A Category of Classical Proofs

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THE UNIVERSITY OF MELBOURNE

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My Aim

To show how *proof terms* for classical propositional logic form a *category*, and to examine some of its properties.

Today's Plan


PROOF TERMS

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

There can be different ways to prove the same thing

$$p \land q \succ p \lor q$$

$$p \land \neg p \succ p \lor \neg p$$

PROOF TERMS FOR CLASSICAL DERIVATIONS

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Abstract: I give an account of *proof terms* for derivations in a sequent calculus for classical propositional logic. The term for a derivation δ of a sequent $\Sigma \succ \Delta$ encodes *how* the premises Σ and conclusions Δ are related in δ. This encoding is many–to–one in the sense that different derivations can have the same proof term, since different derivations may be different ways of representing the same underlying connection between premises and conclusions. However, not all proof terms for a sequent $\Sigma \succ \Delta$ are the same. There may be *different* ways to connect those premises and conclusions.

Proof terms can be simplified in a process corresponding to the elimination of cut inferences in sequent derivations. However, unlike cut elimination in the sequent calculus, each proof term has a *unique normal form* (from which all cuts have been eliminated) and it is straightforward to show that term reduction is strongly normalising—*every* reduction process terminates in that unique normal form. Furthermore, proof terms are *invariants* for sequent derivations in a strong sense—two derivations δ_1 and δ_2 have the same proof term *if and only if* some permutation of derivation steps sends δ_1 to δ_2 (given a rela-

$$\acute{\wedge}x \curvearrowright \acute{\wedge}\,\acute{\vee}y \quad \acute{\wedge}x \curvearrowright \acute{\wedge}\,\grave{\vee}y \quad \acute{\vee}\,\grave{\wedge}x \curvearrowright \grave{\wedge}\,\acute{\vee}y \quad \grave{\vee}\,\grave{\wedge}x \curvearrowright \grave{\wedge}\,\grave{\vee}y$$
$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

Proof Terms as Graphs on Sequents

$$\acute{\wedge}x \frown \acute{\wedge}\acute{\vee}y \quad \acute{\wedge}x \frown \acute{\wedge}\grave{\vee}y \quad \acute{\vee}\grave{\wedge}x \frown \grave{\wedge}\acute{\vee}y \quad \grave{\vee}\grave{\wedge}x \frown \grave{\wedge}\grave{\vee}y$$

$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \vee r)$$

$$(p \wedge q) \vee (p \wedge r)$$

Proof Terms as Graphs on Sequents

$$p \supset q$$

$$(q \supset r) \supset ((p \supset (q \wedge r)))$$

$(p \wedge q) \vee (p \wedge r)$

$(p \wedge q) \vee (p \wedge r)$

$p \wedge (q \vee r)$

$(p \wedge q) \vee (p \wedge r)$

$(p \wedge q) \vee (p \wedge r)$

That's not the Identity Proof

$(p \wedge q) \vee (p \wedge r)$ $\qquad$ $(p \wedge q) \vee (p \wedge r)$

$(p \wedge q) \vee (p \wedge r)$ $\qquad$ $(p \wedge q) \vee (p \wedge r)$

$q \wedge \neg r$

$p \supset q$

$(q \supset r) \supset (p \supset (q \wedge r))$

$q \wedge \neg r$

$(q \supset r) \supset (p \supset (q \wedge r))$

Bounds

$\top$ and $\perp$ are *interesting*.

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They act like $p \vee \neg p$ and $q \wedge \neg q$,
except without internal structure.

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$$p \wedge \neg p \qquad\qquad p$$

$$q \qquad\qquad q \vee \neg q$$

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$$p \wedge \neg p \qquad\qquad p \qquad\qquad \bot$$

$$q \qquad\qquad q \vee \neg q \qquad\qquad q$$

Bounds

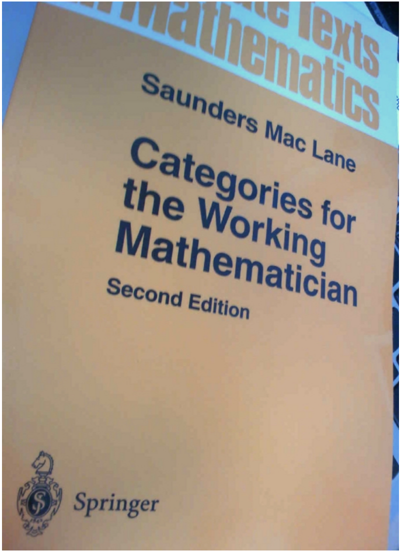
$\top$ and $\bot$ are *interesting*.

