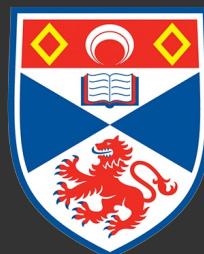


CLASSICAL LOGIC & INTUITIONISTIC LOGIC

looking both ways

GREG RESTALL



University of
St Andrews

6 SEPTEMBER 2022

MY PLAN

1. My Question
2. Logics & Proofs & MODELS
3. TRANSLATIONS
4. WHAT THIS MEANS — for the Classical partisan
5. — for the Intuitionist partisan
6. — for pluralists

1. MY QUESTION

2. LOGICS & PROOFS & MODELS

3. TRANSLATIONS

4. WHAT THIS MEANS — for the Classical partisan

5. — for the Intuitionist partisan

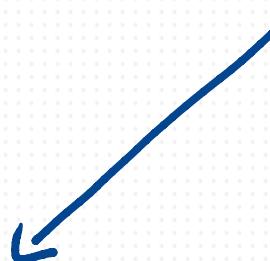
6. — for pluralists

How are we to understand
the relationship between
Classical logic &
intuitionistic logic?

In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

Errett Bishop
foundations of
Constructive Analysis
1967

Discussion of classical mathematical reasoning



In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

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Contrast with intuitionistic reasoning

Errett Bishop
foundations of
Constructive Analysis
1967

In classical mathematics we
can reason like this:

$$\begin{array}{c} x > y \\ \vdots \\ \vdots \\ \vdots \\ \bot \end{array}$$

In classical mathematics we
can reason like this:

$$\begin{array}{c} [x > y] \\ \vdots \\ \perp \\ \hline \neg I \\ \neg(x > y) \end{array}$$

In classical mathematics we
can reason like this:

$$\begin{array}{c} [x > y]' \\ \vdots \\ \perp \\ \hline \neg(x > y) \\ \hline \neg> + \leq \\ x \leq y \end{array}$$

In classical mathematics we
can reason like this:

$$\begin{array}{c} [x > y]' \\ \vdots \\ \perp \\ \hline \neg(x > y) \\ \hline \neg> + \leqslant \end{array}$$

But when you're
reasoning constructively,
this last step is blocked.

Logicians have a **very** good
handle on classical logic

& on intuitionistic logic,
their properties & what it
is to develop a theory in
one or other logic.

Logicians have a **very** good
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But we have less of a consensus
about what to do with this
knowledge.

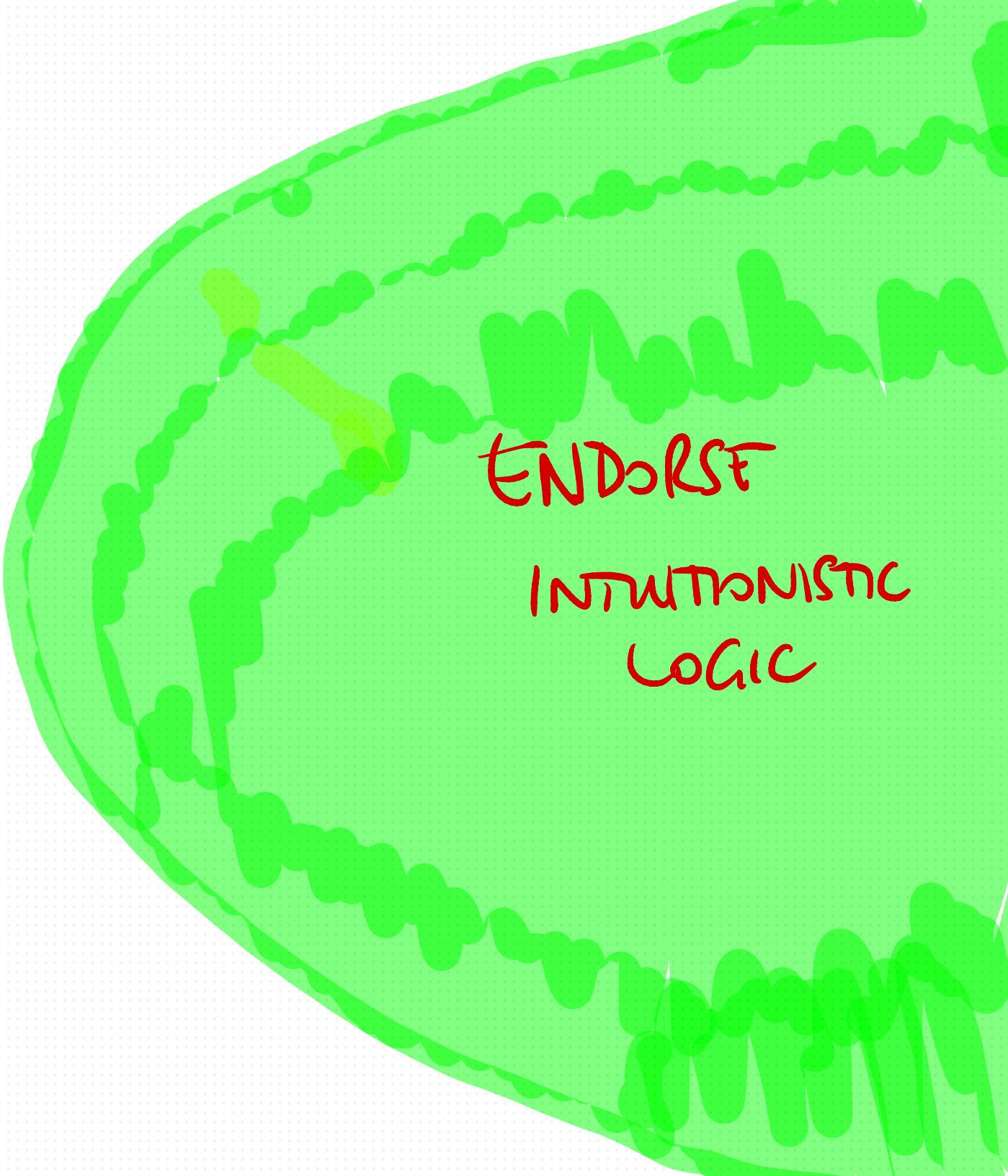
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But we have less of a consensus
about what to do with this
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I want to help us to think
more clearly about this.

