Speech Acts & the Quest for a Natural Account of Classical *Proof*

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THE UNIVERSITY OF MELBOURNE

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https://consequently.org/presentation/2020/speech-acts-for-classical-natural-deduction-berkeley

My Aim

To *introduce* and *defend*
Michel Parigot's λμ-calculus
as an appropriate framework
for *inferentialists* to study
classical logical concepts.

My Plan

Inferentialism & Natural Deduction

Natural Deduction is Opinionated

Other Frameworks

Natural Deduction with Alternatives

Meeting Objections

Going Beyond

INFERENTIALISM & NATURAL DEDUCTION

$$
\begin{prooftree}
\AxiomC{$[\neg(r \vee s)]^4$}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p]^3$}
\RightLabel{$\to E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$q \to s$}
\AxiomC{$[q]^1$}
\RightLabel{$\to E$}
\BinaryInfC{s}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\AxiomC{$[r]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\RightLabel{$\vee E^{1,2}$}
\TrinaryInfC{$r \vee s$}
\RightLabel{$\neg E$}
\BinaryInfC{$\perp$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg p$}
\RightLabel{$\to I^4$}
\UnaryInfC{$\neg(r \vee s) \to \neg p$}
\end{prooftree}
$$

Natural Deduction is ***Beautiful***!

$$\dfrac{\dfrac{\dfrac{[\neg(r \vee s)]^4 \qquad \dfrac{\dfrac{p \to (q \vee r) \qquad [p]^3}{q \vee r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \vee s}\,{\vee}I \qquad \dfrac{[r]^2}{r \vee s}\,{\vee}I}{r \vee s}\,{\vee}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \vee s) \to \neg p}\,{\to}I^4$$

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$$
\dfrac{\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4
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\end{prooftree}
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\dfrac{\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4
$$

$$
\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\dfrac{\dfrac{\bot}{\neg p}\,\neg I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4}\,\neg E
$$

$$
\dfrac{\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4
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$$

$$
\dfrac{
  \dfrac{
    [\neg(r \vee s)]^4
    \qquad
    \dfrac{
      \dfrac{p \to (q \vee r) \qquad [p]^3}{q \vee r}\,{\scriptstyle \to E}
      \qquad
      \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\scriptstyle \to E}}{r \vee s}\,{\scriptstyle \vee I}
      \qquad
      \dfrac{[r]^2}{r \vee s}\,{\scriptstyle \vee I}
    }{r \vee s}\,{\scriptstyle \vee E^{1,2}}
  }{\dfrac{\bot}{\neg p}\,{\scriptstyle \neg I^3}}\,{\scriptstyle \neg E}
}{\neg(r \vee s) \to \neg p}\,{\scriptstyle \to I^4}
$$

The Rules

A

The Rules

$$A$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\ {\to}I^i$$

$$A \qquad\qquad \dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\,{\to}I^i \qquad\qquad \dfrac{\begin{array}{c}\Pi\\ A \to B\end{array}\quad \begin{array}{c}\Pi'\\ A\end{array}}{B}\,{\to}E$$

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$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\ {\to}I^i$$

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$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}\ {\wedge}I$$

The Rules

$$A \qquad \dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}{\to}I^i \qquad \dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B}{\to}E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}{\wedge}I \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{A}{\wedge}E \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{B}{\wedge}E$$

The Rules

$$A \qquad \dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\to I^i \qquad \dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\to E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad \begin{matrix}\Pi' \\ B\end{matrix}}{A \land B}\land I \qquad \dfrac{\begin{matrix}\Pi \\ A \land B\end{matrix}}{A}\land E \qquad \dfrac{\begin{matrix}\Pi \\ A \land B\end{matrix}}{B}\land E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix}}{A \lor B}\lor I \qquad \dfrac{\begin{matrix}\Pi \\ B\end{matrix}}{A \lor B}\lor I$$

The Rules

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$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix} \qquad \begin{matrix}\Pi'\\ B\end{matrix}}{A \wedge B}\,{\wedge}I \qquad\qquad \dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{A}\,{\wedge}E \qquad\qquad \dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{B}\,{\wedge}E$$

$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix}}{A \vee B}\,{\vee}I \qquad\qquad \dfrac{\begin{matrix}\Pi\\ B\end{matrix}}{A \vee B}\,{\vee}I \qquad\qquad \dfrac{\begin{matrix}\Pi\\ A \vee B\end{matrix} \qquad \begin{matrix}[A]^j\\ \Pi'\\ C\end{matrix} \qquad \begin{matrix}[B]^k\\ \Pi''\\ C\end{matrix}}{C}\,{\vee}E$$

$$A \qquad \dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\to I^i \qquad \dfrac{\begin{array}{c}\Pi\\ A \to B\end{array}\quad\begin{array}{c}\Pi'\\ A\end{array}}{B}\to E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}\quad\begin{array}{c}\Pi'\\ B\end{array}}{A \wedge B}\wedge I \qquad \dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{A}\wedge E \qquad \dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{B}\wedge E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}}{A \vee B}\vee I \qquad \dfrac{\begin{array}{c}\Pi\\ B\end{array}}{A \vee B}\vee I \qquad \dfrac{\begin{array}{c}\Pi\\ A \vee B\end{array}\quad\begin{array}{c}[A]^j\\ \Pi'\\ C\end{array}\quad\begin{array}{c}[B]^k\\ \Pi''\\ C\end{array}}{C}\vee E$$

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ \bot\end{array}}{\neg A}\neg I^i$$

The Rules

$$A \qquad \frac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}{\to}I^i \qquad \frac{\begin{matrix}\Pi \\ A \to B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B}{\to}E$$

$$\frac{\begin{matrix}\Pi \\ A\end{matrix} \quad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}\wedge I \qquad \frac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{A}\wedge E \qquad \frac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{B}\wedge E$$

$$\frac{\begin{matrix}\Pi \\ A\end{matrix}}{A \vee B}\vee I \qquad \frac{\begin{matrix}\Pi \\ B\end{matrix}}{A \vee B}\vee I \qquad \frac{\begin{matrix}\Pi \\ A \vee B\end{matrix} \quad \begin{matrix}[A]^j \\ \Pi' \\ C\end{matrix} \quad \begin{matrix}[B]^k \\ \Pi'' \\ C\end{matrix}}{C}\vee E$$

$$\frac{\begin{matrix}[A]^i \\ \Pi \\ \bot\end{matrix}}{\neg A}\neg I^i \qquad \frac{\begin{matrix}\Pi \\ \neg A\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{\bot}\neg E$$

