Natural Deduction with Alternatives

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https://consequently.org/presentation/2020/natural-deduction-with-alternatives

My Aim

To introduce *natural deduction with alternatives*,
a well-behaved single-conclusion natural deduction
framework for a range of logical systems,
including *classical*, *linear*, *relevant* logic and *affine* logic,
by varying the policy for managing discharging of
assumptions and retrieval of alternatives.

My Plan

Inferentialism & Natural Deduction

Classical Sequent Calculus

Assertion, Denial, Negation and Contradiction

Alternatives

Normalisation and its Consequences

Operational Rules as *Definitions*

INFERENTIALISM & NATURAL DEDUCTION

$$\dfrac{\dfrac{\dfrac{[\neg(r\lor s)]^4 \qquad \dfrac{\dfrac{p\to(q\lor r)\qquad [p]^3}{q\lor r}\;{\to}E \qquad \dfrac{\dfrac{q\to s\qquad [q]^1}{s}\;{\to}E}{r\lor s}\;{\lor}I \qquad \dfrac{[r]^2}{r\lor s}\;{\lor}I}{r\lor s}\;{\lor}E^{1,2}}{\bot}\;{\neg}E}{\neg p}\;{\neg}I^3}{\neg(r\lor s)\to\neg p}\;{\to}I^4$$

$$\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\dfrac{\bot}{\neg p}\,\neg I^3}\,\neg E}{\neg(r \lor s) \to \neg p}\,{\to}I^4$$

Natural Deduction is ***Beautiful***!

$$\dfrac{\dfrac{\dfrac{[\neg(r \vee s)]^4 \qquad \dfrac{\dfrac{p \to (q \vee r) \qquad [p]^3}{q \vee r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \vee s}\,{\vee}I \qquad \dfrac{[r]^2}{r \vee s}\,{\vee}I}{r \vee s}\,{\vee}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \vee s) \to \neg p}\,{\to}I^4$$

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$$
\begin{prooftree}
\AxiomC{$[\neg(r \vee s)]^4$}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p]^3$}
\RightLabel{$\to\! E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$q \to s$}
\AxiomC{$[q]^1$}
\RightLabel{$\to\! E$}
\BinaryInfC{s}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\AxiomC{$[r]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\RightLabel{$\vee E^{1,2}$}
\TrinaryInfC{$r \vee s$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg p$}
\RightLabel{$\to\! I^4$}
\UnaryInfC{$\neg(r \vee s) \to \neg p$}
\end{prooftree}
$$

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$$
\dfrac{\dfrac{\dfrac{[\neg(r \vee s)]^4 \qquad \dfrac{\dfrac{p \to (q \vee r) \qquad [p]^3}{q \vee r}{\scriptstyle \to E} \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}{\scriptstyle \to E}}{r \vee s}{\scriptstyle \vee I} \qquad \dfrac{[r]^2}{r \vee s}{\scriptstyle \vee I}}{r \vee s}{\scriptstyle \vee E^{1,2}}}{\bot}{\scriptstyle \neg E}}{\neg p}{\scriptstyle \neg I^3}}{\neg(r \vee s) \to \neg p}{\scriptstyle \to I^4}
$$

$$
\begin{prooftree}
\AxiomC{$[\neg(r \vee s)]^4$}
\AxiomC{$p \to (q \vee r)$}
\AxiomC{$[p]^3$}
\RightLabel{$\to E$}
\BinaryInfC{$q \vee r$}
\AxiomC{$q \to s$}
\AxiomC{$[q]^1$}
\RightLabel{$\to E$}
\BinaryInfC{s}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\AxiomC{$[r]^2$}
\RightLabel{$\vee I$}
\UnaryInfC{$r \vee s$}
\RightLabel{$\vee E^{1,2}$}
\TrinaryInfC{$r \vee s$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^3$}
\UnaryInfC{$\neg p$}
\RightLabel{$\to I^4$}
\UnaryInfC{$\neg(r \vee s) \to \neg p$}
\end{prooftree}
$$

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$$
\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}{\scriptstyle \to E} \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}{\scriptstyle \to E}}{r \lor s}{\scriptstyle \lor I} \qquad \dfrac{[r]^2}{r \lor s}{\scriptstyle \lor I}}{r \lor s}{\scriptstyle \lor E^{1,2}}}{\dfrac{\bot}{\neg p}{\scriptstyle \neg I^3}}{\scriptstyle \neg E}}{\neg(r \lor s) \to \neg p}{\scriptstyle \to I^4}
$$

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\dfrac{\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\bot}\,{\neg}E}{\neg p}\,{\neg}I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4
$$

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\dfrac{\dfrac{\dfrac{[\neg(r \lor s)]^4 \qquad \dfrac{\dfrac{p \to (q \lor r) \qquad [p]^3}{q \lor r}\,{\to}E \qquad \dfrac{\dfrac{q \to s \qquad [q]^1}{s}\,{\to}E}{r \lor s}\,{\lor}I \qquad \dfrac{[r]^2}{r \lor s}\,{\lor}I}{r \lor s}\,{\lor}E^{1,2}}{\bot}\,\neg E}{\neg p}\,\neg I^3}{\neg(r \lor s) \to \neg p}\,{\to}I^4
$$

The Rules

A

The Rules

$$A$$

$$\dfrac{\begin{array}{c}[A]^i \\ \Pi \\ B\end{array}}{A \to B}\ {\to}I^i$$

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$$A$$

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\;{\to}I^i$$

$$\dfrac{\begin{array}{c}\Pi\\ A \to B\end{array} \qquad \begin{array}{c}\Pi'\\ A\end{array}}{B}\;{\to}E$$

The Rules

$$A$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\ {\to}I^i$$

$$\dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\ {\to}E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}\ {\wedge}I$$

The Rules

$$A \qquad \dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\,{\to}I^i \qquad \dfrac{\begin{array}{c}\Pi\\ A \to B\end{array}\quad\begin{array}{c}\Pi'\\ A\end{array}}{B}\,{\to}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}\quad\begin{array}{c}\Pi'\\ B\end{array}}{A \land B}\,{\land}I \qquad \dfrac{\begin{array}{c}\Pi\\ A \land B\end{array}}{A}\,{\land}E \qquad \dfrac{\begin{array}{c}\Pi\\ A \land B\end{array}}{B}\,{\land}E$$

The Rules

$$A \qquad \dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\to I^i \qquad \dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\to E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad \begin{matrix}\Pi' \\ B\end{matrix}}{A \wedge B}\wedge I \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{A}\wedge E \qquad \dfrac{\begin{matrix}\Pi \\ A \wedge B\end{matrix}}{B}\wedge E$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix}}{A \vee B}\vee I \qquad \dfrac{\begin{matrix}\Pi \\ B\end{matrix}}{A \vee B}\vee I$$

$$A$$

$$\dfrac{\begin{array}{c}[A]^i \\ \Pi \\ B\end{array}}{A \to B}\ {\to}I^i$$

$$\dfrac{\begin{array}{c}\Pi \\ A \to B\end{array} \quad \begin{array}{c}\Pi' \\ A\end{array}}{B}\ {\to}E$$

$$\dfrac{\begin{array}{c}\Pi \\ A\end{array} \quad \begin{array}{c}\Pi' \\ B\end{array}}{A \wedge B}\ {\wedge}I \qquad \dfrac{\begin{array}{c}\Pi \\ A \wedge B\end{array}}{A}\ {\wedge}E \qquad \dfrac{\begin{array}{c}\Pi \\ A \wedge B\end{array}}{B}\ {\wedge}E$$

$$\dfrac{\begin{array}{c}\Pi \\ A\end{array}}{A \vee B}\ {\vee}I \qquad \dfrac{\begin{array}{c}\Pi \\ B\end{array}}{A \vee B}\ {\vee}I \qquad \dfrac{\begin{array}{c}\Pi \\ A \vee B\end{array} \quad \begin{array}{c}[A]^j \\ \Pi' \\ C\end{array} \quad \begin{array}{c}[B]^k \\ \Pi'' \\ C\end{array}}{C}\ {\vee}E$$

$$A$$

$$\dfrac{\begin{matrix}[A]^i\\ \Pi\\ B\end{matrix}}{A \to B}\,{\to}I^i$$

$$\dfrac{\begin{matrix}\Pi\\ A \to B\end{matrix}\qquad\begin{matrix}\Pi'\\ A\end{matrix}}{B}\,{\to}E$$

$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix}\qquad\begin{matrix}\Pi'\\ B\end{matrix}}{A \wedge B}\,{\wedge}I$$

$$\dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{A}\,{\wedge}E \qquad \dfrac{\begin{matrix}\Pi\\ A \wedge B\end{matrix}}{B}\,{\wedge}E$$

