOSITIONS

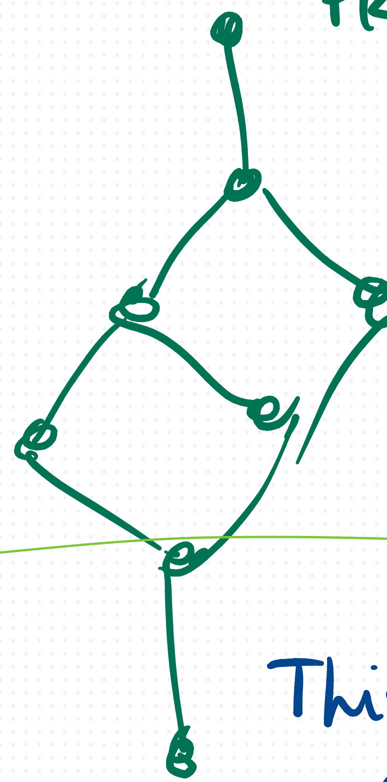
But what are these propositions?

This is a philosophical question & a mathematical one, too...

MODELS

LANGUAGE $\Longrightarrow$

PROPOSITIONS



But what are these propositions?

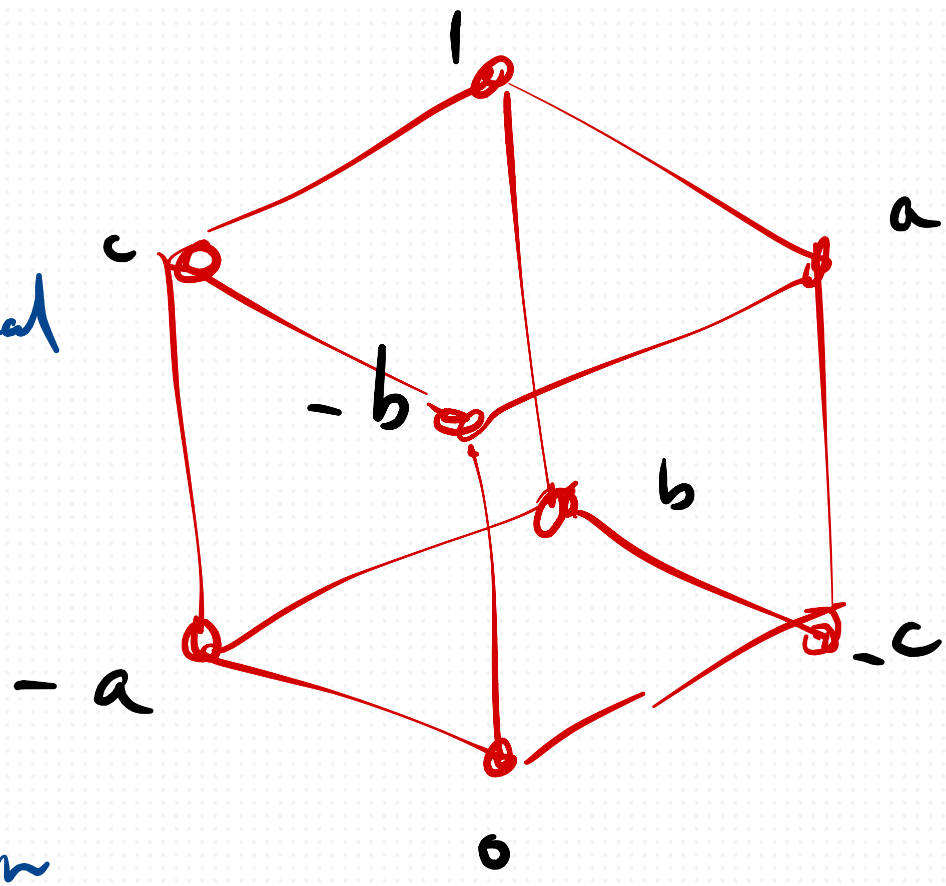
This is a philosophical question & a mathematical one, too...

MY FOCUS TODAY

AN EASY EXAMPLE

A Boolean Algebra is a propositional structure satisfying a simple set of equations on

$\wedge \quad \vee \quad - \quad 0 \quad 1$



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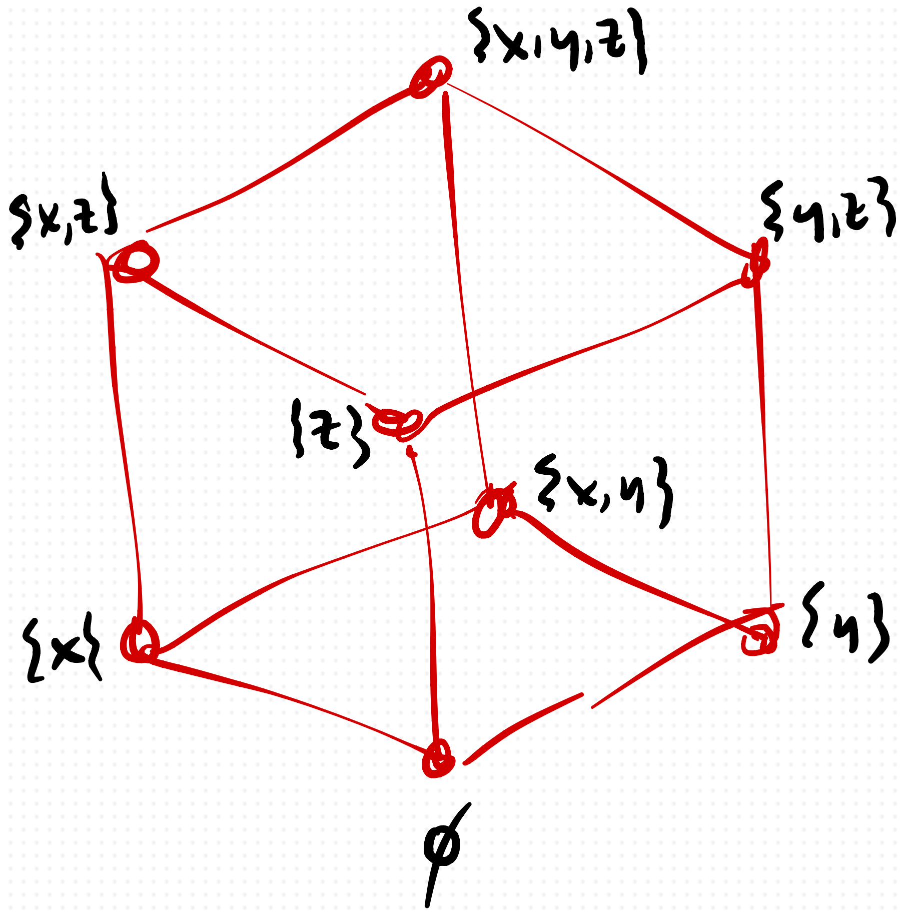

Every Boolean Algebra is isomorphic to a subalgebra of some powerset algebra on a set P

$\cap \ \cup \ \bar{\ } \ \emptyset \ P$

AN EASY EXAMPLE

A Boolean Algebra is a propositional structure satisfying a simple set of equations on

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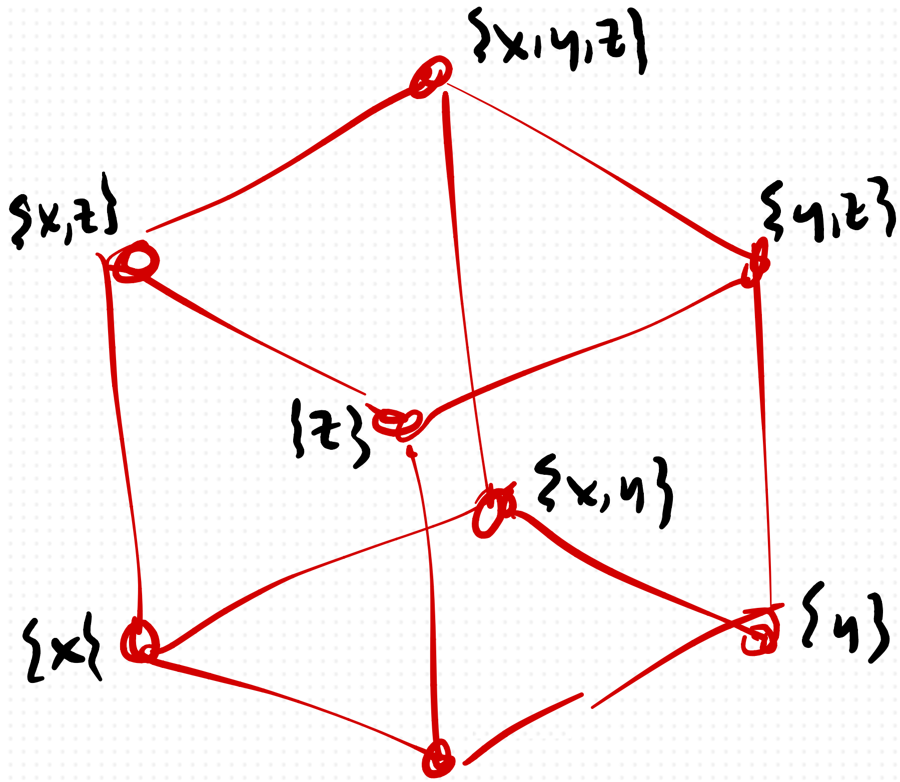