The Rules

$$A \qquad \frac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\,{\to}I^i \qquad \frac{\begin{array}{c}\Pi\\ A \to B\end{array} \quad \begin{array}{c}\Pi'\\ A\end{array}}{B}\,{\to}E$$

$$\frac{\begin{array}{c}\Pi\\ A\end{array} \quad \begin{array}{c}\Pi'\\ B\end{array}}{A \wedge B}\,{\wedge}I \qquad \frac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{A}\,{\wedge}E \qquad \frac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{B}\,{\wedge}E$$

$$\frac{\begin{array}{c}\Pi\\ A\end{array}}{A \vee B}\,{\vee}I \qquad \frac{\begin{array}{c}\Pi\\ B\end{array}}{A \vee B}\,{\vee}I \qquad \frac{\begin{array}{c}\Pi\\ A \vee B\end{array} \quad \begin{array}{c}[A]^j\\ \Pi'\\ C\end{array} \quad \begin{array}{c}[B]^k\\ \Pi''\\ C\end{array}}{C}\,{\vee}E$$

$$\frac{\begin{array}{c}[A]^i\\ \Pi\\ \bot\end{array}}{\neg A}\,{\neg}I^i \qquad \frac{\begin{array}{c}\Pi\\ \neg A\end{array} \quad \begin{array}{c}\Pi'\\ A\end{array}}{\bot}\,{\neg}E \qquad \frac{\begin{array}{c}\Pi\\ \bot\end{array}}{A}\,{\bot}E$$

- Inference is something we can *do*, and can *learn*.

Inferentialists like Natural Deduction

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- The rules are *separated*.

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- Inference is something we can *do*, and can *learn*.
- The rules are *separated*.
- I/E rules play a similar role to 'truth conditions.'
- Proofs *normalise*. (We can straighten out *detours*.)
- Normal proofs are *analytic*.

Normalisation

$$\dfrac{\dfrac{\begin{matrix}[A]^i \\ \Pi_1 \\ B\end{matrix}}{\boxed{A \to B}}{\scriptstyle \to I^i} \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}{\scriptstyle \to E}$$

Normalisation

$$
\dfrac{\dfrac{\begin{matrix}[A]^i \\ \Pi_1 \\ B\end{matrix}}{\boxed{A \to B}}{\scriptstyle \to I^i} \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}{\scriptstyle \to E}
\quad \rightsquigarrow \quad
\begin{matrix}\Pi_2 \\ A \\ \Pi_1 \\ B\end{matrix}
$$

. . . and it's type theory and the λ-calculus under the hood.

$$\dfrac{\dfrac{\begin{matrix}[x : A]^i \\ \Pi_1 \\ t(x) : B\end{matrix}}{\boxed{\lambda x.t(x) : A \to B}}\,{\to}I^i \qquad \begin{matrix}\Pi_2 \\ s : A\end{matrix}}{(\lambda x.t(x))s : B}\,{\to}E \qquad \rightsquigarrow \qquad \begin{matrix}\Pi_2 \\ s : A \\ \Pi_1 \\ t(s) : B\end{matrix}$$

What's not to love?

I try to be a *philosophical logician*, with equal emphasis on '**philosophical**' and '**logician**,' and I try to take both *proof theory* and *model theory* equally seriously for foundational purposes.

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Soundness and *completeness* help me explore the relationship between *inferentialism* and *representationalism*.

NATURAL DEDUCTION IS *OPINIONATED*

We get *inuitionistic* logic

$$\nvdash p \vee \neg p$$

We get *inuitionistic* logic

$$\nvdash p \lor \neg p$$

$$\neg\neg p \nvdash p$$

We get *inuitionistic* logic

$$\nvdash p \vee \neg p$$

$$\neg\neg p \nvdash p$$

$$\nvdash (p \to q) \vee (q \to r)$$

We get *inuitionistic* logic

$$\nvdash p \lor \neg p$$

$$\neg\neg p \nvdash p$$

$$\nvdash (p \to q) \lor (q \to r)$$

$$\nvdash (((p \to q)) \to p) \to p$$

‘Textbook’ natural deduction plugs the gap, but it has no *taste*.

$$\dfrac{\begin{array}{c}\Pi \\ \neg\neg A\end{array}}{A}\,\textit{DNE}$$

‘Textbook’ natural deduction plugs the gap, but it has no *taste*.

$$\dfrac{\begin{matrix}\Pi \\ \neg\neg A\end{matrix}}{A}\;DNE \qquad\qquad \dfrac{\begin{matrix}[\neg A]^i \\ \Pi \\ \bot\end{matrix}}{A}\;\bot E_c$$

‘Textbook’ natural deduction plugs the gap, but it has no *taste*.

$$\dfrac{\begin{array}{c}\Pi\\ \neg\neg A\end{array}}{A}\ DNE \qquad \dfrac{\begin{array}{c}[\neg A]^i\\ \Pi\\ \bot\end{array}}{A}\ \bot E_c \qquad \dfrac{\begin{array}{c}[A]^i\\ \Pi\\ C\end{array} \quad \begin{array}{c}[\neg A]^j\\ \Pi\\ C\end{array}}{C}\ Cases^{i,j}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg p]^2 \qquad \dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[\neg p]^2 \qquad [p]^1}{\bot}\,{\scriptstyle \neg E}}{q}\,{\scriptstyle \bot E}}{p \to q}\,{\scriptstyle \to I^1}}{p}\,{\scriptstyle \to E}}{\bot}\,{\scriptstyle \neg E}}{\neg\neg p}\,{\scriptstyle \neg I^2}}{p}\,{\scriptstyle DNE}}{((p \to q) \to p) \to p}\,{\scriptstyle \to I^3}
$$

OTHER FRAMEWORKS

Gentzen's Sequent Calculus

$$
\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}\;{\to}R}{(p \to q) \to p \succ p, p}\;{\to}L}{\dfrac{(p \to q) \to p \succ p}{\succ ((p \to q) \to p) \to p}\;{\to}R}\;W
$$

Gentzen's Sequent Calculus

$$
\dfrac{p \succ p \qquad \dfrac{\dfrac{p \succ q, p}{\succ p \to q, p}\ {\to}R}{}}{}
$$

$$
\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}\,{\to}R}{(p \to q) \to p \succ p, p}\,{\to}L}{(p \to q) \to p \succ p}\,W}{\succ ((p \to q) \to p) \to p}\,{\to}R
$$

$$
\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R
\qquad
\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L
$$

$$\dfrac{p \succ p \qquad \dfrac{\dfrac{p \succ q, p}{\succ p \to q, p}\,{\to}R}{}}{}$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}{\scriptstyle \to R}}{(p \to q) \to p \succ p, p}{\scriptstyle \to L}}{(p \to q) \to p \succ p}{\scriptstyle W}}{\succ ((p \to q) \to p) \to p}{\scriptstyle \to R}$$

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}{\scriptstyle \neg R}}{\succ p \vee \neg p}{\scriptstyle \vee R}$$

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}{\scriptstyle \neg L}}{p \wedge \neg p \succ}{\scriptstyle \wedge L}$$

Classical • *Separated Rules* • *Normalising* • *Analytic*

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}\,{\to}R}{(p \to q) \to p \succ p, p}\,{\to}L}{(p \to q) \to p \succ p}\,W}{\succ ((p \to q) \to p) \to p}\,{\to}R \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,{\lor}R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,{\land}L$$

Classical • *Separated Rules* • *Normalising* • *Analytic*

… but what does deriving $X \succ Y$ have to do with *proof*?