$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix}}{A \vee B}\,{\vee}I \qquad \dfrac{\begin{matrix}\Pi\\ B\end{matrix}}{A \vee B}\,{\vee}I$$

$$\dfrac{\begin{matrix}\Pi\\ A \vee B\end{matrix}\qquad\begin{matrix}[A]^j\\ \Pi'\\ C\end{matrix}\qquad\begin{matrix}[B]^k\\ \Pi''\\ C\end{matrix}}{C}\,{\vee}E$$

$$\dfrac{\begin{matrix}[A]^i\\ \Pi\\ \bot\end{matrix}}{\neg A}\,{\neg}I^i$$

The Rules

$$A$$

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ B\end{array}}{A \to B}\ {\to}I^i \qquad \dfrac{\begin{array}{c}\Pi\\ A \to B\end{array} \quad \begin{array}{c}\Pi'\\ A\end{array}}{B}\ {\to}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array} \quad \begin{array}{c}\Pi'\\ B\end{array}}{A \wedge B}\ {\wedge}I \qquad \dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{A}\ {\wedge}E \qquad \dfrac{\begin{array}{c}\Pi\\ A \wedge B\end{array}}{B}\ {\wedge}E$$

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array}}{A \vee B}\ {\vee}I \qquad \dfrac{\begin{array}{c}\Pi\\ B\end{array}}{A \vee B}\ {\vee}I \qquad \dfrac{\begin{array}{c}\Pi\\ A \vee B\end{array} \quad \begin{array}{c}[A]^j\\ \Pi'\\ C\end{array} \quad \begin{array}{c}[B]^k\\ \Pi''\\ C\end{array}}{C}\ {\vee}E$$

$$\dfrac{\begin{array}{c}[A]^i\\ \Pi\\ \bot\end{array}}{\neg A}\ {\neg}I^i \qquad \dfrac{\begin{array}{c}\Pi\\ \neg A\end{array} \quad \begin{array}{c}\Pi'\\ A\end{array}}{\bot}\ {\neg}E$$

The Rules

$$A \qquad \dfrac{\begin{matrix}[A]^i\\ \Pi\\ B\end{matrix}}{A \to B}{\to}I^i \qquad \dfrac{\begin{matrix}\Pi\\ A \to B\end{matrix} \quad \begin{matrix}\Pi'\\ A\end{matrix}}{B}{\to}E$$

$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix} \quad \begin{matrix}\Pi'\\ B\end{matrix}}{A \land B}{\land}I \qquad \dfrac{\begin{matrix}\Pi\\ A \land B\end{matrix}}{A}{\land}E \qquad \dfrac{\begin{matrix}\Pi\\ A \land B\end{matrix}}{B}{\land}E$$

$$\dfrac{\begin{matrix}\Pi\\ A\end{matrix}}{A \lor B}{\lor}I \qquad \dfrac{\begin{matrix}\Pi\\ B\end{matrix}}{A \lor B}{\lor}I \qquad \dfrac{\begin{matrix}\Pi\\ A \lor B\end{matrix} \quad \begin{matrix}[A]^j\\ \Pi'\\ C\end{matrix} \quad \begin{matrix}[B]^k\\ \Pi''\\ C\end{matrix}}{C}{\lor}E$$

$$\dfrac{\begin{matrix}[A]^i\\ \Pi\\ \bot\end{matrix}}{\neg A}\neg I^i \qquad \dfrac{\begin{matrix}\Pi\\ \neg A\end{matrix} \quad \begin{matrix}\Pi'\\ A\end{matrix}}{\bot}\neg E \qquad \dfrac{\begin{matrix}\Pi\\ \bot\end{matrix}}{A}\bot E$$

What makes natural deduction *natural deduction*?

- Proofs are *direct*, *from* premise(s) to conclusion(s).

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- Proofs are *direct*, *from* premise(s) to conclusion(s).
- Proofs are structures made out of *formulas*.
- The inferential relationships between those formulas is implicit in the structure of the proof.
- Rules for the connectives are, typically, *separable*.
- Proofs *normalise*. (We can straighten out *detours*.)

Normalisation

$$
\begin{prooftree}
\AxiomC{$[A]^i$}
\noLine
\UnaryInfC{Π_1}
\noLine
\UnaryInfC{B}
\RightLabel{$\to I^i$}
\UnaryInfC{$\boxed{A \to B}$}
\AxiomC{Π_2}
\noLine
\UnaryInfC{A}
\RightLabel{$\to E$}
\BinaryInfC{B}
\end{prooftree}
$$

Normalisation

$$\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ B\end{array}}{\boxed{A \rightarrow B}}{\rightarrow}I^i \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{B}{\rightarrow}E \qquad \rightsquigarrow \qquad \begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ B\end{array}$$

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- A proof from X to A shows show to meet a justification request for A against a background of granting X.
- I/E rules play a similar role to *truth conditions*.
- Normal proofs are *analytic*.

- Sequents (like $X \succ A$) are a good way to 'keep score.'

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- Structural rules, like *identity*, *cut*, *contraction* and *weakening*, are typically *explicit* in a sequent system and *implicit* in natural deduction.

A

Structural Rules

$A \qquad\qquad A \succ A$

Structural Rules

$$\begin{array}{cc} & X \\ & \Pi_1 \\ Y & A \\ \multicolumn{2}{c}{\Pi_2} \\ \multicolumn{2}{c}{B} \end{array}$$

$$\frac{X \succ A \qquad Y, A \succ B}{X, Y \succ B}$$

Structural Rules

$$\begin{array}{ccc} & & X \\ & & \Pi_1 \\ Y & & A \\ & \Pi_2 & \\ & B & \end{array}$$

$$\frac{X \succ A \qquad Y, A \succ B}{X, Y \succ B}$$

$$\begin{array}{ccc} & & X \\ & & \Pi_1 \\ Y & & A \\ & \Pi_2 & \\ & B & \end{array}$$

$$\frac{X \succ A \qquad Y, A \succ B}{X, Y \succ B}$$

$$\dfrac{\dfrac{\begin{matrix} X \quad [A]^i \quad [A]^i \\ \Pi \\ B \end{matrix}}{A \to B}{\to}I^i \qquad A}{B}{\to}E$$

$$\dfrac{X, A, A \succ B}{X, A \succ B}$$

$$
\dfrac{\dfrac{\begin{array}{c} X \quad [A]^i \quad [A]^i \\ \Pi \\ B \end{array}}{A \to B}{\scriptstyle \to I^i} \qquad A}{B}{\scriptstyle \to E}
$$

$$
\frac{X, A, A \succ B}{X, A \succ B}
$$

$$\dfrac{\dfrac{\begin{matrix} X \\ \Pi \\ B \end{matrix}}{A \to B}{\to}I \qquad A}{B}{\to}E$$

$$\dfrac{X \succ B}{X, A \succ B}$$

Structural Rules

$$\dfrac{\dfrac{\begin{matrix} X \\ \Pi \\ B \end{matrix}}{A \to B}{\to}I \qquad A}{B}{\to}E$$

$$\dfrac{X \succ B}{X, A \succ B}$$

	DUPLICATES	NO DUPLICATES
VACUOUS	*Standard*	*Affine*
NO VACUOUS	*Relevant*	*Linear*

Natural deduction is *opinionated*

$$\nvdash p \lor \neg p$$

$$\neg\neg p \nvdash p$$

Natural deduction is *opinionated*

$$\nvdash p \vee \neg p$$

$$\neg\neg p \nvdash p$$

$$\nvdash (p \to q) \vee (q \to r)$$

$$\nvdash (((p \to q)) \to p) \to p$$

‘Textbook’ natural deduction plugs the gap, but it has no *taste*.

$$\dfrac{\begin{matrix}\Pi \\ \neg\neg A\end{matrix}}{A}\ DNE$$

$$\dfrac{\begin{matrix}[\neg A]^i \\ \Pi \\ \bot\end{matrix}}{A}\ \bot E_c$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ C\end{matrix} \qquad \begin{matrix}[\neg A]^j \\ \Pi \\ C\end{matrix}}{C}\ Cases^{i,j}$$

We get *classical logic*, but the rules are no longer separated

$$
\begin{prooftree}
\AxiomC{$[\neg p]^2$}
\AxiomC{$[(p \to q) \to p]^3$}
\AxiomC{$[\neg p]^2$}
\AxiomC{$[p]^1$}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\bot E$}
\UnaryInfC{q}
\RightLabel{$\to I^1$}
\UnaryInfC{$p \to q$}
\RightLabel{$\to E$}
\BinaryInfC{p}
\RightLabel{$\neg E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg\neg p$}
\RightLabel{DNE}
\UnaryInfC{p}
\RightLabel{$\to I^3$}
\UnaryInfC{$((p \to q) \to p) \to p$}
\end{prooftree}
$$

CLASSICAL SEQUENT CALCULUS

$$
\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}\,{\to}R}{(p \to q) \to p \succ p, p}\,{\to}L}{(p \to q) \to p \succ p}\,W}{\succ ((p \to q) \to p) \to p}\,{\to}R
$$

$$\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}\,{\to}R}{(p \to q) \to p \succ p, p}\,{\to}L}{\dfrac{(p \to q) \to p \succ p}{\succ ((p \to q) \to p) \to p}\,{\to}R}\,W$$

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,\lor R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,\land L$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}\,{\to}R}{(p \to q) \to p \succ p, p}\,{\to}L}{(p \to q) \to p \succ p}\,W}{\succ ((p \to q) \to p) \to p}\,{\to}R \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L$$

Classical • *Separated Rules* • *Normalising* • *Analytic*

$$\dfrac{p \succ p \qquad \dfrac{p \succ q, p}{\succ p \to q, p}\,{\to}R}{\dfrac{\dfrac{(p \to q) \to p \succ p, p}{(p \to q) \to p \succ p}\,W}{\succ ((p \to q) \to p) \to p}\,{\to}R}\,{\to}L$$

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,\lor R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,\land L$$

Classical • *Separated Rules* • *Normalising* • *Analytic*

… but what kind of proof does $X \succ Y$ *score*?