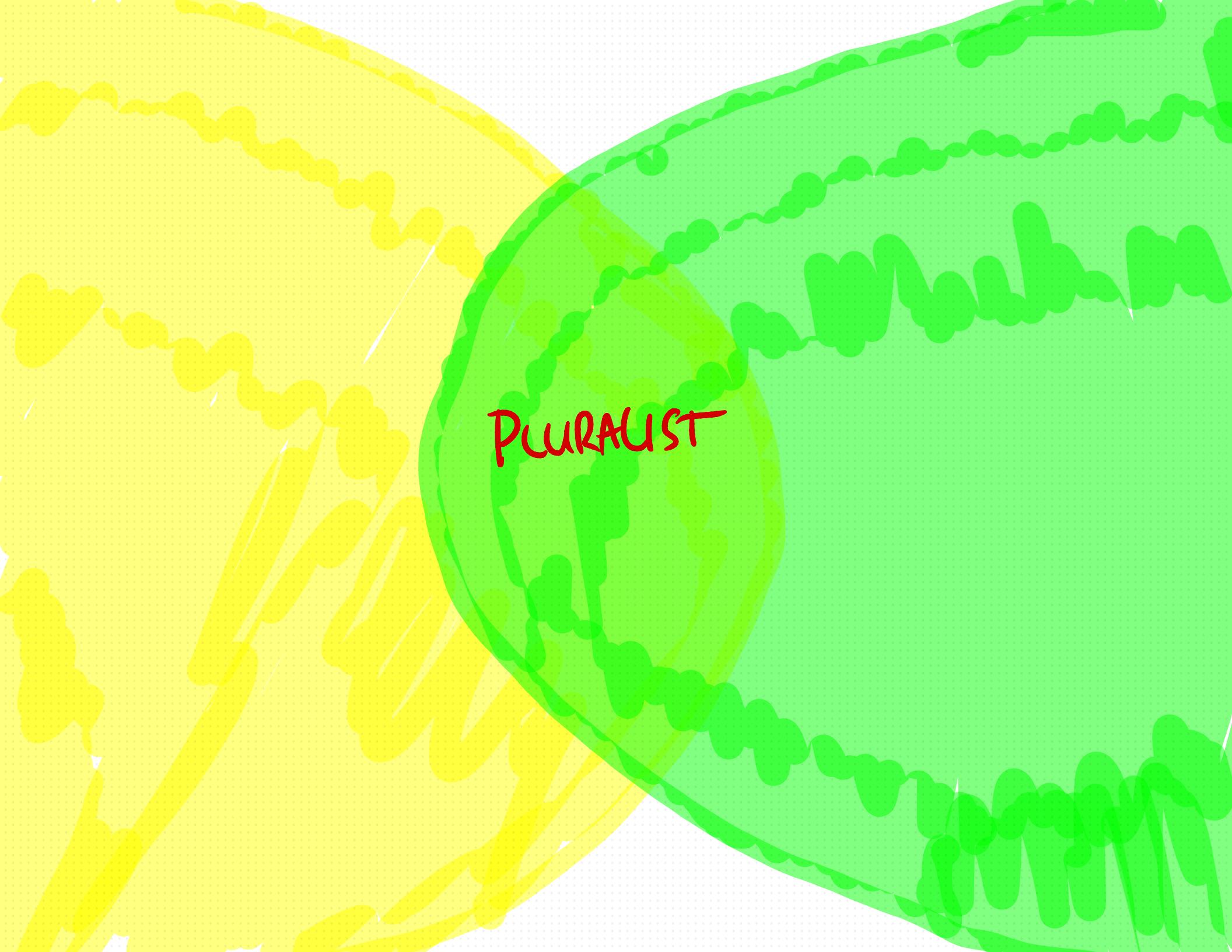
ENDORSE
CLASSICAL
LOGIC



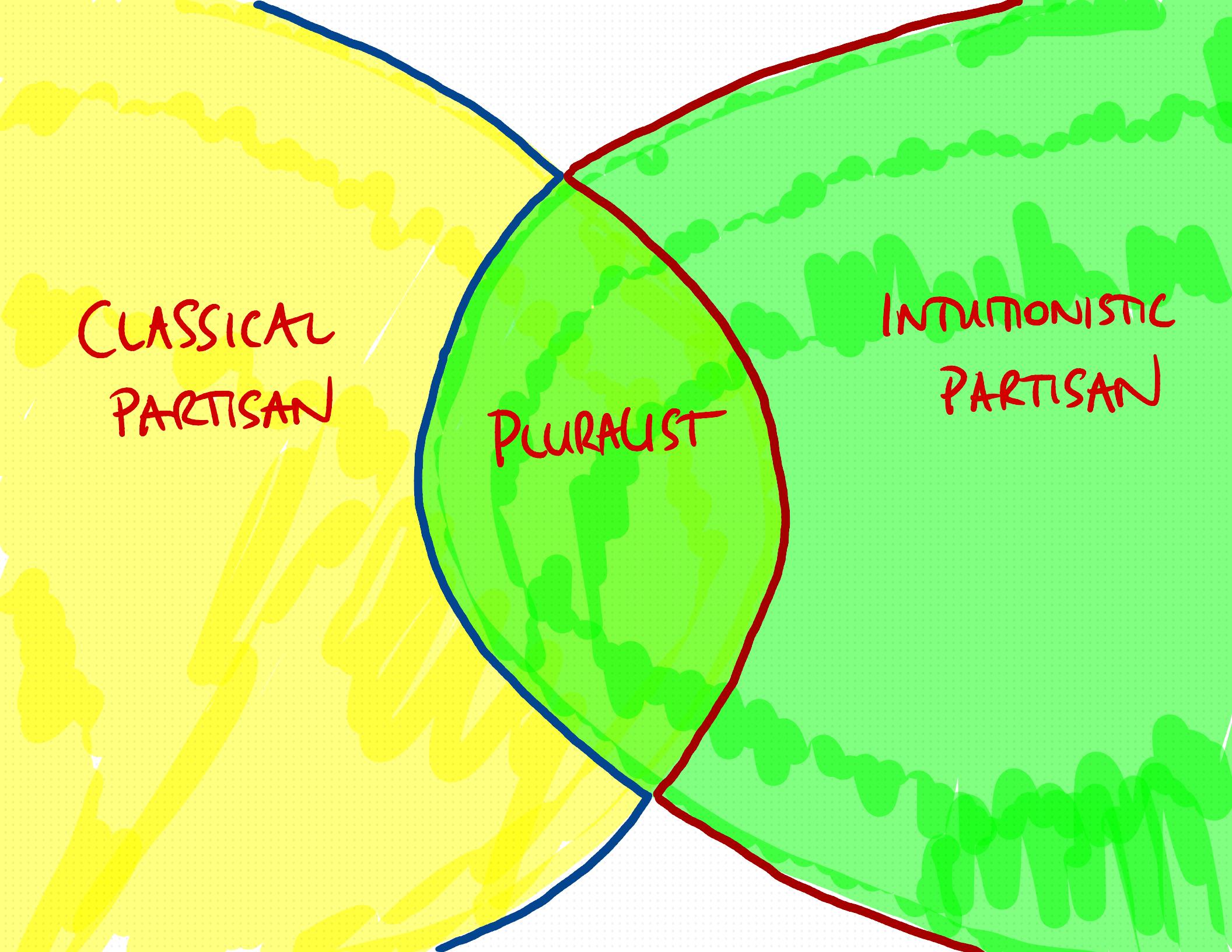
ENDORSE
INTUITIONISTIC
LOGIC

ENDORSE

INTUITIONISTIC
LOGIC



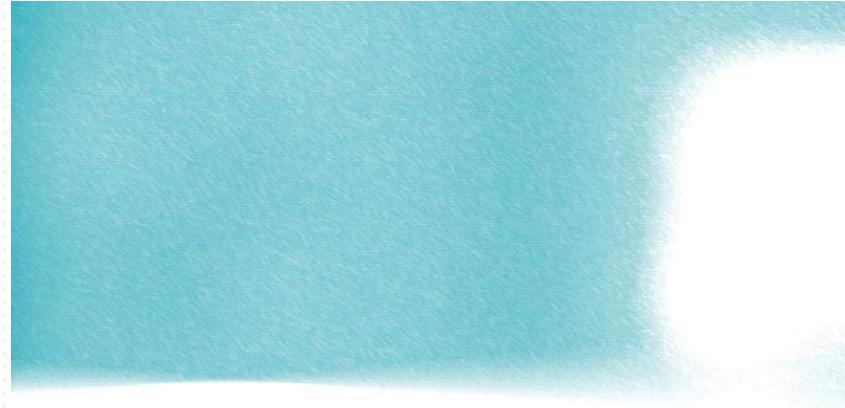
PLURALIST



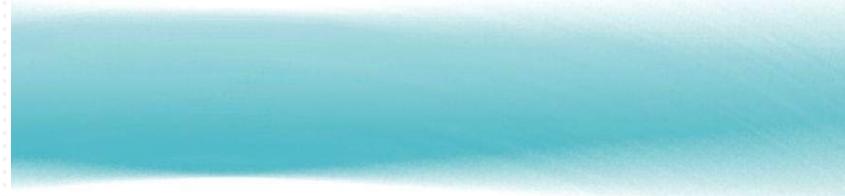
CLASSICAL
PARTISAN

INTUITIONISTIC
PARTISAN

PLURALIST



LogicalPluralism



JC Beall and Greg Restall

THIS IS A SHAMELESS ADVERTISEMENT

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Cambridge
Elements
Philosophy and Logic

Proofs and Models
in Philosophical
Logic

Greg Restall

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THIS IS ANOTHER SHAMELESS ADVERTISEMENT

NATURAL DEDUCTION

20

NATURAL DEDUCTION OF GENTZEN-TYPE

$$\&I) \frac{A \quad B}{A \& B}$$

$$\&E) \frac{A \& B}{\begin{array}{c} A \\ B \end{array}}$$

$$\vee I) \frac{\begin{array}{c} A \\ A \vee B \end{array} \quad \begin{array}{c} B \\ A \vee B \end{array}}{A \vee B}$$

$$\vee E) \frac{A \vee B \quad \begin{array}{c} (A) \quad (B) \\ C \quad C \end{array}}{C}$$

$$\supset I) \frac{\begin{array}{c} (A) \\ B \end{array}}{A \supset B}$$

$$\supset E) \frac{A \quad A \supset B}{B}$$

$$\forall I) \frac{A}{\forall x A_x^a}$$

$$\forall E) \frac{\forall x A}{A_t^i}$$

$$\exists I) \frac{A_t^i}{\exists x A}$$

$$\exists E) \frac{\begin{array}{c} (A_u^x) \\ \exists x A \quad B \end{array}}{B}$$

$$\lambda_1) \frac{\lambda}{A}$$

$$\lambda_c) \frac{\lambda}{\begin{array}{c} (\sim A) \\ A \end{array}}$$

Dag Prawitz
NATURAL DEDUCTION 1965

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$$\vee E) \frac{\begin{array}{c} (A) \quad (B) \\ A \vee B \quad C \quad C \end{array}}{C}$$

$$\supset I) \frac{\begin{array}{c} (A) \\ B \end{array}}{A \supset B}$$

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$$\forall I) \frac{A}{\forall x A_x^a}$$

$$\forall E) \frac{\forall x A}{A_i^x}$$

$$\exists I) \frac{A_i^x}{\exists x A}$$

$$\exists E) \frac{\begin{array}{c} (A_u^x) \\ \exists x A \quad B \end{array}}{B}$$

$$\lambda_1) \frac{\lambda}{A}$$

$$\lambda_c) \frac{\lambda}{(\sim A)}$$

$$\frac{\begin{array}{c} p \wedge \neg r \\ p \rightarrow (q \rightarrow r) \end{array}}{\frac{p}{\neg E}} \wedge E$$

$$\frac{\begin{array}{c} p \wedge \neg r \\ \neg r \end{array}}{\perp} \neg E$$

$$\frac{\begin{array}{c} q \rightarrow r \\ q \end{array}}{\frac{r}{\perp}} \rightarrow E$$

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$$\exists E) \frac{\begin{array}{c} (A_a^i) \\ \exists x A \quad B \\ B \end{array}}{A}$$

$$\lambda_1) \frac{\lambda}{A}$$

$$\lambda_c) \frac{\lambda}{\begin{array}{c} (\sim A) \\ A \end{array}}$$

$$\begin{array}{c}
 \frac{P \wedge \neg r}{P} \wedge E \\
 \frac{P \rightarrow (q \rightarrow r)}{P} \rightarrow E \\
 \frac{\frac{\frac{P \wedge \neg r}{\neg r} \wedge E}{q \rightarrow r} [q]^\perp \rightarrow E}{r} \neg E \\
 \frac{\perp}{\neg q} \neg I^\perp
 \end{array}$$

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$$\vee E) \frac{A \vee B \quad \frac{(A) \quad (B)}{C \quad C}}{C}$$

$$\supset I) \frac{\frac{(A)}{B}}{A \supset B}$$

$$\supset E) \frac{A \quad A \supset B}{B}$$

$$\forall I) \frac{A}{\forall x A_x^a}$$

$$\forall E) \frac{\forall x A}{A_i^a}$$

$$\exists I) \frac{A_i^a}{\exists x A}$$

$$\exists E) \frac{\exists x A \quad \frac{(A_a^x)}{B}}{B}$$

$$\lambda_1) \frac{\lambda}{A}$$

$$\lambda_c) \frac{\lambda}{A} \quad (\sim A)$$

$$\begin{array}{c}
 \frac{[p \wedge \neg r]^2}{\perp} \wedge E \\
 \frac{p \rightarrow (q \rightarrow r) \quad \frac{[p \wedge \neg r]^2}{\perp} \wedge E}{q \rightarrow r} \rightarrow E \\
 \frac{[q \vee]^2 \quad q \rightarrow r}{r} \rightarrow E \\
 \frac{\perp}{\neg q} \neg I^1 \\
 \frac{\perp \quad \frac{\perp}{\neg q}}{(\neg p \wedge \neg r) \rightarrow \neg q} \rightarrow I^2
 \end{array}$$

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NATURAL DEDUCTION 1965

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$$\exists E) \frac{\begin{array}{c} (A_a^i) \\ \exists x A \quad B \\ B \end{array}}{A}$$

$$\lambda_I) \frac{\lambda}{A}$$

$$\lambda_C) \frac{\lambda}{\begin{array}{c} (\sim A) \\ A \end{array}}$$

$$\begin{array}{c}
 \frac{[p \wedge \neg r]^2}{\perp} \wedge E \\
 \frac{p \rightarrow (q \rightarrow r)}{\frac{[p \wedge \neg r]^2}{\neg r} \wedge E} \frac{q \rightarrow r}{\frac{[q]^2}{r} \rightarrow E} \\
 \frac{\perp}{\neg q} \neg I^1 \\
 \frac{\perp}{(p \wedge \neg r) \rightarrow \neg q} \neg I^2
 \end{array}$$

$$p \rightarrow (q \rightarrow r) \vdash (p \wedge \neg r) \rightarrow \neg q$$

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NATURAL DEDUCTION 1965

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$$\vee E) \frac{A \vee B \quad \frac{(A) \quad (B)}{C \quad C}}{C}$$

$$\supset I) \frac{\frac{(A)}{B}}{A \supset B}$$

$$\supset E) \frac{A \quad A \supset B}{B}$$

$$\forall I) \frac{A}{\forall x A_x^a}$$

$$\forall E) \frac{\forall x A}{A_i^a}$$

$$\exists I) \frac{A_i^a}{\exists x A}$$

$$\exists E) \frac{\exists x A \quad \frac{(A_a^x)}{B}}{B}$$

$$\lambda_I) \frac{\lambda}{A}$$

$$\lambda_C) \frac{(\sim A)}{\frac{\lambda}{A}}$$

$$\begin{array}{c}
 [p \wedge \neg r]^2 \qquad \qquad \qquad \text{NE} \\
 \hline
 p \rightarrow (q \rightarrow r) \qquad \qquad \qquad P \\
 \hline
 \frac{[p \wedge \neg r]^2 \quad \frac{q \rightarrow r}{[q]_1^1} \rightarrow E}{r} \neg E \\
 \hline
 \frac{\neg r}{\perp} \qquad \qquad \qquad \neg I^1 \\
 \hline
 \frac{\perp}{\neg q} \qquad \qquad \qquad \neg I^2 \\
 \hline
 (p \wedge \neg r) \rightarrow \neg q \qquad \qquad \qquad \rightarrow I^2
 \end{array}$$

THE ODD RULE OUT!

