Classical Logic & Intuitionistic Logic
looking both ways

Greg Restall

University of St Andrews

6 September 2022

MY PLAN

1. MY QUESTION
2. LOGICS & PROOFS & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists

1. MY QUESTION
2. LOGICS & PROOFS & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists

How are we to understand the relationship between Classical logic & intuitionistic logic?

In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

Errett Bishop

Foundations of Constructive Analysis 1967

Discussion of classical mathematical reasoning

In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

Errett Bishop

Foundations of Constructive Analysis

1967

Discussion of classical mathematical reasoning

In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

Contrast with intuitionistic reasoning

Errett Bishop

Foundations of Constructive Analysis 1967

In classical mathematics we can reason like this:

$$\begin{array}{c} x > y \\ \vdots \\ \bot \end{array}$$

In classical mathematics we can reason like this:

$$\dfrac{\begin{array}{c}[x > y]^1 \\ \vdots \\ \bot\end{array}}{\neg(x > y)}\ \neg I^1$$

In classical mathematics we can reason like this:

$$\dfrac{\dfrac{\begin{array}{c}[x > y]^{1}\\ \vdots\\ \bot\end{array}}{\neg(x>y)}\,\neg I^{1}}{x \leqslant y}\,\neg{>}\text{ to }\leqslant$$

In classical mathematics we can reason like this:

$$\dfrac{\dfrac{\begin{array}{c}[x>y]^{1}\\ \vdots\\ \bot\end{array}}{\neg(x>y)}\,\neg I^{1}}{x\leq y}\,\neg{>}\text{ to }\leq$$

But when you're reasoning constructively, this last step is blocked.

Logicians have a very good handle on classical logic & on intuitionistic logic, their properties & what it is to develop a theory in one or other logic.

Logicians have a very good handle on classical logic & on intuitionistic logic, their properties & what it is to develop a theory in one or other logic.

But we have less of a consensus about what to do with this knowledge.

Logicians have a very good handle on classical logic & on intuitionistic logic, their properties & what it is to develop a theory in one or other logic.

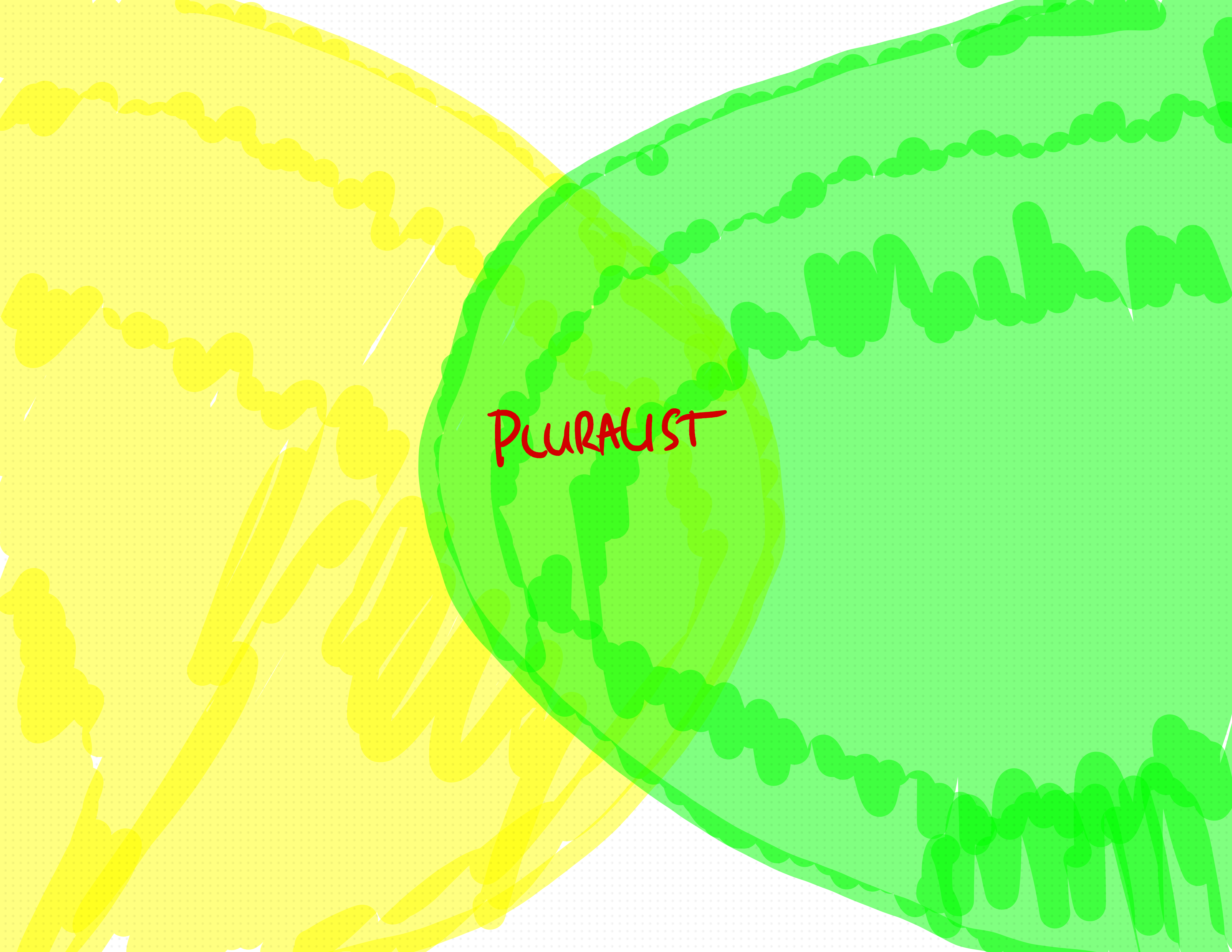
But we have less of a consensus about what to do with this knowledge.

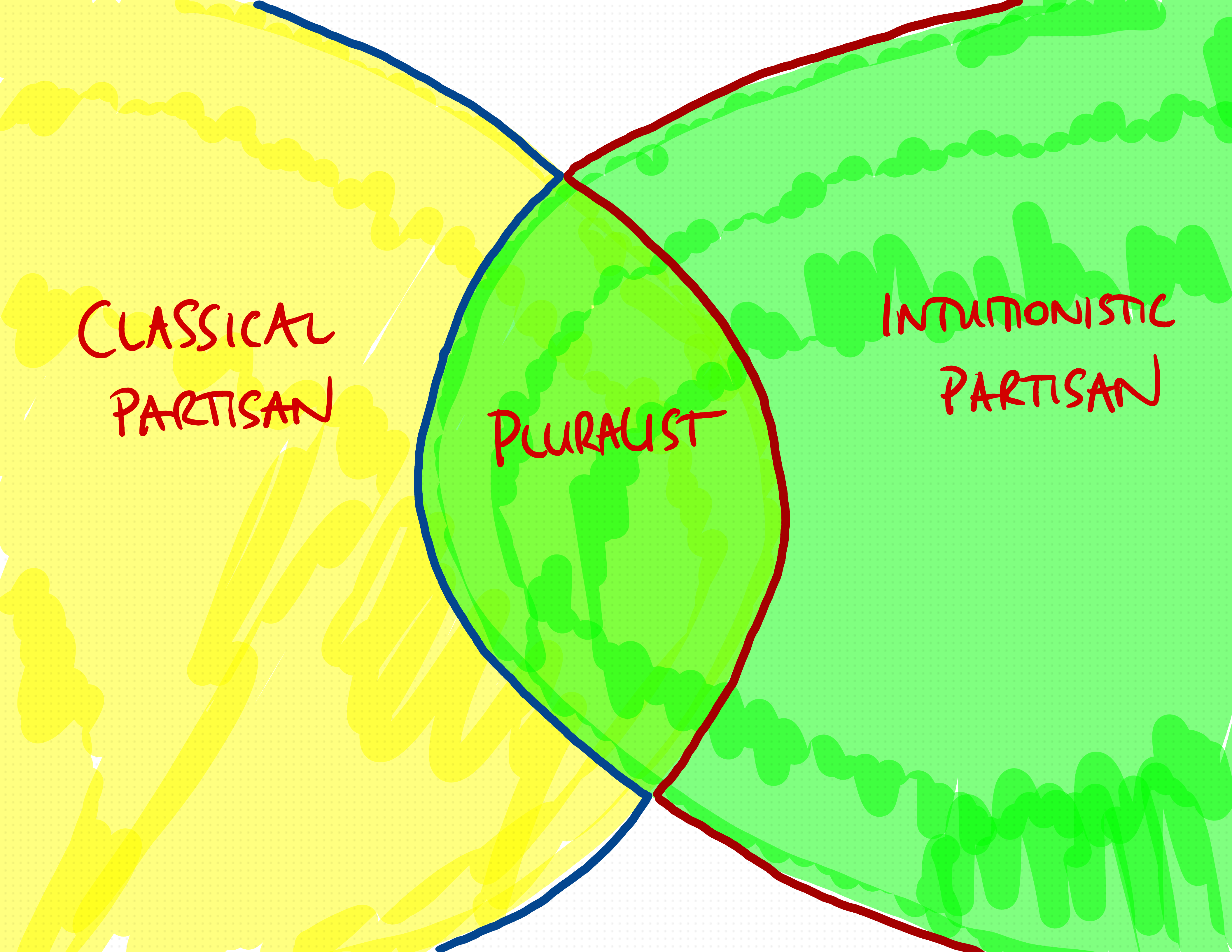
I want to help us to think more clearly about this.

