Proof Theory

NCS 2024

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DAY 2

SEQUENT SYSTEMS

TODAY'S PLAN

SEQUENTS & DERIVATIONS

CORRESPONDENCE BETWEEN PROOFS & DERIVATIONS

CUT & ELIMINATING IT

CONSEQUENCES OF CUT-FREE PROOFS

STRUCTURAL RULES

EXTRA TOPICS

Proofs & Sequents

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p \to (q \to r)]^{3} \quad [p]^{1}}{q \to r}\,{\scriptstyle \to E} \qquad \dfrac{[p \to q]^{2} \quad [p]^{1}}{q}\,{\scriptstyle \to E}}{r}\,{\scriptstyle \to E}}{p \to r}\,{\scriptstyle \to I^{1}}}{(p \to q) \to (p \to r)}\,{\scriptstyle \to I^{2}}}{(p \to (q \to r)) \to ((p \to q) \to (p \to r))}\,{\scriptstyle \to I^{3}}
$$

$p \to (q \to r), p, p \to q, p \vdash r$

$p \to (q \to r), p \to q \vdash p \to r$

$p \to (q \to r) \vdash (p \to q) \to (p \to r)$

$\vdash (p \to (q \to r)) \to ((p \to q) \to (p \to r))$

A CALCULUS FOR SIMPLE SEQUENTS

$$A \succ A \quad \text{IDENTITY}$$

$$\frac{L \succ C \qquad C \succ R}{L \succ R}\ \text{CUT}$$

$$\frac{L \succ A \qquad L \succ B}{L \succ A \wedge B}\ \wedge R \qquad \frac{A \succ R}{A \wedge B \succ R}\ \wedge L_1 \qquad \frac{B \succ R}{A \wedge B \succ R}\ \wedge L_2$$

$$\frac{L \succ A}{L \succ A \vee B}\ \vee R_1 \qquad \frac{L \succ B}{L \succ A \vee B}\ \vee R_2 \qquad \frac{A \succ R \qquad B \succ R}{A \vee B \succ R}\ \vee L$$

A CALCULUS FOR SIMPLE SEQUENTS

$A \succ A$ IDENTITY

$$\frac{L \succ C \quad C \succ R}{L \succ R}\ \text{CUT}$$

$\underbrace{\hspace{10em}}_{\text{STRUCTURAL RULES}}$

$\overbrace{\hspace{10em}}^{\text{OPERATIONAL RULES}}$

$$\frac{L \succ A \quad L \succ B}{L \succ A \wedge B}\ \wedge R \qquad \frac{A \succ R}{A \wedge B \succ R}\ \wedge L_1 \qquad \frac{B \succ R}{A \wedge B \succ R}\ \wedge L_2$$

$$\frac{L \succ A}{L \succ A \vee B}\ \vee R_1 \qquad \frac{L \succ B}{L \succ A \vee B}\ \vee R_2 \qquad \frac{A \succ R \quad B \succ R}{A \vee B \succ R}\ \vee L$$

A CALCULUS FOR SIMPLE SEQUENTS

$$A \succ A$$

$$\frac{L \succ C \quad C \succ R}{L \succ R}$$

Formulas L & R in each rule are said to be PASSIVE (or bystanders) in the inference step, while the others are ACTIVE (or participants).

$$\frac{L \succ A \quad L \succ B}{L \succ A \wedge B}\,\wedge R \qquad \frac{A \succ R}{A \wedge B \succ R}\,\wedge L_1 \qquad \frac{B \succ R}{A \wedge B \succ R}\,\wedge L_2$$

$$\frac{L \succ A}{L \succ A \vee B}\,\vee R_1 \qquad \frac{L \succ B}{L \succ A \vee B}\,\vee R_1 \qquad \frac{A \succ R \quad B \succ R}{A \vee B \succ R}\,\vee L$$

PASSIVE formulas are totally ARBITRARY

If we replace L by L' or R by R' in an inference, we have another instance of the same rule.

(The $L \succ$ and the $\succ R$ are the CONTEXT in which the inference is applied.)

$$\frac{L \succ C \quad C \succ R}{L \succ R}$$

$$\frac{L \succ A \quad L \succ B}{L \succ A \wedge B}\,{\wedge R} \qquad \frac{A \succ R}{A \wedge B \succ R}\,{\wedge L} \qquad \frac{B \succ R}{A \wedge B \succ R}\,{\wedge L}$$

$$\frac{L \succ A}{L \succ A \vee B}\,{\vee R} \qquad \frac{L \succ B}{L \succ A \vee B}\,{\vee R} \qquad \frac{A \succ R \quad B \succ R}{A \vee B \succ R}\,{\vee L}$$

LET'S DERIVE $p \wedge (q \wedge r) \succ (p \wedge q) \wedge r$

$$
\begin{prooftree}
\AxiomC{$p \succ p$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \wedge (q \wedge r) \succ p$}
\AxiomC{$q \succ q$}
\RightLabel{$\wedge L$}
\UnaryInfC{$q \wedge r \succ q$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \wedge (q \wedge r) \succ q$}
\RightLabel{$\wedge R$}
\BinaryInfC{$p \wedge (q \wedge r) \succ p \wedge q$}
\AxiomC{$r \succ r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$q \wedge r \succ r$}
\RightLabel{$\wedge L$}
\UnaryInfC{$p \wedge (q \wedge r) \succ r$}
\RightLabel{$\wedge R$}
\BinaryInfC{$p \wedge (q \wedge r) \succ (p \wedge q) \wedge r$}
\end{prooftree}
$$