They act like $p \lor \neg p$ and $q \land \neg q$,
except without internal structure.

$$\begin{array}{cccc} p \land \neg p & p & \bot & p \\ & & & \\ q & q \lor \neg q & q & \top \end{array}$$


CATEGORIES

Categories as a Unifying Mathematical Vocabulary



Objects, Arrows, Composition and Identities

A category consists of a class of *objects* and *arrows*.

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- For each object A, there is an *identity arrow* $id_A : A \to A$
  - such that $f \circ id_A = f$, when $f : A \to B$ and $id_A \circ g = g$ when $g : C \to A$.

Many properties are definable at this level of generality

$f : A \to B$ is an *isomorphism* if
there is some arrow $g : B \to A$
where $g \circ f = id_A$ and $f \circ g = id_B$.

Many properties are definable at this level of generality

$\mathsf{f} : A \to \mathsf{B}$ is an *isomorphism* if

there is some arrow $\mathsf{g} : \mathsf{B} \to A$

where $\mathsf{g} \circ \mathsf{f} = id_{\mathsf{A}}$ and $\mathsf{f} \circ \mathsf{g} = id_{\mathsf{B}}$.

We write '$\mathsf{f} : A \xrightarrow{\sim} \mathsf{B}$' to show that f is an isomorphism.

Examples of Categories

- Sets & *functions*.

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- Any monoid.
  - *Objects*: one object. *Arrows*: an arrow for each element in the monoid, with the identity arrow as the monoid identity, and composition the monoid operation.

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- Any monoid.
  - *Objects*: one object. *Arrows*: an arrow for each element in the monoid, with the identity arrow as the monoid identity, and composition the monoid operation.
- Formulas & *proofs*.


THE PROOF TERM
CATEGORY

OBJECTS: *Formulas* — ARROWS: *Proof Terms without Cuts*

- $\pi : A \to B$ iff $\pi(x)[y]$ is a *cut-free* proof for $x : A \succ y : B$.
- $id_A : A \to A$ is the identity proof term $x \curvearrowright y$ of type A.
- Composition is chaining proofs & elimination of cuts.
  - If $\pi : A \to B$ and $\tau : B \to C$ then $\tau \circ \pi : A \to C$ is $(\pi(x)[\bullet]\ \tau(\bullet)[y])^*$.
- Composition is associative.
- Identity proofs are identities in the category:
  - $(\pi(x)[\bullet]\ \bullet \curvearrowright y)^* = \pi(x)[y]$, and $(x \curvearrowright \bullet\ \ \pi(\bullet)[y])^* = \pi(x)[y]$, when π is cut-free.

How Identity Proofs Compose

$$p \supset q$$

$$(q \supset r) \supset (p \supset (q \wedge r))$$

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How Identity Proofs Compose

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We have a Category

Proof Terms—in general

- π has type $\Sigma \succ \Delta$.
- Proofs are SET–SET.
- Proofs include *Cuts*.

$$\dfrac{\overset{x \curvearrowright x}{x : A \succ y : A} \qquad \overset{\pi(x)[y]}{x : A \succ y : B}}{\overset{x \curvearrowright \bullet \quad \pi(\bullet)[y]}{x : A \succ y : B}}\ \mathit{Cut}$$

Categories of Cut-Free Terms

- $\pi : A \to B$.
- Proofs are FMLA–FMLA.
- Proofs have no *Cuts*.

$$\dfrac{A \xrightarrow{id_A} A \qquad A \xrightarrow{\pi} B}{A \xrightarrow{\pi \circ id_A \;=\; \pi} B}$$

What is the proof term category *like*?

Cartesian Products

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

Cartesian Products

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Cartesian Products

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Cartesian Products

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\ & C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] & \end{tikzcd}$$

$$\pi_1 \circ \langle f, g \rangle = f \quad \pi_2 \circ \langle f, g \rangle = g$$

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\end{tikzcd}$$

$$\pi_1 \circ \langle f, g \rangle = f \quad \pi_2 \circ \langle f, g \rangle = g$$

This looks a *lot* like conjunction.

Many interesting categories have cartesian products

Final Objects — the empty product

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Final Objects — the empty product

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\top \\
C
\end{tikzcd}
$$

Final Objects — the empty product

$$
\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\top \\
C \arrow[u, dashed]
\end{tikzcd}
$$

Coproducts and Terminal Objects

$$
\begin{tikzcd}
A \arrow[r, "\iota_1"] \arrow[dr, "f"'] & A + B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\
& C &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bot \arrow[d, dashed] \\
C
\end{tikzcd}
$$

$$f : A \times B \to C \qquad \tilde{f} : A \to B \supset C \qquad ev : (B \supset C) \times B \to C$$

$$\begin{tikzcd}
(B \supset C) \times B \arrow[r, "ev"] & C \\
A \times B \arrow[u, "\tilde{f} \times id"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Cartesian Closed Categories...

