SPEECH ACTS & THE QUEST FOR A NATURAL ACCOUNT OF CLASSICAL *PROOF*

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Abstract: It is tempting to take the logical connectives, such as conjunction, disjunction, negation and the material conditional to be defined by the basic inference rules in which they feature. Systems of 'natural deduction' provide the basic framework for studying these inference rules. In natural deduction proof systems, well-behaved rules for the connectives give rise to *intuitionistic logic*, rather than classical logic. Some, like Michael Dummett [31], take this to show that intuitionistic logic is on a sounder theoretical footing than classical logic. Defenders of classical logic have argued that some other framework, such as Gentzen's sequent calculus, or a *bilateralist* system of signed natural deduction, can provide a proof-theoretic justification of classical logic. Such defences of classical logic have significant shortcomings, in that the systems of proof offered are much less natural than existing systems of natural deduction. Neither sequent derivations nor signed natural deduction proofs are good matches for representing the inferential structure of everyday proofs.

In this paper I clarify the shortcomings of existing bilateralist defences of classical proof, and, making use of recent results in the proof theory for classical logic from theoretical computer science [88, 89], I show that the bilateralist can give an account of natural deduction proof that models our everyday practice of proof as well as intuitionist natural deduction, if not better.

1 THE PROBLEM OF CLASSICAL PROOF

There are many ways to design a system of *proofs* for classical logic. Since the rise of proof theory in the 20th Century we have seen a plethora of different systems of proof,

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ranging from axiomatic proof systems in the style of Hilbert [53, 54], tableaux proofs in the manner of Beth [21], Hintikka [55] and Smullyan [115], and Gentzen's influential sequent calculus [44]. No style of proof system, though, has received anywhere near the sustained degree of philosophical attention, either in logic textbooks, or in the research literature, that has been given to systems of *natural deduction*. With origins in the pioneering works of Jáskowski [63] and Gentzen [43], natural deduction proof systems have been studied by Fitch [36], Lemmon [73] and Prawitz [93] in the middle of the 20th Century, and philosophers such as Michael Dummett [31], Dag Prawitz [94–96] and Neil Tennant [121, 122] have placed natural deduction systems at the focus of their accounts of the semantics of logical vocabulary.

For approaches like these, a natural deduction proof system is much more than a convient way to specify the valid arguments. It is a framework in which the rules for each logical constant can be given a well-defined *semantics* [38, 66, 95, 112, 126], by way of a system of introduction and elimination rules, each of which feature one (and *only* one) logical concept at a time, and such that the rules for each connective are appropriately *harmonious*. This means (very roughly speaking) that the elimination rule for a concept allows you to extract all and only the information that was 'put in' to the concept by way of its introduction rule. In such a harmonious natural deduction system, we can see each pair of introduction and elimination rules as, in some sense, *defining* a logical concept uniquely, in such a way that the addition of any such a logical concept to some language by way of these inference rules is a *conservative* extension (not allowing for the addition of new proofs in the old vocabulary), while also affording an increase in expressive power to the language. The harmonious inference rules of the logical concepts give us an answer to Prior's challenge to explain how it is that some sets of inference rules might truly *define* a logical concept [98]. A natural deduction proof system provides a background context of deducibility against which logical concepts can be given 'definitions' which are both conservative and uniquely defining [15]. To use Brandom's terminology, the addition of logical vocabulary like $\to$ (a conditional), $\neg$ (a negation), $\wedge$ (conjunction), $\vee$ (disjunction) or the quantifiers or identity, allow us to *make explicit* [25] what was merely implicit (inferential connections, contradictoriness, generality, etc.) in the original vocabulary. Natural deduction proof systems are philosophically rich as well as pedagogically useful.

Rather than continuing to talk abstractly about natural deduction systems as such, it will be helpful to attend to one particular system of rules, as a focus of our attention. It is simplest, for our purposes, to choose Gentzen's own presentation of natural deduction, as systematised and popularised by Dag Prawitz [93].[1] The basic rules for this

[1]There are significant subtleties involved in the choice of one system or another, especially when it comes the behaviour of the structural rules [105, 117]. It would be worth exploring whether the choices made here in Gentzen–Prawitz-style natural deduction could be adapted to other frameworks. For a good account of the plethora of such frameworks, either of the histories of natural deduction by Jeff

$$A \qquad \dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\,{\to}I^i \qquad \dfrac{\begin{array}{cc}\Pi & \Pi'\\ A \to B & A\end{array}}{B}\,{\to}E$$

$$\dfrac{\begin{array}{cc}\Pi & \Pi'\\ A & B\end{array}}{A \wedge B}\,{\wedge}I \qquad \dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{A}\,{\wedge}E \qquad \dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{B}\,{\wedge}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}}{A \vee B}\,{\vee}I \qquad \dfrac{\begin{array}{c}\Pi\\ B\end{array}}{A \vee B}\,{\vee}I \qquad \dfrac{\begin{array}{ccc} & [A]^j & [B]^k\\ \Pi & \Pi' & \Pi''\\ A \vee B & C & C\end{array}}{C}\,{\vee}E$$

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ \bot\end{array}}{\neg A}\,{\neg}I^i \qquad \dfrac{\begin{array}{cc}\Pi & \Pi'\\ \neg A & A\end{array}}{\bot}\,{\neg}E \qquad \dfrac{\begin{array}{c}\Pi\\ \bot\end{array}}{A}\,{\bot}E$$

Figure 1: NATURAL DEDUCTION PROOFS

system of natural deduction are displayed in Table 1. In this display, A, B and C range over formulas, Π, Π' and Π'' are each *proofs*, whose *conclusion* is the formula written directly below it. If any formula is written above a 'Π', and surrounded in brackets, this means that some number of occurrences of that premise in the proof Π are *discharged* in the next inference. So, in a $\to I$ inference to the conclusion $A \to B$, some number of instances of A occurring as assumptions in the proof Π are discharged, and are no longer active premises of the proof of $A \to B$.[2]

Proofs are inductively defined objects, and so, we include the base case of the induction, the single formula A. This is the smallest proof, in which A as a premise that has been assumed, and that very same formula A is at the very same time its conclusion. These 'atomic' proofs are the seeds from which all proofs grow.

Natural deduction proof systems form an attractive package. The introduction and elimination rules for each logical concept behave rather like the left-to-right and right-to-left components of the truth conditions for sentences in which that concept is dom-

Pelletier (the longer, co-authored with Allen Hazen) are well worth exploring [91, 92].

[2]One surprising, but important fact for these natural deduction proofs is that this number of occurrences of the supposition A to be discharged can be *zero*. There is a one-step proof from the premise p to the conclusion $q \to p$ using the rule $\to I$, discharging *zero* instances of the supposition q.

$$\dfrac{\begin{array}{c}\Pi\\ \neg\neg A\end{array}}{A}\,DNE \qquad \dfrac{\begin{array}{c}[\neg A]^i\\ \Pi\\ \bot\end{array}}{A}\,\bot E_c \qquad \dfrac{\begin{array}{cc}[A]^i & \neg A\\ \Pi & \Pi'\\ C & C\end{array}}{C}\,Cases^{i,j}$$

Figure 2: CLASSICAL PRINCIPLES IN STANDARD NATURAL DEDUCTION

inant. Each rule governs what it takes for the claim to be true, or what follows given that the claim is true.

However, this package is, in its current form, rather *opinionated*. Natural deducion is well suited to intuitionistic logic, and not its older cousin, classical logic. The familiar natural deduction rules for the conditional do not allow for a proof of Peirce's Law $((p \to q) \to p) \to p$, even though this is a tautology of classical logic. The rules for the conditional and disjunction are not enough to supply a proof of $p \lor (p \to q)$, which is also a classical tautology. The rules for negation do not supply a proof from $\neg\neg p$ to p, and neither do we have a proof of the Law of the Excluded Middle, $p \lor \neg p$.

It is simple enough to extend a natural deduction system with rules to plug these gaps. The rules for classical natural deduction as you will find in the usual textbooks [11, 24, 28, 73, 120, for example] do so by adding rules for *negation*. Candidate rules are *Double Negation Elimination*, a classically strengthened $\bot$ Elimination Rule, or a rule that allows reasoning by arbitrary 'Boolean' cases. These are collected together in Figure 2. Each of these rules are natural enough in their own way. (They would not be proposed as *rules* if they struck most people as being *invalid*, after all.) However, the upshot of the need to patch the proof system with rules like these is that the connective rules are no longer harmonious. The rules add to our usual introduction and elimination rules an *extra* rule, governing one connective, negation.[3] The typical results for natural deduction proof theory, including normalisation, and the subformula property and conservative extension results fail to hold or hold only in an eviscerated form in this kind of classical natural deduction [93]. Peirce's Law can be proved only by way of a detour through negation. Figure 3 contains, for example, a proof of Peirce's Law, using a Double Negation Elimination inference. That we have to use proofs like this shows either that theses like Peirce's Law are not *analytic* in the sense of following from the semantic rules governing the conditional alone, or those rules as presented in Figure 1 are at best, incomplete. The rules for the conditional do not adequately capture its meaning. They only do so when supplemented by rules governing some other connective.[4]

[3] Or two concepts, if we wish to count '$\bot$' as a distinct logical concept, as we ought. See Section 4 for a discussion of a slightly different way to treat contradictoriness, which provides a cleaner way for negation to be modelled, without relying on its connection with the contradiction constant $\bot$.

[4] I have chosen negation rules here, as they are the usual textbook additions. Classical logic can be

$$
\begin{prooftree}
\AxiomC{$[\neg p]^2$}
\AxiomC{$[(p \to q) \to p]^3$}
\AxiomC{$[\neg p]^2$}
\AxiomC{$[p]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{q}
\RightLabel{$\to I^1$}
\UnaryInfC{$p \to q$}
\RightLabel{$\to E$}
\BinaryInfC{p}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg p$}
\RightLabel{DNE}
\UnaryInfC{p}
\RightLabel{$\to I^3$}
\UnaryInfC{$((p \to q) \to p) \to p$}
\end{prooftree}
$$

Figure 3: Peirce's Law, proved with Double Negation Elimination

So, 'textbook' classical natural deduction has nothing like the appeal of its intuitionistic cousin. This unsatisfactory state of affairs for classical proof theory is one plank of Dummett's argument in favour of intuitionistic logic over classical logic in *The Logical Basis of Metaphysics* [31]. We have a use-based theory of meaning grounded in well-behaved natural deduction rules for the logical connectives, but those rules give rise to intuitionistic logic, and give no justification for properly classical logical principles, or so the argument goes.

So, for the friend of classical logic the challenge is clear. If you wish to retain your allegiance to classical logic, you must either give up the search for a system of proofs with well-behaved rules and give some other account of the semantics of our logical vocabulary (admittedly, this is the overwhelmingly dominant response to the problem), or you must find a better proof system than textbook classical natural deduction.

If we want a well-behaved proof system for classical logic, we hope to do better. We *must* do better. We *can* do better.

* * *

The philosophical literature contains two dominant contenders for providing a well-behaved proof system for classical logic, with separable rules, normalisation and the subformula property. The most venerable system of this kind is Gentzen's sequent calculus [43]. We shift from considering natural deduction proofs, which are structured

regained by supplementing the rules for the conditional with princples like a Pierce rule (from $(A \to B) \to A$, derive A) or Tarski's conditional reasoning by cases (if C can be derived from $B \to A$ and from $A \to B'$, then C follows). The point we have been making concerning the standard textbook rules apply to these, too. Now we *need* to apply a rule governing the conditional in order to derive p from $\neg\neg p$, breaking the subformula property.

lists or trees of *formulas*, representing the different steps at which claims are assumed, inferred, discharged, etc., to trees of *sequents*, which are not formulas but are themselves collections of formulas. Given this shift, it is straightforward to design separable rules for the classical connectives, which are just as harmonious as the rules in a natural deduction system. The price to pay is that derivations in a sequent calculus are trees of sequents of the form $X \succ Y$, where X and Y are collections[5] of formulas. The upshot is that sequent derivations do not bear as close a relationship to everyday *proofs* as proofs formalised in natural deduction systems.

The more recent contender for a well-behaved proof-theoretic foundation for classical logic is a *signed* natural deduction system. Here, again, there is an added layer of complexity beyond natural deduction proofs. In a signed natural deduction system, formulas are tagged with signs. In the systems proposed for classical logic, there are two signs, one positive, and one negative, and a signed formula represents either a positive or negative *attitude* to a formula (acceptance or rejection) or a positive or negative *speech act* (assertion or denial).[6]

The sequent calculus and signed natural deduction are both very well behaved proof systems, with none of the inelegance of textbook systems for classical natural deduction. The rules for the connectives in these systems have the right kind of harmony to be in contention for providing an account of what it is to *define* logical concepts. However, as we will see, both the sequent calculus and signed natural deduction have shortcomings which mean they are not as suited to the project of providing a semantics for the inferentialist as standard unsigned natural deduction. The proper inferentialist treatment for classical logic has some way, yet, to go.