LET'S DERIVE $p \vee (p \wedge q) \vdash p$

$$p \wedge (q \wedge r) \vdash (p \wedge q) \vee (p \wedge r)$$

??

$$\dfrac{p \vdash p \qquad \dfrac{p \vdash p}{p \wedge q \vdash p}\ \wedge L}{p \vee (p \wedge q) \vdash p}\ \vee L$$

WE COULD REPLACE Identity WITH Atomic Identity

$A \succ A$ Identity

$p \succ p$ Atomic Identity

$$\dfrac{\dfrac{\dfrac{\vdots}{A \succ A}}{A \wedge B \succ A}\,{\scriptstyle \wedge L_1} \qquad \dfrac{\dfrac{\vdots}{B \succ B}}{A \wedge B \succ B}\,{\scriptstyle \wedge L_2}}{A \wedge B \succ A \wedge B}\,{\scriptstyle \wedge R}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{A \succ A}}{A \succ A \vee B}\,{\scriptstyle \vee R_1} \qquad \dfrac{\dfrac{\vdots}{B \succ B}}{B \succ A \vee B}\,{\scriptstyle \vee R_2}}{A \vee B \succ A \vee B}\,{\scriptstyle \vee L}$$

WE CAN REPLACE CUT WITH NOTHING

$$\dfrac{\begin{array}{c}\delta_1 \\ \vdots \\ L \vdash C\end{array} \qquad \begin{array}{c}\delta_2 \\ \vdots \\ C \vdash R\end{array}}{L \vdash R}\;\text{cut}$$

(Let's suppose there is no cut in δ_1 or δ_2.)

We show that this rule application is redundant, by induction on the structure of the cut-formula C.

CUT-FORMULA: Atomic

$$\dfrac{\begin{array}{c}\delta_1 \\ \vdots \\ L \succ p\end{array} \qquad \begin{array}{c}\delta_2 \\ \vdots \\ p \succ R\end{array}}{L \succ R}\ \text{Cut}$$

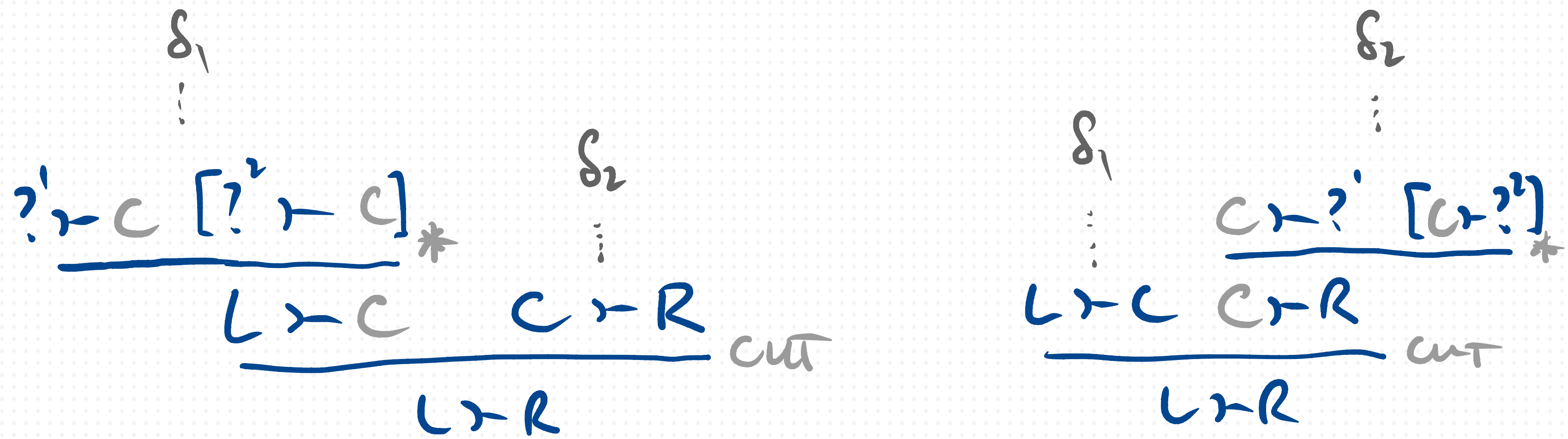
δ_1 & δ_2 contain no CUTS, so each derivation is either an ATOMIC IDENTITY axiom, or p is PASSIVE in the last inference step. (An atom is never ACTIVE in the conclusion of an operational rule.)