ENDORSE
CLASSICAL
LOGIC

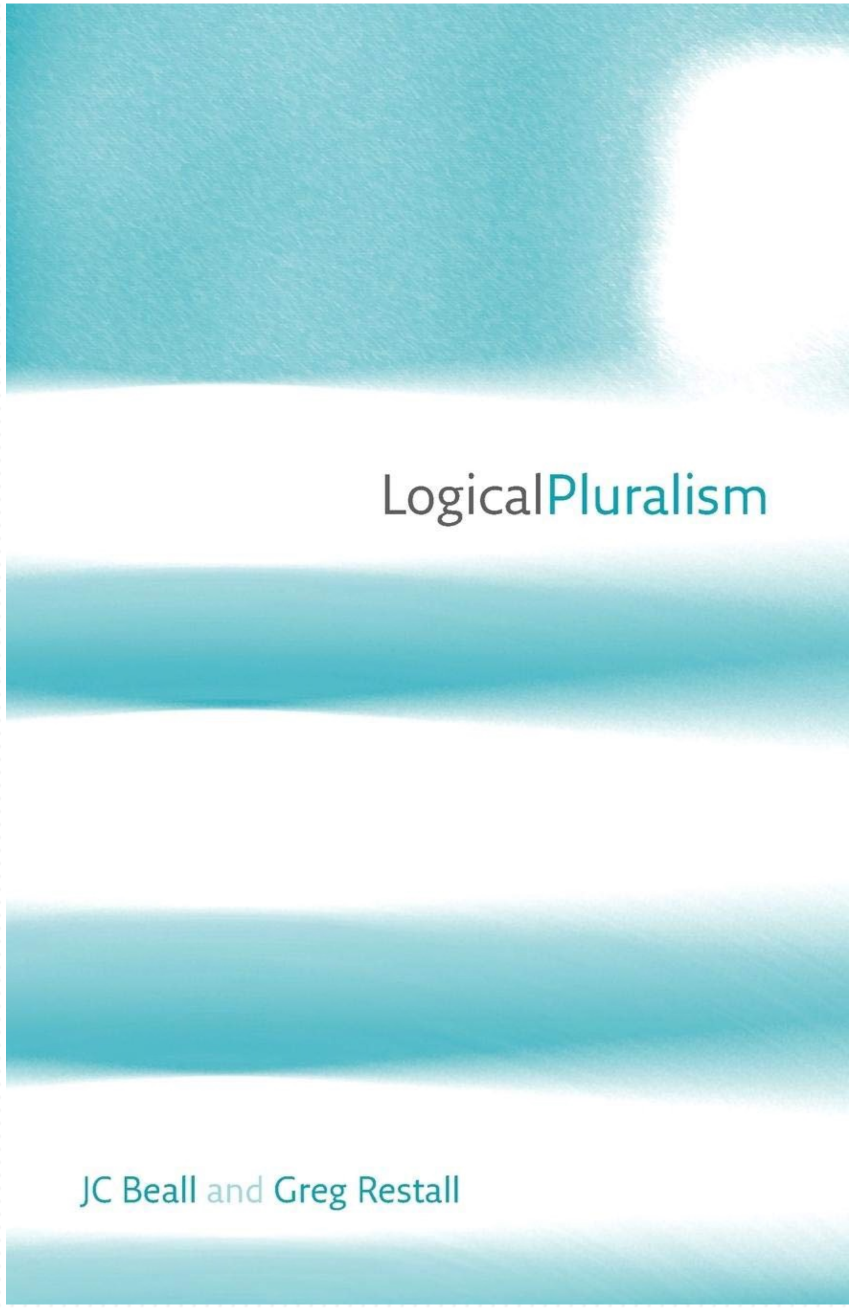
ENDORSE

INTUITIONISTIC
LOGIC


PLURALIST

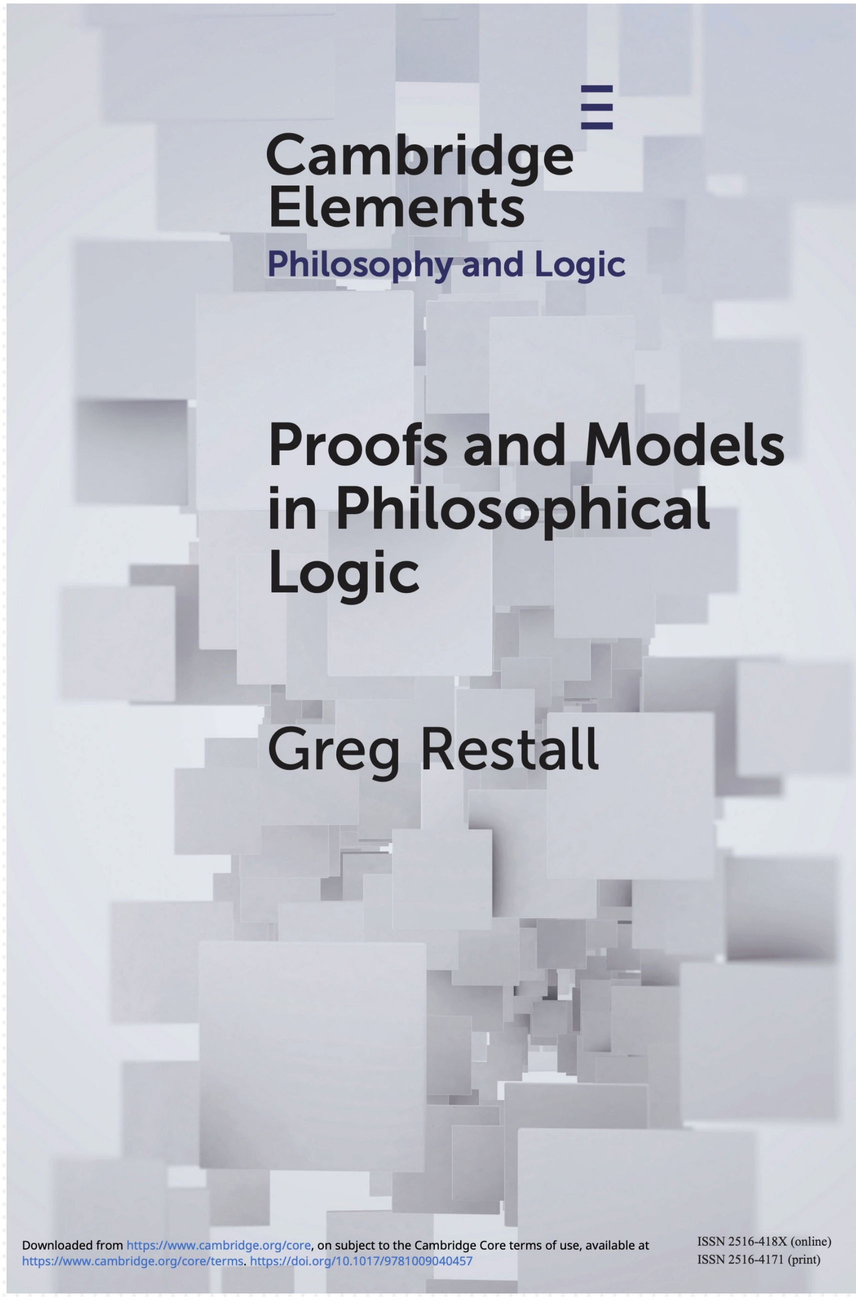


CLASSICAL
PARTISAN
PLURALIST
INTUITIONISTIC
PARTISAN



THIS IS A SHAMELESS ADVERTISEMENT

1. MY QUESTION
2. LOGICS & PROOFS & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists



THIS IS ANOTHER SHAMELESS ADVERTISEMENT

NATURAL DEDUCTION

&I) $\dfrac{A \quad B}{A \,\&\, B}$ &E) $\dfrac{A \,\&\, B}{A} \qquad \dfrac{A \,\&\, B}{B}$

∨I) $\dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \begin{matrix}(A)\\ C\end{matrix} \quad \begin{matrix}(B)\\ C\end{matrix}}{C}$

⊃I) $\dfrac{\begin{matrix}(A)\\ B\end{matrix}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

∀I) $\dfrac{A}{\forall x A^a_x}$ ∀E) $\dfrac{\forall x A}{A^x_t}$

∃I) $\dfrac{A^x_t}{\exists x A}$ ∃E) $\dfrac{\exists x A \quad \begin{matrix}(A^x_a)\\ B\end{matrix}}{B}$

$\curlywedge_I$) $\dfrac{\curlywedge}{A}$ $\curlywedge_C$) $\dfrac{\begin{matrix}(\sim A)\\ \curlywedge\end{matrix}}{A}$

Dag Prawitz

NATURAL DEDUCTION 1965

NATURAL DEDUCTION

&I) $\dfrac{A \quad B}{A \,\&\, B}$ &E) $\dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B}$

∨I) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \begin{array}{c}(A)\\ C\end{array} \quad \begin{array}{c}(B)\\ C\end{array}}{C}$

⊃I) $\dfrac{\begin{array}{c}(A)\\ B\end{array}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

∀I) $\dfrac{A}{\forall x A^a_x}$ ∀E) $\dfrac{\forall x A}{A^x_t}$

∃I) $\dfrac{A^x_t}{\exists x A}$ ∃E) $\dfrac{\exists x A \quad \begin{array}{c}(A^x_a)\\ B\end{array}}{B}$

$\curlywedge_{\mathrm{I}}$) $\dfrac{\curlywedge}{A}$ $\curlywedge_{\mathrm{C}}$) $\dfrac{\begin{array}{c}(\sim A)\\ \curlywedge\end{array}}{A}$

$$\dfrac{\dfrac{p \wedge \neg r}{\neg r}\,\wedge E \qquad \dfrac{\dfrac{p \to (q \to r) \qquad \dfrac{p \wedge \neg r}{p}\,\wedge E}{q \to r}\,\to E \qquad q}{r}\,\to E}{\bot}\,\neg E$$

Dag Prawitz

NATURAL DEDUCTION 1965

NATURAL DEDUCTION

&I) $\dfrac{A \quad B}{A \,\&\, B}$ &E) $\dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B}$

∨I) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \overset{(A)}{C} \quad \overset{(B)}{C}}{C}$

⊃I) $\dfrac{\overset{(A)}{B}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

∀I) $\dfrac{A}{\forall x A^a_x}$ ∀E) $\dfrac{\forall x A}{A^x_t}$

∃I) $\dfrac{A^x_t}{\exists x A}$ ∃E) $\dfrac{\exists x A \quad \overset{(A^x_a)}{B}}{B}$

$\curlywedge_I$) $\dfrac{\curlywedge}{A}$ $\curlywedge_C$) $\dfrac{\overset{(\sim A)}{\curlywedge}}{A}$

$$\dfrac{\dfrac{\dfrac{p\wedge\neg r}{\neg r}\,\wedge E \qquad \dfrac{\dfrac{p\to(q\to r) \quad \dfrac{p\wedge\neg r}{p}\,\wedge E}{q\to r}\,\to E \qquad [q]^1}{r}\,\to E}{\bot}\,\neg E}{\neg q}\,\neg I^1$$

Dag Prawitz

NATURAL DEDUCTION 1965

NATURAL DEDUCTION

&I) $\dfrac{A \quad B}{A \,\&\, B}$ &E) $\dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B}$

∨I) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \begin{matrix}(A)\\ C\end{matrix} \quad \begin{matrix}(B)\\ C\end{matrix}}{C}$

⊃I) $\dfrac{\begin{matrix}(A)\\ B\end{matrix}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

∀I) $\dfrac{A}{\forall x A^a_x}$ ∀E) $\dfrac{\forall x A}{A^x_t}$

∃I) $\dfrac{A^x_t}{\exists x A}$ ∃E) $\dfrac{\exists x A \quad \begin{matrix}(A^x_a)\\ B\end{matrix}}{B}$

⋏$_I$) $\dfrac{\curlywedge}{A}$ ⋏$_C$) $\dfrac{\begin{matrix}(\sim A)\\ \curlywedge\end{matrix}}{A}$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p\wedge\neg r]^2}{\neg r}\,{\scriptstyle \wedge E} \qquad \dfrac{\dfrac{p\to(q\to r) \quad \dfrac{[p\wedge\neg r]^2}{p}\,{\scriptstyle \wedge E}}{q\to r}\,{\scriptstyle \to E} \qquad [q]^1}{r}\,{\scriptstyle \to E}}{\bot}\,{\scriptstyle \neg E}}{\neg q}\,{\scriptstyle \neg I^1}}{(p\wedge\neg r)\to\neg q}\,{\scriptstyle \to I^2}$$

Dag Prawitz

NATURAL DEDUCTION 1965

NATURAL DEDUCTION

20 NATURAL DEDUCTION OF GENTZEN-TYPE

&I) $\dfrac{A \quad B}{A \,\&\, B}$ &E) $\dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B}$

∨I) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \overset{(A)}{C} \quad \overset{(B)}{C}}{C}$

⊃I) $\dfrac{\overset{(A)}{B}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

∀I) $\dfrac{A}{\forall x A^a_x}$ ∀E) $\dfrac{\forall x A}{A^x_t}$

∃I) $\dfrac{A^x_t}{\exists x A}$ ∃E) $\dfrac{\exists x A \quad \overset{(A^x_a)}{B}}{B}$

$\curlywedge_I$) $\dfrac{\curlywedge}{A}$ $\curlywedge_C$) $\dfrac{\overset{(\sim A)}{\curlywedge}}{A}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[p\wedge\neg r]^2}{\neg r}\,\wedge E \qquad \dfrac{\dfrac{p\to(q\to r) \qquad \dfrac{[p\wedge\neg r]^2}{p}\,\wedge E}{q\to r}\,\to E \qquad [q]^1}{r}\,\to E}{\bot}\,\neg E}{\neg q}\,\neg I^1}{(p\wedge\neg r)\to\neg q}\,\to I^2
$$

$$p\to(q\to r) \vdash (p\wedge\neg r)\to\neg q$$

Dag Prawitz

NATURAL DEDUCTION 1965

NATURAL DEDUCTION

&I) $\dfrac{A \quad B}{A \,\&\, B}$ &E) $\dfrac{A \,\&\, B}{A} \quad \dfrac{A \,\&\, B}{B}$