As proponents of the sequent calculus have seen [29, 30, 51, 103], Gentzen's proof system for classical logic is very well suited to its target. The Law of the Excluded Middle ($\succ p \vee \neg p$) and the Law of Non-Contradiction ($p \wedge \neg p \succ$) have derivations that are

[5] Whether the components of a sequent are sets, multisets or lists or some other kind of collection is not important for our purposes.

[6] In Timothy Smiley's influential paper, "Rejection" [114], the attitude interpretation was dominant, and the sign '$*$' was used to tag rejections of formulas, and a sign for accepted formulas was ommitted. In more recent works parity between *pro* and *con* is maintained with the use of two signs, '$+$' and '$-$', and the interpretation centres on speech acts of assertion and denial [61, 110, 114]. I will follow the recent conventions in this paper. As Humberstone has shown [58], this 'rejective' approach to negation was pioneered in the 1970s by Kent Bendall [18, 19]. The use of positively and negatively signed formulas in a proof system for classical logic occurs elsewhere, too. Smullyan [115] uses signed formulas in an analytic tableaux system for classical first-order logic (this work was known and cited by Bendall [19], but is cited neither by Smiley [114] nor Rumfitt [110]). Beisecker [14] discusses an explicitly bilateralist interpretation of such tableaux proof systems, and shows that many of the motivating bilateralist ideas are expressed, at least in nascent form, by Charles Sanders Peirce.

exactly dual to one another:

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\neg R}{\succ p \vee \neg p}\vee R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\neg L}{p \wedge \neg p \succ}\wedge L$$

The structure of sequents, allowing for collections of formulas on the left, and on the right, gives the maximum degree of flexibility in constructing derivations. The left/right symmetry pairs neatly with the inherent true/false duality between conjunction and disjunction in two-valued Boolean valuations.[7] As a formal, structural axiomatisation of valid sequents in classical logic, the classical sequent calculus is unimpeachable. However, this does not mean that it helps to isolate an understanding of classical *proof*. If we grant that a proof is a proof *of some conclusion*, then the flexibility of classical sequents begins to look not so much like a feature, and more like a *bug*. Restall [103] and Ripley [108] have argued that derivations in the sequent calculus can be wedded to our practice as giving us an account of which positions are out of bounds. On their accounts, a derivation of $X \succ Y$ shows how it is that asserting each member of X and denying Y is out of bounds. On this account, the two derivations above show that it is always out of bounds to deny $p \vee \neg p$ (by classical lights, instances of the Law of the Excluded Middle are undeniable), and similarly, the self contradiction $p \wedge \neg p$ is (by classical lights) unassertible.[8]

It is not my place in this paper to take issue with Restall's and Ripley's *bilateralism* in their interpretations of the sequent calculus. Many proponents of classical logic have argued for some form of bilateralism, for which denial is not to be taken as analysed in terms of the assertion of negation, but rather uses denial and its features as a part of the analysis of the significance of negation [97, 114]. A question for the inferentialist classical logician remains. It is one thing to argue that the sequent calculus gives us an account of the bounds for combinations of assertions and denials. It is very much another thing to think that the sequent calculus is itself a calculus in which a derivation

[7] A Boolean valuation is a function assigning to each formula one (and only one) of the truth values true and false, satisfying the expected two-valued truth conditions: a conjunction is assigned true iff both conjuncts are assigned true; a disjunction is assigned false iff both disjuncts are assigned false, a conditional is assigned false iff the antecedent is assigned true and the consequent assigned false; a negation is assigned true iff the negand is assigned false and $\bot$ is assigned false. The Chapter 1 of Humberstone's *The Connectives* contains a comprehensive discussion of valuations and their connections to consequence relations [59, esp. p.54–100].

[8] Note, 'unassertible' here does not merely mean that the assertion is unwarranted. It means that any position in which the assertion is made is *out of bounds* in a much stronger sense than being unwarranted. That is, it involves a clash, in just the same way that any position in which p is asserted and p is denied involves a clash. In fact, the derivation of the sequent $p \wedge \neg p \succ$ shows how the clash involved in asserting $p \wedge \neg p$ arises out of the clash involved in asserting p and asserting $\neg p$, which itself arises out of the clash involved in asserting p and denying p.

of a sequent describes some kind of *proof*.[9] The central plank of Steinberger's argument against the application of the sequent calculus to inferentialist ends is that it violates the *principle of answerability*, to the effect that the only deductive systems that can meet the inferentialist's aims must be "suitably connected" to our ordinary deductive inferential practices [118, p. 335]. So, with this in mind, let us look at the relationship between sequent calculus derivations and what might reasonably be called *proofs*.

It is relatively straightforward to take the derivation of $p \land \neg p \succ$ and to transform it into a natural deduction refutation of $p \land \neg p$, that is, a proof of a contradiction, $\bot$, from $p \land \neg p$.

$$\dfrac{\dfrac{p \land \neg p}{p}\,{\land E} \qquad \dfrac{p \land \neg p}{\neg p}\,{\land E}}{\bot}\,{\neg E}$$

We can read this proof, quite straightforwardly, as explaining *why*, under the assumption of $p \land \neg p$, $\bot$ would follow. It is a proof, with the conclusion, $\bot$, and $p \land \neg p$ as an premise. There is no straightforward way to massage the dual sequent calculus derivation into a proof of the same form. If we were to turn this derivation upside down (by way of visual analogy, where the left–right mirror duality between antecedents and succedents in the sequent calculus becomes an up–down duality between premises and conclusions in proofs) then the result would be a downward branching tree with $\top$ at the leaf, and two nodes at the bottom, both containing $p \lor \neg p$.

$$\dfrac{\top}{\dfrac{\neg p}{p \lor \neg p}\,{\lor I} \qquad \dfrac{p}{p \lor \neg p}\,{\lor I}}\,{\neg I}$$

While such an inferential network may be theoretically elegant, and formally interesting,[10] it is less than compelling as an account of the structure of a *proof* [118]. A derivation of a sequent $X \succ A$ in a single-conclusion sequent system can be seen as a means to construct a natural deduction proof from X to A, but it is less clear how one might understand a derivation of a *classical* sequent $X \succ Y$ is a construction of something that is clearly a proof. The structure above, starting from $\top$ and leading to two "conclusions" $p \lor \neg p$ shares some structural similarities to everyday proofs, but the analogy is strained at best. Typically, at least, a proof is a proof of a *conclusion*, relative to a background (the collection of assumptions) that is taken for granted. This structure is present in

[9] To use MacFarlane's terminology [77], Restall's and Ripley's accounts of the sequent calculus and the bounds is a kind of *normative pragmatism* (the logic is grounded in *rules* for *use*), but they are not *inferentialist*.

[10] Multiple conclusion 'proof structures' of this general shape are a minority tradition in logic, though they are of significant formal interest [23, 26, 27, 39, 64, 100, 104, 109, 113, 124]. Proof structures that are not *trees* have come into focus since the introduction of Jean-Yves Girard's *linear logic* [45], and the introduction of proof nets [40, 46].

tree-style natural deduction proofs, with the assumptions at the leaves and the conclusion at the root. This structure is clearly reflected in sequents of the form $X \succ A$ where the assumptions are collected together in X and the conclusion A is singled out. This structure is not manifest in sequents of the form $X \succ Y$, in which each formula in X and in Y has equal status, and no formula is in focus.[11]

This preference for a single conclusion structure is not merely a conservative longing for the familiar or the everyday. I take it that the point of Steinberger's principle of answerability is not that the deductive practices to be systematised in a theory of proof be everyday and familiar, but that the formal system be appropriately connected to not merely to the features that our practices *happen* to have, but that they answer to the *aim* of that practice. A good system of natural deduction might stand to our deductive inferential practice as Peano Arithmetic corresponds to our counting practice. It would, in a small number of primitive principles, make rigorous and explicit and precise, what is at least implicit or nascent in our everyday deductive inferential practice.

So, let me spell out one of the aims of our ordinary deductive practice that makes the single conclusion nature of proof not an accident, but the central plank of the exercise. One way to understand the function of giving a proof is to provide an answer to a justification request for an assertion. Suppose you make a claim—let it be A, and I ask you to defend it. You do so, by making other claims—say, B and $B \to A$. I could stop there, satisfied, or I could ask you to defend either of those claims, until I am satisfied. If I ask you to defend $B \to A$, one strategy would be for you to ask me to *suppose* B (to grant it for the sake of the argument), and then you will defend A, now appealing to that supposition of A. In this process of claim and defence, at any stage there is a single claim in focus, the current target of the justification request [52, Chapter 7,8]. It is natural to understand the function of proof in just this dialogical fashion [34, 35, 48], and when we do so, we see that the focus on single conclusions in proof structures is not an accident, but is at the heart of the exercise. So, too, is the tree structure of Prawitz–style

[11] Another approach to get something a little more like a natural deduction proof, starting from a sequent derivation, is the sequence conclusion system of Boričić [23, 100]. Here, we allow for a collection of formulas as alternative conclusions, at each step of the proof, while imposing the tree structure familiar to natural deduction. Our proof of $p \vee \neg p$ would become something like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\top}{p, \neg p}\,\neg I}{p \vee \neg p, \neg p}\,\vee I}{p \vee \neg p, p \vee \neg p}\,\vee I}{p \vee \neg p}\,W$$

Such 'proofs' are in a halfway house between the sequent calculus, with a structure of formulas at a step in a proof doing duty for the right hand side of the sequent, and the undischarged leaves doing duty for the left hand side, and the critical comments about downward branching proofs apply to sequence proofs, too.

natural deduction proofs. The sub-tree rooted in some given formula in a proof is naturally understood as that part of the proof that is used to meet the justification request for *that* formula. As we play out the questions and answers, these can be laid out in the shape of a tree, with the leaves being either temporary commitments later discharged, or assumptions granted and left unjustified.

Of course, a proof presented in natural deduction form does not bear the marks of its dialogical origin,[12] but it is natural to think of an inference step, of the form "*A, B, so C*" or "*suppose A, we can show B, so* $A \to B$" as presenting the means to pre-emptively meet justification requests, or to show our working. We not only make the claim C or $A \to B$, but we also supply backup in the form of pointing to what we *would* give as a justification, were we asked. A proof is not merely a way to come to a yes or no answer—an oracle would suffice for that—a proof of A is a way to *show* that A. A sequent derivation of $X \succ Y$, or its rendering in a proof with multiple "conclusions" does not fit this structure anywhere near as well. Much more work would need to be done to explain how sequent derivations relate to proofs, if we use proofs as means of meeting justification requests. This is not to say that sequent calculi (even those with multiple formulas on the right hand side of a sequent) are not of use in their own way as a part of the inferentialist's toolkit—to the contrary, the sequent calculus is an excellent framework for providing *scoreboards* for the state of play in some context—but it is to say they should not, on their own, be the whole account of proof for an inferentialist. After all, designing new ways to manipulate a *scoreboard* does not guarantee that your newfangled scoreboard matches some possible play of some game.[13]

* * *

The second major contender for a proof-theoretical framework for classical logic does not have the shortcoming of not having identifiable conclusions in proof structures. Signed natural deduction systems give us proof structures where, at every stage of development, a proof has a single conclusion. Here is an example of a signed natural de-

[12] Unlike other dialogical formal systems, natural deduction proofs do not have moves marked "proponent" and "opponent" [20, 56, 57, 74–76].

[13] For more on the game/scoreboard metaphor, and what the related question concerning the relationship between the relatively abstract notion of a game, which could be implemented in many different ways, and the essentially embodied notion we might call a *sport*, see Mark Lance's "Some Reflections on the Sport of Language" [71]. This becomes salient when we consider the relation between the dialogical notions of *assertion* and *denial* and the embodied notions of *belief* and *disbelief*.

duction proof, for the Law of the Excluded Middle.

$$\dfrac{\dfrac{\dfrac{\dfrac{[-p\lor\neg p]^1}{-p}\,{-\lor E}}{+\neg p}\,{+\neg I}}{+p\lor\neg p}\,{+\lor I}\qquad [-p\lor\neg p]^2}{+p\lor\neg p}\,RAA^{1,2}$$

In a proof of this form, every formula is signed, with a '+' (marking *assertions*) or a '−' (marking *denials* or *rejections*). For each connective there are introduction and elimination rules, both for assertions of formulas in which that connective is dominant and for denials of those formulas. There are also distinctive *structural rules* such as the *reductio ad absurdum* rule (*RAA*) employed here in the last step. The *RAA* rule allows us to prove a signed formula ($+A$, or $-A$) if under the supposition of its *opposite* ($-A$, or $+A$, respectively) we are able to prove some signed formula ($+B$) and also prove its opposite ($-B$). In the instance of *RAA* in the proof above, in the left branch we proved $+p \lor \neg p$ from $-p \lor \neg p$ (that takes three steps) and we have proved $-p \lor \neg p$ from itself (that was immediate). So, discharging the supposition of $-p \lor \neg p$, we conclude its opposite, $+p \lor \neg p$.