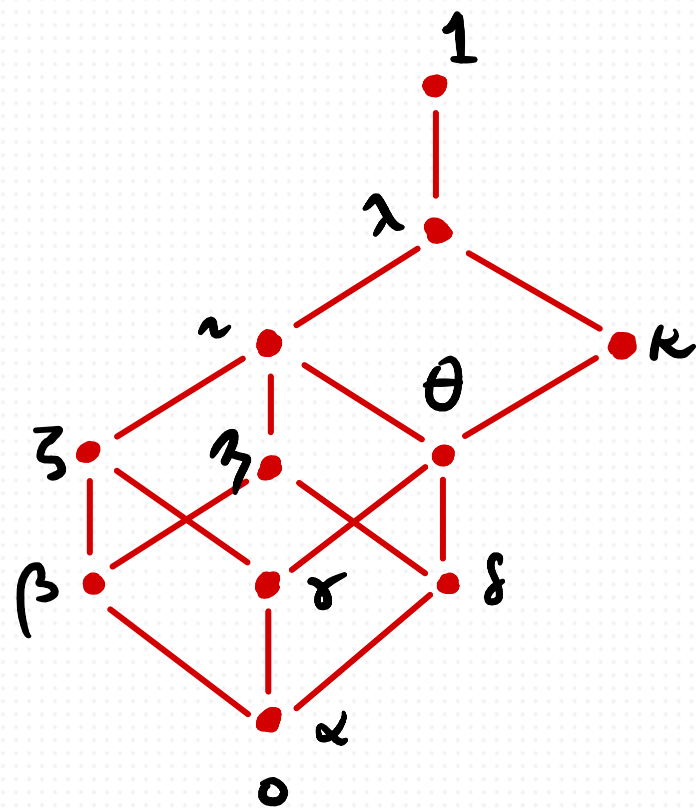
$\emptyset$

Every Boolean Algebra is isomorphic to a subalgebra of some powerset algebra on a set P

$\cap \quad \cup \quad \bar{} \quad \emptyset \quad P$

THIS IS PHILOSOPHICALLY & MATHEMATICALLY SUGGESTIVE

A slightly less EASY EXAMPLE



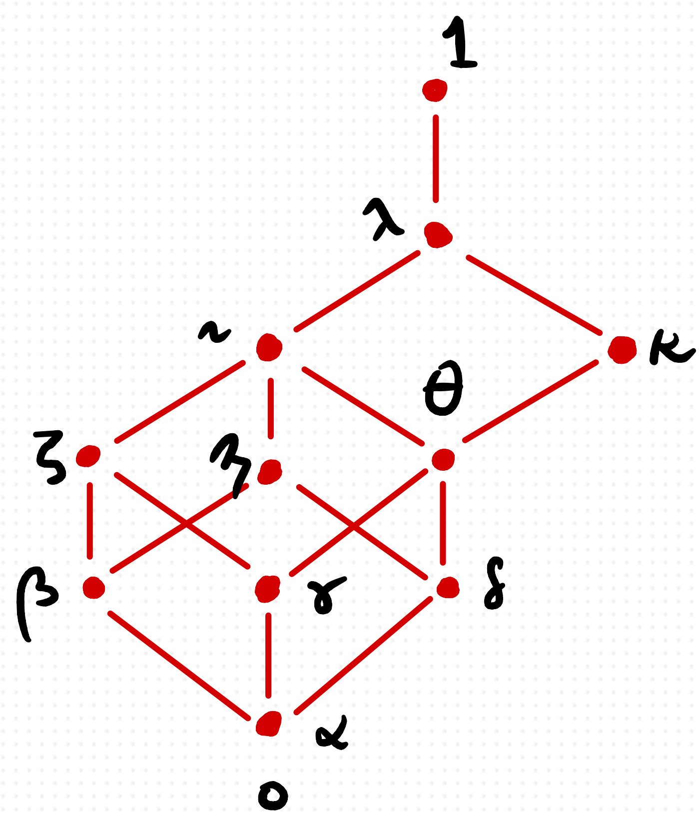
A Heyting Algebra is a propositional structure satisfying a simple set of equations on

$\wedge \quad \vee \quad \rightarrow \quad 0 \quad 1$

A slightly less Easy Example

A Heyting Algebra is a propositional structure satisfying a simple set of equations on

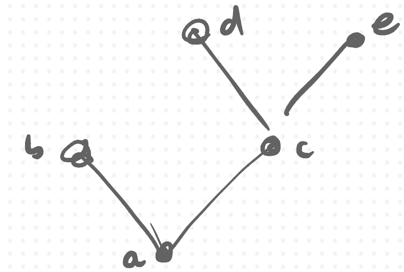
$\wedge \quad \vee \quad \to \quad 0 \quad 1$



Every Heyting Algebra is isomorphic to a subalgebra of open sets on a topological space T

$\cap \quad \cup \quad \Rightarrow \quad \emptyset \quad T$

A Slightly less EASY EXAMPLE



A Heyting Algebra is a propositional structure satisfying a simple set of equations on

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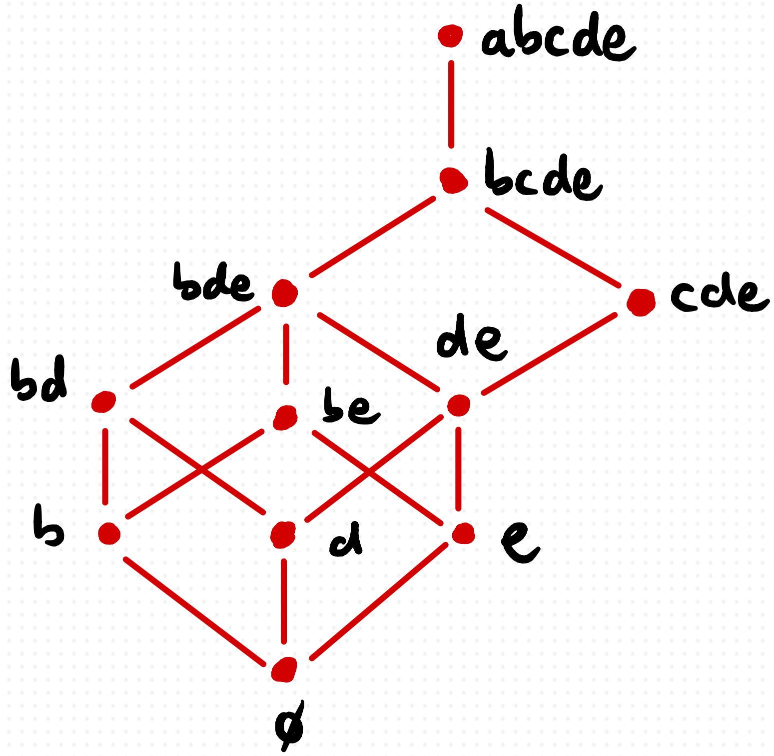
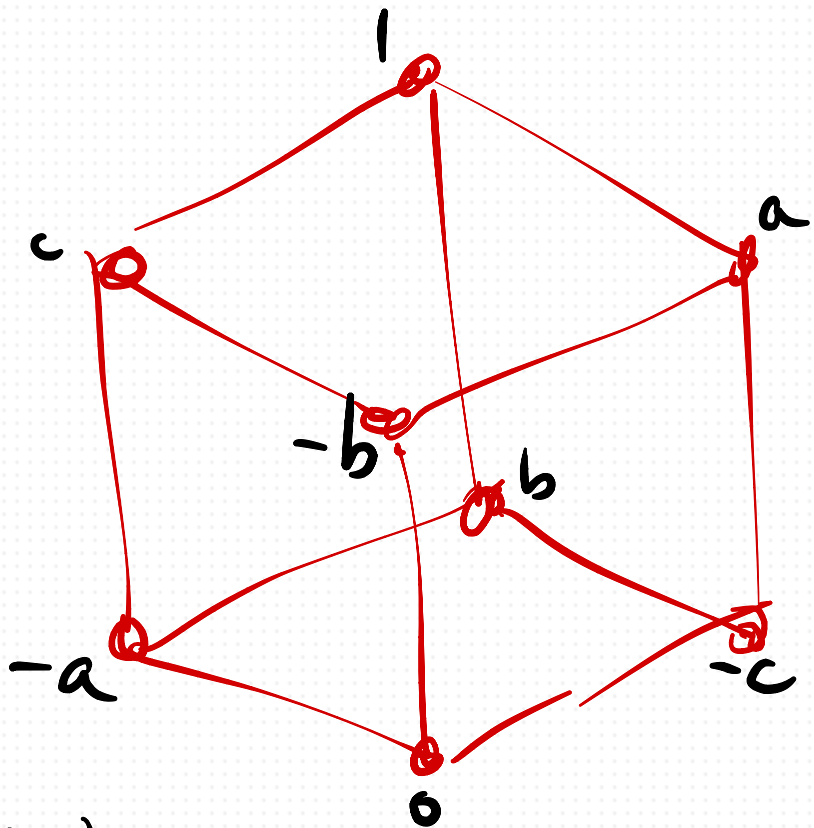


Every Heyting Algebra is also isomorphic to a subalgebra of upward closed sets on a partially ordered set of points P

$\cap \quad \cup \quad \Rightarrow \quad \emptyset \quad P$

THIS IS ALSO PHILOSOPHICALLY & MATHEMATICALLY SUGGESTIVE

YET ANOTHER EXAMPLE



An S4-Algebra is a Boolean Algebra with an operator $\Box$ satisfying

$\Box 1 = 1$

$\Box x \wedge \Box y = \Box(x \wedge y)$

$\Box x \leq x$; $\Box\Box x = \Box x$

α	$\emptyset$	$-a$	$-b$	$-c$	b	c	a	1
$\Box\alpha$	$\emptyset$	$\emptyset$	$-b$	$-c$	$-c$	$-b$	a	1