∨I) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ ∨E) $\dfrac{A \vee B \quad \begin{matrix}(A)\\ C\end{matrix} \quad \begin{matrix}(B)\\ C\end{matrix}}{C}$

⊃I) $\dfrac{\begin{matrix}(A)\\ B\end{matrix}}{A \supset B}$ ⊃E) $\dfrac{A \quad A \supset B}{B}$

∀I) $\dfrac{A}{\forall x A^a_x}$ ∀E) $\dfrac{\forall x A}{A^x_t}$

∃I) $\dfrac{A^x_t}{\exists x A}$ ∃E) $\dfrac{\exists x A \quad \begin{matrix}(A^x_a)\\ B\end{matrix}}{B}$

$\curlywedge_I$) $\dfrac{\curlywedge}{A}$ $\curlywedge_C$) $\dfrac{\begin{matrix}(\sim A)\\ \curlywedge\end{matrix}}{A}$

THE ODD RULE OUT!

$$\dfrac{\dfrac{\dfrac{\dfrac{[p\wedge\neg r]^2}{\neg r}\,\wedge E \qquad \dfrac{\dfrac{p\to(q\to r) \qquad \dfrac{[p\wedge\neg r]^2}{p}\,\wedge E}{q\to r}\,\to E \qquad [q]^1}{r}\,\to E}{\bot}\,\neg E}{\neg q}\,\neg I^1}{(p\wedge\neg r)\to\neg q}\,\to I^2$$

Dag Prawitz

NATURAL DEDUCTION 1965

NATURAL DEDUCTION & TYPE THEORIES

$$\supset \mathrm{I})\quad \dfrac{\begin{matrix} x:(A) \\ t:B \end{matrix}}{\lambda x.t : A \supset B} \qquad\qquad \supset \mathrm{E})\quad \dfrac{t:A \quad s:A\supset B}{(s\,t):B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x:p\wedge\neg r]^2}{\mathrm{snd}\ x:\neg r}\wedge E \qquad \dfrac{\dfrac{y:p\to(q\to r) \quad \dfrac{[x:p\wedge\neg r]^2}{\mathrm{fst}\,x:p}\wedge E}{(y\ \mathrm{fst}\,x):q\to r}\to E \qquad [z:q]^1}{z(y\ \mathrm{fst}\,x):r}\to E}{(\mathrm{snd}\ x)(z(y\ \mathrm{fst}\,x)):\bot}\neg E}{\lambda z.(\mathrm{snd}\ x)(z(y\ \mathrm{fst}\,x)):\neg q}\neg I^1}{\lambda x.\lambda z.(\mathrm{snd}\ x)(z(y\ \mathrm{fst}\,x)):(p\wedge\neg r)\to\neg q}\to I^2$$

NATURAL DEDUCTION & TYPE THEORIES

$$\supset I)\quad \dfrac{\begin{matrix} x:(A) \\ t:B \end{matrix}}{\lambda x.t : A \supset B} \qquad\qquad \supset E)\quad \dfrac{t: A \quad s: A \supset B}{(s\,t): B}$$

You can think of the terms as denoting procedures for construction, verification, or grounding

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x: p\wedge\neg r]^2}{\mathrm{snd}\,x:\neg r}\,\wedge E \qquad \dfrac{\dfrac{y: p\to(q\to r) \qquad \dfrac{[x:p\wedge\neg r]^2}{\mathrm{fst}\,x : p}\,\wedge E}{(y\,\mathrm{fst}\,x): q\to r}\,\to E \qquad [z:q]^1}{z(y\,\mathrm{fst}\,x): r}\,\to E}{(\mathrm{snd}\,x)(z(y\,\mathrm{fst}\,x)):\bot}\,\neg E}{\lambda z.(\mathrm{snd}\,x)(z(y\,\mathrm{fst}\,x)):\neg q}\,\neg I^1}{\lambda x.\lambda z.(\mathrm{snd}\,x)(z(y\,\mathrm{fst}\,x)) : (p\wedge\neg r)\to\neg q}\,\to I^2
$$

NATURAL DEDUCTION & TYPE THEORIES

Classical Proofs as Programs *

Michel Parigot
Equipe de logique — CNRS UA 753
45-55 5ème étage, Université Paris 7
2 place jussieu, 75251 PARIS Cedex 05, FRANCE
e-mail: parigot@logique.jussieu.fr

Abstract. We present an extension of the correspondence between intuitionistic proofs and functional programs to *classical proofs*, and more precisely to second order classical proofs. The advantage of classical logic in this context is that it allows to model *imperative features* of programming languages too (cf [5]). But there is an intrinsic difficulty with classical logic which lies in certain non-determinism of its computational interpretations. The use of a natural deduction system removes a part of this non determinism by fixing the inputs to the left of the sequents (cf [10] and [11]). However a conflict remains between the *confluence* of the computation mechanism and the *uniqueness* of the representation of data (for instance the uniqueness of the representation of the natural number 1). In this paper we develop the solution to this problem proposed in [11]: we show how to *extract the intuitionistic representation of a data from a classical one* using an "output" operator, while keeping a confluent computation mechanism. This result allows to extend in a sound way the proofs-as-programs paradigm to classical proofs in a framework where all the usual theoretical properties of intuitionistic proofs still hold.

(You can extend this correspondence to classical natural deduction, at the cost of losing some nice features of constructive reasoning)

$$x : A^x \vdash A$$

$$\frac{u : \Gamma \vdash B, \Delta}{\lambda x.u : \Gamma / A^x \vdash A \to B, \Delta} \to_i \qquad \frac{t : \Gamma_1 \vdash A \to B, \Delta_1 \quad u : \Gamma_2 \vdash A, \Delta_2}{(t)u : \Gamma_1, \Gamma_2 \vdash B, \Delta_1, \Delta_2}$$

$$\frac{u : \Gamma \vdash A[Y/X], \Delta}{u : \Gamma \vdash \forall X A, \Delta} \forall_i^2 \ (*) \qquad \frac{u : \Gamma \vdash \forall X A, \Delta}{u : \Gamma \vdash A[T/X], \Delta} \forall_e^2$$

$$\frac{u : \Gamma \vdash A[y/x], \Delta}{u : \Gamma \vdash \forall x A, \Delta} \forall_i^1 \ (*) \qquad \frac{u : \Gamma \vdash \forall x A, \Delta}{u : \Gamma \vdash A[\tau/x], \Delta} \forall_e^1$$

$$\frac{t : \Gamma \vdash A, \Delta}{\mu\beta.[\alpha]t : \Gamma \vdash B, (\Delta, A^\alpha)/B^\beta} \mu$$

$$\frac{t : \Gamma \vdash A[\tau/x], \Delta}{t : \Gamma \vdash A[\sigma/x], \Delta} eq \ (**)$$

CLASSICAL MODEL THEORY

A MODEL M for a first-order language is a pair $\langle D, I\rangle$ where D is a nonempty set, called the DOMAIN, and I is a function on the language such that for each name a in the language, $I(a) \in D$, and if F is an n-ary predicate, then $I(F): D^n \mapsto \{0,1\}$, where D^n is the set of sequences of length n of objects from D.

- $I(\neg A, v) = 1$ iff $I(A, v) = 0$
- $I(A \wedge B, v) = 1$ iff $I(A, v) = 1$ and $I(B, v) = 1$
- $I(A \vee B, v) = 1$ iff $I(A, v) = 1$ or $I(B, v) = 1$
- $I(A \rightarrow B, v) = 1$ iff $I(A, v) = 0$ or $I(B, v) = 1$
- $I(\perp, v) = 1$ never
- $I(\forall x A, v) = 1$ iff for every u where $u \sim_x v$, $I(A, u) = 1$
- $I(\exists x A, v) = 1$ iff for some u where $u \sim_x v$, $I(A, u) = 1$

A pair of a model M and an assignment v is a COUNTEREXAMPLE to the argument $X \succ A$ iff $I(B, v) = 1$, for each $B \in X$, and $I(A, v) = 0$.

truth conditional semantics

BHK INTERPRETATION

BROUWER, HEYTING, KOLMOGOROV

p proves $A \& B$	*iff p is a pair* $\langle r, s\rangle$ *and r proves A and s proves B.*
p proves $A \vee B$	*iff p is a pair* $\langle n, q\rangle$, *where* $n=0$ *and q proves A, or* $n=1$ *and q proves B.*
p proves $A \rightarrow B$	*iff p is a rule q which transforms any proof s of A into a proof* $q(s)$ *of B, together with any supplementary information needed to convince us of this property.*
p proves $\perp$	*is impossible.*
p proves $\neg A$	*iff p proves* $A \rightarrow \perp$.
p proves $\exists x \in X\, \phi(x)$	*iff p is a pair* $\langle x, q\rangle$ *where x is a completely presented member of X and q is a proof that* $\phi(x)$.
p proves $\forall x \in X\, \phi(x)$	*iff p is a rule q such that for each completely presented member x of X,* $q(x)$ *is a proof of* $\phi(x)$, *together with any supplementary data needed to convince us of this property.*

The definitions just offered for implication and universal quantification are notoriously imprecise, because of the phrase "any supplementary data needed to convince of this property". This is one reason why we have presented the above as an "explanation" of the logical connectives, and not as a "definition".

Michael Beeson, Foundations of Constructive Mathematics, 1985

BHK INTERPRETATION

BROUWER, HEYTING, KOLMOGOROV

p *proves* $A \& B$	*iff* p *is a pair* $\langle r, s\rangle$ *and* r *proves* A *and* s *proves* B.
p *proves* $A \vee B$	*iff* p *is a pair* $\langle n, q\rangle$, *where* $n=0$ *and* q *proves* A, *or* $n=1$ *and* q *proves* B.
p *proves* $A \to B$	*iff* p *is a rule* q *which transforms any proof* s *of* A *into a proof* $q(s)$ *of* B, *together with any supplementary information needed to convince us of this property.*
p *proves* $\perp$	*is impossible.*
p *proves* $\neg A$	*iff* p *proves* $A \to \perp$.
p *proves* $\exists x \in X\, \phi(x)$	*iff* p *is a pair* $\langle x, q\rangle$ *where* x *is a completely presented member of* X *and* q *is a proof that* $\phi(x)$.
p *proves* $\forall x \in X\, \phi(x)$	*iff* p *is a rule* q *such that for each completely presented member* x *of* X, $q(x)$ *is a proof of* $\phi(x)$, *together with any supplementary data needed to convince us of this property.*

The definitions just offered for implication and universal quantification are notoriously imprecise, because of the phrase "any supplementary data needed to convince of this property". This is one reason why we have presented the above as an "explanation" of the logical connectives, and not as a "definition".

Michael Beeson, Foundations of Constructive Mathematics, 1985

BHK INTERPRETATION

BROUWER, HEYTING, KOLMOGOROV

p proves $A \& B$	*iff p is a pair* $\langle r, s\rangle$ *and r proves A and s proves B.*
p proves $A \vee B$	*iff p is a pair* $\langle n, q\rangle$, *where* $n=0$ *and q proves A, or* $n=1$ *and q proves B.*
p proves $A \rightarrow B$	*iff p is a rule q which transforms any proof s of A into a proof* $q(s)$ *of B, together with any supplementary information needed to convince us of this property.*
p proves $\perp$	*is impossible.*
p proves $\neg A$	*iff p proves* $A \rightarrow \perp$.
p proves $\exists x \in X\, \phi(x)$	*iff p is a pair* $\langle x, q\rangle$ *where x is a completely presented member of X and q is a proof that* $\phi(x)$.
p proves $\forall x \in X\, \phi(x)$	*iff p is a rule q such that for each completely presented member x of X,* $q(x)$ *is a proof of* $\phi(x)$, *together with any supplementary data needed to convince us of this property.*

The definitions just offered for implication and universal quantification are notoriously imprecise, because of the phrase "any supplementary data needed to convince of this property". This is one reason why we have presented the above as an "explanation" of the logical connectives, and not as a "definition".