This signed natural deduction proof of $+p \lor \neg p$ is slightly longer than the swift sequent derivation of $\succ p \lor \neg p$ in the classical sequent calculus (which amounts to only two inferences), but it does have the virtue of having a single signed formula at its conclusion, rather than a sequent. If you squint at the proof, you can perhaps see the shared structure with the inverted natural deduction proof with the two conclusions. There are exactly three inferences using a connective rule. In the signed system, we have one $-\lor E$ and one $+\lor I$ step. These are mirror images of one another. What is traditionally an unsigned inference from B to C may be taken positively as a step from $+B$ to $+C$ or negatively, as a step from $-C$ to $-B$, so the disjunction steps, $-\lor E$ and $+\lor I$ in some sense correspond to the two $\lor I$ steps in the inverted natural deduction 'proof'. The *positive* negation introduction rule corresponds to the negation introduction step in the inverted tree proof, and the remaining step, the *RAA* inference, is structural bookkeeping, wrapping up the result in a single conclusion.

The phenomenon of one structure (signed natural deduction) *rhyming* with another (the sequent calculus) is quite general. Viewed from the perspective of the sequent calculus, or of proof-nets, a *negatively* signed premise in a proof provides a way to represent what would otherwise be an alternative positive conclusion. What is *proved* by the left sub-proof from $-p \lor \neg p$, to $+p \lor \neg p$ would be represented in the sequent calculus as a derivation of $\succ p \lor \neg p, p \lor \neg p$, perhaps with one of the instances singled out in focus as the conclusion. In general, it is straightforward to rewrite a signed natural deduction derivation from positive assumptions $+X$ and negative assumptions $-Y$ to conclusion $+A$ as a derivation of the sequent $X \succ A, Y$, where X and Y are the corresponding collections of *unsigned* formulas. If the conclusion is $-B$, then the sequent derived is $X, B \succ Y$.[14] We have a notational variant of the sequent calculus, with the extra feature that all sequents have one formula singled out as the conclusion. If the formula is on the right of the sequent, we give it a positive sign, if the formula is on the left, we give it a negative sign. The price to pay for this transformation is that instead of introduction and elimination rules for each connective, we have strangely *doubled* pairs of rules. Positive introduction and negative elimination rules (which look uncannily alike) and negative introduction and positive elimination rules (which also look strangely similar).[15] Therefore, the question of harmony for our connective rules transforms from the matter of the match between introduction and elimination rules to the more complex *four* way match between positive introduction, positive elimination, negative introduction and negative elimination rules.

So, it is worth considering if the price of this added complexity is worth paying for the benefit of having proofs in which one signed formula is present as the conclusion. As we have seen, there are good reasons to look for a single conclusion framework, given the fit with our practices of proof. Here, the formal costs of syntactic complexity are noticeable, but are, ultimately, manageable. The more important price to consider is whether the structures that result are any good at representing *proofs*—does the bilateralist framework meet the answerability criterion any better than the sequent calculus?

Signed natural deduction is not a simple *extension* of natural deduction proofs with the addition of some extra rules, but a change to a different framework. Natural deduction proofs, as traditionally understood, contain formulas. Signed natural deduction proofs contain *signed* formulas. So, to understand the costs and benefits of the framework, we need to understand the significance of those signs, and whether they can be understood in ways that are answerable to our inferential practice. How, exactly, are we to understand '+' and '−'?[16] Rumfitt's answer, given when he distinguishes *negation*, as a freely iterating sentence-forming operator on sentences, and '−' the sign of rejection, goes like this:

> The sign of rejection, by contrast, was explained as the formal correlate of the operation of forming an interrogative sentence from a declarative and

[14] This observation is by no means original with me. Humberstone, in "The Revival of Rejective Negation" [58] notes this connection between signed natural deduction proofs and sequent derivations.

[15] The rules are not *quite* mirror images of each other, given the distinctive role played by the conclusion. Rumfitt's rule $+{\to}I$, which is the signed version of the traditional ${\to}I$ rule, is not the mirror image of his $-{\to}E$ rule [110, p. 802], because the former invovles discharging positive assumptions, and there is no way to dualise this, because there is only a single conclusion spot. However, he could well have instead dualised his $-{\to}E$ rule for his $+{\to}I$ rule. Rumfitt's $-{\to}E$ rule is *additive*, while the $+{\to}I$ is *multiplicative*.

[16] This short discussion is indebted to Nils Kürbis' more extensive treatment of bilateralism and speech acts, presented at the 10th European Congress for Analytic Philosophy in August 2020 [69]. In hearing Nils' presentation, I was convinced that my concerns about the interpretation of bilateralist natural deduction were worth attending to. I thank him for his presentation and for our subsequent discussion.

> appending the answer "No", and this operation cannot be iterated. "Is it the case that two is not a prime number? No" makes perfectly good sense, but "Is it the case that is it the case that two is a prime number? No? No?" is gibberish. The sign "$-$", then, does not contribute to propositional content, but indicates the force with which that content is promulgated. Just as one asserts the entire content expressed by A by inscribing $\ulcorner +A \urcorner$, so one expressly rejects that same content by inscribing $\ulcorner -A \urcorner$. [110, p. 802–803]

Rumfitt's understanding of the signs is clear. '+' and '$-$' are force indicators. They cannot be embedded. '+' expresses assertion and '$-$' expresses rejection. Rumfitt's argument is, in part, that '$-$' does not embed, while negation can.[17] Whether our natural language works exactly as Rumfitt claims is not so important for the issue at hand. Rumfitt's understanding of how he takes these signs to be interpreted when they are used in his formalism is what is important for our purposes. Rumfitt's intention is completely clear when they stand alone and signed formulas are not linked together in inference. If I write down a number of sentences on a sheet of paper and prepend some with a '+' and others with a '$-$', it is clear that, following the convention set out, I am asserting those marked with a '+' and denying those with a '$-$'.

The situation becomes more delicate when we move from a collection of disconnected sentences—some asserted, others denied—to the interlocked network of sentences used when using a proof to justify some conclusion. Consider the bilateralist proof, ending in $+A \lor \neg A$ given above. If we follow the convention that $\ulcorner +A \urcorner$ is a sign of asserting the content expressed by $\ulcorner A \urcorner$ and writing down $\ulcorner -A \urcorner$ is a sign of rejecting that content, then when I wrote down that signed proof of $\ulcorner + A \lor \neg A \urcorner$ I contradicted myself, because in the process of writing down that proof, I asserted and rejected the same content. But of course I never actually contradict myself when I write down such a proof. I do not even need to retract any of the assertions that I made along the way. I have not really *rejected* the content $A \lor \neg A$ when I start writing down $\ulcorner - A \lor \neg A \urcorner$ as I compose the left branch of the proof. After all, I am *proving* $A \lor \neg A$, not rejecting it. In a traditional natural deduction proof, with no signs, I can start a branch of my tree by writing down a formula and in doing so, I do not assert that formula, I *suppose* it.[18] Perhaps I suppose it for the sake of the argument. The traditional norms of assertion (whether the truth norm, the knowledge norm, or whatever else you prefer [70, 83]) simply do not apply to supposition. It would be wrong to criticise a supposition, or to say that it in any sense failed in its aim, if it turned out to be untrue, or if we had no evidence for it. It would be wrong to take what is a supposition, for the sake of an

[17] Rumfitt's view is controversial. Textor argues that "No" is never a force marker [123]. For responses to Textor, see Incurvati and Smith [60] and Schang and Trafford [111].

[18] This point is a variant what Geach calls *the Frege point* [42, p. 449]. Just as a content appears *unasserted* when it occurs as the antecedent of a conditional, so it may when it is *supposed* when introduced as a premise, later to be discharged, in a natural deduction proof.

argument, as a license to reassert that content in other contexts. Yet these norms are what we take to be salient when it comes to assertion [78]. Supposition is, of course, related to assertion, but supposition should not be identified with assertion. When we suppose, try a content on for size, working with its downstream consequences, without imposing any of the quality control measures we apply to assertion, or to denial.

This difference between the assertion and supposition is clearly marked in the type theoretic natural deduction formalisms. In systems in which types are annotated with *terms* [47, 80], proofs look like this:

$$\dfrac{\dfrac{[x : p \to q]^1 \qquad t : p}{xt : q}\to E}{\lambda x.xt : (p \to q) \to q}\to I^1$$

Here, the undischarged assumption p is paired with a term t. We can think of t as our ground for the assertion that p, whatever that ground may be. On this view, to assert a content is (at least in part) to present it as having grounds. From p, we aim to conclude $(p \to q) \to q$, to show that it, too, has grounds, by showing how grounds for $(p \to q) \to q$ may be derived from the grounds we posses for p. To do this, we *suppose* $p \to q$. We need not possess any grounds for $p \to q$, and neither do we need to present it as having grounds, so instead of naming any ground for $p \to q$ in the proof, we use a *variable* as a marker to stand in for the grounds were there any. Then, in the $\to E$ step, the *supposed* grounds x are combined with the grounds t we in fact possess for p, to give us xt, which *would* be grounds for q, under the scope of the supposition that x does select some ground for $p \to q$. This hybrid object xt is partly ground, partly placeholder, and is not, in itself, grounds for q. Then, in the final step, the supposition is discharged, the variable x is *bound* in the term $\lambda x.xt$, and we have grounds for the conclusion $(p \to q) \to p$. This ground involves the ground t we possess for p, and it constructs the ground for $(p \to q) \to q$, as a function, which when supplied a ground x for $p \to q$ (if there are any such grounds), returns the result of applying that ground (as a function) to the ground t we possess.

The details of how to interpret terms as grounds do not matter for the point at hand. The important lesson for us in this example is the clear difference between those speech acts as marked by the terms annotating them. *Assertions* in proofs, are annotated by terms in which no variables are free. This represents the fact that they have been supplied grounds, as they depend on no undischarged assumptions. *Suppositions*, which are leaves in the proof tree marked with variables, which are unasserted but granted for the sake of the argument. Proofs contain *other claims* (like the q in this case) which are inferred under the scope of a supposition. These are neither supposed, and nor are they asserted, since they have not presented as having grounds. In the proof we have only specified what *would* be grounds for q, were some value for the variable x to be supplied as grounds for $p \to q$. In this formalism, there is a clear representation of what formulas are *asserted* and what formulas are not, as a proof unfolds.[19] This distinction in the formal system resepects our answerability criterion. In our everyday reasoning practice, what we *assert* we can also *deny*, and those contents we can assert and deny are the same contents that we can *suppose* and also *infer* or *conclude*.

So how might a user of a signed natural deduction system understand the signs in a natural deduction proof if it is not quite correct to understand all positively signed formulas in a proof as asserted and all negatively signed formulas as rejected?

A natural thought might be to think of the supposed positively signed formulas as *hypothetically asserted* and the supposed negatively signed formulas as *hypothetically rejected*. This should not mean that the person who presents the proof supposes *that they assert* (or *reject*) the content in question, for that way lies a confusion. When I can quite coherently suppose that the cat is on the mat but that I do not assert that it is, but it is a contradictory supposition to suppose that I assert that the cat is on the mat and that I do not assert that it is. To suppose that p is not to suppose that you (or that anyone else) assert p.

At this point we might seek guidance from Rumfitt's canonical understanding of signed formulas, quoted above. We are to understand $\ulcorner + A \urcorner$ as the polar question "Is it the case that A?" followed by "Yes". It makes little grammatical sense to embed *this* in a "Suppose" wrapper. However, it seems straightforwardly meaningful to enclose the *answer* in such a wrapper, like this:

> "Is it the case that two is a prime number? *Suppose* no."

