# An Inferentialist Account of Identity and Modality

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To show how *defining rules* in a *hypersequent* setting can give a uniform proof-theoretic semantics of *identity* and *modality*, allowing — equally naturally — for (1) modal operators for which identity statements are *necessary* (if true), and (2) modal operators for which identity statements can be *contingently* true.

My Approach

Defining Identity

Defining Necessity

Combining Them

The Upshot

# MY APPROACH

# *Normative Pragmatic, Semantically Anti-Realist* and *Dialogical*

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- Define concepts by way of rules for their *use*. (That's *normative pragmatism*.)
- Their semantics is not given in the first instance in terms of reference relations to objects. (That's *semantic anti-realism*.)
- The rules are grounded in our *communicative* and *dialogical practice*, involving assertion, denial and inference.

See Catarina Dutilh Novaes, *The Dialogical Roots of Deduction*, 2020.

# *Defining Rules* for Logical Concepts

$\wedge$ **Conjunction**

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ {\scriptstyle \wedge Df}$$

Asserting a conjunction has the
same force as asserting its conjuncts.

$\wedge$ Conjunction

$\rightarrow$ **Conditional**

$$\frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y}\ {\rightarrow}Df$$

Denying a material conditional has the same force
as asserting its antecedent and denying its consequent.

Or equivalently, to *prove* a material conditional
*suppose* its antecedent and *prove* its consequent.

# *Defining Rules* for Logical Concepts

$\wedge$ Conjunction

$\rightarrow$ Conditional

$\forall$ **Universal Quantifier**

$$\frac{X \succ A[x/n], Y}{X \succ \forall x A, Y}\ \forall Df$$

To prove that everything has some feature, prove that $n$ has that feature, making no assumptions about $n$.

Or equivalently, denying that everything isatisfies $A$ has the same force as denying that $n$ satisfies $A$ for some fresh name '$n$'.

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. . . not by finding a paraphrase in the original language, but by 'making explicit' something that was implicit.

In controlled circumstances, we can prove that concepts added by defining rules like these are *conservative* and *uniquely defined*, just like definitions by paraphrase.

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In our everyday vocabulary, we make moves *like these* with our informal concepts ‘and’, ‘not’, ‘if’ and ‘all’.

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In our everyday vocabulary, we make moves *like these* with our informal concepts ‘and’, ‘not’, ‘if’ and ‘all’.

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Defining rules suffice for us to *coordinate* and to *communicate*, especially in those contexts where precision is valuable.

These defining rules give rise to Gentzen-style left/right rules in the presence of *Id* and *Cut*, and the resulting rules allow for the usual *cut elimination* argument.

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# GENERALITY AND EXISTENCE 1: QUANTIFICATION AND FREE LOGIC

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**Abstract.** In this paper, I motivate a cut free sequent calculus for classical logic with first order quantification, allowing for singular terms free of existential import. Along the way, I motivate a criterion for rules designed to answer Prior's question about what distinguishes rules for logical concepts, like conjunction from apparently similar rules for putative concepts like Prior's *tonk*, and I show that the rules for the quantifiers—and the existence predicate—satisfy that condition.

**§1. Sequents and defining rules.** Let's take it for granted for the moment that learning a language involves—at least in part—learning how assertions and denials expressed in that language bear on one another. The basic connection, of course, is that to assert $A$ and to deny $A$ *clash*. When we learn conjunction, we learn that there is a clash involved in asserting $A$, asserting $B$ and denying $A \wedge B$. Similarly, when we learn disjunction, we learn that there is a clash involved in asserting $A \vee B$, denying $A$ and denying $B$.

One way to systematically take account of the kinds of clashes involved in these acts of assertion and denial is through the language of the sequent calculus. Given collections $\Gamma$ and $\Delta$ of sentences from our language $\mathcal{L}$, a *sequent* $\Gamma \succ \Delta$ makes the claim that there is a clash involved in asserting each element of $\Gamma$ and denying each element of $\Delta$. The structural rules of the sequent calculus can be understood in the following way [16]. *Identity*:

$$A \succ A.$$

There is a clash involved in asserting $A$ and denying $A$. *Weakening*[1]:

$$\frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta}\,[KL] \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}\,[KR].$$

# DEFINING IDENTITY

# Identity and Harmony

## *Identity and harmony*

Stephen Read

*1. Harmony*

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960–61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if 'tonk' has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of '$p$ tonk $q$' is given by Prior's rule:

$$\frac{p}{p \text{ tonk } q}\ \text{tonk-I}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{p \text{ tonk } q \qquad \begin{array}{c}(p)\\ r\end{array}}{r}\ \text{tonk-E}$$

# A Defining Rule for Identity

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y} {=}Df$$

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Or equivalently, to prove that $a = b$, prove $Fb$ from the assumption $Fa$ (and *vice versa*), where the predicate $F$ is *arbitrary*.

