Questions, Justification Requests, Inference, and Definition

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ABSTRACT

In this paper, I examine connections between the speech acts of assertion, denial, polar questions and justification requests, and the common ground. When we pay attention to the structure of norms governing polar questions, we can clarify the distinction between strong and weak denial, together with the parallel distinction between strong and weak assertion, and the distinct way that these speech acts interact with the common ground. In addition, once we pay attention to the distinct norms concerning justification requests, we can give a distinctive answer to Carroll's puzzle concerning the force of the logical *must* [6], and the sense in which certain rules for logical concepts can indeed count as definitions.

To truly *prove* a conclusion from some premises is to show how it is that, given the premises, the conclusion must invariably follow. A proof is a very useful tool. If I grant the premises, and you have a proof of some conclusion from those premises, then you can use this proof to *show* me that the conclusion is true, too. Of course, I may not follow your reasoning and so, I might not reach your conclusion. For example, I may not understand some step in the proof, or I might simply lose patience and stop listening to you. And even if I have the capacity to follow and I understand each step, I may nonetheless resist the conclusion. You may have proved, for example, that the premises I have granted are jointly incompatible, and that some absurdity follows from them. So, baulking at the conclusion, I might backtrack and start to question the premises. Nonetheless, to follow a proof is to be

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in some sense *compelled* to its conclusion. In this paper I will explore one way to understand how such a compulsion can arise, and how this might relate to distinct norms governing non-assertoric speech acts, including the asking of polar questions, and the making of justification requests.

This paper elaborates Restall's 2005 paper, "Multiple Conclusions", which argued that the classical sequent calculus can be understood on normative pragmatic grounds. A derivation of a sequent of the form $X \succ Y$ shows how it is that asserting each member of X and denying each member of Y is, in some sense, out of bounds [38]. In *what* sense might such a combination of assertions and denials be out of bounds? One claim of "Multiple Conclusions" is that whatever norm violation is involved is the same kind of violation involved in asserting and denying the very same thing. For example, this derivation

$$
\begin{prooftree}
\AxiomC{$p, q \vee r \succ p \wedge q, q \vee r$}
\RightLabel{$\vee Df$}
\UnaryInfC{$p, q \vee r \succ p \wedge q, r, q$}
\AxiomC{$p \wedge q, q \vee r \succ p \wedge q, r$}
\RightLabel{$\wedge Df$}
\UnaryInfC{$q, p, q \vee r \succ p \wedge q, r$}
\RightLabel{Cut}
\BinaryInfC{$p, q \vee r \succ p \wedge q, r$}
\RightLabel{$\vee Df$}
\UnaryInfC{$p, q \vee r \succ (p \wedge q) \vee r$}
\RightLabel{$\wedge Df$}
\UnaryInfC{$p \wedge (q \vee r) \succ (p \wedge q) \vee r$}
\end{prooftree}
$$

shows how the clashes involving explicit self-refuting positions, in which the one and the same thing is both asserted and denied,[1] make their way to clashes involving logically complex expressions, by way of the sequent rules for conjunction and disjunction.

This story is compelling enough, but it does not answer the question of how it is that a proof can, in some sense, compel us to its conclusion. A derivation of a sequent $X \succ Y$ tells us only to avoid asserting each member of X and denying each member of Y. Even if we have a sequent with a single formula in conclusion position, such as $X \succ C$, a derivation of a sequent shows us that it is out of bounds to assert each member of X and *deny* the conclusion C. If I refuse to grant C, I have not violated *this* norm. Yet, it seems that if I resist the inference to C (at least while remaining explicitly committed to the premises X) I have violated some norm or other. This is one lesson of Carroll's celebrated dialogue between Achilles and the Tortoise [6]. The relevant passage starts here:[2]

> TORTOISE: … let's take a little bit of the argument in that First Proposition — just two steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let's call them A, B, and Z: —
>
> (A) Things that are equal to the same are equal to each other.

[1] In the leaf atop the left branch the clash arises from asserting and denying $q \vee r$, while on the right branch, the clash involves $p \wedge q$. The particular rules in use will be discussed in Sections 3 and 4.

[2] I have added 'SPEAKER:' markers to make each speaker explicit.

(B) The two sides of this Triangle are things that are equal to the same.

(Z) The two sides of this Triangle are equal to each other.

> Readers of Euclid will grant, I suppose, that Z follows logically from A and B, so that any one who accepts A and B as true, must accept Z as true?
>
> ACHILLES: Undoubtedly! The youngest child in a High School — as soon as High Schools are invented, which will not be till some two thousand years later — will grant that.

After a brief exchange in which the Tortoise distinguishes following the inference from A and B to Z from accepting the hypothetical proposition that *if* A and B are true then Z must be true, the Tortoise continues:

> T: And might there not also be some reader who would say 'I accept A and B as true, but I don't accept the Hypothetical'?
>
> A: Certainly there might. . . .
>
> T: And neither of these readers . . . is as yet under any logical necessity to accept Z as true?
>
> A: Quite so . . .
>
> T: Well, now, I want you to consider me as a reader of the second kind, and to force me, logically, to accept Z as true . . .
>
> A: I'm to force you to accept Z, am I? . . . And your present position is that you accept A and B, but you don't accept the Hypothetical —
>
> T: Let's call it C . . .
>
> A: — but you don't accept
>
> (C) If A and B are true, Z must be true.
>
> T: That is my present position . . .
>
> A: Then I must ask you to accept C.
>
> T: I'll do so, . . . as soon as you've entered it in that note-book of yours.

This dialogue can continue indefinitely. In Carroll's version, Achilles asks the Tortoise to grant Z now that he has granted A, B, and the hypothetical $(A \wedge B) \to Z$ (entered into the notebook as C), and the Tortoise raises the issue of how one proceeds from these premises to the conclusion Z. Achilles, having taken the bait once, bites again, and asks the Tortoise to grant the further hypothetical proposition with antecedent $(A \wedge B) \wedge C$ and consequent Z. The Tortoise complies, and now the question arises: how do *these* lead to Z? This raises the question concerning the hypothetical judgement D: $((A \wedge B) \wedge C) \to Z$, which the Tortoise grants, then asking another question: why, having granted all this, must I conclude Z?

No doubt, the Tortoise is violating some kind of norm in pursuing this line of questioning and in delaying the inference to Z. But what norm has been violated? At no point in this dialogue does the Tortoise *deny* the target proposition Z. If the Tortoise had granted A, B, $(A \wedge B) \to Z$ and *denied* Z, that position would be out of bounds. However, the Tortoise has done no such thing. The wholly *negative* norm of "Multiple Conclusions", which places combinations of assertions and denials as out of bounds, is not enough to diagnose the Tortoise's malady. For a more comprehensive diagnostic toolkit, we will need to look beyond the norms governing assertion and denial. Nonetheless, to find the trail that leads us beyond the assertoric speech acts, we will start at home, examining the norms governing assertion and denial more closely.

1. ASSERTION AND DENIAL

In "Multiple Conclusions" [38], valid sequents of the from $X \succ Y$ are interpreted as enjoining us to not assert each member of X and deny each member of Y (to put things in the vocabulary of public speech acts), and not to accept each member of X and reject each member of Y (to use the vocabulary of more private cognitive states), or as claiming that a position $[X : Y]$ in which such a combination of assertions and denials has been made is *out of bounds*.

Since the issue raised by the Tortoise's behaviour in Carroll's dialogue concerns what is *said*, we will focus on the speech acts of assertion and denial. To gain insight into these speech acts and their connection with *inference* we need to start with some clarity on what these speech acts amount to. Even when we restrict our attention to the positive act, *assertion*, we see a very large literature.[3] One sizeable strand in this literature involves characterising assertion as a rule-governed activity, which can be understood in terms of its norms. There is no consensus on exactly what these norms should, fundamentally, be taken to be. Luckily, we need not take a specific side in this debate, but it will help to have an overview of the broader discussion. It is instructive, for our purposes, to cluster approaches to the norms governing assertion around three distinct but related foci. We can attend to the norms governing the *sender*, the one who asserts. Or we can attend to the norms governing the *recipient*. Finally, we can consider norms governing the shared space formed by the sender and recipient together. Let's consider these norms in turn.

SENDER NORMS: The natural first thought when considering norms governing assertion is to focus on the correctness or incorrectness of the asserted statement [33]. There is a natural sense in which one who asserts something that turns out to be untrue has made a mistake. This is the appeal of the so-called *truth* norm of assertion. In addition, if I happen to make a claim that turns out to be correct, but I was not in a position to know that it was correct (say I make a lucky guess), then I may not have made the same kind of mistake as one who asserts a falsehood, but I have overstepped the mark in a different way: I have been at the very least, *unsafe*, and perhaps, *deceptive*. According to Timothy Williamson, "asserting that p without knowing that p is doing something without having the authority to do it, like giving someone a command without having the authority to do so" [54, p. 257]. One

[3] This selection of readings merely scratches the surface [3, 4, 14, 26, 33, 49, 54].

way to spell out the risk an asserter runs when they assert something that outruns their evidence is to consider their response if their audience asks them to *justify* their assertion. Whenever someone makes an assertion, it is at least conceivable that their audience responds with a "Really? Why is that?" If the assertion has some ground, then pointing to that ground can at least begin to address that concern. In Robert Brandom's words, each act of assertion involves the conditional commitment to justify the claim if challenged [3, p. 641–642]. In short, when you assert, you commit to meeting a *justification request*. This raises the normative horizon beyond the sender to take in the recipient, so it is there that we turn next.