This does look like a kind of negative supposition, and a supposition of just this form could well serve as the leaf in a signed natural deduction proof. (Whether the content of what is supposed is the content expressed by "Is it the case that two is a prime number?", which is somehow *negatively* supposed, or if it is better understood as a supposition of "It is *not* the case that two is a prime number", I leave to others to decide.) This strategy, of allowing the supposition to modify the answer indicator, seems to provide some way to make sense of positive and negative supposition. However, it is a strategy that comes with its own costs. If we allow for answer modifiers such as "*suppose* no", then we have opened the gate to forms such as the double negative:

> "Is it the case that two is a prime number? Not *no*."

which seems just as meaningful an answer as 'suppose no'. In saying this, you reject giving the answer "no" to the question. If Rumfitt or any friend of signed natural deduction is going to avail himself of *supposed* rejection, then the issue of *rejected* rejection is on the table, and with it, the question of exactly which of these speech acts should

[19] See J. E. Wiredu's "Deducibility and Inferability" as an older example of the importance of keeping track of the difference between suppositions that may later be discharged in a proof, and assumptions, that are asserted [127].

play a role in our theorising, and what benefit is being played by constructions that do seem to take us quite some distance away from the plain form of everyday reasoning, dealing with speech acts like *assertion*, *denial*, *supposition* and *inference*, each of which take the same content.

So, at the very least, if we are to employ signed natural deduction as an account of the structure of our inferential practice, we should say something about the range of speech acts involved in *using* a signed proof. Restricting our attention to assertion and denial (or rejection) is not enough to give an account of the speech acts involved when we suppose and when we infer. These are also speech acts, and if we are to use a signed natural deduction system, the onus is on us to explain how that system manages to regiment the structure of our everyday inferential practice.

* * *

So, in both the sequent calculus and signed natural deduction systems, the bilateralist has a formalism that is theoretically elegant, with beautiful proof-theoretic features, such as separable rules, normalisation, the subformula property, conservative extension, and the like. However, the price that we have paid for that formal elegance is moving some distance away from a natural understanding of everyday proof, composed of individual speech acts of *supposing* or *granting*, and *inferring* or *concluding*. In the remainder of this paper, I will show how the way is available—for the bilateralist—to take classical proof theory back to its proof-theoretical roots, keeping all of the good formal properties we want out of a natural deduction system, while hewing much more closely to the everyday notion of proof. The tools that we need have already been built for us by our colleagues in theoretical computer science: specifically, we may make use of Michel Parigot's $\lambda\mu$-calculus, which is a single conclusion, normalising natural deduction system for classical logic [88, 89].[20] (Its elegant features extend to second-order predicate logic, but our attention will be restricted to the propositional fragment.) My task, in the rest of this paper, is to show how this readily available system of natural deduction proofs addresses exactly the criticisms that have been laid at the feet of those who would use the sequent calculus or bilateralist signed natural deduction as a means of accounting for *proof*. Proofs in this system are single conclusion. At each inference a single formula is marked out as the conclusion,[21] not a signed formula, not sequences

[20]The $\lambda\mu$-calculus has gone on to receive quite some attention in the literature [5–7, 65, 72, 81, 86, 87], and its intimate connection to the important concept of a *continuation*, which finds its use in the semantics of programming languages [85, 107, 119], as well as scope features in the semantics of natural languages [1, 8–10, 49, 50] is some evidence that the $\lambda\mu$-calculus is not merely a 'hack' designed to solve one particular problem, but is isolating something of independent interest. The connections with continuations in the speech act context will not be spelled out in what follows. How those connections can be made and what we might find is left to further research.

[21]This includes the contradiction marker $\perp$. Later we will consider whether this is best understood as a formula or as something else.

or sets of formulas, and not sequent. The way we make use of a bilateral treatment of assertion and *denial* is that we allow formulas in the *leaves* of our proofs to be signed. A bare formula by itself is taken to be supposed or postively granted for the sake of the argument, and a slashed formula (e.g. $\not A$) is taken to be set aside as an *alternative*, or *negatively granted*, also for the sake of the argument. As we will see, speech acts employed in *using* a $\lambda\mu$-proof to infer a conclusion from a context in which some things have been granted and others have been set aside as alternatives, are speech acts that we use in everyday unformalised proofs.

Before proceeding with the positive account of proofs and their interpretation, it is worth indicating that there is no attempt here to defend classical logic to any constructivists or intuitionists who have their own independent reasons to reject classical deductive practice *tout court*. If your reasons to reject distinctively classical principles such as Peirce's Law, and the Law of the Excluded Middle amount to more than the unavailability of a normalising natural deduction proof system that hews closely to the speech acts we actually make when we prove things, then, likely as not, these reasons will still stand. The aims of this paper are modest. I will show that everyday feasible speech acts can be harnessed in a natural notion of proof, in a way that gives rise to a well-behaved system of rules for classical logic. That is the aim of the rest of this paper, no more, no less.

2 NATURAL DEDUCTION WITH ALTERNATIVES

The bilateralist need not move so far away from the traditional natural deduction format for intuitionistic logic. The Gentzen–Prawitz proofs we have already seen [43, 93] can be kept, unchanged. A proof of a formula is a tree, with that formula (unsigned) situated at its root. The premises of the inference—the assumptions on which the conclusion rests—are among the leaves of the tree. Our only change will be admitting at some of these leaves may be *negatively* marked, as well as the traditional, unmarked *positive* assumptions. The rules for each *connective* are unchanged from the rules we have already seen (see Table 1), and the additions to our proofs are purely structural rules governing the negatively marked leaves—the so-called *alternatives*.

So where does our classical proof system differ from intuitionistic natural deduction? The difference should not, and cannot be located in a rule for this or that connective, because intuitionist logic differs from classical logic across a range of different connectives. Classical logic asks us to provide a proof from $\neg\neg p$ to p. If the rules for each connective are separable, and if we can normalise proofs appropriately, then the only connective rules involved in a properly analytic *normal* proof from $\neg\neg p$ to p will be the negation rules. Classical logic also asks us to provide a proof from no premises to $((p \to q) \to p) \to p$. In a normal proof of Peirce's Law, the conditional rules will be the only connective rules in play. The only rules that could play a part in both proofs—

as the crucial ingredient in supplying *classical* reasoning—would be properly *structural* rules, not those particular to this or that connective.

One pair of structural rules is enough to make the difference between intuitionistic and classical proof. In one sense, the nature of these structural rules are not difficult to understand. Since Gentzen's insight in constructing the sequent calculus, we have seen that adding space in our proof structures so that more than one formula can appear in the *consequent* position of a sequent makes the difference needed classical proofs, to restore symmetry where there was once asymmetry between the many premises and the single conclusion in intuitionistic natural deduction or the intuitionistic sequent calculus.

The thoroughgoing duality between truth and falsity in classical logic seems to call for some measure of duality between premise and conclusion. It is not for nothing that classical proof systems have all involved reintroducing some kind of premise/conclusion duality, whether by expanding making sequents from the form $X \succ A$ to the form $X \succ Y$, or by layering in an extra duality, like the duality between $+$ and $-$, to do that job. These innovations were not for nothing, and they all circle around the one logical phenomenon, the need to have 'locations' in our proof *structures* that allow for more than one formula to be in 'positive' position, in just the same way that all natural deduction proofs allow for more than one formula to be in 'negative' position, as undischarged assumptions in a proof.[22]

The rules that we add will thread a seemingly impossible needle, by allowing for our proof to (*a*) keep track of more than one occurrence of a formula in positive position at any stage of a proof, while (*b*) having at any stage of our proof one and only one current *conclusion*. That is our target.

The inference rules that manage to thread this needle are the *Alternative* rules, presented in Figure 4. At first glance, these rules are—of course—*absurd*, at least to those who are used to traditional natural deduction systems. The *Retrieve* rule looks rather like $\bot E$, and the *Store* seems patently invalid, since using it permits a deduction of a contradiction from an arbitrary conclusion, A. The saving grace for these rules are the side effects, marked by *storing* the conclusion A in a *Store* inference, and possibly *Re-*

[22] The role of *positive* and *negative* position in logical presentations is most clearly set out *algebraically* in J. Michael Dunn's *Gaggle Theory* [22, 32, 33, 101], and *proof theoretically* in Nuel Belnap's *Display Logic* [16, 17, 101, 125]. A position in A a complex formula $C(A)$ is said to be *positive* if $C(A)$ entails the result $C(A')$, of replacing A by a formula A' that A entails. If, on the other hand, when A entails A' we also have $D(A')$ entailing $D(A)$, then the position A inside D is said to be *negative*. In a conditional $A \to B$, whether that conditional be material, strict, relevant, or linear, the antecedent position is *negative* and the consequent position is *positive*. In the same way, it makes sense to think of the assumptions in a proof as being in *negative* position while the conclusion is in *positive* position, for if we have a proof from X, A to B, then we can construct another proof from X, A' to B' when we have a proof *from* A' to A, and when we have a proof from B to B'. In natural deduction proofs, traditionally understood, we have many places (the assumptions) in negative position, and only one (the conclusion) in positive position.

$$\dfrac{\cancel{A} \qquad \begin{array}{c}\Pi\\ A\end{array}}{\bot}\,\uparrow(\textit{Store}) \qquad\qquad \dfrac{\begin{array}{c}[\cancel{B}]^i\\ \Pi\\ \bot\end{array}}{B}\,\downarrow^i(\textit{Retrieve})$$

Figure 4: THE *ALTERNATIVE* RULES

trieving some conclusion that has been stored at some later step, when the proof has reached a dead end. With the addition of these simple rules, of we keep track of the conclusions that have been placed into cold storage and later retrieved, we have a single conclusion, natural deduction system for classical logic with all the nice properties of intuitionist natural deduction.

The *Store* rule can be understood as the bilateralist's analogue to $\neg E$, with the denial of A substituting for the assertion of the negation of A. If concluding A and concluding $\neg A$ leads to a contradiction, so does concluding A and also (for the sake of the argument, perhaps temporarily) *ruling A out*. This is the mild bilateralism that is required for our natural deduction system. When we place a conclusion A in storage (marked by the '$\uparrow$', showing that what was a conclusion is now stored among the leaves of the proof) we temporarily rule it out, setting it aside to consider alternatives. Of course, since we have *proved A*, it is undeniable. To reject it is to close off your options. No matter. Once we have reached that contradiction, as is usual in natural deduction, we turn to the commitments we have undertaken to that point, to see which we we might take back. If we wish to take back something that we have supposed, we can discharge it and deduce the negation of that supposition (in a $\neg I$ step, as usual). On the other hand, we could retrieve one of the claims we have placed in storage, to retrieve it.[23]

So, at any stage of a classical natural deduction proof we keep track of two kinds of claims. As usual in natural deduction proofs, we keep track of the assumptions active at this point of the proof. These may be premises we have granted and which will remain undischarged at the time of the proof's conclusion, or they may be suppositions which have been made and have not yet been discharged. We naturally account for assumptions when we step from inference to inference. For example, when inferring from an assumption A to the conclusion $A \vee B$ in a $\vee I$ step, I do not *forget* that assumption. It is

[23] In Michael and Murdoch Gabbay's "Some Formal Considerations on Gabbay's Restart Rule" [41] these two rule are—in effect—combined into a single rule, which they call *Restart*, which allows the move from A to B, at the cost of adding B as an alternative. As will become clear, that name is not well suited to our intended reading of the rule, especially when the natural deduction proof are read from top to bottom. The B in the conclusion of a *Store*/*Retrieve* pair does not represent in any way a new *start* to a line of reasoning. It is not an extra assumption. Instead, it is an alternative *conclusion*.

retained as the assumption *under which* the conclusion has been proved. If I *discharge* a supposition in a $\to I$ step or in a $\vee E$ step, then the discharged suppositions are removed from the collection of assumptions that are active at this point. When we introduce a conjunction $A \wedge B$ by proving A (from some assumptions) and proving B (from others), then these assumptions are collected together as the assumptions under which $A \wedge B$ has been proved. All this is totally standard. In classical proofs, we not only keep track of assumptions: We also keep track of *alternatives*.[24]

With this in mind, the *Store* and *Retrieve* steps are straightforward. If before a *Store* step, we have proved A from the assumptions X, having also gathered some alternatives Y along the way, then *after* the inference, now proved $\perp$ from X with the former conclusion A added to our collection of alternatives. When we *Retrieve*, the other hand, we have proved a contradiction $\perp$ from some collection of X of suppositions and a collection B, Y of alternatives, among which we find B.[25] After the *Retrieve* step, we have proved B from the suppositions X and alternatives Y. We can present these rules in 'sequent' form as follows:

$$\frac{[X : Y] \succ A}{[X : A, Y] \succ \perp}\ \textit{Store} \qquad \frac{[X : B, Y] \succ \perp}{[X : Y] \succ B}\ \textit{Retrieve}$$

where the left hand side of the sequent separator bears the *context*, the claims ruled in and those ruled out, and the right hand side bears the current conclusion. These may look more familiar presented in the following form, which look nearly indistinguishable from rules of the classical sequent calculus:

$$\frac{X \succ A; Y}{X \succ \perp; A, Y}\ \textit{Store} \qquad \frac{X \succ \perp; B, Y}{X \succ B; Y}\ \textit{Retrieve}$$

The only difference is the presence of a *semicolon* on the right hand side. Our natural deduction proofs are not best modelled by an undifferentiated sequent of the form $X \succ Y$, for our proofs always have one and only one conclusion (that is, either a formula, or $\perp$).[26] To model the state of play at a stage of a proof, the appropriate sequent representation should single out a formula as the conclusion. So, sequents have the form $X \succ C; Y$, where the conclusion C is in 'focus', or in the form we will use henceforth, $[X : Y] \succ C$, indicating that $[X : Y]$ is the background *context* against which the conclusion C has been derived.