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Or equivalently, to prove that $a = b$, prove $Fb$ from the assumption $Fa$ (and *vice versa*), where the predicate $F$ is *arbitrary*.

Identity is a kind of *indistinguishability*.

$$
\begin{prooftree}
\AxiomC{$Fa \succ Fa$}
\AxiomC{$Fa \succ Fa$}
\RightLabel{$=\!Df\!\downarrow$}
\BinaryInfC{$\succ a = a$}
\RightLabel{$\lambda Df\!\downarrow$}
\UnaryInfC{$\succ (\lambda x.x = a)a$}
\AxiomC{$a = b \succ a = b$}
\RightLabel{$=\!Df\!\uparrow$}
\UnaryInfC{$a = b, Fa \succ Fb$}
\RightLabel{$Spec^{F}_{(\lambda x.x=a)}$}
\UnaryInfC{$a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b$}
\RightLabel{$Cut$}
\BinaryInfC{$a = b \succ (\lambda x.x = a)b$}
\RightLabel{$\lambda Df\!\uparrow$}
\UnaryInfC{$a = b \succ b = a$}
\end{prooftree}
$$

## An Example Derivation

$$\dfrac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y} \; {=}Df$$

$$\dfrac{\dfrac{\dfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a = a}\,{=}Df{\downarrow}}{\succ (\lambda x.x = a)a}\,\lambda Df{\downarrow} \qquad \dfrac{\dfrac{a = b \succ a = b}{a = b, Fa \succ Fb}\,{=}Df{\uparrow}}{a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b}\,Spec^{F}_{(\lambda x.x=a)}}{\dfrac{a = b \succ (\lambda x.x = a)b}{a = b \succ b = a}\,\lambda Df{\uparrow}}\,Cut$$

# An Example Derivation

$$\frac{X \succ Y}{X[F/P] \succ Y[F/P]}\;\mathit{Spec}^{F}_{P}$$

$$\cfrac{\cfrac{\cfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a = a}\;{=}\mathit{Df}{\downarrow}}{\succ (\lambda x.x = a)a}\;\lambda\mathit{Df}{\downarrow} \qquad \cfrac{\cfrac{a = b \succ a = b}{a = b, Fa \succ Fb}\;{=}\mathit{Df}{\uparrow}}{a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b}\;\mathit{Spec}^{F}_{(\lambda x.x=a)}}{\cfrac{a = b \succ (\lambda x.x = a)b}{a = b \succ b = a}\;\lambda\mathit{Df}{\uparrow}}\;\mathit{Cut}$$

# An Example Derivation

$$\dfrac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y}\;{=}Df$$

(double line)

$$\dfrac{\dfrac{\dfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a = a}\;{=}Df{\downarrow}}{\succ (\lambda x.x = a)a}\;\lambda Df{\downarrow} \qquad \dfrac{\dfrac{a = b \succ a = b}{a = b, Fa \succ Fb}\;{=}Df{\uparrow}}{a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b}\;Spec^{F}_{(\lambda x.x = a)}}{\dfrac{a = b \succ (\lambda x.x = a)b}{a = b \succ b = a}\;\lambda Df{\uparrow}}\;Cut$$

# An Example Derivation

$$\frac{X \succ A[x/a], Y}{X \succ (\lambda x.A)a, Y}\ \lambda Df$$

$$\cfrac{\cfrac{\cfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a = a}\ {=}Df{\downarrow}}{\succ (\lambda x.x = a)a}\ \lambda Df{\downarrow} \qquad \cfrac{\cfrac{a = b \succ a = b}{a = b, Fa \succ Fb}\ {=}Df{\uparrow}}{a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b}\ \mathit{Spec}^{F}_{(\lambda x.x=a)}}{\cfrac{a = b \succ (\lambda x.x = a)b}{a = b \succ b = a}\ \lambda Df{\uparrow}}\ \mathit{Cut}$$

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$$\frac{X \succ A[x/a], Y}{X \succ (\lambda x.A)a, Y}\ \lambda Df \quad \text{(double line)}$$