...model intuitionistic logic.

...model intuitionistic logic.

They *collapse* when made classical.

So what *is* the proof term category?


IT’S NOT CARTESIAN

$\top$ is not Terminal, $\perp$ is not Initial

T is *terminal* iff for each C there's a *unique* arrow C $\rightarrow$ T.

$\top$ is not Terminal, $\bot$ is not Initial

T is *terminal* iff for each C there's a *unique* arrow C $\to$ T.

$$p \wedge \neg p \qquad\qquad p \wedge \neg p$$

$$\top \qquad\qquad\qquad \top$$

$\top$ is not Terminal, $\bot$ is not Initial

T is *terminal* iff for each C there's a *unique* arrow C → T. $\top$ is not terminal.

$p \wedge \neg p$ $\qquad\qquad$ $p \wedge \neg p$

$\top$ $\qquad\qquad$ $\top$

$\top$ is not Terminal, $\bot$ is not Initial

T is *terminal* iff for each C there's a *unique* arrow $C \to T$. $\top$ is not terminal.

I is *initial* iff for each C there's a *unique* arrow $I \to C$.

$$p \wedge \neg p \qquad\qquad p \wedge \neg p$$

$$\top \qquad\qquad \top$$

T is *terminal* iff for each C there's a *unique* arrow $C \to T$. $\top$ is not terminal.

I is *initial* iff for each C there's a *unique* arrow $I \to C$.

$$p \wedge \neg p \qquad p \wedge \neg p \qquad \bot \qquad \bot$$

$$\top \qquad \top \qquad q \vee \neg q \qquad q \vee \neg q$$

$\top$ is not Terminal, $\bot$ is not Initial

T is *terminal* iff for each C there's a *unique* arrow $C \to T$. $\top$ is not terminal.

I is *initial* iff for each C there's a *unique* arrow $I \to C$. $\bot$ is not initial.

$p \wedge \neg p$	$p \wedge \neg p$	$\bot$	$\bot$
$\top$	$\top$	$q \vee \neg q$	$q \vee \neg q$

Conjunction isn't Cartesian Product

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Conjunction isn't Cartesian Product

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\ & C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g\rangle"] \arrow[ur, "g"'] & \end{tikzcd}$$

We *do* have projection arrows.

$$A \wedge B \qquad\qquad A \wedge B$$

$$A \qquad\qquad\qquad B$$

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A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

We *do* have projection arrows.

And we have a conjunctive arrow.

C

$C \qquad\qquad C$

$A \qquad\qquad B$

$A \wedge B$

$f \star g$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& C \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"] \arrow[ur, "g"'] &
\end{tikzcd}$$

We *do* have projection arrows.

And we have a conjunctive arrow.

$$\begin{array}{ccc} & C & \\ C & & C \\ A & & B \\ & A \wedge B & \end{array}$$

$f \star g$

But their composition need not restore the original arrows f and g.

An Example

$$\begin{array}{cc} \color{darkred}{f} & \color{darkred}{g} \\ p \wedge \neg p & p \wedge \neg p \\ & \\ p & q \end{array}$$

An Example

$$f \qquad\qquad g \qquad\qquad \langle f, g\rangle$$

$$p \land \neg p \qquad p \land \neg p \qquad p \land \neg p$$

$$p \qquad\qquad q \qquad\qquad p \land q$$

f	g	$\langle f, g\rangle$	$\pi_1 \circ \langle f, g\rangle$
$p \wedge \neg p$	$p \wedge \neg p$	$p \wedge \neg p$	$p \wedge \neg p$
			$p \wedge q$
p	q	$p \wedge q$	p

An Example

f	g	$\langle f, g \rangle$	$\pi_1 \circ \langle f, g \rangle$	$\pi_1 \circ \langle f, g \rangle$
$p \wedge \neg p$	$p \wedge \neg p$	$p \wedge \neg p$	$p \wedge \neg p$	$p \wedge \neg p$
			$p \wedge q$	
p	q	$p \wedge q$	p	p

So, what is the category *like*?


IT IS MONOIDAL,
& MORE…

Monoidal Categories

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Tensor product — $\otimes$ — in vector spaces is an important example.

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Tensor product — $\otimes$ — in vector spaces is an important example.

This motivates the definition of a *monoidal* category.

