Comparing Rules for Identity in sequent systems & natural deduction

Greg Restall

THE UNIVERSITY OF MELBOURNE

PROOF THEORY VIRTUAL SEMINAR · 21 APRIL 2021

https://consequently.org/presentation/2021/comparing-identity-rules

My Aim

To explore different rules for an identity predicate in natural deduction and the sequent calculus.

Sequent Calculus & Natural Deduction

Defining Rules

Defining Rules for Identity

Identity *Axioms*

SEQUENT CALCULUS & NATURAL DEDUCTION

A derivation of $X \succ A$
builds a proof from X to A.

$$
\dfrac{
  \dfrac{p \succ p \qquad
    \dfrac{
      \dfrac{\dfrac{q \succ q}{q, \neg q \succ}\,{\neg L}}{q, \neg q \succ r}\,{K}
      \qquad r \succ r
    }{q \lor r, \neg q \succ r}\,{\lor L}
  }{
    \dfrac{p \to (q \lor r), p, \neg q \succ r}{p \to (q \lor r), p \land \neg q \succ r}\,{\land L}
  }\,{\to L}
}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,{\to R}
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\dfrac{
  \dfrac{
    \dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,{\land E}}{q \lor r}\,{\to E}
    \qquad
    \dfrac{\dfrac{\dfrac{[p \land \neg q]^2}{\neg q}\,{\land E} \qquad [q]^1}{\sharp}\,{\neg E}}{r}\,{K}
    \qquad [r]^1
  }{r}\,{\lor E^1}
}{(p \land \neg q) \to r}\,{\to I^2}
$$

$$
\dfrac{
  \dfrac{p \succ p \qquad
    \dfrac{
      \dfrac{
        \dfrac{q \succ q}{q, \neg q \succ}\,\neg L
      }{q, \neg q \succ r}\,K
      \qquad r \succ r
    }{q \lor r, \neg q \succ r}\,\lor L
  }{p \to (q \lor r), p, \neg q \succ r}\,{\to}L
}{
  \dfrac{p \to (q \lor r), p \land \neg q \succ r}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,{\to}R
}\,\land L
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\dfrac{
  \dfrac{
    \dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,\land E}{q \lor r}\,{\to}E
    \qquad
    \dfrac{
      \dfrac{\dfrac{[p \land \neg q]^2}{\neg q}\,\land E \qquad [q]^1}{\sharp}\,\neg E
    }{r}\,K
    \qquad [r]^1
  }{r}\,\lor E^1
}{(p \land \neg q) \to r}\,{\to}I^2
$$

An Example

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$q \succ q$}
\RightLabel{$\neg L$}
\UnaryInfC{$q, \neg q \succ$}
\RightLabel{K}
\UnaryInfC{$q, \neg q \succ r$}
\AxiomC{$r \succ r$}
\RightLabel{$\vee L$}
\BinaryInfC{$q \vee r, \neg q \succ r$}
\RightLabel{$\to L$}
\BinaryInfC{$p \to (q \vee r), p, \neg q \succ r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \to (q \vee r), p \wedge \neg q \succ r$}
\RightLabel{$\to R$}
\UnaryInfC{$p \to (q \vee r) \succ (p \wedge \neg q) \to r$}
\end{prooftree}
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\RightLabel{$\to E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\neg q$}
\AxiomC{$[q]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{K}
\UnaryInfC{r}
\AxiomC{$[r]^1$}
\RightLabel{$\vee E^1$}
\TrinaryInfC{r}
\RightLabel{$\to I^2$}
\UnaryInfC{$(p \wedge \neg q) \to r$}
\end{prooftree}
$$

An Example

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q, \neg q \succ}\,\neg L}{\color{red}{q, \neg q \succ r}}\,K \qquad r \succ r}{q \lor r, \neg q \succ r}\,\lor L}{p \to (q \lor r), p, \neg q \succ r}\,{\to}L}{p \to (q \lor r), p \land \neg q \succ r}\,\land L}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,{\to}R$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,\land E}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{\dfrac{[p \land \neg q]^2}{\color{red}{\neg q}}\,\land E \qquad \color{red}{[q]}^1}{\sharp}\,\neg E}{\color{red}{r}}\,K \qquad [r]^1}{r}\,\lor E^1}{(p \land \neg q) \to r}\,{\to}I^2$$

An Example

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$q \succ q$}
\RightLabel{$\neg L$}
\UnaryInfC{$q, \neg q \succ$}
\RightLabel{K}
\UnaryInfC{$q, \neg q \succ r$}
\AxiomC{$\color{red}{r \succ r}$}
\RightLabel{$\vee L$}
\BinaryInfC{$q \vee r, \neg q \succ r$}
\RightLabel{$\to L$}
\BinaryInfC{$p \to (q \vee r), p, \neg q \succ r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \to (q \vee r), p \wedge \neg q \succ r$}
\RightLabel{$\to R$}
\UnaryInfC{$p \to (q \vee r) \succ (p \wedge \neg q) \to r$}
\end{prooftree}
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\RightLabel{$\to E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\neg q$}
\AxiomC{$[q]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{K}
\UnaryInfC{r}
\AxiomC{$[\color{red}{r}]^1$}
\RightLabel{$\vee E^1$}
\TrinaryInfC{r}
\RightLabel{$\to I^2$}
\UnaryInfC{$(p \wedge \neg q) \to r$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$q \succ q$}
\RightLabel{$\neg L$}
\UnaryInfC{$q, \neg q \succ$}
\RightLabel{K}
\UnaryInfC{$q, \neg q \succ r$}
\AxiomC{$r \succ r$}
\RightLabel{$\vee L$}
\BinaryInfC{$q \vee r, \neg q \succ r$}
\RightLabel{$\to L$}
\BinaryInfC{$p \to (q \vee r), p, \neg q \succ r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \to (q \vee r), p \wedge \neg q \succ r$}
\RightLabel{$\to R$}
\UnaryInfC{$p \to (q \vee r) \succ (p \wedge \neg q) \to r$}
\end{prooftree}
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\RightLabel{$\to E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\neg q$}
\AxiomC{$[q]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{K}
\UnaryInfC{r}
\AxiomC{$[r]^1$}
\RightLabel{$\vee E^1$}
\TrinaryInfC{r}
\RightLabel{$\to I^2$}
\UnaryInfC{$(p \wedge \neg q) \to r$}
\end{prooftree}
$$

