Comparing Rules for Identity in sequent systems & natural deduction

Greg Restall

THE UNIVERSITY OF
MELBOURNE

PROOF THEORY VIRTUAL SEMINAR · 21 APRIL 2021

https://consequently.org/presentation/2021/comparing-identity-rules

My Aim

To explore different rules for an identity predicate in natural deduction and the sequent calculus.

My Plan

Sequent Calculus & Natural Deduction

Defining Rules

Defining Rules for Identity

Identity *Axioms*

SEQUENT CALCULUS & NATURAL DEDUCTION

Intuitionistic Proofs & Derivations

A derivation of $X \succ A$
builds a proof from X to A.

An Example

$$\dfrac{\dfrac{p \succ p \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q, \neg q \succ}\,\neg L}{q, \neg q \succ r}\,K \qquad r \succ r}{q \lor r, \neg q \succ r}\,\lor L}{\dfrac{p \to (q \lor r), p, \neg q \succ r}{p \to (q \lor r), p \land \neg q \succ r}\,\land L}\,{\to}L}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,{\to}R$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,\land E}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{\dfrac{[p \land \neg q]^2}{\neg q}\,\land E \qquad [q]^1}{\sharp}\,\neg E}{r}\,K \qquad [r]^1}{r}\,\lor E^1}{(p \land \neg q) \to r}\,{\to}I^2$$

The one proof can be built in different ways

$$\dfrac{\dfrac{p \succ p \qquad q \lor r \succ q \lor r}{p \to (q \lor r), p \succ q \lor r}\,{\to}L \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q, \neg q \succ}\,\neg L}{q, \neg q \succ r}\,K \qquad r \succ r}{q \lor r, \neg q \succ r}\,{\lor}L}{\dfrac{\dfrac{p \to (q \lor r), p, \neg q \succ r}{p \to (q \lor r), p \land \neg q \succ r}\,{\land}L}{p \to (q \lor r) \succ (p \land \neg q) \to r}\,{\to}R}\,Cut$$

SEQUENT CALCULUS

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{p \to (q \lor r) \qquad \dfrac{[p \land \neg q]^2}{p}\,{\land}E}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{\dfrac{[p \land \neg q]^2}{\neg q}\,{\land}E \qquad [q]^1}{\sharp}\,\neg E}{r}\,K \qquad [r]^1}{r}\,{\lor}E^1}{(p \land \neg q) \to r}\,{\to}I^2$$

Classical derivations build . . . *what*?

$$\dfrac{\dfrac{p \succ p}{\succ \neg p, p}\,{\scriptstyle \neg R}}{\succ \neg p \lor p}\,{\scriptstyle \lor R}$$

$$\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ p, q}{\succ p, p \to q}\,{\scriptstyle \to R}}{(p \to q) \to p \succ p, p}\,{\scriptstyle \to L}}{(p \to q) \to p \succ p}\,{\scriptstyle W}$$

Add *focus*

From $P_1, P_2, P_3 \succ C_1, C_2, C_3$ to $P_1, P_2, P_3 \succ C_1; C_2, C_3$

A *focussed* sequent has the shape $X \succ C; Y$
where C is either a formula or is *empty*,
and X and Y are finite multisets of formulas.

(The empty case corresponds to a proof of a *contradiction*.)

A *proof* for $P_1, P_2, P_3 \succ C_1; C_2, C_3$ is a proof *of* C_1
from the context P_1, P_2, P_3 (*positive*) and C_2^-, C_3^- (*negative*).