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# An Example Derivation

$$\dfrac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y}\;{=}Df \qquad \dfrac{X \succ A[x/a], Y}{X \succ (\lambda x.A)a, Y}\;\lambda Df \qquad \dfrac{X \succ Y}{X[F/P] \succ Y[F/P]}\;Spec^F_P$$

$$\dfrac{\dfrac{\dfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a = a}\;{=}Df{\downarrow}}{\succ (\lambda x.x = a)a}\;\lambda Df{\downarrow} \qquad \dfrac{\dfrac{a = b \succ a = b}{a = b, Fa \succ Fb}\;{=}Df{\uparrow}}{a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b}\;Spec^F_{(\lambda x.x=a)}}{\dfrac{a = b \succ (\lambda x.x = a)b}{a = b \succ b = a}\;\lambda Df{\uparrow}}\;Cut$$

# From *Df* to *L/R*: $=R$ is $=Df{\downarrow}$

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y}\ {=}Df{\downarrow}$$

# From *Df* to *L*/*R*: These are the $=L$ rules

$$\frac{X \succ Pa, Y \qquad X', Pb \succ Y'}{a = b, X, X' \succ Y, Y'}\,{=}L_1 \qquad \frac{X \succ Pb, Y \qquad X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'}\,{=}L_2$$

# From *Df* to *L*/*R*: These are the $=L$ rules

$$\dfrac{X \succ Pa, Y \qquad X', Pb \succ Y'}{a = b, X, X' \succ Y, Y'}\;{=}L_1$$

$$\dfrac{\dfrac{X \succ Pa, Y \qquad \dfrac{\dfrac{a = b \succ a = b}{a = b, Fa \succ Fb}\;{=}Df{\uparrow}}{a = b, Pa \succ Pb}\;Spec^{F}_{P}}{a = b, X \succ Pb, Y}\;Cut \qquad X', Pb \succ Y'}{a = b, X, X' \succ Y, Y'}\;Cut$$

# From *Df* to *L*/*R*: These are the $=L$ rules

$$\dfrac{X \succ Pb, Y \qquad X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'}\;{=}L_2$$

$$\dfrac{\dfrac{X \succ Pb, Y \qquad \dfrac{\dfrac{a = b \succ a = b}{a = b, Fb \succ Fa}\;{=}Df{\uparrow}}{a = b, Pb \succ Pa}\;Spec^{F}_{P}}{a = b, X \succ Pa, Y}\;Cut \qquad X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'}\;Cut$$

# From *Df* to *L*/*R*: These are the $=L$ rules

$$\frac{X \succ Pa, Y \qquad X', Pb \succ Y'}{a = b, X, X' \succ Y, Y'}\,{=}L_1 \qquad \frac{X \succ Pb, Y \qquad X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'}\,{=}L_2$$

# Going Without Some Rules

$$\frac{X \succ Y}{X[F/P] \succ Y[F/P]}\,\mathit{Spec}^{F}_{P} \qquad \frac{X \succ A, Y \qquad X', A \succ Y'}{X, X' \succ Y, Y'}\,\mathit{Cut}$$

# Going Without Some Rules

$$\frac{X \succ Y}{X[F/P] \succ Y[F/P]}\,\mathit{Spec}^{F}_{P}$$

- *Spec* is *height-preserving admissible* in the the system with *L/R* rules + *Cut*.

# Going Without Some Rules

$$\frac{X \succ A, Y \qquad X', A \succ Y'}{X, X' \succ Y, Y'}\ \mathit{Cut}$$

- *Spec* is *height-preserving admissible* in the the system with *L*/*R* rules + *Cut*.
- *Cut* is eliminable in the system with *L*/*R* rules.

# Eliminating *Cut*

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_1\\ X, Fa \succ Fb, Y\end{array} \quad \begin{array}{c}\vdots\,\delta_2\\ X, Fb \succ Fa, Y\end{array}}{X \succ a = b, Y}{=}R \qquad \dfrac{\begin{array}{c}\vdots\,\delta'\\ X' \succ Pa, Y'\end{array} \quad \begin{array}{c}\vdots\,\delta''\\ X'', Pb \succ Y''\end{array}}{a = b, X', X'' \succ Y', Y''}{=}L_1}{X, X', X'' \succ Y, Y', Y''}\,Cut
$$