CUT WITH IDENTITY IS REDUNDANT

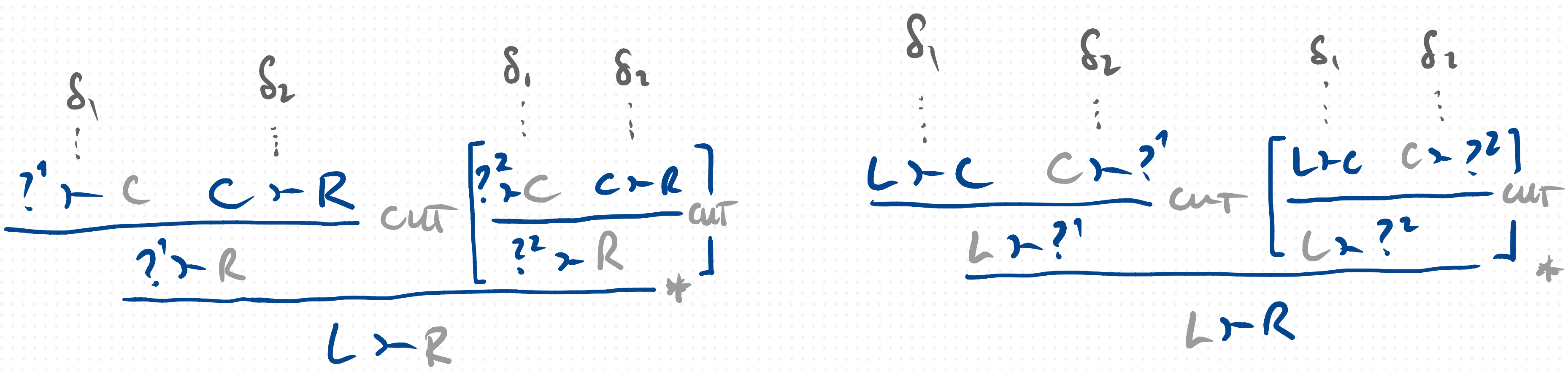
$$\dfrac{C \vdash C \qquad \begin{array}{c}\delta_2\\ \vdots\\ C \vdash R\end{array}}{C \vdash R}\ \text{cut} \qquad \rightsquigarrow \qquad \begin{array}{c}\delta_2\\ \vdots\\ C \vdash R\end{array}$$

$$\dfrac{\begin{array}{c}\delta_1\\ \vdots\\ L \vdash C\end{array} \qquad C \vdash C}{L \vdash C}\ \text{cut} \qquad \rightsquigarrow \qquad \begin{array}{c}\delta_1\\ \vdots\\ L \vdash C\end{array}$$

WHEN A CUT FORMULA ARRIVES AT THE CUT AS A BYSTANDER



In either case, we could have applied the CUT rule before the * rule, since the cut formula is passive in *.



SO, WE ARE LEFT WITH CUTS WHERE THE CUT-FORMULA ARRIVES AS ACTIVE

$$\dfrac{\begin{matrix}\delta_1 \\ \vdots \\ L \vdash C\end{matrix} \qquad \begin{matrix}\delta_2 \\ \vdots \\ C \vdash R\end{matrix}}{L \vdash R}\ \text{cut}$$

So, the only possibilities for the last steps in δ_1 & δ_2 are:

δ_1	δ_2
Identity	Identity
$\wedge R$	$\wedge L$
$\vee R$	$\vee L$

SO, WE ARE LEFT WITH CUTS WHERE THE CUT-FORMULA ARRIVES AS ACTIVE

$$\dfrac{\begin{array}{c}\delta_1\\ \vdots\\ L \vdash C\end{array} \qquad \begin{array}{c}\delta_2\\ \vdots\\ C \vdash R\end{array}}{L \vdash R}\ \text{cut}$$

δ_1	δ_2
~~Identity~~	~~Identity~~
$\wedge R$	$\wedge L$
$\vee R$	$\vee L$

We've seen that cuts with Identity sequents are redundant, so the operational rules remain.

SO, WE ARE LEFT WITH CUTS WHERE THE CUT-FORMULA ARRIVES AS ACTIVE

$$\dfrac{\begin{matrix}\delta_1\\ \vdots\\ L \succ C\end{matrix} \qquad \begin{matrix}\delta_2\\ \vdots\\ C \succ R\end{matrix}}{L \succ R}\;\text{cut}$$

δ_1	δ_2
~~Identity~~	~~Identity~~
$\wedge R$	$\wedge L$
$\vee R$	$\vee L$

If δ_1 ends in $\wedge R$, δ_2 must end in $\wedge L$. We have

$$\dfrac{\dfrac{\begin{matrix}\delta_{11}\\ \vdots\\ L \succ C_1\end{matrix} \qquad \begin{matrix}\delta_{12}\\ \vdots\\ L \succ C_2\end{matrix}}{L \succ C_1 \wedge C_2}\;\wedge R \qquad \dfrac{\begin{matrix}\delta_{2i}\\ \vdots\\ C_i \succ R\end{matrix}}{C_1 \wedge C_2 \succ R}\;\wedge L_i}{L \succ R}\;\text{cut} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\delta_{1i}\\ \vdots\\ L \succ C_i\end{matrix} \qquad \begin{matrix}\delta_{2i}\\ \vdots\\ C_i \succ R\end{matrix}}{L \succ R}\;\text{cut}$$

So, we are left with cuts where the cut-formula arrives as active

$$\dfrac{\begin{matrix}\delta_1\\ \vdots\\ L \vdash C\end{matrix} \qquad \begin{matrix}\delta_2\\ \vdots\\ C \vdash R\end{matrix}}{L \vdash R}\ \text{cut}$$