[24] So, an alternative in a proof is a claim which now falls into the *context*, like the assumptions and suppositions on which the current conclusion rests, rather than another *conclusion*.

[25] See below for a discussion of whether B must have been stored as an alternative, or if we should allow 'vacuous' retrieval, of conjuring an alternative from thin air. To jump ahead, vacuous retrieval is another way to understand the $\perp E$ rule, and this setting gives us a helpful context for understanding why, in proofs, $\perp E$ and vacuous *discharging* of assumptions stand or fall together.

[26] Is that disjunction inclusive or exclusive? Is $\perp$ a formula or not? Our account, so far, is agnostic on this issue. If $\perp$ is not in fact a *formula*, but is a punctuation mark, then proofs ending in $\perp$ do not have a concluding formula.

$$\dfrac{\dfrac{\dfrac{[\not p]^2 \qquad \dfrac{[(p \to q) \to p]^3 \qquad \dfrac{\dfrac{\dfrac{[\not p]^2 \quad [p]^1}{\bot}\uparrow}{q}\bot E}{p \to q}{\to}I^1}{p}{\to}E}{\bot}\uparrow}{p}\downarrow^2}{((p \to q) \to p) \to p}{\to}I^3$$

Figure 5: A direct proof of Peirce's Law

* * *

Let's see how we can use these rules to prove essentially classical theorems. First, Peirce's Law: The straightforward proof is presented in Figure 5. In the right branch of this proof, we suppose p, and immediately set p aside to land in a contradiction, from which we infer q as our conclusion. The state of the proof is, therefore $[p : p] \succ q$. Since we reached this q under the scope of the supposition of p, we discharge that supposition and conclude $p \to q$ using $\to I$, and hence the state is $[\ : p] \succ p \to q$, since we have derived $p \to q$ at the cost of setting p aside as an alternative. Of course, this does not mean that $p \to q$ is true *categorically* (despite the fact that we have discharged our only supposition, p), but it does tell us that $p \to q$ holds if we can rule out the alternative, p. Then, in the left branch we suppose $(p \to q) \to p$, and combining this with our conclusion $p \to q$, we infer q, by $\to E$. The status is $[(p \to q) \to p : p] \succ p$, but that step, having reached the conclusion p again, we absorb the alternative p into our conclusion by first setting p aside in another *Store* step, and thence to retrieve *two* copies of p, so we have $[(p \to q) \to p :\ ] \succ p$. There is now no remaining alternative other than our conclusion p. We have concluded p from our supposition of $(p \to q) \to p$, using the $p \to q$ that we had proved (at the cost of allowing p as an alternative). Discharging that supposition we conclude $((p \to q) \to p) \to p$ on the basis of no remaining assumptions, and no remaining alternatives. We have proved Peirce's Law. There was no need to involve negation in the proof, there was never downward *branching*, though we did, of course, set a conclusion aside, and the proof simply involves assumptions, inferences, and alternatives.[27] (We do encounter the contradiction marker $\bot$ after we store a con-

[27] A natural question arises. Do we need to understand disjunction before we can operate with alternatives? After all, alternatives seem rather *disjunction*-like. We will see at the end of the next section that —given the mildest form of bilateralism—that no more prior grasp of disjunction is required, than a grasp of conjunction is required for operating with the notion of assumptions.

clusion, and we will see, later, that this need not involve a violation of the subformula property for proofs, because there is no need to treat $\perp$ as a *formula*.) Finally, we do not need to decorate each and every formula with a sign. The constituents in this proof are contents, not speech acts. At each step of this proof we suppose, we set aside, we infer and we conclude.

It is instructive to compare this direct proof of Peirce's Law with the proof employing *Double Negation Elimination*, displayed in Figure 3. In the simpler proof using alternatives, in the rightmost branch we suppose p, park it as an alternative, and derive $p \to q$. In the proof in Figure 3 we manage the same effect in the rightmost branches of the proof, at the cost of encoding that alternative p instead as an *assumption* of $\neg p$. (This is an implicit double negation translation, in effect parking our alterative p in an assumption, the only place we can store it, under the cover of a negation, to give it the correct polarity.) This assumption of $\neg p$ plays a role again, to contradict the conclusion p (arising out of the $\to E$ step from the supposed $(p \to q) \to p$ and the derived $p \to q$), to give us the contradiction that we blame on that very assumption. This gives us $\neg\neg p$, which we need to unpack into the desired p. So, the more complex proof in Figure 3 can thus be seen to use negation to approximate the reasoning more directly represented in the proof in Figure 5, which exhibits the subformula property and has no need to make this detour through negation.

What goes for the conditional also goes for the other connectives. Here are two *negation* proofs, one, from $\neg\neg p$ to p for Double Negation Elimination, and the other for the Law of the Excluded Middle.

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[\cancel{p}]^2 \quad [p]^1}{\perp}\uparrow}{\neg p}\neg I^1}{\perp}\neg E}{p}\downarrow^2
\qquad\qquad
\dfrac{\dfrac{[\cancel{p \lor \neg p}]^3 \qquad \dfrac{\dfrac{\dfrac{[\cancel{p \lor \neg p}]^3 \qquad \dfrac{\dfrac{\dfrac{[\cancel{p}]^2 \quad [p]^1}{\perp}\uparrow}{\neg p}\neg I^1}{p \lor \neg p}\lor I}{\perp}\uparrow}{p}\downarrow^2}{p \lor \neg p}\lor I}{\perp}\uparrow}{p \lor \neg p}\downarrow^3$$

These proofs share their initial inferences, from an assumption p, which is immediately set aside for the contradiction $\perp$, at which point the assumption is discharged, giving rise to the conclusion $\neg p$, in the context of the alternative, p. In the proof for double negation elimination, this conclusion $\neg p$ is ruled out by the assumed $\neg\neg p$, leaving the alternative p the only option on the field. In this way, $\neg\neg p$ entails p. The proof for the Law of the Excluded Middle proceeds by deriving $p \lor \neg p$ from $\neg p$, setting this aside, temporarily, to also prove it from the other alternative p, and concluding that $p \lor \neg p$, therefore, holds inevitably. The proof of $p \vee \neg p$, like the proof for $+p \vee \neg p$ in the signed natural deduction system, has two disjunction steps, one negation step, together with some bookkeping structural rules. With alternatives in play, there is no need to decorate formulas with signs, or to involve multiple conclusions, to get a natural deduction proof with the desired structure.

Further examples of distinctively classical proofs could be multiplied endlessly. Instead of exploring more proofs, let's turn to some of the options that are opened up for us once we have alternatives in our repertoire. A natural place to turn is the rules for disjunction and negation, which both have a certain complexity, and are worth revisiting in the light of this new setting for natural deduction proofs.

Consider negation. If we were to attempt to characterise the fundamental principles that govern *negation* as such, there is no doubt that a constructivist or intuitionist would be happy with something like our current $\neg I$ and $\neg E$ rules, but this enthusiasm is not universally shared. Contenders for fundamental rules governing *negation* would be the Law of Non-Contradiction and the Law of the Excluded Middle. Now, as *sequents*, these are natural and simple:

$$\neg A, A \succ \qquad\qquad \succ A, \neg A$$

and the $\neg E$ rule is as good as any representation of the law of non-contradiction in our setting. On the other hand, $\neg I$ does not show its connection with the Law of the Excluded Middle so clearly. However, using $\neg E$, in the presence of alternatives, the connection becomes a little clearer. Consider these two small proofs:

$$\dfrac{\dfrac{\cancel{A} \qquad [A]^1}{\bot}\uparrow}{\neg A}\neg I^1 \qquad\qquad \dfrac{\dfrac{\cancel{\neg A} \qquad \dfrac{\dfrac{[\cancel{A}]^2 \qquad [A]^1}{\bot}\uparrow}{\neg A}\neg I^1}{\bot}\uparrow}{A}\downarrow^2$$

In the first, we have concluded $\neg A$, at the cost of the alternative, A, with no remaining assumptions. In the second, we have concluded A, at the cost of the alternative, $\neg A$, also with no remaining assumptions. Packaging up these small proofs as new *rules*, we have the two following simple principles for reasoning by cases.

$$\dfrac{\cancel{A}}{\neg A}\neg I' \qquad\qquad \dfrac{\cancel{\neg A}}{A}\neg I'$$

We can introduce $\neg A$ as conclusion, paying the price of accepting A as an alternative, or we can introduce A as conclusion, paying the price of keeping $\neg A$ on the books as an alternative. In either case, no positive assumption is required. The price to be paid is wholly in the coin of alternatives.

The $\neg I'$ rules are good candidates for simple rules that bear a more clear connection with the Law of the Excluded Middle than the original $\neg I$ rule does. If we like, we can *replace* $\neg I$ with the $\neg I'$ rule, at no cost. Any appeal to $\neg I$, can be replaced by an appeal to $\neg I'$, as follows:

$$\begin{array}{c} [A]^i \\ \Pi \\ \dfrac{\bot}{\neg A}\,\neg I^i \end{array} \qquad \textit{becomes} \qquad \begin{array}{c} \dfrac{[\cancel{\neg A}]^i}{A}\,\neg I' \\ \Pi \\ \dfrac{\bot}{\neg A}\,\downarrow^i \end{array}$$

So there is no loss of generality or proof power if we use these rules. With alternatives in our proof toolkit, the negation rules could just as well be $\neg E$ (LNC) and $\neg I'$ (LEM) as they are the regular rules.[28]

In the proof for the *Law of the Excluded Middle*, and in the introduction of the $\neg I'$ rule, we appealed to a combination of *Store* and *Retrieve* steps to swap out one alternative and replace it for another. The dance has this form:

$$\dfrac{\dfrac{\cancel{A} \qquad \begin{array}{c}[\cancel{B}]^i \\ \Pi \\ A\end{array}}{\bot}\,\uparrow}{B}\,\downarrow^i$$

The current conclusion A of Π is stored, and B is retrieved from storage, in its place. In such a dance, the last inference here could instead be $\bot E$, in which case the conclusion B comes out of nowhere. In either case, we will abbreviate such a dance as follows, labelling them '*Swap*' and '*Alt*':[29]

$$\dfrac{\cancel{A} \qquad \begin{array}{c}[\cancel{B}]^i \\ \Pi \\ A\end{array}}{B}\,\textit{Swap}^i \qquad\qquad \dfrac{\cancel{A} \qquad \begin{array}{c}\Pi \\ A\end{array}}{B}\,\textit{Alt}$$

Here is another sequence of steps that is worth giving a name:

$$\dfrac{\dfrac{[\cancel{A}]^i \qquad \begin{array}{c}[\cancel{A}]^i \\ \Pi \\ A\end{array}}{\bot}\,\uparrow}{A}\,\downarrow^i$$

[28] There are reasons to make this choice, and there are also reasons to keep the traditional natural deduction rules, which have the virtue of applying equally to intuitionistic proofs—in which we avoid using alternatives—as to classical proofs, where we make free use of alternatives. $\neg I'$ is exclusively classical, and so, cannot be used in settings where we wish to restrict ourselves to constructive scruples. For more on why we might want to do this, see the conclusion.

[29] The *Alt* inference is Gabbay's *Restart* rule in our notation, making alternatives explicit [41].