An Example

$$
\begin{prooftree}
\AxiomC{$\color{red}{p \succ p}$}
\AxiomC{$q \succ q$}
\RightLabel{$\neg L$}
\UnaryInfC{$q, \neg q \succ$}
\RightLabel{K}
\UnaryInfC{$q, \neg q \succ r$}
\AxiomC{$r \succ r$}
\RightLabel{$\vee L$}
\BinaryInfC{$q \vee r, \neg q \succ r$}
\RightLabel{$\to L$}
\BinaryInfC{$p \to (q \vee r), p, \neg q \succ r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \to (q \vee r), p \wedge \neg q \succ r$}
\RightLabel{$\to R$}
\UnaryInfC{$p \to (q \vee r) \succ (p \wedge \neg q) \to r$}
\end{prooftree}
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\color{red}{p}$}
\RightLabel{$\to E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\neg q$}
\AxiomC{$[q]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{K}
\UnaryInfC{r}
\AxiomC{$[r]^1$}
\RightLabel{$\vee E^1$}
\TrinaryInfC{r}
\RightLabel{$\to I^2$}
\UnaryInfC{$(p \wedge \neg q) \to r$}
\end{prooftree}
$$

An Example

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q, \neg q \succ}\,{\neg L}}{q, \neg q \succ r}\,K \qquad r \succ r}{q \lor r, \neg q \succ r}\,{\lor L}}{p \to (q \lor r), p, \neg q \succ r}\,{\to L}}{p \to (q \lor r), p \land \neg q \succ r}\,{\land L}}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,{\to R}$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,{\land E}}{q \lor r}\,{\to E} \qquad \dfrac{\dfrac{\dfrac{[p \land \neg q]^2}{\neg q}\,{\land E} \qquad [q]^1}{\sharp}\,{\neg E}}{r}\,K \qquad [r]^1}{r}\,{\lor E^1}}{(p \land \neg q) \to r}\,{\to I^2}$$

An Example

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q, \neg q \succ}\,\neg L}{q, \neg q \succ r}\,K \qquad r \succ r}{q \lor r, \neg q \succ r}\,\lor L}{p \to (q \lor r), p, \neg q \succ r}\,\to L}{p \to (q \lor r), p \land \neg q \succ r}\,\land L}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,\to R$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,\land E}{q \lor r}\,\to E \qquad \dfrac{\dfrac{\dfrac{[p \land \neg q]^2}{\neg q}\,\land E \qquad [q]^1}{\sharp}\,\neg E}{r}\,K \qquad [r]^1}{r}\,\lor E^1}{(p \land \neg q) \to r}\,\to I^2$$

An Example

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\AxiomC{$q \succ q$}
\RightLabel{$\neg L$}
\UnaryInfC{$q, \neg q \succ$}
\RightLabel{K}
\UnaryInfC{$q, \neg q \succ r$}
\AxiomC{$r \succ r$}
\RightLabel{$\vee L$}
\BinaryInfC{$q \vee r, \neg q \succ r$}
\RightLabel{$\to L$}
\BinaryInfC{$p \to (q \vee r), p, \neg q \succ r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \to (q \vee r), p \wedge \neg q \succ r$}
\RightLabel{$\to R$}
\UnaryInfC{$p \to (q \vee r) \succ (p \wedge \neg q) \to r$}
\end{prooftree}
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{p}
\RightLabel{$\to E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$[p \wedge \neg q]^2$}
\RightLabel{$\wedge E$}
\UnaryInfC{$\neg q$}
\AxiomC{$[q]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{K}
\UnaryInfC{r}
\AxiomC{$[r]^1$}
\RightLabel{$\vee E^1$}
\TrinaryInfC{r}
\RightLabel{$\to I^2$}
\UnaryInfC{$(p \wedge \neg q) \to r$}
\end{prooftree}
$$

The one proof can be built in different ways

$$
\dfrac{
\dfrac{p \succ p \qquad q \lor r \succ q \lor r}{p \to (q \lor r), p \succ q \lor r}\,{\scriptstyle \to L}
\qquad
\dfrac{
\dfrac{\dfrac{q \succ q}{q, \neg q \succ}\,{\scriptstyle \neg L}}{q, \neg q \succ r}\,{\scriptstyle K}
\qquad r \succ r}{q \lor r, \neg q \succ r}\,{\scriptstyle \lor L}
}{
\dfrac{
\dfrac{p \to (q \lor r), p, \neg q \succ r}{p \to (q \lor r), p \land \neg q \succ r}\,{\scriptstyle \land L}
}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,{\scriptstyle \to R}
}\,{\scriptstyle Cut}
$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$
\dfrac{
\dfrac{
\dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,{\scriptstyle \land E}}{q \lor r}\,{\scriptstyle \to E}
\qquad
\dfrac{\dfrac{\dfrac{[p \land \neg q]^2}{\neg q}\,{\scriptstyle \land E} \qquad [q]^1}{\sharp}\,{\scriptstyle \neg E}}{r}\,{\scriptstyle K}
\qquad [r]^1
}{r}\,{\scriptstyle \lor E^1}
}{(p \land \neg q) \to r}\,{\scriptstyle \to I^2}
$$

Classical derivations build . . . *what*?

$$\dfrac{\dfrac{p \succ p}{\succ \neg p, p}\,{\scriptstyle \neg R}}{\succ \neg p \lor p}\,{\scriptstyle \lor R}$$

$$\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ p, q}{\succ p, p \to q}\,{\scriptstyle \to R}}{(p \to q) \to p \succ p, p}\,{\scriptstyle \to L}}{(p \to q) \to p \succ p}\,{\scriptstyle W}$$

Add *focus*

From $P_1, P_2, P_3 \succ C_1, C_2, C_3$ to $P_1, P_2, P_3 \succ C_1; C_2, C_3$

Add *focus*

From $P_1, P_2, P_3 \succ C_1, C_2, C_3$ to $P_1, P_2, P_3 \succ C_1; C_2, C_3$

A *focussed* sequent has the shape $X \succ C; Y$
where C is either a formula or is *empty*,
and X and Y are finite multisets of formulas.

(The empty case corresponds to a proof of a *contradiction*.)

Add *focus*

From $P_1, P_2, P_3 \succ C_1, C_2, C_3$ to $P_1, P_2, P_3 \succ C_1; C_2, C_3$

A *focussed* sequent has the shape $X \succ C; Y$
where C is either a formula or is *empty*,
and X and Y are finite multisets of formulas.

(The empty case corresponds to a proof of a *contradiction*.)

A *proof* for $P_1, P_2, P_3 \succ C_1; C_2, C_3$ is a proof *of* C_1
from the context P_1, P_2, P_3 (*positive*) and C_2^-, C_3^- (*negative*).