Symmetric Monoidal Categories

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad 1 \in Ob(\mathcal{C})$$

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$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad 1 \in Ob(\mathcal{C})$$

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$$

$$\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A \qquad \iota_A : 1 \otimes A \xrightarrow{\sim} A$$

where *associativity* (α), *symmetry* (σ) and *unit* (ι) behave sensibly.

Associativity

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A\otimes B,C,D}"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "\alpha_{A,B,C\otimes D}"] \arrow[d, "id_A \otimes \alpha_{B,C,D}"'] & & ((A \otimes B) \otimes C) \otimes D \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes id_D"']
\end{tikzcd}$$

Associativity

$$\begin{tikzcd}[column sep=small]
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A\otimes B,C,D}"] & \\
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A \otimes ((B \otimes C) \otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes id_D"']
\end{tikzcd}$$

(The 'Pentagon')

Symmetry

$$\begin{tikzcd} A \otimes B \arrow[rr, bend left, "\sigma_{A,B}"] & & B \otimes A \arrow[ll, bend left, "\sigma_{B,A}"] \end{tikzcd}$$

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$$\begin{tikzcd} A \otimes B \arrow[rr, bend left, "\sigma_{A,B}"] & & B \otimes A \arrow[ll, bend left, "\sigma_{B,A}"] \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\sigma_{A,B} \otimes id_C"'] & A \otimes (B \otimes C) \arrow[r, "\sigma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\ (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "id_B \otimes \sigma_{A,C}"'] & B \otimes (C \otimes A) \end{tikzcd}$$

(The 'Hexagon')

Symmetry

$$\begin{tikzcd} A \otimes B \arrow[r, bend left, "\sigma"] & B \otimes A \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha"] \arrow[d, "\sigma\otimes id"'] & A \otimes (B \otimes C) \arrow[r, "\sigma"] & (B \otimes C) \otimes A \arrow[d, "\alpha"] \\ (B \otimes A) \otimes C \arrow[r, "\alpha"'] & B \otimes (A \otimes C) \arrow[r, "id\otimes\sigma"'] & B \otimes (C \otimes A) \end{tikzcd}$$

(The 'Hexagon')

Let's drop subscripts when there is no ambiguity.

Unit

$$
\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[r, "\alpha"] \arrow[d, "\sigma\otimes id"'] & A \otimes (1 \otimes B) \arrow[d, "id\otimes\iota"] \\
(1 \otimes A) \otimes B \arrow[r, "\iota\otimes id"'] & A \otimes B
\end{tikzcd}
$$

(The ‘Square’)

Proof Terms are a Symmetric Monoidal Category under $\wedge/\top$

$$\wedge : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \qquad \top \in Ob(\mathcal{T})$$

Proof Terms are a Symmetric Monoidal Category under $\wedge/\top$

$$\wedge : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T} \qquad \top \in Ob(\mathfrak{T})$$

$$\hat{\alpha} : A \wedge (B \wedge C) \xrightarrow{\sim} (A \wedge B) \wedge C$$

$$\hat{\sigma} : A \wedge B \xrightarrow{\sim} B \wedge A \qquad \hat{\iota} : \top \wedge A \xrightarrow{\sim} A$$

and indeed, *associativity* ($\hat{\alpha}$), *symmetry* ($\hat{\sigma}$) and *unit* ($\hat{\iota}$) behave sensibly.

$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$

$$\hat{\alpha}_{A,B,C}$$

$$A \wedge (B \wedge C)$$

$$(A \wedge B) \wedge C$$

$$\hat{\sigma}_{A,B}$$

$$A \wedge B$$

$$B \wedge A$$

$$\hat{\iota}$$

$$\top \wedge A$$

$$A$$

$\hat{\alpha}$, $\hat{\sigma}$ and $\hat{\iota}$ are isomorphisms

$\hat{\alpha}^{-1} \circ \hat{\alpha}$

$A \wedge (B \wedge C)$

$(A \wedge B) \wedge C$

$A \wedge (B \wedge C)$

id

$A \wedge (B \wedge C)$

$A \wedge (B \wedge C)$

$\hat{\sigma}^{-1} \circ \hat{\sigma}$

$A \wedge B$

$B \wedge A$

$A \wedge B$

id

$A \wedge B$

$A \wedge B$

$\hat{\iota}^{-1} \circ \hat{\iota}$

$\top \wedge A$

A

$\top \wedge A$

$\downarrow$

$=$

id

$\top \wedge A$

$\top \wedge A$

$\downarrow$

$=$

$\hat{\iota} \circ \hat{\iota}^{-1}$

A

$\top \wedge A$

A

$\downarrow$

$=$

id

A

A

The *Pentagon*, *Hexagon*, *Square*, etc., commute

$$\begin{tikzcd}
(A \wedge \top) \wedge B \arrow[r, "\hat{\alpha}"] \arrow[d, "\hat{\sigma} \wedge id"'] & A \wedge (\top \wedge B) \arrow[d, "id \wedge \hat{\iota}"] \\
(\top \wedge A) \wedge B \arrow[r, "\hat{\iota} \wedge id"'] & A \wedge B
\end{tikzcd}$$