Me, in 2005: **Nothing much . . .**

MULTIPLE CONCLUSIONS

“Multiple Conclusions,” in *Logic, Methodology and Philosophy of Science: Proceedings of the Twelfth International Congress*, edited by Petr Hajek, Luis Valdes-Villanueva and Dag Westerstahl, Kings' College Publications, 2005, 189–205.

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I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with multiple premises and multiple conclusions. Gentzen’s multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for classical logic as it does for intuitionistic logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us understand and adjudicate debates between proponents of classical and non-classical logics.

This paper has now been reprinted in *Analysis and Metaphysics*, 6, 2007, 14-34.

`https://consequently.org/writing/multipleconclusions/`

. . . but deriving $X \succ Y$ *does* tell you that it’s out of bounds to assert each member of X and deny each member of Y, and that’s *something*!

Steinberger on the Principle of Answerability

The mistake in this position, however, resides in the idea that any formal game incorporating what appear to be inference rules will confer meanings on its logical symbols. Adherence to inferentialism importantly constrains one's choice of proof-theoretic frameworks and thus requires one to reject Carnap's amoralism about logic: the inferentialist must remain faithful to our ordinary inferential practice. Only those deductive systems that answer to the use we put our logical vocabulary to fit the bill. After all, it is the practice represented, not the formalism as such, that confers meanings. Therefore, the formalism is of meaning-theoretic significance and hence of interest to the inferentialist only if it succeeds in capturing (in a perhaps idealised form) the relevant meaning-constituting features of our practice. It is in this sense, then, that the inferentialist position imposes strict demands on the form deductive systems may take. For future reference, let us refer to these demands as the

> *Principle of answerability* only such deductive systems are permissible as can be seen to be suitably connected to our ordinary deductive inferential practices.

Florian Steinberger, "Why Conclusions Should Remain Single"
JPL (2011) 40:333–355 `https://dx.doi.org/10.1007/s10992-010-9153-3`

This is not just *conservatism*

What is a proof of p?

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A proof of p meets a *justification request* for the assertion of p.

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A proof of p meets a *justification request* for the assertion of p.

(Not every way to meet a justification request is a *proof*, but proofs meet justification requests in a *very* stringent way.)

An Example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \cfrac{\cfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \cfrac{[r]^2}{r \lor s}\,{\lor}I}{[\neg(r \lor s)]^4 \qquad r \lor s}\,{\lor}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4
$$

We've granted $p \to (q \lor r)$ *and* $q \to s$.

An Example

$$
\begin{prooftree}
\AxiomC{$[\neg(r \lor s)]^4$}
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\BinaryInfC{s}
\RightLabel{$\lor I$}
\UnaryInfC{$r \lor s$}
\AxiomC{$[r]^2$}
\RightLabel{$\lor I$}
\UnaryInfC{$r \lor s$}
\RightLabel{$\lor E^{1,2}$}
\TrinaryInfC{$r \lor s$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg p$}
\RightLabel{$\to I^4$}
\UnaryInfC{$\neg(r \lor s) \to \neg p$}
\end{prooftree}
$$

We've granted $p \to (q \lor r)$ *and* $q \to s$. *I assert* $\neg(r \lor s) \to \neg p$, *and you challenge me.*

An Example

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\begin{prooftree}
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\UnaryInfC{$\neg p$}
\RightLabel{$\to I^4$}
\UnaryInfC{$\neg(r \vee s) \to \neg p$}
\end{prooftree}
$$

We've granted $p \to (q \vee r)$ *and* $q \to s$. *I assert* $\neg(r \vee s) \to \neg p$, *and you challenge me. I say, suppose* $\neg(r \vee s)$. *We've got* $\neg p$. *You challenge me again,*

$$
\begin{prooftree}
\AxiomC{$[\neg(r \vee s)]^4$}
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\end{prooftree}
$$

We've granted $p \to (q \vee r)$ *and* $q \to s$. *I assert* $\neg(r \vee s) \to \neg p$, *and you challenge me. I say, suppose* $\neg(r \vee s)$. *We've got* $\neg p$. *You challenge me again, so I say, suppose* p, *and I'll show that this is inconsistent. You ask me to do that,*

$$
\dfrac{
  \dfrac{
    [\neg(r \lor s)]^4
    \qquad
    \dfrac{
      \dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}{\scriptstyle \to E}
      \qquad
      \dfrac{\dfrac{q \to s \qquad [q]^1}{s}{\scriptstyle \to E}}{r \lor s}{\scriptstyle \lor I}
      \qquad
      \dfrac{[r]^2}{r \lor s}{\scriptstyle \lor I}
    }{r \lor s}{\scriptstyle \lor E^{1,2}}
  }{
    \dfrac{\bot}{\neg p}{\scriptstyle \neg I^3}
  }{\scriptstyle \neg E}
}{\neg(r \lor s) \to \neg p}{\scriptstyle \to I^4}
$$

We've granted $p \to (q \lor r)$ *and* $q \to s$. *I assert* $\neg(r \lor s) \to \neg p$, *and you challenge me. I say, suppose* $\neg(r \lor s)$. *We've got* $\neg p$. *You challenge me again, so I say, suppose* p, *and I'll show that this is inconsistent. You ask me to do that, so I'll say we get* $r \lor s$, *which clashes with the* $\neg(r \lor s)$ *we assumed.*

An Example

$$
\begin{prooftree}
\AxiomC{$[\neg(r \lor s)]^4$}
\AxiomC{$p \to (q \lor r)$}
\AxiomC{$[p]^3$}
\RightLabel{$\to E$}
\BinaryInfC{$q \lor r$}
\AxiomC{$q \to s$}
\AxiomC{$[q]^1$}
\RightLabel{$\to E$}
\BinaryInfC{s}
\RightLabel{$\lor I$}
\UnaryInfC{$r \lor s$}
\AxiomC{$[r]^2$}
\RightLabel{$\lor I$}
\UnaryInfC{$r \lor s$}
\RightLabel{$\lor E^{1,2}$}
\TrinaryInfC{$r \lor s$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg p$}
\RightLabel{$\to I^4$}
\UnaryInfC{$\neg(r \lor s) \to \neg p$}
\end{prooftree}
$$

We've granted $p \to (q \lor r)$ *and* $q \to s$. *I assert* $\neg(r \lor s) \to \neg p$*, and you challenge me. I say, suppose* $\neg(r \lor s)$. *We've got* $\neg p$. *You challenge me again, so I say, suppose* p*, and I'll show that this is inconsistent. You ask me to do that, so I'll say we get* $r \lor s$*, which clashes with the* $\neg(r \lor s)$ *we assumed. You ask me how do you do that? I say, we'll, we've got* $q \lor r$*, from our* $p \to (q \lor r)$ *and* p*. So, let's split into two cases.*