Me, in 2005: Not a proof, **but** . . .

MULTIPLE CONCLUSIONS

"Multiple Conclusions," in *Logic, Methodology and Philosophy of Science: Proceedings of the Twelfth International Congress*, edited by Petr Hajek, Luis Valdes-Villanueva and Dag Westerstahl, Kings' College Publications, 2005, 189–205.

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I argue for the following four theses. (1) Denial is not to be analysed as the assertion of a negation. (2) Given the concepts of assertion and denial, we have the resources to analyse logical consequence as relating arguments with multiple premises and multiple conclusions. Gentzen's multiple conclusion calculus can be understood in a straightforward, motivated, non-question-begging way. (3) If a broadly anti-realist or inferentialist justification of a logical system works, it works just as well for classical logic as it does for intuitionistic logic. The special case for an anti-realist justification of intuitionistic logic over and above a justification of classical logic relies on an unjustified assumption about the shape of proofs. Finally, (4) this picture of logical consequence provides a relatively neutral shared vocabulary which can help us understand and adjudicate debates between proponents of classical and non-classical logics.

This paper has now been reprinted in *Analysis and Metaphysics*, 6, 2007, 14-34.

`https://consequently.org/writing/multipleconclusions/`

. . . deriving $X \succ Y$ *does* tell you that it's out of bounds to assert each member of X and deny each member of Y, and that's *something*!

Steinberger on the Principle of Answerability

The mistake in this position, however, resides in the idea that any formal game incorporating what appear to be inference rules will confer meanings on its logical symbols. Adherence to inferentialism importantly constrains one's choice of proof-theoretic frameworks and thus requires one to reject Carnap's amoralism about logic: the inferentialist must remain faithful to our ordinary inferential practice. Only those deductive systems that answer to the use we put our logical vocabulary to fit the bill. After all, it is the practice represented, not the formalism as such, that confers meanings. Therefore, the formalism is of meaning-theoretic significance and hence of interest to the inferentialist only if it succeeds in capturing (in a perhaps idealised form) the relevant meaning-constituting features of our practice. It is in this sense, then, that the inferentialist position imposes strict demands on the form deductive systems may take. For future reference, let us refer to these demands as the

> *Principle of answerability* only such deductive systems are permissible as can be seen to be suitably connected to our ordinary deductive inferential practices.

Florian Steinberger, "Why Conclusions Should Remain Single"
JPL (2011) 40:333–355 `https://dx.doi.org/10.1007/s10992-010-9153-3`

What is a proof of p?

What is a proof of p?

A *proof* of p meets a *justification request* for the assertion of p.

(Not every way to meet a justification request is a *proof*, but proofs meet justification requests in a *very* stringent way.)

A proof of A (in a context)
meets a justification request for A
on the basis of the claims we take for granted.

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at least, not without quite a bit of *work*.

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A sequent calculus derivation doesn't do *that*,
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Is there a way to read a *classical* sequent derivation
as constructing this kind of proof?

$$\dfrac{\dfrac{\dfrac{\dfrac{[-\,p \lor \neg p]^1}{-\,p}\;{\scriptstyle -\lor E}}{+\,\neg p}\;{\scriptstyle +\neg I}}{+\,p \lor \neg p}\;{\scriptstyle +\lor I} \qquad [-\,p \lor \neg p]^2}{+\,p \lor \neg p}\;{\scriptstyle RAA^{1,2}}$$

Decorate your proof with *signs*.

Double up your Rules

$$\dfrac{\begin{array}{c}\Pi\\ +A\end{array}}{+A\vee B}\;{+\vee I} \qquad \dfrac{\begin{array}{c}\Pi\\ +B\end{array}}{+A\vee B}\;{+\vee I} \qquad \dfrac{\begin{array}{c}\Pi\\ +A\vee B\end{array} \quad \begin{array}{c}[+A]^{j}\\ \Pi'\\ \phi\end{array} \quad \begin{array}{c}[+B]^{k}\\ \Pi''\\ \phi\end{array}}{\phi}\;{\vee E^{j,k}}$$

$$\dfrac{\begin{array}{c}\Pi\\ -A\vee B\end{array}}{-A}\;{-\vee E} \qquad \dfrac{\begin{array}{c}\Pi\\ -A\vee B\end{array}}{-B}\;{-\wedge E} \qquad \dfrac{\begin{array}{c}\Pi\\ -A\end{array} \quad \begin{array}{c}\Pi'\\ -B\end{array}}{-A\vee B}\;{-\vee E}$$

$$\dfrac{\begin{array}{c}\Pi\\ -A\end{array}}{+\neg A}\;{+\neg I} \qquad \dfrac{\begin{array}{c}\Pi\\ +\neg A\end{array}}{-A}\;{+\neg E} \qquad \dfrac{\begin{array}{c}\Pi\\ +A\end{array}}{-\neg A}\;{-\neg I} \qquad \dfrac{\begin{array}{c}\Pi\\ -\neg A\end{array}}{+A}\;{-\neg E}$$

Add some 'Structural' Rules

$$\dfrac{\begin{matrix}\Pi \\ \alpha\end{matrix} \qquad \begin{matrix}\Pi' \\ \alpha^*\end{matrix}}{\bot}\ \bot I \qquad\qquad \dfrac{\begin{matrix}[\alpha]^i \\ \Pi \\ \bot\end{matrix}}{\alpha^*}\ \textit{Reductio}^i \qquad\qquad \dfrac{\begin{matrix}[\alpha]^j \\ \Pi' \\ \beta\end{matrix} \qquad \begin{matrix}[\alpha]^k \\ \Pi'' \\ \beta^*\end{matrix}}{\alpha^*}\ SR^{j,k}$$

α and β are signed formulas.

$(-A)^* = +A$ and $(+A)^* = -A$.

The duality of *assertion* and *denial*
are important to the defender of classical logic,
but doubling up *every* connective rule is like
cracking a *small nut* with a *sledgehammer*.

So far …

- Answerability to our practice is a constraint worth meeting.

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- Sequents help *keep score* in a proof.

So far . . .

- Answerability to our practice is a constraint worth meeting.
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- Answerability to our practice is a constraint worth meeting.
- Sequents help *keep score* in a proof.
- *Bilateralism* (paying attention to *assertion* and *denial*) is important to the defender of classical logic.
- Sequent calculus and signed natural deduction do not approach the simplicity of standard natural deduction as an account of *proof*.

ASSERTION, DENIAL, NEGATION AND CONTRADICTION

Are these rules truly *separated*?

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \bot\end{matrix}}{\neg A}\ \neg I^i \qquad \dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{\bot}\ \neg E \qquad \dfrac{\begin{matrix}\Pi \\ \bot\end{matrix}}{A}\ \bot E$$

Are these rules truly *separated*?

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \bot\end{matrix}}{\neg A}\,\neg I^i \qquad \dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{\bot}\,\neg E \qquad \dfrac{\begin{matrix}\Pi \\ \bot\end{matrix}}{A}\,\bot E$$

If $\bot$ is a *formula* we do not have the *subformula property* for normal proofs.

$$\dfrac{\dfrac{\neg p \quad p}{\bot}\,\neg E}{q}\,\bot E$$

In the sequent calculus, it's *structure*, not a *formula*

$$\dfrac{\dfrac{\neg p \qquad p}{\bot}\,\neg E}{q}\,\bot E$$

$$\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,K$$

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In the sequent calculus, it's *structure*, not a *formula*

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,{\neg E}}{q}\,{\sharp E}$$

$$\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,{\neg L}}{\neg p, p \succ q}\,K$$

Following Tennant (*Natural Logic*, 1978),
I'll use "$\sharp$" as a contradiction marker. It's not a formula.

We regain the *subformula property* for normal proofs.

In the sequent calculus, it's *structure*, not a *formula*

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E$$

$$\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,K$$

Following Tennant (*Natural Logic*, 1978),
I'll use "♯" as a contradiction marker. It's not a formula.

We regain the *subformula property* for normal proofs.

