MODAL LOGIC AND CONTINGENT EXISTENCE
(GENERALITY AND EXISTENCE 2)

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Abstract: In this paper, I defend *contingentism*, the natural idea that some things exist contingently. Had my parents not met, I would not have existed. It is perhaps surprising that an everyday idea like contingentism needs defence, but natural reasoning principles concerning possibility and necessity on the one hand, and the existential and universal quantifiers on the other, have led some to *necessitism*, the alternate view, that everything that exists, exists necessarily. ¶ Almost all recent work on the semantics of possibility, necessity and the quantifiers—and its metaphysics—makes essential use of possible worlds models. These models have proved useful for analysing the formal and structural properties of modal logics, but it is less clear that these models help fix the meaning of our modal vocabulary, given that we have no grasp of what counts as a possible world independent of our grasp of what counts as possible. In this paper, I develop an alternate *inferentialist* semantics for the modal and quantificational vocabulary, not as a rival to possible worlds models, but as an explanation of how it is that the concepts we do employ can be modelled using possible worlds. I then use this inferentialist semantics to clarify the contingentist's commitments, and offer answers to necessitist objections.

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We like to think that we understand basic logical concepts like the universal quantifier ('every', $\forall$) and the existential quantifier ('some', $\exists$). Indeed, we teach our introductory logic students how to distinguish valid arguments from invalid arguments in the language of first-order predicate logic. The same goes for the modal concepts of possibility ($\Diamond$) and necessity ($\Box$). While there isn't universal agreement about the correct principles of reasoning governing possibility and necessity, the basic contours of propositional modal logic have been well understood for many years.

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When we combine modal logic with the quantifiers, though, much less is settled. A host of conceptual and metaphysical issues are raised when we attempt to understand the interaction of quantification and possibility. For instance: suppose it is possible that something has feature F (that is, that $\Diamond\exists xFx$) does it follow that there is an existing thing such that it's possible that *it* has feature F ($\exists x\Diamond Fx$)? On some accounts of modal predicate logic [35, 75], the answer is *yes*, it does follow. However, it is not at all obvious that the conclusion follows from the premise. After all, it is possible that, had history gone differently, the dodos would have survived into the 21st Century. That is, it is possible that (had we made better choices as a species), there would have been 21st Century descendants of the dodos that were alive in the 17th Century. (Call such critters *21st Century dodos* for short.) Had things gone differently, there would have been 21st Century dodos. Does it follow from this that there *is* something that is a *possible* 21st Century dodo? What on earth could such a thing be? To say that a merely possible (but not actual) 21st Century dodo *exists* seems like a substantial metaphysical commitment, and it is one that may be contested [41; 49, Chapter 8].

Similar issues arise when we consider the interaction between quantifiers and *temporal* vocabulary: if at some future time there is something that has feature G, does it follow that there is *now* something that will (at that later time) have feature G? A thicket of conceptual and metaphysical issues arise immediately from this one inference, connecting the quantifiers and modal (and temporal) operators.

Since we do not yet have a clear understanding of quantified modal logic, perhaps our grasp of its components—the quantifiers, and the modal operators—was less firm than we might have first thought. Different views can be mapped in a number of different ways. One disagreement is between the *actualists* and the *possibilists* [19, 38, 40]. In recent years, the discussion of these issues has revealed a division between two camps, the *necessitists* (for whom, of necessity, all objects there are exist necessarily) [35, 36, 75] and the *contingentists* [38, 39] (for whom it is contingent what objects there happen to be). A contingentist will say that my merely possible daughter *could* have existed but does not actually exist, thereby rejecting the inference from $\Diamond\exists xFx$ to $\exists x\Diamond Fx$,[1] while the *necessitist* grants the inference, since my merely possible daughter, existing in those circumstances where she is my daughter, must exist in this one too, since it by necessitist lights, it necessary that everything that exists, exists necessarily.

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The most recent extended discussion of the logical and metaphysical issues arising in the combination of quantification and modality is found in Timothy Williamson's *Modal Logic as Metaphysics* [75]. This discussion, like almost every treatment of modal metaphysics since the 1960s, makes substantial use of possible worlds models for modal predicate logic. Possible worlds models have provided a significant conceptual advance in our understanding of modal logics, and they have fuelled an explosion of work in philosophical logic in the last 70 years. We have learned a great deal about the logic of modality, using these tools.

However, it is less clear that possible worlds models help us fix the *meaning* of modal and quantificational vocabulary. Possible worlds models for modal logic feature a collection of 'points' at which statements in our modal language are evaluated. It is, of course, suggestive to think of these as "possible worlds", but when it comes to the truth

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[1] Contingentists reject the inference from $\Diamond\exists xFx$ to $\exists x\Diamond Fx$ if the '$\exists$' quantifier is taken to have existential import, at least.

conditions of formulas in our modal models, these points are a purely structural device. The models have their formal properties, independently of what we take points to be. For the models to be in some way connected to the concepts that we employ in the languages we use, these structures must in some way be connected to our practice. However, it is by no means clear that we have any independent understanding of what a possible world *is*, prior to our grasp of the notions of possibility and necessity. As Arthur Prior famously put it:

> … possible worlds, in the sense of possible states of affairs are not *really* individuals (just as numbers are not *really* individuals). To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case … We understand 'truth in states of affairs' because we understand 'necessarily'; not *vice versa* … To use a distinction I once heard Quine insisting upon, what we have in [a translation between the modal logic s5 and monadic first-order predicate logic] may be a *model* for modal logic, but it is not an *interpretation* of the modal words. [50, p. 54, emphasis his]

Any insight gained from employing models for modal logic is partial, since if we wish to use possible worlds models as a representation of *how things really are* (*and can be*) modally speaking, we must rely either on an identification between the indices of a possible world model and some antecedently given class of *possible worlds* (however these are to be understood) — and it is not at all obvious that this can be achieved in any theoretically or practically satisfying fashion — or we must give some other account of how and why it is that these models accurately represent modal truth. Merely *stating* that a given model represents modal reality does not make it so, and neither does it explain how these models connect to the concepts we actually employ. If we want to use possible worlds models in our modal explanations, we must do the work to explain the relationship between these models and our modal concepts, in some way or another. Otherwise, "possible worlds" as the indices of modal models (and the *possibilia* that inhabit the domains of such worlds) are a theoretical device, whose effectiveness relies on a promissory note, which might be called in at any time.

Of course, the flexibility of modal models has benefits as well as these theoretical costs. What can be interpreted as *possible worlds* can also be understood as *moments of time* in models for the temporal logic of reasoning about the past and the future [29, 62], or as *scenarios* in models of epistemic logic and the information that is open to us [4,23,56]. As we have seen, issues of temporal and epistemic reasoning raise parallel questions to those raised in the interaction of modality and the quantifiers. In what follows, it will be useful to keep the parallels beyond the traditional concepts of possibility and necessity in mind, though the notions of metaphysical modality will be central.[2]

In this paper, we will strike out on a less-frequented track for exploring the landscape of modality and quantification. I will harness the resources from proof theory to address these questions, a framework that does not make direct use of possible worlds

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[2] To specific concerning the scope of my discussion in this paper: I do not have the space to consider questions concerning the identity predicate, or matters of higher order logic. The discussion here will be limited to first order quantified modal logic without identity, though this is not an essential limitation of my approach. Higher order logic and identity can be treated in a similar way to the modal predicate logic discussed here, though to do in sufficient depth so would make what is an already long paper unmanageable.

models.[3] This different perspective will give us fresh insight to interpret our modal and quantificational vocabulary, and a new way to assess what is at stake concerning different principles governing their interaction. Along the way, we might just learn something more about possible worlds and the objects that reside therein.

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So, in the next section (Section 1) I rehearse the treatment of the Barcan formula in models for quantified modal logic, for contingentists and for necessitists. In Section 2, I introduce a *hypersequent* proof theory for the modal logic S5, and explain how it can be used as a formal account of the meaning for modal vocabulary, which does not start with a commitment to an ontology of possible worlds, but that can be used to elucidate patterns in our *use* of modal vocabulary. In Section 3, I introduce the treatment of quantifiers in this hypersequent calculus, and the options for contingentists and necessitists, showing that there is a natural way for the contingentists to interpret the quantifiers in a manner consonant with their commitments. In Section 4, we reexamine variable domain models for contingentist quantified modal logic, and we see that the proof theorist's view of models provides an alternate explanation for why those models accurately give an account of the behaviour of modal concepts without committing the contingentist to a seemingly problematic ontology of merely possible objects. However, the proof theory of quantified modal logic raises its own questions, and in Section 5 I will explore the resources in the proof theory itself to define wider quantifiers—quantifiers that at least *seem* to put pressure on contingentist commitments. However, making use of the concept of *ampliation*, from medieval discussions of modal reasoning, we will see how the wider quantifiers so-defined are intelligible on a properly contingentist basis, without making any concessions to necessitism.

In a short final section, I conclude, taking stock of what we have seen, and sketching avenues for further exploration. *After* the conclusion, there is an extended technical appendix, where I collect together the definitions, and state and prove the soundness and completeness results linking the proof theory and the model theory of the modal predicate logic under discussion, for those readers to whom those technical details matter.

1 MODAL MODEL THEORY AND BARCAN FORMULAS

We have seen that there is little agreement around the interaction between the quantifiers and the modal operators. Two contentious principles have come to be known as *Barcan formulas*, after Ruth Barcan Marcus [5–7], who explicitly introduced discussion of these principles in the 20th Century. These can be stated using the universal quantifier an necessity, or in a dual form, with possibility and the existential quantifier.

$$\forall x \Box Fx \to \Box \forall x Fx \qquad \Diamond \exists x Fx \to \exists x \Diamond Fx$$

The universal/necessity form says that if everything is necessarily F, then it is necessary that everything is F. By necessitist lights, this rings true. If everything that *is* exists

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[3] Here is a parallel to keep in mind: the proof theory of first-order logic (whether in the form of natural deduction [48], sequent calculus [28], tableaux [67], etc.) does not make use of a domain of quantification or the rest of the apparatus of Tarski's semantics. The logic that results is equivalent to that given by Tarski's model theory (that is the point of the soundness and completeness theorems after all), but if you learn first-order predicate logic by way of learning natural deduction, the domain of quantification is nowhere in your instruction manual.

necessarily, then if everything is necessarily F, then no matter how things go, everything would be F, because (even if things had gone differently) if there *were* some thing that is not F, then (as a matter of fact) that thing—which in that other circumstance fails to be F—is something that is not necessarily F.

The contingentist need not be convinced by this reasoning, for she may respond that just because it is *possible* that something fails to be F, any such thing that fails to be F had things gone *that* way, may not actually exist. Perhaps everything (that exists) is necessarily F, but this doesn't preclude the possibility *other* things existing which fail to be F. The Barcan formulas provide one locus of disagreement between contingentists and necessitists, and it sits right at the interaction between quantifiers and modal operators.

If we are to use models to try to clarify this disagreement, we might reason like this: A possible world counterexample to the Barcan formula $\Diamond\exists xFx \to \exists x\Diamond Fx$ will have some world w at which $\Diamond\exists xFx$ holds and at which $\exists x\Diamond Fx$ fails. If $\Diamond\exists xFx$ holds at w, then there is some (accessible[4]) world v at which $\exists xFx$ holds. In these models, this means that there is some value d (taken from the domain of the model) where the open formula Fx holds, at world v, when the variable x is assigned the value d.

On the other hand, for $\exists x\Diamond Fx$ to fail at w, we need there to be no objects available to assign x that would make $\Diamond Fx$ at w. Otherwise, the existentially quantified statement will turn out to be true. Now, since Fx holds at v when x is assigned d, and from the point of view of w, v counts as *possible*, it looks very much like the object d we saw before would be available to do the job of witnessing the truth of $\exists x\Diamond Fx$ at w.

To block this move, the natural response of the contingentist is to say that the object d might not be *available* for substitution at w, since it need not *exist* at w. For a quantified statement $\exists xGx$ to hold at w we need some object e *at* w where Gx holds at w when x is assigned e. The objects available at w to interpret the quantifiers (call them the domain at w: D_w) need not be the same as the objects available at other worlds, such as v. If the object d, which witnesses $\exists xFx$ at v is not available at D_w, we cannot move from $\Diamond\exists xFx$, which says that there is some world featuring an object that is an F, to $\exists x\Diamond Fx$, which commits us to the presence *here* of some object that happens to be an F at some world. If the objects that are present vary from world to world, the order you select *world* and *object* matters, so the order of the quantifier and modal operator matters, too. If the same objects are available for choice at each world, the choices are independent of each other, and the Barcan formulas hold.

So much is quite standard when it comes to the behaviour of variable domain models for modal logics [16; 27; 31, part III; 70]. There are many more details and complications, but this is enough to introduce the key issues. Variable domain models are the well-understood way to give an account of first-order predicate modal logic in which the Barcan formulas can fail, and at which, what exists is a contingent matter.

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I will focus on three distinct issues raised by these varying domain models, following Linsky and Zalta's influential article "In Defense of the Simplest Quantified Modal Logic" [35].

First, if we are to take these models at face value, we are committed, in some sense, to the different domains of objects existing at each world. In any model that is a coun-

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[4] If the modal logic is weaker than S5, or if it is a multi-modal logic, we may need to keep track of an accessibility relation, but in what follows, this is not important, so I will drop consideration of modal accessibility from now on.

terexample to a Barcan formula, we have objects which exist at other worlds and do not at the actual world. So, if we interpret these models in a naïve fashion, taking the domain at a world to literally consist of the objects at that world (and not merely be theoretical devices to represent those objects), then the contingentist's models seem to contain objects that do not exist, by their own lights. This seems to strain credulity, without an account of how non-existent objects can play an explanatory theoretical role. On the other hand, if the contingentist takes their models of other possible worlds and their denizens to be *models*, and not to be taken as a literal description of how things are, then the one who makes use of such models owes us an explanation of work these models manage to do, and what we can learn from them. How is it that models with this structure can, in any way, represent features of the modal and quantificational concepts we employ?