An Example

$$\dfrac{\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \quad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \quad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4$$

We've granted $p \to (q \lor r)$ *and* $q \to s$*. I assert* $\neg(r \lor s) \to \neg p$*, and you challenge me. I say, suppose* $\neg(r \lor s)$*. We've got* $\neg p$*. You challenge me again, so I say, suppose* p*, and I'll show that this is inconsistent. You ask me to do that, so I'll say we get* $r \lor s$*, which clashes with the* $\neg(r \lor s)$ *we assumed. You ask me how do you do that? I say, we'll, we've got* $q \lor r$*, from our* $p \to (q \lor r)$ *and* p*. So, let's split into two cases. In the* q *case, we've got* $r \lor s$*,*

An Example

$$\dfrac{\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}{\to}E}{r \lor s}{\lor}I \qquad \dfrac{[r]^2}{r \lor s}{\lor}I}{r \lor s}{\lor}E^{1,2}}{\bot}\neg E}{\neg p}\neg I^3}{\neg(r \lor s) \to \neg p}{\to}I^4$$

We've granted $p \to (q \lor r)$ *and* $q \to s$. *I assert* $\neg(r \lor s) \to \neg p$, *and you challenge me. I say, suppose* $\neg(r \lor s)$. *We've got* $\neg p$. *You challenge me again, so I say, suppose* p, *and I'll show that this is inconsistent. You ask me to do that, so I'll say we get* $r \lor s$, *which clashes with the* $\neg(r \lor s)$ *we assumed. You ask me how do you do that? I say, we'll, we've got* $q \lor r$, *from our* $p \to (q \lor r)$ *and* p. *So, let's split into two cases. In the* q *case, we've got* $r \lor s$, *and we have it in the* r *case, too.*

An Example

$$
\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}{\scriptstyle \to E} \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}{\scriptstyle \to E}}{r \lor s}{\scriptstyle \lor I} \qquad \dfrac{[r]^2}{r \lor s}{\scriptstyle \lor I}}{r \lor s}{\scriptstyle \lor E^{1,2}}}{\dfrac{\bot}{\neg p}{\scriptstyle \neg I^3}}{\scriptstyle \neg E}}{\neg(r \lor s) \to \neg p}{\scriptstyle \to I^4}
$$

We've granted $p \to (q \lor r)$ *and* $q \to s$. *I assert* $\neg(r \lor s) \to \neg p$, *and you challenge me. I say, suppose* $\neg(r \lor s)$. *We've got* $\neg p$. *You challenge me again, so I say, suppose* p, *and I'll show that this is inconsistent. You ask me to do that, so I'll say we get* $r \lor s$, *which clashes with the* $\neg(r \lor s)$ *we assumed. You ask me how do you do that? I say, we'll, we've got* $q \lor r$, *from our* $p \to (q \lor r)$ *and* p. *So, let's split into two cases. In the* q *case, we've got* $r \lor s$, *and we have it in the* r *case, too. So in either case, we've got* $r \lor s$.

Slogan

A proof of A (in a context)
meets a justification request for A
on the basis of the claims we take for granted.

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A proof of A (in a context)
meets a justification request for A
on the basis of the claims we take for granted.

A sequent calculus derivation doesn't do *that*,
at least, not without quite a bit of *work*.

$$\dfrac{\dfrac{\dfrac{\dfrac{[-\,p \lor \neg p]^1}{-\,p}\,{-\lor E}}{+\,\neg p}\,{+\neg I}}{+\,p \lor \neg p}\,{+\lor I} \qquad [-\,p \lor \neg p]^2}{+\,p \lor \neg p}\,RAA^{1,2}$$

Signed Natural Deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{[-\, p \vee \neg p]^1}{-\, p}\,{\scriptstyle -\vee E}}{+\, \neg p}\,{\scriptstyle +\neg I}}{+\, p \vee \neg p}\,{\scriptstyle +\vee I} \qquad [-\, p \vee \neg p]^2}{+\, p \vee \neg p}\,{\scriptstyle RAA^{1,2}}$$

Decorate your proof with *signs*.

Double up your Rules

$$\frac{\begin{matrix}\Pi\\ +A\end{matrix}}{+A\vee B}\,{+\vee I}\qquad \frac{\begin{matrix}\Pi\\ +B\end{matrix}}{+A\vee B}\,{+\vee I}\qquad \frac{\begin{matrix}\\ \Pi\\ +A\vee B\end{matrix}\quad \begin{matrix}[+A]^j\\ \Pi'\\ \phi\end{matrix}\quad \begin{matrix}[+B]^k\\ \Pi''\\ \phi\end{matrix}}{\phi}\,{\vee E^{j,k}}$$

$$\frac{\begin{matrix}\Pi\\ -A\vee B\end{matrix}}{-A}\,{-\vee E}\qquad \frac{\begin{matrix}\Pi\\ -A\vee B\end{matrix}}{-B}\,{-\wedge E}\qquad \frac{\begin{matrix}\Pi\\ -A\end{matrix}\quad \begin{matrix}\Pi'\\ -B\end{matrix}}{-A\vee B}\,{-\vee E}$$

Double up your Rules

$$\dfrac{\begin{array}{c}\Pi\\ +A\end{array}}{+A\vee B}\ {+\vee I} \qquad \dfrac{\begin{array}{c}\Pi\\ +B\end{array}}{+A\vee B}\ {+\vee I} \qquad \dfrac{\begin{array}{c}\Pi\\ +A\vee B\end{array} \quad \begin{array}{c}[+A]^j\\ \Pi'\\ \phi\end{array} \quad \begin{array}{c}[+B]^k\\ \Pi''\\ \phi\end{array}}{\phi}\ {\vee E^{j,k}}$$

$$\dfrac{\begin{array}{c}\Pi\\ -A\vee B\end{array}}{-A}\ {-\vee E} \qquad \dfrac{\begin{array}{c}\Pi\\ -A\vee B\end{array}}{-B}\ {-\wedge E} \qquad \dfrac{\begin{array}{c}\Pi\\ -A\end{array} \quad \begin{array}{c}\Pi'\\ -B\end{array}}{-A\vee B}\ {-\vee E}$$

$$\dfrac{\begin{array}{c}\Pi\\ -A\end{array}}{+\neg A}\ {+\neg I} \qquad \dfrac{\begin{array}{c}\Pi\\ +\neg A\end{array}}{-A}\ {+\neg E} \qquad \dfrac{\begin{array}{c}\Pi\\ +A\end{array}}{-\neg A}\ {-\neg I} \qquad \dfrac{\begin{array}{c}\Pi\\ -\neg A\end{array}}{+A}\ {-\neg E}$$

Add some 'Structural' Rules

$$\frac{\begin{matrix}\Pi \\ \alpha\end{matrix} \qquad \begin{matrix}\Pi' \\ \alpha^*\end{matrix}}{\bot}\ \bot I \qquad\qquad \frac{\begin{matrix}[\alpha]^i \\ \Pi \\ \bot\end{matrix}}{\alpha^*}\ \mathit{Reductio}^i \qquad\qquad \frac{\begin{matrix}[\alpha]^j \\ \Pi' \\ \beta\end{matrix} \qquad \begin{matrix}[\alpha]^k \\ \Pi'' \\ \beta^*\end{matrix}}{\alpha^*}\ SR^{j,k}$$

α and β are signed formulas.