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\neg A}\,\neg I^i \qquad \dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{\sharp}\,\neg E \qquad \dfrac{\begin{matrix}\Pi \\ \sharp\end{matrix}}{A}\,\sharp E$$

In the sequent calculus, it's *structure*, not a *formula*

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E \qquad\qquad \dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,K$$

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$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\neg A}\,\neg I^i \qquad \dfrac{\begin{matrix}\Pi \\ \neg A\end{matrix} \qquad \begin{matrix}\Pi' \\ A\end{matrix}}{\sharp}\,\neg E \qquad \dfrac{\begin{matrix}\Pi \\ \sharp\end{matrix}}{A}\,\sharp E \qquad \dfrac{\begin{matrix}\Pi \\ \sharp\end{matrix}}{f}\,fI \qquad \dfrac{\begin{matrix}\Pi \\ f\end{matrix}}{\sharp}\,fE$$

Relevance, Vacuous Discharge, and $\sharp E$

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E$$

$$\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,K$$

Relevance, Vacuous Discharge, and $\sharp E$

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E$$

$$\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,K$$

$$\dfrac{\color{red}{p}}{q \to p}\,{\to}I$$

$$\dfrac{\dfrac{\color{red}{p \succ p}}{p, q \succ p}\,K}{p \succ q \to p}\,{\to}I$$

Relevance, Vacuous Discharge, and $\sharp E$

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E$$

$$\dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,K$$

$$\dfrac{p}{q \to p}\,{\to}I$$

$$\dfrac{\dfrac{p \succ p}{p, q \succ p}\,K}{p \succ q \to p}\,{\to}I$$

Relevance, Vacuous Discharge, and $\sharp E$

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$$\dfrac{p}{q \to p}\,{\to}I$$

$$\dfrac{\dfrac{p \succ p}{p, q \succ p}\,K}{p \succ q \to p}\,{\to}I$$

Relevance, Vacuous Discharge, and $\sharp E$

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,\neg E}{q}\,\sharp E \qquad\qquad \dfrac{\dfrac{p \succ p}{\neg p, p \succ}\,\neg L}{\neg p, p \succ q}\,K$$

$$\dfrac{\color{red}{p}}{q \to p}\,{\to}I \qquad\qquad \dfrac{\dfrac{p \succ p}{p, q \succ p}\,K}{p \succ q \to p}\,{\to}I$$

What connects *vacuous discharge* and $\sharp E$?

In the sequent calculus, they are both *weakening*.

But in natural deduction?

Is ♯ genuinely *structural*?

If ♯ is a genuine *structural* feature of proofs,
why does it feature only in the f and ¬ rules?

If $\sharp$ is a genuine *structural* feature of proofs,
why does it feature only in the f and $\neg$ rules?

For *bilateralists*, the notion of a contradiction
is more fundamental than any particular *connective*.

Asserting and denying the one thing
can *also* lead to a dead end . . .

If $\sharp$ is a genuine *structural* feature of proofs,
why does it feature only in the f and $\neg$ rules?

For *bilateralists*, the notion of a contradiction
is more fundamental than any particular *connective*.

Asserting and denying the one thing
can *also* lead to a dead end …

and so can setting aside the current
conclusion, to look for an *alternative*.

ALTERNATIVES

Adding *alternatives*

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad A^{\uparrow}}{\sharp}\,\uparrow$$

Adding *alternatives*

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad A^{\uparrow}}{\sharp}\ \uparrow \qquad\qquad \dfrac{\begin{matrix}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\ \downarrow^{i}$$

Adding *alternatives*

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad A^{\uparrow}}{\sharp}\ {\uparrow}$$

$$\dfrac{\begin{matrix}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\ {\downarrow^{i}}$$

$$\dfrac{[X : Y] \succ A}{[X : A, Y] \succ \sharp}\ {\uparrow}$$

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad A^{\uparrow}}{\sharp}\,\uparrow \qquad\qquad \dfrac{\begin{matrix}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\,\downarrow^{i}$$

$$\dfrac{[X : Y] \succ A}{[X : A, Y] \succ \sharp}\,\uparrow$$

Adding *alternatives*

$$\dfrac{\begin{array}{c}\Pi\\ A\end{array} \quad A^{\uparrow}}{\sharp}\,\uparrow \qquad\qquad \dfrac{\begin{array}{c}[A^{\uparrow}]^{i}\\ \Pi\\ \sharp\end{array}}{A}\,\downarrow^{i}$$

$$\frac{[X : Y] \succ A}{[X : A, Y] \succ \sharp}\,\uparrow \qquad\qquad \frac{[X : A, Y] \succ \sharp}{[X : Y] \succ A}\,\downarrow$$

Adding *alternatives*

$$\dfrac{\begin{array}{c}\Pi \\ A\end{array} \qquad A^{\uparrow}}{\sharp}\,\uparrow \qquad\qquad \dfrac{\begin{array}{c}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{array}}{A}\,\downarrow^{i}$$

$$\dfrac{[X : Y] \succ A}{[X : A, Y] \succ \sharp}\,\uparrow \qquad\qquad \dfrac{[X : A, Y] \succ \sharp}{[X : Y] \succ A}\,\downarrow$$

Adding *alternatives*

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix} \qquad A^{\uparrow}}{\sharp}\ \uparrow \qquad\qquad \dfrac{\begin{matrix}[A^{\uparrow}]^i \\ \Pi \\ \sharp\end{matrix}}{A}\ \downarrow^i$$

$$\dfrac{X \succ A; Y}{X \succ \sharp; A, Y}\ \uparrow \qquad\qquad \dfrac{X \succ \sharp; A, Y}{X \succ A; Y}\ \downarrow$$

Adding *alternatives*

$$\dfrac{\begin{matrix}\Pi & \\ A & A^{\uparrow}\end{matrix}}{\sharp}\,\uparrow \qquad\qquad \dfrac{\begin{matrix}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\,\downarrow^{i}$$

$$\frac{X \succ A; Y}{X \succ \sharp; A, Y}\,\uparrow \qquad\qquad \frac{X \succ \sharp; A, Y}{X \succ A; Y}\,\downarrow$$

We add the *store* and *retrieve* rules and keep the other rules *fixed*.

The store and retrieve rules are the only rules that *manipulate* alternatives.

A minimal set of rules

$$A \qquad \dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad A^{\uparrow}}{\sharp}\,{\uparrow} \qquad \dfrac{\begin{matrix}[A^{\uparrow}]^i \\ \Pi \\ \sharp\end{matrix}}{A}\,{\downarrow}^i$$

$$\dfrac{\begin{matrix}[A]^i \\ \Pi \\ B\end{matrix}}{A \to B}\,{\to}I^i \qquad \dfrac{\begin{matrix}\Pi \\ A \to B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\,{\to}E \qquad \dfrac{\begin{matrix}\Pi \\ \sharp\end{matrix}}{f}\,fI \qquad \dfrac{\begin{matrix}\Pi \\ f\end{matrix}}{\sharp}\,fE$$

$$[(A \to f) \to B : \,] \succ (B \to f) \to A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \to f]^3 \qquad \dfrac{(A \to f) \to B \qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}{\downarrow}}{f}{fI}}{A \to f}{\to}I^1}{B}{\to}E}{f}{\to}E}{\sharp}{fE}}{A}{\downarrow^2}}{(B \to f) \to A}{\to}I^3$$

Example Proof: Contraposition

$$[(A \to f) \to B : \ ] \succ (B \to f) \to A$$

$$
\begin{prooftree}
\AxiomC{$[B \to f]^3$}
\AxiomC{$(A \to f) \to B$}
\AxiomC{$[A]^1$}
\AxiomC{$[A^{\uparrow}]^2$}
\RightLabel{$\downarrow$}
\BinaryInfC{$\sharp$}
\RightLabel{fI}
\UnaryInfC{f}
\RightLabel{$\to I^1$}
\UnaryInfC{$A \to f$}
\RightLabel{$\to E$}
\BinaryInfC{B}
\RightLabel{$\to E$}
\BinaryInfC{f}
\RightLabel{fE}
\UnaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{A}
\RightLabel{$\to I^3$}
\UnaryInfC{$(B \to f) \to A$}
\end{prooftree}
$$

$$[A : \ ] \succ A$$

Example Proof: Contraposition

$$[(A \to f) \to B : \,] \succ (B \to f) \to A$$

$$\cfrac{\cfrac{\cfrac{\cfrac{[B \to f]^3 \qquad \cfrac{(A \to f) \to B \qquad \cfrac{\cfrac{\cfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}\,{\downarrow}}{f}\,fI}{A \to f}\,{\to}I^1}{B}\,{\to}E}{f}\,{\to}E}{\sharp}\,fE}{A}\,{\downarrow}^2}{(B \to f) \to A}\,{\to}I^3$$

$$[A : A] \succ \sharp$$

Example Proof: Contraposition

$$[(A \to f) \to B : \;] \succ (B \to f) \to A$$

$$\dfrac{\dfrac{[B \to f]^3 \qquad \dfrac{(A \to f) \to B \qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}\,\downarrow}{f}\,fI}{A \to f}\,{\to}I^1}{B}\,{\to}E}{\dfrac{\dfrac{f}{\sharp}\,fE}{A}\,\downarrow^2}\,{\to}E}{(B \to f) \to A}\,{\to}I^3$$

$$[A : A] \succ f$$

Example Proof: Contraposition

$$[(A \to f) \to B : \;] \succ (B \to f) \to A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \to f]^3 \qquad \dfrac{(A \to f) \to B \qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}\downarrow}{f}\,fI}{A \to f}\to I^1}{B}\to E}{f}\to E}{\sharp}\,fE}{A}\downarrow^2}{(B \to f) \to A}\to I^3$$