YET ANOTHER EXAMPLE

An S4-Algebra is a Boolean Algebra with an operator $\Box$ satisfying

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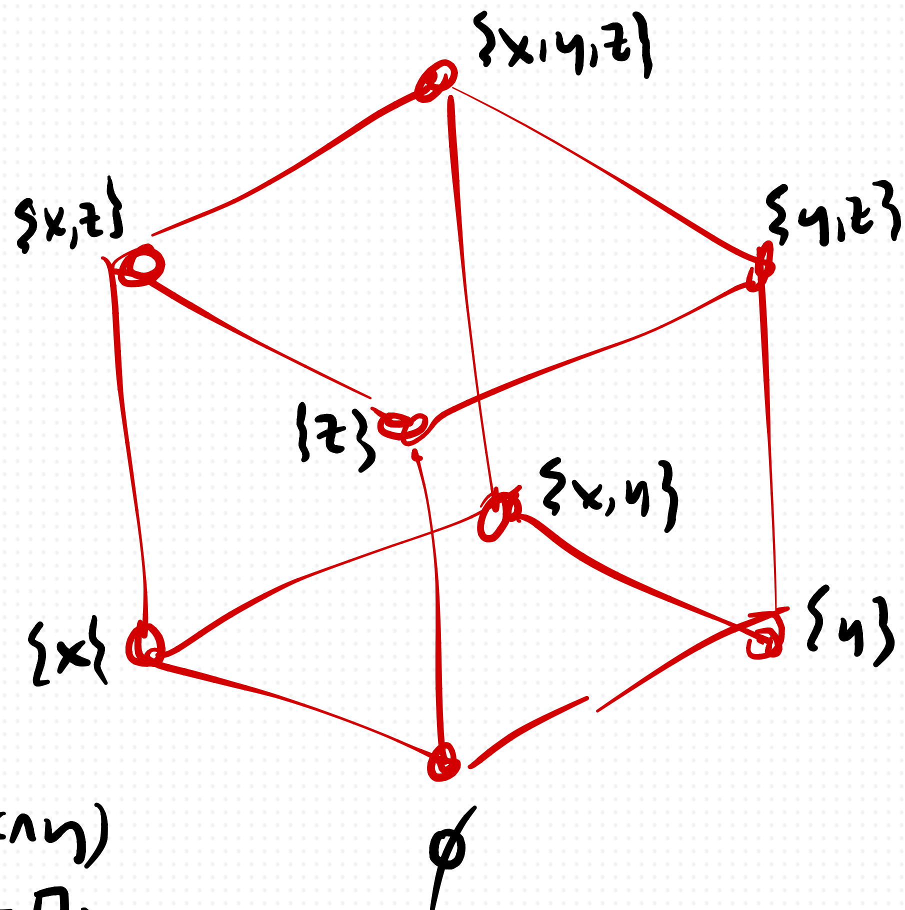

Every S4 algebra is isomorphic to a subalgebra of a powerset algebra equipped with a topology for which $\Box$ is an interior operator.

α	$\emptyset$	x	y	z	xy	xz	yz	xyz
$\Box\alpha$	$\emptyset$	$\emptyset$	y	z	y	z	yz	xyz

YET ANOTHER EXAMPLE

An S4-Algebra is a Boolean Algebra with an operator $\Box$ satisfying

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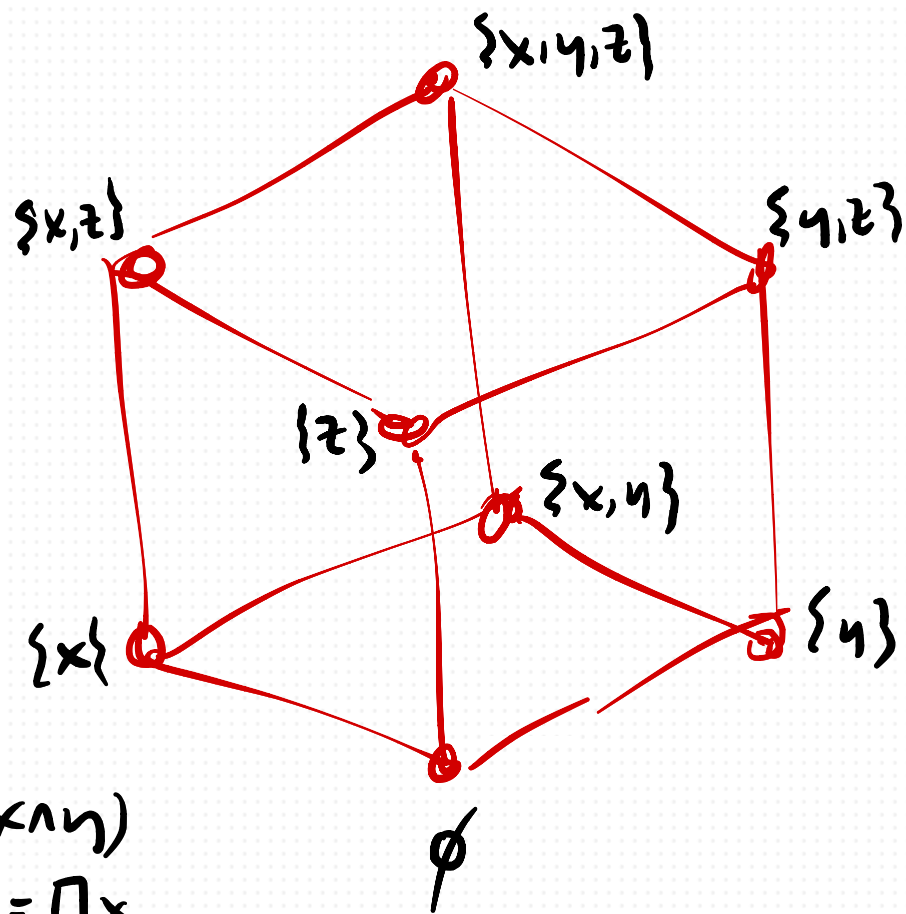
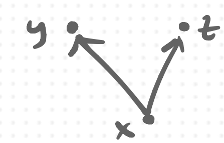


α	ϕ	x	y	z	xy	xz	yz	xyz
$\Box\alpha$	ϕ	ϕ	y	z	y	z	yz	xyz

Every S4 algebra is isomorphic to a subalgebra of a powerset algebra equipped with a topology generated by a partial order on the ground set, where the opens are up-closed sets.



THIS IS ALSO PHILOSOPHICALLY & MATHEMATICALLY SUGGESTIVE

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Relevant Algebras

An RW^{+} algebra is a DISTRIBUTIVE LATTICE $\langle L; \wedge, \vee \rangle$ equipped with an associative & commutative operator $\circ$, residuated by $\rightarrow$ (so $x \circ y \leq z$ iff $x \leq y \rightarrow z$) & with an identity, e (so $e \circ x = x$).

RELEVANT ALGEBRAS

An RW^+ algebra is a DISTRIBUTIVE LATTICE $\langle L; \wedge, \vee \rangle$ equipped with an associative & commutative operator $\circ$, residuated by $\to$ (so $x \circ y \leq z$ iff $x \leq y \to z$) & with an identity, e (so $e \circ x = x$).

These generalise Heyting Algebras, by distinguishing $\circ$ & $\wedge$.

You'd expect to be able to generalise the representation theorem for Heyting Algebras to represent these...

RELEVANT FRAMES

An RW^+ frame is a 4-tuple $\langle P, \sqsubseteq, R, N \rangle$ consisting of a non-empty set P equipped with

- A partial order $\sqsubseteq$ on P
- A 3-place relation R on P
- A subset N of P

RELEVANT FRAMES

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where
- $R{\downarrow}{\downarrow}{\uparrow}$
- $N{\uparrow}$
- $\exists x(Nx \,\&\, Rxyz) \Leftrightarrow y \sqsubseteq z$

RELEVANT FRAMES

An RW⁺ frame is a 4-tuple $\langle P, \sqsubseteq, R, N \rangle$ consisting of a non-empty set P equipped with

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Propositions are sets of points closed upwards along $\sqsubseteq$.

where
- $R\downarrow\downarrow\uparrow$
- $N\uparrow$
- $\exists x(Nx \,\&\, Rxyz) \iff y \sqsubseteq z$

RELEVANT FRAMES

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- $\exists x(Nx \;\&\; Rxyz) \iff y \sqsubseteq z$

- $Rxyz \iff Ryxz$

$R(xy)z\,v$

- $\exists z(Rxyz \;\&\; Rzuv) \iff \exists w(Ryzw \;\&\; Rxwv)$

$Rx(yz)v$

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THIS IS NOT SO PHILOSOPHICALLY OR MATHEMATICALLY SUGGESTIVE

RELEVANT FRAMES

COMES WITH THE TERRITORY, Since every Heyting Lattice is an RW^+ algebra

- A partial order $\sqsubseteq$
- A 3-place relation R ← Used for $\circ, \to$
- A subset N ← Used for e.

where
- $R{\downarrow}{\downarrow}{\uparrow}$
- $N\uparrow$
- $Rxyz \iff Ryxz$
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These make $N, \sqsubseteq, R$
play nicely together.

RELEVANT FRAMES

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- $\exists z(Rxyz \;\&\; Rzuv) \iff$
$\exists w(Ryzw \;\&\; Rxwv)$

Required for Commutativity
& associativity of $\circ$.

UNDERSTANDING RELEVANT FRAMES

- A partial order $\sqsubseteq$
- A 3-place relation R
- A subset N

where
- $R\downarrow\downarrow\uparrow$
- $N\uparrow$
- $\exists x(Nx \,\&\, Rxyz) \iff y \sqsubseteq z$

- $Rxyz \iff Ryxz$
- $\exists z(Rxyz \,\&\, Rzuv) \iff \exists w(Ryzw \,\&\, Rxwv)$

UNDERSTANDING RELEVANT FRAMES

- A partial order R^2
- A 3-place relation R^3
- A subset R^1

HMM...

where
- $R^3{\downarrow}{\downarrow}{\uparrow}$
- $R^1{\uparrow}$
- $\exists x(R^1x \,\&\, R^3xyz) \Leftrightarrow R^2yz$

- $Rxyz \Leftrightarrow Ryxz$
- $\exists z(Rxyz \,\&\, Rzuv) \Leftrightarrow \exists w(Ryzw \,\&\, Rxwv)$

UNDERSTANDING RELEVANT FRAMES

- A partial order R^2
- A 3-place relation R^3
- A subset R^1

where
- $R^3{\downarrow}{\downarrow}{\uparrow}$
- $R^1{\uparrow}$
- $\exists x(R^1x \,\&\, R^3xyz) \Leftrightarrow R^2yz$

HMM...

We also have $R^2{\downarrow}{\uparrow}$, of course, since R^2 is the partial order itself!

UNDERSTANDING RELEVANT FRAMES

- A partial order R^2
- A 3-place relation R^3
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We also have $R^2{\downarrow}{\uparrow}$, of course, since R^2 is the partial order itself!