Focus, *Defocus*; *Retrieve* and *Store*

NATURAL DEDUCTION

$$\begin{array}{c} [A^-] \\ \Pi \\ \sharp \end{array}$$
$$\frac{}{A}\ \textit{Retrieve}$$

SEQUENT CALCULUS

$$\frac{X \succ\ ; A, Y}{X \succ A; Y}\ \textit{Focus}$$

Focus, *Defocus*; *Retrieve* and *Store*

NATURAL DEDUCTION

SEQUENT CALCULUS

$$\dfrac{\begin{matrix}[A^-] \\ \Pi \\ \sharp\end{matrix}}{A}\,\textit{Retrieve}$$

$$\dfrac{X \succ\ ; A, Y}{X \succ A; Y}\,\textit{Focus}$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad A^-}{\sharp}\,\textit{Store}$$

$$\dfrac{X \succ A; Y}{X \succ\ ; A, Y}\,\textit{Defocus}$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{p \succ p;\ \ \dfrac{\dfrac{\dfrac{p \succ p;}{p \succ\ ;p}\ \textit{Defocus}}{p \succ q;p}\ \textit{Focus}}{\succ p \to q;p}\ {\to}R}
      {(p \to q) \to p \succ p;p}\ {\to}L
    }{(p \to q) \to p \succ\ ;p,p}\ \textit{Defocus}
  }{(p \to q) \to p \succ p;}\ \textit{Focus}
}{}
$$

NATURAL DEDUCTION

$$
\dfrac{
  \dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1\ \ [p^-]^2}{\sharp}\ \textit{Store}}{q}\ \textit{Retrieve}}{p \to q}\ {\to}I^1}{p}\ {\to}E
  \qquad [p^-]^2
}{\sharp}\ \textit{Store}
\Bigg/ \ \dfrac{\sharp}{q}\ \textit{Retrieve}
$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$
\dfrac{\dfrac{\dfrac{p \succ p; \qquad \dfrac{\dfrac{\dfrac{p \succ p;}{p \succ\ ; p}\,\textit{Defocus}}{p \succ q; p}\,\textit{Focus}}{\succ p \to q; p}\,{\to}R}{(p \to q) \to p \succ p; p}\,{\to}L}{(p \to q) \to p \succ\ ; p, p}\,\textit{Defocus}}{(p \to q) \to p \succ p;}\,\textit{Focus}
$$

NATURAL DEDUCTION

$$
\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1 \quad [p^-]^2}{\sharp}\,\textit{Store}}{q}\,\textit{Retrieve}}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^-]^2}{\dfrac{\sharp}{q}\,\textit{Retrieve}}\,\textit{Store}
$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$\dfrac{\dfrac{\dfrac{p \succ p; \qquad \dfrac{\dfrac{\dfrac{p \succ p;}{p \succ\ ;p}\,\textit{Defocus}}{p \succ q;p}\,\textit{Focus}}{\succ p \to q;p}\,{\to}R}{(p \to q) \to p \succ p;p}\,{\to}L}{(p \to q) \to p \succ\ ;p,p}\,\textit{Defocus}}{(p \to q) \to p \succ p;}\,\textit{Focus}$$

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1 \quad [p^-]^2}{\sharp}\,\textit{Store}}{q}\,\textit{Retrieve}}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^-]^2}{\sharp}\,\textit{Store}}{q}\,\textit{Retrieve}$$

SEQUENT CALCULUS

$$\dfrac{\dfrac{\dfrac{p \succ p; \qquad \dfrac{\dfrac{\dfrac{p \succ p;}{p \succ\ ;p}\,\textit{Defocus}}{\color{red}{p \succ q;p}}\,\textit{Focus}}{\succ p \to q;p}\,{\to}R}{(p \to q) \to p \succ p;p}\,{\to}L}{(p \to q) \to p \succ\ ;p,p}\,\textit{Defocus}}{(p \to q) \to p \succ p;}\,\textit{Focus}$$

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[\color{red}{p}]^1 \quad [\color{red}{p^-}]^2}{\sharp}\,\textit{Store}}{\color{red}{q}}\,\textit{Retrieve}}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^-]^2}{\sharp}\,\textit{Store}}{q}\,\textit{Retrieve}$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$
\begin{prooftree}
\AxiomC{$p \succ p;$}
\AxiomC{$p \succ p;$}
\RightLabel{\textit{Defocus}}
\UnaryInfC{$p \succ\ ;p$}
\RightLabel{\textit{Focus}}
\UnaryInfC{$p \succ q;p$}
\RightLabel{$\to R$}
\UnaryInfC{$\succ p \to q;p$}
\RightLabel{$\to L$}
\BinaryInfC{$(p \to q) \to p \succ p;p$}
\RightLabel{\textit{Defocus}}
\UnaryInfC{$(p \to q) \to p \succ\ ;p,p$}
\RightLabel{\textit{Focus}}
\UnaryInfC{$(p \to q) \to p \succ p;$}
\end{prooftree}
$$

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$(p \to q) \to p$}
\AxiomC{$[p]^1 \quad [p^-]^2$}
\RightLabel{\textit{Store}}
\UnaryInfC{$\sharp$}
\RightLabel{\textit{Retrieve}}
\UnaryInfC{q}
\RightLabel{$\to I^1$}
\UnaryInfC{$p \to q$}
\RightLabel{$\to E$}
\BinaryInfC{p}
\AxiomC{$[p^-]^2$}
\RightLabel{\textit{Store}}
\BinaryInfC{$\sharp$}
\RightLabel{\textit{Retrieve}}
\UnaryInfC{q}
\end{prooftree}
$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$
\dfrac{
  p \succ p; \qquad
  \dfrac{
    \dfrac{
      \dfrac{p \succ p;}{p \succ \;; p}\,\textit{Defocus}
    }{p \succ q; p}\,\textit{Focus}
  }{\succ p \to q; p}\,{\to}R
}{
  \dfrac{
    \dfrac{(p \to q) \to p \succ p; p}{(p \to q) \to p \succ \;; p, p}\,\textit{Defocus}
  }{(p \to q) \to p \succ p;}\,\textit{Focus}
}\,{\to}L
$$

NATURAL DEDUCTION

$$
\dfrac{
  \dfrac{
    (p \to q) \to p \qquad
    \dfrac{
      \dfrac{
        \dfrac{[p]^1 \quad [p^-]^2}{\sharp}\,\textit{Store}
      }{q}\,\textit{Retrieve}
    }{p \to q}\,{\to}I^1
  }{p}\,{\to}E
  \qquad [p^-]^2
}{\sharp}\,\textit{Store}
\;\Rightarrow\;
\dfrac{\sharp}{q}\,\textit{Retrieve}
$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$
\begin{prooftree}
\AxiomC{$p \succ p;$}
\AxiomC{$p \succ p;$}
\RightLabel{\textit{Defocus}}
\UnaryInfC{$p \succ\ ;p$}
\RightLabel{\textit{Focus}}
\UnaryInfC{$p \succ q;p$}
\RightLabel{$\rightarrow R$}
\UnaryInfC{$\succ p \rightarrow q;p$}
\RightLabel{$\rightarrow L$}
\BinaryInfC{$\color{red}{(p \rightarrow q) \rightarrow p \succ p;p}$}
\RightLabel{\textit{Defocus}}
\UnaryInfC{$(p \rightarrow q) \rightarrow p \succ\ ;p,p$}
\RightLabel{\textit{Focus}}
\UnaryInfC{$(p \rightarrow q) \rightarrow p \succ p;$}
\end{prooftree}
$$

