Defining Rules for Quantifiers & Identity

Greg Restall

University of St Andrews

PPLV @ UCL ☆ February 20, 2025

1. DEFINING RULES
2. QUANTIFIERS & GENERALITY
3. DEFINING RULES for FREE QUANTIFIERS
4. GOING WIDER, TOO...
5. WHAT is A QUANTIFIER?
   on TERMS & VALUES
6. IDENTITY & PROPERTIES

1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

6. IDENTITY & PROPERTIES

(How) Do Sequent Rules Characterise Concepts?

$$\frac{X \succ A, Y}{X \succ A \,\mathrm{tonk}\, B, Y}\,\mathrm{tonk}R \qquad \frac{X, B \succ Y}{X, A \,\mathrm{tonk}\, B \succ Y}\,\mathrm{tonk}L$$

Too strong!

$$\frac{p \succ p \,\mathrm{tonk}\, q \quad p \,\mathrm{tonk}\, q \succ q}{p \succ q}$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \,\mathrm{plink}\, B, Y}\,\mathrm{plink}R \qquad \frac{X, A \succ Y \quad X, B \succ Y}{X, A \,\mathrm{plink}\, B \succ Y}\,\mathrm{plink}L$$

Too weak!

A plink B could be $A \wedge B$ or $A \vee B$ or anything in between!

$$\frac{X, A \succ B, Y}{X \succ A \to B, Y}\to R \qquad \frac{X \succ A, Y \quad X', B \succ Y'}{X, X', A \to B \succ Y, Y'}\to L$$

Just right

A DEFINITION

x is equilateral iff x has three sides & each side has equal length.

A DEFINITION

as an invertible rule

x has three sides   each side of x has equal length.

x is equilateral

DEFINING THE CONDITIONAL

$$\frac{X, A \succ B, Y}{X \succ A \to B, Y}\ {\to}Df$$

This is clearly uniquely defining (if $\to_1$ & $\to_2$ are defined using the same rule, they are interchangeable)

$$\cfrac{\cfrac{X \succ A \to_1 B, Y}{X, A \succ B, Y}\ {\to_1}Df\uparrow}{X \succ A \to_2 B, Y}\ {\to_2}Df\downarrow$$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \to_2 B \succ A \to_2 B}\ Id}{A \to_2 B, A \succ B}\ {\to_2}Df\uparrow}{A \to_2 B \succ A \to_1 B}\ {\to_1}Df\downarrow \qquad X, A \to_1 B \succ Y}{X, A_2 \to B \succ Y}\ Cut$$

DEFINING THE CONDITIONAL

$$\dfrac{X, A \succ B, Y}{X \succ A \to B, Y}\ \to Df$$

This is also conservatively extending — the usual left/right rules can be recovered from $\to Df$ using Cut & ID.

$$\dfrac{X_1 \succ A, Y_1 \qquad \dfrac{\dfrac{}{A \to B \succ A \to B}\ Id}{A \to B, A \succ B}\ \to Df\uparrow}{X_1, A \to B \succ B, Y_1}\ Cut_A \qquad X_2, B \succ Y_2$$

$$\dfrac{X_1, A \to B \succ B, Y_1 \qquad X_2, B \succ Y_2}{X_1, X_2, A \to B \succ Y_1, Y_2}\ Cut_B$$

DEFINING THE CONDITIONAL

$$\cfrac{X, A \succ B, Y}{X \succ A \to B, Y}\ {\to}Df$$

This is also conservatively extending — the usual left/right rules can be recovered from $\to Df$ using Cut & ID, and the principal cut-reduction step unwinds this definition.