Focus, *Defocus*; *Retrieve* and *Store*

NATURAL DEDUCTION

SEQUENT CALCULUS

$$\dfrac{\begin{matrix}[A^{-}]\\ \Pi\\ \sharp\end{matrix}}{A}\,\textit{Retrieve}$$

$$\dfrac{X \succ\ ;A, Y}{X \succ A; Y}\,\textit{Focus}$$

$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix}\qquad A^{-}}{\sharp}\,\textit{Store}$$

$$\dfrac{X \succ A; Y}{X \succ\ ;A, Y}\,\textit{Defocus}$$

Derivations with *focus* build proofs with *alternatives*

SEQUENT CALCULUS

$$\dfrac{\dfrac{\dfrac{p \succ p; \qquad \dfrac{\dfrac{\dfrac{p \succ p;}{p \succ \ ;p}\,\textit{Defocus}}{p \succ q;p}\,\textit{Focus}}{\succ p \to q;p}\,{\to}R}{(p \to q) \to p \succ p;p}\,{\to}L}{(p \to q) \to p \succ \ ;p,p}\,\textit{Defocus}}{(p \to q) \to p \succ p;}\,\textit{Focus}$$

NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1 \quad [p^-]^2}{\sharp}\,\textit{Store}}{q}\,\textit{Retrieve}}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^-]^2}{\sharp}\,\textit{Store}}{q}\,\textit{Retrieve}$$

That’s *Classical logic*

Adding the *Store/Retrieve* rules
to Gentzen–Prawitz Natural Deduction
gives you a well-behaved, normalising
natural deduction system for classical logic.

(It’s basically Michel Parigot’s λμ calculus.)

Interpreting Sequents with Focus

$X \succ A; Y$ — a *proof* of A, from a context where X is asserted and Y is denied.

$X \succ\ ; Y$ — a *refutation* of asserting X and denying Y.

I’ll pass freely between sequent derivations
and natural deduction proofs with alternatives.

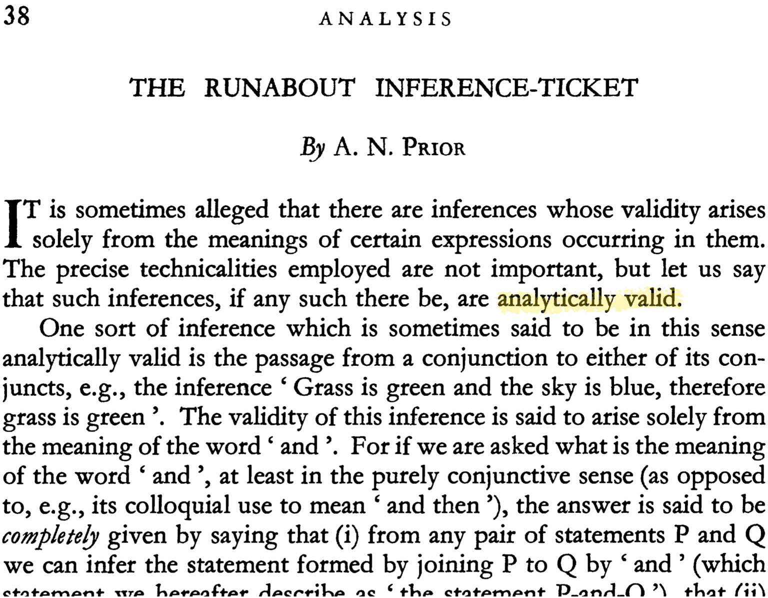
DEFINING RULES

What makes rules *well behaved*?

$$\frac{A \quad B}{A \wedge B}\,{\scriptstyle \wedge I} \qquad \frac{A \wedge B}{A}\,{\scriptstyle \wedge E} \quad \frac{A \wedge B}{B}\,{\scriptstyle \wedge E}$$

$$\frac{A}{A \mathbin{\textit{tonk}} B}\,{\scriptstyle \textit{tonkI}} \qquad \frac{A \mathbin{\textit{tonk}} B}{B}\,{\scriptstyle \textit{tonkE}}$$

Analytic Validity



38 ANALYSIS

THE RUNABOUT INFERENCE-TICKET

By A. N. PRIOR

IT is sometimes alleged that there are inferences whose validity arises solely from the meanings of certain expressions occurring in them. The precise technicalities employed are not important, but let us say that such inferences, if any such there be, are analytically valid.