Second, the logic of these possible worlds models with varying domains, almost invariably, involves making a revision to first-order predicate logic, and a rejection of some important inference principles. Suppose a term t has a value that is present at some worlds, but *not* the world w. The sentence "t exists" (whether this is represented in our language as $\exists x\, x = t$ or in some other way) should turn out to be false at w on contingentist lights. However, the universal generalisation "everything exists" (whether represented as $\forall y \exists x\, x = y$, or in some other way) is true, since every *thing* (the denizens of the domain at w) exists (at w). So, we have a counterexample to the inference from $\forall x\, A(x)$ to $A(t)$, which is traditionally understood as the fundamental rule of universal quantifier elimination.

Similarly the inference from $A(t)$ to $\exists x\, A(x)$ has a counterexample, since at w, t does not exist, but it does not follow that there is *something* (at w) that does not exist. We have wrenched the quantifiers away from their standard classical interpretation, and this, at least according to Linsky and Zalta, comes at a significant theoretical cost.

Third, once we have varying domain possible worlds models, it is straightforward to interpret predicates in such a way that their extension (at a world) goes beyond the domain of objects that exist at that world. After all, the open sentence "x does not exist" is true (at w) of some the denizens of worlds other than w. This violates what has come to be known as *serious actualism*, the thesis that it is not possible for an object to have a *property* without also existing [43]. For the serious actualist, a nonexistent person is *not* a person that also has the property of nonexistence. It does not have the property of being a person, it does not have the property of nonexistence. It does not exist, and hence, has no properties.) Many contingentists are also serious actualists [1, 9, 38, 41], and more work must be done to show either that the use of varying domain models does not violate serious actualist commitments, or that those commitments may be jettisoned.

Linsky and Zalta presented these three issues as considerations against the use of varying domain modal logics and in favour of necessitism and the adoption of the Barcan formula. To these three objections, Williamson has recently added another consideration, in by way of his analysis of logical truth in terms of his notion of metaphysical universality. A formula in the language of modal predicate logic is *metaphysically universal* when its universal generalisation[5] is true on its intended interpretation. $Fa \rightarrow Fa$ is metaphysically universal, because under the intended interpretation, $\forall X \forall x (Xx \rightarrow Xx)$ is true.

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[5] In a higher order logic, for *all* non-logical constants are to be generalised.

So, $\exists y\ a = y$ is metaphysically universal, because $\forall x \exists y\ x = y$ is true. Williamson then argues that an *intended* model structure for modal predicate logic should validate all the metaphysically universal sentences, since on his account, logical truths should be understood as those that are metaphysically universal [75, §3.3; 76]. Williamson then argues that intended model structures validate the Barcan formulas. Since $\exists y\ x = y$ is metaphysically universal, it holds at all worlds in all intended model structures. So, take any d in the domain of w (D_w), and assign x the value d. Since $\exists y\ x = y$ holds at world v too (by metaphysical universality), there must be some e in D_v where when y is assigned the value e, $x = y$ holds at v. This can happen (given a standard semantics for identity) only when e *is* d. The d chosen from D_w was arbitrary, so we have shown that $D_w \subseteq D_v$, for any worlds w and v whatsoever, i.e. the domains are *equal*. This suffices for the Barcan formulas to hold in these models, since varying domains were required to generate a counterexample. So, the contingentist must either reject the criterion of metaphysical universality, or avoid one or other of the platitudes for the interpretation of the logical constants used in this argument.

None of these four considerations are presented as a decisive argument against the rejection of the Barcan formulas or against contingentism — there are a number of different moves the contingentist can make, both concerning metaphysical universality and the details of the semantics for the quantifiers and identity — but it does pose a significant explanatory challenge to the contingentist. To respond to this challenge, and other challenges concerning the status of modal model theory, the contingentist needs resources to understand modal semantics better. Forms of semantics beyond model theory will help, so it is time to turn to *proof theory*, to see if insights from this domain can be used to properly *interpret* our modal vocabulary, and thereby to answer some of the questions raised about modal models with varying domains.

2 SEQUENTS AND HYPERSEQUENTS

In a series of papers [47, 51–57, 59, 60], I have explored an approach to the proof theory of classical propositional logic and its extensions to modal logic, and first and second order predicate logic, in which the proofs are not merely a technical device for demarcating the logically valid formulas, but rather, are understood as articulating the rules for the *use* of the logical vocabulary. The proof rules for $\wedge, \vee, \rightarrow, \neg, \forall, \exists, \Box$ and $\Diamond$ are understood as governing assertions and denials[6] using those concepts—that is, they provide a *semantics*. In this paper, I will use these resources elaborate the scope for a rigorous, coherent, defensible, and hopefully *illuminating* semantic framework in which the Barcan formulas, and the push and pull between contingentism and necessitism may be evaluated.

The central idea in this account is that of a *sequent* $\Gamma \succ \Delta$ consisting of two finite collections of sentences Γ and Δ from our formal language. A sequent can be seen as constraining the *positions* one could take in a discourse or in a reasoning situation—to derive the sequent $\Gamma \succ \Delta$ is to show that position $[\Gamma : \Delta]$ (in which each sentence in Γ

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[6] The proof rules also govern *inferences* involving those concepts, as well as assertion and denial. I do not have time to get into the connection between multiple-premise and multiple-conclusion sequents and *inferences* from a context in which some things are ruled in and some things are ruled out, but the account I have given elsewhere [59, 60], concerning sequent derivations with a singled-out item *in focus*, can be readily applied in this setting, but space forbids developing that connection in this paper, since adding focus to the hypersequent derivations presented below would be to add a technical nicety that is completely orthogonal to the issues closest to hand. Suffice it to say that those niceties could be added with no complications other than having to work through what amount to routine details.

is asserted an each sentence in Δ is denied) is *out of bounds*. The sequent $\Gamma \succ \Delta$ said to be *valid* if the corresponding position $[\Gamma : \Delta]$ is out of bounds.

One way to understand the bounds is to say that the sequent $\Gamma \succ \Delta$ is valid and only if it is impossible for each sentence in Γ to be true and each sentence in Δ to be false. That is true enough, but this explanation uses concepts we wish to *explain*—in particular, a notion of *possibility*, or logical satisfiability, at its heart. Rather, we build up the account of the bounds piece-by-piece, starting from the simplest cases.[7] The most simple case of a valid sequent is an *Identity* sequent like this,

$$A \succ A$$

which says that a position $[A : A]$ in which the one and the same claim (here, A) is both asserted and denied, is out of bounds. Identity sequents are at the core of the notion of a position being out of bounds. At its heart, a position is out of bounds when it is *self-undermining*. When we attempt to give with one hand (by asserting A) and take with the other (by denying it). This condition depends only on the identity of the item asserted and denied, and not on its structure or content. The same holds for some other constraints on the bounds. These are the other so-called *structural rules*. First, *weakening*:

$$\frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta}\,\mathit{KL} \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}\,\mathit{KR}$$

according to which if a position $[\Gamma : \Delta]$ is out of bounds, it remains out of bounds when either more assertions or denials are added. The rule of *contraction*:

$$\frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta}\,\mathit{WL} \qquad \frac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}\,\mathit{WR}$$

makes explicit that repetitions of assertions or denials have no special extra force.[8] The most significant structural rule is the *Cut* rule:

$$\frac{\Gamma \succ A, \Delta \qquad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}\,\mathit{Cut}$$

according to which if a position $[\Gamma : \Delta]$ is *in* bounds (if it isn't out of bounds: that is, there is no clash in asserting each member of Γ and denying each member of Δ) then if there were a clash involved in denying A (if $\Gamma \succ A, \Delta$ is valid), that is, if A is *undeniable* (relative to the background position $[\Gamma : \Delta]$) then there is no clash involved in asserting A (again, given $[\Gamma : \Delta]$). In other words, if A is undeniable given $[\Gamma : \Delta]$ then asserting A is simply making explicit what is already implicit in $[\Gamma : \Delta]$. Adding the assertion of A to is no more out of bounds than $[\Gamma : \Delta]$ itself.[9]

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[7] This recursive account of the bounds has the same kind of structure as the recursive explication of truth in a model from the simplest cases in model theory.

[8] If the 'collections' of assertions and denials were *sets* and not *multisets* or *lists*, this rule would be redundant. It is good to make it explicit, because in most proof theory it is simpler to assume that the premises (left-hand side) and conclusions (right-hand side) of sequents form *multisets* and not sets.

[9] The form of the *Cut* rule I have given is the so-called *additive* cut rule, in which the side formulas Γ and Δ are shared between both premises of the rule. A *multiplicative Cut* rule

$$\frac{\Gamma \succ A, \Delta \qquad \Gamma', A \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,\mathrm{m}\mathit{Cut}$$

is possible, where we allow for a *Cut* on sequents with distinct side-formulas. The multiplicative *Cut* rule is

To go beyond these structural features of the bounds to the distinct behaviour of the connectives and quantifiers, we need to appeal to rules for those connectives and quantifiers. I will examine the behaviour of classical propositional connectives, and then the modal operators, and then in the next, the quantifiers.

The classical connectives can be introduced uniformly with a series of *invertible* rules, which can be applied from top to bottom or from bottom to top.

$$\begin{array}{c}\Gamma \succ A, \Delta \\ \hline\hline \Gamma, \neg A \succ \Delta\end{array}\ \neg Df \qquad \begin{array}{c}\Gamma, A, B \succ \Delta \\ \hline\hline \Gamma, A \wedge B \succ \Delta\end{array}\ \wedge Df \qquad \begin{array}{c}\Gamma \succ A, B, \Delta \\ \hline\hline \Gamma \succ A \vee B, \Delta\end{array}\ \vee Df \qquad \begin{array}{c}\Gamma, A \succ B, \Delta \\ \hline\hline \Gamma \succ A \to B, \Delta\end{array}\ \vee Df$$

These rules *define* the bounds governing judgements using the connectives in terms of the bounds governing judgements involving their constituents. So, we can use these as rules to interpret assertions or denials of our involving the *defined* connective in terms of assertions or denials of their constituents. The classical negation rules, governed by $\neg$*Df* show that the asserting $\neg A$ has the same force as a denial of A. For conjunction, an assertion of $A \wedge B$ has the same force as the assertion of A and the assertion of B. The denial of $A \vee B$ has the same force as the denial of A and denial of B. To deny $A \to B$ has the same force as asserting A and denying B.

These count as 'definitions' because the rules suffice to fix the behaviour of the connectives involved, insofar as two concepts introduced with rules of the same shape (say, for example, $\vee_1$ and $\vee_2$, both disjunctions), then there is no open position where one could assert a 1-disjunction and deny a 2-disjunction, or *vice versa*.

$$\begin{array}{c}A \vee_1 B \succ A \vee_1 B \\ \hline A \vee_1 B \succ A, B \\ \hline A \vee_1 B \succ A \vee_2 B\end{array}\begin{array}{l}\\ \vee_1 Df \\ \vee_2 Df \\ \\ \end{array} \qquad \begin{array}{c}A \vee_2 B \succ A \vee_2 B \\ \hline A \vee_2 B \succ A, B \\ \hline A \vee_2 B \succ A \vee_1 B\end{array}\begin{array}{l}\\ \vee_2 Df \\ \vee_1 Df \\ \\ \end{array}$$

The same goes for the other connectives, too. These defining rules govern the behaviour of the propositional connectives by uniquely characterising them—the rules characterise the concepts, rather than merely describing some constraints they satisfy.[10] A longer argument [57] shows that a language governed by a consequence relation satisfying our rules of identity, weakening, contraction and *Cut* can be *conservatively extended* by the propositional connectives given by these defining rules. That argument uses Gentzen's *cut elimination* argument [28], together with the fact that Gentzen's left and right rules for each connective can be recovered from each defining rule, like this.

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justified by way of the weakening rule and additive *Cut*.

$$\dfrac{\dfrac{\Gamma \succ A, \Delta}{\Gamma, \Gamma' \succ A, \Delta, \Delta'}\,K \qquad \dfrac{\Gamma', A \succ \Delta'}{\Gamma, \Gamma', A \succ \Delta, \Delta'}\,K}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,Cut$$

We can conversely, use contraction and multiplicative *Cut* to retrieve the original additive *Cut* in the same way. In the rest of this paper, I will switch between additive and multiplicative *Cut* rules as needed without further mention.

[10] I have in mind here the distinction between these defining rules and the axioms for a modal operator, such as an S5 necessity. These axioms describe constraints satisfied by the $\Box$ in question without uniquely characterising it. We can have two non-equivalent necessity operators both satisfying the S5 axioms.