$$\cfrac{\cfrac{X_3, A \succ B, Y_3}{X_3 \succ A \to B \succ Y_3}\ {\to}Df{\downarrow} \qquad \cfrac{\cfrac{X_1 \succ A, Y_1 \qquad \cfrac{\cfrac{}{A \to B \succ A \to B}\ Id}{A \to B, A \succ B}\ {\to}Df{\uparrow}}{X_1, A \to B \succ B, Y_1}\ Cut_A \qquad X_2, B \succ Y_2}{X_1, X_2, A \to B \succ Y_1, Y_2}\ Cut_B}{X_1, X_2, X_3 \succ Y_1, Y_2, Y_3}\ Cut_{A \to B}$$

$$
\dfrac{\dfrac{\vdots}{X_1 \succ A, Y_1} \qquad \dfrac{\dfrac{\dfrac{\vdots}{X_3, A \succ B, Y_3}}{X_3 \succ A \to B \succ Y_3}\,{\to}\mathrm{Df}{\downarrow} \qquad \dfrac{\dfrac{}{A \to B \succ A \to B}\,\mathrm{Id}}{A \to B, A \succ B}\,{\to}\mathrm{Df}{\uparrow}}{X_3, A \succ B, Y_3}\,\mathrm{Cut}_{A \to B}}{\dfrac{X_1, X_3 \succ B, Y_3 \qquad \dfrac{\vdots}{X_2, B \succ Y_2}}{X_1, X_2, X_3 \succ Y_1, Y_2, Y_3}\,\mathrm{Cut}_B}\,\mathrm{Cut}_A
$$

$$
\dfrac{\dfrac{\dfrac{\vdots}{X_1 \succ A, Y_1} \qquad \dfrac{\vdots}{X_3, A \succ B, Y_3}}{X_1, X_3 \succ B, Y_3}\,\mathrm{Cut}_A \qquad \dfrac{\vdots}{X_2, B \succ Y_2}}{X_1, X_2, X_3 \succ Y_1, Y_2, Y_3}\,\mathrm{Cut}_B
$$

THIS IS TOTALLY GENERAL

(it works like this for any invertible defining rule, in the presence or absence of contraction & weakening, in different sequent structures.)

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge Df \qquad \frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\ \otimes Df \qquad \frac{X, A \succ Y \quad X \succ B, Y}{X \succ A \sqsupset B, Y}\ \sqsupset Df$$

1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

6. IDENTITY & PROPERTIES

Defining Rules for Quantifiers

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\ \forall Df \qquad \frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y}\ \exists Df$$

The term n must be absent in the lower sequent.

n is an eigenvariable (more on what this means, soon).

Recovering the usual left/right rules

$$\begin{array}{c} X \succ A(n), Y \\ \hline\hline X \succ \forall_x A(x), Y \end{array}\ \forall Df$$

$\forall R$

$$\dfrac{X \succ A(n), Y}{X \succ \forall_x A(x), Y}\ \forall Df{\downarrow}$$

$\forall L$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall_x A(x) \succ \forall_x A(x)}\ Id}{\forall_x A(x) \succ A(n)}\ \forall Df{\uparrow}}{\forall_x A(x) \succ A(t)}\ Spec^n_t \qquad X, A(t) \succ Y}{X, \forall_x A(x) \succ Y}\ Cut$$

(t can be any term at all, not necessarily an eigenvariable)

The Specialisation^n_t Rule

$$\frac{X \succ Y}{[X \succ Y]^n_t}\ \mathit{Spec}^n_t$$

If a sequent obtains concerning some eigenvariable n, it also obtains for the term t.

Eigenvariables are inferentially general among terms.

Spec^n_t is admissible in Gentzen's sequent calculus.

Equivalent Formulations

$$
\begin{tikzcd}[column sep=huge]
\begin{array}{l}\mathrm{Id}\\ \mathrm{Cut}\\ \forall\mathrm{Df}\\ \mathrm{Spec}\end{array}
\arrow[r, bend left, "\text{define } \forall L \text{ by } \forall Df, Spec\ldots"]
&
\begin{array}{l}\mathrm{Id}\\ \mathrm{Cut}\\ \forall L/\forall R\\ \mathrm{Spec}\end{array}
\arrow[l, bend left, "\text{define } \forall Df \text{ by } \forall L, Spec, \ldots"]
\arrow[r, bend left, "\text{Spec elimination}"]
&
\begin{array}{l}\mathrm{Id}\\ \mathrm{Cut}\\ \forall L/\forall R\end{array}
\arrow[l, bend left, "\text{inclusion}"]
\arrow[r, bend left, "\text{Cut elimination}"]
&
\begin{array}{l}\mathrm{Id}\\ \forall L/\forall R\end{array}
\arrow[l, bend left, "\text{inclusion}"]
\end{tikzcd}
$$

[See my "Generality & Existence 1", RSL (2019) for details.]