Dag Prawitz
NATURAL DEDUCTION 1965

NATURAL DEDUCTION & TYPE THEORIES

$$\supset I) \frac{x:(A) \quad t:B}{\lambda x.t:A \supset B}$$

$$\supset E) \frac{t:A \quad s:A \supset B}{(st):B}$$

$$\begin{array}{c}
 \frac{}{x:(P \wedge \neg r)^2} \text{NE} \\
 \frac{y:P \rightarrow (q \rightarrow r) \quad \text{fst } x:P}{\frac{(y \text{ fst } x):q \rightarrow r}{\frac{\frac{z:q}{\text{snd } x:\neg r} \text{NE}}{\frac{z(y \text{ fst } x):r}{\frac{(\text{snd } x)(z(y \text{ fst } x)):\perp}{\frac{\chi_z.(\text{snd } x)(z(y \text{ fst } x)): \neg q}{\frac{\chi_x.\chi_z.(\text{snd } x)(z(y \text{ fst } x)):(P \wedge \neg r) \rightarrow \neg q}{\rightarrow I^2}}}}}} \rightarrow E} \\
 \end{array}$$

NATURAL DEDUCTION & TYPE THEORIES

$$\Rightarrow I) \frac{x:A \quad t:B}{\lambda x.t:A \Rightarrow B}$$

$$\Rightarrow E) \frac{t:A \quad s:A \Rightarrow B}{(st):B}$$

You can think of the terms as denoting procedures for construction, verification, or grounding.

$$\begin{array}{c}
 \frac{x:P \wedge \neg r}{x:(P \wedge \neg r)^2} \text{NE} \quad \frac{y:P \rightarrow (q \rightarrow r) \quad \text{fst } x:P}{(y \text{ fst } x):q \rightarrow r} \rightarrow E \\
 \frac{\frac{snd x:\neg r}{(z(y \text{ fst } x)):r} \neg E}{(snd x)(z(y \text{ fst } x)):\perp} \neg I^1 \\
 \frac{(snd x)(z(y \text{ fst } x)):\perp}{\lambda z.(snd x)(z(y \text{ fst } x)): \neg q} \neg I^2 \\
 \frac{\lambda x.\lambda z.(snd x)(z(y \text{ fst } x)):(P \wedge \neg r) \rightarrow \neg q}{\lambda x.\lambda z.(snd x)(z(y \text{ fst } x)):(P \wedge \neg r) \rightarrow \neg q} \rightarrow I^2
 \end{array}$$

NATURAL DEDUCTION & TYPE THEORIES

Classical Proofs as Programs *

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Abstract. We present an extension of the correspondence between intuitionistic proofs and functional programs to *classical proofs*, and more precisely to second order classical proofs. The advantage of classical logic in this context is that it allows to model *imperative features* of programming languages too (cf [5]). But there is an intrinsic difficulty with classical logic which lies in certain non-determinism of its computational interpretations. The use of a natural deduction system removes a part of this non determinism by fixing the inputs to the left of the sequents (cf [10] and [11]). However a conflict remains between the *confluence* of the computation mechanism and the *uniqueness* of the representation of data (for instance the uniqueness of the representation of the natural number 1). In this paper we develop the solution to this problem proposed in [11]: we show how to *extract the intuitionistic representation of a data from a classical one* using an “output” operator, while keeping a confluent computation mechanism. This result allows to extend in a sound way the proofs-as-programs paradigm to classical proofs in a framework where all the usual theoretical properties of intuitionistic proofs still hold.

(You can extend this correspondence to classical natural deduction, at the cost of losing some nice features of constructive reasoning)

$$x : A^x \vdash A$$

$$\frac{u : \Gamma \vdash B, \Delta}{\lambda x.u : \Gamma/A^x \vdash A \rightarrow B, \Delta} \rightarrow_i$$

$$\frac{u : \Gamma \vdash A[Y/X], \Delta}{u : \Gamma \vdash \forall X A, \Delta} \forall_i^2 (*)$$

$$\frac{u : \Gamma \vdash A[y/x], \Delta}{u : \Gamma \vdash \forall x A, \Delta} \forall_i^1 (*)$$

$$\frac{t : \Gamma \vdash A, \Delta}{\mu\beta.[\alpha]t : \Gamma \vdash B, (\Delta, A^\alpha)/B^\beta} \mu$$

$$\frac{t : \Gamma \vdash A[\tau/x], \Delta}{t : \Gamma \vdash A[\sigma/x], \Delta} eq (**)$$

$$\frac{t : \Gamma_1 \vdash A \rightarrow B, \Delta_1 \quad u : \Gamma_2 \vdash A, \Delta_2}{(t)u : \Gamma_1, \Gamma_2 \vdash B, \Delta_1, \Delta_2}$$

$$\frac{u : \Gamma \vdash \forall X A, \Delta}{u : \Gamma \vdash A[T/X], \Delta} \forall_e^2$$

$$\frac{u : \Gamma \vdash \forall x A, \Delta}{u : \Gamma \vdash A[\tau/x], \Delta} \forall_e^1$$

CLASSICAL MODEL THEORY

A MODEL M for a first-order language is a pair $\langle D, I \rangle$ where D is a nonempty set, called the DOMAIN, and I is a function on the language such that for each name a in the language, $I(a) \in D$, and if F is an n -ary predicate, then $I(F) : D^n \mapsto \{0, 1\}$, where D^n is the set of sequences of length n of objects from D .

- $I(\neg A, v) = 1$ iff $I(A, v) = 0$
- $I(A \wedge B, v) = 1$ iff $I(A, v) = 1$ and $I(B, v) = 1$
- $I(A \vee B, v) = 1$ iff $I(A, v) = 1$ or $I(B, v) = 1$
- $I(A \rightarrow B, v) = 1$ iff $I(A, v) = 0$ or $I(B, v) = 1$
- $I(\perp, v) = 1$ never
- $I(\forall x A, v) = 1$ iff for every u where $u \sim_x v$, $I(A, u) = 1$
- $I(\exists x A, v) = 1$ iff for some u where $u \sim_x v$, $I(A, u) = 1$

A pair of a model M and an assignment v is a COUNTEREXAMPLE to the argument $X \succ A$ iff $I(B, v) = 1$, for each $B \in X$, and $I(A, v) = 0$.

forth Conditional Semantics

BHK INTERPRETATION

Brouwer, Heyting, Kolmogorov

p proves $A \& B$

iff p is a pair $\langle r, s \rangle$ and r proves A and s proves B .

p proves $A \vee B$

iff p is a pair $\langle n, q \rangle$, where $n = 0$ and q proves A , or $n = 1$ and q proves B .

p proves $A \rightarrow B$

iff p is a rule q which transforms any proof s of A into a proof $q(s)$ of B , together with any supplementary information needed to convince us of this property.

p proves \perp

is impossible.

p proves $\neg A$

iff p proves $A \rightarrow \perp$.

p proves $\exists x \in X \phi(x)$

iff p is a pair $\langle x, q \rangle$ where x is a completely presented member of X and q is a proof that $\phi(x)$.

p proves $\forall x \in X \phi(x)$

iff p is a rule q such that for each completely presented member x of X , $q(x)$ is a proof of $\phi(x)$, together with any supplementary data needed to convince us of this property.

The definitions just offered for implication and universal quantification are notoriously imprecise, because of the phrase “any supplementary data needed to convince of this property”. This is one reason why we have presented the above as an “explanation” of the logical connectives, and not as a “definition”.

Michael Beeson, Foundations of Constructive Mathematics, 1985

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KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathcal{G}, \mathcal{R}, \models \rangle$, where \mathcal{G} is a non-empty set, \mathcal{R} is a transitive, reflexive relation on \mathcal{G} , and \models (conveniently read “forces”) is a relation between elements of \mathcal{G} and formulas, satisfying the following conditions:

For any $\Gamma \in \mathcal{G}$

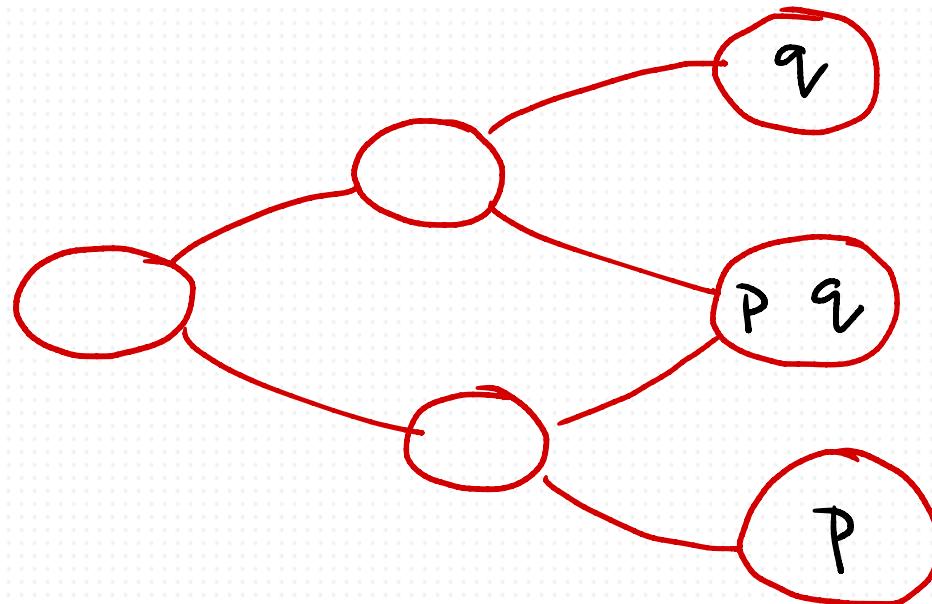
P0. if $\Gamma \models A$ and $\Gamma \mathcal{R} \Delta$ then $\Delta \models A$ (recall A is atomic).