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$\color{red}{(p \rightarrow q) \rightarrow p}$}
\AxiomC{$[p]^1 \quad [\color{red}{p^-}]^2$}
\RightLabel{\textit{Store}}
\UnaryInfC{$\sharp$}
\RightLabel{\textit{Retrieve}}
\UnaryInfC{q}
\RightLabel{$\rightarrow I^1$}
\UnaryInfC{$p \rightarrow q$}
\RightLabel{$\rightarrow E$}
\BinaryInfC{$\color{red}{p}$}
\AxiomC{$[p^-]^2$}
\RightLabel{\textit{Store}}
\BinaryInfC{$\sharp$}
\RightLabel{\textit{Retrieve}}
\UnaryInfC{q}
\end{prooftree}
$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$\dfrac{\dfrac{\dfrac{p \succ p; \qquad \dfrac{\dfrac{\dfrac{p \succ p;}{p \succ\ ; p}\,\textit{Defocus}}{p \succ q; p}\,\textit{Focus}}{\succ p \to q; p}\,{\to}R}{(p \to q) \to p \succ p; p}\,{\to}L}{(p \to q) \to p \succ\ ; p, p}\,\textit{Defocus}}{(p \to q) \to p \succ p;}\,\textit{Focus}$$

NATURAL DEDUCTION

$$\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1 \quad [p^-]^2}{\sharp}\,\textit{Store}}{q}\,\textit{Retrieve}}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^-]^2}{\dfrac{\sharp}{q}\,\textit{Retrieve}}\,\textit{Store}$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$
\begin{prooftree}
\AxiomC{$p \succ p;$}
\AxiomC{$p \succ p;$}
\RightLabel{\textit{Defocus}}
\UnaryInfC{$p \succ \ ;p$}
\RightLabel{\textit{Focus}}
\UnaryInfC{$p \succ q;p$}
\RightLabel{$\to R$}
\UnaryInfC{$\succ p \to q;p$}
\RightLabel{$\to L$}
\BinaryInfC{$(p \to q) \to p \succ p;p$}
\RightLabel{\textit{Defocus}}
\UnaryInfC{$(p \to q) \to p \succ \ ;p,p$}
\RightLabel{\textit{Focus}}
\UnaryInfC{$(p \to q) \to p \succ p;$}
\end{prooftree}
$$

NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$(p \to q) \to p$}
\AxiomC{$[p]^1 \quad [p^-]^2$}
\RightLabel{\textit{Store}}
\UnaryInfC{$\sharp$}
\RightLabel{\textit{Retrieve}}
\UnaryInfC{q}
\RightLabel{$\to I^1$}
\UnaryInfC{$p \to q$}
\RightLabel{$\to E$}
\BinaryInfC{p}
\AxiomC{$[p^-]^2$}
\RightLabel{\textit{Store}}
\BinaryInfC{$\sharp$}
\RightLabel{\textit{Retrieve}}
\UnaryInfC{q}
\end{prooftree}
$$

Adding the *Store/Retrieve* rules
to Gentzen–Prawitz Natural Deduction
gives you a well-behaved, normalising
natural deduction system for classical logic.

Adding the *Store/Retrieve* rules
to Gentzen–Prawitz Natural Deduction
gives you a well-behaved, normalising
natural deduction system for classical logic.

(It’s basically Michel Parigot’s $\lambda\mu$ calculus.)

$X \succ A; Y$ — a *proof* of A, from a context where X is asserted and Y is denied.

$X \succ A; Y$ — a *proof* of A, from a context where X is asserted and Y is denied.

$X \succ \ ; Y$ — a *refutation* of asserting X and denying Y.

I’ll pass freely between sequent derivations
and natural deduction proofs with alternatives.

DEFINING RULES

What makes rules *well behaved*?

$$\frac{A \qquad B}{A \wedge B}\,{\wedge}I \qquad \frac{A \wedge B}{A}\,{\wedge}E \quad \frac{A \wedge B}{B}\,{\wedge}E$$

What makes rules *well behaved*?

$$\frac{A \qquad B}{A \wedge B}\,{\scriptstyle \wedge I} \qquad \frac{A \wedge B}{A}\,{\scriptstyle \wedge E} \quad \frac{A \wedge B}{B}\,{\scriptstyle \wedge E}$$

$$\frac{A}{A \mathbin{\textit{tonk}} B}\,{\scriptstyle \textit{tonkI}} \qquad \frac{A \mathbin{\textit{tonk}} B}{B}\,{\scriptstyle \textit{tonkE}}$$

THE RUNABOUT INFERENCE-TICKET

By A. N. Prior

IT is sometimes alleged that there are inferences whose validity arises solely from the meanings of certain expressions occurring in them. The precise technicalities employed are not important, but let us say that such inferences, if any such there be, are analytically valid.

One sort of inference which is sometimes said to be in this sense analytically valid is the passage from a conjunction to either of its conjuncts, e.g., the inference ' Grass is green and the sky is blue, therefore grass is green '. The validity of this inference is said to arise solely from the meaning of the word ' and '. For if we are asked what is the meaning of the word ' and ', at least in the purely conjunctive sense (as opposed to, e.g., its colloquial use to mean ' and then '), the answer is said to be *completely* given by saying that (i) from any pair of statements P and Q we can infer the statement formed by joining P to Q by ' and ' (which statement we hereafter describe as ' the statement P-and-Q '), that (ii)

One way to be analytically valid is to be a *definition* . . .

One way to be analytically valid is to be a *definition* . . .

. . . but $\wedge I$ and $\wedge E$ don't look much like *definitions*.

Invertible rules look *more* like definitions.

$$\frac{X, A, B \succ Z}{X, A \land B \succ Z}\,{\land}Df \qquad \frac{X, A \succ B; Y}{X \succ A \to B; Y}\,{\to}Df \qquad \frac{X, A \succ\ ; Y}{X \succ \neg A; Y}\,{\neg}Df$$

They charaterise *one* aspect of the behaviour of the introduced concept (positively or negatively). The structural rules settle the rest.

Defining rules *define*

They are *conservative* and *uniquely defining*.