RECIPIENT NORMS: One way to understand the force of the knowledge norm of assertion[4] is to not only consider the sender's debt to a recipient, were the recipient ever to call them up on it, but to consider what that recipient can do with the promissory note granted by the sender. If the recipient were *also* to make the same promise, by repeating the assertion, then she has a means of paying her debts if she were called to justify her claim, by passing the debt on the initial sender. Brandomn puts it like this: "An assertion in force licenses others to re-assert the original claim … *deferring to the author of the original assertion the justificatory responsibility which would otherwise thereby be undertaken*" [3, p. 642, emphasis his]. This is a norm governing the recipient, not *binding* or otherwise limiting them, but rather, *entitling* them to an action, the action of re-asserting. They can pass on the justificatory burden to the initial asserter.

This attention to the recipient, as well as the sender, can help us distinguish assertions from observatives. An observative, such as "Lo, a rabbit!" not only states that there is a rabbit, but may only be used felicitously if the speaker in some way directly perceives a rabbit. Observatives, like assertions, are governed by truth and knowledge norms. (If it turns out to not be a rabbit, but a carefully presented simulacrum, then I have made a mistake. If it turns out that I am seeing a rabbit, but I am in some 'sceptical' scenario where I cannot know that I am seeing a rabbit because I know that my sensory apparatus is regularly being manipulated, then I overstep my bounds in saying "Lo, a rabbit!" in the same sort of way as has I said "That's a rabbit" or "I see a rabbit.") However, they are not assertions, but are a distinct speech act. I do not license you, my audience, to reproduce the claim "Lo, a rabbit!", but only assertoric partners of the observative, such as "There is a rabbit" or "Greg saw a rabbit." My making the claim may motivate you to get into a position where you can make it, too, but only when you see or otherwise perceive the rabbit would your claim be appropriate, and it is only that experience (and not claim) that licenses your claim.

So, the re-assertion license for the *recipient* is one thing that helps distinguish assertions from observatives [25],[5] so it is fruitful to broaden our focus from the

[4]Or perhaps the *warrant* norm, if you think that untrue assertions can nonetheless satisfy this justificatory constraint.

[5]Similarly, if I ask you to pass the salt, this does not, in and of itself, license anyone else to likewise ask you to pass the salt, whether to me, or to them. On the other hand, if I use a declarative "you

sender, to take in the receiver as well.

CONVERSATION NORMS: Once we take in both the sender and the receiver, we have the entire conversation in view, and some norms governing individual acts of assertion are characterised in terms of this shared space. The notion of the *common ground* of a conversation has proved to be a useful theoretical tool. One example is the characterisation of accommodation and presupposition in discourse.

> To presuppose something is to take it for granted, or at least to act as if one takes it for granted, as background information as *common ground* among the participants in the conversation. What is most distinctive about this propositional attitude is that it is a social or public attitude: one presupposes that ϕ only if one presupposes that others presuppose it as well. [51]

To assert p is to bid for the common ground to be updated with the content asserted. If the assertion meets with no resistance and is accepted, then the common ground is updated appropriately, and that content is added to the shared pool which can be taken for granted. In the conversation, the common ground takes shape, and this can be used as a foundation upon which more is built. If you say "that's a rabbit" and we grant your claim, then we can build on that, adding another assertion, like "it's on the move," or by asking a question, such as "where is it going?"

Of course, what is granted at one point may come into question at another, but this must be explicitly marked. It would be very strange to immediately ask "have you seen a rabbit?" after having granted that this is what we saw, but with enough conversational effort, we can dislodge something from the common ground: if I respond, after some time, "Hang on—maybe that was a small hare and not a rabbit ..." then the original claim is no longer taken for granted, it is retracted from the common ground, and the conversation continues.

A crucial feature of the common ground as it plays a role in the semantics and pragmatics of discourse is its use as a *scoreboard*, which is used to keep track of the status of some rule-governed activity [28]. This model of conversation and information update has proved incredibly fertile, in giving an account of the dynamics of discourse and the behaviour of many different linguistic phenomena, such as presupposition, illocutionary mood, parentheticals, and not-at-issue content [13, 35, 47]. A great deal can be learned in the shift to a dynamic perspective, understanding speech acts as moves that update the scoreboard, rather than attending only to the static representational features of individual utterances [53, 56].

I will take for granted the notion that these three perspectives (sender, recipient, and whole conversation) are each important ways to understand the function of assertion (while sidestepping the active debate over whether any or each of these norms *characterise* assertion, and whether any of these perspectives is more

ought to pass the salt" rather than the imperative form, then I license your restatement of that claim.

fundamental than the others). I will take it that if some speech act is governed by these norms, then it is a good candidate for counting as assertion, and if it is *not* so governed, it does not count as assertion. We can already see in the notion of justification requests some hint of the relationship between assertion and other speech acts. This connection will be crucial to the analysis to come.

Let's turn next to the relationship between *assertion* and *denial*. "Multiple Conclusions" [38], does not say much about the relationship between assertion and denial other than that they clash. As Dickie [8] and Incurvarti and Schlöder [22] have pointed out, this needs clarification. The characterisation in terms of the bare notion of a clash does not help with distinguishing denial from other speech acts which also clash or conflict (in some way) with assertion. To *retract* an assertion is, in *some* sense, incompatible with assertion, but it is not the same as denial. Retraction of an assertion aims at reversing the effect of an assertion, so its aim is in some sense directly opposed to the aim of the assertion [5, 24].[6] An assertion that p and a retraction of that assertion are aimed at incompatible outcomes: To be specific, the assertion aims at the content asserted being in the common ground, while the retraction aims at extracting it from the common ground. To take back the assertion that p is not the same as to *deny* p in the sense in question in "Multiple Conclusions", but more can be done to distinguish denial from retraction, and from other speech acts opposed to assertion. To clarify the difference between denial and retraction, we will turn to related speech acts, *polar questions*.

2. POLAR QUESTIONS

Polar (or *yes/no*) questions, and their answers, are distinct speech acts with their own norms.[7] The polar question "is it the case that p?" (abbreviated "p?" in what follows) raises an *issue*. There are two ways to *settle* the issue so raised: to settle it positively (to say YES) and to settle it negatively (to say NO). These two answers to a polar question *clash* in the sense that if I say YES and you say NO to the same polar question p? then we DISAGREE: at least on the straightforward understanding of these matters, there is no shared position incorporating both of our answers.[8]

Other responses to p?, such as *maybe*, *I don't know*, or *I think so* are acceptable *responses* to p?, but they don't answer the question, or settle the issue.

Looking back on the norms of assertion, we see that it is quite plausible that settling answers are governed by these very same norms. These answers are appropriately governed by norms of truth and knowledge in just the same way that

[6] Of course, retracting an assertion also *requires* the earlier assertion, so although the retraction aims at an effect opposed to the *aim* of the assertion, the retraction and the assertion are of course, compatible. For more on the distinctive nature of the retractions of different kinds of speech acts, see the essays in the forthcoming collection *Retraction Matters* [57].

[7] This is only a brief introduction to some of the issues around polar questions and their answers. This is a rich and interesting literature on its own. I have found papers by Bruce, Farkas, Humberstone and Roelofsen particularly helpful here [11, 12, 20, 44].

[8] This is to set aside, for the moment, the issues of paraconsistency and dialetheism [36, see especially Section 20.4].

we judge assertions, and they entitle the audience to repeat the same answer if asked, and to defer to the one who gave the answer initially, if called on to justify their answer.[9] Furthermore, a *settling* answer to a polar question is a bid to add to the common ground of the conversation, just like other assertions. If I answer 'yes' to the polar question p?, and this answer is accepted, then the other participants in the conversation can henceforth take this for granted, and could repeat the answer whenever the question arises again. If asked to justify the answer, they could defer to me. A settling answer to a polar question can also be blocked with the same kinds of reaction as one might make to an assertion. If I respond "yes" to the question p?, you could call that into question with a "why is that?", asking for further elaboration. As far as the norms governing the practice of assertion go, settling answers to polar questions act like assertions in all the ways that seem to matter.

So, if settling answers to polar questions act just like assertions, the question arises: what is the content so asserted? Presumably, a YES answer to p? has the same content as the flat assertion p. If I answer 'yes' to the question 'Is the cat on the mat?' (and if this answer is, indeed, acce the common ground is updated with the information that the cat is on the mat. If, on the other hand, I answer 'no' to that question (and again, the answer is accepted), the common ground is updated with the information that it's not the case that the cat is on the mat.[10] It is plausible to think that what is asserted in a negative answer to a polar question is a negative content, a content that we might describe in negative terms, by negating a statement.[11]

Nothing in what follows here is inconsistent with that view of the relationship between answers to polar questions and assertion. However, there is good reason to leave the way open to understand the content of the speech act of the negative answer to the question p? not as the assertion of a negation (of p), but rather, as the *denial* of p. (The particular sense of denial in play at this point here will be clarified in the rest of this section.) One reason for this is that we may want to have room to consider the speech acts (and cognitive representational capacities) of a community whose members can answer simple polar questions in a restricted vocabulary, where this vocabulary does not include *negation* in the rich sense of an embeddable

[9]Consider a potentially troubling case: a child asks her mother "can I stay up late tonight?"—a polar question. The mother answers YES, thereby giving permission. You, not the child's parent, are not in a position to give permission yourself, but you *are* allowed to repeat the answer to the question, if asked. You can say "yes, you can stay up late tonight", even though you are *not* thereby also giving permission. You are, in this case, reporting on the permission that has been already granted. The subtleties surrounding the norms for restating answers to polar questions in this case seem completely parallel to the subtleties surrounding reproducing assertions and other speech acts.

[10]Let's distinguish this from the claim that the cat is *not* on the mat, to allow for cases where the discourse does not supply a referent for 'the cat' or 'the mat'.