# Eliminating *Cut*

$$\dfrac{\dfrac{\overset{\vdots\,\delta_1}{X, Fa \succ Fb, Y} \quad \overset{\vdots\,\delta_2}{X, Fb \succ Fa, Y}}{X \succ a = b, Y}{=R} \quad \dfrac{\overset{\vdots\,\delta'}{X' \succ Pa, Y'} \quad \overset{\vdots\,\delta''}{X'', Pb \succ Y''}}{a = b, X', X'' \succ Y', Y''}{=L_1}}{X, X', X'' \succ Y, Y', Y''}\mathit{Cut}$$

becomes

$$\dfrac{\dfrac{\overset{\vdots\,\delta'}{X' \succ Pa, Y'} \quad \overset{\vdots\,\delta_1[F/P]}{X, Pa \succ Pb, Y}}{X, X' \succ Pb, Y, Y'}\mathit{Cut} \quad \overset{\vdots\,\delta''}{X'', Pb \succ Y''}}{X, X', X'' \succ Y, Y', Y''}\mathit{Cut}$$

# But eliminating *Cut* hardly seems worth it!

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y} {=}R$$

$$\frac{X \succ Pa, Y \qquad X', Pb \succ Y'}{a = b, X, X' \succ Y, Y'} {=}L_1 \qquad \frac{X \succ Pb, Y \qquad X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'} {=}L_2$$

# But eliminating *Cut* hardly seems worth it!

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y} {=}R$$

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Violations of the subformula property *everywhere*.

# But eliminating *Cut* hardly seems worth it!

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Violations of the subformula property *everywhere*.

For *analytic* derivations, we need different rules.

# From Rules to *Axioms*: From $=R$ to *Refl*

$$\dfrac{}{\succ a = a}\,\textit{Refl}$$

# From Rules to *Axioms*: From $=R$ to *Refl*

$$\dfrac{}{\succ a = a}\ \textit{Refl}$$

$$\dfrac{Fa \succ Fa \qquad Fa \succ Fa}{\succ a = a}\ {=}R$$

Replace this:

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ X, Fa \succ Fb, Y\end{array}\qquad\begin{array}{c}\vdots\,\delta_2\\ X, Fb \succ Fa, Y\end{array}}{X \succ a = b, Y}\;{\scriptstyle =R}$$

Replace this:

$$\dfrac{\dfrac{\vdots\,\delta_1}{X, Fa \succ Fb, Y} \qquad \dfrac{\vdots\,\delta_2}{X, Fb \succ Fa, Y}}{X \succ a = b, Y}\;{=}R$$

With this:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\succ a = a}\,\mathit{Refl}}{\succ \lambda x.(a = x)a}\,\lambda R \qquad \dfrac{\vdots\,\delta_1[F/\lambda x.(a = x)]}{X, \lambda x.(a = x)a \succ \lambda x.(a = x)b, Y}}{X \succ \lambda x.(a = x)b, Y}\,\mathit{Cut}}{X \succ a = b, Y}\,\lambda R$$

# From $=L$ to $=L.ax$ and back.

$$\dfrac{}{a = b, Pa \succ Pb}\ {=}L.ax_1 \qquad \dfrac{}{a = b, Pb \succ Pa}\ {=}L.ax_2$$

# From $=L$ to $=L.ax$ and back.

$$\frac{}{a = b, Pa \succ Pb}{=}L.ax_1 \qquad \frac{}{a = b, Pb \succ Pa}{=}L.ax_2$$

$$\frac{Pa \succ Pa \qquad Pb \succ Pb}{a = b, Pa \succ Pb}{=}L_1 \qquad \frac{Pb \succ Pb \qquad Pa \succ Pa}{a = b, Pb \succ Pa}{=}L_2$$

# From $=L$ to $=L.ax$ and back.

$$\dfrac{}{a = b, Pa \succ Pb}\,{=}L.ax_1 \qquad \dfrac{}{a = b, Pb \succ Pa}\,{=}L.ax_2$$

$$\dfrac{\dfrac{X \succ Pa, Y \qquad \dfrac{}{a = b, Pa \succ Pb}\,{=}L.ax_1}{a = b, X \succ Pb, Y}\,Cut \qquad X', Pb \succ Y'}{a = b, X, X' \succ Y, Y'}\,Cut$$

$$\dfrac{\dfrac{X \succ Pb, Y \qquad \dfrac{}{a = b, Pb \succ Pa}\,{=}L.ax_2}{a = b, X \succ Pa, Y}\,Cut \qquad X', Pa \succ Y'}{a = b, X, X' \succ Y, Y'}\,Cut$$