P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

P2. $\Gamma \models (X \vee Y)$ iff $\Gamma \models X$ or $\Gamma \models Y$.

P3. $\Gamma \models \sim X$ iff for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, $\Delta \not\models X$.

P4. $\Gamma \models (X \supset Y)$ iff for all $\Delta \in \mathcal{G}$ such that $\Gamma \mathcal{R} \Delta$, if $\Delta \models X$, then $\Delta \models Y$.



- truth conditional semantics

KRIPKE FRAMES

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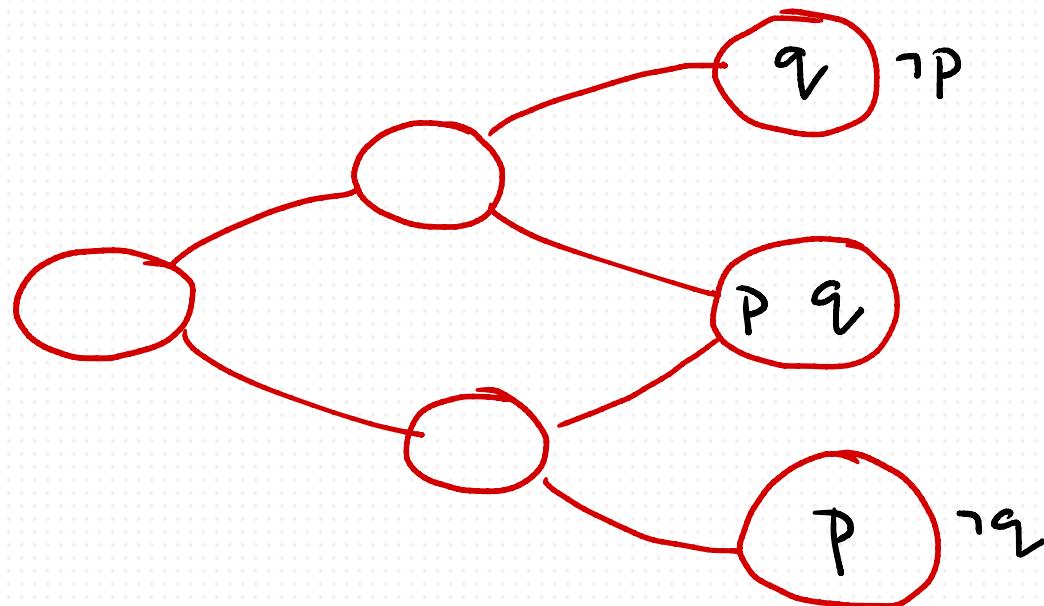
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P1. $\Gamma \models (X \wedge Y)$ iff $\Gamma \models X$ and $\Gamma \models Y$.

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- truth Conditional Semantics

KRIPKE FRAMES

By a (*propositional intuitionistic*) *model* we mean an ordered triple $\langle \mathcal{G}, \mathcal{R}, \models \rangle$, where \mathcal{G} is a non-empty set, \mathcal{R} is a transitive, reflexive relation on \mathcal{G} , and \models (conveniently read “forces”) is a relation between elements of \mathcal{G} and formulas, satisfying the following conditions:

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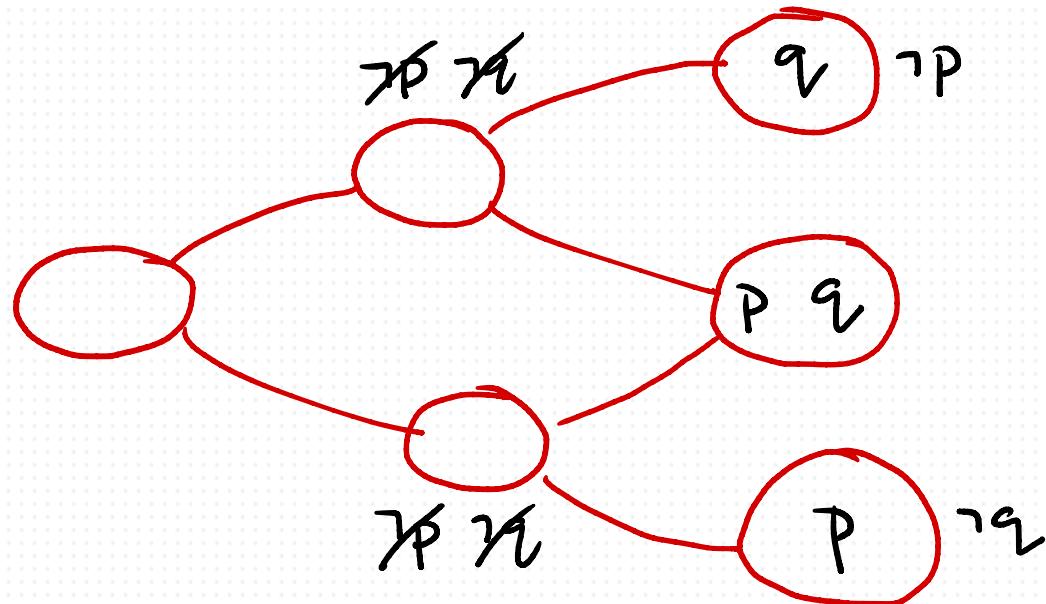
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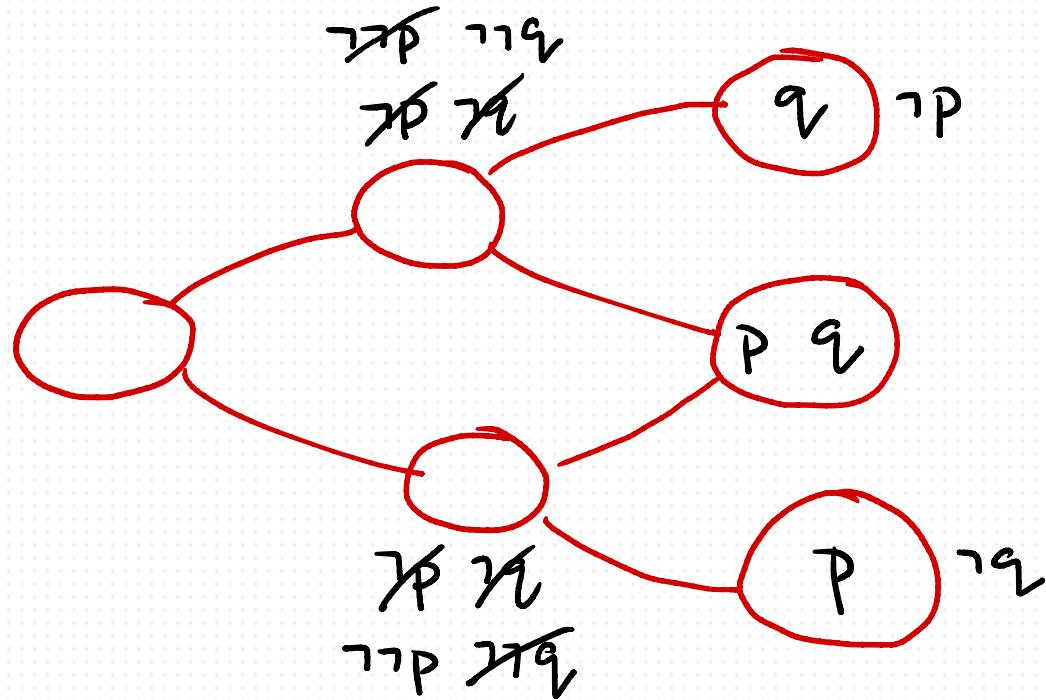
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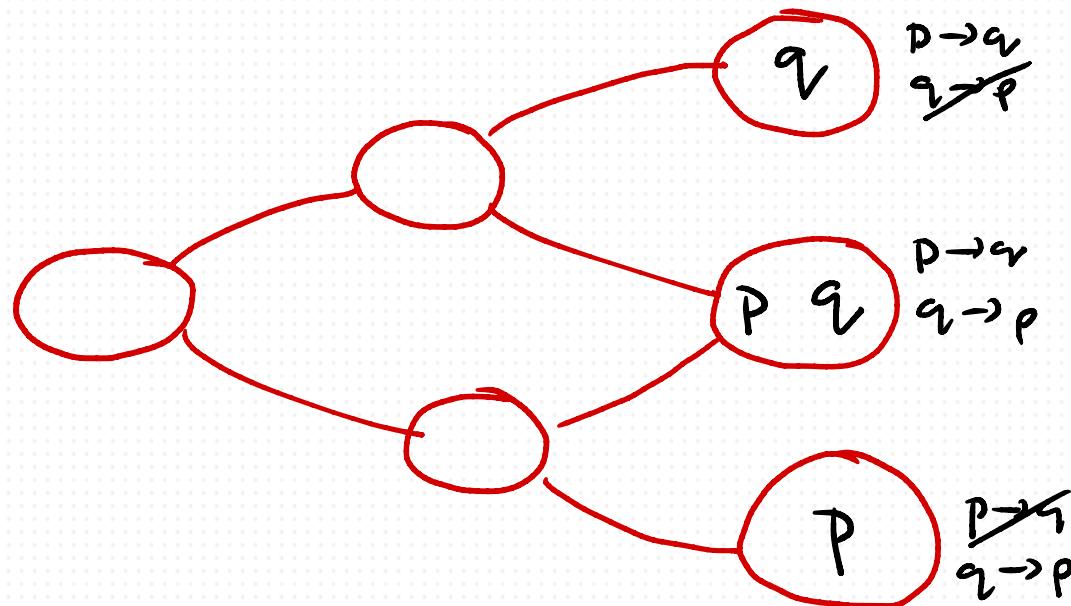
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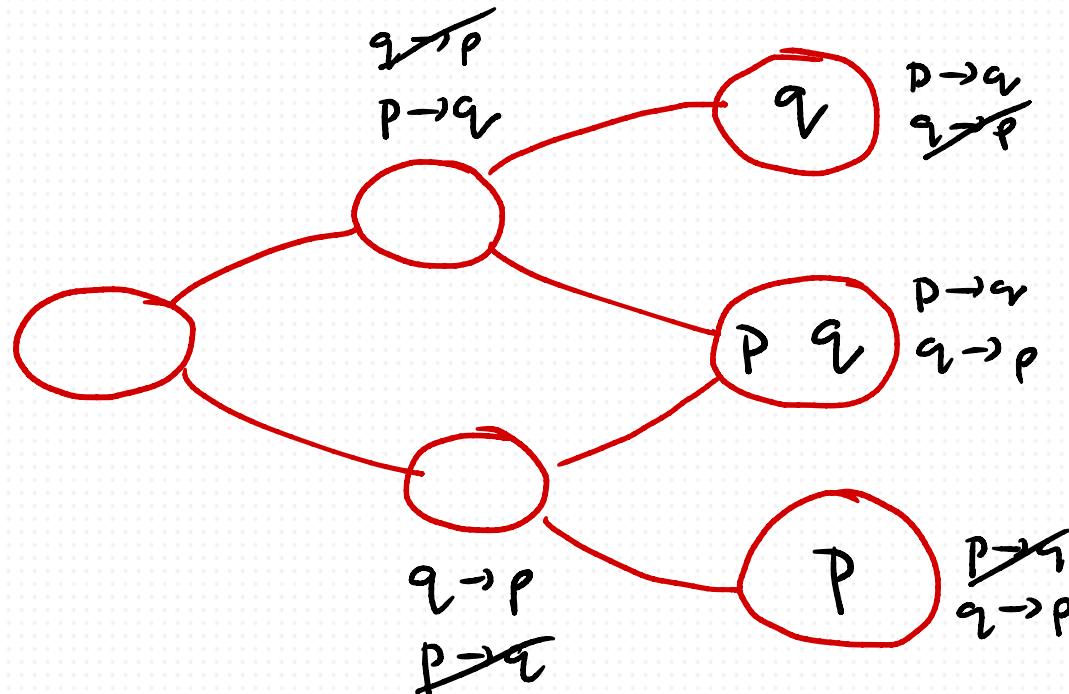
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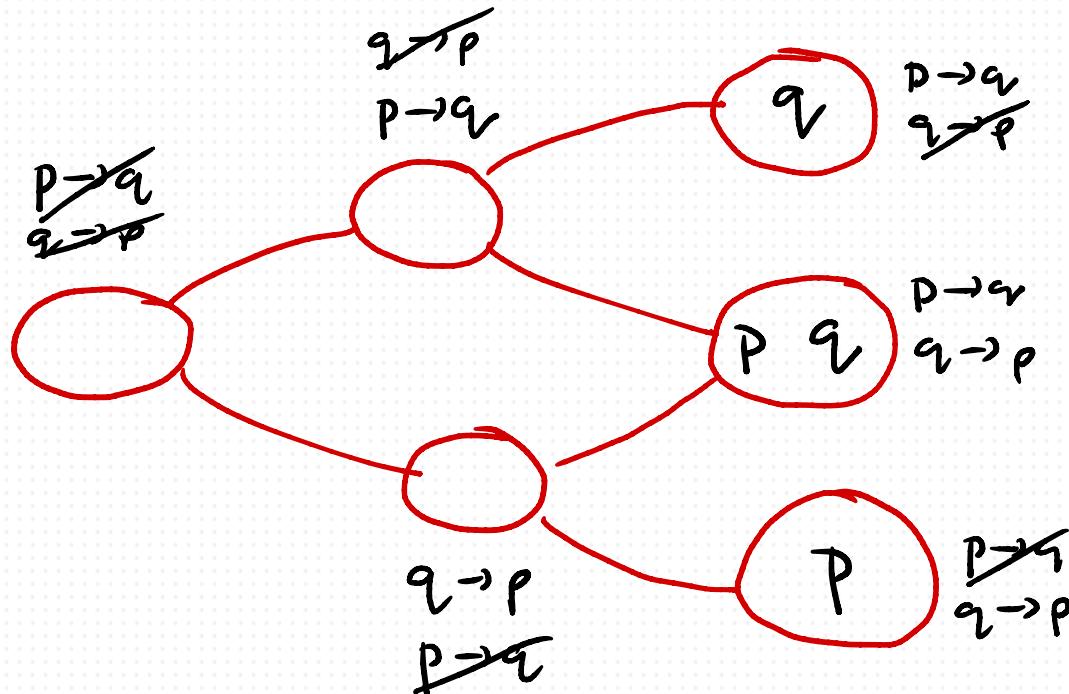
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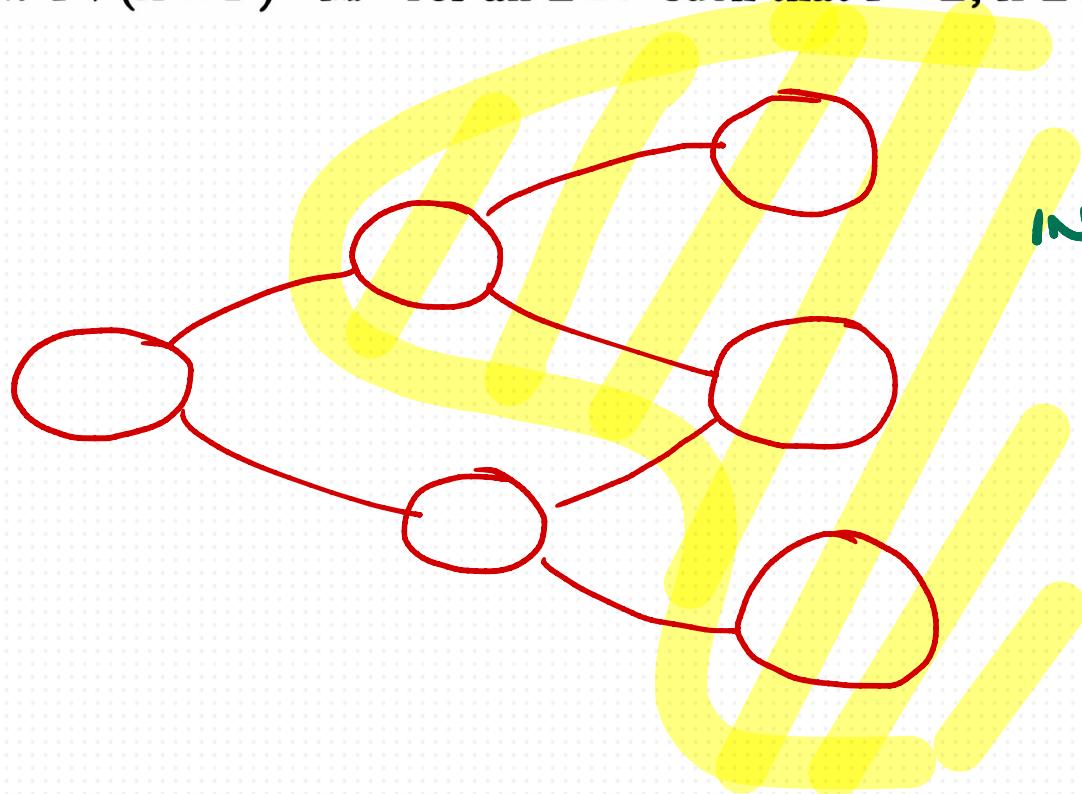
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IN ANY KRIPKE MODEL
A ‘PROPOSITION’ IS AN
UPWARDLY CLOSED
SET OF POINTS