$(-A)^* = +A$ and $(+A)^* = -A$.

An Example

$$
\dfrac{\dfrac{\dfrac{[-\,p]^2 \qquad \dfrac{[+\,(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[-\,p]^2 \quad [+\,p]^1}{\bot}\,{\scriptstyle \bot I}}{+\,q}\,{\scriptstyle \bot E}}{+\,p \to q}\,{\scriptstyle \to I^1}}{+\,p}\,{\scriptstyle \to E}}{\bot}\,{\scriptstyle \bot I}}{+\,p}\,{\scriptstyle \mathit{Reductio}^2}}{+\,((p \to q) \to p) \to p}\,{\scriptstyle \to I^3}
$$

An Example

$$
\begin{prooftree}
\AxiomC{$[-\,p]^2$}
\AxiomC{$[+(p \to q) \to p]^3$}
\AxiomC{$[-\,p]^2$}
\AxiomC{$[+\,p]^1$}
\RightLabel{$\bot I$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{$+\,q$}
\RightLabel{$\to I^1$}
\UnaryInfC{$+\,p \to q$}
\RightLabel{$\to E$}
\BinaryInfC{$+\,p$}
\RightLabel{$\bot I$}
\BinaryInfC{$\bot$}
\RightLabel{$\mathit{Reductio}^2$}
\UnaryInfC{$+\,p$}
\RightLabel{$\to I^3$}
\UnaryInfC{$+\,((p \to q) \to p) \to p$}
\end{prooftree}
$$

Classical • *Separated Rules* • *Normalising* • *Analytic* • *Single Conclusion*

An Example

$$
\dfrac{\dfrac{\dfrac{[-\,p]^2 \qquad \dfrac{[+\,(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[-\,p]^2 \qquad [+\,p]^1}{\bot}\,{\bot I}}{+\,q}\,{\bot E}}{+\,p \to q}\,{\to I^1}}{+\,p}\,{\to E}}{\bot}\,{\bot I}}{+\,p}\,\mathit{Reductio}^2}{+\,((p \to q) \to p) \to p}\,{\to I^3}
$$

Classical • *Separated Rules* • *Normalising* • *Analytic* • *Single Conclusion*

… but what are ‘+’ and ‘−’ really *doing*?

What are these '$+$' and '$-$' doing anyway?

THE OFFICIAL LINE:

$+A$ is an *assertion* of A

$-A$ is a *denial* or *rejection* of A.

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$-A \neq +\neg A$, since denial is a speech act that cannot be embedded in other contexts, while negation modifies content, and can embed.

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$-A \neq +\neg A$, since denial is a speech act that cannot be embedded in other contexts, while negation modifies content, and can embed.

Proofs contain *speech acts*, not *contents*.

A Problem: Supposition $\neq$ Assertion

Natural deduction proofs *already* contain different speech acts.

At the leaves we can *suppose* A to later *discharge* it.

Natural deduction proofs *already* contain different speech acts.

At the leaves we can *suppose* A to later *discharge* it.

Supposing $- A$ is ... *what*, exactly?

The Lessons

- Answerability to our practice is a constraint worth meeting.

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- *Bilateralism* (paying attention to *assertion* and *denial*) is important to the defender of classical logic.

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- Answerability to our practice is a constraint worth meeting.
- *Bilateralism* (paying attention to *assertion* and *denial*) is important to the defender of classical logic.
- Sequent calculus and signed natural deduction do not approach the simplicity of standard natural deduction as an account of *proof*.

NATURAL DEDUCTION WITH ALTERNATIVES

Parigot's λμ-Calculus

Example. $((A \to B) \to A) \to A$

$$\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A \qquad \dfrac{A \vdash A}{\vdash A \to B, A}}{(A \to B) \to A \vdash A}}{\vdash ((A \to B) \to A) \to A}$$

This proof produces a $\lambda\mu$-term as follows:

$$\dfrac{\dfrac{y : (A \to B) \to A^y \vdash (A \to B) \to A \qquad \dfrac{x : A^x \vdash A}{\lambda x.\mu\delta.[\alpha]x : \vdash A \to B, A^\alpha}}{(y\ \lambda x.\mu\delta.[\alpha]x) : (A \to B) \to A^y \vdash A, A^\alpha}}{\lambda y.\mu\alpha.[\alpha](y\ \lambda x.\mu\delta.[\alpha]x) : \vdash ((A \to B) \to A) \to A}$$

Let κ be $\lambda y.\mu\alpha.[\alpha](y\ \lambda x.\mu\delta.[\alpha]x)$. When applied to arguments $u, v_1, \ldots, v_n$, it reduces in the following way:

$(\ldots((\lambda y.\mu\alpha.[\alpha](y\ \lambda x.\mu\delta.[\alpha]x\ )\ u)\ v_1)\ldots v_n)$
$\triangleright(\ldots(\mu\alpha.[\alpha](u\ \lambda x.\mu\delta.[\alpha]x)\ v_1)\ldots v_n)$
$\triangleright\mu\alpha.[\alpha](\ldots((u\ \lambda x.\mu\delta.[\alpha](\ldots(x\ v_1)\ldots v_n))\ v_1)\ldots v_n)$

The term κ has a behaviour close to the one of the *call/cc* operator of the Scheme programming language.

Michel Parigot "λμ-Calculus: an algorithmic interpretation of classical natural deduction"
International Conference on Logic for Programming Artificial Intelligence and Reasoning, 1992

I’ll translate this for an audience of non-specialists,
showing how it meets the answerability criterion
much better than previous efforts, staying
close to our practice of giving a *proof*,
without decorating formulas with signs,
while retaining the good properties
of intuitionistic natural deduction.