[11]None of this is to take sides on the nature of propositions or contents, whether they be structured entities, or sets of worlds or circumstances. If you take a content to be (equivalent to) a set of possible worlds, then none of these is particularly more negative than any other. The point here, though, is that we have specified the content (what is asserted in the negative answer to the question p?) in terms of the negative statement $\neg$p.

sentential modifier. For given judgements p in their vocabulary, they can say 'yes' or 'no' to the question p?, but the *question* $(\neg p)$?, or the issue $\neg p \rightarrow q$, or other constructions in which negation is embedded, do not, and perhaps *cannot* occur to them. There seems to be no bar against this sort of cognitive and communicative stage of development,[12] and so, flexibility and generality motivate allowing for an account of speech acts and common ground that at least allows for limited repertoire of contents to be stated. For such creatures, it seems more appropriate to think of yes/no answers to polar questions, when accepted, as contributing to the common ground by ruling some issues in (given by polar questions answered positively) and other issues out (given by polar questions answered negatively). In the vocabulary of "Multiple Conclusions" [38], the common ground can be modelled by a *position*, a pair of sets of issues $[X : Y]$ where each issue in X has been ruled *in*, while every issue in Y has been ruled *out*. In what follows, we will call the left component of the pair (here, X) the POSITIVE COMMON GROUND and the right component of the pair (here, Y), the NEGATIVE COMMON GROUND.

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The relationship between positive and negative common ground is relatively crisp and clear when it comes to settling answers to polar questions. The situation is less clear when it comes to denials expressed in other contexts. So, with this background in place, let's turn to the relationship between assertion and denial when it comes to declarative utterances in general. To set the scene, consider these two short dialogues, between Abelard and Eloise, concerning their young son, Astralabe,[13] who is hiding from them.

(1) ABELARD: Astralabe is in the study.
ELOISE: No, he's in the kitchen.

(2) ABELARD: Astralabe is in the study.
ELOISE: No, he's *either* in the kitchen *or* the study.

Eloise's "no" is clearly *negative* in both dialogues, but there are differences between these two cases.[14] In both cases, Abelard's assertion has been rejected by Eloise,

[12] Furthermore, there seems to be good reason to think that creatures like us pass through such a stage of development on our way to developing our communicative and cognitive capacities [21].

[13] Abelard and Eloise (often simply abbreviated ∀ and ∃) are characters who often appear in modern treatments of dialogues and games in logic [18, e.g., pp. 23, 24]. Peter Abelard (1079–1142) was a medieval French theologian and logician. Héloïse d'Argenteuil (?–1164) was a French theologian and abbess. They had a troubled relationship, which could variously be described as a forbidden love affair between peers or as sexual abuse of a younger student by an older teacher. Before being forcibly separated, they had a child, whom Héloïse named Astralabe. For more on the life, thought and correspondence of Peter Abelard and Héloïse d'Argenteuil, consult Constant Mews' *Abelard and Heloise* [34]. For more on Astralabe's curious name, and what we know of his life, start on page 13 of William Levitan's *Abelard and Heloise: The Letters and Other Writings* [27].

[14] The structure of this example is due to Paul Grice [15, Lecture 5, page 9]. See Wilson [55, p. 149] and Horn [19, pp. 378, 416] for discussion.

who makes a counterclaim in response. However, the upshot of Eloise's negative response differs between the two cases. In the first case, we can hear Eloise's "no" as indeed a bid to settle the issue of Astralabe's being in the study negatively. However, Eloise's "no" in (2) is not a *denial* of Abelard's claim that sense, because (as the follow-up disjunctive claim indicates), Eloise does not take the issue to be settled one way or the other. On the natural reading of the dialogue, she takes this issue to still be unsettled, with two options (study and kitchen) remaining live. Eloise's "no" acts as a bid for Abelard's claim to be *retracted* from the common ground, if the claim managed to enter it in the first place, or as a bid to block its entry into the common ground. This distinction can be clarified if we consider the variant dialogues where Abelard asks a polar question instead of making the assertion:

(3) ABELARD: Is Astralabe in the study?
ELOISE: No, he's in the kitchen.

(4) ABELARD: Is Astralabe in the study?
ELOISE: *No, he's *either* in the kitchen *or* the study.

(5) ABELARD: Is Astralabe in the study?
ELOISE: *Maybe*, he's *either* in the kitchen *or* the study.

Here, we can clearly see the difference between Eloise's two responses. The "no" in (3) settles the issue negatively. It is a settling (negative) answer to Abelard's polar question. In (4) the "no" is not appropriate, because Eloise (in the natural understanding of this dialogue) does not intend to settle the issue, but rather, to draw Abelard's attention to the kitchen, as well as the study. Eloise is able to give a *partial* answer to the question: hence, her "*maybe*".[15] The "no" answer to (4) is not appropriate if its function is to block the addition of the issue (that Astralabe is in the study) to the positive common ground, because in dialogue (4) the issue is *not* in the positive common ground in the first place, and Abelard has not offered it to be added to the common ground. Abelard merely raised the issue, he did not bid to settle it positively, unlike the assertoric form given in dialogue (2). This distinction motivates the following understanding of the difference between settling an issue negatively (which I will call STRONG DENIAL) and retraction or blocking (which I will call WEAK DENIAL).[16]

- To *strongly deny* p is to bid to add p to the *negative common ground*.

[15] There is a deviant reading of the dialogue, where Eloise's (4*) response is appropriate. Imagine Eloise giving Abelard elementary logic lessons, and is getting him to practice reasoning with disjunctive syllogism. Abelard asks p?. Eloise answers with a 'no' (hence, denying p in the sense of ruling it out) and with the disjunctive claim $p \vee q$, and waits for Abelard to draw q as a conclusion. Though perhaps appropriate in the case of learning basic inferences, this is not a particularly natural reading of the dialogue.

[16] Here I follow Luca Incurvarti and Julian Schlöder, though my understanding of the relationship between strong and weak speech acts differs from theirs, as will become clear in the next section [22, 46].

- To *weakly deny* p is to block the addition of p to the *positive common ground*, or to bid for its retraction if it is already in the positive common ground.

Given this understanding of the two kinds of denial, it follows that both strong and weak denials of p are appropriate responses to an assertion of p, because the assertion of p is a bid to add p to the positive common ground, and this could be opposed either by merely blocking this addition, or by going further and instead bidding to add p instead to the negative common ground. A strong denial of p is also one way to settle the polar question p?, so it is also appropriate in this context. On the other hand, a weak denial of p is not generally an appropriate response to the polar question p?, as this polar question does not place p in the positive common ground, and either the question would be infelicitous while p is in the positive common ground, or if it is a felicitous question, the asking of it takes p out of both the positive and negative common ground, since in asking the question, the questioner is not taking the issue to be currently settled.

This treatment of weak denial is intentionally *very* weak, especially when it comes to its action on the common ground. In particular, on this account, it transforms the common ground by retraction, and not by addition. Why not think that there is another aspect of the common ground, in which a weak denial of p adds p to some other part of the scoreboard, say, the *weakly negative common ground*? On this view, the weak denial of p not only moves p out of the positive common ground, it parks it into an explicitly marked zone, of those things *weakly* denied. Perhaps there might be good reason for treating the common ground in this way, but I think that there is reason to avoid such an expansion of the way we keep track of conversational commitment. Consider, the following expansion of dialogue (2).

(2′) ABELARD: Astralabe is in the study.
ELOISE: No, he's *either* in the kitchen *or* the study.
(*they check the kitchen*)
ABELARD: He's not in the kitchen.
ELOISE: So, he *is* in the study.

Does Eloise, when she concludes that Astralabe is indeed in the study, take *back* anything she said earlier in the dialogue? If what she is doing in her first weak denial is resisting Abelard's bid (presumably on the basis of it being too hasty, since they hadn't yet checked the kitchen), and blocking the addition of the claim to the common ground, then that still stands. Abelard's move was too hasty. The common ground is merely *expanded* as the dialogue continues, first with the disjunction added to the positive common ground by Eloise, then the claim that he is in the kitchen is added to the negative common ground, by Abelard, and finally, Eloise concludes, to add the claim that Astralabe is in the study, to the positive common ground. The informational state, of what is accepted in common is, at one level, continually expanding. Eloise takes nothing back.

If, however, at the time of Eloise's weak denial of Abelard's claim, we model this as Eloise's placing "Astralabe is in the study" in some "weakly negative" common ground, then this needs to be retracted at the final step of the dialogue, when it is asserted, because this assertion places it in the positive common ground. That involves a retraction. There is nothing *wrong* with modelling the dialogue in this way, and perhaps there is some theoretical benefit for doing so, but if we do this, we lose the sense in which the common ground — in the sense of the claims that the participants are together taking for granted — is expanding at each step of the discussion. Now, instead of agreeing that nothing is taken back, the item in the weak negative common ground needs to be retracted.[17] The same goes if we re-tell the story with Eloise saying "*Perhaps* Astralabe is in the kitchen." The semantics of epistemic modals like these is notoriously difficult (try using "perhaps p" as the antecedent or as the consequent of a conditional, or try negating it), but it is plausible to read "perhaps p" as a bid to retract p from the negative common ground. In this dialogue, it looks as if Abelard is taking for granted that Astralabe is not in the kitchen, and if Eloise raises this claim as a possibility using "perhaps", it is plausible to model this by extracting that claim from the negative common ground: it is no longer taken for granted that he isn't there.

Treating "perhaps" in this way, as a bid to retract or block a claim from the negative common ground completes the pattern of strong and weak assertion and denial.[18]

- STRONG DENIAL: add to the *negative common ground*.
- STRONG ASSERTION: add to the *positive common ground*.
- WEAK DENIAL: retract (or block) from the *positive common ground*.
- WEAK ASSERTION: retract (or block) from the *negative common ground*.