Michael Beeson, Foundations of Constructive Mathematics, 1985

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

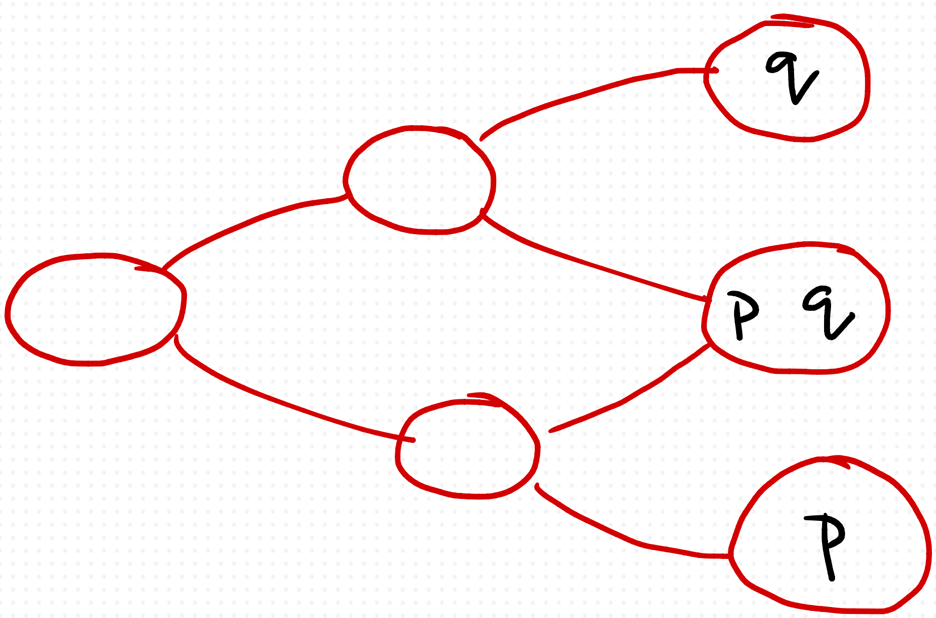
P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



truth conditional semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

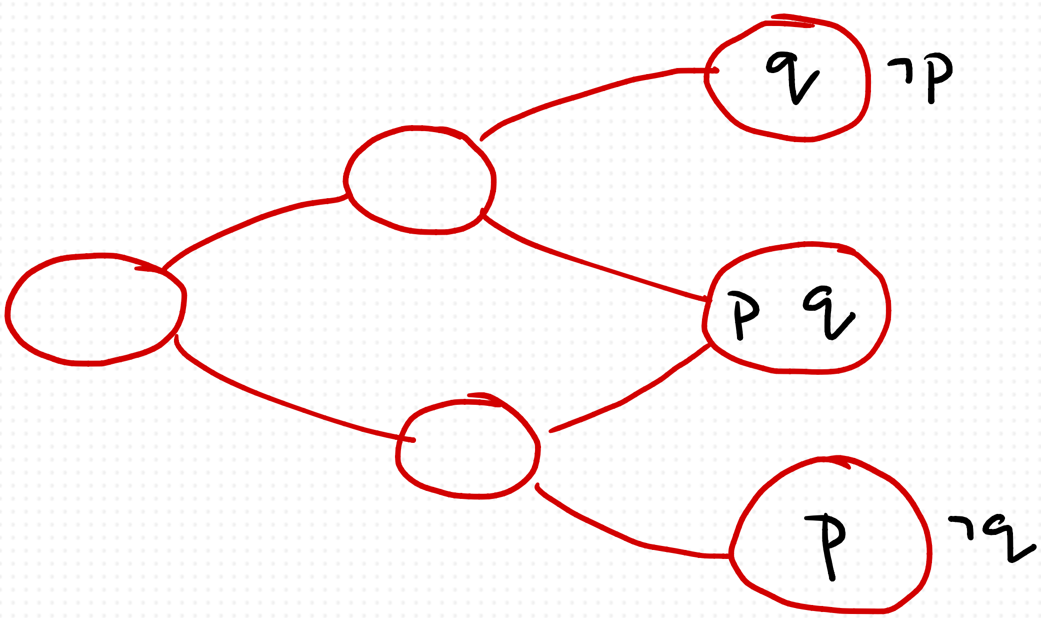
P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



truth conditional semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

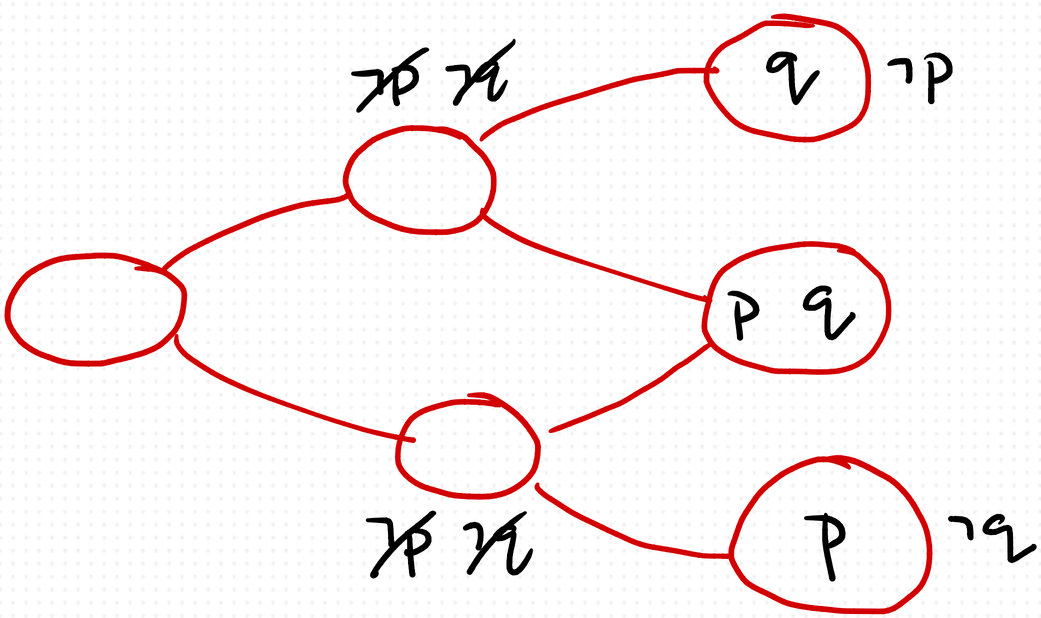
P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



truth conditional semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \vDash \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\vDash$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

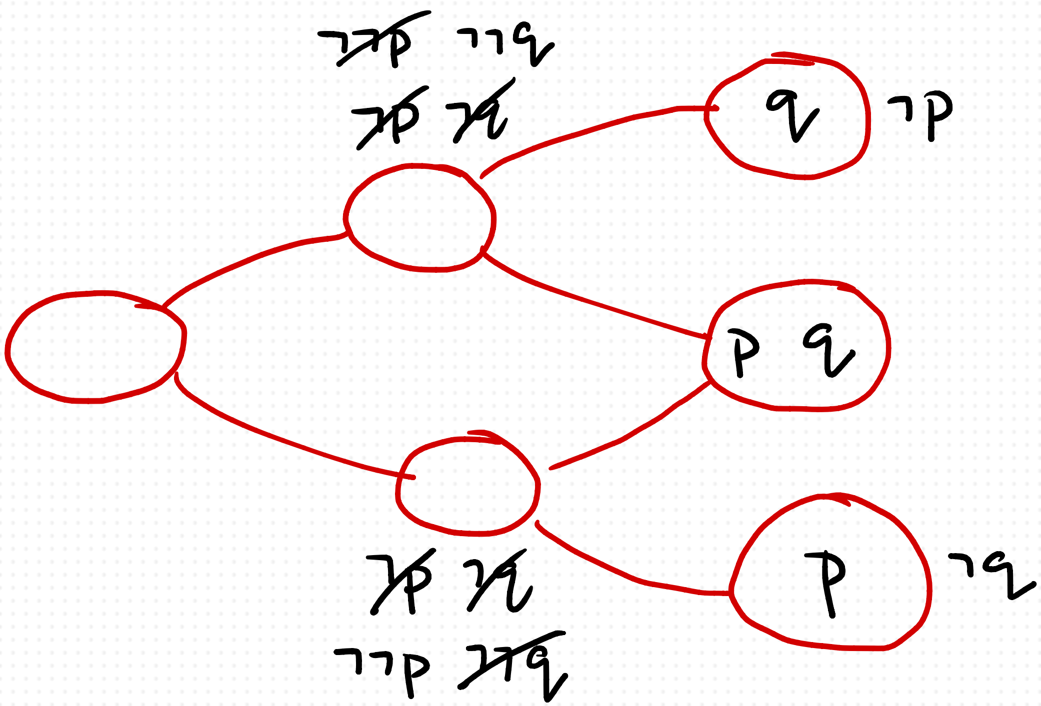
P0. if $\Gamma \vDash A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \vDash A$ (recall A is atomic).

P1. $\Gamma \vDash (X \wedge Y)$ iff $\Gamma \vDash X$ and $\Gamma \vDash Y$.

P2. $\Gamma \vDash (X \vee Y)$ iff $\Gamma \vDash X$ or $\Gamma \vDash Y$.

P3. $\Gamma \vDash \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \nvDash X$.

P4. $\Gamma \vDash (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \vDash X$, then $\Delta \vDash Y$.



truth conditional semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

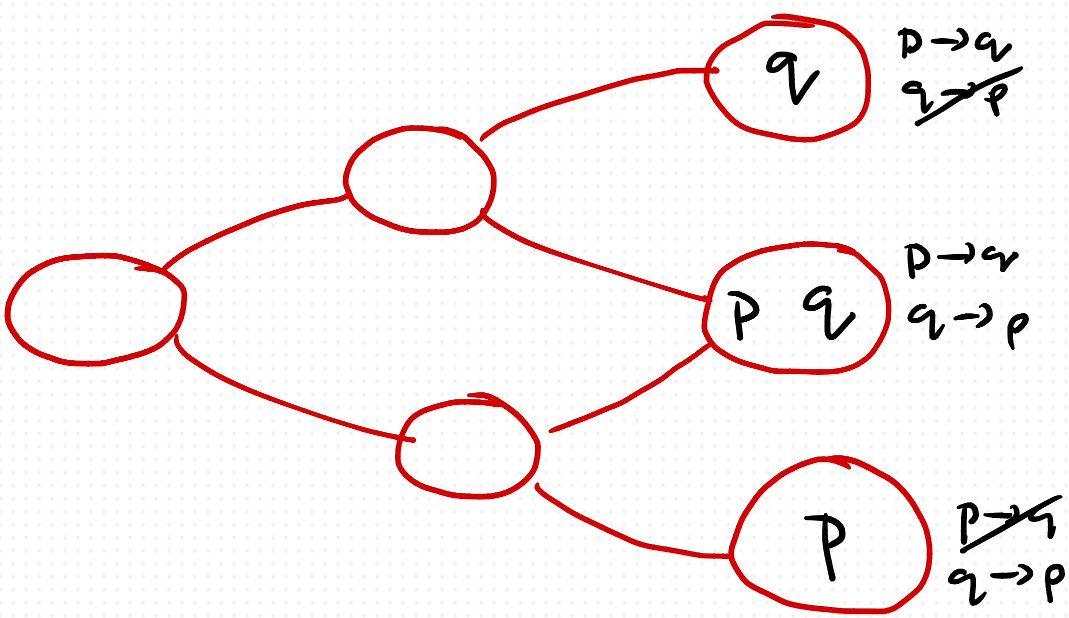
P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