# We can restrict $=$*L.ax* to *primitive* predicates

$$
\dfrac{\dfrac{\dfrac{}{a = b, Pa \succ Pb}{\scriptstyle =L.ax_1}}{a = b, Pa \land Qa \succ Pb}{\scriptstyle \land L} \qquad \dfrac{\dfrac{}{a = b, Qa \succ Qb}{\scriptstyle =L.ax_1}}{a = b, Pa \land Qa \succ Qb}{\scriptstyle \land L}}{\dfrac{a = b, Pa \land Qa \succ Pa \land Qb}{a = b, \lambda x.(Px \land Qx)a \succ \lambda x.(Px \land Qx)b}{\scriptstyle \lambda}}{\scriptstyle \land R}
$$

# We can restrict $=L.ax$ to *primitive* predicates

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a = b, Pb \succ Pa}\;{=}L.ax_2}{a = b, Pb, \neg Pa \succ}\;\neg L}{a = b, \neg Pa \succ \neg Pb}\;\neg R}{a = b, \lambda x.(\neg Px)a \succ \lambda x.(\neg Px)b}\;\lambda
$$

# We can restrict $=$*L.ax* to *primitive* predicates

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{a = b, Pac \succ Pbc}\,{\scriptstyle =L.ax_2}}{a = b, \forall y Pay \succ Pbc}\,{\scriptstyle \neg L}}{a = b, \forall y Pay \succ \forall y Pby}\,{\scriptstyle \neg R}}{a = b, \lambda x.(\forall y Pxy) a \succ \lambda x.(\forall y Pxy) b}\,\lambda
$$

# *Now* eliminate *Cut*

$$\dfrac{\dfrac{}{a = b, Pa \succ Pb}{=}L.ax_1 \qquad \dfrac{}{c = b, Pb \succ Pc}{=}L.ax_2}{a = b, c = b, Pa \succ Pc}\,Cut$$

# *Now* eliminate *Cut*

$$\dfrac{\dfrac{}{a = b, Pa \succ Pb}{=}L.ax_1 \qquad \dfrac{}{c = b, Pb \succ Pc}{=}L.ax_2}{a = b, c = b, Pa \succ Pc}\,Cut$$

becomes

$$\dfrac{}{a = b, c = b, Pa \succ Pc}{=}L??$$

# *Now* eliminate *Cut*

It suffices to close the *axioms* under *Cut*.

$$\dfrac{}{I^a_b, Pa \succ Pb}\ {=}L.ax^*$$

Where $I^a_b$ is any multiset
of identities *linking* $a$ *to* $b$,
and $P$ is any primitive predicate.

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Where $I^a_b$ is any multiset
of identities *linking* $a$ *to* $b$,
and $P$ is any primitive predicate.

(a) The *empty* multiset links $a$ to $a$.

(b) $a = b$ links $a$ to $b$ and $b$ to $a$.

(c) If $X$ links $a$ to $b$ and $Y$ links
$b$ to $c$ then $X, Y$ links $a$ to $c$.

# *Now* eliminate *Cut*

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Where $I^a_b$ is any multiset
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(a) The *empty* multiset links $a$ to $a$.

(b) $a = b$ links $a$ to $b$ and $b$ to $a$.

(c) If $X$ links $a$ to $b$ and $Y$ links
$b$ to $c$ then $X, Y$ links $a$ to $c$.

(We can leave '$Pa$' out if it is $a = a$.)

# Kinds of Identity Rules

$$\frac{X, Fa \succ Fb, Y \qquad X, Fb \succ Fa, Y}{X \succ a = b, Y} {=}Df$$

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- *Refl* and $=$*L.ax*$_*$ are *semantic constraints connecting* primitive predicates.
- They are *equivalent* as far as derivability goes.

To apply identity rules, we need to agree on what counts as a *singular term* and what counts as a *predicate*.

# DEFINING NECESSITY

# Modal reasoning trades on *supposition* and *context shift*

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Oswald shot Kennedy | Oswald didn't shoot Kennedy

The New York Times.

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LATE CITY EDITION

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+ Possible worlds semantics is useful in modelling *both* modal notions.
+ Let's consider both these options in the development of a *proof system*.

# An Analogy

A *modal operator* stands to a kind of *context shift*
as a *quantifier* stands to a category of *terms*.

# Hypersequents

$$\Box A \succ \quad | \quad \succ A$$

# A Defining Rule for Necessity

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The *form* of the rule is the same in either case, though the *content* differs.