$(\hat{\iota} \wedge id) \circ (\hat{\sigma} \wedge id)$	$=$	$(id \wedge \hat{\iota}) \circ \hat{\alpha}$
$(A \wedge \top) \wedge B$	$(A \wedge \top) \wedge B$	$(A \wedge \top) \wedge B$
$(\top \wedge A) \wedge B$		$A \wedge (\top \wedge B)$
$A \wedge B$	$A \wedge B$	$A \wedge B$

The *Pentagon*, *Hexagon*, *Square*, etc., commute

$$
\begin{tikzcd}
& (A \wedge B) \wedge (C \wedge D) \arrow[dr, "\hat{\alpha}"] & \\
A \wedge (B \wedge (C \wedge D)) \arrow[ur, "\hat{\alpha}"] \arrow[d, "id \wedge \hat{\alpha}"'] & & ((A \wedge B) \wedge C) \wedge D \\
A \wedge ((B \wedge C) \wedge D) \arrow[rr, "\hat{\alpha}"'] & & (A \wedge (B \wedge C)) \wedge D \arrow[u, "\hat{\alpha} \wedge id"']
\end{tikzcd}
$$

The *Pentagon*, *Hexagon*, *Square*, etc., commute

$$
\begin{tikzcd}
(A \wedge B) \wedge C \arrow[r, "\hat{\alpha}"] \arrow[d, "\hat{\sigma} \wedge id"'] & A \wedge (B \wedge C) \arrow[r, "\hat{\sigma}"] & (B \wedge C) \wedge A \arrow[d, "\hat{\alpha}"] \\
(B \wedge A) \wedge C \arrow[r, "\hat{\alpha}"'] & B \wedge (A \wedge C) \arrow[r, "id \wedge \hat{\sigma}"'] & B \wedge (C \wedge A)
\end{tikzcd}
$$

Proof Terms are a Symmetric Monoidal Category under $\vee/\bot$

$$\vee : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \qquad \bot \in Ob(\mathcal{T})$$

$$\check{\alpha} : A \vee (B \vee C) \xrightarrow{\sim} (A \vee B) \vee C$$

$$\check{\sigma} : A \vee B \xrightarrow{\sim} B \vee A \qquad \check{\iota} : \bot \vee A \xrightarrow{\sim} A$$

and *associativity* ($\check{\alpha}$), *symmetry* ($\check{\sigma}$) and *unit* ($\check{\iota}$) behave *just as* sensibly.

The operators $\wedge$ and $\vee$ are connected by δ and δ'

$$\delta : A \wedge (B \vee C) \to (A \wedge B) \vee C \qquad \delta' : (A \vee B) \wedge C \to A \vee (B \wedge C)$$

The operators $\wedge$ and $\vee$ are connected by δ and δ'

$$\delta : A \wedge (B \vee C) \to (A \wedge B) \vee C \qquad \delta' : (A \vee B) \wedge C \to A \vee (B \wedge C)$$

If the operators are *symmetric*, then we need only one.

$$\begin{tikzcd}
(A \vee B) \wedge C \arrow[r, "\delta'"] \arrow[d, "\hat{\sigma}"'] & A \vee (B \wedge C) \\
C \wedge (A \vee B) \arrow[d, "id \wedge \check{\sigma}"'] & (B \wedge C) \vee A \arrow[u, "\check{\sigma}"'] \\
C \wedge (B \vee A) \arrow[r, "\delta"'] & (C \wedge B) \vee A \arrow[u, "\hat{\sigma} \vee id"']
\end{tikzcd}$$

