PROOF THEORY, RULES AND MEANING

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AIMS

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1. Accessible presentation of core results in proof theory — for philosophers.

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2. Embracing the duality between proof theory & model theory.

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4. Applying the results to issues in philosophical logic, semantics, metaphysics & beyond.

Accessibility

6.4 | ELIMINATING CUT

Now we will proceed to the second step of our argument, eliminating *Cut*. To make clear the scope of the setting, we need to pay a little more attention to the transition from our starting language $\mathcal{L}_1$ to our new language $\mathcal{L}_2$ with the addition of a newly definable connective $\sharp$, given by a defining rule. To be more general, we will no longer suppose that our starting language has no concepts which allow for the formation of complex formulas out of simpler formulas. We will allow this, with one constraint. These concepts allowing for the formulation of complex formulas will *themselves* be given by defining rules. So, the bounds in $\mathcal{L}_1$ are given by a set of *axiomatic* sequents, themselves closed under *Identity* and *Cut* (and perhaps *Contraction* and *Weakening*, according to taste), and we have already added some number (perhaps zero, perhaps more) of connectives by way of defining rules.

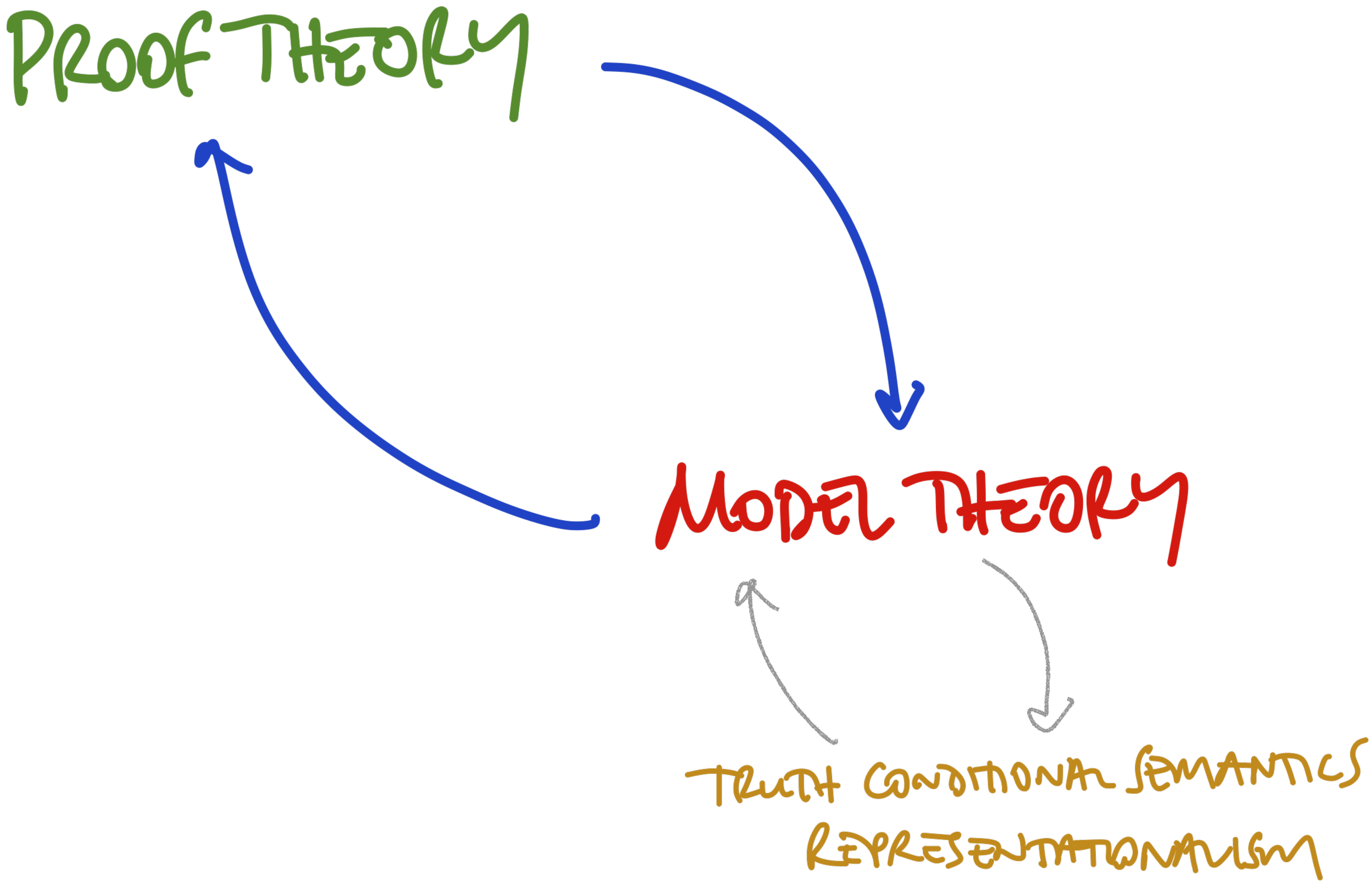
The argument of this section will show that the addition of a new connective $\sharp$ by way of its defining rule $\sharp Df$, to form the language $\mathcal{L}_2$, is conservative, by showing how to eliminate *Cut* from derivations of the bounds in the new language, using the left and right rules for each of the connectives in $\mathcal{L}_2$. We show that any sequent derivable in $\mathcal{L}_2$, using the bounds of $\mathcal{L}_1$, *Identity*, *Cut*, and the left and right rules for the connectives in $\mathcal{L}_2$ can be derived without appeal to the *Cut* rule. Then, inspecting the rules which remain — the original bounds from $\mathcal{L}_1$, including *Identity*, the left and right rules (arising out of defining rules), and perhaps *Contraction* and *Weakening* — we see that all of the rules *introduce* concepts but never *eliminate* them, so the only *Cut*-free derivations of $\mathcal{L}_2$ sequents were already present in $\mathcal{L}_1$. It follows that the addition of