There is further reason to treat weak denials and weak assertions as retraction or blocking moves, rather than additions to the common ground in the manner of their strong siblings. The weak denial and weak assertion are not constrained by all of the same norms as strong denial and strong assertion. Eloise's *No* in dialogue ($2'$) is not appropriately evaluated with respect to the same kind of *truth* norm that would govern a strong denial. There is no sense in which Eloise's move here is incorrect on the ground of truth. She was not saying that Astralabe *wasn't* in the study (that would be the strong denial), she was blocking Abelard's assertion, and insofar as she was doing that, she does not need to take that back when she later grants the content of Abelard's claim, when new information arises. The truth norm does not seem to govern these weak denials or weak assertions, so they differ in another

[17]When characterising the difference between strong and weak denial, Ripley puts it this way: "The kind of denial I am interested in [that is, strong denial, as opposed to weak denial] is different. It is one which, like assertion, adds information to a conversation" [43, p. 47]. For Ripley, as for me, weak denials are best understood as retracting information from the common ground, rather than adding it.

[18]See Incurvarti and Schlöder's "Weak Assertion" [23] for more on weak assertion. Icurvarti and Schlöder's account differs in detail from the account given here (especially on the relationship between logical consequence, strong assertion and weak denial), though there are many commonalities.

respect from strong denials and from strong assertions.

So, we have a picture of the dynamics of the common ground involving updates to the common ground in the form of additions and retractions, to positive and to negative answers to polar questions, as well as to assertions and denials in general. This is one way to understand the relationship between assertion and denial, and how to distinguish strong denial from other negative speech acts. In this way, we can be more explicit about the relationship between assertion and denial and the particular kind of clash found in a valid sequent of the form $X \succ Y$: the central claim of "Multiple Conclusions" is that in a derivation of such a sequent, any position $[X : Y]$ in which X is strongly asserted and Y is strongly denied, involves a clash. That is, such a position is out of bounds.

The structural rules of the sequent calculus cohere well with this interpretation, in terms of strong assertion and strong denial. The structural rules of *Identity* and *Weakening* together give us the axiomatic sequent $X, A \succ A, Y$. Under this interpretation, this would mean that any position of the form $[X, A : A, Y]$ in which the issue A has been both strongly asserted and strongly denied is out of bounds. This is not controversial: it is merely to say that the two ways to settle the issue $A?$ conflict with one another. That is, there is no *available* position that takes both sides of that disagreement.[19] The *Cut* rule, on the other hand

$$\frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\,\textit{Cut}$$

can be understood as saying that there are no *quandries*. In other words, if a position $[X : Y]$ is available (that is, it is not out of bounds), then given the polar question $A?$, then a position extending $[X : Y]$ with at least one of the answers YES or NO is *available*. The *Cut* rule understood in this way is not without its critics [7, 40, 41, 42]. However, if something is undeniable (in the sense that its *strong* denial is out of bounds, is ruled out merely by our other commitments) then there is a natural appeal in concluding that its strong assertion is a matter of making explicit the commitments we have already implicitly undertaken. Consider this passage of reasoning from the characteristically careful Michael Dummett:

> It is unclear whether there is here a genuine disagreement between Gadamer and Davidson. *It is undeniable that* someone may lack a concept that others have, and that we now have many concepts that no one had three hundred years ago. New concepts are continually introduced. They cannot always be deWned in the existing language, but they can be explained by means of it; a study of how we acquire concepts, such as the concept of infinity, that could not even be expressed before their introduction would be highly illuminating. *It is also undeniable that* we can now recognize, of certain concepts that were used

[19] So, if you are tempted to say "yes and no" to a polar question $A?$, the strategy is to disambiguate. To find two closely related issues A_1 and A_2, where you say YES to one and NO to the other.

in some previous age, that they were incoherent or confused. [9, p. 94, emphasis mine]

When Dummett says that it is *undeniable* that someone may lack a concept that others have, he is not saying something *weaker* than the claim that someoine may lack a concept that others have. To grant that is undeniable that p is to grant no less than p — and often it is to grant rather more. We use "it is undeniable that . . ." as an *intensifier*, not as a way to weaken a claim.

So, given this more explicit understanding of the relationship between assertion and denial — distinguishing the strong versions from their weaker cousins — we have a richer picture of the norms governing assertion and denial and the sequent calculus. Could this be expanded further? Now that weak assertion and weak denial are in the picture, is there some connection between the standard sequent calculus and these weaker notions of assertion and denial? To address this, we must consider the structure of the common ground a little more closely.

One consequence of this view of the common ground and this understanding of the role of weak denial as blocking or retracting from the positive common ground is that we have reason to understand the common ground in a very finely grained way: much more finely grained than the traditional Stalnakerian picture according to which the common ground is (modelled by) a the set of all and only those worlds compatible with what has been presupposed [49, 50, 51]. We will not follow Stalnaker in modelling the common ground in this way. Since our aim is to understand the semantics of logical concepts such as conjunction, disjunction, negation and the quantifiers, and to analyse the dynamics of proof and inference and their interaction with the common ground, taking a perspective that erases any distinction between logically equivalent formulations of assertions is to blunt our instruments before beginning to use them.[20] For us, it will matter a great deal that we can presuppose, say, the axioms of geometry, and take those claims as a part of the common ground of a mathematical discourse, without also taking all the consequences of those claims *for granted* in the same way. The point of Euclid's *Elements* was to derive consequences of those first principles, and in doing this, you do not first take those consequences for granted. So, with this in mind, consider the following dialogue, where Abelard is being tutored by Eloise in geometry. He is

[20] David Lewis is helpfully explicit concerning the price of such an assumption. "Likewise, the known proposition that I have two hands may go unrecognised when presented as the proposition that the number of my hands is the least number n such that every even number is the sum of n primes. (Or if you doubt the necessary existence of numbers, switch to an example involving equivalence by logic alone.) These problems of disguise shall not concern us here. Our topic is modal, not hyperintensional, epistemology" [29, p. 552]. In contrast, *our* topic is the semantics of logical connectives, so identifying propositions up to logical equivalence is to work with a coarseness of grain that obscures exactly the features we wish to clarify. So, we will not take the approach of identifying the common ground with a set of worlds. (Thanks to Lloyd Humberstone for reminding me of Lewis' formulation.) Electing to attend to these fine-grained distinctions does not mean, of course, that the coarser perspective is to be rejected *tout court*. Much can be gained by *ignoring* the differences we must attend to here. There is a place for acting fast and loose with propositions, but *this* is not that place.

reasoning about a particular triangle with interior angles of 40, 60 and 80 degrees. He adds up the angles, and notices that they sum to 180°. . .

(6) ABELARD: The interior angles of triangles sum to 180°.
ELOISE: No, *this* triangle's interior angles sum to 180°. Can you prove the general case?

Here, Eloise seems to block from the common ground (to weakly deny) a logical consequence of claims that are already in the common ground (that is, the axioms of geometry), for the same general reason as for other weak denials — Abelard has jumped to a conclusion too fast, without sufficient grounds. Understanding the common ground as a set of worlds would make this analysis impossible. A more finely individuated account of the common ground is of use in cases like these.[21] This is not to say that the only claims that are in the common ground are those that are explicitly ruled in or ruled out in the discourse. There may still be good reason to incorporate many more things as part of the common informational store than that. The point to conclude from this example is that in certain conversational contexts, it is important to not close the common ground under logical consequence. There are contexts in which it is indeed taken for granted that A, where A (as a matter of fact) entails B and where it is by no means taken for granted that B.

This example shows that it is not appropriate to think of the validity of a sequent $X \succ Y$ as ruling out the strong assertion of each member of X with the *weak* denial of each member of Y. Eloise is well within her rights to strongly assert the axioms of geometry (they have been ruled in to the positive common ground) and to weakly deny — at this juncture — Abelard's hasty claim that the interior angles of triangles sum to 180°. Eloise (or anyone else) would be making a mistake to strongly assert the axioms of geometry and to *strongly* deny Abelard's claim. (This is no counterexample to the view that there is a clash in strongly asserting X and strongly denying Y when the sequent $X \succ Y$ is valid.) However, to merely weakly deny the conclusion, for the purposes of slowing down Abelard's reasoning, to "connect the dots" as it were, seems totally appropriate. It violates no conversational norms.

This then motivates the obvious question: when is the assertion of A and the weak denial of some consequence of A out of bounds? Are we not opening our flank to further attack from Lewis Carroll's Achilles and the Tortoise puzzle [6]? I have argued that it is acceptable for Eloise to weakly deny a claim that is nonetheless a logical consequence of other claims present in the common ground. It seems, then, that I will have no room to find anything wrong with the Tortoise's obstinate resistance in drawing an inference, whose conclusion is likewise a logical consequence of the common ground in his discussion with Achilles. To see how these

[21] A more finely individuated account of the conversational score is by no means unprecedented. Understanding the common ground as a pair $[X : Y]$ of claims ruled in and claims ruled out is much closer then to the analysis of *commitment stores* in dialogue, from the work of Hamblin [16, 17] and Mackenzie [30, 31, 32] dating from the 1970s, as well as more recent work on the common ground in some more recent work in the semantics of questions, for which a fine-grained conversational slate is required [11].

cases differ, we will consider another non-assertoric speech act, the making of a *justification request*.

3. JUSTIFICATION REQUESTS

A justification request is another kind of imperative, querying a strong assertion (or a strong denial), temporarily blocking it from entering the common ground, until some kind of justification can be provided for it.[22] The need for justification requests seems transparent: If in asserting I undertake a commitment to give some kind of ground for that assertion, you ought to have some way of calling me on it. A justification request for a strong assertion (or a strong denial) is an attempt to block the proposed addition to the common ground until a *reason* is given. This reason can be given by a number of assertions or denials[23] which must be *granted* (i.e., also included in the common ground) in order for the request to be satisfied. Granting the reason is a *necessary* but not a *sufficient* condition for the justification request to be satisfied and for the original assertion to be added to the common ground. You could, of course, grant the claims which are offered as a reason, without granting that they stand as a reason for the initial claim. Let's illustrate these distinctions in some dialogues:

(7) ABELARD: Astralabe is in the kitchen.
ELOISE: *Really*?
ABELARD: I saw him there five minutes ago.
ELOISE: OK.