truth conditional semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

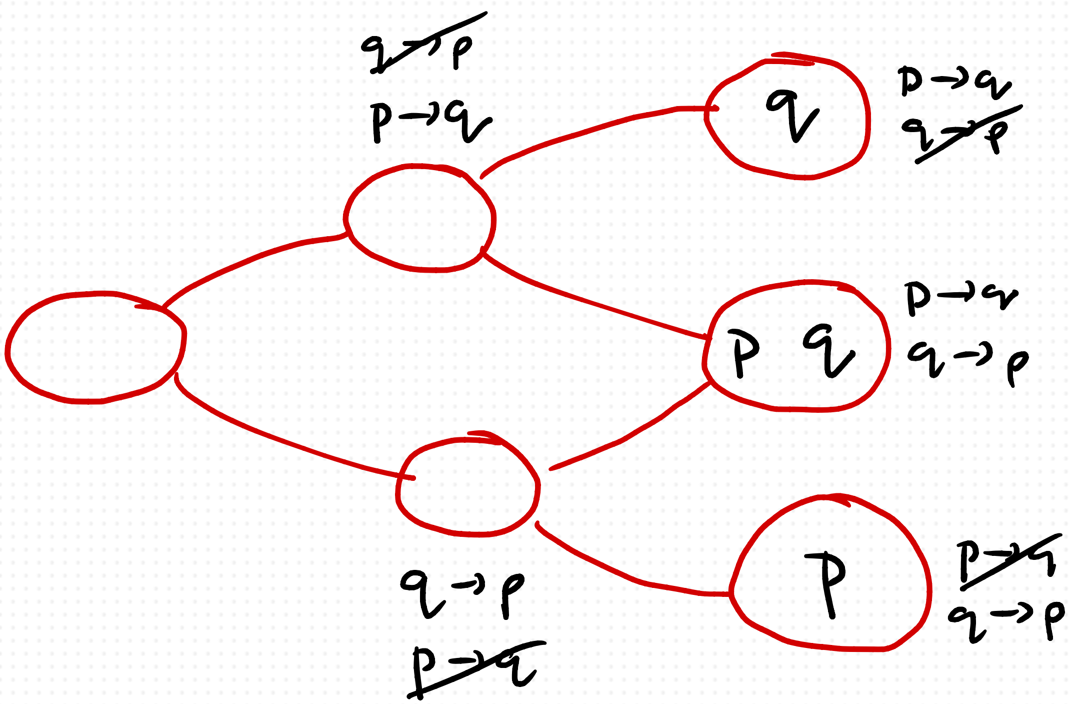
P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



truth conditional semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

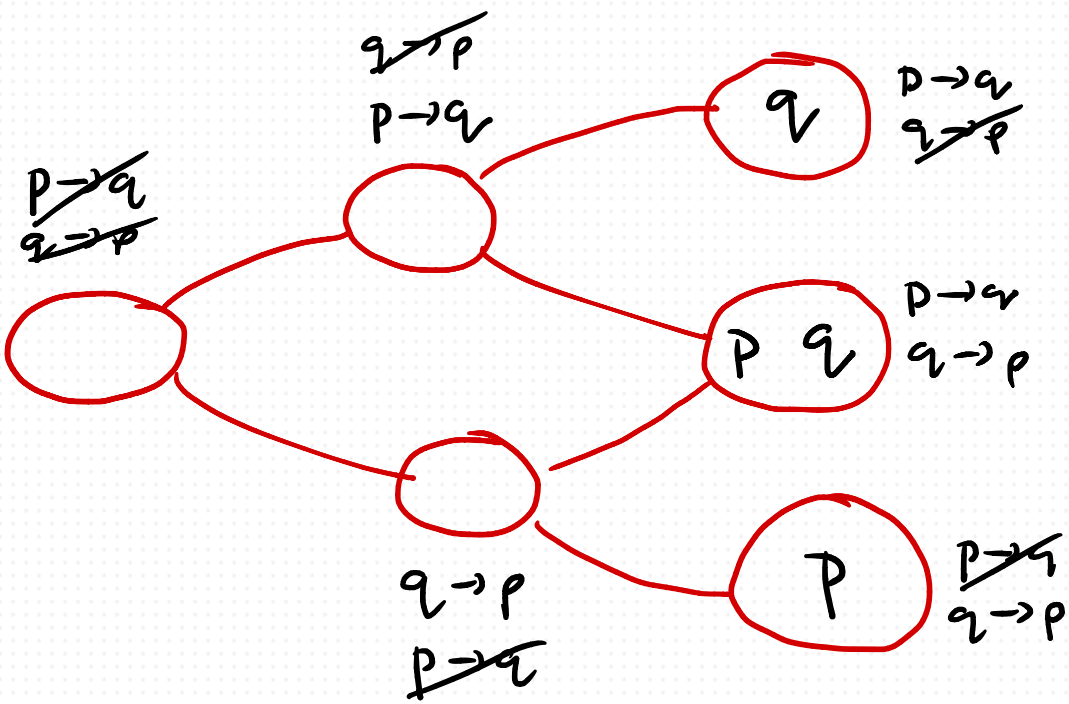
P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



truth conditional semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathscr{G}, \mathscr{R}, \models \rangle$, where $\mathscr{G}$ is a non-empty set, $\mathscr{R}$ is a transitive, reflexive relation on $\mathscr{G}$, and $\models$ (conveniently read "forces") is a relation between elements of $\mathscr{G}$ and formulas, satisfying the following conditions:

For any $\Gamma \in \mathscr{G}$

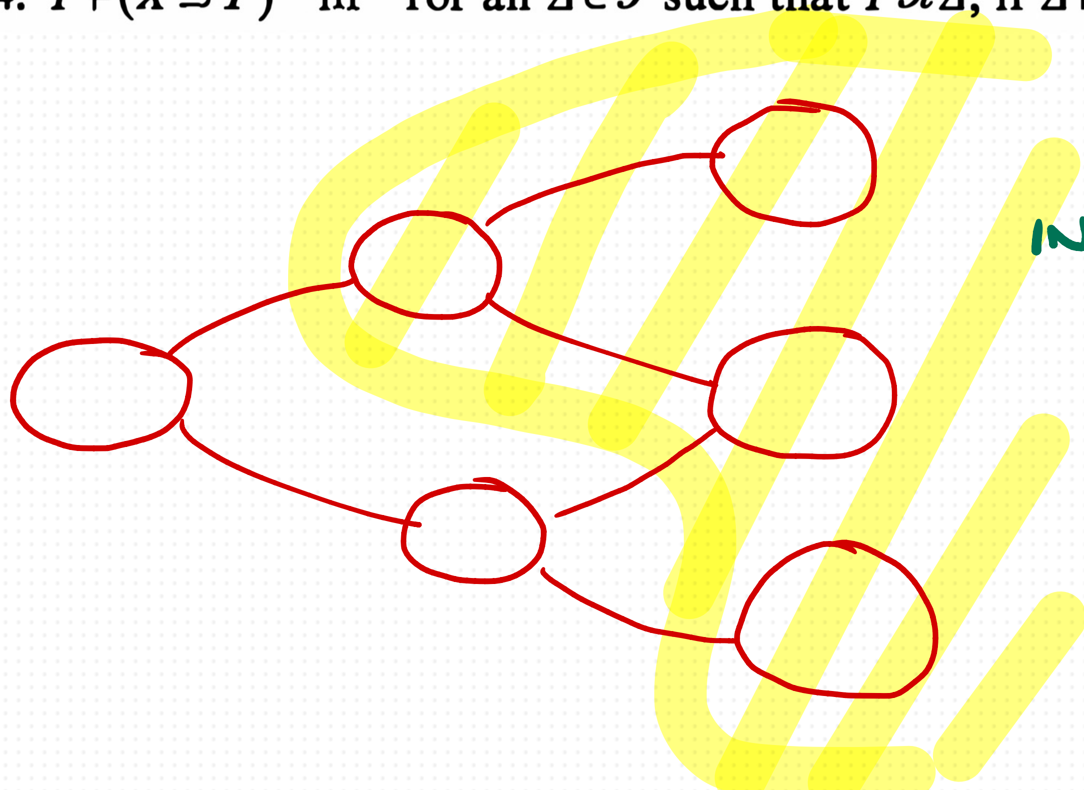
P0. if $\Gamma \models A$ and $\Gamma \mathscr{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathscr{G}$ such that $\Gamma \mathscr{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



IN ANY KRIPKE MODEL A 'PROPOSITION' IS AN UPWARDLY CLOSED SET OF POINTS

truth conditional semantics

TOPOLOGICAL SPACES

TOPOLOGICAL SPACES

Open Sets in a topological space provide a model of intuitionistic logic

$$(x,y) \cap (v,w) = (v,y)$$

$$(x,y) \cup (v,w) = (x,w)$$

$$\neg (x,y) = (-\infty,x) \cup (y,\infty)$$

TOPOLOGICAL SPACES

Open Sets in a topological space provide a model of intuitionistic logic

$$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$$

$$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$$

$$[\![\neg A]\!] = \mathrm{Int}(S \setminus [\![A]\!])$$

$$[\![A \supset B]\!] = \mathrm{Int}((S \setminus [\![A]\!]) \cup [\![B]\!])$$

TOPOLOGICAL SPACES

Open Sets in a topological space form a Heyting Lattice. $\langle H, \wedge, \vee, \to, \bot \rangle$

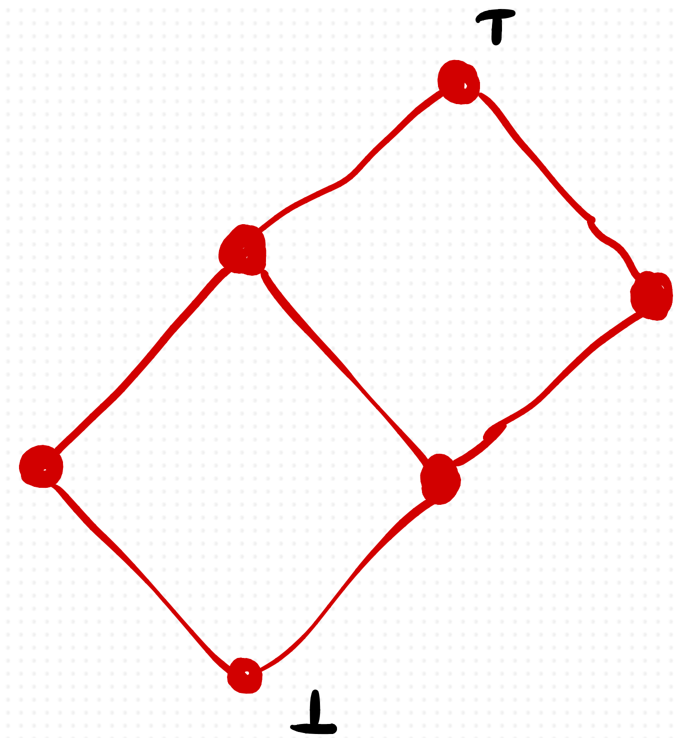
- $\langle H, \wedge, \vee, \bot \rangle$ is a lattice with bottom element $\bot$.

- $\to$ residuates $\wedge$:

  $x \wedge y \leq z$ iff $x \leq y \to z$

- $\neg x$ is $x \to \bot$

- All Boolean Algebras are Heyting Lattices, but not vice versa.



CATEGORICAL MODELS

... in which objects are propositions & arrows are proofs

... or topos models, eg Eff.

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[rr, bend right, "\pi' \circ \pi"'] & B \arrow[r, "\pi'"] & C
\end{tikzcd}$$

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "fst"] \arrow[r, "snd"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \times Y \arrow[d, dotted, "\lambda g \times id_Y"'] \arrow[dr, "g"] & \\
Y \Rightarrow Z \times Y \arrow[r, "eval"'] & Z
\end{tikzcd}$$

What can we learn from all this?