# An example derivation

$$
\dfrac{\dfrac{\dfrac{\Box p \succ \Box p}{\succ p \mid \Box p \succ}\,\Box Df\!\uparrow}{\succ \Box p \mid \Box p \succ}\,\Box Df\!\downarrow \qquad \dfrac{\neg\Box p \succ \neg\Box p}{\Box p, \neg\Box p \succ}\,\neg Df\!\uparrow}{\dfrac{\dfrac{\neg\Box p \succ \mid \Box p \succ}{\neg\Box p \succ \mid \succ \neg\Box p}\,\neg Df\!\downarrow}{\neg\Box p \succ \Box\neg\Box p}\,\Box Df\!\downarrow}\,\mathit{Cut}
$$

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- Our everyday practice is messy and informal. *Formalising* an aspect of it helps us isolate distinctive features of that practice.
- None of this requires an antecedent commitment to an ontology of *worlds*.

# Combining Indicative and Subjunctive Shifts

You can combine the two kinds of context shift
into one hypersequent system easily enough

(see my "A Cut-Free Sequent System for Two-Dimensional Modal Logic")

but we don't need to worry about those details here.

# From $\Box Df$ to $\Box L/\Box R$

$$\dfrac{\succ A \mid X \succ Y \mid \mathcal{S}}{X \succ \Box A, Y \mid \mathcal{S}}\ \Box Df$$

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# From $\Box Df$ to $\Box L/\Box R$

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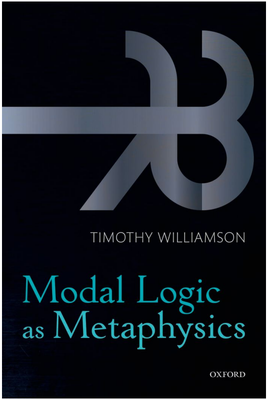
$$\frac{X, A \succ Y \mid \mathcal{S}}{\Box A \succ \ \mid X \succ Y \mid \mathcal{S}}\ \Box L$$

# COMBINING THEM

# Doesn't identity become necessary, *automatically*?

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\succ a = a}\,\mathit{Refl}}{\succ \Box(a = a)}\,\Box I}{\succ \lambda x.\Box(a = x)a}\,\lambda Df{\downarrow} \qquad \dfrac{}{a = b, \lambda x.\Box(a = x)a \succ \lambda x.\Box(a = x)b}\,{=}L.ax_1}{\dfrac{a = b \succ \lambda x.\Box(a = x)b}{a = b \succ \Box(a = b)}\,\lambda Df{\uparrow}}\,\mathit{Cut}
$$

# Willamson *defends* the necessity of identity



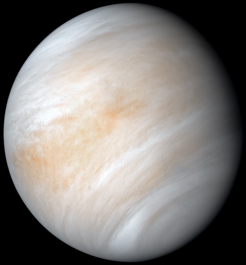
# Williamson on the necessity of identity

> . . . to mess with the modal or temporal logic of identity in order to avoid ontological inflation would be a lapse of methodological good taste, or good sense, for it means giving more weight to ontology than to the vastly better developed and more successful discipline of logic.
>
> More specifically, the classical modal or temporal logic is a strong, simple, and elegant theory. To weaken, complicate, and uglify it without overwhelming reason to do so merely in order to block the derivation of the necessity or permanence of identity would be as retrograde and wrong-headed a step in logic and metaphysics as natural scientists would consider a comparable sacrifice of those virtues in a physical theory.
>
> — Timothy Williamson, *Modal Logic as Metaphysics*, pp. 26, 27.

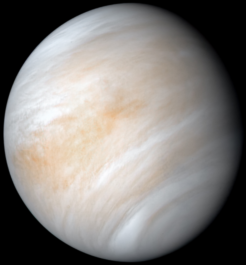
# Modal Logic with Contingent Identity

Giovanna Corsi “Counterpart Semantics” 2002

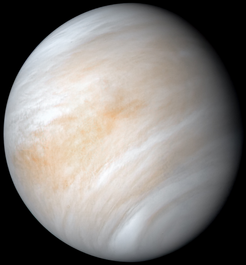

Maria Aloni “Individual Concepts in Modal Predicate Logic” 2005



Hesperus



Phosphorous



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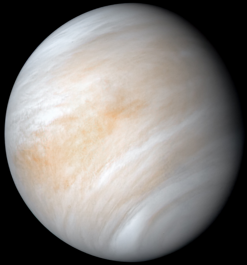

Phosphorous

$$h = p \mid h \neq p$$

# Identity and Context Shift

$$a = b \succ \quad | \; Fa \succ Fb$$

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(We *learned* that $h = p$. The possibility that $h \neq p$ was open to us.)