(8) ABELARD: Astralabe is in the kitchen.
ELOISE: *Really*?
ABELARD: I saw him there five minutes ago.
ELOISE: *Are you sure*? He's been in the study with me for the last half hour.

(9) ABELARD: Astralabe is in the kitchen.
ELOISE: *Really*?
ABELARD: I saw him there five minutes ago.
ELOISE: Yes, but he was in the study two minutes ago.

In these dialogues we see three different ways that an answer to a justification request might be processed. In (7), Eloise's justification request (her "*Really*?") is answered by Abelard, Eloise accepts this answer, and so both Abelard's original assertion, and the reason he gives for it, are added to the common ground. In

[22] For the literature on justification requests, resolution requests and related dialogical moves, see especially the pioneering literature of Charles Hamblin [16, 17] and Jim Mackenzie [30, 31, 32].

[23] The making of further assertions (or denials) is not the only way to meet a justification request. I might make a claim about what I see. You might issue a justification request. In response, I gesture at the scene and tell you to look. If, when you look, you grant my claim, the justification request is also met, but I have not made any further assertions or denials.

(8), Abelard's answer to Eloise's first justification request is met by another justification request (her "*Are you sure?*"), followed by another assertion which seems to undercut Abelard's proffered justification. Unless further justification is given, Abelard's original assertion is blocked from the common ground, because the assertion forming the answer to the justification request is also blocked from the common ground. Finally, in (9), the original claim is also blocked from the common ground, but Abelard's answer is nonetheless granted (Eloise's "Yes" indicates that she concedes Abelard's assertion), but it is immediately followed by a "but . . ." with a further claim showing that Eloise does not take Abelard's answer to meet her original request for justification.

It is important to note that in these dialogues, Eloise does not need to have a worked-out theory of justification, of evidence, or of grounds for assertions. Whenever she says "*Really?*" she is merely indicating that she does not yet grant Abelard's claim, and is asking him to say more. She may accept the reason he gives, even though it is not a very good reason, in some objective sense. (If Astralabe is a fast moving toddler, the fact that he was in the kitchen five minutes ago is not a decisive reason to believe that he is still there now.) Of course, there is *no* sense in which a justification request is always a request for a decisive or deductively valid reason for the initial claim. These are, fundamentally, conversational moves that function to help coordinate the dialogue partners' construction of the common ground. Instead of purely *blocking* the claim of another, a justification request gives the proposer of an item of information a second chance in getting that information added to the common store.

When you meet a justification request you proceed from the item justified, let's say some assertion, to its justification. It is possible, though, to present the justification before making the target assertion, to preempt a justification request. This is one way to understand inference. If I say "A, so B," I not only assert A and assert B, but I offer the assertion of A as meeting the justification request for the assertion of B. We can see this, because the possible responses to the presentation of the inference from A to B parallel precisely the possible responses to the offering of A as a meeting a justification request for B. The hearer could accept the whole package (granting A, B and allowing A as meeting the justification request for B), or she could question A (thereby blocking both A and B from the common ground, at least temporarily), or she could grant A (allowing it into the common ground) but resist the inference to B by asking for further justification for B. In this way, the public communicative use of inferential vocabulary (with terms like *so*, *therefore*, or *it follows that*, and the like) works alongside the more explicitly dialogical back-and-forth of justification requests.[24]

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[24] There is much more to be made in the connection between proof and inference and dialogue. A very good place to start exploring the connection is Catarina Dutilh Novaes' *The Dialogical Roots of Deduction* [10].

With this account of justification requests and the connection to inference in discourse in mind, we return to the dialogue between Achilles and the Tortoise. We now see that the dialogue does not consist only of assertions and denials. The Tortoise grants the claims A, B and $(A \wedge B) \rightarrow Z$ (each of which are accepted into the positive common ground), but she does not resist the conclusion Z to the extent of denying it, but she does ask for a justification request for the assertion of Z. In particular, she does not accept A, B and $(A \wedge B) \rightarrow Z$ as jointly sufficing as meeting the justification request for Z. This helps us pinpoint the Tortoise's deviant behaviour — behaviour that displays a deficient grasp of the meaning of the conditional or of the meaning of conjunction. We would expect that someone who understands what "$\rightarrow$" means would grant that A and $A \rightarrow Z$ are (when they are already accepted) sufficient grounds to justify the conclusion Z.[25] But what kind of norm is violated by the Tortoise? We could just grant that there is a *sui generis* condition connecting justification requests and the competent use of logical concepts: that *modus ponens* be accepted, to the extent that A and $A \rightarrow Z$ are to be accepted as meeting a justification request for Z.[26] However, to resort to this answer, without exploring *how* such constraints on understanding arise, would give us no particular insight either into the logical connectives or our ways of grasping their meaning. We should hope for a more clarifying answer than this.

Instead, I will start with the notion that the logical rules, like these:

$$\frac{X, A, B \succ Y}{\overline{\overline{X, A \wedge B \succ Y}}}\,\wedge Df \qquad \frac{X, A \succ B, Y}{\overline{\overline{X \succ A \rightarrow B, Y}}}\,\rightarrow Df$$

can be used to *define* the connectives involved (here, conjunction and the material conditional), and whether there is any connection between *definition* and norms governing justification requests. In taking this approach, my aim is quite modest. I will argue that *if* we conceive of the logical vocabulary as defined or explicated by rules such as these, *then* we can pinpoint the Tortoise's norm violation in the dialogue with Achilles. There is no need, of course, to treat the logical vocabulary as defined or explicated in just this way. Proponents of some other account of the semantics of the connectives will have to provide some other account of the Tortoise's norm violation.

Before we explore the sense in which defining rules such as $\wedge Df$ and $\rightarrow Df$ can act as definitions, consider an alternate dialogue between Achilles and the Tortoise, featuring a simpler definition:

ACHILLES: So . . . this is an *equilateral* triangle.

TORTOISE: I'm sorry, I don't follow, my heroic friend. I've not heard that word before: what does '*equilateral*' mean?

[25] In what follows we will focus on the simple one-step inference from A and $A \rightarrow Z$ to Z, rather than the slightly longer deduction from A, B and $(A \wedge B) \rightarrow Z$, but the point will remain the same.

[26] Or, slightly more generally, that there is some kind of relation of *immediate consequence* where the premises of an immediate consequence answer a justification request for its conclusion [30].

A: Oh, that's easy to explain. '*Equilateral*' means having sides of the same length. An *equilateral* triangle is a triangle in which all three sides have the same length.

T: I understand: You may continue with your reasoning.

A: Well, as I was saying, the sides of this triangle are all one cubit in length, so it is an equilateral triangle.

T: Perhaps you will forgive me, Achilles, but I still do not follow. I grant to you that the sides of this triangle all have the same length. I fail to see, however, that it *follows* that it is an equilateral triangle. Could you explain why it is?

As before, the Tortoise is infuriatingly obtuse, but in this dialogue it is simpler to pinpoint a specific norm that the Tortoise violates, and this norm connects definitions and justification requests:

> If I accept the definition $A =_{\mathrm{df}} B$, then if I have granted A, then this is enough to meet a justification request for the assertion of B — nothing further is needed. Similarly, if I have ruled A out, then this is enough to meet a justification request for the denial of B. Similarly, if I have granted B (or ruled B out), nothing further is needed to meet a justification request for the assertion (or denial) of A. If I ask for anything more, this is a sign that I have not yet mastered the definition of A.

In other words, to accept a definition and to be competent with it, involves not merely granting a statement (such as $A \leftrightarrow B$, which may not even be in your vocabulary, because you may not have mastered "if and only if") but in competently passing from the *definiens* to the *definiendum* and vice versa in dialogue — and in particular, in justification requests. If I grant A and take it that B is still in question (and this being in question doesn't thereby open up the question for A), then I am not treating A as being defined as B.

The wording given above for this norm is important. It does *not* mean that if I define A in terms of B then when the question of B arises, then granting A will be enough to finalise the issue. If we treat A and B as definitionally equivalent, then in any context in which A is at issue can be extended (by citing this equivalence) into one in which B may also be at issue. The converse holds too: I can *settle* the issue of A by settling the issue of B. If I am seriously wondering whether I have an equilateral triangle before me, then (if I have mastered the definition of the term) then settling the question of whether the triangle has sides of equal length is enough for me to settle the original issue. There is no further question to be addressed. If I take it as settled that the triangle has sides of equal length, and ask for further justification of the claim that it is equilateral, I have not mastered the definition.

Another point will help clarify the significance of this norm. This does not mean that the *only* way a defined term may competently be used in justification requests is by way of passing woodenly between *definiens* and *definiendum*. Competent users make bigger leaps in reasoning. Against a background context in which it is already granted that two sides of a triangle are one cubit in length, if I measure the third side and learn that it is one cubit in length, it may be quite legitimate for me to conclude that the traingle is equilateral, even if I do not explicitly collect the three measurements together to first form the judgement that the three sides are all of the same length, and only then, transition to the conclusion that the triangle is equilateral. It is *possible* to slow the reasoning down to make each of these steps explicit, but it would make the back-and-forth of assertions (or denials) and justification requests more tedious than necessary. The norm for definitions and justification requests is a necessary condition for competent use (and perhaps a baseline condition for adequate use), not a comprehensive account of all of the ways we might legitimately use a definition in concert with justification requests.