The derivation below

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \lor B \succ A \lor B}\,\mathit{Id}}{A \lor B \succ A, B}\,{\lor}\mathit{Df}}{\Gamma, A \lor B \succ A, B, \Delta}\,K \qquad \dfrac{\Gamma, A \succ \Delta}{\Gamma, A \succ B, \Delta}\,K}{\Gamma, A \lor B \succ B, \Delta}\,\mathit{Cut} \qquad \dfrac{\Gamma, B \succ \Delta}{\Gamma, A \lor B, B \succ \Delta}\,K}{\Gamma, A \lor B \succ \Delta}\,\mathit{Cut}$$

Shows how the traditional disjunction left rule:

$$\dfrac{\Gamma, A \succ \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \lor B \succ \Delta}\,{\lor}L$$

can be justified in terms of *Id*, $\lor$*Df*, *K* and *Cut*, and as with unique definability, this conservative extension argument works for each connective in the vocabulary [57]. We can justify the usual Gentzen rules for negation conjunction and the conditional

$$\dfrac{\Gamma, A \succ \Delta}{\Gamma \succ \neg A, \Delta}\,\neg R \qquad \dfrac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \land B, \Delta}\,{\land}R \qquad \dfrac{\Gamma \succ A, \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \to B \succ \Delta}\,{\to}L$$

in terms of the defining rules, given *Id* and *Cut*. The rules $\lor L$, $\land R$ and $\to L$ differ from the defining rules for the connectives. They are not invertible—and neither should we expect them to be. The rule $\lor L$ tells us some of what is involved in asserting the disjunction $A \lor B$. We should not expect this to be equivalent to any combination of assertions and denials involving A and B. After all, there are some things we can *do* with a disjunction that we could not do without it. Having a single item we can deny whose denial has the same significance as the denial of A and the denial of B gives us something *new*. Its *assertion* allows us to say something we may not have been able to say without that concept. The defining rule $\lor$*Df* defines disjunction in the sense that it uniquely characterises the bounds for assertions and denials of disjunctions.

In this way, we have a *semantics* for a vocabulary involving the classical propositional connectives, in that we have defined rules for the coherence of positions involving assertions and denials in that vocabulary. The resulting relation of coherence is exactly the same as that delivered by truth tables for propositional logic, but we have not started with the notion of truth. Gentzen's sequent calculus, which we have defined in a roundabout way, is sound and complete for classical propositional logic. A sequent $\Gamma \succ \Delta$ is derivable if and only if there is no evaluation which assigns each member of Γ true and each member of Δ false.

This is not to say that the notion of *truth* is altogether absent from this style of proof theory. If we move from the referee's position, where we stand apart and judge positions like $\Gamma \succ \Delta$ for coherence, to the *player's* standpoint, where we *make* those assertions and denials, we see that someone who asserts Γ and denies Δ is (in some sense[11]) taking each member in Γ to be *true* and each member of Δ to be false. Further, if we have taken up the position $[\Gamma : \Delta]$ and if $\Gamma \succ A, \Delta$ is valid, then there is a sense in which A, too, is taken to be *true* in $[\Gamma : \Delta]$, since it is undeniable—the only coherent option that takes a stand on it is to assert it, and that option *is* coherent if the position $[\Gamma : \Delta]$

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[11] Since we can 'try on' assertions and denials under suppositions, or when taking someone else's position as a starting point in our reasoning, this 'taking to be true' need not involve *belief* or a *commitment* any wider than the scope of the dialogue or that supposition.

is coherent.[12] This does not go far enough to ground a full-blooded and robust notion of truth, but we can go so far as to draw the connection between truth in a position and truth in a model, and that connection becomes very tight when we move from finite positions to *refined* positions for which we fill Γ and Δ out to take a stand on *more* sentences of the language [54,57].[13] None of this moves us beyond truth in a model to truth *per se*, because nothing tells us which of these idealised positions counts as the truth. However, we can say a little more. To *take* the position $[\Gamma : \Delta]$—to assert each member of Γ and to deny each member of Δ is to take each member of Γ to be true and each member of Δ to be false, and to take the whole truth (in that language) to be given by one of the ideal positions extending $[\Gamma : \Delta]$. What we have taken to be true in asserting Γ and denying Δ is whatever is true in each of those ideal positions extending $\Gamma \succ \Delta$. *Which* of those positions is the truth? To single one out is to move beyond $[\Gamma : \Delta]$ to a stronger position, adding more assertions or denials, and choosing between some of the ideal positions extending $[\Gamma : \Delta]$.

This understanding of proof theory and its use in semantics is fit for the *normative pragmatist* [13, 14], who takes a semantic theory to be formulated in rules for use. The account is *pragmatist* in the sense that the theory governs acts—in this case, acts of assertion and denial (or of accepting and rejecting)—and it is *normative* in that the theory gives *rules* or *norms* governing those acts—in this case, the norms governing positions, combinations of assertions and denials, and the outer boundary of the space of such positions. There is more to say concerning the norms governing assertion (and denial), and the role of such norms in semantics, but this is enough to go on with for the moment.[14] It is time to consider the modal operators.

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Let's consider modal reasoning, and the norms governing the modal operators $\Box$ and $\Diamond$. Consider this simple stretch of deductive reasoning featuring the modal operator $\Diamond$, for possibility:

> Suppose it's possible that either A or B (i.e., suppose $\Diamond(A \lor B)$). So, in some possibility, we have either A or B (i.e., there we have $A \lor B$). So, there are two cases, A, and B. In the first case, since here we have A, it follows that where we started, it is possible that A (i.e., $\Diamond A$). In the other case, since here we have B, it follows that where we started, it's possible that B (i.e., $\Diamond B$). In either case, therefore, we have that it's either possible that A or it's possible that B (that is, $\Diamond A \lor \Diamond B$). So, we have shown that if it is possible that either A or B, then either A is possible or B is possible (i.e. $\Diamond(A \lor B) \to (\Diamond A \lor \Diamond B)$).

That was a small piece of modal reasoning, deriving the complex claim $\Diamond(A \lor B) \to (\Diamond A \lor \Diamond B)$. We moved from the supposition $\Diamond(A \lor B)$ to reason with the constituent claim $A \lor B$—which we granted, for the sake of the argument, but not in the same way

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[12] This reasoning undergirds the validity of the *Cut* rule, and it has its critics [21,63,64]. One way to quickly defend it is to acknowledge that we are limiting our attention to the kinds of issues that are expressed in polar questions. When we inquire as to whether or not A holds (for well-posed issues A), to rule out one 'yes' or 'no' option is to leave the other [61].

[13] See the appendix of this paper for some of the details on how to fully refine a position, and the connection between fully refined positions and models.

[14] For example, I have said *nothing* concerning norms governing correct assertion, and the large literature discussing these norms [12, 17, 33, 37, 72, 73]. The fact that I have not discussed these does not mean that I take them to be unimportant or unrelated to the bounds of assertion and denial discussed here.

that we supposed $\Diamond(A \vee B)$, in order to prove the conditional. We asserted $A \vee B$ 'in some possibility'. With that $A \vee B$ granted, we split into two different cases. In the A case, back in the home context, we concluded $\Diamond A$. In the B case, back in the home context, we concluded $\Diamond B$. So in either case, we have $\Diamond A \vee \Diamond B$, and discharging our original supposition that $\Diamond(A \vee B)$ we derived the conditional $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$.

In modal reasoning, we typically transform a modalised statement $\Box A$ or $\Diamond A$ into its constituent claim A—with some given shift in context. Given that the word 'context' is used in many different ways in semantics, I will reserve the 'zone' for the different discourse regions, introduced by the modal transitions characteristic of this sort of reasoning. In stretch of reasoning given above, the supposition of $\Diamond(A \vee B)$, and the conclusion of $\Diamond A \vee \Diamond B$ occurs in one zone, while the case-split into the A case and the B case occurs in another. The transition from the first zone to the second occurs where we say "so, in some possibility, we have either A or B."

The proof theory of modal logic can take these zone shifts very seriously, using *hypersequents* allowing for formulas to be asserted and denied not just in a single zone, but in many. A simple hypersequent (or simply, a *hypersequent*)[15] has the form:

$$\Gamma_1 \succ \Delta_1 \mid \cdots \mid \Gamma_n \succ \Delta_n$$

A hypersequent is a nonempty multiset of sequents. Where a single sequent $\Gamma \succ \Delta$ represents a bound on combinations of assertions and denials taken together, a *hyper*sequent represents a bound on combinations of assertions and denials distributed across a number of different zones in a discourse. While there is a clash between asserting A and denying A in the one zone, there need be no clash between asserting A in one zone and denying A in another. It might be altogether coherent to grant that A *is* the case, but to concede to an alternative scenario a circumstance in which A fails, so we should expect a position, split into two zones, $[A : \mid : A]$ may well be coherent.

A key insight in the hypersequent proof theory of modal logics is that cross-zone connections are facilitated by the modal operators. It *is* out of bounds to grant, in one zone, that A is *necessary*, and to deny A in another. Similarly, it is out of bounds to grant A in one zone, and to deny that A is *possible* in another. That is, these two hypersequents are valid:

$$\Box A \succ \mid \succ A \qquad\qquad A \succ \mid \succ \Diamond A$$

In the kind of modal reasoning we will focus on, concerning bare counterfactual possibility and necessity, we keep track only of the different zones, and not any notion of 'relative possibility' or 'nearness'. For the hypersequent proof theory for S5 we need only keep track of a number of different zones, not anything more than that.

In referring to hypersequents we have some new syntax. We will use $\mathcal{H}$ as a variable ranging over for hypersequents,[16] and $\Gamma \succ \Delta \mid \mathcal{H}$ is the hypersequent $\mathcal{H}$ with the sequent $\Gamma \succ \Delta$ added, analogously to Γ, A being the multiset Γ with the formula A added. Before looking at how the structural rules are to be understood in this setting, we start with the defining rules for necessity and possibility:

$$\frac{\Gamma \succ \Delta \mid \ \succ A \mid \mathcal{H}}{\Gamma \succ \Box A, \Delta \mid \mathcal{H}}\,\Box Df \qquad \frac{\Gamma \succ \Delta \mid A \succ \ \mid \mathcal{H}}{\Gamma, \Diamond A \succ \Delta \mid \mathcal{H}}\,\Diamond Df$$

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[15] I say 'simple' hypersequent in contrast to *tree* hypersequents, that have a more complex modal structure [2, 20, 46, 47], fit for a wider range of modal logics.

[16] The formal definition (see the Appendix) allows for $\mathcal{H}$ to be empty, in the statement of rules like this, while, hypersequents themselves must contain at least one sequent.

The idea is straightforward: to deny $\Box A$ in a zone of a discourse is out of bounds (given the other commitments in $\Gamma \succ \Delta$ and $\mathcal{H}$) iff denying A in *some* zone is out of bounds (relative to those commitments). That is, $\Box A$ is undeniable in a given zone iff A is undeniable in *any* zone. To assert $\Diamond A$ in a zone of a discourse is coherent iff asserting A in *some* zone is coherent. Possibility and necessity trade on this shift of zones.

The hypersequent structure in the proof theory plays a role in fixing the interpretation of the modal operators. Possibility and necessity are *not* logical constants in the same sense as the classical propositional connectives. They are not constant—not only in the sense that there are many different modal logics, but in the stronger sense that even if we fix on one logic as the correct account of necessity, one can have a multi-modal logic in which there is more than one 'necessity' operator satisfying that logic. The axioms and theorems governing necessity are not enough to fix its meaning. In a multi-modal logic (say, given by a model with two different accessibility relations governing each necessity operator). Nonetheless, these defining rules *define* the connectives, relative to the hypersequent structure. If we agree to interpret $\Box$ using $\Box$*Df*, then you and I agree on the interpretation of $\Box$, even though we could have very different views on what formulas of the form $\Box A$ are true.[17] Agreement on modal operators trades on coordination on the zone shifts used in our reasoning with them.

The structural rules for hypersequents can be motivated in the same sort of way as in the nonmodal case, except that we keep track of zones. *Identity* is as before

$$A \succ A$$

since an assertion of A clashes with a denial of A—in the same zone. Of course, there need be no clash between an assertion of A in one zone and a denial of A in another. Weakening comes in more forms, allowing for internal and *external* weakening.

$$\frac{\Gamma \succ \Delta \mid \mathcal{H}}{\Gamma, A \succ \Delta \mid \mathcal{H}}\,KL \qquad \frac{\Gamma \succ \Delta \mid \mathcal{H}}{\Gamma \succ A, \Delta \mid \mathcal{H}}\,KR \qquad \frac{\mathcal{H}}{\Gamma \succ \Delta \mid \mathcal{H}}\,KE$$

For *KE*, if a position is out of bounds, then adding extra zones (in which other things are asserted and denied) is not going to help. The *contraction* rules can be understood both internally and externally too:

$$\frac{\Gamma, A, A \succ \Delta \mid \mathcal{H}}{\Gamma, A \succ \Delta \mid \mathcal{H}}\,WL \qquad \frac{\Gamma \succ A, A, \Delta \mid \mathcal{H}}{\Gamma \succ A, \Delta \mid \mathcal{H}}\,WR \qquad \frac{\Gamma \succ \Delta \mid \Gamma \succ \Delta \mid \mathcal{H}}{\Gamma \succ \Delta \mid \mathcal{H}}\,WE$$

If it is incoherent to assert Γ and deny Δ in two different zones of the discourse, it's incoherent to assert Γ and deny Δ in one. For asserting Γ and denying Δ in two different zones of the discourse commits you to nothing more than you are committed to in asserting Γ and denying Δ in one, if there is nothing different in the two zones—and there isn't, since we have individuated those zones purely in terms of what is asserted and denied in them.[18]

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[17] This is completely analogous to agreement about $\vee$, given that we agree to guide our use of $\vee$ by way of the defining rule $\vee$*Df*. There, agreement depends only upon the comma in the sequent structure. For $\Box$ and $\Diamond$, we need to agree on more—on the hypersequent separator "|". Once you and I agree on what counts as an alternative zone to what, in a given a discourse, we can fix on the interpretation of $\Box$ and $\Diamond$, even if we disagree on what statements involving $\Box$ and $\Diamond$ are *true*.