δ and δ' are *obvious* proof terms

δ

$$A \wedge (B \vee C)$$

$$(A \wedge B) \vee C$$

δ'

$$(A \vee B) \wedge C$$

$$A \vee (B \wedge C)$$

Linear Distributivity Conditions

$$\begin{tikzcd}
\top \wedge (A \vee B) \arrow[d, "\delta"'] \arrow[dr, "\hat{\iota}"] & \\
(\top \wedge A) \vee B \arrow[r, "\hat{\iota} \vee id"'] & A \vee B
\end{tikzcd}$$

$$\begin{tikzcd}
(A \wedge B) \wedge (C \vee D) \arrow[r, "\hat{\alpha}"] \arrow[dd, "\delta"'] & A \wedge (B \wedge (C \vee D)) \arrow[d, "id \wedge \delta"] \\
 & A \wedge ((B \wedge C) \vee D) \arrow[d, "\delta"] \\
((A \wedge B) \wedge C) \vee D \arrow[r, "\hat{\alpha} \vee id"'] & (A \wedge (B \wedge C)) \vee D
\end{tikzcd}$$

$$\begin{tikzcd}
((A \vee B) \wedge C) \vee D \arrow[d, "\delta' \vee id"'] & (A \vee B) \wedge (C \vee D) \arrow[l, "\delta"'] \arrow[r, "\delta'"] & A \vee (B \wedge (C \vee D)) \arrow[d, "id \vee \delta"] \\
(A \vee (B \wedge C)) \vee D \arrow[rr, "\alpha"'] & & A \vee ((B \wedge C) \vee D)
\end{tikzcd}$$

Linear Distributivity Conditions

$$
\begin{tikzcd}
\top \wedge (A \vee B) \arrow[d, "\delta"'] \arrow[dr, "\hat{\iota}"] & \\
(\top \wedge A) \vee B \arrow[r, "\hat{\iota} \vee id"'] & A \vee B
\end{tikzcd}
$$

$$
\begin{tikzcd}
(A \wedge B) \wedge (C \vee D) \arrow[r, "\hat{\alpha}"] \arrow[dd, "\delta"'] & A \wedge (B \wedge (C \vee D)) \arrow[d, "id \wedge \delta"] \\
& A \wedge ((B \wedge C) \vee D) \arrow[d, "\delta"] \\
((A \wedge B) \wedge C) \vee D \arrow[r, "\hat{\alpha} \vee id"'] & (A \wedge (B \wedge C)) \vee D
\end{tikzcd}
$$

$$
\begin{tikzcd}
((A \vee B) \wedge C) \vee D \arrow[d, "\delta' \vee id"'] & (A \vee B) \wedge (C \vee D) \arrow[l, "\delta"'] \arrow[r, "\delta'"] & A \vee (B \wedge (C \vee D)) \arrow[d, "id \vee \delta"] \\
(A \vee (B \wedge C)) \vee D \arrow[rr, "\alpha"'] & & A \vee ((B \wedge C) \vee D)
\end{tikzcd}
$$

(These diagrams *clearly* commute in the proof term category.)

There are a number of ways to define *Star-Autonomous* Categories.

We have a $\neg A$ for each object A, and two sets of arrows.

$$\gamma_A : A \wedge \neg A \to \bot \qquad \tau_A : \top \to \neg A \vee A$$

There are a number of ways to define *Star-Autonomous* Categories.

We have a $\neg A$ for each object A, and two sets of arrows.

$$\gamma_A : A \wedge \neg A \to \bot \qquad \tau_A : \top \to \neg A \vee A$$

These arrows have natural proof terms.

γ $\qquad\qquad$ τ

$A \wedge \neg A$ $\qquad\qquad$ $\top$

$\bot$ $\qquad\qquad$ $\neg A \vee A$

These Diagrams Must Commute

$$\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}$$

$$\begin{tikzcd}
(\neg A \vee A) \wedge \neg A \arrow[r, "\delta'"] & \neg A \vee (A \wedge \neg A) \arrow[r, "id \vee \gamma"] & \neg A \vee \bot \arrow[d, "\check{\iota}"] \\
\top \wedge \neg A \arrow[u, "\tau \wedge id"] \arrow[rr, "\hat{\iota}"'] & & \neg A
\end{tikzcd}$$

These Diagrams Must Commute

$$\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}$$

$$\begin{tikzcd}
(\neg A \vee A) \wedge \neg A \arrow[r, "\delta'"] & \neg A \vee (A \wedge \neg A) \arrow[r, "id \vee \gamma"] & \neg A \vee \bot \arrow[d, "\check{\iota}"] \\
\top \wedge \neg A \arrow[u, "\tau \wedge id"] \arrow[rr, "\hat{\iota}"'] & & \neg A
\end{tikzcd}$$

These *aren't* so obviously commutative as proof terms.