I take it that this connection between treating a term as defined and its use in justification requests is relatively uncontroversial. What is more controversial is the connection between definition and the logical connectives and quantifiers, and any thought that this connection might play a role in the use of those logical concepts in inference or in proof. While it has long been an appealing idea that the inference rules for logical concepts are in some sense *analytically valid*,[27] if these inference rules are truly *definitions* in some sense, they cannot all be understood as abbreviative definitions, where we introduce some new concept as a shorthand for a concept already in our repertoire. We *could*, of course define the conditional $A \rightarrow B$ as shorthand for $\neg(A \wedge \neg B)$, treating one connective as a complex of connectives present in our language, but this is not the structure of defining rules such as $\wedge Df$ or $\rightarrow Df$. These rules are more ambitious. They dictate the behaviour of claims involving the introduced concepts in terms of the behaviour of those claims' constituents. If they count as definitions, they are not abbreviative.

So, how could we treat $\wedge Df$ and $\rightarrow Df$ like abbreviative definitions, even though they are different? Consider $\wedge Df$ first:

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge Df$$

If we think of sequents as solely governing assertions and denials, this states that a position featuring an assertion of $A \wedge B$ is out of bounds if and only if the same position which instead involves separate assertions of A and of B is also out of bounds. How could this connect with justification requests? If we think of this defining rule as telling us that an assertion of $A \wedge B$ is somehow intimately related to the assertions of A and of B, then it is plausible to think that it should give rise to the following norm:

> It is a mistake to grant A and grant B and to look for something more to discharge a justification request for an assertion of $A \wedge B$, if you take $\wedge Df$ as a definition.

[27] This view is the target of Arthur Prior's infamous "The Runabout Inference Ticket" [37].

Indeed, if someone had granted A and granted B and thought that there was a further question as to whether or not $A \land B$, this would be a very good sign that this person was not treating '$\land$' as defined by $\land Df$. Consider now $\to Df$:

$$\genfrac{}{}{}{}{X, A \succ B, Y}{\overline{X \succ A \to B, Y}}\ {\to}Df$$

It seems reasonable to expect this rule should give rise to a norm like this:

> It is a mistake to rule A in and rule B out and to look for something more to discharge a justification request for a denial of $A \to B$ if you accept $\to Df$ as a definition.

Here, the defined concept is on the right side of the sequent, so it is the denial of the conditional $A \to B$ that is tied to the assertion of the antecedent A and the denial of the consequent B. This seems correct enough. If by '$\to$' you mean anything like the material conditional, then in a context in which you have asserted A and denied B, there is nothing more to do to justify denying $A \to B$. This connects justification and the denial of a material conditional. What about assertion? It would also seem reasonable to expect that there is some error in expecting more to be required to justify the assertion of Z in a context in which we have already granted A and $A \to Z$. (This is, after all, the Tortoise's error.) We could leave things here, making a bespoke list of connections between individual connectives and contexts for justification. This is Mackenzie's strategy, relying on a primitive relation of *immediate consequence*, where it is given by *fiat* that Z is an immediate consequence of A and $A \to Z$; that $A \land B$ is an immediate consequence of A and B, and so on [30]. A justification request for some claim A can be answered by appeal to some collection of premises, where the conclusion is an immediate consequence of those premises. Such an approach seems correct as far as it goes, and it goes far enough to provide some kind of diagnosis of the Tortoise's error. However, it would be good to press further for an answer to our question, by inquiring more deeply into how such immediate consequences might arise, and in particular how any given rule, when adopted as a *definition*, might give rise to distinctive norms for competent use of that rule in processes of justification. My task in the rest of this paper is to sketch out a systematic account of how sequent rules for the connectives can be adopted as definitions which can play this role in the answering of justification requests. To do this, we will add one piece of structure to the familiar sequents of the form $X \succ Y$. We will take such sequents and place a single formula in the sequent in the spotlight, to form sequents *with focus*.

4. SEQUENTS WITH FOCUS

A focused sequent is a regular sequent in which one formula is in focus, whether on the left ($X, \boxed{A} \succ Y$) or the right ($X \succ \boxed{B}, Y$). The intended reading of a sequent $X \succ \boxed{B}, Y$ in which the indicated 'B' is in focus, is that in any available position extending $[X : Y]$ (that is, a position in which each member of X is ruled in and each member of Y is ruled out) a justification request for an assertion of B can be met, purely on the basis of what has already been granted. When the formula in focus is in the left, in a sequent like $X, \boxed{A} \succ Y$, we can meet a justification request for the denial of A, in any available position extending $[X : Y]$. To put this bluntly $X, \boxed{A} \succ Y$ says that given $[X : Y]$, the claim A is ruled *out*, while $X \succ \boxed{B}, Y$ says that B is ruled *in*.[28]

Focused sequents are a means for attending to norms governing justification requests. A justification request is made for an individual assertion (or a denial) against a background context of claims taken for granted. With focussed sequents in play, we can reconsider defining rules like $\to Df$. The obvious question arises: Which formulas should be in focus in such a rule? The sequent $X \succ \boxed{A \to B}, Y$ concerns justifying an assertion $A \to B$ in the context $[X : Y]$, so let's place the introduced conditional formula in focus. A natural choice for finding focus in the premise sequent $X, A \succ B, Y$ would be to focus on B, to give the following *Focused* defining rule for the conditional:

$$\frac{X, A \succ \boxed{B}, Y}{\overline{X \succ \boxed{A \to B}, Y}} \;{\to}FDf$$

The choice of focus in the upper sequent is natural because if we read this rule from top to bottom, we have the conditional introduction rule from natural deduction. If I can meet a justification request for B in a context in which A is taken for granted, then (no longer appealing to that A) I can justify the assertion of $A \to B$. Reading the rule from bottom to top, we have a form of *modus ponens*. How can I justify the assertion of B in a context where I have granted A, together with the context $[X : Y]$? One way is to show $A \to B$, from that background context. This *focussed* defining rule has a natural reading in terms of justification requests.[29]

It makes sense to think of the focused defining rule $\to FDf$ as a kind of *definition* of '$\to$', analagously to abbreviative definitions. To adopt it as a definition, you are to use '$\to$' in the manner stipulated by the rule. That is, when it comes to justification requests, you are to take a justification request for an assertion of $A \to B$ to be met when you can justify B on the basis of granting A, and *vice versa*. To resist at either of these points is a sign that you have not yet mastered the definition.[30]

[28] One pleasing side-effect of attending to sequents with focus is a ready answer to Steinberger's pointed criticism of multiple-premise multiple-conclusion sequent calculus [52]. A derivation of a sequent with focus has a unique conclusion (the formula in focus), while the background context $[X : Y]$ in which that conclusion is reached contains both formulas in positive position (those in X) and negative position (those in Y).

[29] We could have chosen to focus on A in the upper sequent, instead of B. Had the upper sequent been $X, \boxed{A} \succ B, Y$, then to show $A \to B$ (from the context $[X : Y]$) you would not temporarily rule A in, to prove B — you would instead temporarily rule B *out*, in order to *refute* A. As we will see, this would ultimately have the same upshot, but it would be a different formulation of the fundamental rule.

[30] This does not address the question of whether this definition is any sense a *good* one. Unlike with abberviative definitions, there is no quick guarantee that anything posed as a definition in this way

This has the general shape of a definition, in that the newly introduced concept (appearing on one side of the double-line rule) is introduced in terms of the prior vocabulary (the concept does not appear on the other side of the rule). This is not, though, a simple abbreviative definition in which the '$\rightarrow$' in '$A \rightarrow B$' may be translated away. To justify $A \rightarrow B$, you justify B, but against a different context — while granting A. This shift of perspective means you need not have something in your original vocabulary that does the job of the material conditional. When adding it to your vocabulary, you may expand your conceptual repertoire.

Notice, though, that this way of specifying norms governing the material conditional does not immediately address the Tortiose's obduracy, since this norm explicitly addresses how to justify the assertion of a conditional, not what we may use a conditional to *justify*. However, in the presence of other commitments concerning justification, the answer is given in one short step:

$$\frac{A \rightarrow B \succ \boxed{A \rightarrow B}}{A \rightarrow B, A \succ \boxed{B}} \rightarrow\!FDf$$

So, if we can answer a justification request for $A \rightarrow B$ in a context in which $A \rightarrow B$ is already taken for granted (presumably by just pointing out that we have already granted $A \rightarrow B$, deferring to the justification given for that), then we may appeal to the definition of '$\rightarrow$' to conclude B in any context in which $A \rightarrow B$ and A are both taken for granted. So, if we adopt $\rightarrow$*FDf*, *this* is how we can meet the justification request for the assertion of B in the context where $A \rightarrow B$ and A have been granted. We first recognise that having granted $A \rightarrow B$ we can indeed prove $A \rightarrow B$ from that (most often, this goes without saying, of course!), and so, having shown $A \rightarrow B$, following $\rightarrow$*FDf*, we recognise that under the assumption of A, which we have, in fact, already granted, B follows.[31]

The derivation starts from $A \rightarrow B \succ \boxed{A \rightarrow B}$. This is a focussed version of the identity axiom. The general shape of sequents like this is as follows:

$$X, A \succ \boxed{A}, Y \qquad X, \boxed{A} \succ A, Y$$

is a conservative extension of your previous commitments, or that a unique concept is so defined. To do that, you must do more work [2]. I have done this work for *unfocussed* defining rules of the shape under consideration here [39]. We will later see that the conservative extension and unique definability results in the unfocussed sequent calculus transfer directly to this focussed setting, so these rules act like definitions in a very strong sense.