1. MY QUESTION
2. LOGICS & PROOFS & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists

$$
\begin{tikzcd}
\text{Classical Logic} \arrow[r, bend left] & \text{Intuitionistic Logic} \arrow[l, bend left]
\end{tikzcd}
$$

DOUBLE NEGATION TRANSLATION

finding classical logic inside intuitionistic logic

$$t(p) = \neg\neg p$$
$$t(A \wedge B) = t(A) \wedge t(B)$$
$$t(A \vee B) = \neg(\neg t(A) \wedge \neg t(B))$$
$$t(A \to B) = t(A) \to t(B)$$
$$t(\neg A) = \neg t(A)$$
$$t(\forall x A) = \forall x\, t(A)$$
$$t(\exists x A) = \neg \forall x \neg t(A)$$

↑

LANGUAGE of a "CLASSICAL" SPEAKER

DOUBLE NEGATION TRANSLATION

finding classical logic inside intuitionistic logic

$$t(p) = \neg\neg p$$
$$t(A \wedge B) = t(A) \wedge t(B)$$
$$t(A \vee B) = \neg(\neg t(A) \wedge \neg t(B))$$
$$t(A \to B) = t(A) \to t(B)$$
$$t(\neg A) = \neg t(A)$$
$$t(\forall x A) = \forall x\, t(A)$$
$$t(\exists x A) = \neg \forall x \neg t(A)$$

LANGUAGE of a "CONSTRUCTIVE" speaker

LANGUAGE of a "CLASSICAL" SPEAKER

DOUBLE NEGATION TRANSLATION

finding classical logic inside intuitionistic logic

$$t(p) = \neg\neg p$$
$$t(A \wedge B) = t(A) \wedge t(B)$$
$$t(A \vee B) = \neg(\neg t(A) \wedge \neg t(B))$$
$$t(A \to B) = t(A) \to t(B)$$
$$t(\neg A) = \neg t(A)$$
$$t(\forall x A) = \forall x\, t(A)$$
$$t(\exists x A) = \neg \forall x \neg t(A)$$

FACT.

$$\boxed{X \vdash_{\text{Classical}} A \quad \text{iff} \quad t(X) \vdash_{\text{Int.}} t(A)}$$

DOUBLE NEGATION TRANSLATION

finding classical logic inside intuitionistic logic

$$t(p) = \neg\neg p$$
$$t(A \wedge B) = t(A) \wedge t(B)$$
$$t(A \vee B) = \neg(\neg t(A) \wedge \neg t(B))$$
$$t(A \to B) = t(A) \to t(B)$$
$$t(\neg A) = \neg t(A)$$
$$t(\forall x A) = \forall x\, t(A)$$
$$t(\exists x A) = \neg \forall x \neg t(A)$$

FACT.

$$\boxed{X \vdash_{\text{Classical}} A \text{ iff } t(X) \vdash_{\text{Int.}} t(A)}$$

$\Rightarrow$: Induction on the structure of the classical proof.

$\Leftarrow$: If $t(X) \vdash_{\text{Int.}} t(A)$ then $t(X) \vdash_{\text{class}} t(A)$, and it's easy to see that $t(A) \dashv\vdash_{\text{class.}} A$.

MODAL TRANSLATION

finding intuitionistic logic inside classical logic + an S4 necessity

$m(p) = \Box p$

Classical speaker with a concept of 'necessity'.

$m(A \wedge B) = m(A) \wedge m(B)$

$m(A \vee B) = m(A) \vee m(B)$

$m(A \to B) = \Box(m(A) \to m(B))$

$m(\neg A) = \Box \neg m(A)$

$m(\forall x A) = \Box \forall x\, m(A)$

$m(\exists x A) = \exists x\, m(A)$

INTUITIONISTIC SPEAKER

MODAL TRANSLATION

finding intuitionistic logic inside classical logic + an S4 necessity

$$m(p) = \Box p$$

$$m(A \wedge B) = m(A) \wedge m(B)$$

$$m(A \vee B) = m(A) \vee m(B)$$

$$m(A \to B) = \Box(m(A) \to m(B))$$

$$m(\neg A) = \Box \neg m(A)$$

$$m(\forall x A) = \Box \forall x\, m(A)$$

$$m(\exists x A) = \exists x\, m(A)$$

FACT.

$$\boxed{X \vdash_{Int} A \quad \text{iff} \quad m(X) \vdash_{S4} m(A)}$$

MODAL TRANSLATION

finding intuitionistic logic inside classical logic + an S4 necessity

$$m(p) = \Box p$$

$$m(A \wedge B) = m(A) \wedge m(B)$$

$$m(A \vee B) = m(A) \vee m(B)$$

$$m(A \to B) = \Box(m(A) \to m(B))$$

$$m(\neg A) = \Box\neg m(A)$$

$$m(\forall x A) = \Box \forall x\, m(A)$$

$$m(\exists x A) = \exists x\, m(A)$$

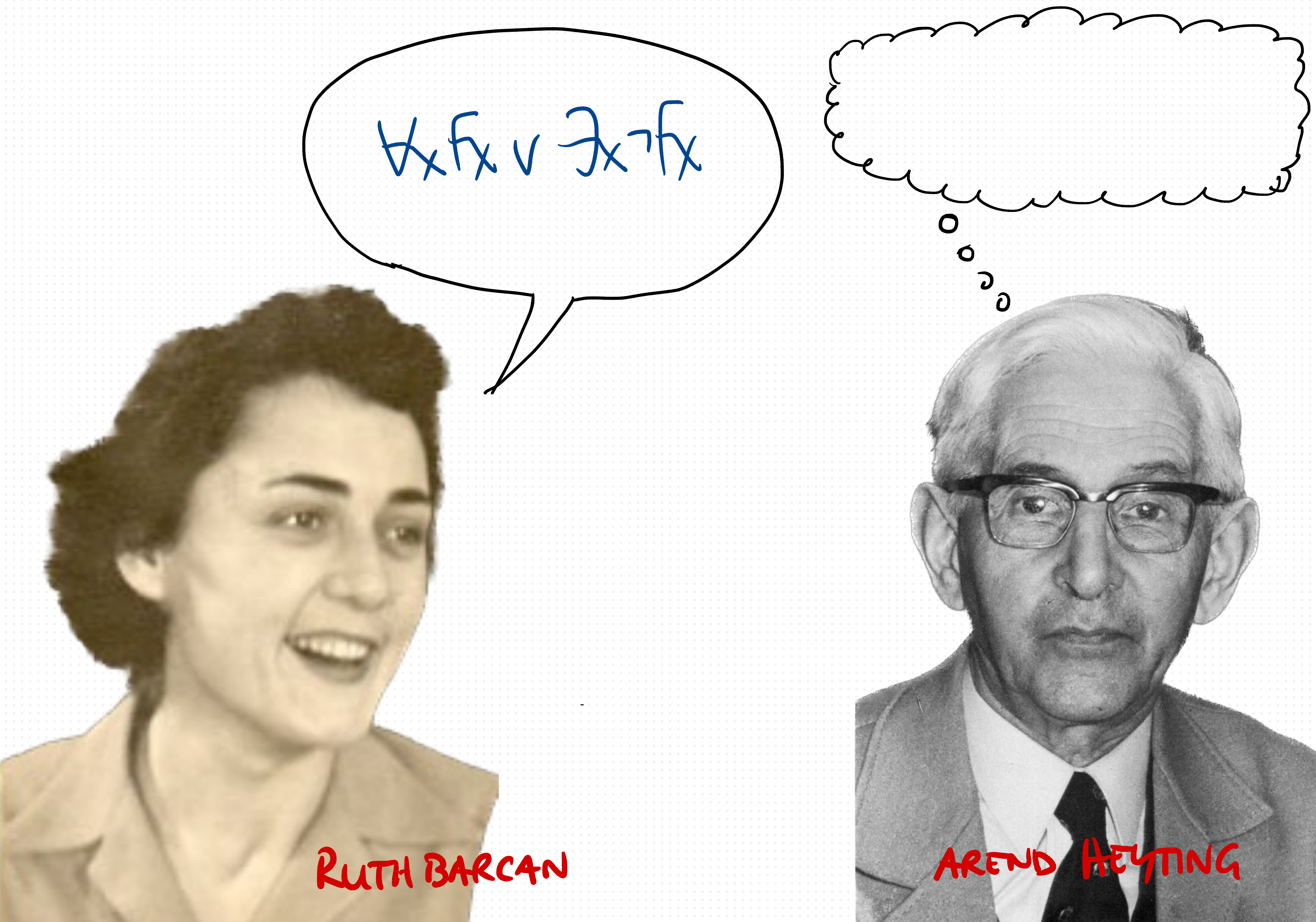
FACT.

$$\boxed{X \vdash_{Int} A \text{ iff } m(X) \vdash_{S4} m(A)}$$

One proof: Intuitionistic Kripke models are S4 Kripke models. Up-closed sets of points are exactly the points of the form $\Box A$.

Another: Topological models for intuitionistic logic are topological models for S4, where $\Box$ is modelled by the interior operator.

These translations
change the
meanings of terms


$\forall x Fx \vee \exists x \neg Fx$
RUTH BARCAN
AREND HEYTING

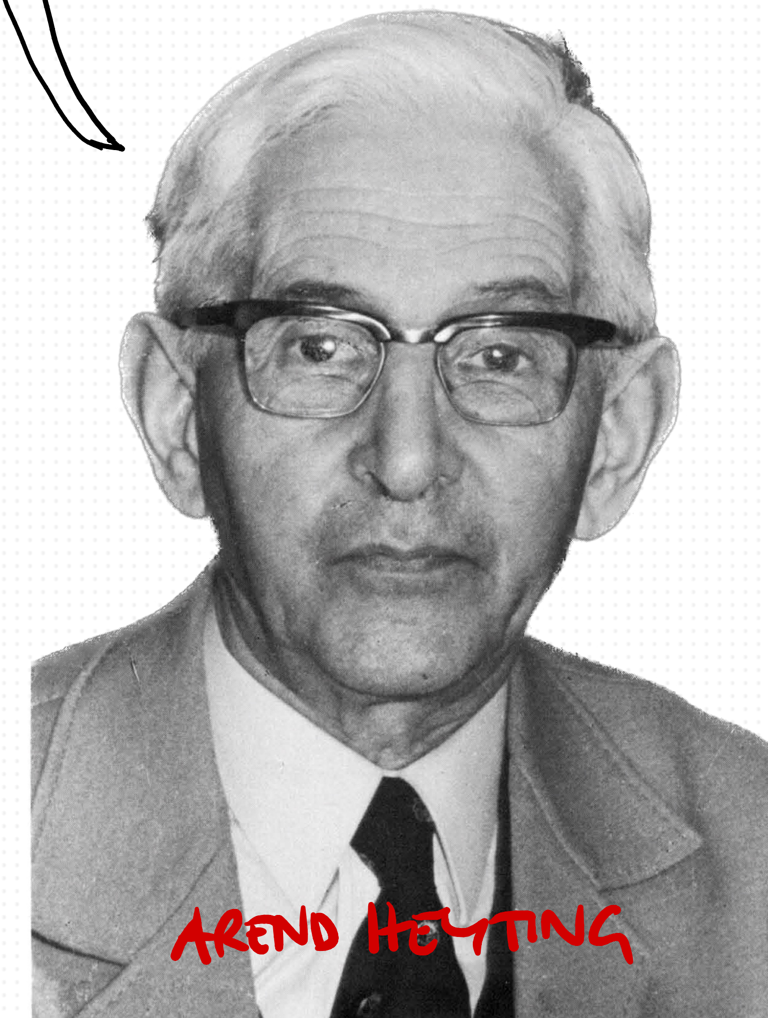
$$\forall x Fx \vee \exists x \neg Fx$$

$$\neg(\neg\forall x \neg\neg Fx \wedge \neg\neg\forall x \neg\neg\neg\neg Fx)$$


$\nvdash \forall x Fx \vee \exists x \neg Fx$
RUTH BARCAN
AREND HEYTING

$$\nvdash \Box\forall x \Box Fx \vee \exists x \Box \neg \Box Fx$$

$$\nvdash \forall x Fx \vee \exists x \neg Fx$$



This doesn't make sense of Bishop's reasoning.