SPEC ELIMINATION

$$\dfrac{\begin{array}{c}\delta \\ \vdots \\ X \succ Y\end{array}}{[X \succ Y]^n_t}\;\mathrm{Spec}^n_t \quad \rightsquigarrow \quad \begin{array}{c}\delta^n_t \\ \vdots \\ [X \succ Y]^n_t\end{array}$$

It's easy to see that each of the rules in Gentzen's system are closed under specialisation — ie there is nothing inferentially special in an eigenvariable as conclusion.

($\forall Df\uparrow$ violates this constraint — hence the requirement to impose Spec.)

DEFINING RULES

... specify concepts that are available (conservative) & determinate (unique) over a basic structural context

.... & for quantifiers, these rules use prior notions of substitution & inferential generality.

1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

6. IDENTITY & PROPERTIES

FREE LOGIC, DEFINED & UNDEFINED TERMS

n is a number.

n/m is not necessarily a number.

n/m is not defined when $m=0$.
(does not exist)

'n/m' is a term, whatever values n & m take.

$$\forall x(\neg \exists y(y = n/x) \leftrightarrow x=0)$$

THE BASIC STRUCTURAL CONTEXT

RULING TERMS IN & RULING THEM OUT

$$s, t, A, B \succ C, D, s', t'$$

[defined true false undefined]

for a counterexample to the sequent

ACCEPT REJECT

$$\frac{}{t, X \succ Y, t}\ t\text{-Id} \qquad\qquad \frac{X \succ Y, t \qquad t, X' \succ Y'}{X, X' \succ Y, Y'}\ t\text{-Cut}$$

DEFINING DEFINEDNESS

$$\frac{X \succ Y, t}{X \succ Y, t\downarrow}\ \text{(double line)}\ \downarrow Df$$

& equivalently, ...

$$\frac{X \succ Y, t}{X \succ Y, t\downarrow}\ \downarrow R \qquad \frac{X, t \succ Y}{X, t\downarrow \succ Y}\ \downarrow L$$

POSSIBLE CONDITIONS ON PREDICATES & FUNCTION SYMBOLS

$$\frac{t_i, X \succ Y}{Ft_1 \ldots t_n, X \succ Y}\ FL \qquad\qquad \frac{t_i, X \succ Y}{ft_1 \ldots t_n, X \succ Y}\ fL$$

(If we grant these, then nonexistence is not a predicate in this sense, since we have $\neg \frac{1}{0}{\downarrow}$, while $\frac{1}{0}$ is not defined.)

DEFINING RULES FOR $\forall/\exists$ with existential commitment

$$\frac{n, X \succ Y, A(n)}{X \succ Y, \forall_x A(x)}\ \forall Df \qquad\qquad \frac{n, A(n), X \succ Y}{\exists_x A(x), X \succ Y}\ \exists Df$$

$$\dfrac{X \succ Y, t \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall_x A(x) \succ \forall_x A(x)}\ Id}{n, \forall_x A(x) \succ A(n)}\ \forall Df\uparrow}{t, \forall_x A(x) \succ A(t)}\ Spec^n_t \qquad X', A(t) \succ Y'}{t, X', \forall_x A(x) \succ Y'}\ Cut}{X, X', \forall_x A(x) \succ Y, Y'}\ t\,Cut$$