[31]This reasoning, spelling things out in such detail, is most likely *not* what everyday reasoners do with a conditional in their vocabulary. This is not a mark against taking $\rightarrow$*FDf* to provide a definition of a concept which might match one way that we use the conditional. Consider the axioms for addition in Peano Arithmetic. To take addition on the natural numbers to be axiomatised by the principles $\forall x\, x + 0 = 0$ and $\forall x \forall y\, (x + s(y)) = s(x + y)$ is not to say that whenever you or I work out the value of $5 + 7$ we must apply the axiom $\forall x \forall y\, (x + s(y)) = s(x + y)$ seven times, and $\forall x\, x + 0 = x$ once, to eventually result in 12. The Peano axioms have value because they provide a simple basis from which a great deal the behaviour of successor, addition, and multiplication, and the natural numbers logically follows. The same can hold for this defining rule: it is a simple primitive invertible rule that can be treated as a definition, and that fixes the behaviour of $\rightarrow$ in a very strong sense.

The intended interpretation is clear: in any position in which A is in the positive common ground, a justification request for a further assertion of A can be met, by pointing us to whatever grounded the assertion of A in the first place. If raising the question concerning A was enough to *eject* A from the common ground, then of course the justification request might no longer be able to be met, but then A is no longer in the background collection of resources from which to draw. What goes for assertions goes also for denials, so it is possible for the formula in focus to occur on the other side of the sequent, too, with the same sort of justification, except we talk about *denials* rather than assertions.

However, there is another place that the focus could go on a sequent of the form $X, A \succ A, Y$. That is, on a different formula selected from the context. This formula might also be on the left or on the right. This gives us focussed sequents of the following form.

$$X, A \succ A, \boxed{B}, Y \qquad X, A, \boxed{B} \succ A, Y$$

There is no doubt that these sequents do not seem quite as straightforwardly *correct* as the previous cases. However, there is reason to admit them as basic focussed sequents. One reason is the result concerning the *Swap* rule we will see below. We will show that if we can derive a sequent with the focus in one position, we can move it around into any other position, too. (This is — in *general* at least — an appealing rule. If $X, A \succ \boxed{B}, Y$ and we take a context in which each member of X is ruled in, each member of Y is ruled out and B is ruled out, too, we can rule A out, since had A held, we could have used it to justify B, which clashes with it not holding.) Instead of taking *Swap* as a primitive rule, we will show that it is derivable from the other rules, but to do so in full generality, we need to admit as axiomatic sequents these instances of the identity axioms, where the focus hits a side formula, rather than a formula that occurs on both sides of the sequent, since these would arise from applying *Swap* to a straightforwardly valid axiom.

The second reason for allowing these sequents is the delicate way we have interpreted focussed sequents. Clearly in any *available* position extending $[X, A : A, Y]$ we could meet a justification request for an assertion of B (or a denial of B for that matter), for there is no available position extending $[X, A : A, Y]$! It is out of bounds, and so are each of its extensions. With this understanding of the task at hand in play, there is no problem with deriving A (or ruling A out) from this inconsistent starting place, because the condition is vacuously true.[32]

That is how one can add focus to the axioms of the sequent calculus. Now consider the structural rules. There are four different *natural* ways to add focus to the

[32]Of course, it seems worth developing a theory of justification requests which does not go so far as to endorse *Swap* in its full generality, and in which out-of-bounds positions are nonetheless discriminating concerning justification requests. Such a *relevance*-preserving focussed sequent calculus would be interesting, but it would take us some way away from classical logic. Exploring such an account of justification need not be opposed to the account developed here: it may simply amount to developing an account of proof and inference that meets a higher standard of correctness than the standard classical account. A pluralist account of logic could make room for both standards [1].

Cut rule, corresponding to the different positions in focus in the premises and the conclusion. Each of these four rules corresponds to the process of meeting one justification request in this process of meeting another justification request.

$$\frac{X \succ \boxed{A}, Y \quad X, A \succ \boxed{B}, Y}{X \succ \boxed{B}, Y}\,\mathit{FCut}^{R}_{R} \qquad \frac{X \succ \boxed{A}, Y \quad X, A, \boxed{B} \succ Y}{X, \boxed{B} \succ Y}\,\mathit{FCut}^{R}_{L}$$

$$\frac{X \succ \boxed{B}, A, Y \quad X, \boxed{A} \succ Y}{X \succ \boxed{B}, Y}\,\mathit{FCut}^{L}_{R} \qquad \frac{X, \boxed{B} \succ A, Y \quad X, \boxed{A} \succ Y}{X, \boxed{B} \succ Y}\,\mathit{FCut}^{L}_{L}$$

Consider the first of the four formulations, FCut^{R}_{R}. Reading it from top to bottom, it tells us that when we can (on the basis of the position $[X : Y]$) meet a justification request for an assertion of A, and when we can (on the basis of the position $[X, A : Y]$) meet a justification request for the assertion of B, then we could have met that justification request for the assertion of B on the basis of $[X : Y]$ alone. It is clear how to do this: first, assert A (addressing its justification request, as $X \succ \boxed{A}, Y$ indicates we can), and so, the context is now $[X, A : Y]$, against which B can be asserted, and justified. The same kind of process works for the other three formulations of *Cut*, which correspond to different choices for whether the intermediate statement A and ultimate target for justification B is asserted or denied.[33]

Then we must consider the contraction rules. Not only contraction *outside* what is in focus, but contraction of an unfocussed item with a focussed item.

$$\frac{X, A, \boxed{A} \succ Y}{X, \boxed{A} \succ Y}\,\mathit{FW}_{L} \qquad \frac{X \succ A, \boxed{A}, Y}{X \succ \boxed{A}, Y}\,\mathit{FW}_{R}$$

Notice that in the presence of focussing, we need to include contraction (*W*) rules, even though the LHS and RHS of a sequent is still a *set*. These structural rules suffice for us to *swap* the focus from one point to another in a sequent, for example, like this:

$$\frac{\dfrac{X \succ \boxed{A}, B, Y \quad X, A \succ A, \boxed{B}, Y}{X \succ A, B, \boxed{B}, Y}\,\mathit{FCut}^{R}_{R}}{X \succ A, \boxed{B}, Y}\,\mathit{FW}_{R}$$

Given *Cut* and contraction (and the axiomatic sequents in which the focus is on some side formula) all four of the following *Swap* rules are derivable:

$$\frac{X \succ \boxed{A}, B, Y}{X \succ A, \boxed{B}, Y}\,\mathit{Swap}^{R}_{R} \qquad \frac{X, \boxed{A}, B \succ Y}{X, A, \boxed{B} \succ Y}\,\mathit{Swap}^{L}_{L}$$

[33] It is well-known that the *Cut* rule faces pressure in the light of paradox, which raises the spectre of counterexamples in which the *Cut* formula may be neither coherently asserted nor coherently denied [7, 41, 42, 40]. At least, this issue is raised for the reading of sequents in terms of assertion and denial alone. It seems significant that even in the presence of antinomies (if indeed there are any antinomies so badly behaved), the *Cut* rule, in its focussed guise, seems blameless. It is the *contraction* rules, described below, which seem more ripe for reconsideration.

$$\frac{X, \boxed{A} \succ B, Y}{X, A \succ \boxed{B}, Y}\,\mathit{Swap}^{L}_{R} \qquad \frac{X, A \succ \boxed{B}, Y}{X, \boxed{A} \succ B, Y}\,\mathit{Swap}^{R}_{L}$$

With such rules, we can move the focus around a sequent freely. This means that, provided that the *connective* rules of our focused calculus have the same shape (if we discard the focus) as the rules in an unfocused calculus — even if they are quite restrictive concerning where the focus should be — then any derivation of an unfocused sequent can be *decorated* into a focused derivation. Before we see an example of this, let's introduce the remaining connective rules:

$$\frac{X, \boxed{A}, B \succ Y}{X, \boxed{A \wedge B} \succ Y}\,{\wedge}\mathit{FDf} \qquad \frac{X \succ \boxed{A}, B, Y}{X \succ \boxed{A \vee B}, Y}\,{\vee}\mathit{FDf} \qquad \frac{X, \boxed{A} \succ Y}{X \succ \boxed{\neg A}, Y}\,{\neg}\mathit{FDf}$$

As with the structural rules, and the case of $\rightarrow\mathit{Df}$, so here: There are a number of different ways to add focus to an unfocussed rule. In these cases, I have used the constraints that the introduced formula in the conclusion be in focus, and that one of its subformulas be in focus in the premise of the rule. For $\neg\mathit{FDf}$ the rule is very natural: to prove $\neg A$, show that A must be denied. The rule $\vee\mathit{FDf}$ is curious: to prove $A \vee B$, it suffices to prove A in a context in which B is ruled out. The rule $\wedge\mathit{FDf}$ is the most curious: to *refute* a conjunction, you refute the first conjunct under the supposition of the second. This is, of course, an unfamiliar way to reason with conjunction, but it is, of course, valid. Furthermore, using an axiomatic sequent, and an application of the defining rule for conjunction and a *Swap* we can justify the straightforward claim that to justify a conjunction, you need nothing more than the conjuncts:

$$\dfrac{\dfrac{\boxed{A \wedge B} \succ A \wedge B}{\boxed{A}, B \succ A \wedge B}\,{\wedge}\mathit{FDf}}{A, B \succ \boxed{A \wedge B}}\,\mathit{Swap}^{L}_{R}$$

Similarly, the *focused* rules for negation and disjunction make sense under this interpretation. To prove a disjunction, it sufices to prove the first disjunct, under the assumption that the second fails. To prove a negation, it suffices to refute the negand of that negation.

As mentioned above, with these rules, any derivation of a sequent $X \succ A, Y$ can be transformed into a focussed derivation of $X \succ \boxed{A}, Y$, and any derivation of a sequent $X, A \succ Y$ can be transformed into a focussed derivation of $X, \boxed{A} \succ Y$.