Subjunctive shifts (*planning*): This is much less plausible.

($h = p$. Were I to travel to $h$, then of necessity, I am going to $p$.)

# More from Williamson

. . . we are not interested in epistemic readings of ‘it is possible that’.

— Timothy Williamson, *Modal Logic as Metaphysics*, p. 11, fn. 17.

# Subjunctive Context Shifts: *Necessary* Identity Rules

$$\frac{X \succ Y \mid X', Fa \succ Fb, Y' \mid \mathcal{S} \qquad X \succ Y \mid X', Fb \succ Fa, Y' \mid \mathcal{S}}{X \succ a = b, Y \mid X' \succ Y' \mid \mathcal{S}} =\!Df$$

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$I^a_b$ is a multiset of identies, linking $a$ and $b$, distributed across the zones of the hypersequent, in the LHS of each sequent.

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(And again, we can leave $Pa$ out if it is $a = a$.)

To deny $a = b$ is to take $a$ and $b$ to be distinguish*able*.

# A Cut-Free Derivation

$$\dfrac{\dfrac{}{a = b \succ \ \mid\ \succ a = b}{=}L.ax_*}{a = b \succ \Box a = b}\Box R$$

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  - After all, it's no argument against $h = p$ that it's epistemically necessary that $h = h$ and not epistemically necessary that $h = p$.

# Two Kinds of Predicates

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- Feature predicates are closed under the classical connectives and quantifiers and $\lambda$.
- Applying $\lambda$ into $\Box_E$ contexts creates a *general* predicate.
- There are two kinds of *Spec* rule: one for feature predicates, and one for general predicates.

# Contingent Identity Rules

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(The rule $=Df$ is drawn with a double line.)

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# Two Dimensions — with *actuality*

$$\begin{array}{ccccccc}
X^1_1 \succ_{@} Y^1_1 & \mid & X^1_2 \succ Y^1_2 & \mid & \cdots & \mid & X^1_{m_1} \succ Y^1_{m_1} & \| \\
X^2_1 \succ_{@} Y^2_1 & \mid & X^2_2 \succ Y^2_2 & \mid & \cdots & \mid & X^2_{m_2} \succ Y^2_{m_2} & \| \\
\vdots & & \vdots & & & & \vdots & \\
X^n_1 \succ_{@} Y^n_1 & \mid & X^n_2 \succ Y^n_2 & \mid & \cdots & \mid & X^n_{m_n} \succ Y^n_{m_n} &
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The zones in correspond to *worlds*,
the formulas in the LHS are *true*
and those in the RHS are *false*.

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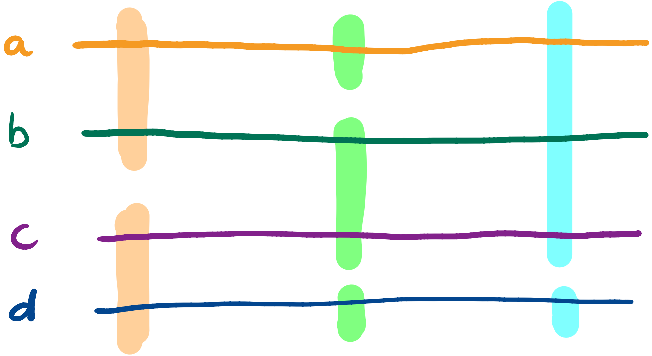
- Thes are regular models for constant domain S5 with necessary identity.
- (The usual rules for quantifiers work, too.)

# *Contingent* Identity Models

$$a = b, c = d \succ b = c \mid b = c \succ a = d \mid a = b, b = c \succ a = d$$

$$a = b, c = d \succ b = c \mid b = c \succ a = d \mid a = b, b = c \succ a = d$$



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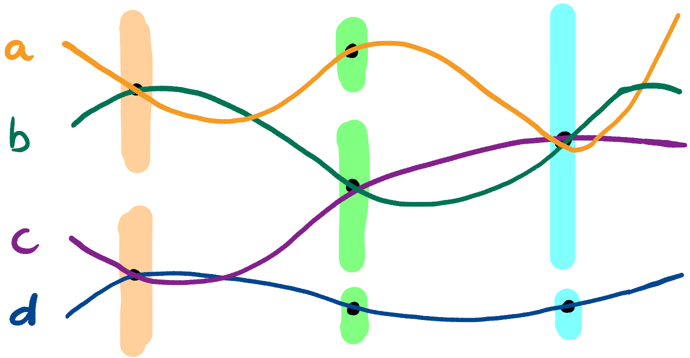