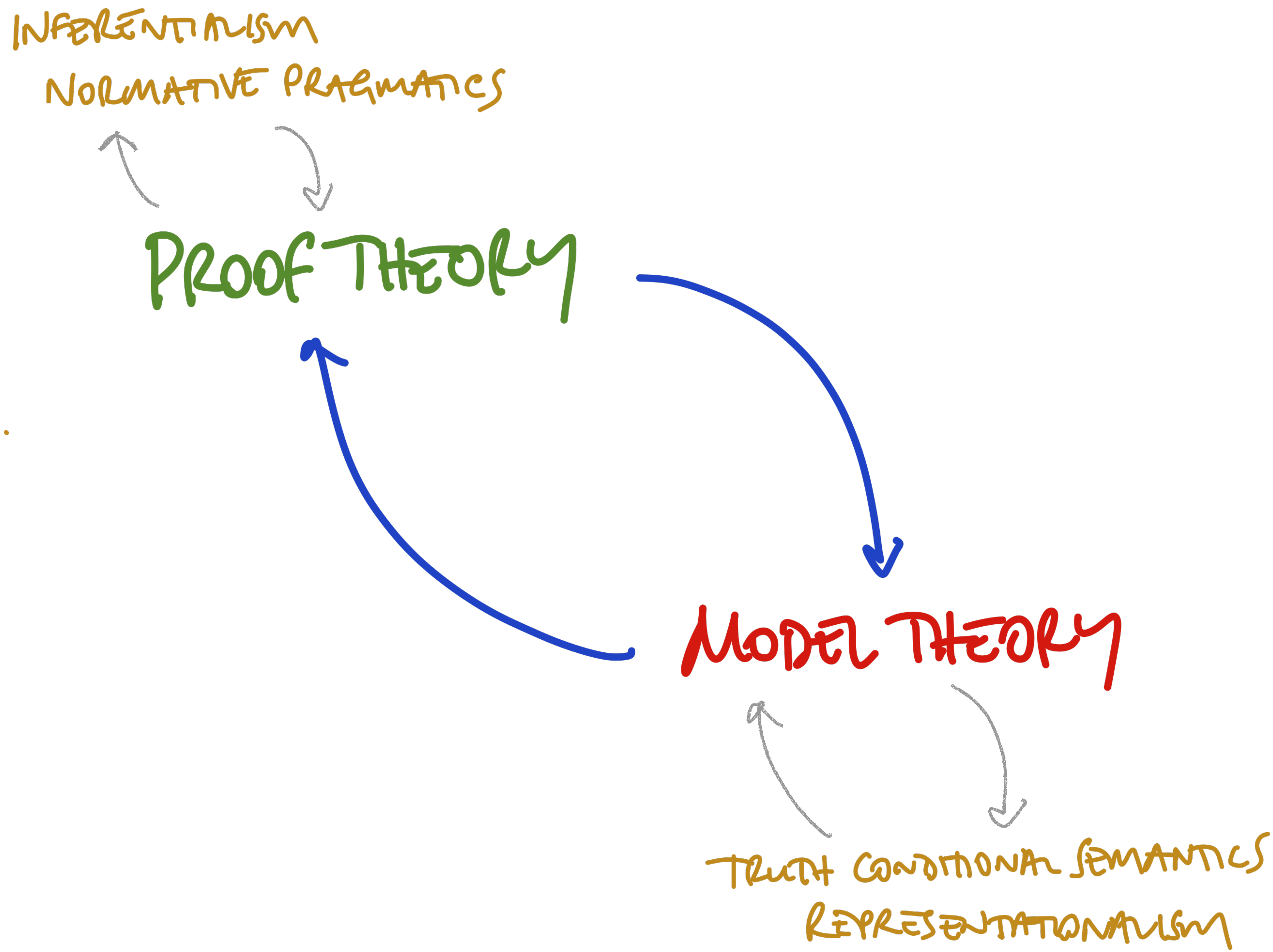
> One can see … main technical limitations in current proof-theory: The lack in *modularity*: in general, neighbouring problems can be attacked by neighbouring methods; but it is only exceptionally that one of the problems will be a corollary of the other … Most of the time, a completely new proof will be necessary (but without any new idea). This renders work in the domain quite long and tedious. For instance, if we prove a cut-elimination theorem for a certain system of rules, and then consider a new system including just a new pair of rules, then it is necessary to make a complete new proof from the beginning. Of course 90% of the two proofs will be identical, but it is rather shocking not to have a reasonable «modular» approach to such a question: a main theorem, to which one could add various «modules» corresponding to various directions. Maybe this is inherent in the subject; one may hope that this only reflects the rather low level of our conceptualization!
>
> — Jean-Yves Girard, *Proof Theory and Logical Complexity*, Vol. 1, pp. 16–17 [92]