Here, we have taken a proof Π *of* A from a context in which A is stored as an alternative, and with another *store/retrieve* pair, we store A again, and retrive *each* instance immediately, to prove A now without appealing to A as an alternative. This is a form of *reductio* reasoning: if I can prove A while having set A aside, I can prove A regardless.

$$\dfrac{\begin{matrix}[\cancel{A}]^i\\ \Pi\\ A\end{matrix}}{A}\,\textit{Reductio}^i$$

Just as alternatives provide scope for different, more familiar rules for negation, we can do the same for disjunction. The traditional disjunction elimination rule

$$\dfrac{\begin{matrix}\\ \Pi\\ A\vee B\end{matrix}\qquad\begin{matrix}[A]^i\\ \Pi'\\ C\end{matrix}\qquad\begin{matrix}[B]^j\\ \Pi''\\ C\end{matrix}}{C}\,\vee E$$

has a feature not shared by the other usual natural deduction rules, the repeated conclusion parameter, here C. We can replace this rule with structurally simpler rules, using alternatives:

$$\dfrac{\begin{matrix}\Pi\\ A\vee B\end{matrix}\quad \cancel{B}}{A}\,\vee E' \qquad \dfrac{\begin{matrix}\Pi\\ A\vee B\end{matrix}\quad \cancel{A}}{B}\,\vee E'$$

Using the $\vee E'$ rule, we infer a disjunct from a disjunction, provided that we pay by storing the other disjunct as an alternative, to be dealt with later. These rules are negation-free renderings of a kind of *disjunctive syllogism* principle, and as such, they have a claim to being principles which are at least as cognitively fundamental for reasoning with disjunction than $\vee E$, if not more so.[30] Given alternatives, these inferences are implicit in the traditional $\vee E$ rule:

$$\dfrac{\begin{matrix}\Pi\\ A\vee B\end{matrix}\quad [A]^1\quad \dfrac{[B]^2\quad \cancel{B}}{A}\,\textit{Alt}}{A}\,\vee E^{1,2} \qquad \dfrac{\begin{matrix}\Pi\\ A\vee B\end{matrix}\quad \dfrac{[A]^1\quad \cancel{A}}{B}\,\textit{Alt}\quad [B]^2}{B}\,\vee E^{1,2}$$

[30]Disjunctive syllogism was one of the Stoic Logicians' *Indemonstrables*, those fundamental principles that brook no further demonstration. Chrysippus went so far as to argue that even *dogs* reason in accordance with disjunctive syllogism: "[Chrysippus] declares that the dog makes use of the fifth complex indemonstrable syllogism when, on arriving at a spot where three ways meet..., after smelling at the two roads by which the quarry did not pass, he rushes off at once by the third without stopping to smell. For, says the old writer, the dog implicitly reasons thus: "The animal went either by this road, or by that, or by the other: but it did not go by this or that, therefore he went the other way."" [37]. It is harder to imagine that the dog directly employs $\vee E$.

On the other hand, given the $\vee E'$ rules, we can recover the original $\vee E$ rule like this:

$$\dfrac{\begin{matrix}\Pi\\ A\vee B\end{matrix} \quad \begin{matrix}[A]^i\\ \Pi'\\ C\end{matrix} \quad \begin{matrix}[B]^j\\ \Pi''\\ C\end{matrix}}{C}\,\vee E \qquad \textit{becomes} \qquad \begin{matrix}\dfrac{\begin{matrix}\dfrac{[\cancel{C}]^j \quad \begin{matrix}\dfrac{\begin{matrix}\Pi\\ A\vee B\end{matrix}\quad [\cancel{B}]^i}{A}\,\vee E'\\ \Pi'\\ C\end{matrix}}{B}\,\textit{Swap}^i\\ \Pi''\\ C\end{matrix}}{C}\,\textit{Reductio}^j\end{matrix}$$

So we could start instead the simpler rules, with no loss of expressive power in constructing our proofs.

Another rule that is worth examining is the *Retrieve* rule. We have seen that $\bot E$ and *Retrieve* have the same shape, with the only difference that *Retrieve* takes a formula out of the storehouse of former conclusions, while $\bot E$ conjurs its conclusion by sleight of hand. In fact, $\bot E$ is not a different rule at all. It is better understood as a *vacuous* case of the *Retrieve* rule, which stands to alternatives as the case of vacuous *discharging*—that is, discharging *zero* occurrences of a formula—does to assumptions. The 'irrelevant' behaviour of our contradiction marker, according to which it entails anything and everything, is just as much a structural rule as any 'irrelevant' behaviour of the conditional.[31]The contradiction marker, $\bot$, is not so much a substantial conclusion but a marker that we have reached a contradiction. In the sequent calculus, whether we allow multiple conclusions or not, $\bot$ is represented most naturally by an *empty* right hand side.[32] Then, in a separate step of weakening, we allow the move from an empty conclusion to an arbitrary one. Weakening, in premise and in conclusion position, corresponds to vacuous discharge and vacuous retrieval, respectively.[33]

[31] In fact, it is most natural to have the irrelevant behaviour either with *both* alternatives and assumptions, or with *neither*. We can smuggle in an 'irrelevant' assumption q in into a proof of p by way of a $\bot E$ step (that is, a vacuous retrieval), like this:

$$\dfrac{\dfrac{\dfrac{[q]^2 \qquad \dfrac{\dfrac{[\cancel{p}]^1 \quad p}{\bot}\uparrow}{\neg q}\,\bot E}{\bot}\,\neg E}{p}\downarrow^1}{q\to p}\to I^2$$

Here, the irrelevant q is smuggled in as an assumption by the adding of $\neg q$ as an (irrelevant) conclusion. If we wish to avoid the smuggling in of irrelevant assumptions, we need to avoid the addition of irrelevant alternatives, too.

[32] For more on the significance of $\bot$ in natural deduction proofs, see the discussion in Section 4.

[33] If we wish to live without vacuous retrieval, a more discriminating approach would be to introduce

* * *

This is our natural deduction system. It is not new. It is Michel Parigot's $\lambda\mu$-calculus [88, 90], stripped of proof terms, and presented in the natural deduction garb of Gentzen and Prawitz. It is a well-behaved single conclusion proof system for classical propositional logic. In the remaining sections, I will explain why this proof system is well suited for the inferentialist's aims, and show how it avoids the criticisms that have been laid at the feet of signed natural deduction, the sequent calculus and other multiple conclusion proof formalisms.

In this discussion, the criterion of *answerability* will play a significant role, as it has so far. It is worth pausing for a moment to clarify what this condition *is* and what it *isn't*. To say that our account of proofs is answerable to our everyday practice is not to say that the account is a simple-minded reiteration of that practice. There are aspects of Gentzen–Prawitz natural deduction — take vacuous discharging[34] — that seem downright weird or unnatural. Yet, the defender of natural deduction can argue that this account of supposition and discharging is at the very least *implicit* in our everyday practice,[35] and that structuring the rules in such a way as to allow vacuous discharging is a way to specify a well-behaved *explication* of our practice that plays a useful theoretical role. The particular choices of the rules for connectives or the structural rules may seem alien at first, but if we can show that every aspect of the system finds its roots in our practice (and so is answerable to that practice in that sense), and we can account for

two false constants. The intensional or multiplicative false constant f could be used as a contradiction marker, corresponding to the empty right hand side of a sequent. The extensional or additive false constant F can still entail everything, if such a constant is desired.

[34] That is, the practice of allowing *zero* instances of an assumption to be discharged in an ${\to}I$ or $\vee E$ inference.

[35] We can get the effect of vacuously discharging the hypothesis A by *laundering* it through a pair of $\wedge I$ and $\wedge E$ steps, like this:

$$\textit{Replace}\qquad \dfrac{\begin{matrix}\Pi\\ B\end{matrix}}{A \to B}\,{\to}I \qquad \textit{with} \qquad \dfrac{\dfrac{\dfrac{[A]^i \qquad \begin{matrix}\Pi\\ B\end{matrix}}{A \wedge B}\,\wedge I}{B}\,\wedge E}{A \to B}\,{\to}I^i$$

The assumption A is now present in the second proof but it plays no significant role in deriving B. The obvious normalisation step for this detour through $A \wedge B$ is to replace the detour with the vacuous dischrarge, so any self-respecting proof system that allows for the premise-laundering $\wedge I/\wedge E$ move of the detour also allows vacuous discharge. This is not to say that there is nothing to be learned by exploring, and perhaps adopting, treatments of conditionals which disallow vacuous discharge [2, 3, 45, 79, 102]. It is just to say that our everyday inferential practices are messy and unsystematic, that $\wedge I$, $\wedge E$ and ${\to}I$ are compelling rules in their own rights, and admitting vacuous discharge with all that involves, is one way to settle on a practice for regimenting proof that has its roots in that messy practice, and which has proved useful and clarifying.

significant properties of the logical concepts in terms of a small number of rules, each governing one and only one concept, then we will have met our aims.

3 FROM FORMAL PROOFS TO SPEECH ACTS

As with regular natural deduction proofs, there are two ways to use a proof with alternatives to guide a process of reasoning. We can read the proof *forwards* (that is, we read from the leaves down to the conclusion), and we can read it *backwards* (that is, starting at the conclusion and reading upwards to the top of the tree). In this section, we will see how natural deduction proofs with alternatives can be read in either way. A proof from assumptions X to conclusion A, with alternatives Y is a way to truly *prove* the conclusion A in a context where each member of X has been ruled in, and each member of Y has been ruled out. When we read the proof from top to bottom, we will construct a chain of reasoning leading *from* the commitments we allow—ruling X in and ruling Y out—*to* the conclusion, A. When we read the proof from bottom to top, we construct a chain of answers to questions, starting from the conclusion, and stopping in the commitments we take for granted—again, ruling X in and ruling Y out. We will see that in the process we stay close to our everyday proof practice, and need employ no exotic speech acts.

One piece of notation will be useful, as we proceed explaining how both ways to read a formal proof can be effected. That is the notation '$[X : Y]$' for a *context*, the background commitments in place as a dialogue progresses. The left component X consists of all of the claims that we have ruled in, and the right component Y consists of all those claims we have ruled out. This is one way to represent the *common ground* [4, 116] in a conversation, where our aim is to keep track of finely-grained distinctions, such as in the dynamics of a *proof* where we don't take for granted that if A has been asserted, then everything that is a logical consequence of A follows in its train without further elaboration of its own. When we make an assertion, it is a bid to rule the claim in, or to add it to the left set in the context. When we deny, it is a bid to rule the claim out, or to add it to the right sent in the context. When we *suppose* A, we temporarily add A to the left set 'for the sake of the argument', with an aim to withdraw it, later, after this supposition is discharged. In the same way, when I set a claim aside, to consider an alternative, I add the original claim, temporarily, to the *right* set of the context, with an aim to withdraw it, later.

* * *

When we read a proof from top to bottom, we start with formulas at the leaves, some of which will be discharged in inferences as the reasoning progresses. If the formula at the leaf, A is not slashed, and not later discharged, when we *employ* the proof, we *assert* A. If the formula is slashed, and is also not later discharged, we *deny* it when we employ the proof. If, on the other hand, the formula (slashed or not) is later discharged, then we assert or deny it *for the sake of the argument*. Wether positive or negative, the claim is entered into the current context in the appropriate component, left for an assertion, and right for a denial. If there are no further steps in our proof, then the conclusion is an assumption, and in this case, the only leaf is positive. We have the limit case of having *shown* our conclusion *from* the context. Represented in sequent form we have this:

$$[X : Y] \succ A \text{ where } A \in X$$

where on the left we keep track of $[X : Y]$, the context current this stage of the proof. The simplest way to *prove* A, relative to the current context is to find it explictly granted in that context.

As a sanity check, we can verify that the assumption rule is indeed satisfied by all Boolean valuations. We say that a sequent $[X : Y] \succ A$ is *Boolean-valid* if every Boolean valuation v that verifies (assigns the value true) to each member of X and falsifies (assigns the value false) to each member of Y verifies A. This is trivially satisfied in the case where $A \in X$. We will see this form of reasoning again and again in what follows, and we will call valuations that verify each member of X and falsify each member of Y '$[X : Y]$-valuations' for short.