# *Contingent* Identity Models

Fa, Fb    Fc, Fd    Fa, Fd    Fb, Fc    Fa, Fb, Fc    Fd

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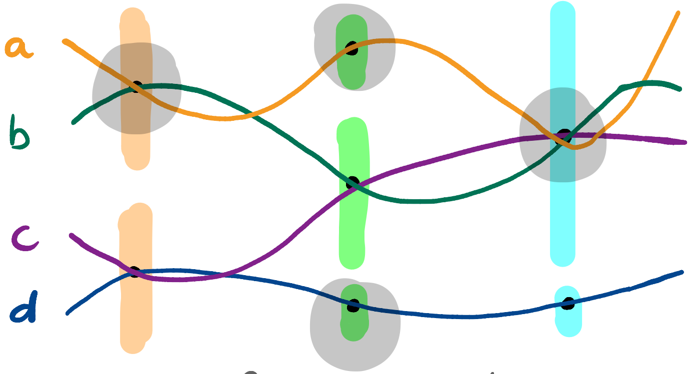
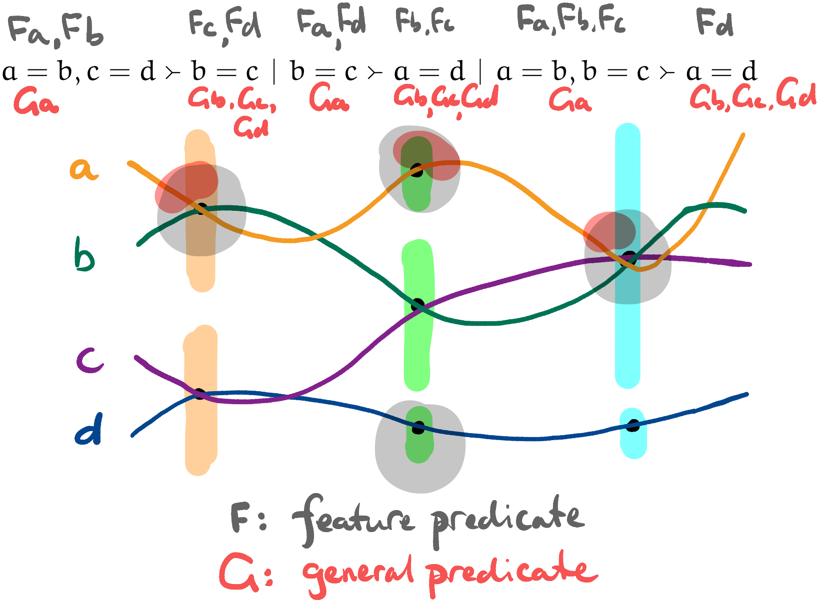



F: feature predicate

# *Contingent* Identity Models

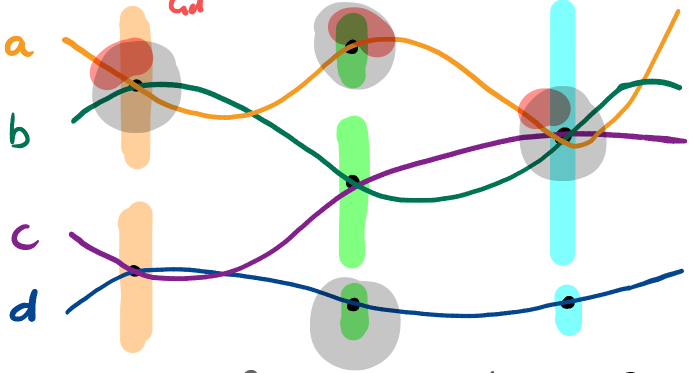


# *Contingent* Identity Models

Fa, Fb | Fc, Fd | Fa, Fd | Fb, Fc | Fa, Fb, Fc | Fd

$$a = b, c = d \succ b = c \mid b = c \succ a = d \mid a = b, b = c \succ a = d$$

Ga | Gb, Gc, Gd | Ga | Gb, Gc, Gd | Ga | Gb, Gc, Gd




F: feature predicate

G: general predicate

$Gx \equiv \Box Fx$

# THE UPSHOT

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- These distinctions can be explained and motivated in terms of our dialogical practice, and its aims. The result is a uniform treatment of epistemic and metaphysical modality and their distinctive interactions with identity.

# There is More to Do

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- Quantifiers
- Sortals, Conceptual Covers
- Existence
- Comparisons with other systems

# THANK YOU!