DUALITY

PROOF THEORY

MODEL THEORY


PROOF THEORY
MODEL THEORY
TRUTH CONDITIONAL SEMANTICS
REPRESENTATIONALISM


INFERENTIALISM
NORMATIVE PRAGMATICS
PROOF THEORY
MODEL THEORY
TRUTH CONDITIONAL SEMANTICS
REPRESENTATIONALISM

I. Tools

II. The Core Argument

III. Insights

I. Tools

1. Natural Deduction

Proofs for Conditionals

Normal Proofs

Strong Normalisation & Terms

2. Sequent Calculus

Derivations for $\wedge/\vee$ $(A \succ B)$

Eliminating Id & Cut

$X \succ A$, $X \succ Y$ Sequents for $\wedge, \vee, \otimes, \oplus, \rightarrow, \neg, t, f, \top, \bot$

Consequences of Cut Elimination

3. From Proofs to Models

Positions & Valuations

Soundness & Completeness

Cut Admissibility

The Significance of Valuations

II. THE CORE ARGUMENT

4. TONK

Prior's Challenge

What Could Count as a Response?

Answering with Model Theory

Conservative Extension

Uniqueness

Harmony

5. POSITIONS

Language

Assertion & its Norms

Assertion, Denial & Other Speech Acts

Positions & Structural Rules

Bounds, Cut & Inference

Challenges

6. DEFINING RULES

Defining a Biconditional

Defining Rules Defined

Defining Rules & I/R Rules

Eliminating Cut

Answering Prior's Question

III. INSIGHTS

7. MEANING & PROOF

Connectives

Proof & Meaning

Necessity

Warrant

8. QUANTIFIERS & OBJECTS

Generality

Defining Rules for Quantifiers

Identity

Positions & Models

Arithmetic, Realism & Anti-Realism

9. MODALITY & WORLDS

Hypersequents

Positions & Worlds

Solving Prior's other Problem

Quantifiers & Identity

Two Dimensions

I. Tools

1. Natural Deduction

- Proofs for Conditionals
- Normal Proofs
- Strong Normalisation & Terms

2. Sequent Calculus

- Derivations for $\wedge/\vee$ $(A \succ B)$
- Eliminating Id & Cut
- $X \succ A$, $X \succ Y$ Sequents for $\wedge, \vee, \otimes, \oplus, \rightarrow, \neg, t, f, \top, \bot$
- Consequences of Cut Elimination