The Rules

$$A$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\ {\to}I^i \qquad \dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\ {\to}E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}\ {\wedge}I \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{A}\ {\wedge}E \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{B}\ {\wedge}E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix}}{A \vee B}\ {\vee}I \qquad \dfrac{\begin{matrix}\Pi \\ B\end{matrix}}{A \vee B}\ {\vee}I \qquad \dfrac{\begin{matrix}\Pi \\ A \vee B\end{matrix} \quad \begin{matrix}[A]^j \\ \Pi' \\ C\end{matrix} \quad \begin{matrix}[B]^k \\ \Pi'' \\ C\end{matrix}}{C}\ {\vee}E$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \bot\end{matrix}}{\neg A}\ {\neg}I^i \qquad \dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{\bot}\ {\neg}E \qquad \dfrac{\begin{matrix}\Pi \\ \bot\end{matrix}}{A}\ {\bot}E \qquad \dfrac{\begin{matrix}\Pi \\ A\end{matrix}}{B}\ Alt, {\downarrow}A$$

The Rules

$$A$$

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\;{\to}I^i$$

$$\dfrac{\begin{array}{c}\Pi\\ A \to B\end{array}\qquad\begin{array}{c}\Pi'\\ A\end{array}}{B}\;{\to}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}\qquad\begin{array}{c}\Pi'\\ B\end{array}}{A \wedge B}\;{\wedge}I$$

$$\dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{A}\;{\wedge}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{B}\;{\wedge}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}}{A \vee B}\;{\vee}I$$

$$\dfrac{\begin{array}{c}\Pi\\ B\end{array}}{A \vee B}\;{\vee}I$$

$$\dfrac{\begin{array}{c}\Pi\\ A \vee B\end{array}\qquad\begin{array}{c}[A]^j\\ \Pi'\\ C\end{array}\qquad\begin{array}{c}[B]^k\\ \Pi''\\ C\end{array}}{C}\;{\vee}E$$

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ \bot\end{array}}{\neg A}\;{\neg}I^i$$

$$\dfrac{\begin{array}{c}\Pi\\ \neg A\end{array}\qquad\begin{array}{c}\Pi'\\ A\end{array}}{\bot}\;{\neg}E$$

$$\dfrac{\begin{array}{c}\Pi\\ \bot\end{array}}{A}\;{\bot}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}}{B}\;\mathit{Alt},\ {\downarrow}A$$

The Rules

$$A \qquad \dfrac{\begin{matrix}[A]^i\\ \Pi\\ B\end{matrix}}{A \to B}\,{\to}I^i, {\color{blue}\uparrow A \to B} \qquad \dfrac{\begin{matrix}\Pi & & \Pi'\\ A \to B & & A\end{matrix}}{B}\,{\to}E, {\color{blue}\uparrow B}$$

$$\dfrac{\begin{matrix}\Pi & & \Pi'\\ A & & B\end{matrix}}{A \wedge B}\,{\wedge}I, {\color{blue}\uparrow A \wedge B} \qquad \dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{A}\,{\wedge}E, {\color{blue}\uparrow A} \qquad \dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{B}\,{\wedge}E, {\color{blue}\uparrow B}$$

$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix}}{A \vee B}\,{\vee}I, {\color{blue}\uparrow A \vee B} \qquad \dfrac{\begin{matrix}\Pi\\ B\end{matrix}}{A \vee B}\,{\vee}I, {\color{blue}\uparrow A \vee B} \qquad \dfrac{\begin{matrix} & & [A]^j & & [B]^k\\ \Pi & & \Pi' & & \Pi''\\ A \vee B & & C & & C\end{matrix}}{C}\,{\vee}E, {\color{blue}\uparrow C}$$

$$\dfrac{\begin{matrix}[A]^i\\ \Pi\\ \bot\end{matrix}}{\neg A}\,{\neg}I^i, {\color{blue}\uparrow \neg A} \qquad \dfrac{\begin{matrix}\Pi & & \Pi'\\ \neg A & & A\end{matrix}}{\bot}\,{\neg}E \qquad \dfrac{\begin{matrix}\Pi\\ \bot\end{matrix}}{A}\,{\bot}E, {\color{blue}\uparrow A} \qquad \dfrac{\begin{matrix}\Pi\\ A\end{matrix}}{B}\,\mathit{Alt}, {\color{red}\downarrow A}, {\color{blue}\uparrow B}$$

Add *just* one rule: the *Alternative* Rule

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix}}{B}\; {\scriptstyle Alt, \downarrow A}$$

$$\dfrac{X \succ A; Y}{X \succ B; A, Y}$$

Add *just* one rule: the *Alternative* Rule

$$\dfrac{\begin{array}{c}\Pi \\ A\end{array}}{B}\ \textit{Alt}, {\downarrow}A, {\uparrow}B$$

$$\dfrac{X \succ A; B, Y}{X \succ B; A, Y}$$

Add *just* one rule: the *Alternative* Rule

$$\frac{\begin{array}{c}\Pi \\ A\end{array}}{B}\ \mathit{Alt}, \downarrow\! A$$

$$\frac{[X : Y] \succ A}{[X : A, Y] \succ B}$$

Add *just* one rule: the *Alternative* Rule

$$\dfrac{\begin{array}{c}\Pi \\ A\end{array}}{B}\ \textit{Alt}, \downarrow A, \uparrow B$$

$$\dfrac{[X : B, Y] \succ A}{[X : A, Y] \succ B}$$

Example Proof (Peirce's Law)

$$
\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{[p]^1}{q}\,{\scriptstyle \mathit{Alt}, \downarrow p^2}}{p \to q}\,{\scriptstyle \to I^1}}{\dfrac{p}{((p \to q) \to p) \to p}\,{\scriptstyle \to I^3}}\,{\scriptstyle \to E, \uparrow p^2}
$$

Example Proof (Peirce's Law)

$$\cfrac{\cfrac{[(p \to q) \to p]^3 \qquad \cfrac{\cfrac{[\color{red}{p}]^1}{q}\;\mathit{Alt}, {\downarrow} p^2}{p \to q}\;{\to}I^1}{p}\;{\to}E, {\uparrow} p^2}{((p \to q) \to p) \to p}\;{\to}I^3$$

$$[p : \ ] \succ p$$

Example Proof (Peirce's Law)

$$
\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{[p]^1}{\color{red}{q}}\,{\scriptstyle Alt,\,\downarrow p^2}}{p \to q}\,{\scriptstyle \to I^1}}{\dfrac{p}{((p \to q) \to p) \to p}\,{\scriptstyle \to I^3}}\,{\scriptstyle \to E,\,\uparrow p^2}
$$

$$[p : p] \succ q$$

Example Proof (Peirce's Law)

$$
\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{[p]^1}{q}\,{\scriptstyle Alt,\downarrow p^2}}{\color{red}{p \to q}}\,{\scriptstyle \to I^1}}{\dfrac{p}{((p \to q) \to p) \to p}\,{\scriptstyle \to I^3}}\,{\scriptstyle \to E,\uparrow p^2}
$$

$$[\ : p] \succ p \to q$$

Example Proof (Peirce's Law)

$$\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{[p]^1}{q}\,Alt, {\downarrow}p^2}{p \to q}\to I^1}{\dfrac{\color{red}{p}}{((p \to q) \to p) \to p}\to I^3}\to E, {\uparrow}p^2$$