– truth conditional semantics

TOPOLOGICAL SPACES

TOPOLOGICAL SPACES

Open Sets in a topological space provide a model of intuitionistic logic



$$(x, y) \cap (v, w) = (v, y)$$

$$(x, y) \cup (v, w) = (x, w)$$

$$\neg(x, y) = (-\infty, x) \cup (y, \infty)$$

TOPOLOGICAL SPACES

Open Sets in a topological space provide a model of intuitionistic logic

$$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$$

$$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]$$

$$[\![\neg A]\!] = \text{Int}(\mathcal{S} \setminus [\![A]\!])$$

$$[\![A \supset B]\!] = \text{Int}((\mathcal{S} \setminus [\![A]\!]) \cup [\![B]\!])$$

TOPOLOGICAL SPACES

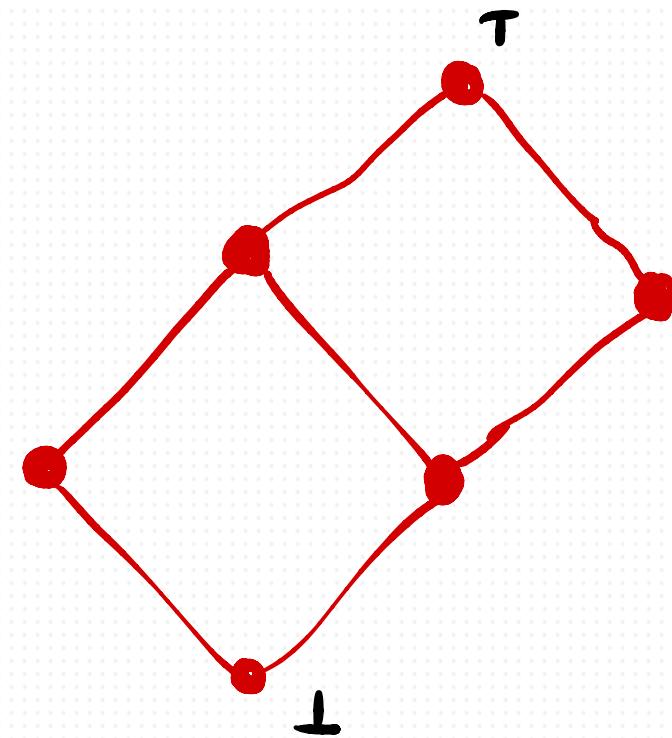
Open Sets in a topological space
form a Heyting lattice. $\langle H, \wedge, \vee, \rightarrow, \perp \rangle$

- $\langle H, \wedge, \vee, \perp \rangle$ is a lattice with bottom element \perp .

- \rightarrow residuates \wedge :
 $x \wedge y \leq z$ iff $x \leq y \rightarrow z$