[18] Consider the analogy: If $Fa, Fb, \Gamma \succ \Delta$ is out of bounds, where Γ and Δ say nothing more about a or b, then so is $Fa, \Gamma \succ \Delta$ since nothing in the position $Fa, Fb, \Gamma \succ \Delta$ says that a and b must be different

The additive *Cut* rule is a straightforward generalisation of the rule in the sequent context:

$$\dfrac{\Gamma \succ A, \Delta \mid \mathcal{H} \qquad \Gamma, A \succ \Delta \mid \mathcal{H}}{\Gamma \succ \Delta \mid \mathcal{H}}\,\mathrm{a}\mathit{Cut}$$

according to which if A is *undeniable* in first zone in the coherent context $\Gamma \succ \Delta \mid \mathcal{H}$, then adding it as an assertion in that zone is coherent. The multiplicative variant of *Cut*, in which contexts are merged

$$\dfrac{\Gamma \succ A, \Delta \mid \mathcal{H} \qquad \Gamma', A \succ \Delta' \mid \mathcal{H}'}{\Gamma, \Gamma' \succ \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,\mathrm{m}\mathit{Cut}$$

is equivalent to the additive variant in the presence of contraction and weakening:

$$\dfrac{\dfrac{\dfrac{\Gamma \succ A, \Delta \mid \mathcal{H} \qquad \Gamma, A \succ \Delta \mid \mathcal{H}}{\Gamma, \Gamma \succ \Delta, \Delta \mid \mathcal{H} \mid \mathcal{H}}\,\mathrm{m}\mathit{Cut}}{\Gamma, \Gamma \succ \Delta, \Delta \mid \mathcal{H}}\,\mathit{WE}}{\Gamma \succ \Delta \mid \mathcal{H}}\,\mathit{WL/WR}$$

$$\dfrac{\dfrac{\dfrac{\Gamma, \succ A, \Delta \mid \mathcal{H}}{\Gamma, \Gamma' \succ A, \Delta, \Delta' \mid \mathcal{H}}\,\mathit{KL/KR}}{\Gamma, \Gamma' \succ A, \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,\mathit{KE} \qquad \dfrac{\dfrac{\Gamma', A \succ \Delta' \mid \mathcal{H}'}{\Gamma, \Gamma', A \succ \Delta, \Delta' \mid \mathcal{H}'}\,\mathit{KL/KR}}{\Gamma, \Gamma', A \succ \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,\mathit{KE}}{\Gamma, \Gamma' \succ \Delta, \Delta' \mid \mathcal{H} \mid \mathcal{H}'}\,\mathrm{a}\mathit{Cut}$$

So, as before, we have a suite of structural rules. We extend them with the modal rules, and the other connective rules, generalising the account to hypersequents rather than sequents, but keeping the rules as before. So here, for example, are the rules for conjunction, disjunction and negation:

$$\begin{array}{c} \Gamma, A, B \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, A \land B \succ \Delta \mid \mathcal{H} \end{array}\,{\land}\mathit{Df} \qquad \begin{array}{c} \Gamma \succ A, B, \Delta \mid \mathcal{H} \\ \hline\hline \Gamma \succ A \lor B, \Delta \mid \mathcal{H} \end{array}\,{\lor}\mathit{Df} \qquad \begin{array}{c} \Gamma, A \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma \succ \neg A, \Delta \mid \mathcal{H} \end{array}\,{\neg}\mathit{Df}$$

As before, these rules are uniquely defining and conservatively extending, once we have moved to the setting of hypersequent positions. As before, two connectives introduced with rules of the same shape are interderivable, and hence, indistinguishable as far as the bounds of positions are concerned. Similarly, rules of the form of Gentzen's *left* and *right* rules for each connective may be defined in terms of our defining rules, *identity* and *Cut*, and a Cut elimination argument proved for he resulting system. The result is a conservative extension fact, showing that any position ruled out of bounds may be done so on the basis of the concepts occurring in that position. Adding new concepts governed by defining rules does not interfere with the bounds for positions in the prior vocabulary. Concepts given by defining rules are free additions to our vocabulary in the sense that they are uniquely defined (relative to the hypersequent structure) and they do not interfere with any prior positions.

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things. So if $Fa, Fb, \Gamma \succ \Delta$ is out of bounds, so is $Fa, \Gamma \succ \Delta$. In the same way, in taking up the position $\Gamma \succ \Delta \mid \Gamma \succ \Delta \mid \mathcal{H}$ we are simply committing ourselves to the possibility of everything in Γ holding and everything in Δ failing, and the possibility of everything in Γ holding and everything in Δ failing, and everything in $\mathcal{H}$. That is no more and no less than the possibility of everything in Γ holding and everything in Δ failing, and everything in $\mathcal{H}$, which is what is said by $\Gamma \succ \Delta \mid \mathcal{H}$.

Here is an example derivation, using defining rules and the structural rules.

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vee B \succ A \vee B}{A \vee B \succ A, B}\,{\scriptstyle \vee Df} \qquad \cfrac{\Diamond A \succ \Diamond A}{A \succ \;\mid\; \succ \Diamond A}\,{\scriptstyle \Diamond Df}}{A \vee B \succ B \mid\; \succ \Diamond A}\,{\scriptstyle mCut} \qquad \cfrac{\Diamond B \succ \Diamond B}{B \succ \;\mid\; \succ \Diamond B}\,{\scriptstyle \Diamond Df}}{\cfrac{A \vee B \succ \;\mid\; \succ \Diamond A \mid\; \succ \Diamond B}{A \vee B \succ \;\mid\; \succ \Diamond A, \Diamond B}\,{\scriptstyle K,W}}\,{\scriptstyle mCut}}{\cfrac{\Diamond(A \vee B) \succ \Diamond A, \Diamond B}{\Diamond(A \vee B) \succ \Diamond A \vee \Diamond B}\,{\scriptstyle \vee Df}}\,{\scriptstyle \Diamond Df}$$

This derivation gives us another account of how to get from $\Diamond A \vee \Diamond B$ from $\Diamond(A \vee B)$—it has a similar structure to the everyday reasoning given at the introduction to this section, though the particulars are different. Take the intermediate hypersequent in the derivation

$$A \vee B \succ \;\mid\; \succ \Diamond A \mid\; \succ \Diamond B$$

The fact that this hypersequent is derivable means that asserting $A \vee B$ (in one zone), while denying $\Diamond A$ (in another) and denying $\Diamond B$ (in another) is out of bounds. We have given an account of a proof theory for the modal logic S5 in which the connectives are defined by way of rules for use, governing assertion and denial of modal formulas—in different zones, as one would expect in modal reasoning. There is more to say about hypersequent proof theory for propositional modal logic [8,45,52], but instead of staying here, we will move at last, to the quantifiers.

3 QUANTIFIERS, DEFINEDNESS AND THE BARCAN FORMULAS

Combining the hypersequent defining rules for modal operators with the natural rules for classical quantifiers is a recipe for delivering the *Barcan formulas*. Defining rules for the classical quantifiers are simple to extend to the hypersequent settings [57].

$$\begin{array}{c} \Gamma \succ A(n), \Delta \mid \mathcal{H} \\ \hline\hline \Gamma \succ \forall x A(x), \Delta \mid \mathcal{H} \end{array}\,{\scriptstyle \forall Df} \qquad \begin{array}{c} \Gamma, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, \exists x A(x) \succ \Delta \mid \mathcal{H} \end{array}\,{\scriptstyle \exists Df}$$

Denying a universally quantified judgement is out of bounds just when it is out of bounds to deny an arbitrary instance. Asserting an existentially quantified judgement is out of bounds just when it is out of bounds to assert an arbitrary instance. Here, arbitrariness and generality is governed by the side condition implicit in these rules: the singular term n must be absent from the premise hypersequent, except for its use in the formula $A(n)$. In the context of standard sequents, these rules suffice for classical predicate logic: We can derive all of the first-order classical validities, as you would expect.

The extension to the hypersequent setting ensures that the inference can be applied in any zone in any modal reasoning context. This means, however, that we can also prove the Barcan formulas, interleaving the defining rules for the quantifiers and the modal operators in this way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\Box Fx \succ \forall x\Box Fx}{\forall x\Box Fx \succ \Box Fn}\,\forall Df}{\forall x\Box Fx \succ \quad\mid\quad \succ Fn}\,\Box Df}{\forall x\Box Fx \succ \quad\mid\quad \succ \forall xFx}\,\forall Df}{\forall x\Box Fx \succ \Box\forall xFx}\,\Box Df}{\succ \forall x\Box Fx \to \Box\forall xFx}\,{\to}Df \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists x\Diamond Fx \succ \exists x\Diamond Fx}{\Diamond Fn \succ \exists x\Diamond Fx}\,\exists Df}{Fn \succ \quad\mid\quad \succ \exists x\Diamond Fx}\,\Diamond Df}{\exists xFx \succ \quad\mid\quad \succ \exists x\Diamond Fx}\,\exists Df}{\Diamond\exists xFx \succ \exists x\Diamond Fx}\,\Diamond Df}{\succ \Diamond\exists xFx \to \exists x\Diamond xFx}\,{\to}Df$$

If we are contingentists, then these derivations should not strike us as compelling, and it is worth taking our time to consider why this is so, and where the inference breaks down.

While it would be a mistake to assert $\forall x\Box Fx$ and deny it at the same time (the first sequent indeed is out of bounds), is it a mistake to assert $\forall x\Box Fx$ and to deny $\Box Fn$? This depends on the status of the term n, and the status of singular terms as we shift from zone to zone is exactly what is at issue when it comes to understanding the interaction between modal and quantificational judgements. If we assume, at the outset, that, as a matter of necessity, all singular terms must denote, this is to set the terms of the debate decisively in favour of the necessitist. In the interest of keeping our theoretical options open, we would do well to at least countenance the notion that singular terms might denote in one zone of a discourse while fail to denote in another. To require terms to operate in the same way across zones is to import necessitist assumptions at the outset.

Excursus on names: Before continuing with this train of thought, it is worth saying something about the status of the terms such as n used in the defining rules for the quantifiers. I will call them *names*, chiefly to distinguish them on the one hand from *variables* (we can reserve variables for use with quantifiers in formulas) and from arbitrary *terms* which may include, for example, function symbols. The important feature of these names is that they are *inferentially general*. They satisfy the condition that if some (hyper)sequent featuring the name n is derivable, then so is the result of globally replacing n by any other term of the same general syntactic category. That is the crucial feature in the traditional Gentzen *Left*/*Right* sequent rules, or natural deduction *introduction* and *elimination* rules for the quantifiers. Inferential generality ensures that if I have, on the one hand, a proof of $\forall xA(x)$ (which I derived from a proof of $A(n)$, where I have made no other assumptions from n) and I infer $A(t)$ from this, for some term t, then I could have used the original proof of $A(n)$ while replacing the name n everywhere by t. This will only work, in general, if the result is still a proof. For this, we need the norms governing n to be satisfied by t, too. That is, we require there to be no inferential norms specific to n. It is *inferentially general*. This is the feature of names which is important in what follows, and it is what underwrites the semantics of the quantifiers. *End of excursus*

Returning to the putative proofs of the Barcan formulas, we see that the proofs break down if we move to a *free logic* in which we not only allow terms to fail to denote (so much is standard [26, 57]), but we allow denotation failure to vary from zone to zone. This is exactly the shift the contingentist desires, for whether a term counts as suitable to substitute for a quantifier *should* differ from zone to zone. What exists here might fail to exist there (and what exists now might not have existed before now, and might cease to be, in the future). So, it is natural to generalise the defining rule for the quantifiers for free logic to the hypersequent case like this:

$$\begin{array}{c} \Gamma, n \succ A(n), \Delta \mid \mathcal{H} \\ \hline\hline \Gamma \succ \forall xA(x), \Delta \mid \mathcal{H} \end{array}\,\forall Df \qquad \begin{array}{c} \Gamma, n, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, \exists xA(x) \succ \Delta \mid \mathcal{H} \end{array}\,\exists Df$$

Now we extend our zones in our hypersequents to not only keep track of those formulas asserted (on the left) and denied (on the right) but we have added the term n to the left hand side of a zone. We allow terms in sequents to keep track of which terms are suitable substitutions for the quantifiers. To rule a term *in* as suitable for substitution in some zone, it is added left hand side of the sequent. To rule it *out* we add it to the right. Then the defining rules for the quantifier are then motivated on contingentist lines. To *deny* $\forall x A(x)$ is to take there to *be* something (which we call n) that doesn't satisfy $A(x)$. That is, in this zone we rule n *in*, and deny A of *it*. To *assert* $\exists x A(x)$ is to take there to be something (again, call it n) that satisfies $A(x)$. We rule n in and assert A of it.