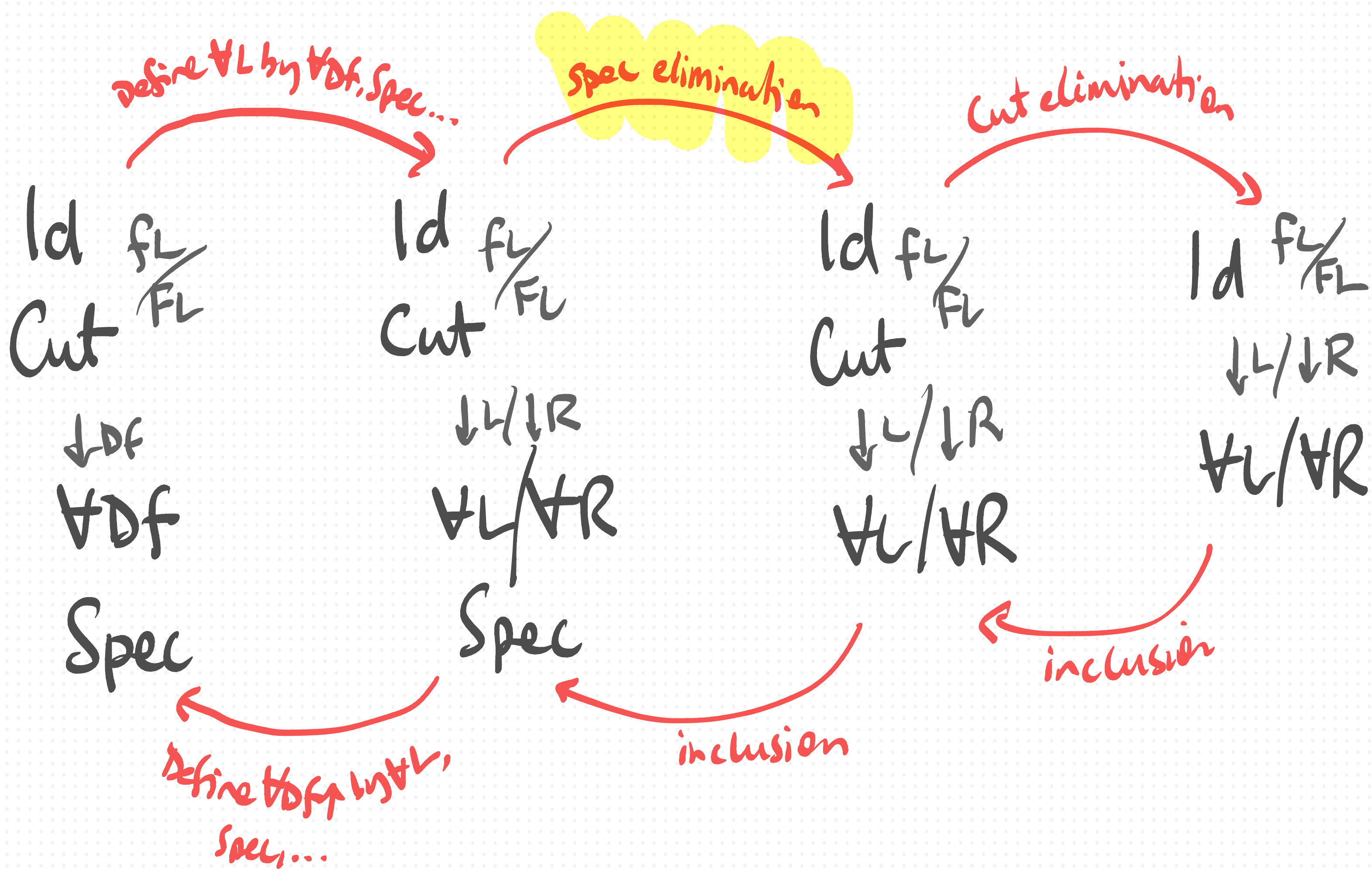
Defining Rules for $\forall/\exists$ with existential commitment

$$\dfrac{n, X \succ Y, A(n)}{X \succ Y, \forall x A(x)}\ \forall Df \qquad \dfrac{n, A(n), X \succ Y}{\exists x A(x), X \succ Y}\ \exists Df$$

(double lines)

$$\dfrac{X \succ Y, t \qquad \dfrac{X' \succ A(t), Y' \qquad \dfrac{\dfrac{\dfrac{}{\exists x A(x) \succ \exists x A(x)}\ Id}{n, A(n) \succ \exists x A(x)}\ Df\uparrow}{t, A(t) \succ \exists x A(x)}\ Spec^{n}_{t}}{t, X' \succ \exists x A(x), Y'}\ Cut}{X, X' \succ \exists x A(x), Y, Y'}\ t\,Cut$$

THE EQUIVALENCES STILL HOLD...