$$[(p \to q) \to p : p] \succ p$$

Example Proof (Peirce's Law)

$$\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{[p]^1}{q}\;{\scriptstyle \mathit{Alt}, \downarrow p^2}}{p \to q}\;{\scriptstyle \to I^1}}{p}\;{\scriptstyle \to E, \textcolor{red}{\uparrow p^2}}}{((p \to q) \to p) \to p}\;{\scriptstyle \to I^3}$$

$$[(p \to q) \to p : \;] \succ p$$

Example Proof (Peirce's Law)

$$
\dfrac{\dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{[p]^1}{q}\,{\scriptstyle Alt,\, \downarrow p^2}}{p \to q}\,{\scriptstyle \to I^1}}{p}\,{\scriptstyle \to E,\, \uparrow p^2}}{\color{red}{((p \to q) \to p) \to p}}\,{\scriptstyle \to I^3}
$$

$$
[\ :\ ] \succ ((p \to q) \to p) \to p
$$

Another Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^1}{\bot}\,Alt,\!\downarrow p^2}{\neg p}\,\neg I^2}{p \lor \neg p}\,\lor I}{p}\,Alt,\!\downarrow p \lor \neg p^3, \uparrow p^2}{p \lor \neg p}\,\lor I, \uparrow p \lor \neg p^3
$$

Another Proof

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[p]^1}{\bot}\,Alt, \downarrow p^2}{\neg p}\,\neg I^2}{p \vee \neg p}\,\vee I}{p}\,Alt, \downarrow p \vee \neg p^3, \uparrow p^2}{p \vee \neg p}\,\vee I, \uparrow p \vee \neg p^3$$

$$[p : \;] \succ p$$

Another Proof

$$
\begin{array}{c l}
[p]^1 & \\ \hline
\bot & \mathit{Alt}, \downarrow p^2 \\ \hline
\neg p & \neg I^2 \\ \hline
p \vee \neg p & \vee I \\ \hline
p & \mathit{Alt}, \downarrow p \vee \neg p^3, \uparrow p^2 \\ \hline
p \vee \neg p & \vee I, \uparrow p \vee \neg p^3
\end{array}
$$

$$[p : p] \succ \bot$$

$$
\begin{prooftree}
\AxiomC{$[p]^1$}
\RightLabel{$\mathit{Alt}, \downarrow p^2$}
\UnaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\color{red}{\neg p}$}
\RightLabel{$\vee I$}
\UnaryInfC{$p \vee \neg p$}
\RightLabel{$\mathit{Alt}, \downarrow p \vee \neg p^3, \uparrow p^2$}
\UnaryInfC{p}
\RightLabel{$\vee I, \uparrow p \vee \neg p^3$}
\UnaryInfC{$p \vee \neg p$}
\end{prooftree}
$$

$$[\ : p] \succ \neg p$$

Another Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\mathrm{p}]^1}{\bot}\,Alt,\!\downarrow\!\mathrm{p}^2}{\neg\mathrm{p}}\,\neg I^2}{\color{red}{\mathrm{p}\vee\neg\mathrm{p}}}\,\vee I}{\mathrm{p}}\,Alt,\!\downarrow\!\mathrm{p}\vee\neg\mathrm{p}^3,\uparrow\!\mathrm{p}^2}{\mathrm{p}\vee\neg\mathrm{p}}\,\vee I,\uparrow\!\mathrm{p}\vee\neg\mathrm{p}^3$$

$$[\ :\mathrm{p}]\succ\mathrm{p}\vee\neg\mathrm{p}$$

Another Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^1}{\bot}\;\mathit{Alt},{\downarrow}p^2}{\neg p}\;\neg I^2}{p \vee \neg p}\;\vee I}{\color{red}{p}}\;\mathit{Alt},{\downarrow}p \vee \neg p^3, {\uparrow}p^2}{p \vee \neg p}\;\vee I, {\uparrow}p \vee \neg p^3$$

$$[\ : p \vee \neg p] \succ p$$

Another Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\mathrm{p}]^1}{\bot}\;\mathit{Alt},{\downarrow}\mathrm{p}^2}{\neg\mathrm{p}}\;\neg I^2}{\mathrm{p}\lor\neg\mathrm{p}}\;\lor I}{\mathrm{p}}\;\mathit{Alt},{\downarrow}\mathrm{p}\lor\neg\mathrm{p}^3,{\uparrow}\mathrm{p}^2}{\color{red}{\mathrm{p}\lor\neg\mathrm{p}}}\;\lor I,{\uparrow}\mathrm{p}\lor\neg\mathrm{p}^3$$

$$[\ :\ ] \succ \mathrm{p} \lor \neg\mathrm{p}$$

Alternative Formulations of the Rules: Negation

$$\dfrac{\dfrac{[A]^1}{\bot}\,Alt,{\downarrow}A}{\neg A}\,\neg I^1$$

Alternative Formulations of the Rules: Negation

$$\dfrac{\dfrac{[A]^1}{\bot}\,Alt,\!\downarrow\! A}{\neg A}\,\neg I^1 \qquad\qquad \dfrac{\dfrac{\dfrac{[A]^1}{\bot}\,Alt,\!\downarrow\! A^2}{\neg A}\,\neg I^1}{A}\,Alt,\!\downarrow\!\neg A,\uparrow\! A^2$$

Alternative Formulations of the Rules: Negation

$$\frac{}{\neg A}\ \neg I', \downarrow A \qquad\qquad \frac{}{A}\ \neg I', \downarrow\neg A$$

Alternative Formulations of the Rules: Negation

$$\cfrac{}{\neg A}\ \neg I', \downarrow A \qquad\qquad \cfrac{}{A}\ \neg I', \downarrow\neg A$$

$$\begin{array}{c} [A]^i \\ \Pi \\ \cfrac{\perp}{\neg A}\ \neg I^i \end{array} \quad \textit{becomes} \quad \begin{array}{c} \cfrac{}{A}\ \neg I', \downarrow\neg A^i \\ \Pi \\ \cfrac{\perp}{\neg A}\ \perp E, \uparrow\neg A^i \end{array}$$

Alternative Formulations of the Rules: Disjunction

$$
\dfrac{\begin{array}{c}\Pi \\ A \vee B\end{array} \qquad [A]^1 \qquad \dfrac{[B]^2}{A}\,{\scriptstyle \mathit{Alt}, \downarrow B}}{A}\,{\scriptstyle \vee E^{1,2}}
\qquad\qquad
\dfrac{\begin{array}{c}\Pi \\ A \vee B\end{array} \qquad \dfrac{[A]^1}{B}\,{\scriptstyle \mathit{Alt}, \downarrow A} \qquad [B]^2}{B}\,{\scriptstyle \vee E^{1,2}}
$$

Alternative Formulations of the Rules: Disjunction

$$\dfrac{\begin{matrix}\Pi \\ A \lor B\end{matrix}}{A}\ {\lor}E', {\downarrow}B \qquad\qquad \dfrac{\begin{matrix}\Pi \\ A \lor B\end{matrix}}{B}\ {\lor}E', {\downarrow}A$$