3. From Proofs to Models

- Positions & Valuations
- Soundness & Completeness
- Cut Admissibility
- The Significance of Valuations

II. The Core Argument

4. Tonk

- Prior's Challenge
- What Could Count as a Response?
- Answering with Model Theory
- Conservative Extension
- Uniqueness
- Harmony

5. Positions

- Language
- Assertion & its Norms
- Assertion, Denial & Other Speech Acts
- Positions & Structural Rules
- Bounds, Cut & Inference
- Challenges

6. Defining Rules

- Defining a Biconditional
- Defining Rules Defined
- Defining Rules & L/R Rules
- Eliminating Cut
- Answering Prior's Question

III. Insights

7. Meaning & Proof

- Connectives
- Proof & Meaning
- Necessity
- Warrant

8. Quantifiers & Objects

- Generality
- Defining Rules for Quantifiers
- Identity
- Positions & Models
- Arithmetic, Realism & Anti-Realism

9. Modality & Worlds

- Hypersequents
- Positions & Worlds
- Solving Prior's other Problem
- Quantifiers & Identity
- Two Dimensions

ANSWER

PRIOR

$$\frac{p \quad q}{p \& q} \qquad \frac{p \& q}{p} \qquad \frac{p \& q}{q}$$

$$\frac{p \quad q}{p \& q} \qquad \frac{p \& q}{p} \qquad \frac{p \& q}{q}$$

$$\frac{p}{p \text{ tonk } q} \qquad \frac{p \text{ tonk } q}{q}$$

Conservative

Uniquely Defining

POSITIONS

$$[X : Y]$$

ASSERTED

DENIED

Derive $X \succ Y \iff$

Show that $[X:Y]$ is out of bounds.

Id

$$A \succ A$$

Cut

$$\frac{X \succ A, Y \qquad X, A \succ Y}{X \succ Y}$$

Weakening

$$\frac{X \succ Y}{X, A \succ Y} \qquad \frac{X \succ Y}{X \succ A, Y}$$

Contraction

$$\frac{X, A, A \succ Y}{X, A \succ Y} \qquad \frac{X \succ A, A, Y}{X \succ A, Y}$$