Here is an example, adding focus to the derivation given at the start of the paper:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p, q \vee r \succ p \wedge q, \boxed{q \vee r}}{p, q \vee r \succ p \wedge q, r, \boxed{q}}\;\vee FDf \qquad \dfrac{\boxed{p \wedge q}, q \vee r \succ p \wedge q, r}{\boxed{p}, q, q \vee r \succ p \wedge q, r}\;\wedge FDf}{\boxed{p}, q \vee r \succ p \wedge q, r}\;FCut^{R}_{L}}{\boxed{p \wedge (q \vee r)} \succ p \wedge q, r}\;\wedge FDf}{p \wedge (q \vee r) \succ \boxed{p \wedge q}, r}\;Swap^{L}_{R}}{p \wedge (q \vee r) \succ \boxed{(p \wedge q) \vee r}}\;\vee FDf
$$

This focussed derivation results from taking the unfocussed derivation, adding focus at each step. Here, there was only one step (the unfocussed sequent $p \wedge (q \vee r) \succ p \wedge q, r$) where we needed to employ a *Swap* rule, since $\wedge$*FDf* delivers a sequent with the conjunction $p \wedge (q \vee r)$ in focus, and the next connective rule, $\vee$*FDf* requires a sequent with the disjunct $p \wedge q$ in focus. The *Swap* step is required to transform the sequent to the required format, and the result is a focused derivation. It is clear that this procedure is totally general. Any unfocused derivation may be transformed into a focused derivation, at will, since for every inference falling under an unfocused rule, there is at least *one* way to add focus to that results in an instance of a focused rule, and *Swap* can be used to shift focus to any position in a sequent, as needed.

This example is a relatively straightforward proof of distribution, a simple sequent. Let's work through another example of a focused derivation, this time, showing how the different structural and connective rules can correspond explicitly to steps of a dialogue, in which justification requests are made and met, and the common ground shifts along the way. Start with this focussed derivation of the classical sequent $\succ ((p \to q) \to p) \to p$, *Peirce's Law*, which is classically valid but is not valid in intuitionistic logic.

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p, \boxed{q}}{\succ p, \boxed{p \to q}}\;\to FDf \qquad \dfrac{(p \to q) \to p \succ \boxed{(p \to q) \to p}}{(p \to q) \to p, p \to q \succ \boxed{p}}\;\to FDf}{(p \to q) \to p \succ p, \boxed{p}}\;FCut^{R}_{R}}{(p \to q) \to p \succ \boxed{p}}\;FW_{R}}{\succ \boxed{((p \to q) \to p) \to p}}\;\to FDf
$$

The conclusion sequent of this focussed derivation is $\succ \boxed{((p \to q) \to p) \to p}$, so the derivation gives us a way to meet a justification request for an assertion of $((p \to q) \to p) \to p$. Let's see an example of how a dialogue following the structure of this derivation might go. The dialogue partners are, as usual, Abelard and Eloise. I will punctuate that dialogue with the common ground at each stage of the dialogue, indicating what is taken for granted (even at least temporarily), for the sake of the discussion.

ELOISE: $((p \to q) \to p) \to p$.

ABELARD: *Really*? I can never understand deeply left-associated conditionals. Why on earth is *that* true?

E: Let's grant $(p \to q) \to p$. I'll now show p.

A: OK, granted.

CG: The common ground is now $[(p \to q) \to p : \quad]$.

E: To show p, let's first rule it out, and if we can show p then, it follows regardless.

A: If you think that'll help, I'll let you grant it, though it seems like ruling p out would make it *harder* to prove, not easier.

CG: $[(p \to q) \to p : p]$

E: Now we can prove $p \to q$, since if we also *assume* p the position is out of bounds, so there are no available positions in which to check for q. (This step uses the vacuous $p \succ p, \boxed{q}$, and then discharges the assumption of p — the left branch of the derivation, using $\to$*Df*.)

A: I grant that.

CG: $[(p \to q) \to p, p \to q : p]$

E: We've granted $(p \to q) \to p$ and $p \to q$. You can see where we're going now, can't you? It follows that p.

A: I see that. This is *modus ponens*.

CG: $[(p \to q) \to p, p \to q, p : p]$

A: But hang on!? I'm feeling a bit queasy now. Haven't we just asserted p *and* denied it? Aren't we out of bounds?

E: That's right. But remember, we ruled p out merely for the sake of the argument. We've managed to show that p is *unavoidable*: even when we tried to deny it, it came back. So, we can discharge that assumption. We've shown p.

CG: $[(p \to q) \to p, p : \quad]$ (We remove the denial of p and the other things derived under the scope of that assumption — here, just the assertion $p \to q$ — leaving the final p.)

A: Phew. That feels better. But what were we trying to prove? I forgot.

E: You asked me about my claim that $((p \to q) \to p) \to p$. I said I'd prove it by assuming $(p \to q) \to p$ and showing that p is true.

A: That's right, I remember now.

E: But we've done it! Notice, we've proved p.

A: You're right. We granted $(p \to q) \to p$, and using this, we showed that p. It follows that *if* $(p \to q) \to p$ *then* p.

E: Which was what you asked me to prove.

CG: $[((p \to q) \to p) \to p : \quad]$.

What goes for this derivation can go for any focussed derivation using the rules I have set. A derivation of $X, \boxed{A} \succ Y$ [of $X \succ \boxed{B}, Y$] in the signed sequent calculus gives you the means to answer a justification request for a denial of A [for an assertion of B] in any available context extending [X, Y], appealing only to the defining rules for the connectives involved in the derivation.

So, the focused proof system provides answers to our original concerns about the relationship between the multiple conclusion classical sequent calculus and proofs and *inference*. If we understand a conclusion of a proof as the formula under the focus in a focused sequent derivation, then we can see why the sequent calculus (with the addition of *focus*) allows us to represent *inference*, while the rest of the sequent structure represents the common ground (both positive and negative) against which that conclusion is drawn. Since both assertions and denials can be the target of a justification request, this single conclusion can be in the *right* or the *left* of a sequent.[34] A derivation of an unsigned sequent $X \succ B, Y$ [or of $X, A \succ Y$] can be transformed into a *procedure* for meeting a justification request for an assertion of B [or a denial of A] in any available position, appealing only to what is granted in [X : Y], and to the focused defining rules used in that derivation.

These defining rules have a special status. The sequent calculus provides a systematic way to represent these connective rules as *definitions*. The defining rules can be shown to share some of the key properties of abbreviative definitions: they are conservatively extending and uniquely defining [39]. If we add focus to such defining rules, the resulting *focussed* defining rules remain conservatively extending and uniquely defining, and they make explicit what it would take to *apply* these rules in reasoning practices which involve not only assertion and denial, but making justification requests and answering them. If Achilles and the Tortoise had been employing their logical vocabulary in compliance with these defining rules, then the Tortoise shows, by their behaviour, that they have not quite mastered the rule $\to$*FDf*. When the Tortoise asks for more to justify Z than A and $A \to Z$, the best response for Achilles would be to step back in an attempt to establish what the Tortoise *does* mean by '*if* … *then* …'. What does follow from *if* A *then* Z, in the Tortoise's vocabulary? Does anything follow from *if* A *then* Z and A? If the Tortoise intends to mean our '$\to$,' then practice is needed to get competent with its use. If, on the other hand, the Tortise has some *other* sense in mind, then Achilles could attempt to learn that. If the Tortoise does mean something *else* by their use of conditional vocabulary, then there is not necessarily any surprise that the Tortoise refrains from concluding Z from A and *if* A *then* Z. If the Tortoise means 'A *or* Z' by '*if* A *then* Z,' after all, they would be irrational to conclude Z, and they are

[34] So, there is a connection between focused sequents and fully signed natural deduction. Just as an intutionist derivation of a sequent of the form $X \succ A$ can represent a natural deduction proof from assumptions in X to the conclusion A. A focussed sequent derivation of $X \succ \boxed{B}, Y$ [or of $X, \boxed{A} \succ Y$] can represent a signed natural deduction proof of $+B$ [or of $-A$] from positively signed formulas from X and negatively signed formulas from Y [45, 48]. While this paper could have been presented from the perspective of signed natural deduction, the connection between invertible defining rules and their role as *definitions* would have been less explicit, and the connection with the unsigned sequent calculus and its treatment in the terms of norms for assertion and denial would be less direct.

well within their rights to resist that conclusion. We have no reason to believe that the Tortoise has *such* a deviant understanding of '*if* ... *then* ...', but given that the meaning of '$A \rightarrow Z$' as given by the defining rule comes with some justificatiory connections to assertions and denials of A and Z, we may at least hope that there are some connections there to be explored, or if not, that the Tortoise may instead come to adopt a concept like '$\rightarrow$', governed by an explicit definition, such as $\rightarrow$*FDf*.

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As Arthur Prior pointed out, many years ago, it is very tempting to think of inference rules governing connectives to be in some sense, definitions, or instructions for how to *use* these concepts in our thought and talk. If the argument in this paper is right, there is a way to spell out this tempting idea. A definition is an instruction for how to use the defined concept in assertion, denial, and in the making and answering of justification requests. Just as proposing a definition is a non-assertoric speech act — it is an imperative — the crucial and distinctive role that definitions play in our reasoning does not show itself in its interaction with assertion and denial alone. The making and answering of justification requests are non-assertoric speech acts in which the distinctive impact of treating a rule as a *definition* is most clearly seen. In this way, we see that it's by paying attention to the non-assertoric speech acts of asking — and answering — polar questions, and making justification requests — and answering them — we can gain a more well rounded perspective on the classical sequent calculus, and its interpretation.

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