From Defining Rules to Left/Right Rules …

$$\frac{X, A, B \succ Z}{X, A \land B \succ Z}\,{\scriptstyle \land Df\!\downarrow} \qquad \frac{X, A, B \succ Z}{X, A \land B \succ Z}\,{\scriptstyle \land L}$$

$\land L$ is one half of $\land Df$

$$\frac{X, A, B \succ Z}{X, A \land B \succ Z}\ {\scriptstyle \land Df\downarrow} \qquad \frac{X, A, B \succ Z}{X, A \land B \succ Z}\ {\scriptstyle \land L}$$

$\land L$ is one half of $\land Df$

$$\dfrac{X \succ A; Y \qquad \dfrac{X' \succ B; Y' \qquad \dfrac{\dfrac{}{A \land B \succ A \land B;}\ {\scriptstyle Id}}{A, B \succ A \land B;}\ {\scriptstyle \land Df\uparrow}}{X', A \succ A \land B; Y'}\ {\scriptstyle Cut}}{X, X' \succ A \land B; Y, Y'}\ {\scriptstyle Cut}$$

From Defining Rules to Left/Right Rules …

$$\dfrac{X, A, B \succ Z}{X, A \wedge B \succ Z}\,{\scriptstyle \wedge Df\downarrow} \qquad \dfrac{X, A, B \succ Z}{X, A \wedge B \succ Z}\,{\scriptstyle \wedge L}$$

$\wedge L$ is one half of $\wedge Df$

$$\dfrac{X \succ A; Y \quad \dfrac{X' \succ B; Y' \quad \dfrac{\dfrac{}{A \wedge B \succ A \wedge B;}\,{\scriptstyle Id}}{A, B \succ A \wedge B;}\,{\scriptstyle \wedge Df\uparrow}}{X', A \succ A \wedge B; Y'}\,{\scriptstyle Cut}}{X, X' \succ A \wedge B; Y, Y'}\,{\scriptstyle Cut} \qquad \dfrac{X \succ A; Y \quad X' \succ B; Y'}{X, X' \succ A \wedge B; Y, Y'}\,{\scriptstyle \wedge R}$$

$\wedge R$ is formed from the other half, using *Id* and *Cut*.

$$\dfrac{\dfrac{\overline{A \succ A;}\;Id \qquad \overline{B \succ B;}\;Id}{A, B \succ A \wedge B;}\;\wedge R \qquad X, A \wedge B \succ Z}{X, A, B \succ Z}\;Cut \qquad\qquad \dfrac{X, A \wedge B \succ Z}{X, A, B \succ Z}\;\wedge Df{\uparrow}$$

We can recover $\wedge Df{\uparrow}$ from $\wedge R$, given *Id* and *Cut*, and $\wedge Df{\downarrow}$ is $\wedge L$.

L/R rules given in this way admit elimination of principal *Cuts*

$$
\dfrac{\dfrac{\dfrac{\Delta}{X \succ A; Y} \quad \dfrac{\Delta'}{X' \succ B; Y'}}{X, X' \succ A \land B; Y, Y'}\,{\scriptstyle \land R} \qquad \dfrac{\dfrac{\Delta''}{X'', A, B \succ Z''}}{X'', A \land B \succ Z''}\,{\scriptstyle \land L}}{X, X', X'' \succ Z'', Y, Y'}\,{\scriptstyle Cut}
$$

Unpacks into . . .

L/R rules given in this way admit elimination of principal $Cuts$

$$\dfrac{\dfrac{\begin{matrix}\Delta \\ X \succ A; Y\end{matrix} \qquad \begin{matrix}\Delta' \\ X' \succ B; Y'\end{matrix}}{X, X' \succ A \wedge B; Y, Y'}{\scriptstyle \wedge R} \qquad \dfrac{\begin{matrix}\Delta'' \\ X'', A, B \succ Z''\end{matrix}}{X'', A \wedge B \succ Z''}{\scriptstyle \wedge L}}{X, X', X'' \succ Z'', Y, Y'}{\scriptstyle Cut}$$

Unpacks into . . .

$$\dfrac{\dfrac{\begin{matrix}\Delta \\ X \succ A; Y\end{matrix} \qquad \dfrac{\begin{matrix}\Delta' \\ X' \succ B; Y'\end{matrix} \qquad \dfrac{\dfrac{}{A \wedge B \succ A \wedge B;}{\scriptstyle Id}}{A, B \succ A \wedge B;}{\scriptstyle \wedge Df\uparrow}}{X', A \succ A \wedge B; Y'}{\scriptstyle Cut}}{X, X' \succ A \wedge B; Y, Y'}{\scriptstyle Cut} \qquad \dfrac{\begin{matrix}\Delta'' \\ X'', A, B \succ Z''\end{matrix}}{X'', A \wedge B \succ Z''}{\scriptstyle \wedge Df\downarrow}}{X, X', X'' \succ Z'', Y, Y'}{\scriptstyle Cut}$$

L/*R* rules given in this way admit elimination of principal *Cuts*

$$\dfrac{\dfrac{\overset{\Delta}{X \succ A; Y} \qquad \overset{\Delta'}{X' \succ B; Y'}}{X, X' \succ A \land B; Y, Y'}\,{\scriptstyle \land R} \qquad \dfrac{\overset{\Delta''}{X'', A, B \succ Z''}}{X'', A \land B \succ Z''}\,{\scriptstyle \land L}}{X, X', X'' \succ Z'', Y, Y'}\,{\scriptstyle Cut}$$

Unpacks into . . .

$$\dfrac{\dfrac{\overset{\Delta}{X \succ A; Y} \qquad \dfrac{\overset{\Delta'}{X' \succ B; Y'} \qquad \dfrac{\dfrac{}{A \land B \succ A \land B;}\,{\scriptstyle Id}}{A, B \succ A \land B;}\,{\scriptstyle \land Df\uparrow}}{X', A \succ A \land B; Y'}\,{\scriptstyle Cut}}{X, X' \succ A \land B; Y, Y'}\,{\scriptstyle Cut} \qquad \dfrac{\overset{\Delta''}{X'', A, B \succ Z''}}{X'', A \land B \succ Z''}\,{\scriptstyle \land Df\downarrow}}{X, X', X'' \succ Z'', Y, Y'}\,{\scriptstyle Cut}$$

L/*R* rules given in this way admit elimination of principal *Cuts*

Permuting the *Cuts*, this becomes . . .

$$
\dfrac{\begin{matrix}\Delta \\ X \succ A; Y\end{matrix} \qquad \dfrac{\begin{matrix}\Delta' \\ X' \succ B; Y'\end{matrix} \qquad \dfrac{\dfrac{\dfrac{}{A \land B \succ A \land B;}\,{\scriptstyle Id}}{A, B \succ A \land B;}\,{\scriptstyle \land Df\uparrow} \qquad \dfrac{\begin{matrix}\Delta'' \\ X'', A, B \succ Z''\end{matrix}}{X'', A \land B \succ Y''}\,{\scriptstyle \land Df\downarrow}}{X'', A, B \succ Z''}\,{\scriptstyle Cut}}{X', A \succ Z'', Y'}\,{\scriptstyle Cut}}{X, X', X'' \succ Z'', Y, Y'}\,{\scriptstyle Cut}
$$