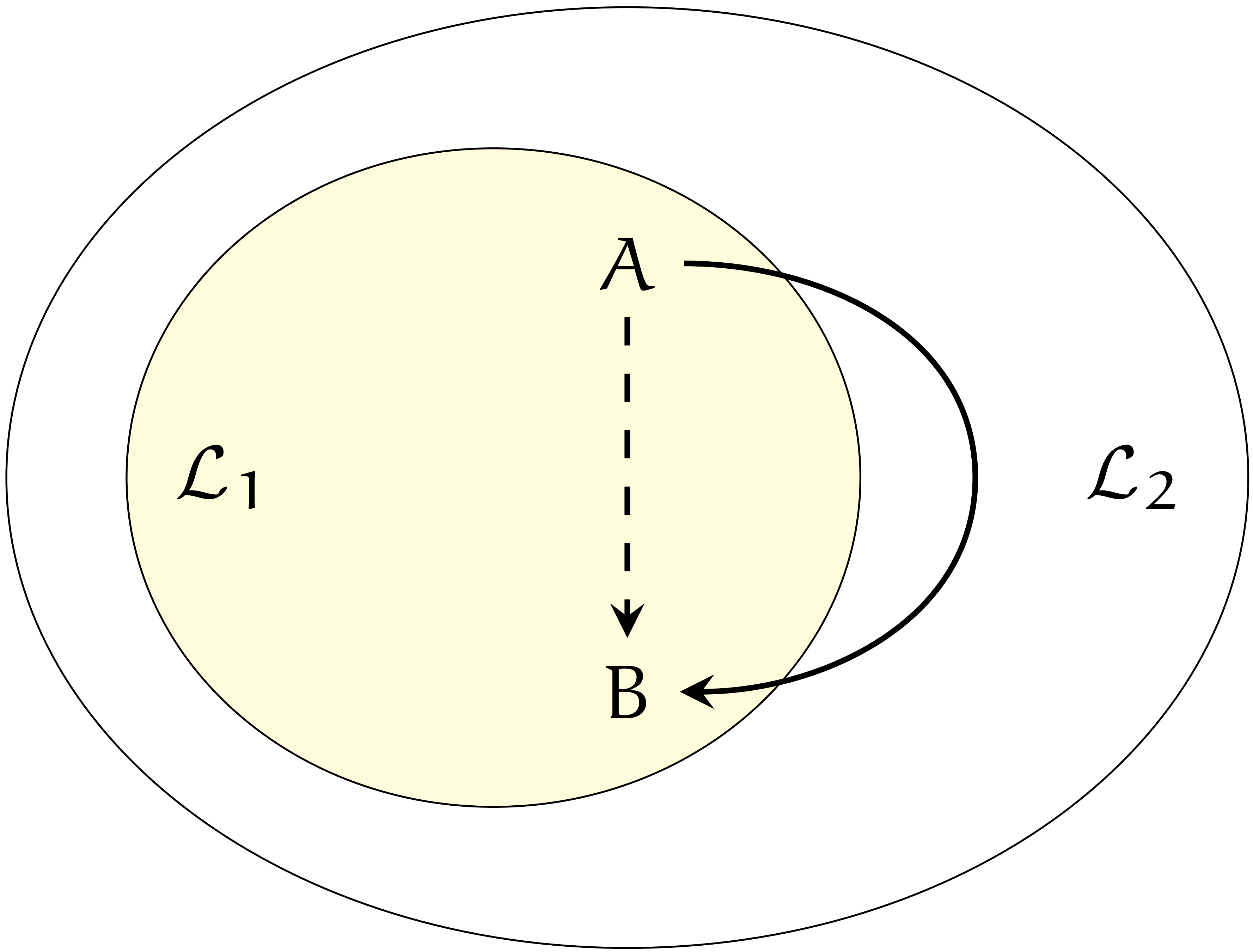
Id $\quad A \succ A$

Cut $\quad \dfrac{X \succ A, Y \qquad X, A \succ Y}{X \succ Y}$

Weakening $\quad \dfrac{X \succ Y}{X, A \succ Y} \qquad \dfrac{X \succ Y}{X \succ A, Y}$

Contraction $\quad \dfrac{X, A, A \succ Y}{X, A \succ Y} \qquad \dfrac{X \succ A, A, Y}{X \succ A, Y}$

$$\mathcal{L}_1 \xrightarrow[\&?]{\text{tonk?}} \mathcal{L}_2$$


A
B
$\mathcal{L}_1$
$\mathcal{L}_2$

$$\dfrac{X, A, B \vdash Y}{X, A \otimes B \vdash Y} \otimes Df$$

Asserting $A \otimes B$

$\equiv$

Asserting A,
Asserting B

$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y} \; \otimes Df$$

Denying $A \otimes B$?

$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y} \; \otimes Df$$

Denying $A \otimes B$?

$$
\begin{prooftree}
\AxiomC{$X, A, B \succ Y$}
\doubleLine
\RightLabel{$\otimes Df$}
\UnaryInfC{$X, A \otimes B \succ Y$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$X' \succ B, Y'$}
\AxiomC{$X \succ A, Y$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \otimes B \succ A \otimes B$}
\RightLabel{$\otimes Df$}
\UnaryInfC{$A, B \succ A \otimes B$}
\RightLabel{$mCut$}
\BinaryInfC{$X, B \succ A \otimes B, Y$}
\RightLabel{$mCut$}
\BinaryInfC{$X, X' \succ A \otimes B, Y, Y'$}
\end{prooftree}
$$

Denying $A \otimes B$?

$$\frac{X, A, B \succ Y}{\overline{X, A \otimes B \succ Y}}\ \otimes Df$$

$$\dfrac{X' \succ B, Y' \qquad \dfrac{X \succ A, Y \qquad \dfrac{\dfrac{}{A \otimes B \succ A \otimes B}\ Id}{A, B \succ A \otimes B}\ \otimes Df}{X, B \succ A \otimes B, Y}\ mCut}{X, X' \succ A \otimes B, Y, Y'}\ mCut$$