One sort of inference which is sometimes said to be in this sense analytically valid is the passage from a conjunction to either of its conjuncts, e.g., the inference ‘ Grass is green and the sky is blue, therefore grass is green ’. The validity of this inference is said to arise solely from the meaning of the word ‘ and ’. For if we are asked what is the meaning of the word ‘ and ’, at least in the purely conjunctive sense (as opposed to, e.g., its colloquial use to mean ‘ and then ’), the answer is said to be *completely* given by saying that (i) from any pair of statements P and Q we can infer the statement formed by joining P to Q by ‘ and ’ (which statement we hereafter describe as ‘ the statement P-and-Q ’), that (ii)

One option . . .

One way to be analytically valid is to be a *definition* . . .

. . . but $\wedge I$ and $\wedge E$ don't look much like *definitions*.

Invertible rules look *more* like definitions.

$$\frac{X, A, B \succ Z}{\overline{X, A \land B \succ Z}}\,{\land}Df \qquad \frac{X, A \succ B; Y}{\overline{X \succ A \rightarrow B; Y}}\,{\rightarrow}Df \qquad \frac{X, A \succ\ ; Y}{\overline{X \succ \neg A; Y}}\,{\neg}Df$$

They charaterise *one* aspect of the behaviour of the introduced concept (positively or negatively). The structural rules settle the rest.

Defining rules *define*

They are *conservative* and *uniquely defining*.

From Defining Rules to Left/Right Rules …

$$\frac{X, A, B \succ Z}{X, A \land B \succ Z}\,{\land}\mathit{Df}{\downarrow} \qquad\qquad \frac{X, A, B \succ Z}{X, A \land B \succ Z}\,{\land}L$$

${\land}L$ is one half of ${\land}\mathit{Df}$

$$\dfrac{X \succ A; Y \qquad \dfrac{X' \succ B; Y' \qquad \dfrac{\dfrac{}{A \land B \succ A \land B;}\,\mathit{Id}}{A, B \succ A \land B;}\,{\land}\mathit{Df}{\uparrow}}{X', A \succ A \land B; Y'}\,\mathit{Cut}}{X, X' \succ A \land B; Y, Y'}\,\mathit{Cut} \qquad\qquad \frac{X \succ A; Y \quad X' \succ B; Y'}{X, X' \succ A \land B; Y, Y'}\,{\land}R$$

${\land}R$ is formed from the other half, using *Id* and *Cut*.

… and back

$$\dfrac{\dfrac{\dfrac{}{A \succ A;}\,\mathit{Id} \qquad \dfrac{}{B \succ B;}\,\mathit{Id}}{A, B \succ A \wedge B;}\,{\wedge}R \qquad X, A \wedge B \succ Z}{X, A, B \succ Z}\,\mathit{Cut} \qquad\qquad \dfrac{X, A \wedge B \succ Z}{X, A, B \succ Z}\,{\wedge}\mathit{Df}{\uparrow}$$

We can recover $\wedge \mathit{Df}{\uparrow}$ from $\wedge R$, given *Id* and *Cut*, and $\wedge \mathit{Df}{\downarrow}$ is $\wedge L$.

L/*R* rules given in this way admit elimination of principal *Cuts*

$$
\begin{prooftree}
\AxiomC{Δ}
\noLine
\UnaryInfC{$X \succ A; Y$}
\AxiomC{Δ'}
\noLine
\UnaryInfC{$X' \succ B; Y'$}
\RightLabel{$\wedge R$}
\BinaryInfC{$X, X' \succ A \wedge B; Y, Y'$}
\AxiomC{Δ''}
\noLine
\UnaryInfC{$X'', A, B \succ Z''$}
\RightLabel{$\wedge L$}
\UnaryInfC{$X'', A \wedge B \succ Z''$}
\RightLabel{Cut}
\BinaryInfC{$X, X', X'' \succ Z'', Y, Y'$}
\end{prooftree}
$$

Unpacks into . . .