We take ruling terms in and out as suitable substitutions for our quantifiers as the basic constituents of sequents and hypersequents. It is natural, though, to consider representing this basic act of ruling a term in or out as expressible in the object language in a formula. This is the job of the so-called 'existence' predicate. We will follow Feferman [26] in representing this with a postfix downward arrow, connoting *denotation*.

$$\frac{\Gamma, t \succ \Delta \mid \mathcal{H}}{\overline{\Gamma, t{\downarrow} \succ \Delta \mid \mathcal{H}}}\ {\downarrow}Df$$

The free logic that results is a simple extension of first-order predicate logic allowing for nondenoting terms. We cannot pass from $\forall x A(x)$ to $A(t)$ without making explicit the extra assumption that the term t indeed does take a value. (If 't' does not pick out a *thing* then the fact that everything is A does not mean that $A(t)$, after all.) Making that assumption explicit is simple:

$$\cfrac{\cfrac{\cfrac{\cfrac{\forall x A(x) \succ \forall x A(x)}{\forall x A(x), n \vdash A(n)}\ \forall Df}{\forall x A(x), n{\downarrow} \succ A(n)}\ {\downarrow}Df}{\forall x A(x) \land n{\downarrow} \succ A(n)}\ {\land}Df}{\succ (\forall x A(x) \land n{\downarrow}) \to A(n)}\ {\to}Df$$

With the modified quantifier rules, the derivation of the Barcan formulas break down, and they do so in an informative way. We can proceed this far with the derivations:

$$\cfrac{\cfrac{\forall x \Box Fx \succ \forall x \Box Fx}{n, \forall x \Box Fx \succ \Box Fn}\ \forall Df}{n, \forall x \Box Fx \succ \quad \mid \quad \succ Fn}\ \Box Df$$

but to go further, to generalise on the n in the zone containing Fn to conclude $\forall x Fx$, we need our hypersequent to contain n on the left in *that* zone, not in the other: we would need to have derived $\forall x \Box Fx \succ \quad \mid n \succ Fn$ because then we could continue:

$$\cfrac{\cfrac{\forall x \Box Fx \succ \quad \mid n \succ Fn}{\forall x \Box Fx \succ \quad \mid \quad \succ \forall x Fx}\ \forall Df}{\forall x \Box Fx \succ \Box \forall x Fx}\ \Box Df$$

In the absence of a rule that would allow terms to migrate from zone to zone, like this:

$$\frac{t, \Gamma \succ \Delta \mid \Gamma' \succ \Delta' \mid \mathcal{H}}{\Gamma \succ \Delta \mid t, \Gamma' \succ \Delta' \mid \mathcal{H}}\ t\ \mathit{Migration}$$

the standard derivation of the $\forall/\Box$ Barcan formula is blocked, and it is blocked in a principled way. In fact, in the system with the standard structural rules and the defining rules for $\Box$ and the free quantifiers $\forall$ and $\exists$ there is no derivation of the Barcan formula. This hypersequent, for example, cannot be derived:

$$a, Fa, \Box Fa, \forall x\Box Fx \succ b, Fb, \Box\forall xFx \mid a, b, Fa \succ Fb, \forall xFx$$

so, the *hyper*position, consisting of two zones like this, is not out of bounds:

$$\big[\, a, Fa, \Box Fa, \forall x\Box Fx : b, Fb, \Box\forall xFx \mid a, b, Fa : Fb, \forall xFx \,\big]$$

There is no clash involved in first asserting $\forall x\Box Fx$, and taking the term a to denote, and so, also asserting $\Box Fa$ and Fa, while at the same time taking the term b to fail to denote, denying Fb, and also denying $\Box\forall xFx$, while second, granting the alternative possibility according to which a and b *both* denote, granting Fa, while still denying Fb, and in this possibility denying $\forall xFx$. Such a package of commitments coheres with the meaning rules for the quantifiers and modal operators, and there is no clash.

In fact, this is this hyperposition is, in an important sense, *fully refined* (see Definition 5 on page 36 for details). For example, in any zone in which a universally quantified formula is asserted, each instance is asserted (so, since $\forall x\Box Fx$ is in the first zone, $\Box Fa$ is in that zone, because a denotes there). In any zone in which a universally quantified sentence is denied, some instance is denied (since $\forall xFx$ is denied in the second zone, there is some instance, namely Fb—since b denotes in this zone—that is denied there). If there is some zone at which $\Box A$ is denied, then A is denied in some zone, and if $\Box A$ is asserted in some zone, A is asserted in every zone. The 'downward' consequences of each formula (its consequences concerning its subformulas) are spelled out in a comprehensive way.

This fully refined hyperposition neatly corresponds to a variable domain model with two worlds, one of which has as domain $\{a\}$ (where bears property F in that world) and the other, domain $\{a, b\}$ (where the extension of F is $\{a\}$ alone). In the first world, $\forall x\Box Fx$ is true since $\Box Fa$ is true (and this world's domain is $\{a\}$), while $\Box\forall xFx$ is *false* since in the second world $\forall xFx$ fails, since there, Fb fails (and b is in the domain at this world).

This construction of a model from a fully refined hyperposition is perfectly general. The procedure of filling out an underivable hypersequent by decomposing complex formulas and supplying instances for quantified formulas, and adding zones to witness modal formulas, results in a systematic and canonical model construction for a variable domain quantified S5, as I spell out, in the Appendix. A model, then, corresponds to an available hyperposition. What start off as inference rules in the proof system then can be understood as the truth conditions for the connectives, operators and quantifiers of the language.

◊ ♦ ◊

This hypersequent system of rules for the connectives, quantifiers and modal operators provides a well-behaved *semantics*, consistent with contingentist's motivation in rejecting the Barcan formulas—even though the free logic modifications to quantification used here[19] are originally motivated by very different concerns [26]. Once we admit that it is coherent that a singular term might actually refer but also *fail* to refer

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[19] ... and explored in my paper "Generality and Existence 1" [57].

had things gone differently, then the hypersequent calculus provides us an apt interpretation, fit for giving a normative pragmatic semantics for a language governed by contingentist norms.

The result is a quantified modal logic with a possible worlds 'semantics' in which the variable domain models for a contingentist modal logic have their place, but they are explained by and grounded in something more fundamental, the norms we employ to govern our use of the modal operators and the quantifiers.[20] To return to the distinction drawn by Arthur Prior, mentioned above (see p. 3), the proof rules provide an *interpretation* of the modal vocabulary, along with the quantifiers and the connectives. The possible worlds 'semantics' provides a way to model this vocabulary, but as for fixing what that vocabulary means, we must look elsewhere.

The inference rules fix meaning in another important sense. If you and I employ vocabulary in a way governed by those inference rules, then the rules will allow us to communicate. As we saw above (page 9), an "or" out of your mouth is equivalent to an "or" out of mine, at least as far as the bounds of discourse (and entailment, equivalence, etc.) go, if we apply the same rules. The same goes for the quantifiers, provided we agree on the predicate/term structure of the language we are using. We no more need to agree on whether this or that term *denotes*, than we need to agree on whether this or that claim is *true*, for us to coordinate on the meaning of the quantifiers.

$$\dfrac{\dfrac{\forall_1 xA(x) \succ \forall_1 xA(x)}{\forall_1 xA(x) \succ A(n)}\,{\scriptstyle \forall_1 Df}}{\forall_1 xA(x) \succ \forall_2 xA(x)}\,{\scriptstyle \forall_2 Df} \qquad \dfrac{\dfrac{\forall_2 xA(x) \succ \forall_2 xA(x)}{\forall_2 xA(x) \succ A(n)}\,{\scriptstyle \forall_2 Df}}{\forall_2 xA(x) \succ \forall_1 xA(x)}\,{\scriptstyle \forall_1 Df}$$

Similarly, we do not need to agree on what things are possible and what things are necessary to coordinate on the meaning of the modal operators: we need only to agree on when we are making discourse shifts from one modal zone to another.

$$\dfrac{\dfrac{\Box_1 A \succ \Box_1 A}{\Box_1 A \succ \mid \succ A}\,{\scriptstyle \Box_1 Df}}{\Box_1 A \succ \Box_2 A}\,{\scriptstyle \Box_2 Df} \qquad \dfrac{\dfrac{\Box_2 A \succ \Box_2 A}{\Box_2 A \succ \mid \succ A}\,{\scriptstyle \Box_2 Df}}{\Box_2 A \succ \Box_1 A}\,{\scriptstyle \Box_1 Df}$$

Coordinating on zone shifts and the defining rules enough structure on which to leverage disagreement on matters of what is possible and what is necessary. Once we are able to *modalise*, to not only engage in the practice of asserting and denying in the sense of flatly describing how we take things to be, but to also apply those assertion and denial norms in situations of *planning*, considering *options* for future action, and even ruminating on *what could have been*, we find ourselves engaging in just these sort of zone shifts, and modal vocabulary can take root [34].[21]

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[20] Of course, this is not to say that our everyday modal or quantificational notions are *identical* to the concepts picked out by these precise definitions, any more than the material conditional '$\rightarrow$' corresponds precisely to our everyday use of 'if'. Rather, the sharply delineated concepts introduced here by way of a defining rule are the kinds of things we *can* freely define (against the background of an assertoric practice with a selected notion of zone shift), and which recognisably do the same kind of work we accomplish with our everyday notions of necessity and possibility. Our everyday modal concepts may or may not *exactly* comply with the defining rules for $\Diamond$ and $\Box$, but insofar as they *do*, we can explain the fit with possible worlds models by appealing to these norms for using those concepts.

[21] Mark Lance and Heath White's "Stereoscopic Vision" [34] makes the case that the conceptual structure arising out of human agency, involves two forms of modal zone shift. *Subjunctive* modalizing involves alternative ways things could go or could have gone. It arises when we plan the future, or retrospectively evaluate

Furthermore, we not only have an inferentialist semantics for the quantified modal language, we also have an explanation for why possible worlds models actually manage to correctly model the truth-conditional behaviour of the language we use. The soundness and completeness theorem of the Appendix spells out, in detail, how the natural variable-domain possible worlds models have the structure that represents the pre-existing semantics of a language with concepts governed by these defining rules. When we attend to the detail of the completeness proof and the models that are generated, we see that notion of a 'possible world' as motivated by that proof is neither a mere uninterpreted algebraic device in some mathematical structure, nor an unanalysed primitive metaphysical commitment nor a theoretical posit required for an explanatory theory. The 'possible worlds' that arise naturally from the construction of a model from a given hyperposition are the result of starting with the commitments incurred in a discourse where we utilised a number of different modal zones, and treating those commitments, together, as describing some *structure*. The issues that arise in the introduction of models therefore bring us back to the perceived shortcomings and theoretical weakness of possible worlds semantics with varying domains, and of contingentism more generally, so let us turn to these questions in the next section.

4 POSSIBLE WORLDS MODELS AND *POSSIBILIA*

In Section 1, we discussed Linsky and Zalta's three complaints about varying domain models for modal first-order predicate logics. Now, in the light of the *semantics* for the modal operators and quantifiers, and this new view on varying domain models, we can consider these objections in turn. *First*, recall that Linsky and Zalta claimed that if we are to take these models seriously, we are then committed to the existence of these putatively non-existent *possibilia*. *Second*, they argued that rejecting classical first-order predicate logic for a free logic comes at severe theoretical cost. *Third*, they stated that varying domain models make it all too easy to violate the constraints of serious actualism, by allowing for true predications (statements of the form Ft) without existential commitment. So, statements like $Ft \land \neg t{\downarrow}$ (t has property F, despite not existing) can be satisfied in varying domain models, so this lacuna should be addressed, at least for contingentists with serious actualist scruples. To these three objections, we added a fourth, from Williamson: that the logic of varying domain models cannot be metaphysically universal, since they allow for rejection of the Barcan formulas.

In this section, I will start by responding to the first objection, explaining how it is that a contingentist, utilising the inferentialist semantics given above, can help themselves to the use of varying domain models, while incurring no existential commitment to mere *possibilia*. Then we will address head-on the shift to free logic and allowing for non-denoting terms, addressing Linsky and Zalta's second objection. This response will bring along with the means to respond to Williamson's argument concerning metaphysical universality. Then, we will finally address Linsky and Zalta's third objection, concerning the putative tension between variable domain models for contingentism and serious actualism.

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the difference our choices and actions make. *Epistemic* modalizing involves different views of how things actually *are*. It arises when we consider different perspectives on what is the case, and when we attempt to resolve disagreements. For social creatures like us, who attempt to *act* on the basis of *shared views*, it is to be expected that we consider alternatives in these two different ways. The technical results in this paper can apply equally to both kinds of zone shift, but the focus is on the subjunctive 'metaphysical' modality in most of the literature on necessitism and contingentism. However, it is a strength of the hypersequent approach that it allows for a uniform treatment of both metaphysical and epistemic modals [56].

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How does the ontological commitment arising out of the use of possible worlds models with varying domains look when viewed from an inferentialist perspective? If we think of modal models as reifications of fully refined available positions, it is tempting to think that the only ontological commitment that ensues is that incurred in the use of the abstract formal or mathematical machinery used to construct models. If you think of models as mathematical constructions that witness various formal properties, then theoretical commitment to the kinds of abstract constructions of models from their components is the price you must pay.[22] However, *that* is not the issue with commitment to the mere possibilia inhabiting the non-actual worlds of our models. The point of these models is not that they merely represent logical space and abstract properties. The issue is that these models purport to represent *how things are*. To put things in a different way: truth in a model is taken to provide a model of *truth, simpliciter* [30]. So, let us take seriously the commitment we incur when we think of the kinds of models that arise when we attempt to model *truths* in our modal and quantificational vocabulary.

So, we start with a given an interpreted language $\mathcal{L}$, including the quantifiers and modal operators, defined on contingentist lines, and let's start from a position $[\Gamma : \Delta]$, where Γ contains sentences and terms that we rule in, and Δ contains sentences and terms we rule out. To make the task as stringent as possible, let's start with Γ containing *all* the sentences in $\mathcal{L}$ that are, in fact, true, and all the terms in $\mathcal{L}$ that denote, while Δ contains all the sentences that fail to be true and all the terms that fail to denote. What does a model arising out of this position look like? If there is no sense in which we can endorse that model and the theoretical and ontological commitments that ensue, then this is indeed a mark against the variable-domain model theory.