. . . which (since the *Id*/*Df*↑/*Df*↓/*Cut* detour is redundant) simplifies to:

Permuting the *Cuts*, this becomes ...

$$\dfrac{\begin{matrix}\Delta \\ X \succ A; Y\end{matrix} \quad \dfrac{\begin{matrix}\Delta' \\ X' \succ B; Y'\end{matrix} \quad \dfrac{\dfrac{\dfrac{}{A \wedge B \succ A \wedge B;}\,Id}{A, B \succ A \wedge B;}\,{\wedge}Df{\uparrow} \quad \dfrac{\begin{matrix}\Delta'' \\ X'', A, B \succ Z''\end{matrix}}{X'', A \wedge B \succ Y''}\,{\wedge}Df{\downarrow}}{X'', A, B \succ Z''}\,Cut}{X', A \succ Z'', Y'}\,Cut}{X, X', X'' \succ Z'', Y, Y'}\,Cut$$

... which (since the *Id*/*Df*↑/*Df*↓/*Cut* detour is redundant) simplifies to:

$$\dfrac{\begin{matrix}\Delta \\ X \succ A; Y\end{matrix} \quad \dfrac{\begin{matrix}\Delta' \\ X' \succ B; Y'\end{matrix} \quad \begin{matrix}\Delta'' \\ X'', A, B \succ Z''\end{matrix}}{X', A \succ Z'', Y'}\,Cut}{X, X', X'' \succ Z'', Y, Y'}\,Cut$$

$\wedge Df$ in Natural Deduction

SEQUENT CALCULUS

$$\frac{X, A, B \succ Z}{X, A \wedge B \succ Z}\,\wedge Df{\downarrow}$$

NATURAL DEDUCTION

$$\dfrac{A \wedge B \qquad \begin{array}{c}[A, B] \\ \Pi \\ C\end{array}}{C}\,\wedge E$$

$\wedge Df$ in Natural Deduction

SEQUENT CALCULUS

$$\frac{X, A, B \succ Z}{X, A \wedge B \succ Z}\ \wedge Df{\downarrow}$$

$$\frac{X, A \wedge B \succ Z}{X, A, B \succ Z}\ \wedge Df{\uparrow}$$

NATURAL DEDUCTION

$$\dfrac{A \wedge B \qquad \begin{matrix}[A, B] \\ \Pi \\ C\end{matrix}}{C}\ \wedge E$$

$$\begin{matrix}\dfrac{A \qquad B}{A \wedge B}\ \wedge I \\ \Pi\end{matrix}$$

Cut Elimination and the *Subformula Property* for *rules other than Cut* gives *Conservative Extension*.

Cut Elimination and the *Subformula Property* for *rules other than Cut* gives *Conservative Extension*.

The shape of the *defining rules* gives *Uniqueness*.

Defining Rules and *Generality*

$$\frac{X \succ A(n); Y}{X \succ \forall x A(x); Y}\ \forall Df$$

$$\frac{X \succ A(n); Y}{X \succ \forall x A(x); Y}\ \forall Df$$

n is absent from the lower sequent,
and it must be *inferentially general*.

Specification as a Rule

$$\dfrac{\dfrac{\forall xFx \succ \forall xFx;}{\forall xFx \succ Fn;}\,\forall Df\!\uparrow}{\forall xFx \succ Ft;}$$

Specification as a Rule

$$\dfrac{\dfrac{\forall xFx \succ \forall xFx;}{\forall xFx \succ Fn;}\,\forall Df{\uparrow}}{\forall xFx \succ Ft;}\,Spec^{n}_{t}$$

Specification as a Rule

$$\dfrac{\dfrac{\forall x Fx \succ \forall x Fx;}{\forall x Fx \succ Fn;}\,\forall Df{\uparrow}}{\forall x Fx \succ Ft;}\,Spec^n_t$$

$$\dfrac{X \succ Z}{X[n/t] \succ Z[n/t]}\,Spec^n_t$$

Spec, like *Id* and *Cut*, are primitive rules in the system with *Df* rules.

Spec, like *Id* and *Cut*, are primitive rules in the system with *Df* rules.

Spec is admissible (*height preserving* admissible, in fact) as are *Id* (for complex formulas) and *Cut* in the system with *L*/*R* rules.

DEFINING RULES FOR IDENTITY

Identity and harmony

Stephen Read

1. Harmony

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960–61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if 'tonk' has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of 'p tonk q' is given by Prior's rule:

$$\frac{p}{p \text{ tonk } q}\,\text{tonk-I}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{p \text{ tonk } q \qquad \begin{matrix}(p)\\ r\end{matrix}}{r}\,\text{tonk-E}$$

A Defining Rule for Identity

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y}{=}Df$$

(Here, F is *inferentially general*, and absent from the lower sequent.)

A Defining Rule for Identity

$$\dfrac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} =\!Df$$

(Here, F is *inferentially general*, and absent from the lower sequent.)

Denying $a = b$ has the same significance as taking there to be *some* feature F that holds of a but not b, or *vice versa*.

A Defining Rule for Identity

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} =Df$$

(Here, F is *inferentially general*, and absent from the lower sequent.)

Denying $a = b$ has the same significance as taking there to be *some* feature F that holds of a but not b, or *vice versa*.

Or equivalently, to prove that $a = b$, prove Fb from the assumption Fa (and *vice versa*), where the predicate F is *arbitrary*.

A Defining Rule for Identity

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} {=}Df$$

(Here, F is *inferentially general*, and absent from the lower sequent.)

Denying $a = b$ has the same significance as taking there to be *some* feature F that holds of a but not b, or *vice versa*.

Or equivalently, to prove that $a = b$, prove Fb from the assumption Fa (and *vice versa*), where the predicate F is *arbitrary*.

Identity is a kind of *indistinguishability*.