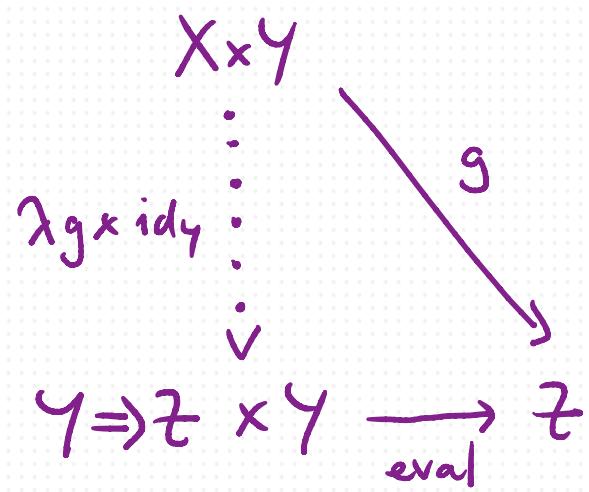
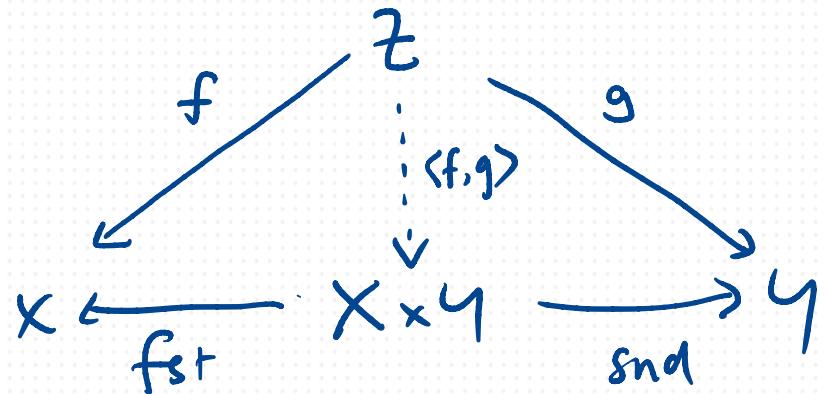
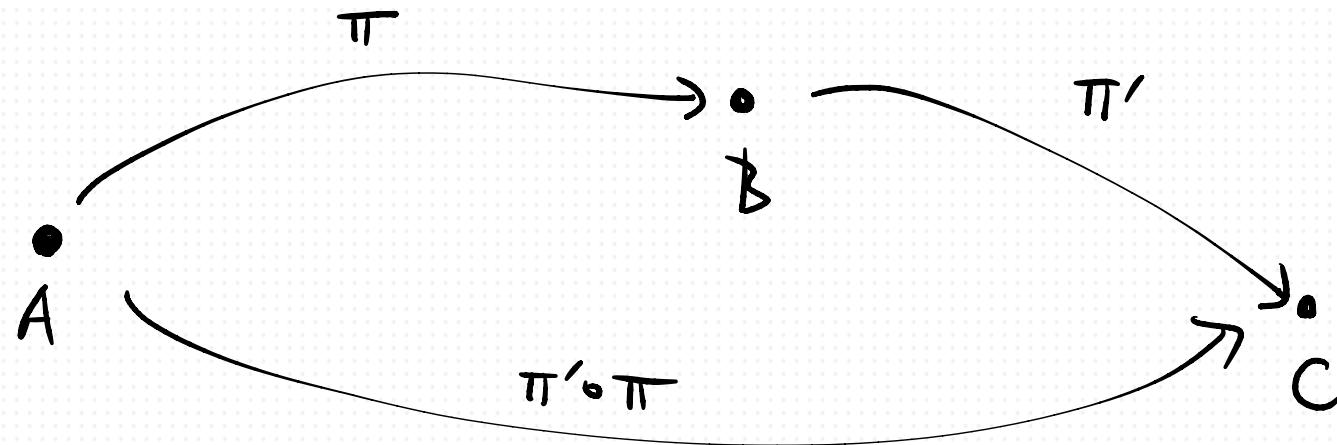
- $\neg x$ is $x \rightarrow \perp$

- All Boolean Algebras are Heyting lattices,
but not vice versa.



CATEGORICAL MODELS

- ... in which objects are propositions & arrows are proofs
- ... or topos models, e.g. Eff.



What can we learn
from all this?

1. MY QUESTION

2. LOGICS & PROOFS & MODELS

3. TRANSLATIONS

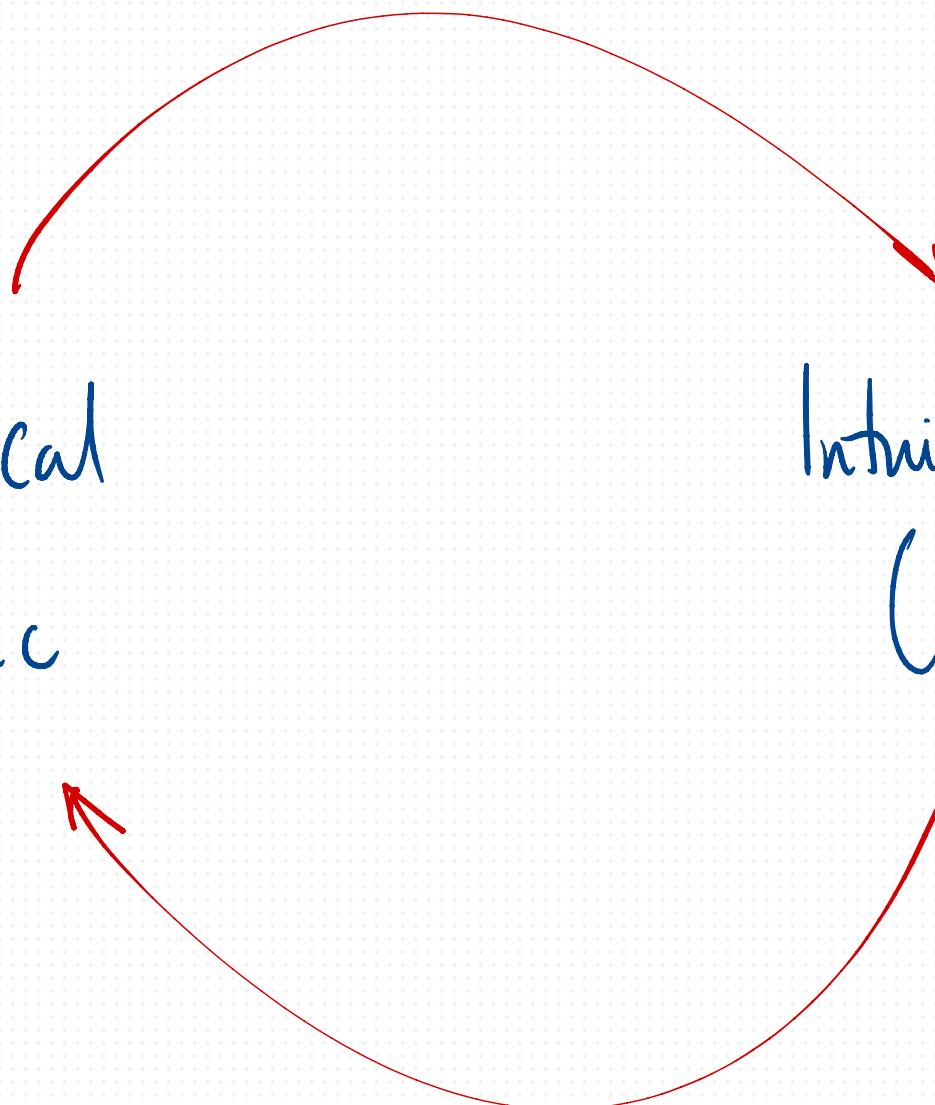
4. WHAT THIS MEANS — for the Classical partisan

5. — for the Intuitionist partisan

6. — for pluralists

Classical
logic

Intuitionistic
logic



DOUBLE NEGATION TRANSLATION

finding classical logic inside intuitionistic logic

$$t(P) = \neg\neg P$$

$$t(A \wedge B) = t(A) \wedge t(B)$$

$$t(A \vee B) = \neg(\neg t(A) \wedge \neg t(B))$$

$$t(A \rightarrow B) = t(A) \rightarrow t(B)$$

$$t(\neg A) = \neg t(A)$$

$$t(\forall x A) = \forall x t(A)$$

$$t(\exists x A) = \neg\forall x \neg t(A)$$



LANGUAGE of a
"CLASSICAL" SPEAKER

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LANGUAGE of a
"CONSTRUCTIVE"
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FACT.

$$X \vdash A \text{ iff } t(X) \vdash_{\text{Int.}} t(A)$$

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FACT.

$$X \vdash_{\text{classical}} A \quad \text{iff} \quad t(X) \vdash_{\text{Int.}} t(A)$$

\Rightarrow : Induction on the structure
of the classical proof.

\Leftarrow : If $t(X) \vdash_{\text{Int.}} t(A)$ then

$$t(X) \vdash_{\text{class.}} t(A),$$

and it's easy to see that

$$t(A) \vdash_{\text{class.}} A.$$

MODAL TRANSLATION

finding intuitionistic logic inside
classical logic + on S4 necessity

$$m(p) = \Box p \quad \text{Classical Speaker}\leftarrow\text{with a concept}$$

$$m(A \wedge B) = m(A) \wedge m(B) \quad \text{of 'necessity'}$$

$$m(A \vee B) = m(A) \vee m(B)$$

$$m(A \rightarrow B) = \Box(m(A) \rightarrow m(B))$$

$$m(\neg A) = \Box \neg m(A)$$

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INTUITIONISTIC
SPEAKER

MODAL TRANSLATION

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$$X \vdash_{\text{Int}} A \quad \text{iff} \quad m(X) \vdash_{S4} m(A)$$

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One proof: Intuitionistic Kripke models
are S4 Kripke models. Up-closed
sets of points are exactly the points
of the form $\Box A$.

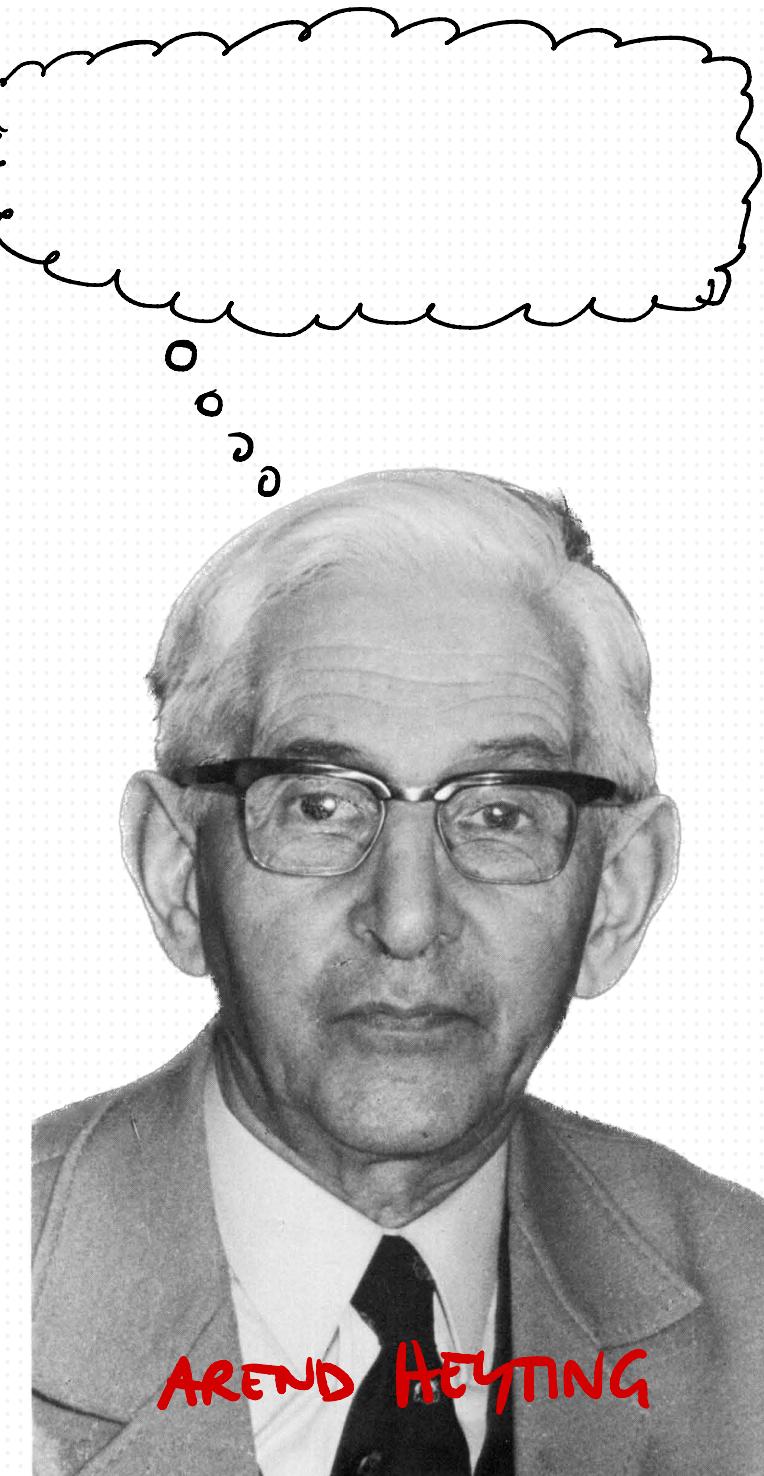
Another: Topological models for intuitionistic
logic are topological models for S4, where
 \Box is modelled by the interior operator.