OR

$$\dfrac{X \succ A, Y \qquad X' \succ B, Y'}{X, X' \succ A \otimes B, Y, Y'}\ \otimes R$$

$$\frac{X, A, B \succ Y}{\overline{X, A \otimes B \succ Y}}\ \otimes Df$$

$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\ \otimes L \qquad \otimes R\ \frac{X \succ A, Y \quad X' \succ B, Y'}{X, X' \succ A \otimes B, Y, Y'}$$

$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y} \; \otimes Df$$

(double line rule)

$\otimes Df$		$\otimes L/R$
Id	$\Longleftrightarrow$	Id
Cut		Cut

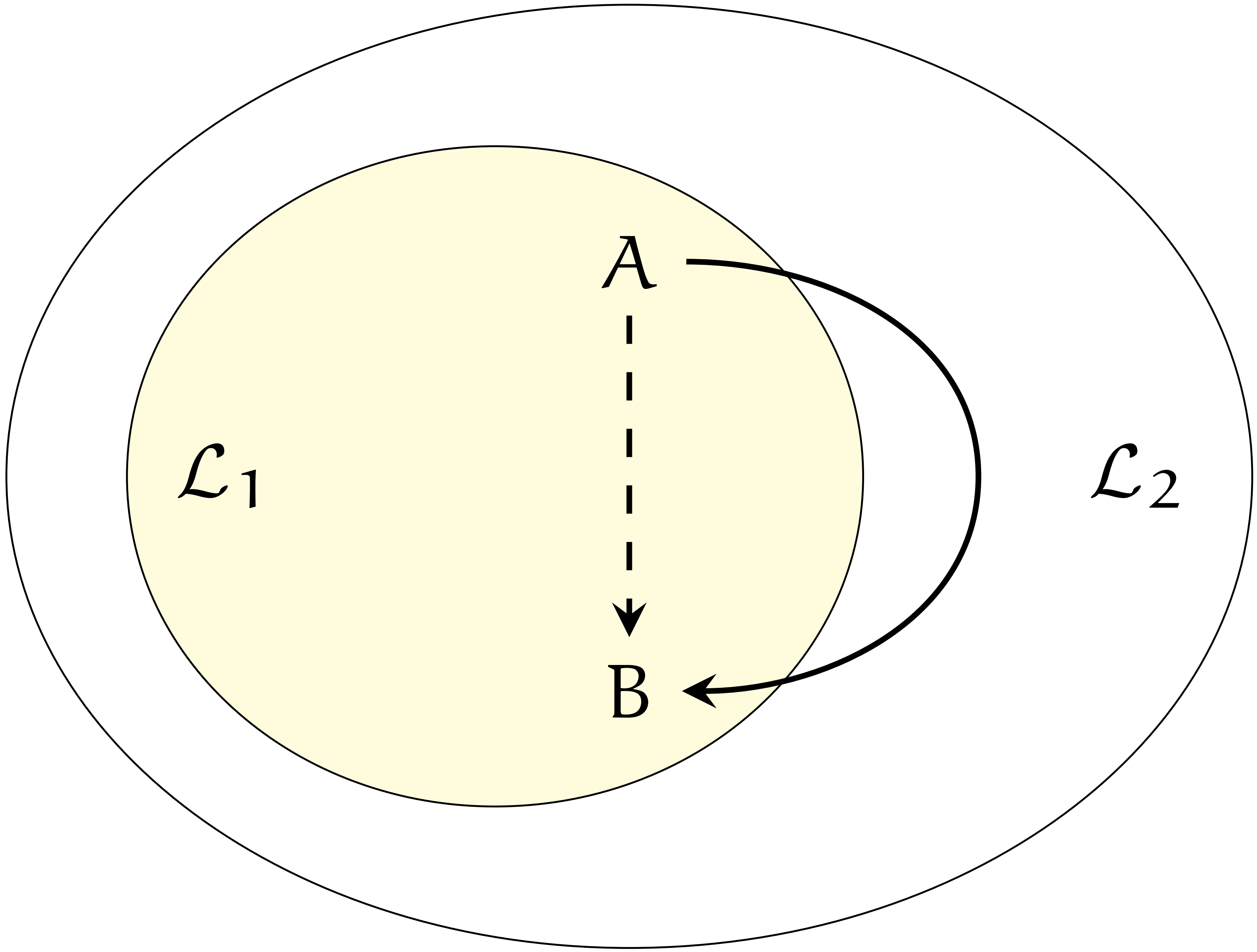
$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\ \otimes Df$$

$$\begin{array}{ccccc} \otimes Df & & \otimes C/R & & \otimes C/R \\ Id & \Leftrightarrow & Id & \Leftrightarrow & \\ Cut & & Cut & & Cut \end{array}$$

$$\dfrac{\dfrac{A \succ A \qquad B \succ B}{A, B \succ A \otimes B}\ \otimes R}{A \otimes B \succ A \otimes B}\ \otimes C$$