δ_1: ~~Identity~~, ~~$\wedge R$~~, $\vee R$

δ_2: ~~Identity~~, ~~$\wedge L$~~, $\vee L$

If δ_1 ends in $\wedge R$, δ_2 must end in $\wedge L$. We have

$$\dfrac{\dfrac{\begin{matrix}\delta_{11}\\ \vdots\\ L \vdash C_1\end{matrix} \qquad \begin{matrix}\delta_{12}\\ \vdots\\ L \vdash C_2\end{matrix}}{L \vdash C_1 \wedge C_2}\ \wedge R \qquad \dfrac{\begin{matrix}\delta_{2i}\\ \vdots\\ C_i \vdash R\end{matrix}}{C_1 \wedge C_2 \vdash R}\ \wedge L_i}{L \vdash R}\ \text{cut} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\delta_{1i}\\ \vdots\\ L \vdash C_i\end{matrix} \qquad \begin{matrix}\delta_{2i}\\ \vdots\\ C_i \vdash R\end{matrix}}{L \vdash R}\ \text{cut}$$

SO, WE ARE LEFT WITH CUTS WHERE THE CUT-FORMULA ARRIVES AS ACTIVE

$$\dfrac{\begin{matrix}\delta_1 \\ \vdots \\ L \succ C\end{matrix} \qquad \begin{matrix}\delta_2 \\ \vdots \\ C \succ R\end{matrix}}{L \succ R}\;\text{cut}$$

δ_1: ~~Identity~~, ~~$\wedge R$~~, ~~$\vee R$~~

δ_2: ~~Identity~~, ~~$\wedge L$~~, ~~$\vee L$~~

If δ_1 ends in $\vee R$, δ_2 must end in $\vee L$. We have

$$\dfrac{\dfrac{\begin{matrix}\delta_{1i} \\ \vdots \\ L \succ C_i\end{matrix}}{L \succ C_1 \vee C_2}\;\vee R_i \qquad \dfrac{\begin{matrix}\delta_{21} \\ \vdots \\ C_1 \succ R\end{matrix} \quad \begin{matrix}\delta_{22} \\ \vdots \\ C_2 \succ R\end{matrix}}{C_1 \vee C_2 \succ R}\;\vee L}{L \succ R}\;\text{cut} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\delta_{1i} \\ \vdots \\ L \succ C_i\end{matrix} \qquad \begin{matrix}\delta_{2i} \\ \vdots \\ C_i \succ R\end{matrix}}{L \succ R}\;\text{cut}$$

ELIMINATING CUTS FROM DERIVATIONS

$$\dfrac{\begin{matrix}\delta_1 \\ \vdots \\ L \succ C\end{matrix} \qquad \begin{matrix}\delta_2 \\ \vdots \\ C \succ R\end{matrix}}{L \succ R} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\text{A derivation with} \\ \text{a lower cut-complexity} \\ \vdots\end{matrix}}{L \succ R}$$

(this process terminates in a cut-free derivation)

Cut Complexity: $\langle c_n, c_{n-1}, \ldots, c_1, c_0 \rangle$

c_i = number of cut formulas of complexity i.

n = complexity of largest cut formula.

CUT-FREE DERIVATIONS

$$A \vdash A \;\; \text{Identity} \qquad\qquad \cancel{\dfrac{L \succ C \quad C \succ R}{L \succ R}\,\text{Cut}}$$

$$\dfrac{L \succ A \quad L \succ B}{L \succ A \wedge B}\,\wedge R \qquad\qquad \dfrac{A \succ R}{A \wedge B \succ R}\,\wedge L_1 \qquad \dfrac{B \succ R}{A \wedge B \succ R}\,\wedge L_2$$

$$\dfrac{L \succ A}{L \succ A \vee B}\,\vee R_1 \qquad \dfrac{L \succ B}{L \succ A \vee B}\,\vee R_1 \qquad\qquad \dfrac{A \succ R \quad B \succ R}{A \vee B \succ R}\,\vee L$$

In any inference, the premise formulas are present in the conclusion.
The conclusion is built out of the premises.

Cut-free derivations are ANALYTIC.
& this is proved by inspecting the rules.

INVERTIBLE RULES

$$A \vdash A \ \text{Identity} \qquad\qquad \dfrac{L \vdash C \quad C \vdash R}{L \vdash R}\ \text{Cut}$$

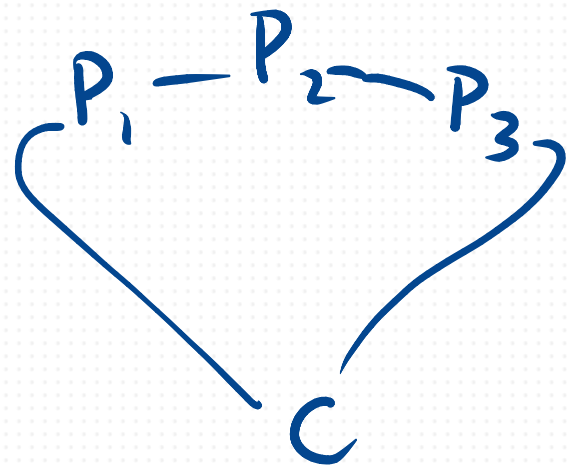
$$\boxed{\dfrac{L \vdash A \quad L \vdash B}{L \vdash A \wedge B}\ \wedge R} \qquad \dfrac{A \vdash R}{A \wedge B \vdash R}\ \wedge L_1 \qquad \dfrac{B \vdash R}{A \wedge B \vdash R}\ \wedge L_2$$

$$\dfrac{L \vdash A}{L \vdash A \vee B}\ \vee R_1 \qquad \dfrac{L \vdash B}{L \vdash A \vee B}\ \vee R_1 \qquad \boxed{\dfrac{A \vdash R \quad B \vdash R}{A \vee B \vdash R}\ \vee L}$$