$$
\begin{prooftree}
\AxiomC{Δ}
\noLine
\UnaryInfC{$X \succ A; Y$}
\AxiomC{Δ'}
\noLine
\UnaryInfC{$X' \succ B; Y'$}
\AxiomC{}
\RightLabel{Id}
\UnaryInfC{$A \wedge B \succ A \wedge B;$}
\RightLabel{$\wedge \mathit{Df}\!\uparrow$}
\UnaryInfC{$A, B \succ A \wedge B;$}
\RightLabel{Cut}
\BinaryInfC{$X', A \succ A \wedge B; Y'$}
\RightLabel{Cut}
\BinaryInfC{$X, X' \succ A \wedge B; Y, Y'$}
\AxiomC{Δ''}
\noLine
\UnaryInfC{$X'', A, B \succ Z''$}
\RightLabel{$\wedge \mathit{Df}\!\downarrow$}
\UnaryInfC{$X'', A \wedge B \succ Z''$}
\RightLabel{Cut}
\BinaryInfC{$X, X', X'' \succ Z'', Y, Y'$}
\end{prooftree}
$$

L/*R* rules given in this way admit elimination of principal *Cuts*

Permuting the *Cuts*, this becomes . . .

$$
\dfrac{\overset{\Delta}{X \succ A; Y} \qquad \dfrac{\overset{\Delta'}{X' \succ B; Y'} \qquad \dfrac{\dfrac{\dfrac{}{A \land B \succ A \land B;}\,\mathit{Id}}{A, B \succ A \land B;}\,{\land}\mathit{Df}{\uparrow} \qquad \dfrac{\overset{\Delta''}{X'', A, B \succ Z''}}{X'', A \land B \succ Y''}\,{\land}\mathit{Df}{\downarrow}}{X'', A, B \succ Z''}\,\mathit{Cut}}{X', A \succ Z'', Y'}\,\mathit{Cut}}{X, X', X'' \succ Z'', Y, Y'}\,\mathit{Cut}
$$

. . . which (since the *Id*/*Df*↑/*Df*↓/*Cut* detour is redundant) simplifies to:

$$
\dfrac{\overset{\Delta}{X \succ A; Y} \qquad \dfrac{\overset{\Delta'}{X' \succ B; Y'} \qquad \overset{\Delta''}{X'', A, B \succ Z''}}{X', A \succ Z'', Y'}\,\mathit{Cut}}{X, X', X'' \succ Z'', Y, Y'}\,\mathit{Cut}
$$

$\wedge Df$ in Natural Deduction

SEQUENT CALCULUS

$$\dfrac{X, A, B \succ Z}{X, A \wedge B \succ Z}\ \wedge Df{\downarrow}$$

$$\dfrac{X, A \wedge B \succ Z}{X, A, B \succ Z}\ \wedge Df{\uparrow}$$

NATURAL DEDUCTION

$$\dfrac{A \wedge B \qquad \begin{array}{c}[A, B]\\ \Pi \\ C\end{array}}{C}\ \wedge E$$

$$\begin{array}{c}\dfrac{A \qquad B}{A \wedge B}\ \wedge I \\ \Pi\end{array}$$

Conservativity and Unique Definability

Cut Elimination and the *Subformula Property* for *rules other than Cut* gives *Conservative Extension*.

The shape of the *defining rules* gives *Uniqueness*.

Defining Rules and *Generality*

$$\frac{X \succ A(n); Y}{X \succ \forall x A(x); Y} \; \forall Df$$

n is absent from the lower sequent,
and it must be *inferentially general*.

Specification as a Rule

$$\dfrac{\dfrac{\forall xFx \succ \forall xFx;}{\forall xFx \succ Fn;}\,\forall Df\!\uparrow}{\forall xFx \succ Ft;}\,Spec^{n}_{t}$$

$$\dfrac{X \succ Z}{X[n/t] \succ Z[n/t]}\,Spec^{n}_{t}$$

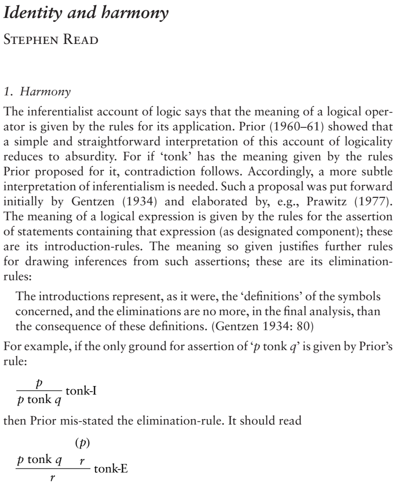
The Status of *Spec*

Spec, like *Id* and *Cut*, are primitive rules in the system with *Df* rules.