The negation diagrams commute in the proof term category

$$
\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}
$$

The negation diagrams commute in the proof term category

$$\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}$$

$(id \wedge \tau)$

$A \wedge \top$

$A \wedge (\neg A \vee A)$

The negation diagrams commute in the proof term category

$$\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}$$

$\delta \circ (id \wedge \tau)$

$A \wedge \top$

$A \wedge (\neg A \vee A)$

$(A \wedge \neg A) \vee A$

The negation diagrams commute in the proof term category

$$
\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}
$$

$(\gamma \vee id) \circ \delta \circ (id \wedge \tau)$

$A \wedge \top$

$A \wedge (\neg A \vee A)$

$(A \wedge \neg A) \vee A$

$\bot \vee A$

$$\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}$$

$\check{\iota} \circ (\gamma \vee id) \circ \delta \circ (id \wedge \tau)$

$$A \wedge \top$$

$$A \wedge (\neg A \vee A)$$

$$(A \wedge \neg A) \vee A$$

$$\bot \vee A$$

$$A$$

The negation diagrams commute in the proof term category

$$
\begin{tikzcd}
A \wedge (\neg A \vee A) \arrow[r, "\delta"] & (A \wedge \neg A) \vee A \arrow[r, "\gamma \vee id"] & \bot \vee A \arrow[d, "\check{\iota}"] \\
A \wedge \top \arrow[u, "id \wedge \tau"] \arrow[rr, "\hat{\iota}"'] & & A
\end{tikzcd}
$$

$\check{\iota} \circ (\gamma \vee id) \circ \delta \circ (id \wedge \tau)$ $\hat{\iota}$

$A \wedge \top$ $A \wedge \top$

$A \wedge (\neg A \vee A)$

$(A \wedge \neg A) \vee A$

$\bot \vee A$

A A

These categories model the multiplicative fragment of linear logic.

I won't pause now to explain how $A \supset B$, definable as $\neg A \vee B$ (or as $\neg(A \wedge \neg B)$, to which it's isomorphic) is a right adjoint to $\wedge$.

We can do more

Our proof terms allow *contraction* and *weakening*.

Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \to A$$

$$\overset{\perp}{\beta}_A : \perp \to A$$

∇_A $\qquad\qquad$ $\overset{\perp}{\beta}_A$

$A \vee A$ $\qquad\qquad$ $\perp$

A $\qquad\qquad$ A

Weakening and Contraction Monoids and Comonoids

$$\nabla_A : A \vee A \to A \qquad\qquad \Delta_A : A \to A \wedge A$$

$$\overset{\perp}{\beta}_A : \perp \to A \qquad\qquad \overset{\top}{\beta}_A : A \to \top$$

∇_A	$\overset{\perp}{\beta}_A$	δ_A	$\overset{\top}{\beta}_A$
$A \vee A$	$\perp$	A	A
A	A	$A \wedge A$	$\top$

Contraction and Weakening Conditions

$$
\begin{tikzcd}
(A \vee A) \vee A \arrow[rr, "\check{\alpha}"] \arrow[d, "\nabla \vee id"'] & & A \vee (A \vee A) \arrow[d, "id \vee \nabla"] \\
A \vee A \arrow[r, "\nabla"'] & A & A \vee A \arrow[l, "\nabla"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \vee \perp \arrow[r, "id \vee \overset{\perp}{\beta}"] \arrow[d, "\check{\sigma}"'] & A \vee A \arrow[d, "\nabla"'] & \perp \vee A \arrow[l, "\overset{\perp}{\beta} \vee id"'] \arrow[dl, "\check{\iota}"] \\
\perp \vee A \arrow[r, "\check{\iota}"] & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \vee A \arrow[rr, "\check{\sigma}"] \arrow[dr, "\nabla"'] & & A \vee A \arrow[dl, "\nabla"] \\
& A &
\end{tikzcd}
$$

Structurality for ∇ and $\overset{\perp}{\beta}$

$$
\begin{tikzcd}
(A \vee B) \vee (A \vee B) \arrow[r, "\alpha"] \arrow[ddd, "\nabla"] & A \vee (B \vee (A \vee B)) \arrow[r, "id \vee \alpha"] & A \vee ((B \vee A) \vee B) \arrow[d, "id \vee (\sigma \vee id)"] \\
& & A \vee ((A \vee B) \vee B) \arrow[d, "id \vee \alpha"] \\
& & A \vee (A \vee (B \vee B)) \arrow[d, "\alpha"] \\
A \vee B & & (A \vee A) \vee (B \vee B) \arrow[ll, "\nabla \vee \nabla"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\perp \arrow[rr, "\check{\iota}"] \arrow[dr, "\overset{\perp}{\beta}"'] & & \perp \vee \perp \arrow[dl, "\overset{\perp}{\beta} \vee \overset{\perp}{\beta}"] \\
& A \vee B &
\end{tikzcd}
$$