HMM...

where
- $R^3{\downarrow}{\downarrow}{\uparrow}$
- $R^1{\uparrow}$
- $\exists x(R^1x \ \& \ R^3xyz) \Leftrightarrow R^2yz$

HMM...

This generalises, too!

e.g. $\exists x(R^2x'x \ \& \ R^3xyz) \Leftrightarrow R^3x'yz$

- A subset R^1
- A partial order R^2
- A 3-place relation R^3

$R^1 \uparrow$ — no source

$R^2 \underbrace{\downarrow}\uparrow$ — single source

$R^3 \underbrace{\downarrow\downarrow}\uparrow$ — pair source

Source

Target

- A subset R^1 — $R^1 \uparrow$ — no source
- A partial order R^2 — $R^2 \underbrace{\downarrow} \uparrow$ — single source
- A 3-place relation R^3 — $R^3 \underbrace{\downarrow\downarrow} \uparrow$ — pair source

Source

Target

How is the source structured?

- A subset R^1
- A partial order R^2
- A 3-place relation R^3

$R^1 \uparrow$	—	no source
$R^2 \downarrow\uparrow$	—	single source
$R^3 \downarrow\downarrow\uparrow$	—	pair source

Source

Target

How is the source structured?

Recall:
- $R^3xyz \iff R^3yxz$ — the order does not matter
- $\exists z(R^3xyz \,\&\, R^3zuv) \iff \exists w(R^3yzw \,\&\, R^3xwv)$

This has something to do with associativity

- A subset R^1 $R^1 \uparrow$ — no source
- A partial order R^2 $R^2 \underbrace{\downarrow}\uparrow$ — single source
- A 3-place relation R^3 $R^3 \underbrace{\downarrow\downarrow}_{\text{Source}}\uparrow_{\text{Target}}$ — pair source

How is the source structured?

Recall:
- $R^3xyz \iff R^3yxz$ — the order does not matter
- $\exists z(R^3xyz \,\&\, R^3zuv) \iff \exists w(R^3yzw \,\&\, R^3xwv)$

  This has something to do with associativity

The source is just one thing: a multiset of points!!

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MULTISET RELATIONS

R is a multiset relation on P iff $R \subseteq \mathcal{M}(P) \times P$.

That is, it relates finite multisets of points to points.

- A subset $R^1 z$ iff $R[\,]z$
- A partial order $R^2 xz$ iff $R[x]z$
- A 3-place relation $R^3 xyz$ iff $R[x,y]z$

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The commutativity postulate $R^3xyz \Leftrightarrow R^3yxz$ comes for free.

For the other frame postulates, we need some kind of condition on R analogous to the requirement for a partial order in Heyting frames.

- R is reflexive iff $[x]Rx$ for every x.

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Recall this condition: $\exists x(R^2x'x \,\&\, R^3xyz) \Leftrightarrow R^3x'yz$

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ie: $\exists x(R[x']x \,\&\, R[x,y]z) \Longleftrightarrow R[x',y]z$

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Recall this condition: $\exists x(R^2x'x \;\&\; R^3xyz) \Leftrightarrow R^3x'yz$

ie: $\exists x(R[x']x \;\&\; R[x,y]z) \Leftrightarrow R[x',y]z$

generalising on y: $\exists x(R[x']x \;\&\; R([x]\cup Y)z) \Leftrightarrow R([x']\cup Y)z$

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ie: $\exists x(R[x']x \ \& \ R[x,y]z) \Leftrightarrow R[x';y]z$

generalising on y: $\exists x(R[x']x \ \& \ R([x]\cup Y)z) \Leftrightarrow R([x']\cup Y)z$

generalising on x': $\exists x(RXx \ \& \ R([x]\cup Y)z) \Leftrightarrow R(X\cup Y)z$

- R is **reflexive** iff $[x]Rx$ for every x.

Recall this condition: $\exists x(R^2x'x \;\&\; R^3xyz) \Leftrightarrow R^3x'yz$

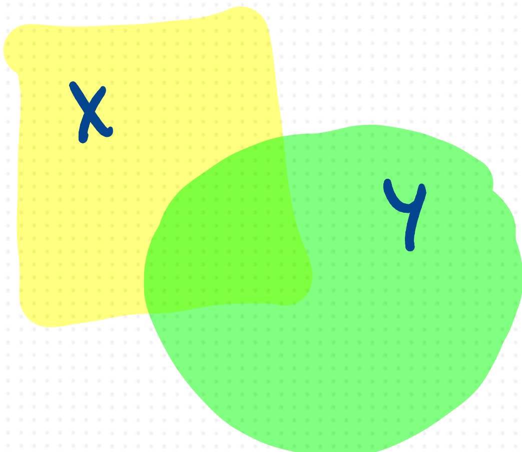
ie: $\exists x(R[x']x \;\&\; R[x,y]z) \Leftrightarrow R[x';y]z$

generalising on y: $\exists x(R[x']x \;\&\; R([x]\cup Y)z) \Leftrightarrow R([x']\cup Y)z$

generalising on x': $\exists x(RXx \;\&\; R([x]\cup Y)z) \Leftrightarrow R(X\cup Y)z$

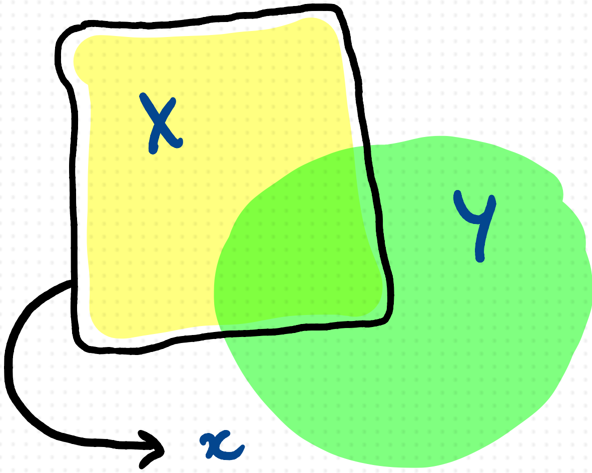
- R is **compositional** iff $\exists x(RXx \;\&\; R([x]\cup Y)z) \Leftrightarrow R(X\cup Y)z$

COMPOSITIONALITY



$$\exists x(RXx \;\&\; R([x] \cup Y)z) \iff R(X \cup Y)z$$

COMPOSITIONALITY



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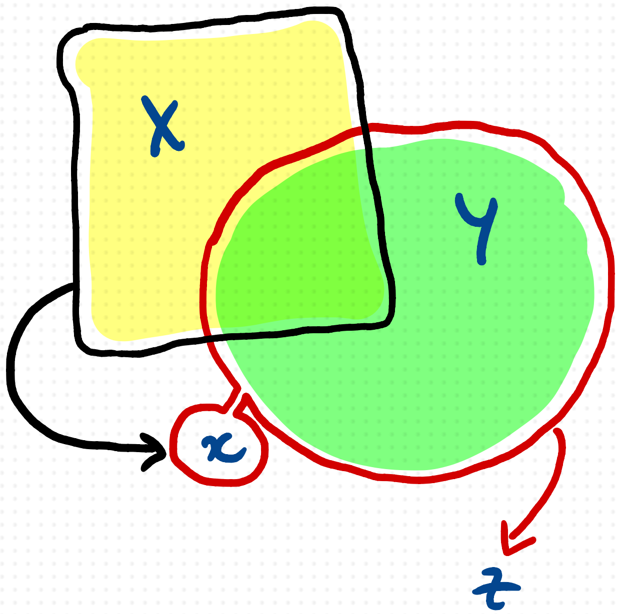