Spec is admissible (*height preserving* admissible, in fact) as are *Id* (for complex formulas) and *Cut* in the system with *L*/*R* rules.

DEFINING RULES FOR IDENTITY

Identity and Harmony



Identity and harmony

STEPHEN READ

1. *Harmony*

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960–61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if ‘tonk’ has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

> The introductions represent, as it were, the ‘definitions’ of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of ‘p tonk q’ is given by Prior’s rule:

$$\frac{p}{p \text{ tonk } q}\ \text{tonk-I}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{p \text{ tonk } q \qquad \begin{array}{c}(p)\\ r\end{array}}{r}\ \text{tonk-E}$$

A Defining Rule for Identity

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y}\; {=}Df$$

(Here, F is *inferentially general*, and
absent from the lower sequent.)

Denying $a = b$ has the same significance as taking there to be *some* feature F that holds of a but not b, or *vice versa*.

Or equivalently, to prove that $a = b$, prove Fb
from the assumption Fa (and *vice versa*),
where the predicate F is *arbitrary*.

Identity is a kind of *indistinguishability*.

$=Df{\uparrow}$ in Natural Deduction

SEQUENT CALCULUS

$$\dfrac{\dfrac{X \succ a = b; Y}{X, Fa \succ Fb; Y}\,{=}Df{\uparrow}_1}{X, Pa \succ Pb; Y}\,Spec^{F}_{P} \qquad \dfrac{\dfrac{X \succ a = b; Y}{X, Fb \succ Fa; Y}\,{=}Df{\uparrow}_2}{X, Pb \succ Pa; Y}\,Spec^{F}_{P}$$

NATURAL DEDUCTION

$$\dfrac{\begin{matrix}\Pi \\ a = b\end{matrix} \qquad FaPa}{FbPb}\,{=}E_1 \qquad \dfrac{\begin{matrix}\Pi \\ a = b\end{matrix} \qquad FbPb}{FaPa}\,{=}E_2$$

An Example Derivation

$$\begin{array}{c} X, Fa \succ Fb; Y \quad X, Fb \succ Fa; Y \\ \hline\hline X \succ a = b; Y \end{array}\ {=}\mathit{Df} \qquad \begin{array}{c} X \succ A[x/a]; Y \\ \hline\hline X \succ (\lambda x.A)a; Y \end{array}\ \lambda\mathit{Df} \qquad \begin{array}{c} X \succ Z \\ \hline X[F/P] \succ Z[F/P] \end{array}\ \mathit{Spec}^{F}_{P}$$

$$\begin{array}{c} \begin{array}{c} \begin{array}{c} Fa \succ Fa; \quad Fa \succ Fa; \\ \hline \succ a = a; \end{array}\ {=}\mathit{Df}{\downarrow} \\ \hline \succ (\lambda x.x = a)a; \end{array}\ \lambda\mathit{Df}{\downarrow} \quad \begin{array}{c} \begin{array}{c} a = b \succ a = b; \\ \hline a = b, Fa \succ Fb; \end{array}\ {=}\mathit{Df}{\uparrow} \\ \hline a = b, (\lambda x.x = a)a \succ (\lambda x.x = a)b; \end{array}\ \mathit{Spec}^{F}_{(\lambda x.x=a)} \\ \hline \begin{array}{c} a = b \succ (\lambda x.x = a)b; \\ \hline a = b \succ b = a; \end{array}\ \lambda\mathit{Df}{\uparrow} \end{array}\ \mathit{Cut}$$