In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

Errett Bishop
Foundations of Constructive Analysis
1967

This doesn't make sense of Bishop's reasoning.

In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

Bishop is not using a translation. He is saying that this inference is classically OK & intuitionistically invalid.

Errett Bishop
Foundations of Constructive Analysis
1967

Classical logic

is also

Intuitionistic logic

plus extra principles, e.g.

$$\forall x \phi(x) \vee \exists x \neg \phi(x)$$

Classical logic

is also

Intuitionistic logic

plus extra principles, e.g.

$$\forall x\,\phi(x) \vee \exists x\,\neg\phi(x)$$

Terminal nodes in a Kripke frame act completely classically

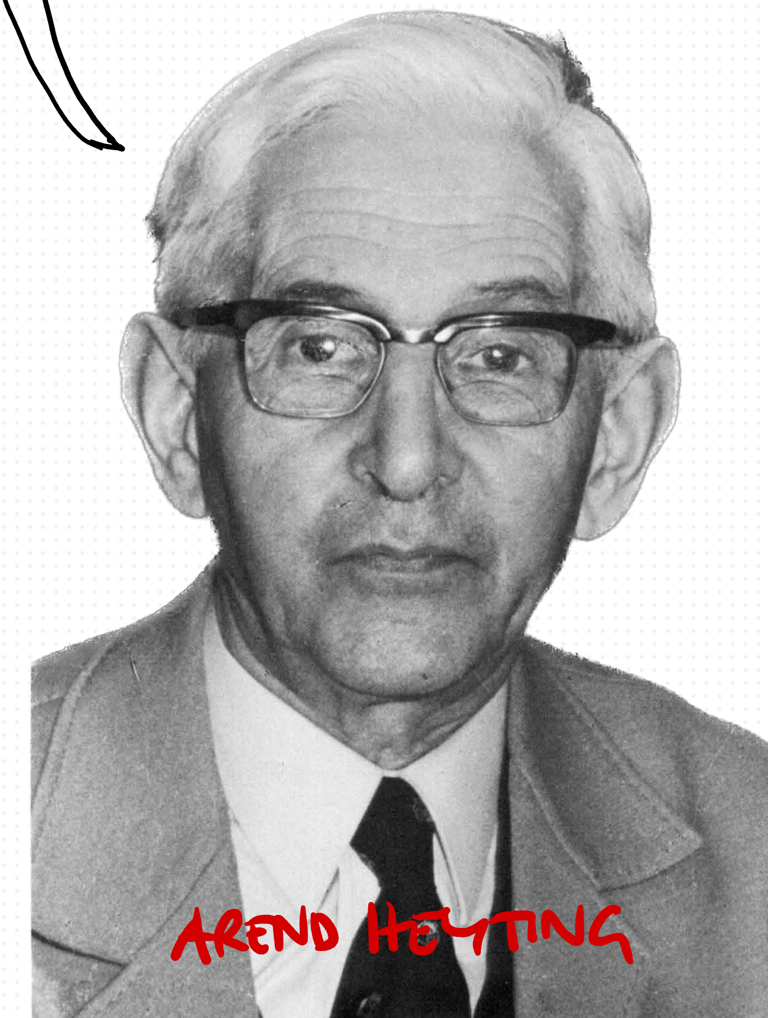
— no translation necessary!

1. MY QUESTION
2. LOGICS & PROOFS & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists

$\nvdash \Box\forall x \Box Fx \vee \exists x \Box\neg\Box Fx$

$\nvdash \forall x Fx \vee \exists x \neg Fx$

STEP 1



At least the Intuitionistic practice is coherent.

But so far, it's deviant regarding the meaning of some connectives.

STEP 2: Learning to speak the language

- Explore some of the models
  (Topos models as mathematical universes are particularly powerful for this, but other models work, too.)

STEP 2: Learning to speak the language

- Explore some of the models
  (Topos models as mathematical universes are particularly powerful for this, but other models work, too.)

  ▷ The internal vocabulary of these models is "governed" by intuitionistic logic.

STEP 2: Learning to speak the language

- Explore some of the models
  (Topos models as mathematical universes are particularly powerful for this, but other models work, too.)

  ▹ The internal vocabulary of these models is "governed" by intuitionistic logic.

- Attempt to live "inside" these models.

STEP 2: Learning to speak the language

- Explore some of the models
  (Topos models as mathematical universes are particularly powerful for this, but other models work, too.)

  ▷ The internal vocabulary of these models is "governed" by intuitionistic logic.

- Attempt to live "inside" these models.

  ▷ This involves restricting your inferences & adopting a stricter standard of proof, making finer distinctions.

A CONCERN (for the friend of classical logic)

- Isn't this changing the meaning of your vocabulary?

- I, a classical logician, can prove $A \vee \neg A$.
  You're now asking me to deny it — how does that make sense?

A CONCERN (for the friend of classical logic)

- Isn't this changing the meaning of your vocabulary?

- I, a classical logician, can prove $A \vee \neg A$.
  You're now asking me to deny it — how does that make sense?

▷ This is weak denial, not asserting $\neg(A \vee \neg A)$,* but rejecting $A \vee \neg A$ on the grounds of not being constructively warranted.

* Caveat: there are constructive models, e.g. models of SMOOTH INFINITESIMAL ANALYSIS, in which classical inconsistencies do hold, e.g. $\neg \forall x (Fx \vee \neg Fx)$ for some F.

A CONCERN (for the friend of classical logic)

- Isn't this changing the meaning of your vocabulary?

- I, a classical logician, can prove $A \vee \neg A$.
  You're now asking me to deny it — how does that make sense?

▷ This is weak denial, not asserting $\neg(A \vee \neg A)$,* but rejecting $A \vee \neg A$ on the grounds of not being constructively warranted.

▷ This is a stricter norm for assertion than one constrained by classical consequence, but given the development of constructive mathematics it is coherent & useful.

* Caveat: there are constructive models, e.g. models of SMOOTH INFINITESIMAL ANALYSIS, in which classical inconsistencies do hold, e.g. $\neg \forall x (Fx \vee \neg Fx)$ for some F.

A CONCERN (for the friend of classical logic)

- Isn't this changing the meaning of your vocabulary?

- I, a classical logician, can prove $A \vee \neg A$. You're now asking me to deny it — how does that make sense?

▷ This is weak denial, not asserting $\neg(A \vee \neg A)$,* but rejecting $A \vee \neg A$ on the grounds of not being constructively warranted.

▷ This is a stricter norm for assertion than one constrained by classical consequence, but given the development of constructive mathematics it is coherent & useful.

▷ There is no problem in CODE SWITCHING between stricter & more lax standards of assertion, without changing semantics (eg legal contexts.)

* Caveat: there are constructive models, eg. models of SMOOTH INFINITESIMAL ANALYSIS, in which classical inconsistencies do hold, eg $\neg\forall x(Fx \vee \neg Fx)$ for some F.

Martin Escardo
@EscardoMartin

I like to work within "neutral" mathematics, as opposed to classical or constructive mathematics.
(I said so here, among other places:
math.andrej.com/2021/05/18/com...)
1/

7:23 PM · Jul 12, 2022 · Twitter Web App

8 Retweets **1** Quote Tweet **59** Likes

Tweet your reply

Reply

Martin Escardo @EscardoMartin · Jul 12
Replying to @EscardoMartin
This idea was probably proposed first by Bishop. But also it is what happens when working in the internal language of an arbitrary topos. Both took place at about the same time, in the late 1960's, and independently.
2/

1 7

Martin Escardo @EscardoMartin · Jul 12
You don't say that the principle of excluded middle is false, for example. You keep it open. You keep it deliberately undecided. You just don't use it.
3/

4 1 7

Martin Escardo @EscardoMartin · Jul 12
In the case of Bishop mathematics, the aim was that every theorem of his constructive mathematics is also a theorem of classical mathematics.

1. MY QUESTION
2. LOGICS & PROOFS & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists

$\forall x Fx \vee \exists x \neg Fx$

$\neg(\neg\forall x \neg\neg Fx \wedge \neg\neg\forall x \neg\neg\neg\neg Fx)$

Step 1

Conversely, this shows to the constructivist that classical practice is coherent.

But so far, it's deviant regarding the meaning of some connectives.

Use your favoured proofs/models

to motivate the coherence of classical

rules $\rightsquigarrow$ proofs ~ call/cc $\lambda\mu$

Kripke classical models

models ~ endpoints in Kripke Models

... etc..

What is the "evidence" for $A \vee \neg A$?

A weight line with nonconstructive reasoning, trading in some of the nice features of constructive evidence, for easier results!

STEP 2

- Using your favoured system of proofs/models, explore the "constructive" meaning of classical rules

▷ In proofs, call/cc $\lambda\mu$ models; Krivine classical realizability
In Kripke models, classical logic holds at endpoints.

- In admitting the possibility of "evidence" for $A \vee \neg A$, you don't need to erase all your constructive distinctions.
You, too, can learn to live with two standards for assertion, your strict, constructive standards, & more lax, classical norms.

▷ The upside is you get to have all those classical results!

Intuitionistic logic
is preserved in more places.

It has more models.

Classical logic
can help you go further.

It has more proofs.

Intuitionistic logic
is preserved in more places.

It has more models.

Classical logic
can help you go further.

It has more proofs.

You can do more, if you have both
classical & intuitionistic logic in
your conceptual toolkit.

1. MY QUESTION
2. LOGICS & PROOFS & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists

WHEREVER YOU STARTED FROM...

If you have taken STEP 2
you are some kind of Pluralist.

But there are different kinds
of logical pluralist.

THEORY PLURALISM

for a theory pluralist, the logic you use is relative to the domain of investigation, to different mathematical theories.

INQUIRY, 2017
https://doi.org/10.1080/0020174X.2017.1357495

Routledge
Taylor & Francis Group

Check for updates

Logical pluralism and normativity

Teresa Kouri Kissel[‡] and Stewart Shapiro

Department of Philosophy, Ohio State University, Columbus, OH, USA

ABSTRACT
We are logical pluralists who hold that the right logic is dependent on the domain of investigation; different logics for different mathematical theories. The purpose of this article is to explore the ramifications for our pluralism concerning normativity. Is there any normative role for logic, once we give up its universality? We discuss Florian Steingerger's "Frege and Carnap on the Normativity of Logic" (*Synthese* 94: 143–162) as a source for possible types of normativity, and then turn to our own proposal, which postulates that various logics are constitutive for thought within particular practices, but none are constitutive for thought as such.

ARTICLE HISTORY Received 1 January 2017; Accepted 7 July 2017

KEYWORDS Logic; pluralism; normativity; Frege; Carnap

THEORY PLURALISM

For a theory pluralist, the logic you use is relative to the domain of investigation, to different mathematical theories.

- This is undoubtedly well motivated. E.g.