$\dfrac{P_1 \ [P_2]}{C}$ is invertible iff whenever C holds so does P_1 [and P_2].

WHAT ABOUT MORE COMPLEX SEQUENTS?

$P_1, P_2, P_3 \succ C$ corresponds to a proof



$X \succ A$ perhaps X is a set, a multiset, a list, or another kind of structure.

X can be empty.

MULTISET-FORMULA SEQUENTS

(corresponding to natural deduction proofs)

??

$$X, A \succ A \quad \text{Identity} \qquad A$$

$$\frac{X \succ C \quad Y, C \succ R}{X, Y \succ R}\,\text{Cut}$$

$\begin{array}{c}\Pi_1\\ C\\ \Pi_2\\ R\end{array}$ (composition)

$$\frac{X \succ A \quad Y \succ B}{X, Y \succ A \wedge B}\,\wedge R \qquad \frac{A \quad B}{A \wedge B}$$

$$\frac{X, A \succ R}{X, A \wedge B \succ R}\,\wedge L_1 \qquad \frac{A \wedge B}{A}$$

$$\frac{X, B \succ R}{X, A \wedge B \succ R}\,\wedge L_2 \qquad \frac{A \wedge B}{B}$$

$$\frac{X \succ A}{X \succ A \vee B}\,\vee R_1 \qquad \frac{A}{A \vee B}$$

$$\frac{X \succ B}{X \succ A \vee B}\,\vee R_2 \qquad \frac{B}{A \vee B}$$

$$\frac{X, A \succ R \quad Y, B \succ R}{X, Y, A \vee B \succ R}\,\vee L \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots\\ R\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ R\end{array}}{R}$$

WEAKENING, CONTRACTION & THE CONNECTIVE RULES

$$\dfrac{\dfrac{p \qquad q}{p \wedge q}\,{\scriptstyle \wedge I}}{q}\,{\scriptstyle \wedge E_2}$$

$$\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,{\scriptstyle \wedge R} \qquad \dfrac{q \succ q}{p \wedge q \succ q}\,{\scriptstyle \wedge L_2}}{p, q \succ q}\,{\scriptstyle \text{CUT}}$$

(p is irrelevent to q)

$$\dfrac{\dfrac{p \wedge q}{p}\,{\scriptstyle \wedge E_1} \qquad \dfrac{p \wedge q}{q}\,{\scriptstyle \wedge E_2}}{p \wedge q}\,{\scriptstyle \wedge I}$$

$$\dfrac{\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,{\scriptstyle \wedge L_1} \qquad \dfrac{q \succ q}{p \wedge q \succ q}\,{\scriptstyle \wedge L_2}}{p \wedge q, p \wedge q \succ p \wedge q}\,{\scriptstyle \wedge R}}{p \wedge q \succ p \wedge q}\,{\scriptstyle \text{CONTRACT}}$$

We need to contract to recover the identity fact.

STRUCTURAL RULES IN SEQUENTS

$$\frac{X \succ R}{X, A \succ R}\ \text{WEAKENING} \qquad \frac{X, A, A \succ R}{X, A \succ R}\ \text{CONTRACTION}$$

Here, the weakened or contracted formulas can also be treated as parameters.

$$
\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \succ C\end{matrix} \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\\ X' \succ R\end{matrix}}{X', C \succ R}\,\text{WEAKENING}}{X, X' \succ R}\,\text{cut}
\quad \rightsquigarrow \quad
\dfrac{\begin{matrix}\delta_2\\ \vdots\\ X' \succ R\end{matrix}}{X, X' \succ R}\,\text{WEAKENINGS}
$$

$$
\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \succ C\end{matrix} \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\\ X', C, C \succ R\end{matrix}}{X', C \succ R}\,\text{CONTRACTION}}{X, X' \succ R}\,\text{cut}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \succ C\end{matrix} \qquad \dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \succ C\end{matrix} \quad \begin{matrix}\delta_2\\ \vdots\\ X', C, C \succ R\end{matrix}}{X, X', C \succ R}\,\text{Cut}}{X, X', X' \succ R}\,\text{Cut}}{X, X' \succ R}\,\text{CONTRACTIONS}
$$

Adjusting the $\wedge$ & $\vee$ Rules

$$\dfrac{A \quad B}{A \otimes B}\,\otimes I \qquad\qquad \dfrac{A\otimes B \quad \begin{matrix}[A][B] \\ \vdots \\ R\end{matrix}}{R}\,\otimes E$$

$$\dfrac{X \succ A \quad Y \succ B}{X, Y \succ A \otimes B}\,\otimes R \qquad\qquad \dfrac{X, A, B \succ R}{X, A \otimes B \succ R}\,\otimes L$$