From Defining Rules to *L*/*R* rules: $=Df{\downarrow}$ is $=R$

$$\frac{X, Fa \succ Fb; Y \quad X, Fb \succ Fa; Y}{X \succ a = b; Y}{\scriptstyle =Df\downarrow}$$

Deriving $=L$ rules

$$\dfrac{\dfrac{X \succ PaPb; Y \qquad \dfrac{\dfrac{a = b \succ a = b;}{a = b, Fa \succ FbFb \succ Fa;}\,{\scriptstyle =\!Df\uparrow}}{a = b, Pa \succ PbPb \succ Pa;}\,{\scriptstyle Spec^{F}_{P}}}{a = b, X \succ PbPa; Y}\,{\scriptstyle Cut} \qquad X', PbPa \succ Z'}{a = b, X, X' \succ Z', Y}\,{\scriptstyle Cut}$$

$$\dfrac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\,{\scriptstyle =\!L_1} \qquad \dfrac{X \succ Pb; Y \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y}\,{\scriptstyle =\!L_2}$$

Comparing *L*/*R* rules and *I*/*E* rules

SEQUENT CALCULUS

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y}\,{=}L$$

$$\frac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\,{=}L_1$$

$$\frac{X \succ Pb; Y \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y}\,{=}L_2$$

NATURAL DEDUCTION

$$\frac{\begin{array}{cc} [Fa] & [Fb] \\ \Pi & \Pi' \\ Fb & Fa \end{array}}{a = b}\,{=}I$$

$$\begin{array}{c} \dfrac{a = b \qquad \begin{array}{c} \Pi \\ Pa \end{array}}{Pb}\,{=}E_2 \\ \Pi' \end{array}$$

$$\begin{array}{c} \dfrac{a = b \qquad \begin{array}{c} \Pi \\ Pb \end{array}}{Pa}\,{=}E_2 \\ \Pi' \end{array}$$

Our Symmetry Derivation in Natural Deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{[Fa] \qquad [Fa]}{a = a}\,{=}I}{(\lambda x.x = a)a}\,\lambda I \qquad a = b}{(\lambda x.x = a)b}\,{=}E}{b = a}\,\lambda E$$

With these Left/Right Rules . . .

Spec is height-preserving admissible.

We can eliminate *Cut*, as usual.

But eliminating *Cut* hardly seems worth it!

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} {=}R$$

$$\frac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y} {=}L_1 \qquad \frac{X \succ Pb; Y \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y} {=}L_2$$

Each rule breaks the subformula property.

$=R$ might be excusable (by analogy with $\forall R/\exists L$),

but in $=L$, P can be *any* predicate,

primitive or complex.

For *analytic* rules, we must look elsewhere.

IDENTITY *AXIOMS*

The Power of Reflexivity

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y}{=}R$$

$=R$ says that we have $a = b$ if we can transport a-features to b (and *vice versa*).

So we are in a position to transport a-features to b, and if we already knew that *being identical to* a was an a-feature, then that's enough show that a is identical to b.

From Rules to *Axioms*: from $=R$ to *Refl*

$$\frac{}{\succ a = a;}\,\textit{Refl}$$

$$\frac{Fa \succ Fa; \quad Fa \succ Fa;}{\succ a = a;}\,{=}R$$

And *Refl* is enough to recover $=R$

Replace this:

$$\dfrac{\begin{array}{c}\Delta_1\\ X, Fa \succ Fb; Y\end{array} \qquad \begin{array}{c}\Delta_2\\ X, Fb \succ Fa; Y\end{array}}{X \succ a = b; Y}\;{=}R$$

With this:

$$\dfrac{\dfrac{\dfrac{}{\succ a = a;}\;\mathit{Refl}}{\succ \lambda x.(a = x)a;}\;\lambda R \qquad \begin{array}{c}\Delta_1[F/\lambda x.(a = x)]\\ X, \lambda x.(a = x)a \succ \lambda x.(a = x)b; Y\end{array}}{\dfrac{X \succ \lambda x.(a = x)b; Y}{X \succ a = b; Y}\;\lambda R}\;\mathit{Cut}$$

The Problem with $=L$

$$\frac{X \succ Pa; Y \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}{\scriptstyle =L_1} \qquad \frac{X \succ Pb; Y \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y}{\scriptstyle =L_2}$$

This looks just like a *Cut* on Pa/Pb, at the cost of granting $a = b$.