$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\ \otimes Df$$

$\otimes Df$ Id Cut	$\Longleftrightarrow$	$\otimes L/R$ Id Cut	$\Longleftrightarrow$	$\otimes L/R$ Cut	$\Longleftrightarrow$	$\otimes L/R$


A
B
$\mathcal{L}_1$
$\mathcal{L}_2$

$$\begin{array}{c} X, A, B \succ Y \\ \hline\hline X, A \otimes B \succ Y \end{array} \qquad \begin{array}{c} X, A, B \succ Y \\ \hline\hline X, A \& B \succ Y \end{array}$$

$$\begin{array}{c} X, A \& B \succ Y \\ \hline\hline X, A, B \succ Y \\ \hline\hline X, A \otimes B \succ Y \end{array}$$

APPLICATIONS

III. INSIGHTS

7. MEANING & PROOF

Connectives

Proof & Meaning

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8. QUANTIFIERS & OBJECTS

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Two Dimensions

$$\frac{X \succ A|^{x}_{n}, Y}{X \succ \forall x A, Y}\ \forall Df$$

(n not present in X, A, Y)

$$\dfrac{X \succ A|^x_n, Y}{X \succ \forall x A, Y} \;\forall Df$$

(n not present in X, A, Y)

Can this mean "all" — unrestrictedly?

Models

$\langle D, E \rangle$

Extensions of predicates & function symbols

Domain

Models $\langle D, E \rangle$

D: Domain

E: Extensions of predicates & function symbols

Models for predicate logic are modest

— D is a set, it doesn't contain everything*

Models $\langle D, E \rangle$

Domain

Extensions of predicates & function symbols

Models for predicate logic are modest

— D is a set, it doesn't contain everything*

* Yes there are set theories with a universal set, both classical & non-classical, but proving that any such are models satisfying the usual truth conditions is a significant problem.

A <u>model</u> arises out of a limit position

$$[X : Y] \subseteq [X_1 : Y_1] \subseteq [X_2, Y_2] \subseteq \cdots$$

relative to a given language $\mathcal{L}$.

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If the defining rules fix the meaning, the smallness of models is no bar to absolute generality.

A <u>model</u> arises out of a limit position

$$[X ; Y] \subseteq [X_1 ; Y_1] \subseteq [X_2 , Y_2] \subseteq \cdots$$

relative to a given language $\mathcal{L}$.

If the defining rules fix the meaning, the smallness of models is no bar to absolute generality. Models are just that: MODELS.

… possible worlds, in the sense of possible states of affairs are not *really* individuals (just as numbers are not *really* individuals).

To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case …

We understand 'truth in states of affairs' because we understand 'necessarily'; not *vice versa*.
— "Worlds, Times and Selves" (1969)

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Well great….
but how?

$$\begin{array}{c} X \succ Y \mid \succ A \\ \hline\hline X \succ \Box A, Y \end{array}$$

$$\frac{\mathcal{H} \mid X \succ Y \mid \ \succ A}{\mathcal{H} \mid X \succ \Box A, Y}$$

$$
\begin{array}{c}
\mathcal{H} \mid X \succ Y \mid \;\succ A \\
\hline\hline
\mathcal{H} \mid X \succ \Box A, Y
\end{array}
\;\Box Df
$$

This rule — and structural rules for hypersequents — give $\Box$ the standard S5 behaviour.

$$\frac{\mathcal{H} \mid X \succ Y \mid \ \succ A}{\mathcal{H} \mid X \succ \Box A, Y} \ \Box Df$$

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$$\frac{\mathcal{H} \mid X \succ Y \mid \;\succ A}{\mathcal{H} \mid X \succ \Box A, Y}\ \Box Df$$

We coordinate on $\Box$ because we coordinate on what counts as subjunctive supposition.

THANKS!

http://consequently.org

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