For a start, if disjunction in $\mathcal{L}$ indeed has the semantics given by $\vee Df$, then we can be sure that $A \vee B \in \Gamma$ if and only if either $A \in \Gamma$ or $B \in \Gamma$, since $A \vee B \in \Delta$ if and only if $A, B \in \Delta$, and our position is a partition: $A \in \Gamma$ iff $A \notin \Delta$. The quantifiers, however, are not necessarily so well behaved in $[\Gamma : \Delta]$. We might have an existentially quantified sentence $\exists x A(x)$ in Γ while the language supplies no term t where $A(t)$ and t are both in Γ. For example, our language might supply a sentence like "there is a uranium atom on the surface of Pluto" without supplying a singular term t such that "t is a Uranium atom on the surface of Pluto" is a true sentence in $\mathcal{L}$. This is the point, in the model construction, where we extend our language with a stock of singular terms, to be supplied as witnesses for the truth of existential quantifications and the untruth of universal quantifications. If we find ourselves committed to $\exists x A(x)$ without endorsing $A(t)$ for any available term t, then we extend our language with a fresh term n, and we grant $A(n)$, while ruling n in as having a value in this zone. *Which* value does n take? In our case of the putative uranium atom on Pluto, we have no way of specifying *which* uranium atom we mean, and so, the singular term n does not act very much like a *name* in the traditional sense. There is no particular given object (whether on Pluto or elsewhere) such that I am committed to *that* item being a uranium atom on Pluto, and if there are many such atoms, there is nothing to say that the term n takes one as its referent over the other. The existentially quantified claim can be satisfied in any number of different ways, by different witnesses. However, extending our language with a term serving as a witness for that existentially quantified statement, is in some sense, a way of spelling out the commitment incurred in the existential claim. We may not be able to name such an item, or identify it in any informative way, but it

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[22] The view of possible worlds as abstract representations is affirmed in different ways by figures such as Alvin Plantinga [42] and Robert Stalnaker [68, 69].

to deny that we can supply a term for just such an item (calling the suspect X, for the sake of the argument) would be to reject the original claim. To say that there is some uranium atom on Pluto is to say that there is some object to take as the value for this given name, were we in a position to name it. So, let us consider that our position has indeed been refined so far as to contain witnesses for each quantified statement, at the cost of extending our language with sufficiently many fresh terms, as needed. The existential commitment incurred by adding these singular terms to our vocabulary is no more and no less than that incurred by the original existentially quantified claim we started out with.

The case of witnesses for the truth of existentials and the untruth of universals is parallel to the case of the truth of possibility and untruth of necessity claims. Whenever $\Diamond A$ is in our set Γ of truths, to *refine* our hyperposition, we may need to add a fresh zone in which A is asserted. This zone provides a witness for the possibility claim in just the same way that a fresh term is added as a witness to a quantified claim. Adding such a zone does not incur only the ontological cost of the ordered pair of sets of sentences and terms involved in the new zone. We started out, granting that $\Diamond A$ is *true*—we granted that it is possible that A, and it is this commitment that is spelled out by the addition of a fresh zone, at which A is asserted. If, for example, I supposed A (in this fresh zone, in the salient sense for modal reasoning), and continued reasoning, to finally arrive at some absurd conclusion, this would show that my original commitment, to the possibility of A, was mistaken. To grant that A is possible comes with its own costs. The commitment incurred must result in an available position, and if it clashes with my other commitments, including any other commitments concerning what is possible and what is necessary, something must give. So, the spelling out of the original commitment to $\Diamond A$, in terms of the opening up of a new zone—in which the consequences of commitment to A (as a possibly counterfactual alternative) can be tested—has its own risks. This commitment to a zone at which A is granted is not, however, commitment to the truth (*simpliciter*) of A. It is only commitment to A's possibility, and, implicitly, a commitment to whatever else the possibility of A involves.

When we move from talk of refined positions (with their zones and claims and terms ruled in and ruled out), to thinking of *models* with their worlds, and domains and extensions of predicates at each world, the same question seems to arise with witnesses to possibility claims as arises with witnesses to existential claims. When it came to witnesses for $\exists x A(x)$, and the shift to $A(n)$, the question of *which* object n is meant naturally arises. When we consider the parallel issue for zones and worlds, the potential difference between commitment to worlds and commitment to objects is clarified. When we move from $\Diamond A$ to consider a 'possible world' at which A is true, we have a similar issue arising as we did for uranium atoms on Pluto. *Which* A-world do we mean? There may well be very many possible worlds at which A obtains. As far as zones and full refinement goes, our language can get very specific, and full refinement will force our zones to take one side or other for every disjunction they assert, and which provide witnesses for each existentially quantified statement. So, each zone gives rise to a world which settles each sentence in the language under discussion (see Definition 11 and Lemma 5 on page 42). However, this need not determine a *world* in any metaphysically rich sense, since if we expand the language with new predicates or terms, there is no guarantee that what was specific enough to count as a world in the original language is a world in the expanded language. If we have a new predicate F, our original fully refined zone will not settle whether each item counts as an F or not, and what was specific enough to be a unique world at one level of analysis is now a whole family of worlds, when viewed more closely. Our level of commitment for our modal vocabulary is more suited to talking about *possibilities* than possible worlds [32], since there is nothing in our language that allows us to *identify* worlds, let alone, to *count* them.[23]

Fully refined positions, and the models we might take them to describe, involve not only quantification and modality, but their interaction. In the general case (such as when we refine a starting position which affirms $\Diamond\exists xFx$ but denies $\exists x\Diamond Fx$), we introduce fresh terms in fresh zones, which may not be taken to denote in our original starting zone.

$$\big[a, \Diamond Fb, \Diamond\exists xFx : b, Fb, Fa, \Diamond Fa, \exists x\Diamond Fx \;\mid\; a, b, Fb, \exists xFx : Fa, \Diamond Fa\big]$$

Here the starting zone (the first of the two in the hyperposition) takes the object a to exist, and to fail to have property F, both in fact, and in the alternative possibility described by the second zone, where b is also taken to exist. These are the only two zones in this hyperposition, so $\Diamond Fa$ can be denied in both positions, since Fa is ruled out everywhere. This a is the only object countenanced in the first zone, so $\exists x\Diamond Fx$ may be denied in the first zone, too. However, since $\Diamond\exists xFx$ is affirmed in the first zone, the second was introduced to affirm $\exists xFx$. For this, we added a fresh term b, as a witness for the quantifier in this zone. Here, b is taken to exist, and Fb is granted. This means that $\Diamond Fb$ cannot be denied in the original zone (that would clash with affirming Fb in the second), but this is compatible with the denial of $\exists x\Diamond Fx$, since b can be taken to fail to denote in the first zone, and to therefore not be a suitable substitution for the existential quantifier there.

This is one way to spell out the theoretical commitments of denying the Barcan formula, and in this way, the mere *possibilia* of items, such as b, must be allowed to at least this extent. There is a kind of theoretical commitment here, above and beyond the commitment to the raw uninterpreted syntax. The construction of the refined position, and any model it may be taken to describe, is a way of spelling out what is involved in the joint commitment to $\Diamond\exists xFx$ and to $\neg\exists x\Diamond Fx$.

However, it does not follow that there is, in the commitment undertaken in upholding this hyperposition any sense in which claims made in the second zone must be treated as be treated as flat descriptions of some scenario that obtains. Claims made in that zone are not taken to be *true* (recall, we introduce other zones not by supposing how things *are*, but by supposing things go otherwise), but we are committed to taking these claims to be jointly possible. The kind of counterfactual commitment is analogous to a present description of how things were in some past historical period. Commitment to a present description of past events does not make those past events

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[23] So, at least in first-order logic with identity, if I have a singular term t, and I consider a model for a language $\mathcal{L}$ involving t, there is nothing to determine one single correct way to extend that model to interpret a language with a fresh predicate F. Perhaps Ft should turn out to be true in the extension, perhaps it should turn out to be false. However, this is a determinate issue. Of *that* object (named t), the question of whether it has property F or not is something to be settled by extended model. In some extensions it does, and in other extensions it doesn't. Even if there are two objects in the new model, one with property F, and one without, and each of which satisfies all the same (non-identity) predicates as the object that is t in the original model, one of them will satisfy the sentence $x = t$, while the other will not. The corresponding issue does not arise for modal models, at least in our vocabulary, since in this language there is no 'identity' predicate for worlds. A starting model can be expanded to a new model where the original world w splits into two variants w^+ and w^-, where a given fresh proposition p is true at w^+ and false at w^-, and there is no fact of the matter concerning which of w^+ and w^- is 'the same world' as w in the original model. Is this difference in expressive power between the first-order and modal vocabulary a *bug* or a *feature* of modal language? If you think it is a shortcoming to be overcome, then *hybrid logic*, which extends the standard modal logic with more expressive power, is the natural next step [3, 11, 15].

actually occur in the present. Using the term b *as if* it denoted an object does not need to be identified with using the term b to in fact denote an '*as if* object'.

Of course, if we treat the different zones as descriptions of different places, then it is natural to think of the term b in some counterfactual zone as *referring* to an actually existing object which is a member of the domain of the world described by that zone. Such a reading of the modal model theory is natural and understandable, and it has the same conceptual form as a model of a temporal logic that conceives of the different moments of time as points on a line distributed in space, each present at the same time. The difference between different times, ordered from earlier to later has the same structure as the points in a line oriented in space. The past moments of time are not literally present now, any more than counterfactual scenarios actually obtain. Treating different zones as if they described worlds that actually obtain but are elsewhere in modal space is to recast the *counterfactual* modality in spatial terms. The metaphor can be very useful indeed, in just the same way that a present-to-us-in-the-moment timeline can represent features in space all at once features that only actually obtain over a period of time. However, that structural analogy remains an analogy, and the terms that would have denoted had counterfactual scenarios obtain no more denote any existing thing than terms that *did* denote past objects must be treated as denoting objects still exist *now* [62].

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Let's turn to the second objection from Linsky and Zalta: that the rejection of classical quantification theory and adoption of a free logic comes at too great a theoretical cost. The appropriate reply to this objection to deny that the cost is in any way significant. The free logic underlying this account is natural, straightforward, and independently motivated on theoretical and scientific grounds [26]. Williamson, in *Modal Logic as Metaphysics* makes a case to the effect that employing a language that countenances non-denoting terms is unscientific.

> For a restriction to completely free logic undermines the application of scientific method by permitting one to hold on to a universal generalization after one of its instances has been refuted: one denies Ga but still asserts $\forall x Gx$ by also denying $\exists y\ a = y$, still retaining the constant a in the language. We assume that the formal languages under consideration in this chapter are well designed in the relevant sense, so that metaphysical universality implies truth. For our present aim is neither to model natural languages, for example in their use of fictional and mythological names. . . nor to stick to what is knowable *a priori* in some sense, which might exclude whether some names refer. Rather, our business is to clarify the structure of metaphysical universality in a broadly scientific spirit. Non-referring uses of 'Pegasus' have no more place in such an enquiry than they have in physics or zoology. Of course, the term 'phlogiston' did occur in scientific language, but if it failed to refer (rather than referring to an empty kind) then its presence in any scientific theory was a defect in that theory. Consequently, we should not distort our formal language by allowing for such a term. [75, pages 131–132]

There is a lot here, but a response seems at hand for the contingentist and the defender of free logic. The use of a free logic does not undermine the application of scientific method, because there is an ambiguity in the expression "hold on to a universal generalization after one of its instances has been refuted." We assert $\forall x\ x{\downarrow}$ (everything exists) while denying $\frac{1}{0}{\downarrow}$. Have we held on to the generalisation and denied one of its instances? In the bare *grammatical* sense, yes, where $G(a)$ is an instance of $\forall x G(x)$ for any singular term a. However, there is another sense in which $\forall x\ x{\downarrow}$ is *not* refuted by $\neg(\frac{1}{0}{\downarrow})$, since the term $\frac{1}{0}$ is ruled out as an appropriate substitution for the quantifier any discourse in which $\frac{1}{0}$ is ruled out as nondenoting. There, we take $\frac{1}{0}$ to not refer, and so, it is not a counterexample to the universal quantifier—it is not an instance at all.

More serious is Williamson's appeal to scientific discourse in his rejection of free logic and his defence of the constraint of metaphysical universality. This, it seems to me, is the core of his defence of necessitism, and the appeal to scientific discourse seems to me misplaced, at least when it comes to *mathematics*, which surely counts as a science if anything is. Mathematical discourse is shot through with what the mathematicians take to be non-referring terms, like these:

$$\frac{1}{0} \qquad \{x : x \notin x\} \qquad \lim_{x \to 0} \frac{\sin x}{x}$$

Then there are expressions like these, which are found in the most scientific of texts (whether pure mathematics, applied mathematics or one of the other sciences), and these expressions sometimes refer, and sometimes do not, depending the behaviour of their components.

$$\lim_{n \to \infty} a_n \qquad \sum_{n=0}^{\infty} a_n \qquad f'(x) \qquad \int_a^b f(x)\, dx \qquad \{x : \phi(x)\}$$

The same can be said for recursion theory, computer science, and other theoretical disciplines. Reasoning about termination of algorithms and the definedness of functions is widespread. Now, of course it is in some sense *possible* for us to strip our mathematical and scientific discourse of such terms—or rather, it's possible for us to stop doing mathematics in the usual manner, and restrict ourselves to a much more limited language, free of undefined and non-denoting terms. However, it is by no means clear that the language which admits of partial functions and non-denoting terms is in *any* sense more defective than a language which manages to do away with them. The free logic of these non-denoting terms is thoroughly classical, and straightforward to work with [26].

If non-denoting terms have their use in *these* contexts, it seems no more problematic to allow for the possibility that terms which *do* denote (like names for seemingly contingently existing objects) might have failed to denote had things gone differently, and analogously, we may employ terms which do not denote, but *might* have denoted in different scenarios. The hypersequent calculus treats non-denoting mathematical terms and names for contingently existing objects in a uniform same manner. The discipline of ruling terms *in* our *out* in zones in a hypersequent keeps track of those terms that are appropriate instances for the quantifiers in those zones, and it is done in a way which respects the strictures and conventions of scientific discourse.

Since we have not seen any objection to the use of free quantification other than a conservative appeal to retain non-free classical logic, and a rejection of the language of mathematics as it is actually used, combined with an idiosyncratic characterisation of the logical validity with metaphysical universality,[24] we can pass over this objection, and turn to the one objection that remains.