$$[\; : A] \succ A \to f$$

$$[(A \to f) \to B : \ ] \succ (B \to f) \to A$$

$$\dfrac{\dfrac{\dfrac{[B \to f]^3 \qquad \dfrac{(A \to f) \to B \qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}\,{\downarrow}}{f}\,fI}{A \to f}\,{\to}I^1}{B}\,{\to}E}{\dfrac{f}{\sharp}\,fE}\,{\to}E}{A}\,{\downarrow}^2}{(B \to f) \to A}\,{\to}I^3$$

$$[(A \to f) \to B : A] \succ B$$

Example Proof: Contraposition

$$[(A \to f) \to B : \;] \succ (B \to f) \to A$$

$$\dfrac{\dfrac{\dfrac{[B \to f]^3 \qquad \dfrac{(A \to f) \to B \qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}\,{\downarrow}}{f}\,fI}{A \to f}\,{\to}I^1}{B}\,{\to}E}{\dfrac{\dfrac{f}{\sharp}\,fE}{A}\,{\downarrow}^2}\,{\to}E}{}}{(B \to f) \to A}\,{\to}I^3$$

$$[B \to f, (A \to f) \to B : A] \succ f$$

Example Proof: Contraposition

$$[(A \to \mathsf{f}) \to B : \ ] \succ (B \to \mathsf{f}) \to A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \to \mathsf{f}]^3 \qquad \dfrac{(A \to \mathsf{f}) \to B \qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}\downarrow}{\mathsf{f}}\,\mathsf{f}I}{A \to \mathsf{f}}\to I^1}{B}\to E}{\mathsf{f}}\to E}{\sharp}\,\mathsf{f}E}{A}\downarrow^2}{(B \to \mathsf{f}) \to A}\to I^3$$

$$[B \to \mathsf{f}, (A \to \mathsf{f}) \to B : A] \succ \sharp$$

Example Proof: Contraposition

$$[(A \to f) \to B : \,] \succ (B \to f) \to A$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[B \to f]^3 \qquad \dfrac{(A \to f) \to B \qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad [A^{\uparrow}]^2}{\sharp}\,\downarrow}{f}\,fI}{A \to f}\,{\to}I^1}{B}\,{\to}E}{f}\,{\to}E}{\sharp}\,fE}{A}\,\downarrow^2}{(B \to f) \to A}\,{\to}I^3
$$

$$[B \to f, (A \to f) \to B : \,] \succ A$$

Example Proof: Contraposition

$$[(A \to f) \to B : \ ] \succ (B \to f) \to A$$

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\UnaryInfC{$A \to f$}
\RightLabel{$\to E$}
\BinaryInfC{B}
\RightLabel{$\to E$}
\BinaryInfC{f}
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\UnaryInfC{$\sharp$}
\RightLabel{$\downarrow^2$}
\UnaryInfC{A}
\RightLabel{$\to I^3$}
\UnaryInfC{$(B \to f) \to A$}
\end{prooftree}
$$

$$[(A \to f) \to B : \ ] \succ (B \to f) \to A$$

$$[(p \to q) \to p : \,] \succ p$$

$$\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [p^{\uparrow}]^2}{\sharp}\,\uparrow}{q}\,\downarrow}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^{\uparrow}]^2}{\dfrac{\sharp}{p}\,\downarrow^2}\,\uparrow$$

Example Proof: Peirce's Law

$$[(p \to q) \to p : \,] \succ p$$

$$\dfrac{\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [p^{\uparrow}]^2}{\sharp}\,\uparrow}{q}\,\downarrow}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^{\uparrow}]^2}{\sharp}\,\uparrow}{p}\,\downarrow^2$$

This proof exhibits both *duplicate* and *vacuous* retrieval.

$$[(p \to q) \to p : \ ] \succ p$$

$$\dfrac{\dfrac{\dfrac{(p \to q) \to p \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [p^{\uparrow}]^2}{\sharp}\,\uparrow}{q}\,\downarrow}{p \to q}\,{\to}I^1}{p}\,{\to}E \qquad [p^{\uparrow}]^2}{\sharp}\,\uparrow}{p}\,\downarrow^2$$

This proof exhibits both *duplicate* and *vacuous* retrieval.

The Unity of Relevance

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,{\scriptstyle \neg E}}{q}\,{\scriptstyle \sharp E} \qquad\qquad \dfrac{p}{q \to p}\,{\scriptstyle \to I}$$

$\sharp E$ is nothing other than *vacuous retrieval*.

$$\dfrac{\dfrac{\neg p \qquad p}{\sharp}\,{\scriptstyle \neg E}}{q}\,{\color{red}\downarrow} \qquad\qquad \dfrac{p}{q \to p}\,{\color{red}{\to}I}$$

1. COMPLETENESS:

Completeness and Soundness — for classical logic

1. COMPLETENESS: Trivial. It's intuitionistic logic + Peirce's Law

1. COMPLETENESS: Trivial. It's intuitionistic logic + Peirce's Law
2. SOUNDNESS: Easy induction. If we have a proof for $[X : Y] \succ A$ then in any Boolean valuation v where $v(X) = 1$ and $v(Y) = 0$ then $v(A) = 1$.

When I set a current conclusion aside as an *alternative*, I temporarily (for the sake of the argument) deny it, to consider a different option in its place.

This is very *mildly* bilateral, but not so much that it litters every formula in a proof with a sign.

Classical (and *Linear*, *Relevant*, and *Affine*, too)

Separated Rules • *Normalising*

Analytic • *Single Conclusion* • *Answerable*

NORMALISATION AND ITS CONSEQUENCES

Flattening Local Peaks: $\to I / \to E$

$$\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ B\end{array}}{A \to B}\,{\to}I^i \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{B}\,{\to}E$$

Flattening Local Peaks: $\to I/\to E$

$$\dfrac{\dfrac{\begin{matrix}[A]^i \\ \Pi_1 \\ B\end{matrix}}{\boxed{A \to B}}\to I^i \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}\to E \qquad \rightsquigarrow_{\to I/\to E} \qquad \begin{matrix}\Pi_2 \\ A \\ \Pi_1 \\ B\end{matrix}$$

$$\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ B\end{array}}{\boxed{A \to B}}{\scriptstyle \to I^i} \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{B}{\scriptstyle \to E} \qquad \leadsto_{\to I/\to E} \qquad \begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ B\end{array}$$

NOTE: if the original proof satisfies a given discharge policy, so does its reduction. To show this we need to ensure that duplicate/vacuous discharge is banned whenever duplicate/vacuous retrieval is banned.

Flattening Local Peaks: $\mathsf{f}I/\mathsf{f}E$

$$\cfrac{\cfrac{\begin{array}{c}\Pi\\ \sharp\end{array}}{\boxed{\mathsf{f}}}\;\mathsf{f}I}{\sharp}\;\mathsf{f}E$$

Flattening Local Peaks: $\mathsf{f}I/\mathsf{f}E$

$$
\dfrac{\dfrac{\begin{array}{c}\Pi\\ \sharp\end{array}}{\boxed{\mathsf{f}}}\,\mathsf{f}I}{\sharp}\,\mathsf{f}E
\quad\rightsquigarrow_{\mathsf{f}I/\mathsf{f}E}\quad
\begin{array}{c}\Pi\\ \sharp\end{array}
$$

Flattening Local Peaks: $\downarrow/\uparrow$

$$
\dfrac{\dfrac{\begin{matrix}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\,{\downarrow^{i}} \qquad A^{\uparrow}}{\sharp}\,{\downarrow}
$$

Flattening Local Peaks: ↓/↑

$$\dfrac{\dfrac{\begin{array}{c}[A^{\uparrow}]^{i}\\ \Pi\\ \sharp\end{array}}{\boxed{A}}\,{\downarrow^{i}} \qquad A^{\uparrow}}{\sharp}\,{\downarrow} \qquad \rightsquigarrow_{\downarrow/\uparrow} \qquad \begin{array}{c}A^{\uparrow}\\ \Pi\\ \sharp\end{array}$$

One more case to consider …

What about $\to I/{\uparrow}/{\downarrow}/{\to}E$ sequences?