Alternative Formulations of the Rules: Disjunction

$$\dfrac{\begin{matrix}\Pi\\ A \lor B\end{matrix}}{A}\ \lor E', \downarrow B \qquad\qquad \dfrac{\begin{matrix}\Pi\\ A \lor B\end{matrix}}{B}\ \lor E', \downarrow A$$

$$\dfrac{\begin{matrix}\Pi\\ A \lor B\end{matrix} \quad \begin{matrix}[A]^i\\ \Pi'\\ C\end{matrix} \quad \begin{matrix}[B]^j\\ \Pi''\\ C\end{matrix}}{C}\ \lor E \qquad \textit{becomes} \qquad \begin{matrix}\dfrac{\begin{matrix}\Pi\\ A \lor B\end{matrix}}{A}\ \lor E', \downarrow B^i\\ \Pi'\\ \dfrac{C}{B}\ \mathit{Alt}, \downarrow C^j \uparrow B^i\\ \Pi''\\ C \quad \uparrow C^j\end{matrix} \qquad \textit{or} \qquad \begin{matrix}\dfrac{\begin{matrix}\Pi\\ A \lor B\end{matrix}}{B}\ \lor E', \downarrow A^i\\ \Pi''\\ \dfrac{C}{A}\ \mathit{Alt}, \downarrow C^j \uparrow A^i\\ \Pi'\\ C \quad \uparrow C^j\end{matrix}$$

Completeness and Soundness

1. COMPLETENESS:

1. COMPLETENESS: Trivial. This is intuitionistic logic + LEM.

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2. SOUNDNESS: Easy induction. If we have a proof for $[X : Y] \succ A$ then in any Boolean valuation v where $v(X) = 1$ and $v(Y) = 0$ then $v(A) = 1$.

MEETING OBJECTIONS

This is *so* much closer to our everyday *proof* practice than either the sequent calculus or a signed system.

(In the *paper* I show top-down *or* bottom-up readings of proofs use only straightforward speech acts.)

There are multiple conclusion sequents $X \succ A; Y$ just lurking under the surface, after all.

Is this *really* a single conclusion system?

There are multiple conclusion sequents $X \succ A; Y$ just lurking under the surface, after all.

Of course, but in the sequent $[X : Y] \succ A$, the X and Y (the assumptions and the alternatives) are the background *context* and the A is what we have *proved* against that background.

Bilateralism does some work for us

When I put a current conclusion aside as an *alternative*, I temporarily (for the sake of the argument) deny it, to consider a different option in its place.

This is very *mildly* bilateral, but not so much that it litters every formula in a proof with a sign.

Benefits

Classical • *Separated Rules* • *Normalising*

Analytic • *Single Conclusion* • *Answerable*

GOING BEYOND

We can put the $\lambda\mu$ terms back into our proofs, and explore what this means for *grounds*.

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A proof of A from $[X : Y]$ constructs a ground *for* A from grounds *for* each member of X and grounds *against* each member of Y.

We can put the λμ terms back into our proofs, and explore what this means for *grounds*.

A proof of A from $[X : Y]$ constructs a ground *for* A from grounds *for* each member of X and grounds *against* each member of Y.

The flourishing tradition of "classical computation" using λμ terms, constructions and closures is worth exploring by those philosophical logicians interested in the epistemic *power* of proof.

Substructural Logics

$$\dfrac{\begin{array}{c}\Pi \\ B\end{array}}{A \to B} \to I$$

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$$\dfrac{\begin{array}{c}\Pi \\ B\end{array}}{A \to B}\,{\to}I$$

$$\dfrac{\dfrac{\dfrac{[A]^i \qquad \begin{array}{c}\Pi \\ B\end{array}}{A \land B}\,{\land}I}{B}\,{\land}E}{A \to B}\,{\to}I^i$$

Substructural Logics

$$\dfrac{\begin{array}{c}\Pi \\ B\end{array}}{A \to B}\;{\to}I$$

$$\dfrac{\dfrac{\dfrac{[A]^i \qquad \begin{array}{c}\Pi \\ B\end{array}}{A \land B}\;{\land}I}{B}\;{\land}E}{A \to B}\;{\to}I^i$$

$$\dfrac{A \qquad B}{A \otimes B}\;{\otimes}I$$

$$\dfrac{A \otimes B \qquad \begin{array}{c}[A]^i\ [B]^j \\ \Pi \\ C\end{array}}{C}\;{\otimes}E^{i,j}$$

Substructural Logics

$$\frac{A}{B}\ {\scriptstyle Alt,\downarrow A} \qquad \text{looks } \textit{fishy}$$

$$\frac{A}{B}\ \mathit{Alt},{\downarrow}A$$ looks *fishy*

So, let's split it up into more basic parts.

$$\frac{A}{\bot}\ \mathit{Alt},{\downarrow}A \qquad \frac{\bot}{B}\ \bot E,{\uparrow}B \qquad \frac{\bot}{B}\ \bot E,\ \mathit{vacuous}$$

Vacuous absorption is *just like* vacuous discharge

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p}{\bot}\;\mathit{Alt},\!\downarrow p^{1}}{\neg q}\;\bot E,\ \mathit{vacuous} \qquad [q]^{2}}{\bot}\;\neg E}{p}\;\bot E,\!\uparrow p^{1}}{q \to p}\;\to I^{2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p}{\bot}\,{\scriptstyle Alt,\downarrow p^1}}{\neg q}\,{\scriptstyle \bot E,\ vacuous} \qquad [q]^2}{\bot}\,{\scriptstyle \neg E}}{p}\,{\scriptstyle \bot E,\uparrow p^1}}{q \to p}\,{\scriptstyle \to I^2}$$

You get well-behaved proof systems for ‘classical’ *relevant*, *affine* and *linear* logics by restricting discharge and absorption in the natural ways, and it ‘just works.’

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p}{\bot}\,{\scriptstyle Alt,\downarrow p^{1}}}{\neg q}\,{\scriptstyle \bot E,\ vacuous} \qquad [q]^{2}}{\bot}\,{\scriptstyle \neg E}}{p}\,{\scriptstyle \bot E,\uparrow p^{1}}}{q \to p}\,{\scriptstyle \to I^{2}}$$

You get well-behaved proof systems for ‘classical’ *relevant*, *affine* and *linear* logics by restricting discharge and absorption in the natural ways, and it ‘just works.’

This seems like good evidence that this technique is worth exploring, and isn’t just a ‘hack’ cooked up to solve just one single problem.

THANK YOU!

Thank you!

SLIDES: `https://consequently.org/presentation/2020/speech-acts-for-classical-natural-deduction-berkeley`

FEEDBACK: `@consequently` on *Twitter*,

or *email* at `restall@unimelb.edu.au`