SEQUENT CALCULUS

$$\frac{X \succ a = b; Y}{X, Fa \succ Fb; Y}\,{=}Df{\uparrow}_1 \qquad \frac{X \succ a = b; Y}{X, Fb \succ Fa; Y}\,{=}Df{\uparrow}_2$$

$=Df{\uparrow}$ in Natural Deduction

SEQUENT CALCULUS

$$\frac{X \succ a = b; Y}{X, Fa \succ Fb; Y}\ {=}Df{\uparrow}_1 \qquad \frac{X \succ a = b; Y}{X, Fb \succ Fa; Y}\ {=}Df{\uparrow}_2$$

$$\frac{\begin{matrix}\Pi \\ a = b\end{matrix} \quad Fa}{Fb}\ {=}E_1 \qquad \frac{\begin{matrix}\Pi \\ a = b\end{matrix} \quad Fb}{Fa}\ {=}E_2$$

NATURAL DEDUCTION

$=Df\!\uparrow$ in Natural Deduction

SEQUENT CALCULUS

$$\dfrac{\dfrac{X \succ a = b; Y}{X, Fa \succ Fb; Y}{\scriptstyle =Df\uparrow_1}}{X, Pa \succ Pb; Y}{\scriptstyle Spec^F_P} \qquad \dfrac{\dfrac{X \succ a = b; Y}{X, Fb \succ Fa; Y}{\scriptstyle =Df\uparrow_2}}{X, Pb \succ Pa; Y}{\scriptstyle Spec^F_P}$$

$$\dfrac{\begin{matrix}\Pi\\ a = b\end{matrix} \qquad Pa}{Pb}{\scriptstyle =E_1} \qquad \dfrac{\begin{matrix}\Pi\\ a = b\end{matrix} \qquad Pb}{Pa}{\scriptstyle =E_2}$$

NATURAL DEDUCTION

An Example Derivation

$$\frac{X, Fa \succ Fb; Y \quad X, Fb \succ Fa; Y}{X \succ a = b; Y} {=}Df \qquad \frac{X \succ A[x/a]; Y}{X \succ (\lambda x.A)a; Y} \lambda Df \qquad \frac{X \succ Z}{X[F/P] \succ Z[F/P]} Spec^F_P$$

An Example Derivation

$$\dfrac{\dfrac{X, Fa \succ Fb; Y \quad X, Fb \succ Fa; Y}{}}{X \succ a = b; Y}\,{=}Df \qquad \dfrac{\dfrac{X \succ A[x/a]; Y}{}}{X \succ (\lambda x.A)a; Y}\,\lambda Df \qquad \dfrac{X \succ Z}{X[F/P] \succ Z[F/P]}\,Spec^F_P$$

$$\dfrac{\dfrac{\dfrac{Fa \succ Fa; \qquad Fa \succ Fa;}{\succ a = a;}\,{=}Df{\downarrow}}{\succ (\lambda x.x = a)a;}\,\lambda Df{\downarrow} \qquad \dfrac{\dfrac{a = b \succ a = b;}{a = b, Fa \succ Fb;}\,{=}Df{\uparrow}}{a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b;}\,Spec^F_{(\lambda x.x=a)}}{\dfrac{a = b \succ (\lambda x.x = a)b;}{a = b \succ b = a;}\,\lambda Df{\uparrow}}\,Cut$$

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y}\,{=}Df\!\downarrow$$

$$
\dfrac{\dfrac{X \succ Pa; Y \qquad \dfrac{\dfrac{a = b \succ a = b;}{a = b, Fa \succ Fb;}\,{=}Df{\uparrow}}{a = b, Pa \succ Pb;}\,Spec^{F}_{P}}{a = b, X \succ Pb; Y}\,Cut \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\,Cut
$$

$$
\dfrac{\dfrac{X \succ Pa; Y \qquad \dfrac{\dfrac{a = b \succ a = b;}{a = b, Fa \succ Fb;}\,{=}\mathit{Df}{\Uparrow}}{a = b, Pa \succ Pb;}\,\mathit{Spec}^{F}_{P}}{a = b, X \succ Pb; Y}\,\mathit{Cut} \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\,\mathit{Cut}
$$

$$
\dfrac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\,{=}L_1
$$

$$
\dfrac{\dfrac{X \succ Pa; Y \qquad \dfrac{\dfrac{a = b \succ a = b;}{a = b, Fa \succ Fb;}\,{\scriptstyle =Df\!\uparrow}}{a = b, Pa \succ Pb;}\,{\scriptstyle Spec^{F}_{P}}}{a = b, X \succ Pb; Y}\,{\scriptstyle Cut} \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\,{\scriptstyle Cut}
$$

$$
\dfrac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\,{\scriptstyle =L_1} \qquad \dfrac{X \succ Pb; Y \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y}\,{\scriptstyle =L_2}
$$

Spec is height-preserving admissible.

We can eliminate *Cut*, as usual.

But eliminating *Cut* hardly seems worth it!

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} {=}R$$

$$\frac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y} {=}L_1 \qquad \frac{X \succ Pb; Y \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y} {=}L_2$$

Each rule breaks the subformula property.

$=R$ might be excusable (by analogy with $\forall R/\exists L$),

but in $=L$, P can be *any* predicate,

primitive or complex.

But eliminating *Cut* hardly seems worth it!

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} {=}R$$

$$\frac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y} {=}L_1 \qquad \frac{X \succ Pb; Y \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y} {=}L_2$$

Each rule breaks the subformula property.

$=R$ might be excusable (by analogy with $\forall R/\exists L$),

but in $=L$, P can be *any* predicate,

primitive or complex.

For *analytic* rules, we must look elsewhere.

IDENTITY *AXIOMS*

From Rules to *Axioms*: From $=R$ to *Refl*

$$\frac{}{\succ a = a;}\,\textit{Refl}$$

From Rules to *Axioms*: From $=R$ to *Refl*

$$\dfrac{}{\succ a = a;}\ \mathit{Refl}$$

$$\dfrac{Fa \succ Fa; \qquad Fa \succ Fa;}{\succ a = a;}\ {=}R$$

From *Refl* to $=R$

Replace this:

$$\dfrac{\begin{array}{c}\Delta_1\\ X, Fa \succ Fb; Y\end{array} \qquad \begin{array}{c}\Delta_2\\ X, Fb \succ Fa; Y\end{array}}{X \succ a = b; Y}\ {=}R$$

Replace this:

$$\dfrac{\dfrac{\Delta_1}{X, Fa \succ Fb; Y} \qquad \dfrac{\Delta_2}{X, Fb \succ Fa; Y}}{X \succ a = b; Y}\ {=}R$$

With this:

$$\dfrac{\dfrac{\dfrac{}{\succ a = a;}\ \mathit{Refl}}{\succ \lambda x.(a = x)a;}\ \lambda R \qquad \dfrac{\Delta_1[F/\lambda x.(a = x)]}{X, \lambda x.(a = x)a \succ \lambda x.(a = x)b; Y}}{\dfrac{X \succ \lambda x.(a = x)b; Y}{X \succ a = b; Y}\ \lambda R}\ \mathit{Cut}$$

From $=L$ to $=L.ax$ and back.

$$\cfrac{}{a = b, Pa \succ Pb;}\ {=}L.ax_1 \qquad \cfrac{}{a = b, Pb \succ Pa;}\ {=}L.ax_2$$

From $=L$ to $=L.ax$ and back.