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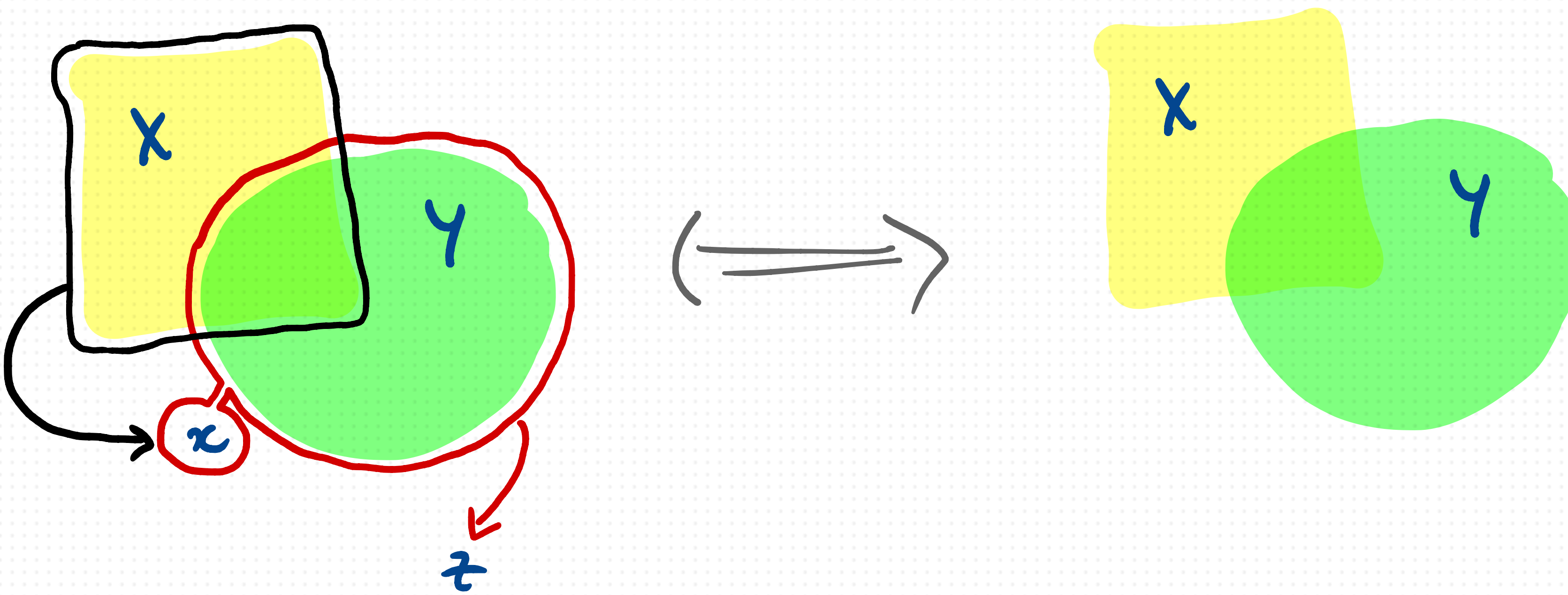
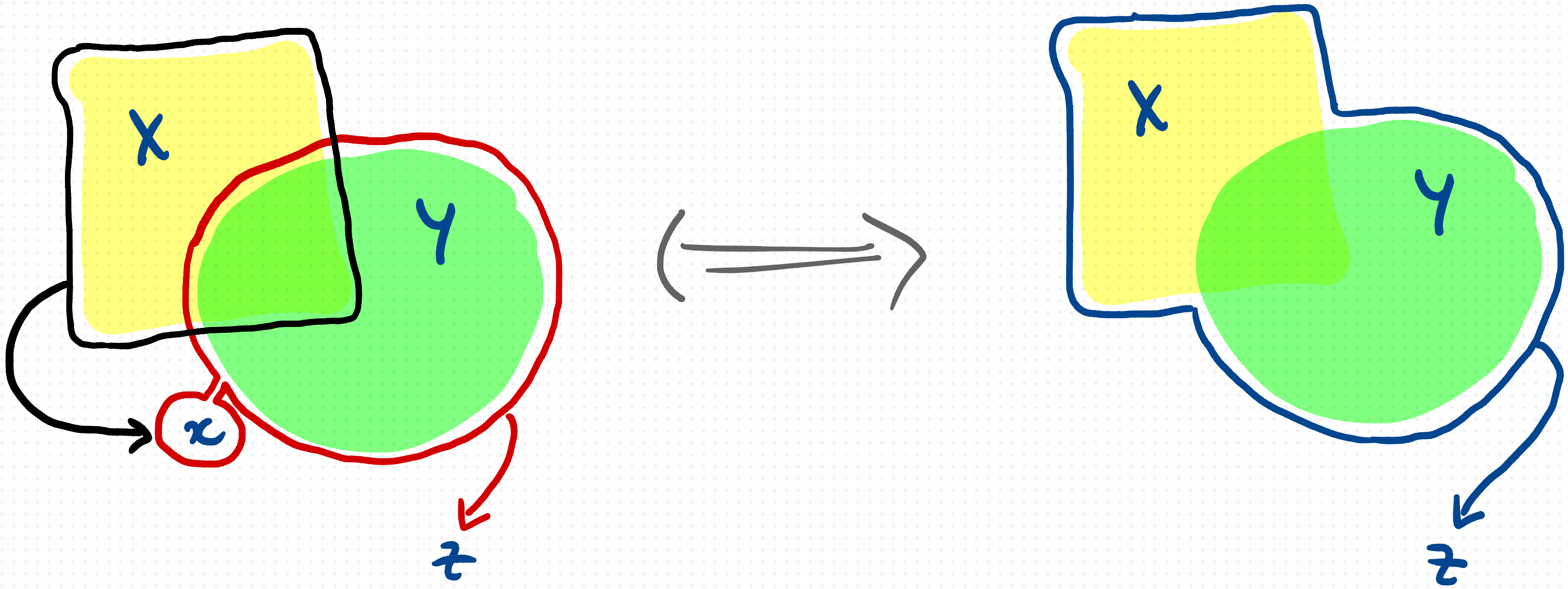
COMPOSITIONALITY



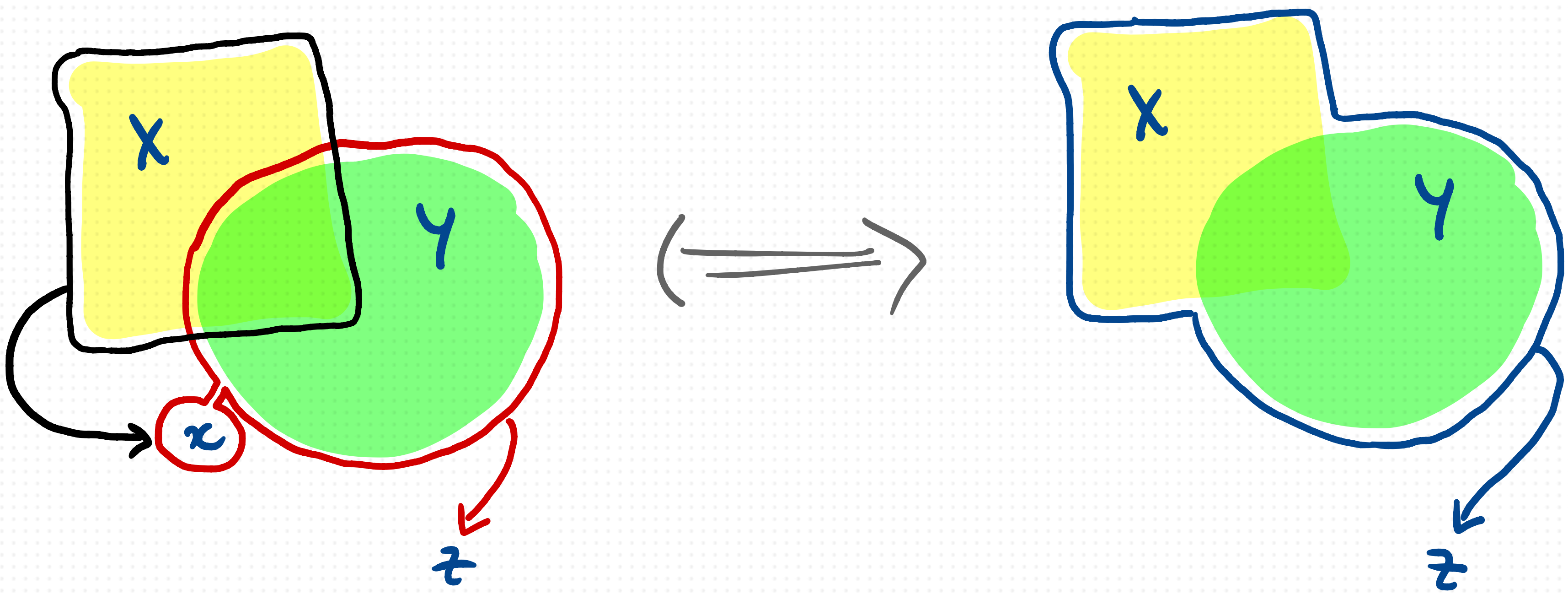
$$\exists x\,(R X_x \,\&\, R([x] \cup Y)z) \iff R(X \cup Y)z$$

COMPOSITIONALITY



$$\exists x \left(R X x \;\&\; R([x] \cup Y) z \right) \iff R(X \cup Y) z$$

Compositionality



$$\exists x \left(R X_x \;\&\; R([x] \cup Y) z \right) \iff R(X \cup Y) z$$

when $|X| = n$, $|Y| = m$, call this Compositionality$^{n}_{m}$

FUN FACT #1: This is conceptually way simpler

TERNARY FRAME LANGUAGE

- Nx
- $x \sqsubseteq y$
- $Rxyz$

MULTISET FRAME LANGUAGE

- $R[\,]x$
- $R[x]y$
- $R[x,y]z$

FUN FACT #1: This is conceptually way simpler

TERNARY FRAME LANGUAGE	MULTISET FRAME LANGUAGE
Nx	$R[\,]x$
$x \sqsubseteq y$	$R[x]y$
$Rxyz$	$R[x,y]z$
$\sqsubseteq$ Reflexive	R reflexive

Fun Fact #1: This is conceptually way simpler

Ternary Frame Language

- Nx
- $x \sqsubseteq y$
- $Rxyz$
- $\sqsubseteq$ Reflexive
- $\sqsubseteq$ Transitive

Multiset Frame Language

- $R[\,]x$
- $R[x]y$
- $R[x,y]z$
- R reflexive
- R compositional.

FUN FACT #1: This is conceptually way simpler

TERNARY FRAME LANGUAGE	MULTISET FRAME LANGUAGE
• Nx	• $R[\,]x$
• $x \sqsubseteq y$	• $R[x]y$
• $Rxyz$	• $R[x,y]z$
• $\sqsubseteq$ Reflexive	• R reflexive
• $\sqsubseteq$ Transitive	• R compositional$_1$
• $N\uparrow$	• R compositional$_2$

Fun Fact #1: This is conceptually way simpler

TERNARY FRAME LANGUAGE

- Nx
- $x \sqsubseteq y$
- $Rxyz$
- $\sqsubseteq$ Reflexive
- $\sqsubseteq$ Transitive
- $N\uparrow$
- $R\downarrow\downarrow\uparrow$

MULTISET FRAME LANGUAGE

- $R[\,]x$
- $R[x]y$
- $R[x,y]z$
- R reflexive
- R compositional$_1$
- R compositional$_2$
- R compositional$_3$

FUN FACT #1: This is conceptually way simpler

TERNARY FRAME LANGUAGE

- Nx
- $x \sqsubseteq y$
- $Rxyz$
- $\sqsubseteq$ Reflexive
- $\sqsubseteq$ Transitive
- $N\uparrow$
- $R\downarrow\downarrow\uparrow$
- $\exists x(Nx \,\&\, Rxyz) \Leftrightarrow y \sqsubseteq z$

MULTISET FRAME LANGUAGE

- $R[\,]x$
- $R[x]y$
- $R[x,y]z$
- R reflexive
- R compositional$_1$
- R compositional$_2$
- R compositional$_3$
- R compositional$_4$

FUN FACT #1: This is conceptually way simpler

TERNARY FRAME LANGUAGE	MULTISET FRAME LANGUAGE
• Nx	• $R[\,]x$
• $x \sqsubseteq y$	• $R[x]y$
• $Rxyz$	• $R[x,y]z$
• $\sqsubseteq$ Reflexive	• R reflexive
• $\sqsubseteq$ Transitive	• R compositional$^1_0$
• $N\uparrow$	• R compositional$^0_0$
• $R\downarrow\downarrow\uparrow$	• R compositional$^1_1$
• $\exists x(Nx \;\&\; Rxyz) \iff y \sqsubseteq z$	• R compositional$^0_1$
• $Rxyz \iff Ryxz$	• —

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Ternary Frame Language

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- $\exists z(Rxyz \,\&\, Rzuv) \iff \exists w(Ryzw \,\&\, Rxwv)$

Multiset Frame Language

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- $R[x]y$
- $R[x,y]z$
- R reflexive
- R compositional$^1_0$
- R compositional$^0_0$
- R compositional$^1_1$
- R compositional$^0_1$
- —
- R compositional$^2_1$

Fun Facts #2 & #3

★ Every RW+ ternary frame $\langle P; R, \sqsubseteq, N \rangle$ uniquely determines a multiset frame $\langle P; \mathbb{R} \rangle$, where $\mathbb{R}$ is reflexive & compositional.