  HA (Heyting Arithmetic) &
  PA (Peano Arithmetic)

  Are different theories with different properties & different logics

INQUIRY, 2017
https://doi.org/10.1080/0020174X.2017.1357495

Routledge
Taylor & Francis Group

Check for updates

Logical pluralism and normativity

Teresa Kouri Kissel[‡] and Stewart Shapiro

Department of Philosophy, Ohio State University, Columbus, OH, USA

ABSTRACT
We are logical pluralists who hold that the right logic is dependent on the domain of investigation; different logics for different mathematical theories. The purpose of this article is to explore the ramifications for our pluralism concerning normativity. Is there any normative role for logic, once we give up its universality? We discuss Florian Steingerger's "Frege and Carnap on the Normativity of Logic" (*Synthese* 94: 143–162) as a source for possible types of normativity, and then turn to our own proposal, which postulates that various logics are constitutive for thought within particular practices, but none are constitutive for thought as such.

ARTICLE HISTORY Received 1 January 2017; Accepted 7 July 2017

KEYWORDS Logic; pluralism; normativity; Frege; Carnap

THEORY PLURALISM

Theory pluralism is ambiguous.

Two different mathematical theories can have the same domain of investigation.

INQUIRY, 2017
https://doi.org/10.1080/0020174X.2017.1357495

Routledge
Taylor & Francis Group

Check for updates

Logical pluralism and normativity

Teresa Kouri Kissel[‡] and Stewart Shapiro

Department of Philosophy, Ohio State University, Columbus, OH, USA

ABSTRACT

We are logical pluralists who hold that the right logic is dependent on the domain of investigation; different logics for different mathematical theories. The purpose of this article is to explore the ramifications for our pluralism concerning normativity. Is there any normative role for logic, once we give up its universality? We discuss Florian Steingerger's "Frege and Carnap on the Normativity of Logic" (*Synthese* 94: 143–162) as a source for possible types of normativity, and then turn to our own proposal, which postulates that various logics are constitutive for thought within particular practices, but none are constitutive for thought as such.

ARTICLE HISTORY Received 1 January 2017; Accepted 7 July 2017

KEYWORDS Logic; pluralism; normativity; Frege; Carnap

THEORY PLURALISM

Theory pluralism is ambiguous.

Two different mathematical theories can have the same domain of investigation.

Both HA & PA investigate the natural numbers.
One, constructively, the other, classically.

INQUIRY, 2017
https://doi.org/10.1080/0020174X.2017.1357495

Routledge
Taylor & Francis Group

Logical pluralism and normativity

Teresa Kouri Kissel‡ and Stewart Shapiro

Department of Philosophy, Ohio State University, Columbus, OH, USA

ABSTRACT
We are logical pluralists who hold that the right logic is dependent on the domain of investigation; different logics for different mathematical theories. The purpose of this article is to explore the ramifications for our pluralism concerning normativity. Is there any normative role for logic, once we give up its universality? We discuss Florian Steingerger's "Frege and Carnap on the Normativity of Logic" (*Synthese* 94: 143–162) as a source for possible types of normativity, and then turn to our own proposal, which postulates that various logics are constitutive for thought within particular practices, but none are constitutive for thought as such.

ARTICLE HISTORY Received 1 January 2017; Accepted 7 July 2017

KEYWORDS Logic; pluralism; normativity; Frege; Carnap

THEORY PLURALISM

Theory pluralism is ambiguous.

Two different mathematical theories can have the same domain of investigation.

Both HA & PA investigate the natural numbers.
One, constructively, the other, classically.

INQUIRY, 2017
https://doi.org/10.1080/0020174X.2017.1357495

Routledge
Taylor & Francis Group

Logical pluralism and normativity

Teresa Kouri Kissel[‡] and Stewart Shapiro

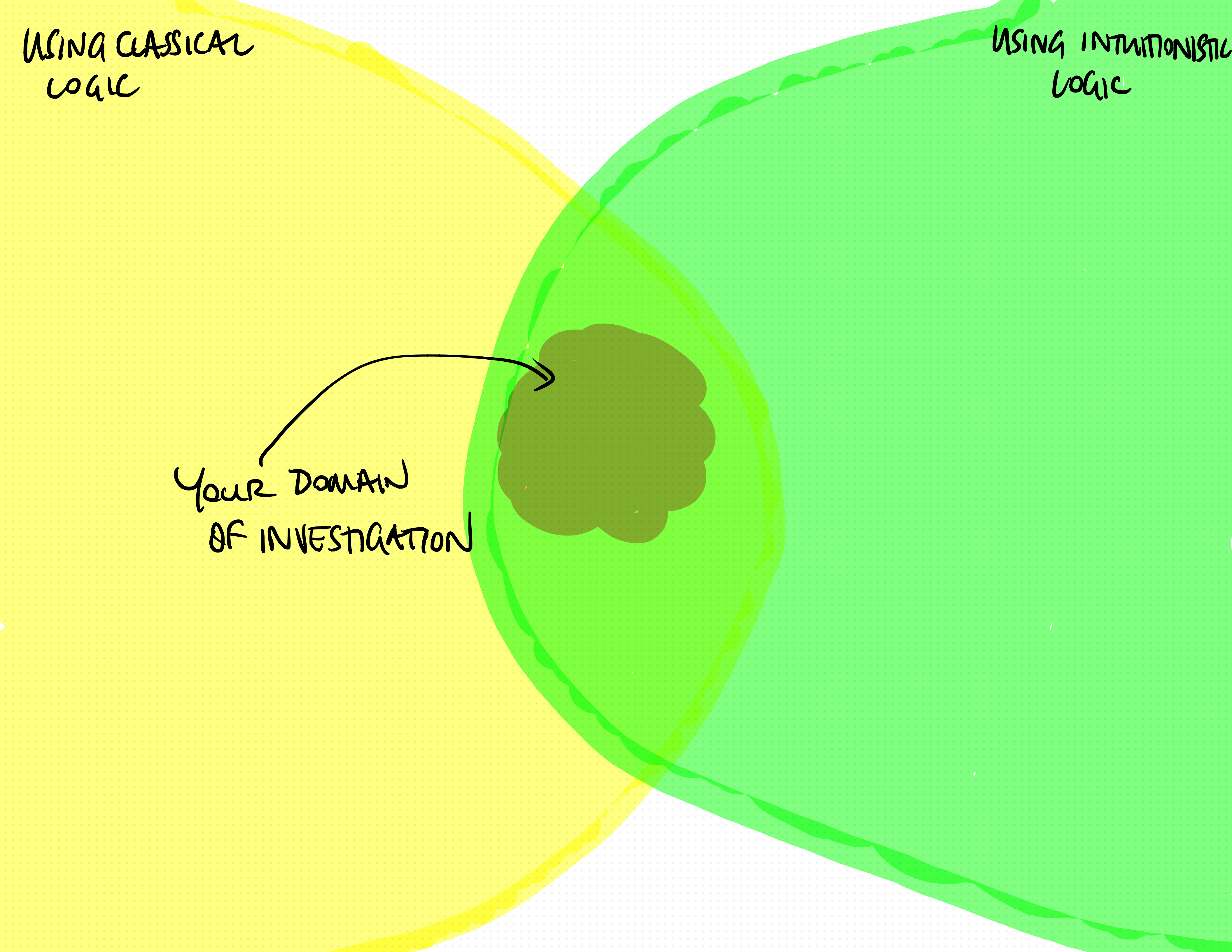
Department of Philosophy, Ohio State University, Columbus, OH, USA

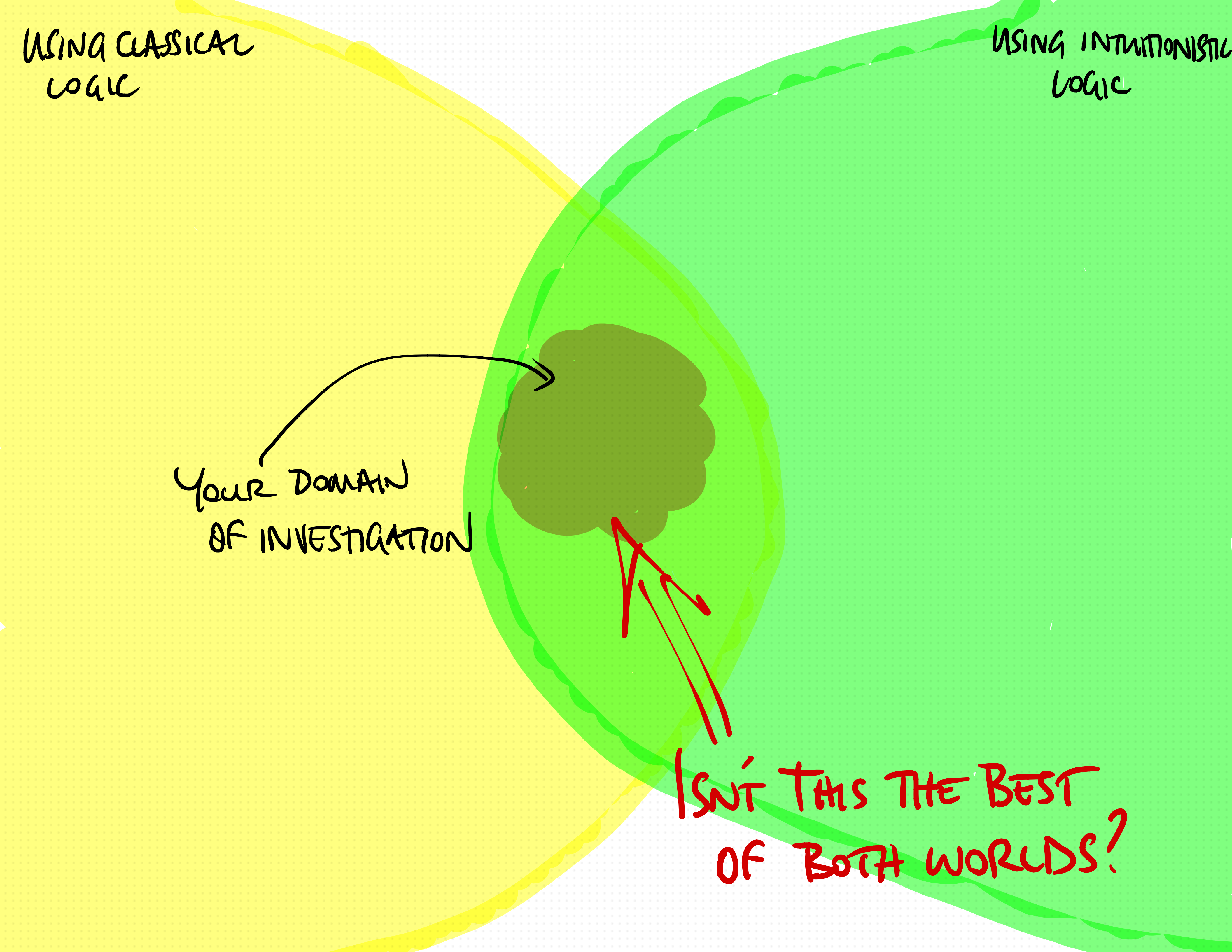
ABSTRACT
We are logical pluralists who hold that the right logic is dependent on the domain of investigation; different logics for different mathematical theories. The purpose of this article is to explore the ramifications for our pluralism concerning normativity. Is there any normative role for logic, once we give up its universality? We discuss Florian Steingerger's "Frege and Carnap on the Normativity of Logic" (*Synthese* 94: 143–162) as a source for possible types of normativity, and then turn to our own proposal, which postulates that various logics are constitutive for thought within particular practices, but none are constitutive for thought as such.

ARTICLE HISTORY Received 1 January 2017; Accepted 7 July 2017

KEYWORDS Logic; pluralism; normativity; Frege; Carnap

A STRONG THEORY PLURALIST takes there to be different theories — investigating the same domain — with different logics.


USING CLASSICAL LOGIC
USING INTUITIONISTIC LOGIC
YOUR DOMAIN OF INVESTIGATION


USING CLASSICAL LOGIC
USING INTUITIONISTIC LOGIC
YOUR DOMAIN OF INVESTIGATION
ISN'T THIS THE BEST OF BOTH WORLDS?

Thank You!

https://consequently.org/