$$\frac{}{a = b, Pa \succ Pb;}{=}L.ax_1 \qquad \frac{}{a = b, Pb \succ Pa;}{=}L.ax_2$$

$$\frac{Pa \succ Pa; \qquad Pb \succ Pb;}{a = b, Pa \succ Pb;}{=}L_1 \qquad \frac{Pb \succ Pb; \qquad Pa \succ Pa;}{a = b, Pb \succ Pa;}{=}L_2$$

From $=L$ to $=L.ax$ and back.

$$\dfrac{}{a = b, Pa \succ Pb;}{\scriptstyle =L.ax_1} \qquad \dfrac{}{a = b, Pb \succ Pa;}{\scriptstyle =L.ax_2}$$

$$\dfrac{\dfrac{X \succ Pa; Y \qquad \dfrac{}{a = b, Pa \succ Pb;}{\scriptstyle =L.ax_1}}{a = b, X \succ Pb; Y}{\scriptstyle Cut} \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{X \succ Pb; Y \qquad \dfrac{}{a = b, Pb \succ Pa;}{\scriptstyle =L.ax_2}}{a = b, X \succ Pa; Y}{\scriptstyle Cut} \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y}{\scriptstyle Cut}$$

We can restrict $=L.ax$ to *primitive* predicates

$$
\cfrac{
\cfrac{\cfrac{}{a = b, Pa \succ Pb;}{=}L.ax_1}{a = b, Pa \wedge Qa \succ Pb;}\wedge L
\qquad
\cfrac{\cfrac{}{a = b, Qa \succ Qb;}{=}L.ax_1}{a = b, Pa \wedge Qa \succ Qb;}\wedge L
}{
\cfrac{a = b, Pa \wedge Qa \succ Pa \wedge Qb;}{a = b, \lambda x.(Px \wedge Qx)a \succ \lambda x.(Px \wedge Qx)b;}\lambda
}\wedge R
$$

We can restrict $=$*L.ax* to *primitive* predicates

$$
\dfrac{\dfrac{\dfrac{\overline{a = b, Pb \succ Pa;}\;{=}L.ax_2}{a = b, Pb, \neg Pa \succ \;;}\;\neg L}{a = b, \neg Pa \succ \neg Pb;}\;\neg R}{a = b, \lambda x.(\neg Px)a \succ \lambda x.(\neg Px)b;}\;\lambda
$$

We can restrict $=L.ax$ to *primitive* predicates

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a = b, Pac \succ Pbc;}{=}L.ax_2}{a = b, \forall y Pay \succ Pbc;}\neg L}{a = b, \forall y Pay \succ \forall y Pby;}\neg R}{a = b, \lambda x.(\forall y Pxy)a \succ \lambda x.(\forall y Pxy)b;}\lambda$$

Identity *axioms* in Natural Deduction

SEQUENT CALCULUS

$$\dfrac{}{\succ a = a;}\,\textit{Refl}$$

NATURAL DEDUCTION

$$a = a$$

Identity *axioms* in Natural Deduction

SEQUENT CALCULUS

NATURAL DEDUCTION

$$\dfrac{}{\succ a = a;}\,\textit{Refl}$$

$$a = a$$

$$\dfrac{}{a = b, Pa \succ Pb;}\,\textit{L.ax}$$

$$\dfrac{a = b \qquad Pa}{Pb}\,{=}E$$

Now eliminate *Cut*

Now that the identity rules are axioms,
Cut elimination proceeds as before.

$$\dfrac{\dfrac{}{a = b, Pa \succ Pb;}{=}L.ax_1 \qquad \dfrac{}{c = b, Pb \succ Pc;}{=}L.ax_2}{a = b, c = b, Pa \succ Pc;}\,Cut$$

Now eliminate *Cut*

Now that the identity rules are axioms,
Cut elimination proceeds as before.

$$\dfrac{\dfrac{}{a = b, Pa \succ Pb;}{=}L.ax_1 \qquad \dfrac{}{c = b, Pb \succ Pc;}{=}L.ax_2}{a = b, c = b, Pa \succ Pc;}\,Cut$$

becomes

$$\dfrac{}{a = b, c = b, Pa \succ Pc;}{=}L??$$

Now eliminate *Cut*

It suffices to close the *axioms* under *Cut*.

$$\frac{}{I^a_b, Pa \succ Pb;}{=}L.ax^*$$

Where I^a_b is any multiset
of identities *linking* a *to* b,
and P is any primitive predicate.

Now eliminate *Cut*

It suffices to close the *axioms* under *Cut*.

$$\overline{\mathrm{I}^a_b, Pa \succ Pb;}\;{=}L.ax^*$$

Where I^a_b is any multiset
of identities *linking* a *to* b,
and P is any primitive predicate.

(a) The *empty* multiset links a to a.

(b) $a = b$ links a to b and b to a.

(c) If X links a to b and Y links
b to c then X, Y links a to c.

Now eliminate *Cut*

It suffices to close the *axioms* under *Cut*.

$$\frac{}{I^a_b, Pa \succ Pb;}{=}L.ax^*$$

Where I^a_b is any multiset
of identities *linking* a *to* b,
and P is any primitive predicate.

(a) The *empty* multiset links a to a.

(b) $a = b$ links a to b and b to a.

(c) If X links a to b and Y links
b to c then X, Y links a to c.

(We can leave 'Pa' out if it is $a = a$.)

Kinds of Identity Rules

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} {=}Df$$

- $=$*Df* defines identity by giving conditions under which $a = b$ may be proved. We're in a position to prove $a = b$ iff we're in a position to transfer Fa to Fb (and back) for arbitrary F.

Kinds of Identity Rules

$$\dfrac{}{\succ a = a;}\ \textit{Refl} \qquad \dfrac{}{I^a_b, Pa \succ Pb;}\ {=}L.ax_*$$

- $=$*Df* defines identity by giving conditions under which $a = b$ may be proved. We're in a position to prove $a = b$ iff we're in a position to transfer Fa to Fb (and back) for arbitrary F.
- *Refl* and $=L.ax_*$ are *semantic constraints connecting* primitive predicates.

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} {=}Df$$

(the rule $=Df$ is drawn with a double line)

$$\frac{}{\succ a = a;}\ \textit{Refl} \qquad \frac{}{I^a_b, Pa \succ Pb;}\ {=}L.ax_*$$

- $=Df$ defines identity by giving conditions under which $a = b$ may be proved. We're in a position to prove $a = b$ iff we're in a position to transfer Fa to Fb (and back) for arbitrary F.
- *Refl* and $=L.ax_*$ are *semantic constraints connecting* primitive predicates.
- These two characterisations are *equivalent* as far as derivability goes.

THANK YOU!

Thank you!

SLIDES: `https://consequently.org/presentation/2021/comparing-identity-rules`

FEEDBACK: `@consequently` on *Twitter*,
or *email* `atrestall@unimelb.edu.au`