Flattening Local Peaks: $\downarrow/{\to}E$

$$\dfrac{\dfrac{\begin{matrix}[A \to B^{\uparrow}]^i \\ \Pi_1 \\ \sharp\end{matrix}}{\boxed{A \to B}}\,{\downarrow^i} \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}\,{\to}E$$

Flattening Local Peaks: $\downarrow/{\to}E$

$$\dfrac{\dfrac{\begin{matrix}[A \to B^{\uparrow}]^i \\ \Pi_1 \\ \sharp\end{matrix}}{A \to B}\,{\downarrow^i} \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}\,{\to}E \qquad \rightsquigarrow_{\downarrow/\to E} \qquad \dfrac{\begin{matrix}[B^{\uparrow}]^j \\ \Pi_1^* \\ \sharp\end{matrix}}{B}\,{\downarrow^j}$$

Where Π_1^* is the proof:

$$\Pi_1\left[\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad A \to B^{\uparrow}}{\sharp}\,{\uparrow^i} := \dfrac{\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}\,{\to}E \qquad B^{\uparrow}}{\sharp}\,{\uparrow^j}\right]$$

$$
\dfrac{\dfrac{\begin{matrix}[A \rightarrow B^{\uparrow}]^{i} \\ \Pi_1 \\ \sharp\end{matrix}}{\boxed{A \rightarrow B}}\ {\downarrow^{i}} \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}\ {\rightarrow}E
\quad \rightsquigarrow_{\downarrow/\rightarrow E} \quad
\dfrac{\begin{matrix}[B^{\uparrow}]^{j} \\ \Pi_1^{*} \\ \sharp\end{matrix}}{B}\ {\downarrow^{j}}
$$

Where Π_1^* is the proof:

$$
\Pi_1\left[\dfrac{\begin{matrix}\vdots \\ A \rightarrow B\end{matrix} \qquad A \rightarrow B^{\uparrow}}{\sharp}\ {\uparrow^{i}} \;:=\; \dfrac{\dfrac{\begin{matrix}\vdots \\ A \rightarrow B\end{matrix} \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}\ {\rightarrow}E \qquad B^{\uparrow}}{\sharp}\ {\uparrow^{j}}\right]
$$

NOTE: if the original proof satisfies a given discharge policy, so does its reduction. To show this we need to ensure that duplicate/vacuous retrieval is banned whenever duplicate/vacuous discharge is banned.

Normalisation and *Strong* Normalisation

It's straightforward to show that any proof Π can be transformed, in some finite series of reduction steps, into a proof Π', to which no reduction applies.

Such a proof is *normal*.

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Normalisation and *Strong* Normalisation

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Such a proof is *normal*.

It's less straightforward to show that *any* sequence of reduction steps applied to a proof Π will terminate, after finitely many steps. (Parigot *JSL* 1997)

Normal proofs are *analytic*. Every formula in a normal proof from $[X : Y]$ to A is a subformula of sone formula in X, Y or A.

Further reductions: $\to E / \to I$

$$\dfrac{\dfrac{\dfrac{\Pi}{A \to B} \qquad [A]^i}{\boxed{B}}\ {\to}E}{A \to B}\ {\to}I^i$$

Further reductions: $\rightarrow E/\rightarrow I$

$$\dfrac{\dfrac{\overset{\Pi}{A \rightarrow B} \qquad [A]^i}{\boxed{B}}\;{\scriptstyle \rightarrow E}}{A \rightarrow B}\;{\scriptstyle \rightarrow I^i} \quad \rightsquigarrow_{\rightarrow E/\rightarrow I} \quad \overset{\Pi}{A \rightarrow B}$$

Further reductions: $\mathsf{f}E/\mathsf{f}I$

$$\dfrac{\dfrac{\begin{array}{c}\Pi \\ \mathsf{f}\end{array}}{\boxed{\sharp}}\;\mathsf{f}E}{\mathsf{f}}\;\mathsf{f}I$$

Further reductions: $\mathsf{f}E/\mathsf{f}I$

$$\dfrac{\dfrac{\begin{matrix}\Pi \\ \mathsf{f}\end{matrix}}{\sharp}\,\mathsf{f}E}{\mathsf{f}}\,\mathsf{f}I \quad \rightsquigarrow_{\mathsf{f}E/\mathsf{f}I} \quad \begin{matrix}\Pi \\ \mathsf{f}\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ A\end{matrix}\qquad [A^{\uparrow}]^{i}}{\sharp}\,\uparrow}{A}\,\downarrow^{i}$$

Further reductions: ↑/↓

$$
\cfrac{\cfrac{\begin{array}{c}\Pi\\ A\end{array} \qquad [A^{\uparrow}]^{i}}{\sharp}\ \uparrow}{A}\ \downarrow^{i}
\qquad \rightsquigarrow_{\uparrow/\downarrow} \qquad
\begin{array}{c}\Pi\\ A\end{array}
$$

The rules

$$A \qquad \dfrac{\begin{array}{cc}\Pi & \\ A & A^{\uparrow}\end{array}}{\sharp}\,\uparrow \qquad \dfrac{\begin{array}{c}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{array}}{A}\,\downarrow^{i}$$

$$\dfrac{\begin{array}{c}[A]^{i} \\ \Pi \\ B\end{array}}{A \to B}\,{\to}I^{i} \qquad \dfrac{\begin{array}{cc}\Pi & \Pi' \\ A \to B & A\end{array}}{B}\,{\to}E \qquad \dfrac{\begin{array}{c}\Pi \\ \sharp\end{array}}{f}\,fI \qquad \dfrac{\begin{array}{c}\Pi \\ f\end{array}}{\sharp}\,fE$$

Defining Negation: $\neg A =_{df} A \to \mathsf{f}$

$$\dfrac{\dfrac{A \to \mathsf{f} \qquad A}{\mathsf{f}}\,{\to}E}{\sharp}\,\mathsf{f}E \qquad\qquad \dfrac{\neg A \qquad A}{\sharp}\,\neg E$$

Defining Negation: $\neg A =_{\text{df}} A \to \mathsf{f}$

$$\cfrac{\cfrac{A \to \mathsf{f} \qquad A}{\cfrac{\mathsf{f}}{\sharp}\,\mathsf{f}E}}{}\;{\to}E$$

$$\frac{\neg A \qquad A}{\sharp}\;\neg E$$

$$\cfrac{\cfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\mathsf{f}}\;\mathsf{f}I}{A \to \mathsf{f}}\;{\to}I^i$$

$$\cfrac{\begin{matrix}[A]^i \\ \Pi \\ \sharp\end{matrix}}{\neg A}\;\neg I^i$$

Negation Reductions: $\neg I/\neg E$

$$\dfrac{\dfrac{\begin{matrix}[A]^i \\ \Pi_1 \\ \sharp\end{matrix}}{\neg A}\,\neg I^i \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{\sharp}\,\neg E \qquad \leadsto_{\neg I/\neg E} \qquad \begin{matrix}\Pi_2 \\ A \\ \Pi_1 \\ \sharp\end{matrix}$$

Negation Reductions: $\neg I/\neg E$

$$\dfrac{\dfrac{\begin{matrix}[A]^{i}\\ \Pi_1\\ \sharp\end{matrix}}{\neg A}\,\neg I^{i} \qquad \begin{matrix}\Pi_2\\ A\end{matrix}}{\sharp}\,\neg E \qquad \rightsquigarrow_{\neg I/\neg E} \qquad \begin{matrix}\Pi_2\\ A\\ \Pi_1\\ \sharp\end{matrix}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[A]^{i}\\ \Pi_1\\ \sharp\end{matrix}}{\mathsf{f}}\,\mathsf{f}I}{A \to \mathsf{f}}\,{\to}I^{i} \qquad \begin{matrix}\Pi_2\\ A\end{matrix}}{\mathsf{f}}\,{\to}E}{\sharp}\,\mathsf{f}E$$

$$\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ \sharp\end{array}}{\neg A}\,\neg I^i \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{\sharp}\,\neg E \qquad \rightsquigarrow_{\neg I/\neg E} \qquad \begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ \sharp\end{array}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ \sharp\end{array}}{\mathsf{f}}\,\mathsf{f}I}{A \to \mathsf{f}}\,{\to}I^i \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{\dfrac{\mathsf{f}}{\sharp}\,\mathsf{f}E}\,{\to}E \qquad \rightsquigarrow_{\to I/\to E} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ \sharp\end{array}}{\mathsf{f}}\,\mathsf{f}I}{\sharp}\,\mathsf{f}E$$

$$\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ \sharp\end{array}}{\neg A}\,{\neg I^i} \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{\sharp}\,{\neg E} \qquad \rightsquigarrow_{\neg I/\neg E} \qquad \begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ \sharp\end{array}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \Pi_1\\ \sharp\end{array}}{\mathsf{f}}\,{\mathsf{f}I}}{A \to \mathsf{f}}\,{\to I^i} \qquad \begin{array}{c}\Pi_2\\ A\end{array}}{\dfrac{\mathsf{f}}{\sharp}\,{\mathsf{f}E}}\,{\to E} \qquad \rightsquigarrow_{\to I/\to E} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ \sharp\end{array}}{\mathsf{f}}\,{\mathsf{f}I}}{\sharp}\,{\mathsf{f}E} \qquad \rightsquigarrow_{\mathsf{f}I/\mathsf{f}E} \qquad \begin{array}{c}\Pi_2\\ A\\ \Pi_1\\ \sharp\end{array}$$

$$\dfrac{\neg A \rightarrow B \qquad \dfrac{\dfrac{[A]^1 \qquad A^{\uparrow}}{\sharp}\,{\uparrow}}{\neg A}\,{\neg I^1}}{B}\,{\rightarrow E}$$

$$\dfrac{A \oplus B \qquad A^{\uparrow}}{B}\,{\oplus E}$$

Defining Disjunction: $A \oplus B =_{df} \neg A \to B$

$$\dfrac{\neg A \to B \qquad \dfrac{\dfrac{[A]^1 \qquad A^{\uparrow}}{\sharp}\,\uparrow}{\neg A}\,\neg I^1}{B}\,\to E$$

$$\dfrac{A \oplus B \qquad A^{\uparrow}}{B}\,\oplus E$$

$$\dfrac{\dfrac{\dfrac{[\neg A]^3 \qquad \dfrac{\dfrac{\begin{matrix}[A^{\uparrow}]^1 \\ \Pi \\ B\end{matrix} \qquad [B^{\uparrow}]^2}{\sharp}\,\uparrow}{A}\,\downarrow^1}{\sharp}\,\neg E}{B}\,\downarrow^2}{\neg A \to B}\,\to I^3$$

$$\dfrac{\begin{matrix}[A^{\uparrow}]^1 \\ \Pi \\ B\end{matrix}}{A \oplus B}\,\oplus I^1$$