Multiplicative Conjunction

$$\dfrac{\dfrac{\begin{matrix}\delta_{11} \\ \vdots \\ X \succ A\end{matrix} \quad \begin{matrix}\delta_{12} \\ \vdots \\ Y \succ B\end{matrix}}{X, Y \succ A \otimes B}\,\otimes R \quad \dfrac{\begin{matrix}\delta_2 \\ \vdots \\ X', A, B \succ R\end{matrix}}{X', A \otimes B \succ R}\,\otimes L}{X, Y, X' \succ R}\,cut \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\delta_{11} \\ \vdots \\ X \succ A\end{matrix} \quad \dfrac{\begin{matrix}\delta_{12} \\ \vdots \\ Y \succ B\end{matrix} \quad \begin{matrix}\delta_2 \\ \vdots \\ X', A, B \succ R\end{matrix}}{X', A, Y \succ R}\,cut}{X, Y, X' \succ R}\,cut$$

Adjusting the $\wedge$ & $\vee$ Rules

$$\frac{A\wedge B}{A} \qquad \frac{A\wedge B}{B} \qquad ???$$

$$\frac{X, A \succ R}{X, A\wedge B \succ R}\,\wedge L_1 \qquad \frac{X, B \succ R}{X, A\wedge B \succ R}\,\wedge L_2 \qquad \frac{X \succ A \quad X \succ B}{X \succ A\wedge B}\,\wedge R$$

Additive Conjunction

$$\dfrac{\dfrac{\overset{\delta_{11}}{\vdots}\atop X \succ A \qquad \overset{\delta_{12}}{\vdots}\atop X \succ B}{X \succ A\wedge B}\,\wedge R \qquad \dfrac{\overset{\delta_{21}}{\vdots}\atop X', A \succ R}{X', A\wedge B \succ R}\,\wedge L_1}{X, X' \succ R}\,\mathrm{Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\delta_{11}}{\vdots}\atop X \succ A \qquad \overset{\delta_{21}}{\vdots}\atop X', A \succ R}{X, X' \succ R}\,\mathrm{Cut}$$

ADJUSTING THE ∧ & ∨ RULES

$\dfrac{A}{A\vee B}\vee I$ $\qquad$ $\dfrac{B}{A\vee B}\vee I$ $\qquad$???

$$\dfrac{X \succ A}{X \succ A\vee B}\vee R_1 \qquad \dfrac{X \succ B}{X \succ A\vee B}\vee R_2 \qquad \dfrac{X, A \succ R \quad X, B \succ R}{X, A\vee B \succ R}\vee L$$

ADDITIVE DISJUNCTION

$$\dfrac{\dfrac{\overset{\delta_{11}}{\vdots}}{X \succ A}}{X \succ A\vee B}\vee R_1 \quad \dfrac{\overset{\delta_{21}}{\vdots}\qquad\quad \overset{\delta_{22}}{\vdots}}{\dfrac{X', A \succ R \quad X', B \succ R}{X', A\vee B \succ R}\vee L}$$

$$\dfrac{X \succ A\vee B \qquad X', A\vee B \succ R}{X, X' \succ R}\mathit{Cut} \quad \rightsquigarrow \quad \dfrac{\overset{\delta_{11}}{\vdots}\qquad\quad \overset{\delta_{21}}{\vdots}}{\dfrac{X \succ A \qquad X', A \succ R}{X, X' \succ R}\mathit{Cut}}$$

LET'S DERIVE

$$p \otimes (q \vee r) \vdash (p \otimes q) \vee (p \otimes r)$$

$$(p \otimes q) \vee (p \otimes r) \vdash p \otimes (q \vee r)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p \quad q \vdash q}{p, q \vdash p \otimes q}}{p, q \vdash (p \otimes q) \vee (p \otimes r)} \qquad \dfrac{\dfrac{p \vdash p \quad r \vdash r}{p, r \vdash p \otimes r}}{p, r \vdash (p \otimes q) \vee (p \otimes r)}}{p, q \vee r \vdash (p \otimes q) \vee (p \otimes r)}}{p \otimes (q \vee r) \vdash (p \otimes q) \vee (p \otimes r)}$$

$$\dfrac{\dfrac{\dfrac{p \vdash p \quad \dfrac{q \vdash q}{q \vdash q \vee r}}{p, q \vdash p \otimes (q \vee r)}}{p \otimes q \vdash p \otimes (q \vee r)} \qquad \dfrac{\dfrac{p \vdash p \quad \dfrac{r \vdash r}{r \vdash q \vee r}}{p, r \vdash p \otimes (q \vee r)}}{p \otimes r \vdash p \otimes (q \vee r)}}{(p \otimes q) \vee (p \otimes r) \vdash p \otimes (q \vee r)}$$