1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

6. IDENTITY & PROPERTIES

But we are free to define more...

$$\frac{n, X \succ Y, A(n)}{X \succ Y, \forall x A(x)} \; \forall Df$$

With Existential Commitment

$$\frac{X \succ Y, A(n)}{X \succ Y, \Pi x A(x)} \; \Pi Df$$

without Existential Commitment

$$\dfrac{\dfrac{\dfrac{\dfrac{X \succ Y, \forall x A(x)}{n, X \succ Y, A(n)} \; \forall Df}{n{\downarrow}, X \succ Y, A(n)} \; {\downarrow}Df}{X \succ Y, n{\downarrow} \to A(n)} \; {\to}Df}{X \succ Y, \Pi x (x{\downarrow} \to A(n))} \; \Pi Df$$

BUT WE ARE FREE TO DEFINE MORE...

$$\dfrac{n, A(n), X \vdash Y}{\exists_x A(x), X \vdash Y}\ \exists\mathrm{Df} \qquad\qquad \dfrac{A(n), X \vdash Y}{\Sigma_x A(x), X \vdash Y}\ \Sigma\mathrm{Df}$$

$$\dfrac{\dfrac{\Sigma_x \neg x{\downarrow} \vdash \Sigma_x \neg x{\downarrow}}{\neg n{\downarrow} \vdash \Sigma_x \neg x{\downarrow}}\ \Sigma\mathrm{Df}{\uparrow}}{\neg \tfrac{1}{0}{\downarrow} \vdash \Sigma_x \neg x{\downarrow}}\ \mathrm{Spec}^{n}_{1/0}$$

Are outer quantifiers just as acceptable as inner quantifiers?

Do they involve the same sort of

$\left\{\begin{array}{l}\text{ontological}\\ \text{ideological}\\ \text{theoretical}\end{array}\right\}$ commitment?

Not necessarily

There are differences...

1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER? on TERMS & VALUES

6. IDENTITY & PROPERTIES

LOOKING CLOSER...

$$\frac{n, X \succ Y, A(n)}{X \succ Y, \forall x A(x)} \; \forall Df \qquad \frac{X \succ Y, A(n)}{X \succ Y, \Pi x A(x)} \; \Pi Df$$

this n is $\begin{cases} \text{an eigenvariable} \\ \text{an inferentially general singular term.} \end{cases}$

WHAT IS A VARIABLE?

What is the meaning of 'x' in $Qx\, A(x)$?

"Everything is A..." "For every value x can take..."

What is the meaning of 'x' in $A(x)$?

- A variable of some sort?
- A pronoun? a demonstrative expression?
- A singular term of some kind?

Should these be connected?

IN FAVOUR OF CONNECTION...

$$\begin{array}{c} x, X \succ Y, A(x) \\ \hline\hline X \succ Y, \forall x A(x) \end{array}\ \forall Df \qquad\qquad \begin{array}{c} x, A(x), X \succ Y \\ \hline\hline \exists x A(x), X \succ Y \end{array}\ \exists Df$$

The defining rules become compositional, just like the other defining rules.

BUT THEN...

$$\frac{x, X \succ Y, A(x)}{X \succ Y, \forall x A(x)} \; \forall Df \qquad \frac{x, A(x), X \succ Y}{\exists x A(x), X \succ Y} \; \exists Df$$

THIS SEEMS REDUNDANT!

What would it mean for the variable x to be undefined in this context?

Isn't using a variable tantamount to treating it as having some value or other?

(E.g. Feferman takes $x\downarrow$ to be a theorem.)

ALTERNATE FORMULATION

- Variables are always treated as defined

$$\frac{x,\ X \succ Y}{X \succ Y}\ \text{Var Drop}$$

- They are no longer inferentially general but are general among the defined terms.

$$\frac{X \succ Y}{t,\ [X \succ Y]^{x}_{t}}\ \text{Spec}^{x}_{t}$$

- Variables are used for quantification, not general singular terms.

$$\frac{X \succ Y,\ A(x)}{X \succ Y,\ \forall x A(x)}\ \forall\text{Df}_x$$

EQUIVALENCE

Derivations in the two systems are inter-translatable

Id, Cut

F_L f_L

Spec^{n}_{t}

Connective Df...

$\forall Df_{n}$ $\exists Df_{n}$

Id, Cut

F_L f_L

Spec^{x}_{t} $x\!\downarrow$

Connective Df...