▸ Define
- $[\,]\mathbb{R}z := Nz$
- $[x,y]\mathbb{R}z := Rxyz$
- $[x]\mathbb{R}z := x \sqsubseteq z$
- $([x] \cup Y)\mathbb{R}z := \exists y (Y\mathbb{R}z \;\&\; Rxyz)$

FUN FACTS #2 & #3

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★ Every multiset frame $\langle P; \mathbb{R} \rangle$, where $\mathbb{R}$ is reflexive & compositional, determines an equivalent ternary frame $\langle P; R, \sqsubseteq, N \rangle$.

(Just reverse the process!)

NOTICE SOMETHING INTERESTING?

TERNARY FRAME LANGUAGE

- Nx
- $x \sqsubseteq y$
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MULTISET FRAME LANGUAGE

- $R[\,]x$
- $R[x]y$
- $R[x,y]z$
- R reflexive ???
- R compositional$^1_0$ ✓
- R compositional$^0_0$ ✓
- R compositional$^1_1$ ✓
- R compositional$^0_1$ ✓
- —
- R compositional$^2_1$ ✓

Why is this here?

MY PLAN

1. WHAT ARE MODELS FOR?
2. THE COMPLEXITY OF ROUTLEY/MEYER FRAMES
3. THREE INTO ONE
4. BEYOND REFLEXIVITY
5. EXAMPLES
6. OPEN QUESTIONS

REFLEXIVITY of $\mathbb{R}$ is not NEEDED for RW^+!

A multiset frame $\langle P; \mathbb{R} \rangle$ for RW^+ is a non-empty set P equipped with a compositional multiset relation R on P.

That's it!

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A multiset frame $\langle P; \mathbb{R} \rangle$ for RW^+ is a non-empty set P equipped with a compositional multiset relation R on P.

That's it!

This is much more mathematically suggestive, at least!

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EXAMPLE #1 MULTIPLICATION on ω

$[a_1, \ldots, a_n] R\, b$ iff $a_1 \times \cdots \times a_n = b$

$[\ ] R\, b$ iff $1 = b$

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Compositionality:

$$\exists x \left(\Pi X = x \ \& \ \Pi([x] \cup Y) = z \right)$$

$$\Leftrightarrow \quad \Pi([\Pi X] \cup Y) = z$$

$$\Leftrightarrow \quad \Pi(X \cup Y) = z$$

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EXAMPLE #2 MULTIPLICATION on ω

with DIVISION

$$[a_1, \ldots, a_n] \mathrel{R} b \text{ iff } a_1 \times \cdots \times a_n \mid b$$

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$x \mid y$ iff

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Compositionality: $\exists x \left(\prod X \mid x \ \& \ \prod([x] \cup Y) \mid z\right)$

$$\Leftrightarrow \exists n>0 \left(\prod([n \textstyle\prod X] \cup Y) \mid z\right)$$

$$\Leftrightarrow \prod(X \cup Y) \mid z$$

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Compositionality: $\exists x (\Pi X \mid x \ \& \ \Pi([x] \cup Y) \mid z)$

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$$\Leftrightarrow \Pi(X \cup Y) \mid z$$

Ordering: $[x] R y$ iff $x \mid y$

$x \mid y$ iff $(\exists n > 0)\, nx = y$

EXAMPLE #3 MULTIPLICATION on ω

with DIVISION — backwards!

$[a_1, \ldots, a_n] \, R \, b$ iff $b \mid a_1 \times \cdots \times a_n$

$[\,] \, R \, b$ iff $b \mid 1$, ie

Compositionality: $\exists x (\Pi X \mid x \;\&\; \Pi([x] \cup Y) \mid z)$

$\leftrightarrow \exists n > 0 (\Pi([n \Pi X] \cup Y) \mid z)$

$\leftrightarrow \Pi(X \cup Y) \mid z$

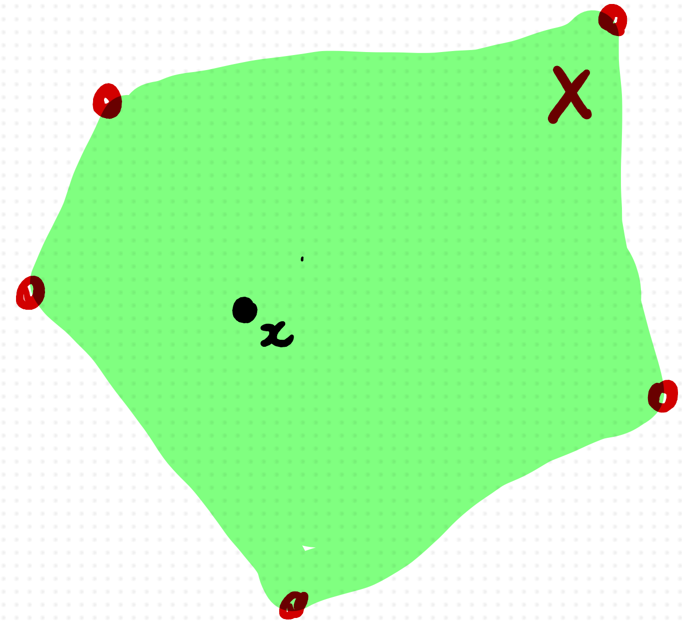
Ordering: $[x] \, R \, y$ iff $y \mid x$

$x \mid y$ iff

$(\exists n > 0)\, nx = y$

EXAMPLE #4? — CONVEX SHAPES IN $\mathbb{R}^2$

$X \mathcal{R} y$ iff y is in the shape bounded by the points X.



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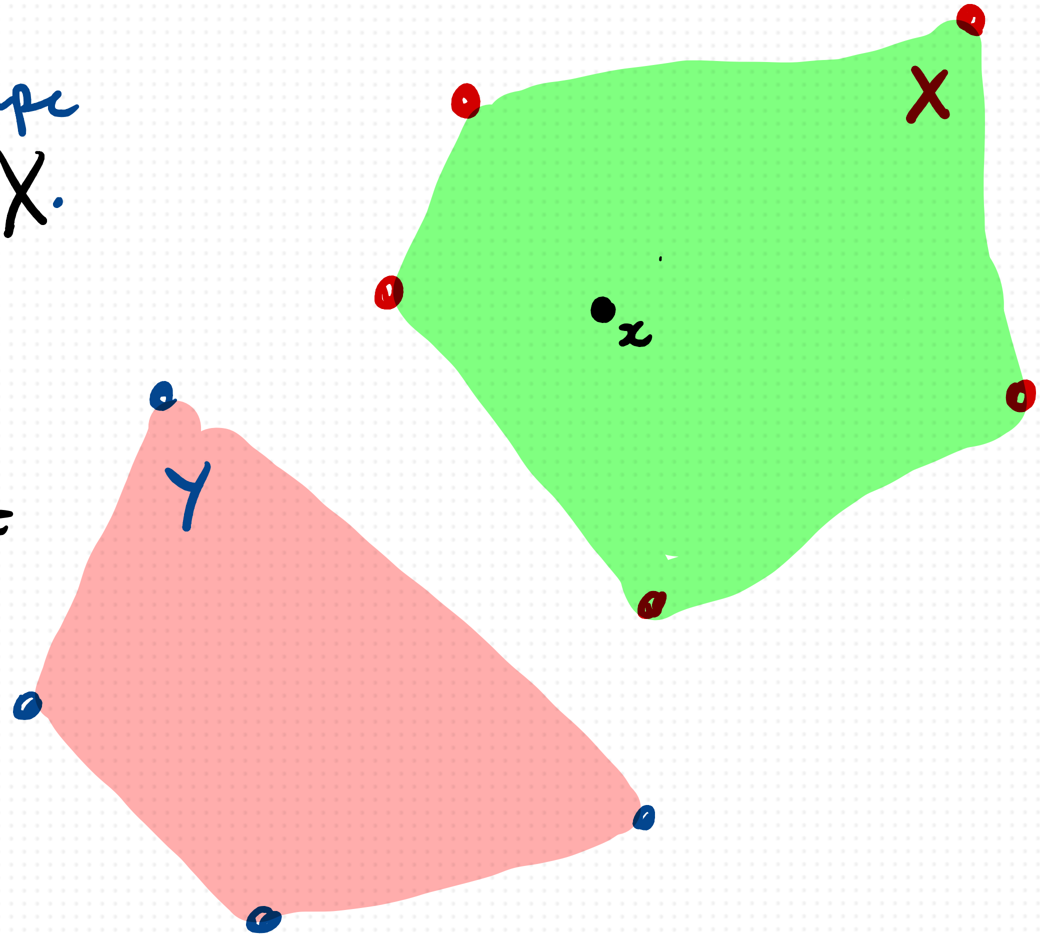
If $X \mathbb{R} x$ & $[x] \cup Y \mathbb{R} z$ then $(X \cup Y) \mathbb{R} z$