ROOM FOR NEW CONNECTIVES

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}\,{\to}I \qquad\qquad \dfrac{A \to B \qquad \overset{\pi_1}{A}}{\begin{matrix}B \\ \pi_2 \\ R\end{matrix}}\,{\to}E$$

$$\dfrac{X, A \succ B}{X \succ A \to B}\,{\to}R \qquad\qquad \dfrac{X \succ A \qquad Y, B \succ R}{X, A \to B, Y \succ R}\,{\to}L$$

$$\dfrac{\dfrac{\begin{matrix}\delta_1 \\ \vdots \\ X, A \succ B\end{matrix}}{X \succ A \to B}\,{\to}R \qquad \dfrac{\begin{matrix}\delta_{21} \\ \vdots \\ X' \succ A\end{matrix} \qquad \begin{matrix}\delta_{22} \\ \vdots \\ Y', B \succ R\end{matrix}}{X', A \to B, Y' \succ R}\,{\to}L}{X, X', Y' \succ R}\,\text{cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\begin{matrix}\delta_{21} \\ \vdots \\ X' \succ A\end{matrix} \qquad \begin{matrix}\delta_1 \\ \vdots \\ X, A \succ B\end{matrix}}{X', X \succ B}\,\text{cut} \qquad \begin{matrix}\delta_{22} \\ \vdots \\ Y', B \succ R\end{matrix}}{X, X', Y \succ R}\,\text{cut}$$

CUT ELIMINATION WORKS AS BEFORE

(Notice that cuts can still be pushed through parameters — though at the cost of duplicating subderivations — and principle cuts simplify to cuts on subformulas)

MORE CONNECTIVES, MORE SEQUENTS

$$\frac{X, A \succ}{X \succ \neg A}\ \neg R \qquad \begin{array}{c}[A] \\ \vdots \\ \frac{\#}{\neg A}\ \neg I\end{array}$$

$$\frac{X \succ A}{X, \neg A \succ}\ \neg L \qquad \begin{array}{c}\vdots \\ \frac{\neg A \quad A}{\#}\ \neg E\end{array}$$

(Careful: now that the RHS can be empty: now all rules in which the RHS formula is a parameter include the empty case.)

$$\frac{\dfrac{\begin{array}{c}\delta_1 \\ \vdots \\ X, A \succ\end{array}}{X \succ \neg A}\ \neg R \qquad \dfrac{\begin{array}{c}\delta_2 \\ \vdots \\ X' \succ A\end{array}}{X', \neg A \succ}\ \neg L}{X, X' \succ}\ Cut \quad \rightsquigarrow \quad \frac{\begin{array}{c}\delta_2 \\ \vdots \\ X' \succ A\end{array} \qquad \begin{array}{c}\delta_1 \\ \vdots \\ X, A \succ\end{array}}{X, X' \succ}\ Cut$$

WEAKENING ON THE RHS?

$$\frac{X \succ}{X \succ A}\ \text{weakening R}$$

This gives you $p, \neg p \succ q$. & it is possible to have weakening on the left without weakening on the right. The result is MINIMAL LOGIC (when you have contraction in addition to weakening).

CLASSICAL SEQUENTS

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\neg L}{p \succ \neg\neg p}\neg R \quad \rightsquigarrow \quad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\neg R}{\neg\neg p \succ p}\neg L$$

DUALISE!

THE NEW RULES feature Multiset/Multiset Sequents

$$\dfrac{X, A \succ Y}{X \succ \neg A, Y}\neg R \qquad \dfrac{X \succ A, Y}{X, \neg A \succ Y}\neg L$$

THE RULES IN THE EXTENDED SETTING

$$A \succ A \quad \text{Identity} \qquad \frac{X \succ C, V \quad Y, C \succ W}{X, Y \succ V, W}\,\text{Cut}$$

$$\frac{X \succ Y}{X, A \succ Y}\,\text{WkL} \qquad \frac{X \succ Y}{X \succ A, Y}\,\text{WkR} \qquad \frac{X, A, A \succ Y}{X, A \succ Y}\,\text{ContL} \qquad \frac{X \succ A, A, Y}{X \succ A, Y}\,\text{ContR}$$

In a classical sequent setting, it is hard to have weakening on the left without weakening on the right

$$\dfrac{\succ A_1, A_2 \qquad \dfrac{X \succ Y}{X, A_1 \succ Y}\,\text{WkL}}{X \succ A_2, Y}\,\text{cut}$$

THE RULES IN THE EXTENDED SETTING

MULTIPLICATIVE CONJUNCTION

$$\frac{X \succ A, V \quad Y \succ B, W}{X, Y \succ A \otimes B, V, W}\ \otimes R \qquad \frac{X, A, B \succ W}{X, A \otimes B \succ W}\ \otimes L$$

ADDITIVE CONJUNCTION

$$\frac{X, A \succ V}{X, A \wedge B \succ V}\ \wedge L_1 \qquad \frac{X, B \succ V}{X, A \wedge B \succ V}\ \wedge L_2 \qquad \frac{X \succ A, V \quad X \succ B, V}{X \succ A \wedge B, V}\ \wedge R$$

ADDITIVE DISJUNCTION

$$\frac{X \succ A, V}{X \succ A \vee B, V}\ \vee R_1 \qquad \frac{X \succ B, V}{X \succ A \vee B, V}\ \vee R_2 \qquad \frac{X, A \succ V \quad X, B \succ V}{X, A \vee B \succ V}\ \vee L$$