Now let us turn to a representative sample of the rules of our proof system, to show how these are to be understood as speech acts that we employ when following a proof, how they manipulate the local context as they are used, and that they, too, produce only Boolean-valid sequents. The rules we will consider are $\to E$, $\to I$, *Store* and *Retrieve*, as they display all the distinctive features of our proof system.[36] Here are the conditional

[36]And indeed, the other connectives are not only definable in terms of $\to$ and $\bot$, but when we use those definitions, the standard *rules* for the defined connectives can be seen as defined rules, arising out of the rules we have given here. For example, if we define $A \vee B$ as $(A \to \bot) \to B$, then we can replace the proof steps $\vee I$ and $\vee E$ from the standard suite in Figure 1 by these proofs:

$$\dfrac{\dfrac{\dfrac{[A \to \bot]^i \qquad \begin{matrix}\Pi \\ A\end{matrix}}{\bot}{\to}E}{B}\,\bot E}{(A \to \bot) \to B}{\to}\not{I}^i \qquad \dfrac{\begin{matrix}\Pi \\ B\end{matrix}}{(A \to \bot) \to B}{\to}\not{I}^j \qquad \begin{matrix} \dfrac{[\cancel{C}]^k \qquad \begin{matrix} \dfrac{\begin{matrix}\Pi \\ (A \to \bot) \to B\end{matrix} \quad \dfrac{\dfrac{[\cancel{A}]^j \ [A]^i}{\bot}\uparrow}{A \to \bot}{\to}I}{B}{\to}E \\ \Pi'' \\ C \end{matrix}}{\dfrac{\bot}{A}\downarrow^j}\uparrow \\ \Pi' \\ \dfrac{C}{C}\,\mathit{Reductio}^k \end{matrix}$$

In the same way, the rules for conjunction and negation can be encoded by $\to I$, $\to E$ and the alternative rules, so there is absolutely no loss of generality in attending only to these particular rules.

rules, written out in sequent form, in which we make the contexts explicit:

$$\frac{[X : Y] \succ A \to B \qquad [X' : Y'] \succ A}{[X, X' : Y, Y'] \succ B}\,{\to}E_{seq} \qquad \frac{[X, A : Y] \succ B}{[X : Y] \succ A \to B}\,{\to}I_{seq}$$

Attending to the conditional elimination rule, we see that if we have proved $A \to B$ (from context $[X : Y]$) and A (from context $[X' : Y']$), then we can go on to infer B, provided that we collect together the contexts appealed-to in the proof of $A \to B$ and in the proof of A. If no assumptions were made, and the two proofs were given in the same context, this is trivially satisfied. The local context is the shared background of claims we take for granted. On the other hand, if we made some assumptions in the proof of $A \to B$, or in the proof of A, then the new conclusion B is in the scope of those suppositions. If, on the other hand, we set aside some conclusion as an alternative in the course of proving $A \to B$, or in the course of proving A then this alternative remains in play as we derive B. So, our claim of B may have the force of an *assertion* if it depends on no other local assumptions later to be discharged. That is, if it depends on a context set $[X : Y]$ which contains only those commitments we are prepared to take for granted. Otherwise, it is a claim made under supposition, for the sake of the argument.

Of course, if any $[X : Y]$-valuation verifies $A \to B$, and any $[X' : Y']$-valuation verifies B, then any $[X, X' : Y, Y']$-valuation will verify both $A \to B$ and A and hence, by elementary features of the valuation conditions for the material conditional, such a valuation verifies B, as desired.

The conditional introduction rule works in the same way, except now, we have the opportunity to discharge an assumption. If I have proved B against the context of assuming A alongside the members of X (and setting aside the members of Y), then when I *discharge* that supposition, I can conclude $A \to B$, as usual with $\to I$. Now the nominated assumptions of A in our proof are marked off as discharged, and they are no longer in the local context. (The number of discharged instances can be *zero*, of course, if the discharge is vacuous.) We have proved $A \to B$ from the context $[X : Y]$. Again, the speech acts involved are straightforward.

As we expect, if any $[X : A, Y]$-valuation also verifies B, then it follows that every $[X : Y]$-valuation must verify $A \to B$, by the usual behaviour of the material conditional on Boolean valuations.

Consider now the *Store* rule. Here it is, in sequent form:

$$\frac{[X : Y] \succ A}{[X : A, Y] \succ \bot}\,{\uparrow}_{seq}$$

If we have proved A in the context $[X : Y]$, we can go on to conclude $\bot$, at the cost of parking A among the alternatives. We have already explained how this is be effected in everyday reasoning. Having concluded A, in the context $[X : Y]$, I simply point to the

A newly added as a *denial* (whether for the sake of the argument, or as something to be kept in the context at the end of our proof), and conclude that, in this context, we have reached a contradiction.

This rule is, of course, truth preserving. If any $[X : Y]$-valuation verifies A, then any $[X : A, Y]$-valuation (*per impossibile*) verifies $\perp$, as desired.

The remaining rule in our toolkit is *Retrieve*, but this is straightforward, too.

$$\frac{[X : A, Y] \succ \perp}{[X : Y] \succ A}\ {\downarrow_{seq}}$$

If, relative to my context $[X : A, Y]$, I have proved $\perp$, I have shown that my commitments (at least locally incurred) are unsatisfiable. I have reduced this context to a contradiction. This is the point at which we release the tension by taking back one of the denials we have temporarily made. We retrieve the option A from the local denial context, concluding A from the narrower context $[X : Y]$. As with *Store*, *Retrieve* is truth preserving.

We have seen that when we read proofs from top to bottom, we can understand the background context $[X : Y]$ of claims ruled in and ruled out as shifting as assumptions are granted and discharged. There are no exotic speech acts involved in reasoning in this way, and the moves we make find their roots in our everyday reasoning practices. We have seen, along the way, that the rules of classical natural deduction preserve Boolean validity, so our proof system is *sound* for classical propositional logic. That the rules are *complete* for classical logic is trivial. We know already that adding any of the classical principles from Figure 2 suffice to generate all of classical logic, and each of these are derivable using the rules of natural deduction with alternatives.

* * *

We will end this section by briefly considering another way to understand our proofs, as a *bottom-up* procedure to be understood in a more dialogical manner. Again, we will consider dialogical readings of the rules for the conditional and the alternative rules as examples of how a dialogical formulation for *all* our rules could be given.

Again, we will proceed by induction on the construction of proofs, to show that if we have a proof for $[X : Y] \succ A$, we can read our proof as providing us a systematic way to meet a justification request for the assertion of the claim A in a local context where each member of X is taken as ruled in, and each member of Y has been ruled out. Again, bare assumption proofs are straightforward. If, relative to the local context $[X : Y]$, the claim A has been made, one way to meet the justification request is to point back to the context if in that context A has been taken for granted.

Consider a proof ending in $\rightarrow E$. If I make the claim B in the local context $[X : Y]$ and I am asked to justify that claim, then one straightforward way to do this is to make *two* claims, $A \to B$ and A. If my interlocutor is willing to take those for granted, all is well and good. If, however, she presses on, I will then attempt to prove $A \to B$ from the local context and do the same for A. Given that I have proofs of $A \to B$ and A, these proofs will guide my justifications.

Consider, on the other hand, a proof ending in $\to I$. If I have made the claim $A \to B$ against the background of the commitments $[X : Y]$, and I am asked to justify this claim, I will ask my interlocutor to *suppose* A, adding it to the local context, temporarily, and I will then, for the sake of this argument, claim B under the scope of that assumption. If I am asked to further justify this claim, I will proceed by following the guidance of the rest of this proof, for the rest of my proof indeed is a proof of B from $[X, A : Y]$.

Next, consider the *Store* rule. If I have been asked to prove a contradiction from the context $[X : A, Y]$, one way to do this would be to attempt to *prove* A, and then point to the fact that in our context A is (at least, for the sake of the argument) denied, and so we are out of options: we have reached a contradiction.

The *Retrieve* rule reverses this process. If I have been asked to prove A from the context $[X : Y]$, one way to attempt this would be to show that *denying* A is off the table—that is, to prove a contradiction from the larger context $[X : A, Y]$. Should we manage this, we can then discharge that local assumption, to conclude A, as desired.

So, it is possible also to give a bottom-up dialogical understanding of our proof rules as means to justify claims when pressed, or as a prospective top-down manner of unfolding the commitments made in our background context of claims we take for granted. The proof techniques are *clearly* the kinds of moves we find in dialogue, or in our everyday reasoning. Yes, they are *classical*, through and through: the justification of the *retrieve* rule makes this manifest. For someone whose tastes are of a more restricted kind, the alternative moves may be unpalatable. However, they cannot be impeached on the grounds of being alien to our proof practices, or of being uninterpretable in the coin of everyday speech acts.

So, we see that the speech acts involved in reading classical proofs with alternatives are reasonable and appropriate to everyday proof. Along the way we have made use of bilateralism: We take the notion of ruling *in* and ruling *out* as fundamental, and ruling out $\ulcorner A \urcorner$ is not to be identified, in the first instance, with ruling in $\ulcorner \neg A \urcorner$. That much we have taken for granted. The task was to find a natural home for the bilateralist *inferentialist*.

* * *

To end this section, notice that in either of these ways to read formal proofs, that there is no implicit 'disjunctive' conclusion at any point of our reasoning. Not only is there one and only one identified conclusion at each step of the proof, but the alternatives, at any stage, do not need to be (implicitly or explicitly) collected together in a disjunction. In general, given a proof for $[X : Y] \succ A$ we have grounds for A whenever we have grounds for each member of X and grounds against each member of Y. There is no requirement that the alternatives Y be understood *disjunctively*, any more than the assumptions X be understood *conjunctively*. As Steinberger points out [118, p. 348], the inferentialist—whether bilateralist or not—is free to note that assertion, in some sense, distributes over conjunction. To assert A and to assert B is, (at least implicitly) to assert their conjunction. In the same way, the bilateralist will note that to deny A and to deny B is, (at least implicitly) to deny their disjunction. So, it is understandable that we might be moved to identify the alternatives in a context with their disjunction, in the same way that we might be moved to identify the assumptions with their conjunction. Nonetheless, this identification is not required, and no prior understanding of disjunction is needed in order to employ natural deduction with alternatives, any more than an understanding of conjunction is needed to employ natural deduction with assumptions.

4 CATEGORICITY

There is one supposed advantage for signed natural deduction over traditional unsigned systems which merits attention. This is its *categoricity* [62, 84, 99]. Call a function v from the sentences in the language to the truth values true and false a *valuation*. We made use of Boolean valuations in the previous section, to verify our proof systems' *soundness*. Not all valuations are Boolean. A valuation is *Boolean* if it satisfies the expected valuation conditions: $v(A \to B)$ is false if and only if $v(A)$ is true and $v(B)$ is false, $v(\bot)$ is false, and so on. There is the gullible valuation v_{true}, which assigns true to every formula whatsoever, including $\bot$. The skeptical valuation v_{false} assigns *false* to every fortmula, and is the tautology valuation $v_{\top}$, which assigns true to every tautology, and false to every other formula. Most valuations are not Boolean.

We say that a valuation v is *compatible* with a traditional natural deduction system of proofs if whenever there is a proof from premises X to the conclusion A, if v assigns true to each member of X, it also assigns true to A. If that proof system is sound for classical valuations, then every Boolean valuation is compatible with it. Not all valuations are compatible with every proof system, of course. The skeptical valuation v_{false} is ruled out as incompatible with standard natural deduction, because we can prove formulas from zero premises. Since we can prove $p \to p$ against an empty context, if v is compatible with this proof system, we require that v at the very least assign true to $p \to p$. So, v_{false} is ruled out as incompatible.

The categoricity problem for a proof system (at least, for a proof system intended to capture classical logic) is to show that *only* Boolean valuations are compatible with that proof system. This is a desirable feature for a proof system to have if you (a) think that the proof system should be doing semantic work and (b) you think that Boolean valuations are the appropriate valuations for the language of propositional logic. This is a problem for traditional (unsigned) natural deduction in which the proofs lead from some collection X of premises to a single conclusion A. Any such proof system, no matter *what* rules it employs, *cannot* rule out v_{true} as incompatible, because the gullible valuation is compatible, vacuously, with every sequent of the form $X \succ A$. For good measure, the tautology valuation $v_{\top}$ is also compatible with any sequent $X \succ A$ that is classically provable, for if the assumptions X are all tautologies, so, too, is the conclusion A. That is the categoricity problem for a single conclusion proof system, and it is a serious problem for anyone who wishes to make the claim that the norms for proof are somehow fundamental, who the truth conditions of valuations are in any sense, derivative, and who takes valuations such as v_{true} and $v_{\top}$ to be defective.

Categoricity is *not* a problem for signed natural deduction [62], and this is a significant point in its favour. Given a signed proof from signed formulas as assumptions to a signed formula as a conclusion, the natural requirement for a valuation be compatible with that proof is that if the valuation assigns true to every formula signed positively as an assumption formula, and false to every formlua signed negatively as an assumption, then it assigns true to the conclusion formula if it is signed positively, and false if it is signed negatively. And indeed, with this understanding of compatibility, only Boolean valuations are compatible with every argument with a signed natural deduction proof. The gullible valuation is ruled out, since we have a proof from $+\neg p$ to $-p$, which means that any valuation that assigns true to $\neg p$ must assign false to p. So, the gullible valuation fails on that score. This proof also shows that the tautology valuation $v_{\top}$ is incompatible, because this assigns false to every atom, and also to every negated atom.

What about classical natural deduction with alternatives? At first blush, our proofs fail the categoricity constraint in exactly the same way that standard single conclusion proofs do. The natural way to read proofs as constraining valuations is to say that if we have a proof from $[X : Y]$ to A then any valuation that assigns true to every member of X and false to every member of Y assigns true to A.