From =*L* to =*L.ax* and back.

$$\dfrac{}{a = b, Pa \succ Pb;}\;{=}L.ax_1 \qquad \dfrac{}{a = b, Pb \succ Pa;}\;{=}L.ax_2$$

$$\dfrac{Pa \succ Pa; \qquad Pb \succ Pb;}{a = b, Pa \succ Pb;}\;{=}L_1 \qquad \dfrac{Pb \succ Pb; \qquad Pa \succ Pa;}{a = b, Pb \succ Pa;}\;{=}L_2$$

$$\dfrac{\dfrac{X \succ Pa; Y \qquad \dfrac{}{a = b, Pa \succ Pb;}\;{=}L.ax_1}{a = b, X \succ Pb; Y}\;\mathit{Cut} \qquad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y}\;\mathit{Cut}$$

$$\dfrac{\dfrac{X \succ Pb; Y \qquad \dfrac{}{a = b, Pb \succ Pa;}\;{=}L.ax_2}{a = b, X \succ Pa; Y}\;\mathit{Cut} \qquad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y}\;\mathit{Cut}$$

We can restrict $=$*L.ax* to *primitive* predicates

$$
\dfrac{\dfrac{\dfrac{}{a = b, Pa \succ Pb;}{=}L.ax_1}{a = b, Pa \wedge Qa \succ Pb;}\wedge L \qquad \dfrac{\dfrac{}{a = b, Qa \succ Qb;}{=}L.ax_1}{a = b, Pa \wedge Qa \succ Qb;}\wedge L}{\dfrac{a = b, Pa \wedge Qa \succ Pa \wedge Qb;}{a = b, \lambda x.(Px \wedge Qx)a \succ \lambda x.(Px \wedge Qx)b;}\lambda}\wedge R
$$

We can restrict $=$*L.ax* to *primitive* predicates

$$\dfrac{\dfrac{\dfrac{\overline{a = b, Pb \succ Pa;}\;{\scriptstyle =L.ax_2}}{a = b, Pb, \neg Pa \succ\ ;}\;{\scriptstyle \neg L}}{a = b, \neg Pa \succ \neg Pb;}\;{\scriptstyle \neg R}}{a = b, \lambda x.(\neg Px)a \succ \lambda x.(\neg Px)b;}\;\lambda$$

We can restrict $=$*L.ax* to *primitive* predicates

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a = b, Pac \succ Pbc;}\;{=}L.ax_2}{a = b, \forall y Pay \succ Pbc;}\;\neg L}{a = b, \forall y Pay \succ \forall y Pby;}\;\neg R}{a = b, \lambda x.(\forall y Pxy)a \succ \lambda x.(\forall y Pxy)b;}\;\lambda$$

Identity *axioms* in Natural Deduction

SEQUENT CALCULUS	NATURAL DEDUCTION
$\dfrac{}{\succ a = a;}\,\mathit{Refl}$	$a = a$
$\dfrac{}{a = b, Pa \succ Pb;}\,\mathit{L.ax}_1$	$\dfrac{a = b \qquad Pa}{Pb}\,{=}E$
$\dfrac{}{a = b, Pb \succ Pa;}\,\mathit{L.ax}_2$	$\dfrac{a = b \qquad Pb}{Pa}\,{=}E$

Now eliminate *Cut*

Now that identity is given by axioms,
Cut elimination proceeds largely
like the system *without* identity.

$$\dfrac{\dfrac{}{a = b, Pa \succ Pb;}\,{=}L.ax_1 \qquad \dfrac{}{c = b, Pb \succ Pc;}\,{=}L.ax_2}{a = b, c = b, Pa \succ Pc;}\,Cut$$

becomes

$$\dfrac{}{a = b, c = b, Pa \succ Pc;}\,{=}L??$$

Now eliminate *Cut*

It suffices to close the *axioms* under *Cut*.

$$\dfrac{}{I^a_b, Pa \succ Pb;}{\scriptstyle =L.ax^*}$$

Where I^a_b is any multiset
of identities *linking* a *to* b,
and P is any primitive predicate.