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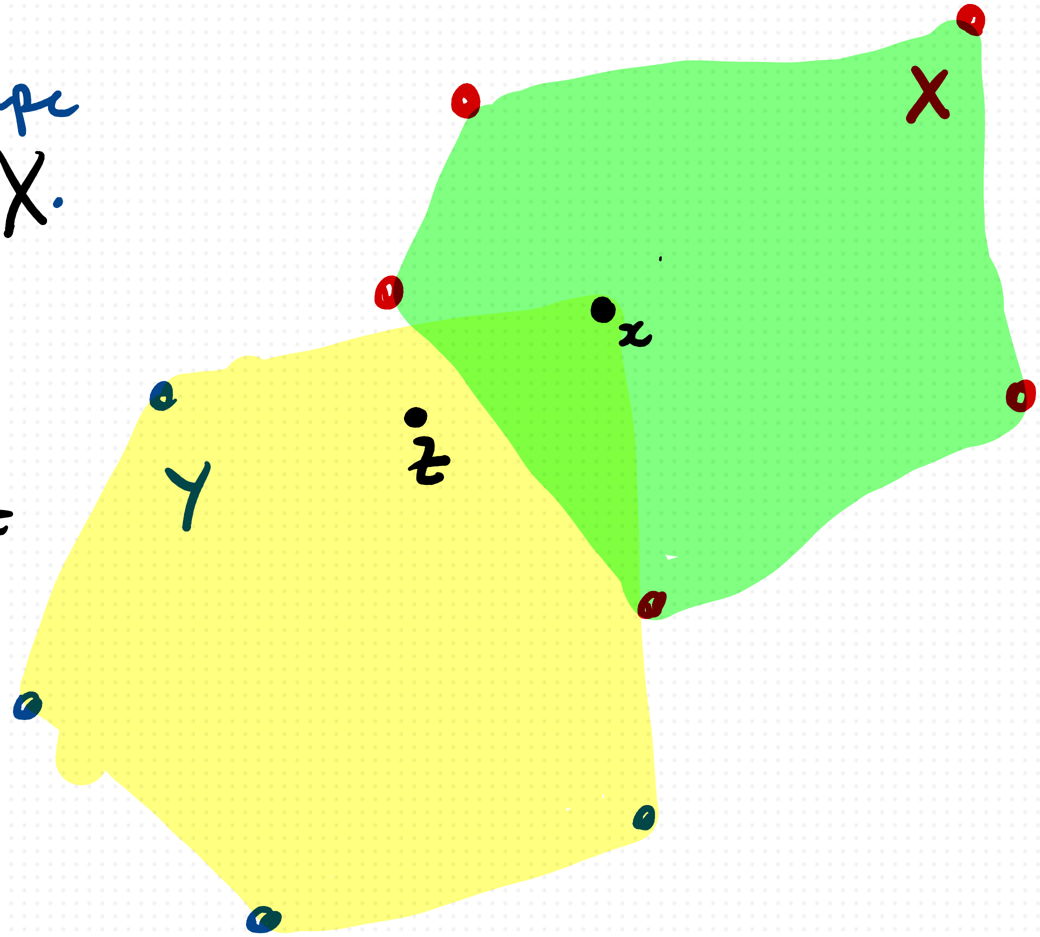

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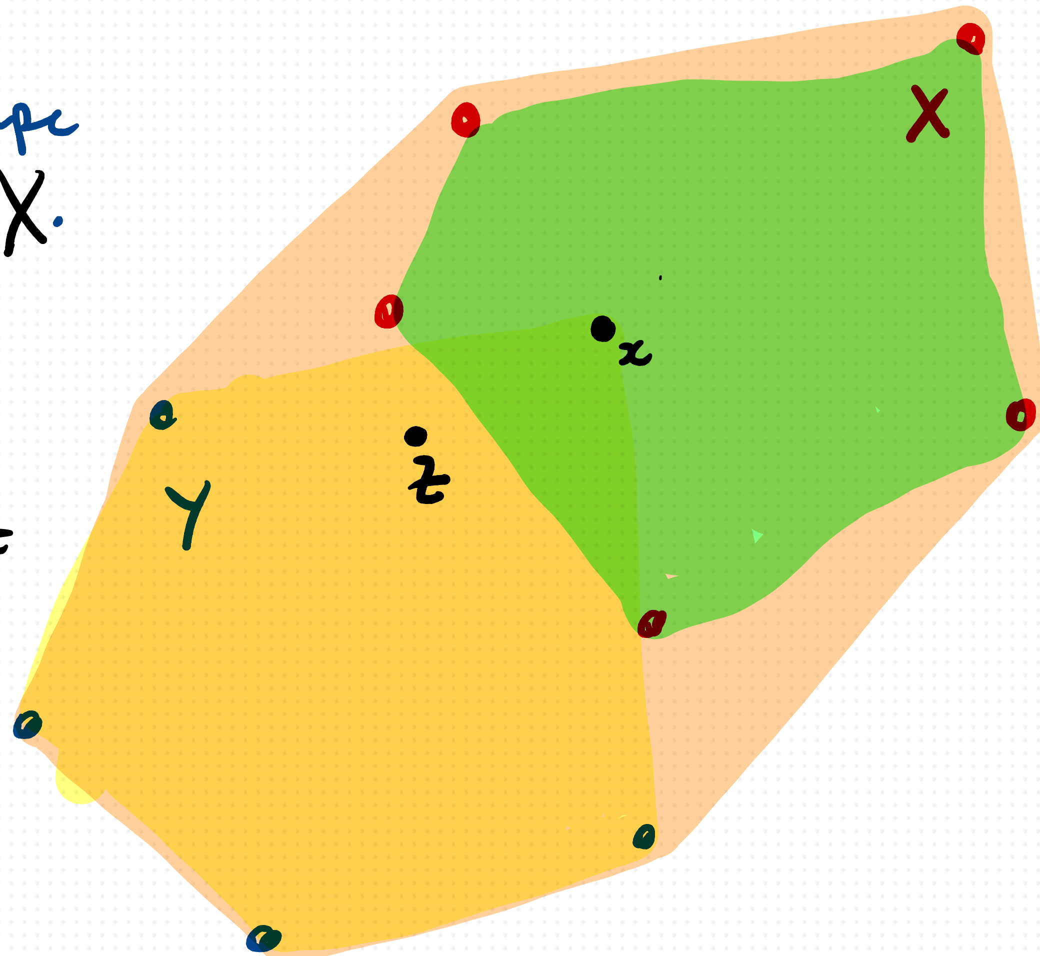

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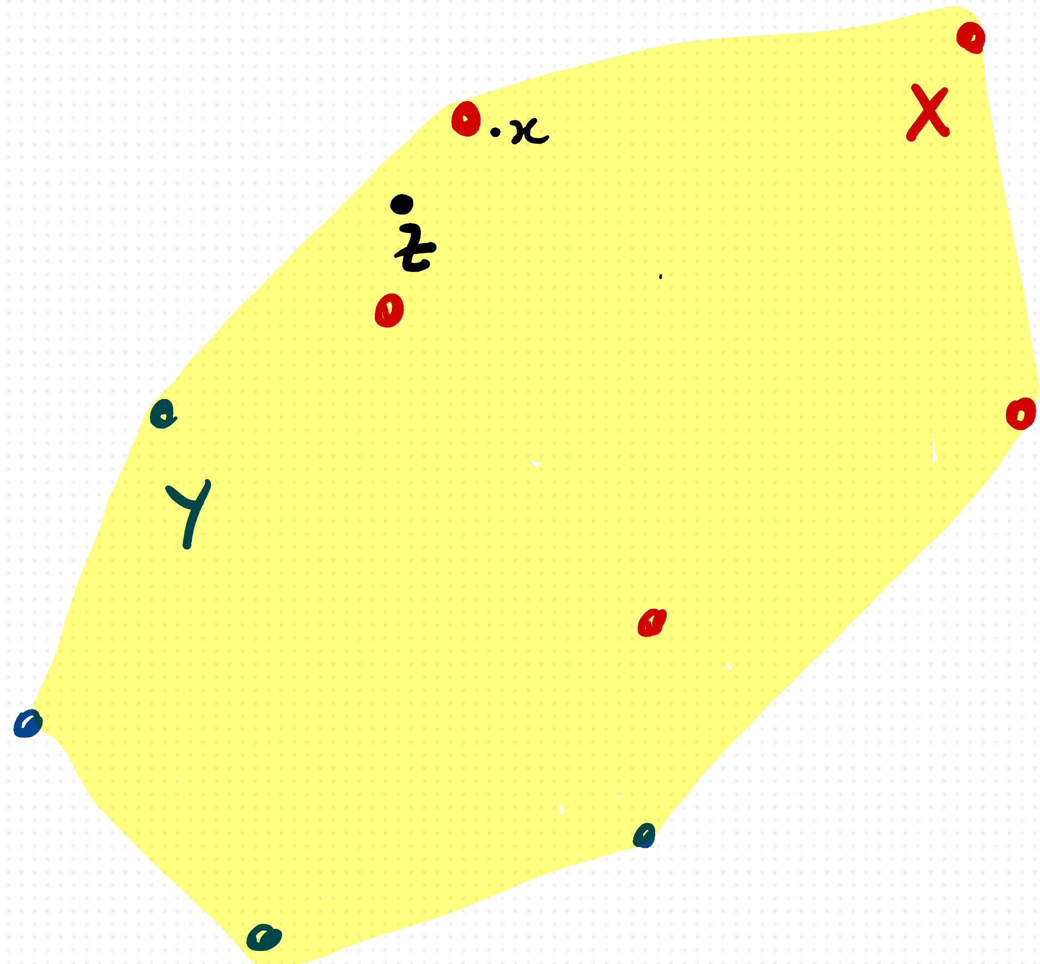