Disjunction Reductions: $\oplus I/\oplus E$

$$\dfrac{\dfrac{\begin{matrix}[A^{\uparrow}]^1 \\ \Pi \\ B\end{matrix}}{\boxed{A \oplus B}}\oplus I^1 \qquad A^{\uparrow}}{B}\oplus E \quad \rightsquigarrow_{\oplus I/\oplus E} \quad \begin{matrix}A^{\uparrow} \\ \Pi \\ B\end{matrix}$$

Disjunction Reductions: $\oplus I/\oplus E$

$$\dfrac{\dfrac{\begin{matrix}[A^{\uparrow}]^1\\ \Pi\\ B\end{matrix}}{A \oplus B}{\scriptstyle \oplus I^1} \qquad A^{\uparrow}}{B}{\scriptstyle \oplus E} \quad \rightsquigarrow_{\oplus I/\oplus E} \quad \begin{matrix}A^{\uparrow}\\ \Pi\\ B\end{matrix}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg A]^3 \qquad \dfrac{\dfrac{\begin{matrix}[A^{\uparrow}]^1\\ \Pi\\ B\end{matrix} \qquad [B^{\uparrow}]^2}{\sharp}{\scriptstyle \uparrow}}{A}{\scriptstyle \downarrow^1}}{\sharp}{\scriptstyle \neg E}}{B}{\scriptstyle \downarrow^2}}{\neg A \to B}{\scriptstyle \to I^3} \qquad \dfrac{\dfrac{[A]^4 \qquad A^{\uparrow}}{\sharp}{\scriptstyle \uparrow}}{\neg A}{\scriptstyle \neg I^4}}{B}{\scriptstyle \to E} \quad \rightsquigarrow_{\to I/\to E}$$

Disjunction Reductions: $\oplus I/\oplus E$

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^4 \qquad A^{\uparrow}}{\sharp}\,\uparrow}{\neg A}\,\neg I^4 \qquad \dfrac{\dfrac{\begin{matrix}[A^{\uparrow}]^1\\ \Pi \\ B\end{matrix} \qquad [B^{\uparrow}]^2}{\sharp}\,\uparrow}{A}\,\downarrow^1}{\sharp}\,\neg E}{B}\,\downarrow^2 \qquad \rightsquigarrow_{\neg I/\neg E}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{matrix}[A^{\uparrow}]^1\\ \Pi \\ B\end{matrix} \qquad [B^{\uparrow}]^2}{\sharp}\,\uparrow}{A}\,\downarrow^1 \qquad A^{\uparrow}}{\sharp}\,\uparrow}{B}\,\downarrow^2 \qquad \rightsquigarrow_{\downarrow/\uparrow}$$

Disjunction Reductions: $\oplus I/\oplus E$

$$\dfrac{\dfrac{\begin{array}{c}A^{\uparrow}\\ \Pi\\ B\end{array} \qquad [B^{\uparrow}]^2}{\sharp}\,\uparrow}{B}\,\downarrow^2 \qquad \rightsquigarrow_{\uparrow/\downarrow}$$

$$\cfrac{\cfrac{\begin{array}{c}A^{\uparrow}\\ \Pi\\ B\end{array} \qquad [B^{\uparrow}]^{2}}{\sharp}\,\uparrow}{B}\,\downarrow^{2} \qquad \rightsquigarrow_{\uparrow/\downarrow} \qquad \begin{array}{c}A^{\uparrow}\\ \Pi\\ B\end{array}$$

Defining Conjunction: $A \otimes B =_{\mathrm{df}} \neg(A \to \neg B)$

$$\dfrac{\dfrac{\dfrac{[A \to \neg B]^1 \qquad A}{\neg B}\,{\to}E \qquad B}{\sharp}\,\neg E}{\neg(A \to \neg B)}\,\neg I^1$$

$$\dfrac{A \qquad B}{A \otimes B}\,\otimes I$$

Defining Conjunction: $A \otimes B =_{\mathrm{df}} \neg(A \to \neg B)$

$$\dfrac{\dfrac{\dfrac{[A \to \neg B]^1 \qquad A}{\neg B}{\to}E \qquad B}{\sharp}\neg E}{\neg(A \to \neg B)}\neg I^1$$

$$\dfrac{A \qquad B}{A \otimes B}\otimes I$$

$$\dfrac{\dfrac{\neg(A \to \neg B) \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[A]^1 \quad [B]^2 \\ \Pi \\ C\end{matrix} \qquad [C^{\uparrow}]^3}{\sharp}\uparrow}{\neg B}\neg I^2}{A \to \neg B}{\to}I^1}{\sharp}\neg E}{C}\downarrow^3$$

$$\dfrac{A \otimes B \qquad \begin{matrix}[A]^1 \quad [B]^2 \\ \Pi \\ C\end{matrix}}{C}\otimes E^{1,2}$$

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\end{matrix} \qquad \begin{matrix}\Pi_2\\ B\end{matrix}}{A \otimes B}\,\otimes I \qquad \begin{matrix}[A]^1 & & [B]^2\\ & \Pi & \\ & C & \end{matrix}}{C}\,\otimes E^{1,2}
\qquad \rightsquigarrow_{\otimes I/\otimes E} \qquad
\begin{matrix}\Pi_1 & & \Pi_2\\ A & & B\\ & \Pi & \\ & C & \end{matrix}
$$

Conjunction Reductions: $\otimes I/\otimes E$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\end{matrix}\qquad\begin{matrix}\Pi_2\\ B\end{matrix}}{A\otimes B}\otimes I\qquad\begin{matrix}[A]^1\quad[B]^2\\ \Pi\\ C\end{matrix}}{C}\otimes E^{1,2}\quad\rightsquigarrow_{\otimes I/\otimes E}\quad\begin{matrix}\begin{matrix}\Pi_1\\ A\end{matrix}\qquad\begin{matrix}\Pi_2\\ B\end{matrix}\\ \Pi\\ C\end{matrix}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A\to\neg B]^1\qquad\begin{matrix}\Pi_1\\ A\end{matrix}}{\neg B}\to E\qquad\begin{matrix}\Pi_2\\ B\end{matrix}}{\sharp}\neg E}{\neg(A\to\neg B)}\neg I^1\qquad\dfrac{\dfrac{\dfrac{\begin{matrix}[A]^1\quad[B]^2\\ \Pi\\ C\end{matrix}\qquad[C^{\uparrow}]^3}{\sharp}\uparrow}{\neg B}\neg I^2}{A\to\neg B}\to I^1}{\sharp}\neg E}{C}\downarrow^3$$

$$
\begin{prooftree}
\AxiomC{Π_1}
\noLine
\UnaryInfC{A}
\AxiomC{Π_2}
\noLine
\UnaryInfC{B}
\RightLabel{$\otimes I$}
\BinaryInfC{$A \otimes B$}
\AxiomC{$[A]^1 \quad [B]^2$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{C}
\RightLabel{$\otimes E^{1,2}$}
\BinaryInfC{C}
\end{prooftree}
\quad \rightsquigarrow_{\otimes I/\otimes E} \quad
\begin{matrix} \Pi_1 & & \Pi_2 \\ A & & B \\ & \Pi & \\ & C & \end{matrix}
$$

$$
\begin{prooftree}
\AxiomC{$[A \to \neg B]^1$}
\AxiomC{Π_1}
\noLine
\UnaryInfC{A}
\RightLabel{$\to E$}
\BinaryInfC{$\neg B$}
\AxiomC{Π_2}
\noLine
\UnaryInfC{B}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^1$}
\UnaryInfC{$\neg(A \to \neg B)$}
\AxiomC{$[A]^1 \quad [B]^2$}
\noLine
\UnaryInfC{Π}
\noLine
\UnaryInfC{C}
\AxiomC{$[C^{\uparrow}]^3$}
\RightLabel{$\uparrow$}
\BinaryInfC{$\sharp$}
\RightLabel{$\neg I^2$}
\UnaryInfC{$\neg B$}
\RightLabel{$\to I^1$}
\UnaryInfC{$A \to \neg B$}
\RightLabel{$\neg E$}
\BinaryInfC{$\sharp$}
\RightLabel{$\downarrow^3$}
\UnaryInfC{C}
\end{prooftree}
$$

This reduces as desired, using $\neg I/\neg E$, $\to I/\to E$, $\neg I/\neg E$ and $\uparrow/\downarrow$ reductions.