Structurality for ∇ and $\overset{\perp}{\beta}$

$$\begin{tikzcd} \perp \vee \perp \arrow[r, bend left, "\nabla_\perp"] \arrow[r, bend right, "\overset{\vee}{\iota}_\perp"'] & \perp \end{tikzcd} \qquad \begin{tikzcd} \perp \arrow[r, bend left, "\overset{\perp}{\beta}_\perp"] \arrow[r, bend right, "id_\perp"'] & \perp \end{tikzcd}$$

Structurality for ∇ and $\overset{\perp}{\beta}$

$$\begin{tikzcd} \perp \vee \perp \arrow[r, bend left, "\nabla_\perp"] \arrow[r, bend right, "\overset{\vee}{\iota}_\perp"'] & \perp \end{tikzcd} \qquad \begin{tikzcd} \perp \arrow[r, bend left, "\overset{\perp}{\beta}_\perp"] \arrow[r, bend right, "id_\perp"'] & \perp \end{tikzcd}$$

All these conditions are straightforward to verify for proof terms.

And dually for Δ and $\overset{\top}{\beta}$.

$$
\begin{array}{ccc}
 & A & \\
A & & A \\
B & & B \\
 & B &
\end{array}
$$

$f \quad g$

Blend

$$A \qquad\qquad\qquad\qquad A$$

$$A \qquad A$$

$$B \qquad B$$

$$B \qquad\qquad\qquad\qquad B$$

$f\ g \qquad\qquad f \cup g$

$$\begin{array}{ccc} & A & \\ A & & A \\ B & & B \\ & B & \end{array} \qquad\qquad \begin{array}{c} A \\ \\ \\ B \end{array}$$

$f \quad g$ $\qquad\qquad$ $f \cup g$

$\cup$ is a semilattice join on $\mathrm{Hom}(A, B)$.

$$(f \cup f') \circ g = (f \circ g) \cup (f' \circ g) \qquad f \circ (g \cup g') = (f \circ g) \cup (f \circ g')$$

A $\qquad\qquad\qquad\qquad\qquad\qquad$ A

$A \qquad A$

$B \qquad B$

B $\qquad\qquad\qquad\qquad\qquad\qquad$ B

$f \quad g$ $\qquad\qquad\qquad\qquad$ $f \cup g$

$\cup$ is a semilattice join on $\mathrm{Hom}(A, B)$.

$$(f \cup f') \circ g = (f \circ g) \cup (f' \circ g) \qquad f \circ (g \cup g') = (f \circ g) \cup (f \circ g')$$

The term category $\mathcal{T}$ is *enriched in* SLat.

Classical categories are
star autonomous categories
with *structural monoids* and *comonoids*,
enriched in SLat.

Isomorphisms

$$A \wedge \top \cong A \qquad A \wedge B \cong B \wedge A \qquad A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$$

$$A \vee \bot \cong A \qquad A \vee B \cong B \vee A \qquad A \vee (B \vee C) \cong (A \vee B) \vee C$$

$$\neg(A \wedge B) \cong (\neg A \vee \neg B) \qquad \neg(A \vee B) \cong (\neg A \wedge \neg B) \qquad \neg\neg A \cong A$$

$$\neg\top \cong \bot \qquad \neg\bot \cong \top \qquad \top \vee \top \cong \top \qquad \bot \wedge \bot \cong \bot$$

Isomorphisms

$$A \wedge \top \cong A \qquad A \wedge B \cong B \wedge A \qquad A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$$

$$A \vee \bot \cong A \qquad A \vee B \cong B \vee A \qquad A \vee (B \vee C) \cong (A \vee B) \vee C$$

$$\neg(A \wedge B) \cong (\neg A \vee \neg B) \qquad \neg(A \vee B) \cong (\neg A \wedge \neg B) \qquad \neg\neg A \cong A$$

$$\neg\top \cong \bot \qquad \neg\bot \cong \top \qquad \top \vee \top \cong \top \qquad \bot \wedge \bot \cong \bot$$

$$p \ncong p \wedge p \qquad p \ncong p \vee p \qquad p \wedge (q \vee r) \ncong (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (p \vee q) \ncong p \vee (p \wedge q) \qquad p \vee \neg p \ncong \top \qquad p \wedge \neg p \ncong \bot$$


FURTHER WORK

To Do List

- Finish the completeness proof, to the effect that $\mathcal{T}_{\mathcal{L}}$ is the free classical category on $\mathcal{L}$.
- Explore other examples of classical categories.
- Extend to first order predicate logic.

THANK YOU!

http://consequently.org/presentation/2017/a-category-of-classical-proofs-logicmelb

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