These translations
change the
meanings of terms



RUTH BARCAN

$\forall x f(x) \vee \exists x \neg f(x)$



ARENDE HEYTING



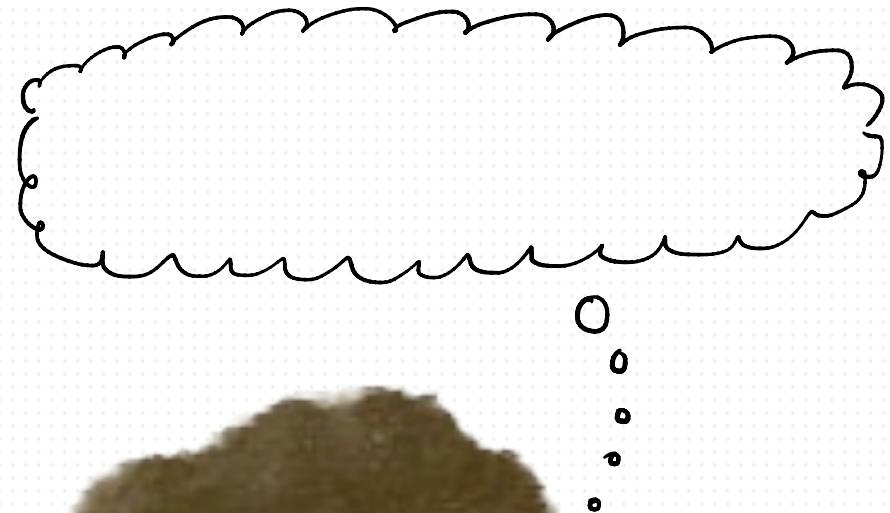
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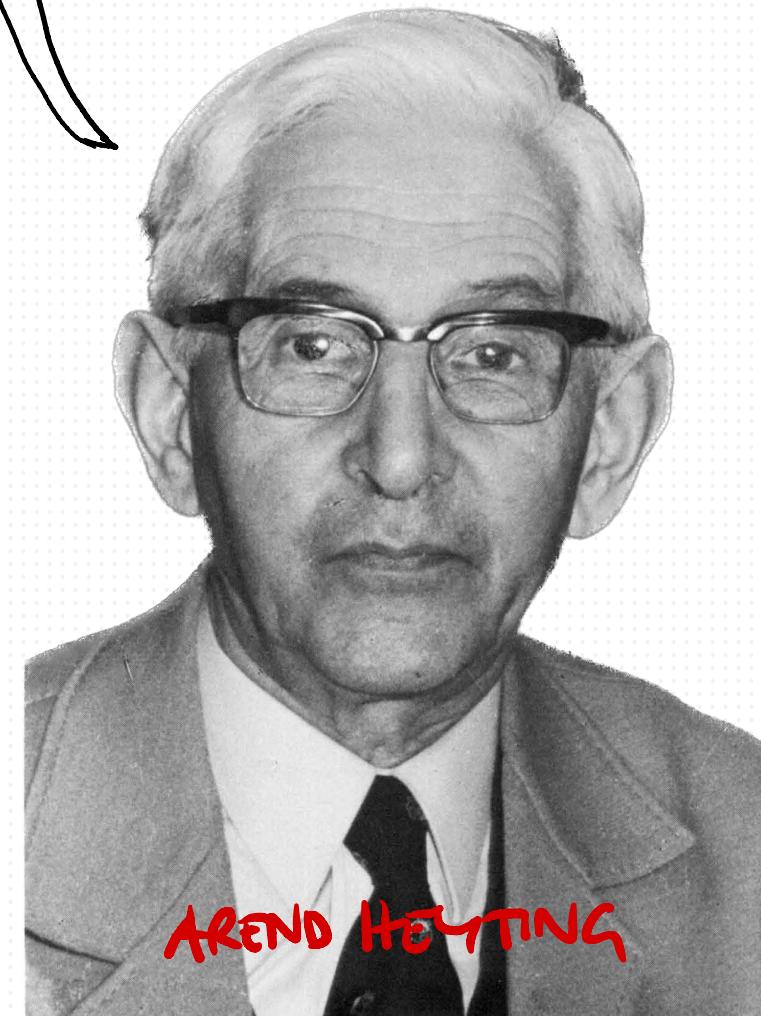
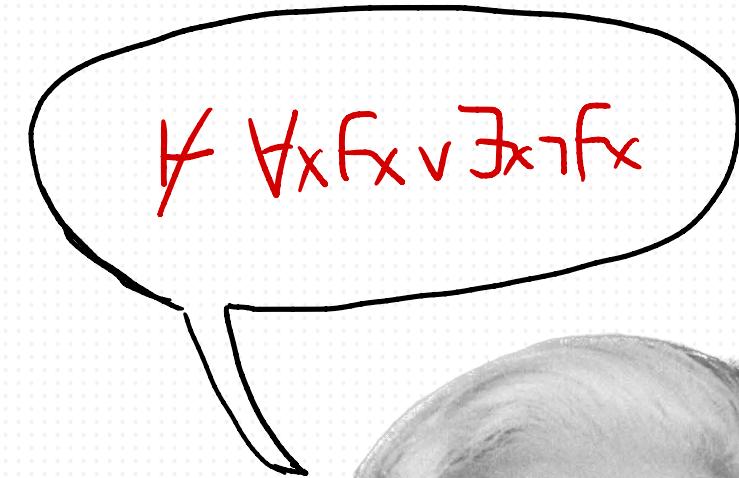
(ג'רדר אלקס ו ג'רלד אלסן)



ARENDE HEYTING



RUTH BARCAN



ARENDE HEYTING

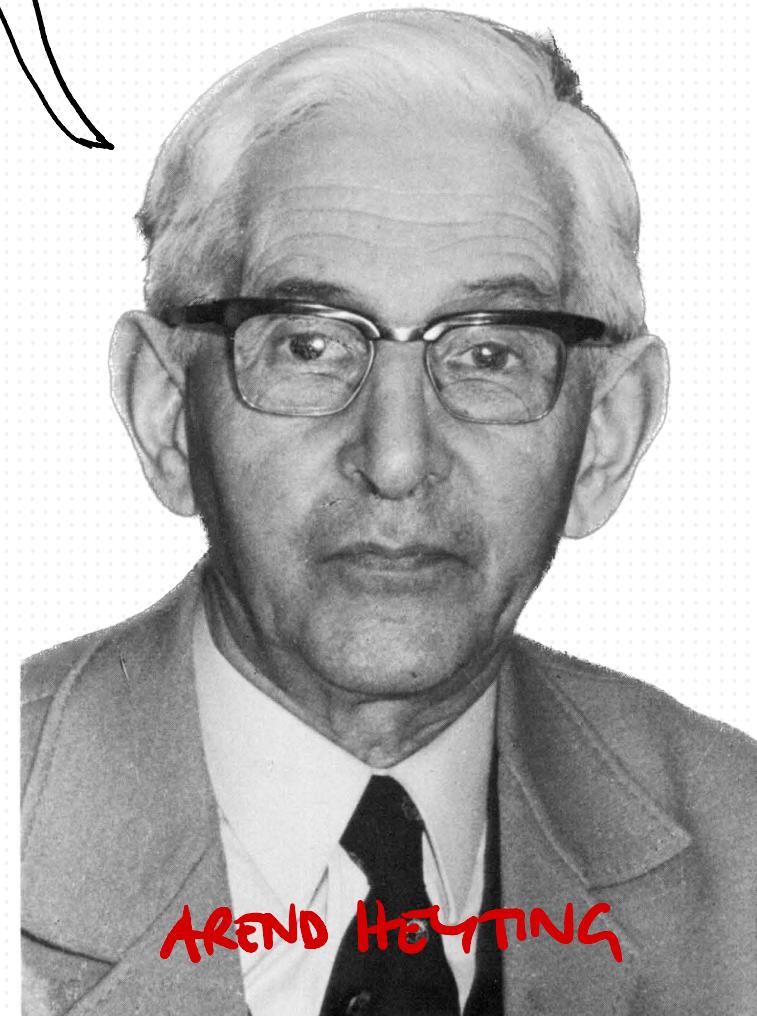


RUTH BARCAN

$\vdash \Box \forall x \Box Fx \vee \exists x \Box \neg \Box Fx$

...

$\vdash \forall x Fx \vee \exists x \neg Fx$



ARENDE HEYTING

This doesn't make sense of
Bishop's reasoning.

In classical mathematics, to assume an inequality $x > y$, derive a contradiction, and conclude that therefore $x \leq y$ (as in Lemma 5) is called an indirect proof. It is necessary to use caution in thinking of Lemma 5 in this way. Although $x \leq y$ is equivalent to the negation of $x > y$, it is *not* true that $x > y$ is equivalent to the negation of $x \leq y$.

Errett Bishop
foundations of
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Bishop is **not** using a
translation. He is saying
that this inference is
classically OK
& intuitionistically invalid.

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Intuitionistic logic

plus extra principles, e.g.

$$\forall x \phi(x) \vee \exists x \neg \phi(x)$$

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Terminal nodes in a Kripke frame

act completely classically
— no translation necessary!