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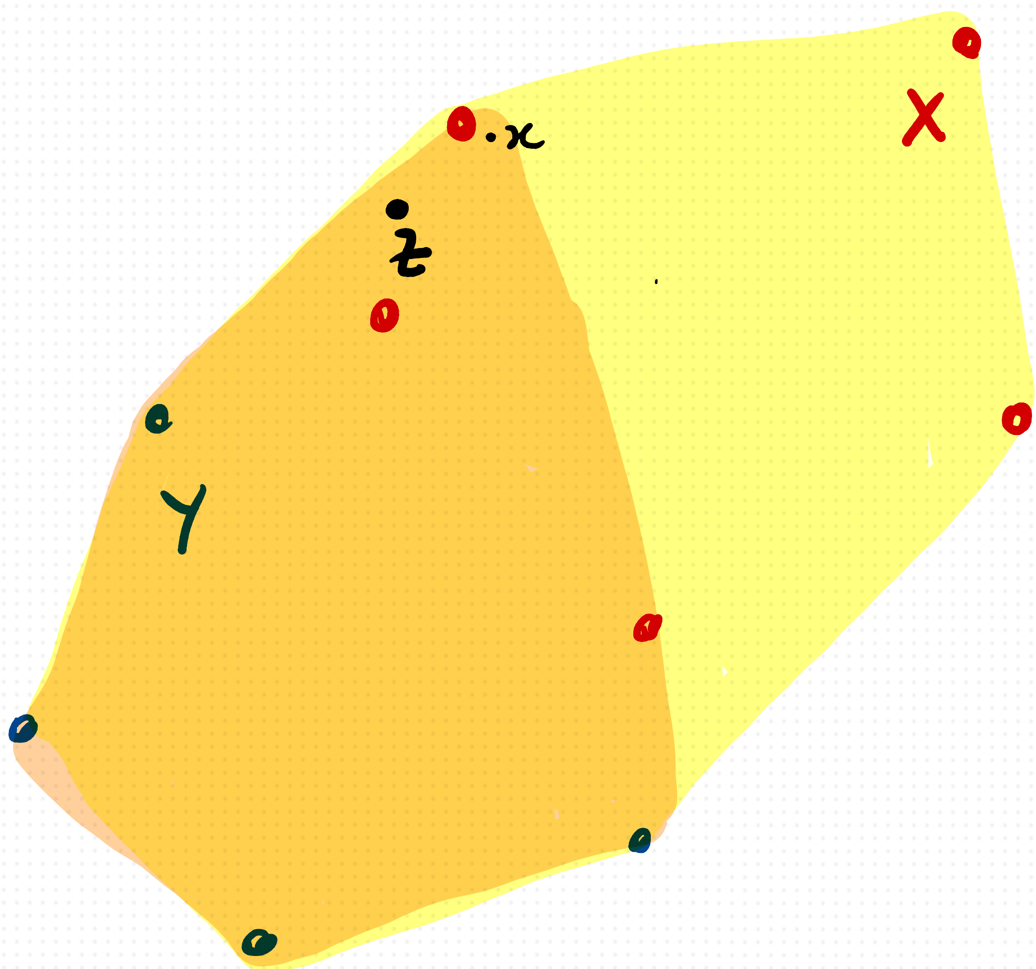

EXAMPLE #4? — CONVEX SHAPES IN $\mathbb{R}^2$

$X \mathrel{R} y$ iff y is in the shape bounded by the points X.

If $X \mathrel{R} x$ & $[x] \cup Y \mathrel{R} z$

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If $(X \cup Y) \mathrel{R} z$ then

$\exists x\ X \mathrel{R} x$ & $([x] \cup Y) \mathrel{R} z$

if X is nonempty!

If X is empty, there's <u>no</u> x where $X \mathrel{R} x$!

membership of convex shape is a compositional relation on <u>inhabited</u> multisets (or sets) on $\mathbb{R}^2$.

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MEMBERSHIP in a convex shape is a compositional relation on inhabited multisets (or sets) of points in $\mathbb{R}^2$

$\mathbb{R}$, a relation on $\mathcal{M}^+(P) \times P$ is compositional iff

inhabited multisets

$$\forall X \in \mathcal{M}^+(P)\ \forall Y \in \mathcal{M}(P)\ \forall z \left(\exists x\ X \mathbb{R} x \ \&\ ([x] \cup Y) \mathbb{R} z \right) \leftrightarrow (X \cup Y) \mathbb{R} z]$$

inhabited

R, a relation on $\mathcal{M}^+(P) \times P$ is compositional iff

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$$\forall X \in \mathcal{M}^+(P)\ \forall Y \in \mathcal{M}(P)\ \forall z \left(\exists x\ X R x\ \&\ ([x] \cup Y) R z\right) \Leftrightarrow (X \cup Y) R z]$$

inhabited

- Interior of convex shapes is a compositional relation on inhabited multisets (or sets) in $\mathbb{R}^2$.

R, a relation on $\mathcal{M}^+(P) \times P$ is **compositional** iff

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$$\forall X \in \mathcal{M}^+(P)\ \forall Y \in \mathcal{M}(P)\ \forall z \big(\exists x\ X R x \ \&\ ([x] \cup Y) R z \big) \Leftrightarrow (X \cup Y) R z \,]$$

inhabited

- Interior of Convex shapes is a compositional relation on inhabited multisets (or sets) in $\mathbb{R}^2$.
- Multiset (or set) membership is also compositional on inhabited sets.

INHABITED MULTISET RELATIONS

Nx	$[\,]Rx$
$x \sqsubseteq y$	$[x]Ry$
$Rxyz$	$[x,y]Rz$

INHABITED MULTISET RELATIONS

$\cancel{Nx}$	$\cancel{[\,]Rx}$
$x \sqsubseteq y$	$[x]Ry$
$Rxyz$	$[x,y]Rz$

These model the logic without t.

They represent validity of sequents, but not theoremhood, as there is nowhere for theorems to hold.

WHAT IF COLLECTIONS ARE SETS & NOT MULTISETS?

$$Rxyz \text{ iff } \{x,y\} R z$$

This means $Rxxy$ iff $\{x\} R y$ iff $x \sqsubseteq y$

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This means $Rxxy$ iff $\{x\}Ry$ iff $x \sqsubseteq y$

This delivers $p \to (p \to q) \vdash p \to q$.

Set frames model R^+ (at least).

WORRY

$$\{x,x\} R y \iff \{x\} R y \iff x \sqsubseteq y$$

looks a lot like $p \circ p \dashv\vdash p$,

which delivers $p \vdash p \to p$.

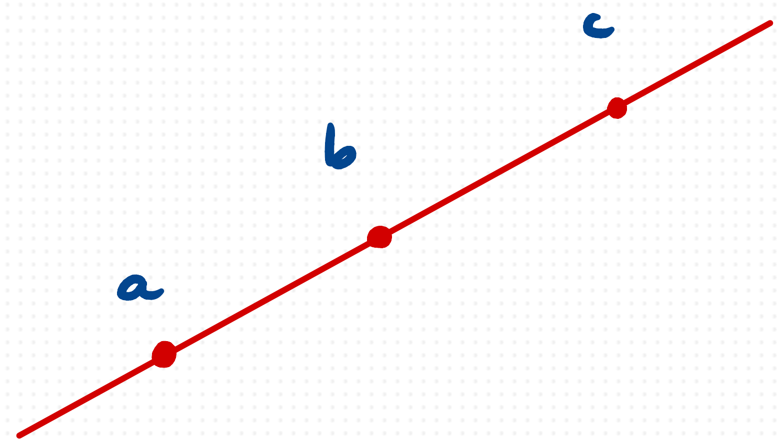
the MINGLE rule,

which doesn't hold in R^+.

MINGLE CAN FAIL IN SET FRAMES

Take the $\mathbb{R}^2$ frame with XRy iff y is in the convex shape bound by X.

Let a, b, c be three distinct points on a line, with b between a and c.



The relation R is compositional only on inhabited subsets of $\mathbb{R}^2$.
You could add a point at infinity, ∞ & extend R to $R^\times$ on $\mathbb{R}^2 \cup \{\infty\}$ by setting $X R^\times z$ iff $\begin{cases} z = \infty & \text{if } X = \{\} \text{ or } \{\infty\} \\ (X \setminus \infty) R z & \text{otherwise.} \end{cases}$

$R^\times$ is compositional on all finite sets.

MINGLE CAN FAIL IN SET FRAMES

Take the $\mathbb{R}^2$ frame with XRy iff y is in the convex shape bound by X.

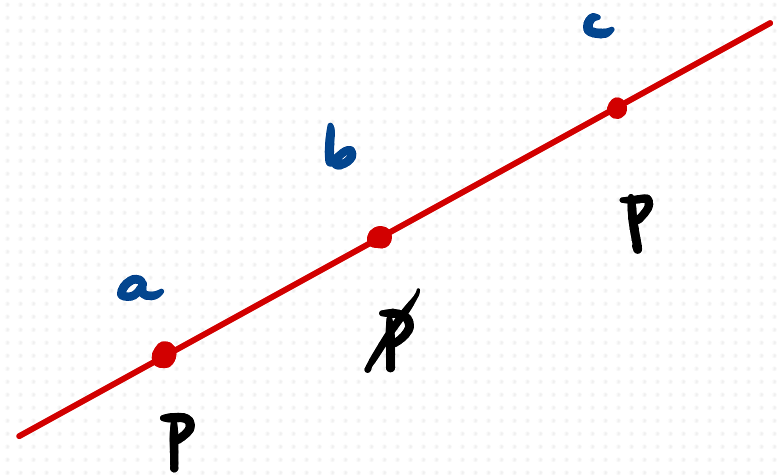
Let a, b, c be three distinct points on a line, with b between a and c.

Let $[\![p]\!] = \{a,c\}$

$a \nVdash p \to p$, since $c \Vdash p$

& $\{a,c\} R b$ & $b \nVdash p$.

So $p \nvdash p \to p$.



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CONJECTURE: SET FRAMES ARE COMPLETE FOR R^+.

Fact: Some R^+ multiset frames (RW^+ frames where $[x,x]Rx$) do not satisfy $[x,x]Ry \Leftrightarrow [x]Ry$.

Tiny Case:

R^2	0	a	b
0	0	a	b
a	a	ab	0ab
b	b	0ab	ab

But — can any such frame be embedded into some R^+ set frame?

Thank You!

https://consequently.org/