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[24] I say 'idiosyncratic' because to characterise the logical truths as the metaphysically universal statements means that we must countenance truths like $\exists x \exists y \exists z(x \neq y \wedge y \neq z \wedge x \neq z)$ as logically true. Taking each statement of the form "there are at least n things" to be *logically* valid is by far a greater revision of classical first-order logic than the admission of non-denoting terms into the vocabulary.

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Linsky and Zalta are correct to note that models for variable-domain quantified modal logics *allow* for violations of serious actualist scruples. The rules I have employed so far do not rule a position like this

$$[Fa : a]$$

out of bounds. We can *grant* Fa yet rule a out as non-denoting. Yet how can a bear the property F without existing?

We will see that there is a way to develop the semantics presented here in harmony with serious actualism, but we must be careful. Making positions *like* this is essential to the project of contingentism, whether actualist or not. After all, if the term a does not denote, then the sentence $a{\downarrow}$ is false, and hence, its negation is true. So, *this* position had better remain available:

$$[\neg a{\downarrow} : a]$$

It should be coherent to say that a does not exist, while at the same time, ruling a out as nondenoting. Serious actualism cannot be understood as the claim that whenever a sentence $A(t)$ involving a term t is true, then t must denote, since $\neg\frac{1}{0}{\downarrow}$ is a true sentence involving the nondenting term $\frac{1}{0}$. We respect serious actualist scruples by requiring that only *objects* are the bearers of properties, and nonexistence, for example, is not a property. However, if we wish to constrain our language so that primitive predicates always predicate *properties* then indeed our rules can be extended to respect this constraint. We simply impose the following connection between predication and denotation:

$$\frac{t_i, \Gamma \succ \Delta \mid \mathcal{H}}{Ft_1 \cdots t_n, \Gamma \succ \Delta \mid \mathcal{H}}\ FL$$

If, in a given position it would be out of bounds to take t_i to denote, then in that position it would also be out of bounds to predicate F of the tuple $t_1, \ldots, t_n$. If this is thought to be a constraint on the predication of properties, then it is straightforward to express this constraint in the hypersequent calculus at the level of predication for primitive properties, and this constraint is imposed on the system discussed in the Appendix.

However, this might not be the end of the story concerning serious actualism, since perhaps our language might have means of expressing genuine properties or relations beyond the device of predication with an atomic n-place predicate. Lambda abstraction is often taken to be one such way of forming complex predicates. For any open sentence $A(x)$, we can treat $(\lambda x.A(x))$ as a complex predicate expression, where $(\lambda x.A(x))t$ is taken to be equivalent to $A(t)$, whenever the term t is free for x in $A(x)$. We cannot take *every* such λ term to be a predicate expressing a property on serious actualist lights, at least if such an equivalence holds universally, since $(\lambda x.\neg x{\downarrow})$ is a perfectly acceptable term, and the direct equivalence has $(\lambda x.\neg x{\downarrow})\frac{1}{0}$ equivalent to $\neg\frac{1}{0}{\downarrow}$. $\frac{1}{0}$ does not exist. We would like to not be forced back into saying that $\frac{1}{0}$ has the complex property of non-existence.

We can have the benefit of λ abstraction, retaining the idea that all λ abstracts denote properties that are existence-entailing, in the way the serious actualist wants, by modifying the way that abstraction is evaluated. The idea is familiar. The traditional conversion of $(\lambda x.A(x))t$ by directly syntactically substituting the term t inside the context $A(x)$, to return $A(t)$ is a call-by-name evaluation strategy, where the substitution occurs purely syntactically, replacing the term t in the context $A(\cdot)$. This may be contrasted with a call-by-*value* evaluation strategy, which asks us to evaluate the term t first, and then substitute the *result* of the evaluation into the context $A(x)$ [44]. In free logic, this can be expressed using the identity predicate, variables and the existential quantifier. To evaluate $(\lambda x.A(x))t$, we find a value for t and substitute *that* into the context. This holds if and only if $\exists x(x = t \land A(x))$. If we use a call-by-value evaluation strategy for λ abstraction, then we have a uniform way of showing that *all* such predications are indeed existence entailing, whether simple or complex. It is straightforward to see why $\frac{1}{0}$ cannot have the property of nonexistence, since there is no thing that is $\frac{1}{0}$ to be the bearer of such a property. There is no bar, on this account, for serious actualist scruples to be respected, in a systematic and thoroughgoing way.

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This ends my discussion of these objections to contingentism and varying domain semantics.[25] The upshot is that modal models are useful representations which can give us insight into the behaviour of modal and quantificational concepts. Given that the semantic labour of interpreting the connectives, quantifiers and modal operators is discharged by the proof theory and not the model theory, and the model theory is also grounded in the proof theory, we have an independent explanation of how it is—and why it is—that Kripke models can give us insight into the interaction between modal concepts and quantifiers. There is nothing in such an explanation that leads the normative pragmatist away from any contingentist sympathies they may have had at the outset. The ontology of domains of non-actual worlds does not, in itself, lead the normative pragmatist to necessitism.

However, the success of the inferentialist semantics for contingentist modal logic does *not* mean that the logical language we have defined in the proof theory and the model theory is a stable stopping point. There are reasons to explore *wider* quantifiers, allowing for more substitutions into open sentences that the world-bound in-zone existence-entailing quantifiers $\exists$ and $\forall$. It is easy to see this in the model theory, since it is natural to take the union of the domains across all worlds in a model as a trans-world domain available for quantification everywhere, but the attraction of interpreting wider quantifiers is not restricted to model theory: it has an equally compelling proof-theoretic motivation. In fact, the motivation to allow for substitution of terms available in one zone into statements formed in another has been considered by philosophers for quite some time, long before the development of possible worlds semantics and varying domains. In the next section we will turn first to medieval discussions of *ampliation*, and what this means for quantified modal logic, contingentism and necessitism.

5 AMPLIATION AND POSSIBILIST QUANTIFICATION

Reflection on the logic of our thought and talk of what exists and what does not exist is not a new phenomenon. Medieval logicians, in particular, thought deeply about many of these issues [24, 25, 71]. These logicians did not have the same account of quantification, of scope and binding that we take for granted today, and as a result, many of their analyses seem foreign to us. However, they were aware of subtle issues in meaning, and drew sophisticated distinctions in order to account of the structure of our thought and talk, and we can learn from what they have to say.

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[25] That is not to say, of course, that all questions about contingentism have been answered, and all objections silenced. This paper, however, must come to an end somewhere, and there is more to do before we get to that point. See the conclusion for a discussion of further points to explore.

To be specific, let's fix on the work of John Buridan (c. 1300–1358/61) [18, 77]. For Buridan, claims like these:

Socrates is human   *Plato is human*   *Socrates is teaching Plato*

were all true while Socrates and Plato were contemporaries and while the first was teaching the second. Some time afterward, Socrates died. After Socrates died, it would no longer be correct to say *Socrates is human*, since at these later times, Socrates no longer exists [18, Intro. §3.4]. If, at this later time we were to take a census of every human, we would not find Socrates on that list. To be human is to exist, and to die is to cease to exist, so since Socrates has died, it is not the case that he is human. Similarly, after Socrates' passing, it is no longer true to say that Socrates is teaching Plato, since for teaching to occur, the teacher and the student must both exist.

However, for Buridan and other medievals, it remains true to say that Socrates *was* human. This is true the later time, not because "Socrates is human" was true at the earlier time (that would be a modern tense-logical analysis of the truth conditions of the claim), but rather, because the introduction of the past tense "was" *ampliates* the predicate "is human" so that it takes in more items: it not only supposits for things that *are* human, it also takes *past* humans. Once the predicate is ampliated, past objects fall under it, as well as present objects, and the statement *Socrates was human* is true, since Socrates is a no-longer-existing item, that was, indeed, human.

Similarly, the past tense of the term *died* also ampliates for past objects, so it is correct to now say that Socrates died, even though Socrates is not present now to have the property of *having died*. Socrates is not present now at all. It is the *past* Socrates who died, not some present item who used to be the person Socrates. Similarly, it is true now to say that Socrates *taught* Plato, because the past tense ampliates the term "teaches" to range over more than just the presently existing teacher-student pairs. The introduction of tensed vocabulary modifies the range of application of predicates in this way.

What holds for the use of names like *Socrates* also holds for what we now recognise to be quantificational judgements. On this account, the statement

All people are alive

is true, because no past (or future) tense is applied, and so the terms "people" and "alive" have their original senses. Since all the currently existing people are, by definition of what it is to be a person, in fact, alive, this statement is true. However, for Buridan, the claim

Some people have died

(and so, those people are *not* alive) is *also* true, because the past tense in "have died" ampliates "Some people" to include not only present people but *past* people, too. Some people have died is true because Socrates has died, and Socrates *was* a person. This does not conflict with the truth that all people are alive, because although Socrates *was* a person, being dead, he is not a person any more, and so, is no counterexample to the general truth that all people are alive. In just the same way, for Buridan it is correct to say that

Someone taught Plato

since the past tense in *taught* ampliates "Someone" to include not only present people but past people, and Socrates again fits that bill.

To translate these notions into our current vocabulary—and to consider the modal analogy to the temporal case—we can see that *some* of these claims are expressible in the language of contingentist modal predicate logic. If we consider two zones, one at the time Socrates (s) and Plato (p) were both alive, and the other after Socrates had died, and the claim that Socrates teaches Plato (Tsp), we have the following terms and atomic statements ruled in and ruled out in each zone:

$$\big[s, p, Tsp : \quad \mid \ p : s, Tsp\big]$$

In the first zone, s, t and Tsp are ruled in, while in the second, s (having died) is ruled out, and so, Tsp goes with it. However the modal vocabulary allows for us to recover the fact that Socrates *taught* Plato, with a zone-shifting $\Diamond$ operator. We have

$$\big[s, p, Tsp : \quad \mid \ p, \Diamond Tsp : s, Tsp\big]$$

In the second zone we can make the modal claim, since in the first zone, it is affirmed that Socrates is teaching Plato, even though Socrates is not present in the second zone. The zone-shift is marked in modern vocabulary with an explicit modal (or tense) operator on the judgement, rather than as a shift in the range of application of the predicate, but the technique is not that different.

The claim *someone teaches Plato* (let's formalize it for the moment as $\exists x\, Txp$, ignoring the *one* in someone) is implicitly granted in the first zone, while it can be either affirmed or denied in the second. What we know in the second is that *Socrates* cannot be a witness to $\exists x\, Txp$, since he is ruled out in that zone, and the existential quantifier is existentially committing: it restricts its attention to terms ruled in.

The same goes for the claim *someone could teach Plato* if we formalise that as $\exists x\, \Diamond Txp$, where the existential quantifier has wider scope that the modal operator. Socrates cannot count as a witness for *this* claim in the second zone, because again, Socrates is ruled out of contention. However, if we were to substitute s for x in $\Diamond Txp$, the result would be true, since we have already granted $\Diamond Tsp$. Medievals such as Buridan had no such qualms. Even though Socrates no longer exists, *someone taught Plato* and *someone could teach Plato* count as true since the past tense (and possibility, in the modal case) widen out the range of "teach" to past and merely possible objects, and the quantifier (to use the anachronistic term) "someone" can pick out non-existent objects when ampliated.

Ampliation, understood in this way, is a form of *extension* of meaning from a home context to a wider context, marked by the introduction of tenses or modal operators. This kind of extension and transfer happens across temporal and modal shifts in a number of different ways. We have already seen cases like this:

Socrates did exist. Now, he doesn't.

Where the term Socrates, taking a value in the past context, is pulled forward to the present context, where the value is no longer to be found, but the term is taken to be meaningful nonetheless. The pronoun 'he' is subordinate to the original 'Socrates', across the temporal shift. Similar dependence relations obtain between anaphoric pronouns and their antecedents across *modal* shifts, like this:

I could have had a pet dodo. However, it doesn't exist.

This phenomenon is called *modal subordination* [65, 66]. On a natural reading of these claims, the pronoun *she* occurs outside the scope of the possibility claim that serves as its antecedent.[26] This is of a piece with claims like *Socrates did exist, now he doesn't*, where Socrates is available for reference in under the scope of the "did" in *did exist* and is then pulled forward outside that scope to be referred to in the present One way to think of these claims is to think of the quantification or variable selection as occurring *first* in the alternate zone, and then some shifting operator, moves that variable, to where it can apply in another zone, so claim about me having a dodo is then rendered:

$$\Diamond\exists x(Dxg \land \neg @x{\downarrow})$$

which takes the choice for the value of x to occur under the scope of the possibility operator, and *then* inside that scope, we use an actuality operator (@) to break out of that context, to return the actual world.[27] However, such a rendering finds a modal operator where the grammar of the claims made do not include it. There is no explicit shift in the *however*. The second claim is rather, outside the scope of the possibility operator in the first. An alternative reading is worth exploring, and if we are not employing a dynamic semantics and syntax (which is more natural in representing the discourse shifts in play in the dialogue) then to render the scope interactions appropriately we make the choice for the variable x is made outside the possibility operator. For that, to work, though, we need choose as witness for the quantifier, my *merely possible* dodo, since at the site of choice it does not exist. The syntax is something like this:

$$\exists^{\Diamond}x(\Diamond Dxg \land {\sim}x{\downarrow})$$

There is a choice for x where, had things been different, x would have been my dodo, but it is not the case that x exists. This avoids the need for a silent scope-breaking actuality operator, and it is clear what the intended semantics for these wider *possibilist* quantifiers must be:

$$\begin{array}{c} n \succ \ \mid\ \Gamma \succ A(n), \Delta \ \mid\ \mathcal{H} \\ \hline\hline \Gamma \succ \forall^{\Diamond}xA(x), \Delta \ \mid\ \mathcal{H} \end{array}\ \forall^{\Diamond}\mathit{Df} \qquad \begin{array}{c} n \succ \ \mid\ \Gamma, A(n) \succ \Delta \ \mid\ \mathcal{H} \\ \hline\hline \Gamma, \exists^{\Diamond}xA(x) \succ \Delta \ \mid\ \mathcal{H} \end{array}\ \exists^{\Diamond}\mathit{Df}$$

We simply relax the requirement that the fresh term n take a value in the zone of application to a weaker requirement that the name be defined *somewhere*. So, we have genuinely possibilist quantifiers define $\forall^{\Diamond}$ and $\exists^{\Diamond}$.[28] These are defining rules for the quantifiers, with exactly the same properties of unique definability and conservative extension as the contingentist quantifiers. If defining rules suffice to give meaning to an expression for a normative pragmatist, then these quantifiers make as much sense as the narrower, existence-entailing ones.