CONDITIONAL (MULTIPLICATIVE)

$$\frac{X, A \succ B, V}{X \succ A \to B, V} \qquad \frac{X \succ A, V \quad Y, B \succ W}{X, Y, A \to B \succ V, W}$$

NEGATION

$$\frac{X, A \succ V}{X \succ \neg A, V} \qquad \frac{X \succ A, V}{X, \neg A \succ V}$$

THE RULES IN THE EXTENDED SETTING

MULTIPLICATIVE CONJUNCTION

$$\frac{X \succ A, V \qquad Y \succ B, W}{X, Y \succ A \otimes B, V, W}\;\otimes R \qquad \frac{X, A, B \succ W}{X, A \otimes B \succ W}\;\otimes L$$

ADDITIVE CONJUNCTION

$$\frac{X, A \succ V}{X, A \wedge B \succ V}\;\wedge L_1 \qquad \frac{X, B \succ V}{X, A \wedge B \succ V}\;\wedge L_2 \qquad \frac{X \succ A, V \qquad X \succ B, V}{X \succ A \wedge B, V}\;\wedge R$$

ADDITIVE DISJUNCTION

$$\frac{X \succ A, V}{X \succ A \vee B, V}\;\vee R_1 \qquad \frac{X \succ B, V}{X \succ A \vee B, V}\;\vee R_2 \qquad \frac{X, A \succ V \qquad X, B \succ V}{X, A \vee B \succ V}\;\vee L$$

MULTIPLICATIVE DISJUNCTION

$$\frac{X \succ A, B, W}{X \succ A \oplus B, W}\;\oplus R \qquad \frac{X, A \succ V \qquad Y, B \succ W}{X, Y, A \oplus B \succ V, W}\;\oplus L$$

Room for two formulas on the right

CONDITIONAL (MULTIPLICATIVE)

$$\frac{X, A \succ B, V}{X \succ A \to B, V} \qquad \frac{X \succ A, V \qquad Y, B \succ W}{X, Y, A \to B \succ V, W}$$

NEGATION

$$\frac{X, A \succ V}{X \succ \neg A, V} \qquad \frac{X \succ A, V}{X, \neg A \succ V}$$

COMPLICATIONS in CUT ELIMINATION

$$\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \vdash V\end{matrix}}{X \vdash C, V}\,\text{wk} \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\\ Y \vdash W\end{matrix}}{Y, C \vdash W}\,\text{wk}}{X, Y \vdash V, W}\,\text{cut}$$

which normal form?

$$\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \vdash V\end{matrix}}{X, Y \vdash V, W}\,\text{weakenings}$$

$$\dfrac{\begin{matrix}\delta_2\\ \vdots\\ Y \vdash W\end{matrix}}{X, Y \vdash V, W}\,\text{weakenings}$$

Complications in Cut Elimination

$$\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \vdash C, C, V\end{matrix}}{X \vdash C, V}\,\text{Cont} \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\\ Y, C, C \vdash W\end{matrix}}{Y, C \vdash W}\,\text{Cont}}{X, Y \vdash V, W}\,\text{Cut}$$

$$\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \vdash C, C, V\end{matrix} \qquad \begin{matrix}\delta_2\\ \vdots\\ Y, C, C \vdash W\end{matrix}}{X, Y \vdash V, W}\,\text{mix}$$

$\rightsquigarrow$ Push δ_2 up δ_1

$\hookrightarrow$ Push δ_1 up δ_2

QUANTIFIER RULES

$$\dfrac{X, A(t) \succ V}{X, \forall x A(x) \succ V}\,\forall L \qquad\qquad \dfrac{X \succ A(n), V}{X \succ \forall x A(x), V}\,\forall R^{*}$$

* n is absent from the conclusion.

$$\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X, A(t) \succ V\end{matrix}}{X, \forall x A(x) \succ V}\,\forall L \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\\ Y \succ A(n), W\end{matrix}}{Y \succ \forall x A(x), W}\,\forall R}{X, Y \succ V, W}\,Cut \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\delta_1\\ \vdots\\ X, A(t) \succ V\end{matrix} \qquad \begin{matrix}\delta_2(n := t)\\ \vdots\\ Y \succ A(t), W\end{matrix}}{X, Y \succ V, W}\,Cut$$

for this to be ok n must be appropriately general

we appeal to the side condition to ensure this substitution has the right conclusion.

THERE ARE MORE DELICATE ISSUES WITH EIGENVARIABLES & SUBSTITUTION

— permuting cuts over parameters.

$$\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ X \succ A(n), C, V\end{matrix}}{X \succ \forall x A(x), C, V}\,\forall R \qquad \begin{matrix}\delta_2\\ \vdots\\ Y, C \succ W\end{matrix}}{X, Y \succ \forall x A(x), V, W}\,\mathrm{cut}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\begin{matrix}\delta_1(n := n')\\ \vdots\\ X \succ A(n'), C, V\end{matrix} \qquad \begin{matrix}\delta_2\\ \vdots\\ Y, C \succ W\end{matrix}}{X, Y \succ A(n'), V, W}\,\mathrm{Cut}}{X, Y \succ \forall x A(x), V, W}\,\forall R$$

Choose an eigenvariable n' which is also absent from Y & W.