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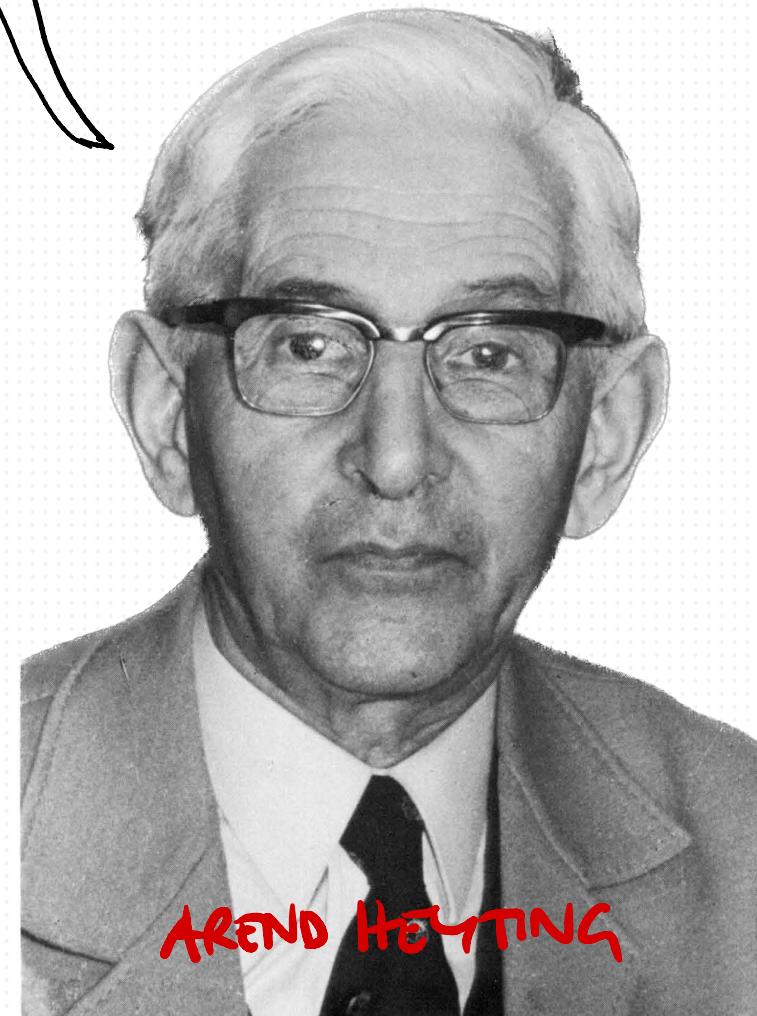
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...

STEP 1

RUTH BARCAN

$\vdash \forall x Fx \vee \exists x \neg Fx$



ARENDE HEYTING

At least the intuitionistic
practice is coherent.

But so far, it's deviant regarding
the meaning of some connectives.

STEP 2 : Learning to speak the language

- Explore some of the models
(Topos models as mathematical universes are particularly powerful for this, but other models work, too.)

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STEP 2 : Learning to speak the language

- Explore some of the models
 - ▷ Topos models as mathematical universes are particularly powerful for this, (but other models work, too.)
- Attempt to live "inside" these models.
 - ▷ This involves restricting your inferences & adopting a stricter standard of proof, making finer distinctions.

A CONCERN (for the friend of classical logic)

- Isn't this changing the meaning of your vocabulary?
- I, a classical logician, can prove $A \vee \neg A$.
You're now asking me to deny it — how does that make sense?

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- ▷ There is no problem in CODE SWITCHING between stricter & more lax standards of assertion, without changing semantics (cf. legal contexts.)

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Martin Escardo
@EscardoMartin

...

I like to work within "neutral" mathematics, as opposed to classical or constructive mathematics.
(I said so here, among other places:
math.andrej.com/2021/05/18/com...)

1/

7:23 PM · Jul 12, 2022 · Twitter Web App

8 Retweets 1 Quote Tweet 59 Likes



Tweet your reply

Reply



Martin Escardo @EscardoMartin · Jul 12

...

Replies to @EscardoMartin

This idea was probably proposed first by Bishop. But also it is what happens when working in the internal language of an arbitrary topos. Both took place at about the same time, in the late 1960's, and independently.

2/



Martin Escardo @EscardoMartin · Jul 12

...

You don't say that the principle of excluded middle is false, for example. You keep it open. You keep it deliberately undecided. You just don't use it.

3/



Martin Escardo @EscardoMartin · Jul 12

...

In the case of Bishop mathematics, the aim was that every theorem of his constructive mathematics is also a theorem of classical mathematics.

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$\forall x f(x) \vee \exists x \neg f(x)$

STEP 1

RUTH BARCAN

ג'רלד אHEYTING
ג'רלד אHEYTING



ARENDE HEYTING

Conversely, this shows to the
Constructivist that Classical
practices is Coherent.

But so far, it's deviant regarding
the meaning of some connectives.

STEP 2

- Using your favoured system of proofs/models, explore the "constructive" meaning of classical rules
 - ▷ In proofs, call/cc $\lambda\mu$ models; Krivine classical realizability
In Kripke models, classical logic holds at endpoints.
- By admitting the possibility of "evidence" for $A \vee \neg A$, you don't need to erase all your constructive distinctions.
You, too, can learn to live with two standards for assertion, your strict, constructive standards, & more lax, classical norms.
- ▷ The upside is you get to have all those classical results!

Intuitionistic logic
is preserved in more places.

It has more models.

Classical logic
can help you go further.

It has more proofs.

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Classical logic
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It has more proofs.

You can do more, if you have both
classical & intuitionistic logic in
your conceptual toolkit.

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6. — for pluralists

WHEREVER YOU STARTED FROM...-

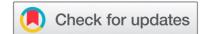
If you have taken STEP 2
you are some kind of Pluralist.

But there are different kinds
of logical pluralist.

THEORY PLURALISM

for a theory pluralist, the logic you use is relative to the domain of investigation, + different mathematical theories.

INQUIRY, 2017
<https://doi.org/10.1080/0020174X.2017.1357495>



Logical pluralism and normativity

Teresa Kouri Kissel[†] and Stewart Shapiro

Department of Philosophy, Ohio State University, Columbus, OH, USA

ABSTRACT

We are logical pluralists who hold that the right logic is dependent on the domain of investigation; different logics for different mathematical theories. The purpose of this article is to explore the ramifications for our pluralism concerning normativity. Is there any normative role for logic, once we give up its universality? We discuss Florian Steingerger's "Frege and Carnap on the Normativity of Logic" (*Synthese* 94: 143–162) as a source for possible types of normativity, and then turn to our own proposal, which postulates that various logics are constitutive for thought within particular practices, but none are constitutive for thought as such.

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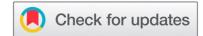
KEYWORDS Logic; pluralism; normativity; Frege; Carnap

THEORY PLURALISM

for a theory pluralist, the logic you use is relative to the domain of investigation, + different mathematical theories.

- This is undoubtedly well motivated. E.g.
 - HA (Heyting Arithmetic) &
 - PA (Peano Arithmetic)Are different theories with different properties & different logics

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Two different mathematical theories can have the same domain of investigation.

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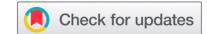
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Both HA & PA investigate the natural numbers.
One, constructively, the other, classically.

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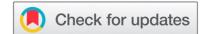
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A STRONG THEORY PLURALIST takes there to be different theories — investigating the same domain — with different logics.

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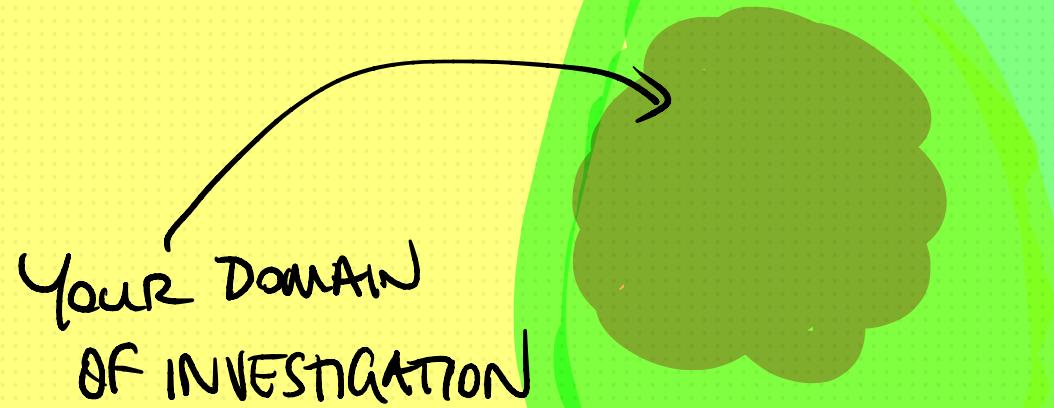
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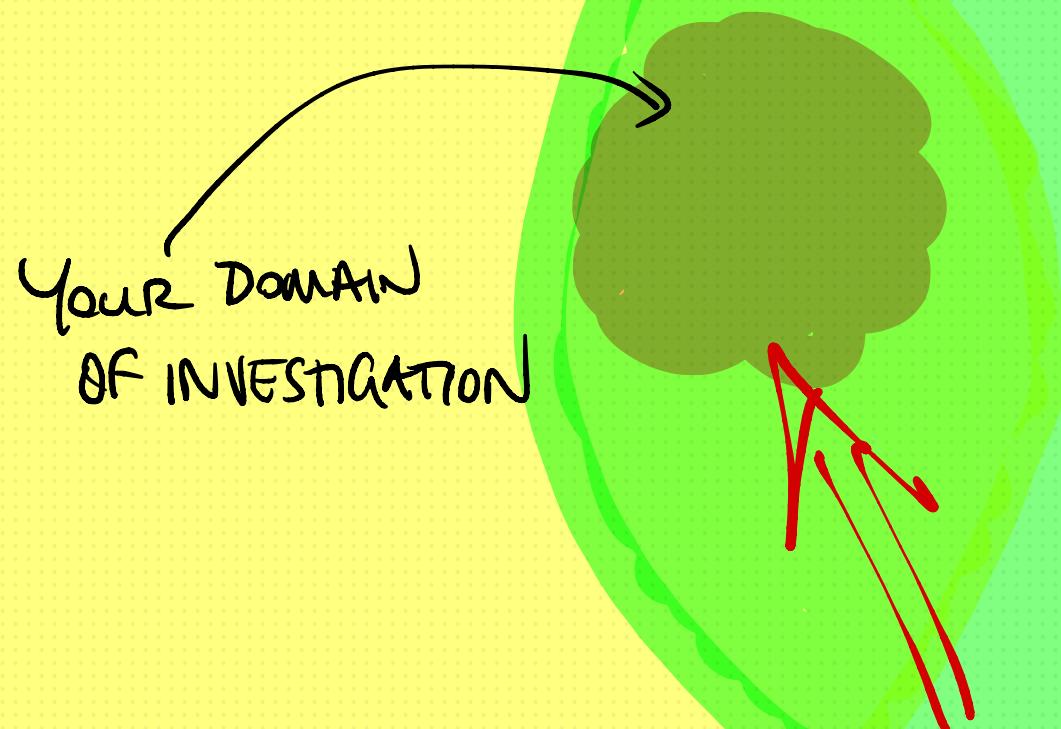
USING CLASSICAL
LOGIC

USING INTUITIONISTIC
LOGIC



USING CLASSICAL
LOGIC

USING INTUITIONISTIC
LOGIC



ISN'T THIS THE BEST
OF BOTH WORLDS?

Thank You /
.

<https://consequently.org/>