Reduction Steps for the full vocabulary

$\to I/\to E$, $\mathsf{f}I/\mathsf{f}E$, $\downarrow/\uparrow$, $\downarrow/\to E$ and $\uparrow/\downarrow$

OPERATIONAL RULES AS *DEFINITIONS*

$$A \qquad \dfrac{\begin{matrix}\Pi \\ A\end{matrix} \quad A^{\uparrow}}{\sharp}\,\uparrow \qquad \dfrac{\begin{matrix}[A^{\uparrow}]^{i} \\ \Pi \\ \sharp\end{matrix}}{A}\,\downarrow^{i}$$

$$\dfrac{\begin{matrix}[A]^{i} \\ \Pi \\ B\end{matrix}}{A \rightarrow B}\,{\rightarrow}I^{i} \qquad \dfrac{\begin{matrix}\Pi \\ A \rightarrow B\end{matrix} \quad \begin{matrix}\Pi' \\ A\end{matrix}}{B}\,{\rightarrow}E \qquad \dfrac{\begin{matrix}\Pi \\ \sharp\end{matrix}}{f}\,fI \qquad \dfrac{\begin{matrix}\Pi \\ f\end{matrix}}{\sharp}\,fE$$

In what sense can the I/E rules for a concept be understood as *defining* it?

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GENERALITY AND EXISTENCE 1: QUANTIFICATION AND FREE LOGIC

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Abstract. In this paper, I motivate a cut free sequent calculus for classical logic with first order quantification, allowing for singular terms free of existential import. Along the way, I motivate a criterion for rules designed to answer Prior's question about what distinguishes rules for logical concepts, like conjunction from apparently similar rules for putative concepts like Prior's *tonk*, and I show that the rules for the quantifiers—and the existence predicate—satisfy that condition.

`https://consequently.org/writing/generality-and-existence-1/`

Examples of Defining Rules

$$\frac{X \succ A, B, Y}{X \succ A \oplus B, Y}\ \oplus Df \qquad \frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\ \otimes Df \qquad \frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg Df$$

$$\frac{X \succ Y}{X \succ \mathsf{f}, Y}\ \mathsf{f}Df \qquad \frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y}\ {\rightarrow}Df \qquad \frac{X, A \succ B, Y}{X, A - B \succ Y}\ {-}Df$$

This gives the conditions under which an assertion [*left side*] or denial [*right side*] of the formula is out of bounds.

Each Defining Rule (using *Cut*/*Id*) gives rise to Left/Right Rules

$$\begin{array}{c} X \succ A, Y \\ \hline\hline X, \neg A \succ Y \end{array}\ {\neg Df} \qquad \frac{X \succ A, Y}{X, \neg A \succ Y}\ {\neg L} \qquad \frac{X, A \succ Y}{X \succ \neg A, Y}\ {\neg R}$$

$$\begin{array}{c} X, A, B \succ Y \\ \hline\hline X, A \otimes B \succ Y \end{array}\ {\otimes Df} \qquad \frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\ {\otimes L} \qquad \frac{X \succ A, Y \qquad X' \succ B, Y'}{X, X' \succ A \otimes B, Y, Y'}\ {\otimes R}$$

$$\begin{array}{c} X \succ A, B, Y \\ \hline\hline X \succ A \oplus B, Y \end{array}\ {\oplus Df} \qquad \frac{X, A \succ Y \qquad X', B \succ Y'}{X, X', A \oplus B \succ Y, Y'}\ {\oplus L} \qquad \frac{X \succ A, B, Y}{X \succ A \oplus B, Y}\ {\oplus R}$$

The Left/Right rules arising in this way admit a straightforward *Cut*-elimination proof.

Conservative Extension

$$\mathrm{Df} + \mathit{Cut} + \mathit{Id} \quad \leftrightarrow \quad \mathrm{L/R} + \mathit{Cut} + \mathit{Id} \quad \leftrightarrow \quad \mathrm{L/R} + \mathit{Id}$$

Adding Focus

$$\frac{X \succ A; Y}{X, \neg A \succ \sharp; Y} \neg FDf \qquad \frac{X \succ A; Y}{X, \neg A \succ \sharp; Y} \neg FL \qquad \frac{X, A \succ \sharp; Y}{X \succ \neg A; Y} \neg FR$$

$$\frac{X, A, B \succ C; Y}{X, A \otimes B \succ C; Y} \otimes FDf \qquad \frac{X, A, B \succ C; Y}{X, A \otimes B \succ C; Y} \otimes FL \qquad \frac{X \succ A; Y \quad X' \succ B; Y'}{X, X' \succ A \otimes B; Y, Y'} \otimes FR$$

$$\frac{X \succ A; B, Y}{X \succ A \oplus B; Y} \oplus FDf \qquad \frac{X, A \succ \sharp; Y \quad X', B \succ \sharp; Y'}{X, X', A \oplus B \succ \sharp; Y, Y'} \oplus FL \qquad \frac{X \succ A; B, Y}{X \succ A \oplus B; Y} \oplus FR$$

$$\frac{X \succ B; A, Y}{X \succ A \oplus B; Y} \oplus FDf' \qquad \frac{X, A \succ \sharp; Y \quad X', B \succ \sharp; Y'}{X, X', A \oplus B \succ \sharp; Y, Y'} \oplus FL' \qquad \frac{X \succ B; A, Y}{X \succ A \oplus B; Y} \oplus FR'$$

There is more than one way to add focus to a defining rule or a pair of left/right rules.

Conservative Extension

$$\begin{array}{ccccc}
\mathrm{Df} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Id} \\
\updownarrow & & \updownarrow & & \updownarrow \\
F\mathrm{Df} + F\mathit{Cut} + F\mathit{Id} + \uparrow/\downarrow & & F\mathrm{L}/F\mathrm{R} + F\mathit{Cut} + F\mathit{Id} + \uparrow/\downarrow & & F\mathrm{L}/F\mathrm{R} + F\mathit{Id} + \uparrow/\downarrow
\end{array}$$

FL/*FR* Rules can be read as *I*/*E* rules

$$\frac{X \succ A; Y}{X, \neg A \succ \sharp; Y}\ \neg FL \qquad \frac{X, A \succ \sharp; Y}{X \succ \neg A; Y}\ \neg FR$$

$$\frac{\neg A \qquad \begin{array}{c}\Pi \\ A\end{array}}{\sharp}\ \neg E \qquad \frac{\begin{array}{c}[A]^i \\ \Pi \\ \sharp\end{array}}{\neg A}\ \neg I^i$$

… and the *E* rules have the major premise as an *assumption*.

The proofs generated from these rules (without using *Cut*) are *normal*.

Conservative Extension

$$
\begin{array}{ccccc}
\mathrm{Df} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Id} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathit{F}\mathrm{Df} + \mathit{FCut} + \mathit{FId} + \uparrow/\downarrow & & \mathit{F}\mathrm{L}/\mathit{F}\mathrm{R} + \mathit{FCut} + \mathit{FId} + \uparrow/\downarrow & & \mathit{F}\mathrm{L}/\mathit{F}\mathrm{R} + \mathit{FId} + \uparrow/\downarrow \\
 & & \updownarrow & & \updownarrow \\
 & & \mathit{I/E}\ \text{Proofs} & & \mathit{Normal}\ \mathit{I/E}\ \text{proofs}
\end{array}
$$

Conservative Extension

$$\begin{array}{ccccc}
\mathrm{Df} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Id} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathit{FDf} + \mathit{FCut} + \mathit{FId} + \uparrow/\downarrow & & \mathit{FL/FR} + \mathit{FCut} + \mathit{FId} + \uparrow/\downarrow & & \mathit{FL/FR} + \mathit{FId} + \uparrow/\downarrow \\
 & & \updownarrow & & \updownarrow \\
 & & \mathit{I/E}\ \text{Proofs} & & \mathit{Normal}\ \mathit{I/E}\ \text{proofs}
\end{array}$$

We have a *systematic* conservative extension result
showing how any concept given by a *defining rule*
can be given *I/E* rules that admit normalisation.

This means new concepts conservatively extend the old vocabulary.

Conservative Extension

$$\begin{array}{ccccc}
\mathrm{Df} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Cut} + \mathit{Id} & \leftrightarrow & \mathrm{L/R} + \mathit{Id} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathit{FDf} + \mathit{FCut} + \mathit{FId} + \uparrow/\downarrow & & \mathit{FL/FR} + \mathit{FCut} + \mathit{FId} + \uparrow/\downarrow & & \mathit{FL/FR} + \mathit{FId} + \uparrow/\downarrow \\
 & & \updownarrow & & \updownarrow \\
 & & \mathit{I/E}\ \text{Proofs} & & \mathit{Normal}\ \mathit{I/E}\ \text{proofs}
\end{array}$$

We have a *systematic* conservative extension result showing how any concept given by a *defining rule* can be given *I/E* rules that admit normalisation.

This means new concepts conservatively extend the old vocabulary.

HOMEWORK: Prove this *directly*, without the detour through the sequent calculus.

How (good) operational rules can *define*

Defining rules *with focus* settle the bounds for assertion and denial of the concepts they govern, and they also show in a systematic way how to meet *justification requests* for judgements involving those concepts.

They do this in a way that this *conservative* and *uniquely defining*.

What more could you want?

THANK YOU!

References and Further Reading

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Thank you!

SLIDES: `https://consequently.org/presentation/2020/natural-deduction-with-alternatives`

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