(a) The *empty* multiset links a to a.

(b) $a = b$ links a to b and b to a.

(c) If X links a to b and Y links
b to c then X, Y links a to c.

(We can leave 'Pa' out if it is $a = a$.)

Different Sequent Systems

- $=$*Df* + *Cut* + *Spec*
  - It's easy to show that $=$*Df* is *uniquely defining*.
- $=$*L/R* + *Cut* + *Spec*
  - Straightforward translation between $=$*Df* and $=$*L/R*.
- $=$*L/R* + *Cut*
  - Since *Spec* is height-preserving admissible.
- $=$*L/R*
  - $=$*L/R* rules don't have the subformula property.
- $=$*L.ax* + *Refl* + *Cut*
  - Easy translation between $=$*L/R* and $=$*L.ax* + *Refl*, using *Cut*.
- $=$*L.ax** + *Refl*
  - $=$*L.ax** + *Refl* are *analytic* and *conservatively extending*.

Kinds of Identity Rules

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{\overline{X \succ a = b; Y}} {=}Df$$

$$\frac{}{\succ a = a;}\,Refl \qquad \frac{}{I^a_b, Pa \succ Pb;}\,{=}L.ax_*$$

- $=$*Df* defines identity by giving conditions under which $a = b$ may be proved. We're in a position to prove $a = b$ iff we're in a position to transfer Fa to Fb (and back) for arbitrary F.

- *Refl* and $=L.ax_*$ are *semantic constraints connecting* primitive predicates.

- These two characterisations are *equivalent* as far as derivability goes.

THANK YOU!

References: Natural Deduction and Sequent Systems

ARTHUR N. PRIOR
“The Runabout Inference-Ticket”
Analysis 21:38–39, 1960

NUEL D. BELNAP
“Tonk, Plonk and Plink”
Analysis 22:130–134, 1962

SARA NEGRI AND JAN VON PLATO
Structural Proof Theory
Cambridge University Press, 2001

MICHEL PARIGOT
“λμ-Calculus: An Algorithmic Interpretation of Classical Natural Deduction”
International Conference on Logic for Programming Artificial Intelligence and Reasoning, LNAI 624, 190–201, 1992

GREG RESTALL
“Speech Acts & the Quest for a Natural Account of Classical Proof”
Presentation at the Berkeley Logic Group, September 2020
https://consequently.org/presentation/

References: Defining Rules, Specification and Identity

DANA SCOTT

"Rules and Derived Rules"

Logical Theory and Semantic Analysis, Springer, 147–161, 1974

IAN HACKING

"What is Logic?"

Journal of Philosophy, 76:285–319, 1979

KOSTA DOŠEN

"Logical Constants as Punctuation Marks"

Notre Dame Journal of Formal Logic 30:362–381, 1989

ARNON AVRON

"Gentzen-type systems, Resolution and Tableaux"

Journal of Automated Reasoning 10:265–281, 1993

G. SAMBIN, G. BATTILOTTI AND C. FAGGIAN

"Basic logic: Reflection, Symmetry, Visibility"

Journal of Symbolic Logic 65:979–1013, 2014

GREG RESTALL

"Generality and Existence 1: Quantification and Free Logic"

Review of Symbolic Logic, 12:1–29, 2019

STEPHEN READ

"Identity and Harmony"

Analysis, 64(2):113–115, 2004

Thank you!

SLIDES: `https://consequently.org/presentation/2021/comparing-identity-rules`

FEEDBACK: `@consequently` on *Twitter*,
or *email* `atrestall@unimelb.edu.au`