$\forall Df_{x}$ $\exists Df_{x}$

x, y.. only ever bound.

Eigenvariables n occur free.

x, y... occur free

No eigenvariables

Eigenvariable with existential commitment $\rightleftarrows$ Variable

THE UPSHOT

This formulation provides a context in which inner quantification is definable, using the resources available, but outer quantification (defined in the obvious way) uses extra resources.

This is not to say that outer quantification is meaningless, of course!

1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER? on TERMS & VALUES

6. IDENTITY & PROPERTIES

WHAT IS IDENTITY?

F is a predicate variable, absent from the conclusion.

$$\frac{X, Fs \succ Ft, Y \qquad X, Ft \succ Fs, Y}{X \succ Y, s = t} \; {=}Df$$

Identity as Indistinguishability

$$\frac{X \succ A(t), Y}{X \succ (\lambda x A(x))t, Y} \; \lambda Df$$

λ as a predicate former.

IDENTITY DERIVATIONS

$$\dfrac{\dfrac{\dfrac{s = t \vdash s == t}{s == t, Ft \vdash Fs}\ {=}{=}\mathrm{Df}}{s == t, \lambda x(x == u)t \vdash \lambda x(x == u)s}\ \mathrm{Spec}^{F}_{\lambda x(x == u)}}{s == t, t == u \vdash s == u}\ \lambda\mathrm{Df} \times 2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x == t \vdash x == t}{Fx, x == t \vdash Ft}\ {=}{=}\mathrm{Df}}{x, x == t \vdash t}\ \mathrm{Spec}^{F}_{-}}{x == t \vdash t}\ \mathrm{Var\ Drop}}{x == t \vdash t{\downarrow}}\ {\downarrow}\mathrm{Df}}{\exists x(x == t) \vdash t{\downarrow}}\ \exists\mathrm{Df}$$

$$\dfrac{\dfrac{\dfrac{Ft \vdash Ft \ \times 2}{\vdash t == t}\ {=}{=}\mathrm{Df} \qquad \dfrac{\dfrac{\exists x(x == t) \vdash \exists x(x == t)}{x == t \vdash \exists x(x == t)}\ \exists\mathrm{Df}}{t, t == t \vdash \exists x(x == t)}\ \mathrm{Spec}^{x}_{t}}{t \vdash \exists x(x == t)}\ \mathrm{Cut}}{t{\downarrow} \vdash \exists x(x == t)}\ {\downarrow}\mathrm{Df}$$

Should Identity be existence-entailing?

$\frac{1}{0} == \frac{1}{0}$ ✓ $\qquad\qquad$ $\frac{1}{0} = \frac{1}{0}$ ✗

$$\frac{X \succ s, Y \quad X, Fs \succ Ft, Y \quad X, Ft \succ Fs, Y}{X \succ Y, s = t} =\!Df$$

$$s = t \leftrightarrow s == t \wedge s\downarrow$$

PROPERTIES & λ ABSTRACTION

$\neg\frac{1}{0}\downarrow$ & so, we have $(\lambda x\,\neg x\downarrow)\frac{1}{0}$, according to the call-by-name semantics of λ abstraction

$$\frac{X \succ A(t), Y}{X \succ (\lambda x\, A(x))t, Y}\ \lambda Df$$

Perhaps $(\lambda x\, A(x))$ isn't the best way to think of the property of being an x such that $A(x)$.

PROPERTIES & λ ABSTRACTION

$$\dfrac{X \succ t, Y \qquad X \succ A(t), Y}{X \succ (\lambda^{\downarrow} x\, A(x))t, Y}\ \lambda^{\downarrow}Df$$

(The inference is written with a double line.)

$\lambda^{\downarrow}$ is obviously existence-entailing (by design), and may be a better fit for some notions of having a property.

LESSONS & FURTHER QUESTIONS

- Quantification involves a number of connected issues
  SUBSTITUTION, GENERALITY, DEFINEDNESS, VARIABLES/VALUES, ETC.
- Choosing a structural context for deduction means taking sides on these issues.
- Quantification into sentence position? predicate position?
  — do similar issues apply there?
- Modal calculi?
  — What of singular terms that take values in some worlds and not others?

Thanks!