We can see, immediately, that this definition fails to rule out v_{true} and $v_{\top}$ as incompatible. The gullible valuation v_{true} is vacuously compatible with any sequent of the form $[X : Y] \succ A$, and $v_{\top}$ is compatible with any classically provable sequent of this form. So, it seems that *signed* natural deduction has one virtue that classical natural deduction with alternatives lacks.

But not so fast.

* * *

It is true that in the system we have discussed so far, there is nothing stopping us from contemplating valuations like v_{true} and $v_{\top}$, and taking them to be compatible with all

$$
\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ \#\end{array}}{\neg A}\,\neg I^i
\qquad
\dfrac{\begin{array}{c}\Pi\\ \neg A\end{array}\quad\begin{array}{c}\Pi'\\ A\end{array}}{\#}\,\neg E
\qquad
\dfrac{\cancel{A}\quad\begin{array}{c}\Pi\\ A\end{array}}{\#}\,\uparrow
\qquad
\dfrac{\begin{array}{c}[\cancel{A}]^i\\ \Pi\\ \#\end{array}}{A}\,\downarrow^i
\qquad
\dfrac{\begin{array}{c}\Pi\\ \#\end{array}}{\bot}\,\bot I
\qquad
\dfrac{\begin{array}{c}\Pi\\ \bot\end{array}}{\#}\,\bot E
$$

Figure 6: REFUTATION RULES

the constraints we have considered. However there is one very natural option available for anyone who favours an unsigned natural deduction system. This is to learn a lesson from the sequent calculus, and to look again at $\bot$. In the sequent calculus the role played by $\bot$ is taken by something other than a formula: the *empty* right hand side. In the sequent calculus, whether for classical or intuitionistic logic, we derive $A, \neg A \succ$, where this sequent has an empty right hand side. If there is no formula to assign a value on the right hand side, the natural way to understand compatibility of a valuation with a sequent $X \succ$ is to require that there is no valuation which assigns each member of X the value true. It has long been known that *sequents*, whether multiple conclusion or single-and-zero conclusion sequents, render deviant valuations like v_{true} incompatible.

We are free to take a leaf out of the sequent calculus' book and to treat our proofs of a *contradiction* not as proofs of a particular conclusion *formula*, but as refutations of the commitments we have already made. Rather than leaving the conclusion spot *blank* in refutations, we will allow for a new conclusion marker #—not a formula!—to take the spot in a conclusion of a proof.[37] We are free to retain $\bot$ as a distinctive *formula constant*, if we wish, but now, # plays the role in proofs not of some *content* that is asserted, but as the cry 'that's a contradiction!' we make when we reach such a spot in a proof. Revising our rules just a little, we get new formulations for the rules for negation, for weakening in a conclusion where there was none before, and new rules for $\bot$, to make it the object language correlate of the punctuation mark '#.' The new rules are collected together in Figure 6. From a proof-theoretical prerspective, these rules embody an even greater separation of powers and clarity of function. The negation rule no longer involves a distinct *formula*, but is connected only with the structure of proofs. The weakening in of a conclusion after a contradiction has been proved is also a feature of the structure of proofs, and is not connected to the behaviour of some formula with the mysterious power of being strong enough to entail everything. If we wish, though, we can reintroduce such a formula with the $\bot$ introduction and elimination rules. The $\bot I$ rule is redundant in the system for classical logic which allows for vacuous discharge. In the *Retrieve* ($\downarrow$) rule, if we discharge *zero* instances of a stored formula, we could

[37] This is the technique, and the notation, used by Neil Tennant in his *Natural Logic* [121].

conclude it from the contradiction marker #, even if that formula is $\bot$. So, any formula, including $\bot$, follows from the contradiction #.

Such a refactoring of our rules is, in fact, well motivated by our everyday inferential practice. We do not, as a habit, have a particular contradiction $\bot$ in mind when we notice that we have proved A and $\neg A$, and go on to say that we have reached a contradiction, and then discharge one of our premises. Better to say that when we have proved A and $\neg A$, we no longer have a conclusion formula to our proof but we have reached a contradiction. This 'proof' has no conclusion formula at all. It is better understood as a refutation of the assumptions leading up to it, consisting of formulas as premises (and perhaps as alternatives), but no conclusion at the root.

As a pleasing side-benefit of this refactoring of our rules, we can answer the categoricity objection. Given that # is not a formula at all, and there is no place in a valuation to assign a value to *it*, if we have a proof from $[X : Y]$ to #, that is, a *refutation* of $[X : Y]$, the natural requirement for valuations is that for it to be compatible with this refutation, that it not be an $[X : Y]$-valuation, that is, that it not verify each member of X and falsify each member of Y. And if we have a proof from $[X : Y]$ to A, if v is to be compatible with this proof, as usual we require that if it is an $[X : Y]$-valuation, then it must verify A.

With that understanding of compatibility, we have an answer to the categoricity objection in just the same form as it is available to the signed natural deduction bilateralist. Since there is a proof from $[p, \neg p : \ ]$ to # (that is, a refutation of $[p, \neg p : \ ]$) any valuation that assigns true to p and to $\neg p$ is incompatible, so v_{true} is ruled out of court. Similarly, since there is a proof from $[\ : p]$ to $\neg p$, any valuation that assigns false to p must assign true to $\neg p$, so $v_\top$ is incompatible with natural deduction with alternatives.

Two examples alone are not enough to demonstrate the fact that *only* Boolean valuations are compatible with our proofs. The demonstration of the general fact is not hard to come by. Consider the connectives, one by one. For conjunction, we have proofs for the following sequents:

$$[A, B : \ ] \succ A \land B \qquad [A \land B : \ ] \succ A \qquad [A \land B : \ ] \succ B$$

which together ensure that if v is a compatible valuation then v verifies $A \land B$ if and only if it verifies A and verifies B. For disjunction, we have

$$[A : \ ] \succ A \lor B \qquad [B : \ ] \succ A \lor B \qquad [A \lor B : A] \succ B$$

which ensure that for any such v, it verifies $A \lor B$ iff it verifies A or verifies B. For the conditional we have

$$[\ : A] \succ A \to B \qquad [B : \ ] \succ A \to B \qquad [A \to B, A : \ ] \succ B$$

which ensure that v verifies $A \to B$ iff it either falsifies A or verifies B. For negation we have

$$[A, \neg A\ :\ ] \succ \# \qquad [\ :A] \succ \neg A$$

which ensure that v verifies $\neg A$ iff it falsifies A, and finally:

$$[\bot\ :\ ] \succ \#$$

ensures that $\bot$ is not verified in any compatible valuation. So, any valuation compatible with these twelve sequent schemes must be Boolean. Since these are all provable in natural deduction with alternatives, this proof system, too, is categorical for Boolean valuations.

Categoricity, therefore, is not a *distinctive* benefit of signed natural deduction. Classical natural deduction with alternatives—given the use of *refutations*—is also categorical.

5 IN CONCLUSION

Let's take stock of where we have come, and look up to the horizon, to see where further adventures may lead us.

In the first instance, we have an answer to the objection that there is no genuinely inferential account of classical proof. We have shown how a bilateralist, with very modest commitments concerning assertion and denial, can give an account of classical proof that is answerable to our inferential practice, and which does not invoke exotic proof structures, and does not unnecessarily decorate every formula in a proof with signs.

The resulting proof system is remarkably well behaved. Although I have not spent time on this, natural deduction with alternatives not only has a *normalisation* result (in which introduction/elimination detours can be normalised away), the proof system is *strongly* normalising, in the sense that *any* process of reducing detours will terminate in a finite number of steps [90]. This result holds not only for *propositional* logic, but holds under the addition of rules for the quantifiers—even the *second order* quantifiers. We have all the benefits of a normalising deduction system, with seprable rules, and the conservative extension results that these entail.

Although we have presented Parigot's $\lambda\mu$-calculus shorn of its proof terms, terms can be added to our proofs when it makes sense to consider the information they carry. Each proof from the context $[X : Y]$ to the conclusion A can be seen as generating a *term* t constructed from *variables* annotating members of X and *co-variables* annotating members of Y. There is no need for us to work through the details of the term calculus here. (Parigot's work is a clear introduction [88–90] in itself.) Suffice it to say, just as there is a straightforward interpretation of traditional λ-terms in terms of procedures to generate verifications or grounds for a conclusion of a proof from grounds for its premises [96], the way is open for us to do the same here for *classical* proof. A proof from the context $[X : Y]$ to the conclusion A gives us a way to generate grounds for A from grounds *for* each member of X and grounds *against* each member of Y.

To be sure, since this is *classical* proof we are talking about, we will have the means to construct grounds—by way of the obvious proofs—for disjunctions like $p \lor \neg p$, where logic alone is no help at generating grounds for p or grounds for $\neg p$, in general. Our grounds will fail to be *prime*. We will have means to decide *that* $p \lor \neg p$ is true, while having, in general, no insight into which of p and $\neg p$ is true. Grounds, constructed with the full power of classical proof, do not have all the features of properly *constructive* grounds. Canonical constructive grounds of disjunctions give us ways to pinpoint grounds for one or other disjunct. Canonical constructive grounds of existentially quantified statements give us the means to find some witness for that statement and to construct a ground for the claim that this is indeed a witness. All that specificity is lost when we move to constructing *classical* proofs and their grounds.

Whether this ability to construct non-prime grounds should count as a *bug* or a *feature* is very much a matter of considerations other than pure logic. A thoroughgoing realist would say that this is a feature, but, despite this defence of the coherence and elegance of classical proof theory, I am not so sure that the friend of classical logic need be so sanguine that the constructivist is missing the point in her desire for constructive proofs and prime grounds. The reasoner who proves things constructively *has* something that the classical reasoner lacks. Kürbis makes the point [68] in favour of the intuitionist to the effect that adopting the intuitionist rules allows for one to have a clean, simple account of *decidability*. The intuitionist can make the straightforward claim that to treat A as decidable is to assume $A \lor \neg A$. Clearly, this move is not open to the friend of classical proof theory, since we can *prove* $A \lor \neg A$, for any formula A at all, whether decidable or not. But clearly, not everything is decidable, and it would be churlish of the friend of classical proof to think that the constructivist is missing the point in paying attention to intuitionist strictures for proofs and grounds.

It is fortunate, then, that I was too swift when I said, in the previous paragraph, that the "reasoner who proves things constructively *has* something that the classical reasoner lacks." That was overstating things. The classical reasoner has *everything* that the constructive reasoner has, *and more*.[38] In adopting classical natural deduction with

[38] This is, perhaps, the starkest difference between natural deduction with alternatives and signed natural deduction systems. Those systems are so thoroughgoingly bilateralist in every aspect that restricting ourselves to constructive reasoning seems well-nigh impossible. The signed rules for negation veritably hard-code the equivalence of $\neg\neg A$ with A. This is very much *unlike* the situation with everyday mathematical proof, where it is not difficult for the alert mathematician to understand when a proof is constructive, and when it uses non-constructive principles. To restrict your attention to constructive proofs, you eschew just a few principles. See Kürbis' "Some Comments on Ian Rumfitt's Bilateralism" [67] for an extended discussion of different signed natural deduction systems and the different commitments to classicality in those different presentations.

alternatives, we do not need to change any of the intuitionistic rules, though we have seen that we *could* replace some of these rules with simpler variants that use alternatives, should we wish. This means that we can do more than the intuitionist when it comes to proof, than the intuitionist, not less. The way is open to be *pluralist* about the canons of proof [12, 13, 106]. Is no problem, for the friend of classical proof, with one and the same set of connective rules for constructive logic as for classical logic, to pay attention to whether or not alternatives are invoked in a proof. If we don't use the alternative rules, the proof is constructive. Any ground we construct for our conclusion is a constructive ground, with all the discriminating properties the intuitionist wants. Let's call those grounds *strong*, in view of their extra discriminating properties.

To be sure, the pluralist cannot say, with Kürbis, that A is decidable when $A \vee \neg A$ is *true*, since she thinks that $A \vee \neg A$ is always true, since she can prove it. However, she can say that A is decidable when we have *strong* grounds for A, and this is a discriminating claim, since the standard proofs for $A \vee \neg A$ produce only weak grounds, not the stronger stuff. That form of words is not quite as stark as the direct statement of an intuitionist, but it goes some way to show how the pluralist can, in an ecumenical spirit, adopt the classical structural rules as the *wide* ambient space in which proofs in general find their home, and see the virtues of constructing more discriminating proofs satisfying the structures laid down by her constructivist comrades.

That seems to me to be an account of classical proof worth having.

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