Furthermore, these new quantifiers to not commit us to reject the contingentism that motivated variable domain semantics and the original quantifiers. These remain,

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[26] This is akin to donkey sentence "every farmer who owns a donkey beats it", where the pronoun *it* is an anaphor for "donkey" which is under the scope of the quantification in the antecedent.

[27] Although I do not present the semantics of the actuality operator in terms of inference rules here, it is not hard to adapt the system in the Appendix to include it. Simply (optionally) mark one zone in each hypersequent as an 'actual' zone, with the intended interpretation that assertions and denials in this zone commit the speaker to taking those assertions and denials actually hold. Then, an assertion of $@A$ in any given zone is out of bounds iff the assertion of A is out of bounds in the zone marked as actual [56].

[28] I use "$\forall^{\Diamond}$" and "$\exists^{\Diamond}$" for these quantifiers, since from the contingentist's perspective, the requirement for n to be an appropriate substitution into $\forall^{\Diamond}xA(x)$ is that is defined *somewhere*, that is, it *possibly* denotes. From the contingentist's perspective, they are possibilist quantifiers.

and still have the interpretation we originally gave them. The wider quantifiers $\forall^{\lozenge}$ and $\exists^{\lozenge}$ are motivated by a more generous interpretation of what it is to be an appropriate substitution instance: a term is appropriate to substitute into a quantifier if it is at least *minimally* defined, that is, if it takes a value in some zone or other. Considerations of ampliation and modal subordination motivate the introduction of these wider quantifiers, not as a rejection of the contingentists commitments, but rather, as a motivation to increase the expressive power of the logical vocabulary.[29] Adding possibilist quantifiers to your conceptual arsenal does not mean a rejection of your contingentist commitments, but it does give you vocabulary with greater expressive power.

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Adding possibilist quantifiers gives rise to a puzzle for the contingentist. With the quantifiers defined by these more liberal rules, new *Barcan formulas* are now derivable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall^{\lozenge}x\Box Fx \succ \forall^{\lozenge}x\Box Fx}{n \succ \mid \forall^{\lozenge}x\Box Fx \succ \Box Fn}\,\forall^{\lozenge}\mathit{Df}}{n \succ \mid \forall^{\lozenge}x\Box Fx \succ \quad \mid \quad \succ Fn}\,\Box\mathit{Df}}{\forall^{\lozenge}x\Box Fx \succ \quad \mid \quad \succ \forall^{\lozenge}xFx}\,\forall^{\lozenge}\mathit{Df}}{\forall^{\lozenge}x\Box Fx \succ \Box\forall^{\lozenge}xFx}\,\Box\mathit{Df}}{\succ \forall^{\lozenge}x\Box Fx \to \Box\forall^{\lozenge}xFx}\,{\to}\mathit{Df}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists^{\lozenge}x\lozenge Fx \succ \exists x^{\lozenge}\lozenge Fx}{n \succ \mid \lozenge Fx \succ \exists x^{\lozenge}\lozenge Fx}\,\exists^{\lozenge}\mathit{Df}}{n \succ \mid Fn \succ \mid \succ \exists x^{\lozenge}\lozenge Fx}\,\lozenge\mathit{Df}}{\exists^{\lozenge}xFx \succ \mid \succ \exists x^{\lozenge}\lozenge Fx}\,\exists^{\lozenge}\mathit{Df}}{\lozenge\exists^{\lozenge}xFx \succ \exists x^{\lozenge}\lozenge Fx}\,\lozenge\mathit{Df}}{\succ \lozenge\exists^{\lozenge}xix \to \exists x^{\lozenge}\lozenge Fx}\,{\to}\mathit{Df}$$

Of course, *these* versions of the Barcan formulas need cause no trouble for the contingentist. We have granted that it is possible that (had things gone differently) I have a pet dodo. So, we could say of it that had things gone differently, *it* would have been my pet dodo. In other words, there 'is' something—in the wider sense, not of something that *exists*, but in the sense of some'thing' that *could* have existed—such that *it* is my possible pet dodo. The wider $\exists^{\lozenge}$ possibilist quantifier no more entails existence than a corresponding 'eternalist' quantifier that ranges over past, present and future existences must collect everything that it quantifies over into a domain of things that each exist *now*.[30] The motivation of the failure of the Barcan formulas, with their original existentially committing reading, remains.

If we have both $\forall^{\lozenge}$ and $\forall$ around, which is more suited to be considered the genuinely *universal* quantifier? In some sense, $\forall^{\lozenge}$ is the more *universal* of the quantifiers, since it has wider scope. We can see this, because we may use the existence predicate $\downarrow$ as a scope restrictor to define the narrower contingentist quantifiers $\forall x$ and $\exists x$ in terms of their possibilist cousins, $\forall^{\lozenge}x$ and $\exists^{\lozenge}x$. $\forall xA(x)$ is equivalent to $\forall^{\lozenge}x(x{\downarrow} \to A(x))$, and $\exists xA(x)$ is equivalent to $\exists^{\lozenge}x(x{\downarrow} \land A(x))$. These derivations show that we can replace

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[29] Compare Cresswell, in Chapter 7 "Possibilist Quantification" in his *Entities and Indices* [22], where these wider quantifiers are introduced. As with Cresswell, the argument here is that possibilist quantification can make sense of a range of linguistic phenomena. Various things we can *say* have a natural possibilist reading, which do not have a natural reading when all quantifiers are interpreted narrowly.

[30] Of course, the contingentist who is not troubled by this is the contingentist who has made peace with rejecting Quine's dictum that to be is to be the value of a bound variable. My merely possible dodo is a potential value of a variable in a possibilist quantification, by virtue of her possible existence, but she, nonetheless, still stubbornly fails to exist.

one by the other on one side of a hypersequent.[31]

$$\begin{array}{c}
\Gamma \succ \forall x A(x), \Delta \mid \mathcal{H} \\ \hline\hline
\Gamma, n \succ A(n), \Delta \mid \mathcal{H} \\ \hline\hline
n \succ \mid \Gamma, n \succ A(n), \Delta \mid \mathcal{H} \\ \hline\hline
n \succ \mid \Gamma, n{\downarrow} \succ A(n), \Delta \mid \mathcal{H} \\ \hline\hline
n \succ \mid \Gamma \succ n{\downarrow} \rightarrow A(n), \Delta \mid \mathcal{H} \\ \hline\hline
\Gamma \succ \forall^{\Diamond} x(x{\downarrow} \rightarrow A(x)), \Delta \mid \mathcal{H}
\end{array}
\begin{array}{l}
\\ \forall Df \\ K,W \\ {\downarrow}Df \\ {\rightarrow}Df \\ \forall^{\Diamond} Df \\
\end{array}
\qquad
\begin{array}{c}
\Gamma, \exists x A(x) \succ \Delta \mid \mathcal{H} \\ \hline\hline
\Gamma, n, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline
n \succ \mid \Gamma, n, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline
n \succ \mid \Gamma, n{\downarrow}, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline
n \succ \mid \Gamma \succ n \wedge A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline
\Gamma, \exists^{\Diamond} x(x{\downarrow} \wedge A(x)) \succ \Delta \mid \mathcal{H}
\end{array}
\begin{array}{l}
\\ \exists Df \\ K,W \\ {\downarrow}Df \\ {\wedge}Df \\ \exists^{\Diamond} Df \\
\end{array}$$

Conversely, there is no way to reconstruct $\forall^{\Diamond}$ in terms of the vocabulary not using the possibilist quantifiers. So, if we were after economy of conceptual resources, we could take $\forall^{\Diamond}$ or $\exists^{\Diamond}$ as primitive and $\forall$ and $\exists$ as defined using the existence predicate $\downarrow$. This should not, though, be confused with conceptual priority. The account spelled out here shows how we could think of the zone-based quantifiers $\forall$ and $\exists$ as primary, and the possibilist quantifiers are introduced by processes of ampliation, where we take the *primary* (in-zone) semantics of ruling in and ruling out (whether of terms or of sentences) as fundamental, and cross-zone application of terms is allowed when necessary, by a process of ampliation. These remarks are not offered as a knock-down argument against the treatment of outer, possibilist quantifiers as secondary, but as a suggestion for the kinds of lines of development that the inferentialist semantics for modality and quantification opens up for us.

6 FURTHER QUESTIONS AND CONCLUDING THOUGHTS

As I indicated above, this discussion is only the beginning of the exploration a proof-first treatment of first-order modal logic and of its significance for semantics and metaphysics. In this brief concluding section, I will point to further developments, raise some questions, and take stock.

The *logic* I have discussed is simply first-order modal logic, with inner and outer quantification. I have said nothing here about the identity predicate, and nothing about second- or higher-order (cf. [74, 75]). Neither have I considered other natural enrichments of the modal language, such as formalisations that allow for a more explicit object-language reference to worlds, such as is found in hybrid logic. Each of these issues can be explored in an inferentialist manner [53, 56, 58]. In particular, proof systems for hybrid logic [15] can be given a natural inferentialist treatment, by way of labelling zones, so these directions of research are natural avenues to explore the norms for *use* for a richer and more expressive language.

Once we consider identity, we have more questions concerning the divergence between the treatment of epistemic and metaphysical or subjunctive modalities. There are good reasons to think treat identity statements (between names or variables, at least) are necessary in a metaphysical or subjunctive sense, but they are by *no* means necessary in the epistemic sense. These considerations can be developed inferentially in ways that allow for the common treatment of what *can* be unified (metaphysical and epistemic alternativeness are treated as zone shifts), and distinctive treatment of what must remain distinct (the rules for the identity predicate treat the different kinds of

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[31] And, in general, if A can be replaced by A' on one side of a hypersequent, a judicious use of *Cut* and *Id* shows that you can replace them on the other side, too.

zone shift in different ways) [56]. There is much scope for a unified treatment of identity, which must wait for some careful and patient work, but the outline of the development seems clear enough [58].

Above and beyond the questions of how to develop the logic, metaphysical issues remain too. If $\downarrow$ is understood as an existence predicate then the outer quantifier ranges over things that do not exist, violating Quine's dictum that to *be* is to be the value of a variable, if you think of existence as identified with *being*. Here, the temporal analogy seems apt. There are plenty of things that did exist but no longer do, and if we talk of them (and quantify over them), we are not committed to their present existence: we picked them out, after all, as things that *did* exist but no longer do. We must also consider the thought that it must be the widest quantifier as the most significant one that is the most existentially committing. A contingentist who makes use of possibilist quantification cannot agree with this judgement, and perhaps some succour can be found in the fact that the inferentialist proof system here allows for the definition of even *wider* quantifiers that omit the definedness condition completely:

$$\begin{array}{c} \Gamma \succ A(n), \Delta \mid \mathcal{H} \\ \hline\hline \Gamma \succ \Pi x A(x), \Delta \mid \mathcal{H} \end{array} \,\Pi Df \qquad \begin{array}{c} \Gamma, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, \Sigma x A(x) \succ \Delta \mid \mathcal{H} \end{array} \,\Sigma Df$$

These quantifiers are even wider than the possibilist quantifiers: $\forall^{\Diamond}$ and $\exists^{\Diamond}$ are found by restricting Π and Σ to *possible* existents. These more general quantifiers allow for statements like: $\Sigma x \Box \neg x\downarrow$ (some things necessarily do not exist), which follows from the truth $\Box\neg\frac{1}{0}\downarrow$. Since quantifiers like this are definable, there seems to be reason to push against the conclusion that the widest quantifiers are the most important for questions of what exists.

Another metaphysical question that we have not addressed is the issue of the *truthmakers* of modal claims. Here, neither the model theory nor the proof theory for modal logic gives much of an answer as to what in the world might make modal statements true. If you think that every truth deserves a truthmaker, then the question is wide open what makes true a claim of the form $\Diamond A$ or $\Box A$. Since there is no real agreement about what might count as the truthmaker of the quantified claims $\forall x A(x)$ or $\exists x A(x)$, it is not clear that either the proof theory or the model theory will be of much help in this inquiry, since both the proof theory and the model theory give some kind of account of truth *conditions* but nothing akin to truth *makers*. However, perhaps the proof theory may give an alternative perspective on these issues.

Finally, questions around *serious actualism* arise concerning the difference between predicates that express properties (that are existence entailing) and those predicates that do not [43]. These scruples are encoded in the predicate rules: the only way to have property F (in a given context) is to exist (at that context). As we saw, not every open sentence determines a property in that sense. (Pegasus does not exist. Pegasus does not have the *property* of nonexistence in the sense that matters for the actualist.) A similar kind of distinction between predicates holds for identity and indiscernibility and *epistemic* modality. If a and b are the same thing, then any *feature* of a is a feature of b. However, epistemic modalities do not necessarily respect identity: being known to be identical to a is not a feature something can have, since it may well be that a is the same thing as b, but a is known to be identical to a and b is not known to be identical to a. So, in an epistemic modal logic with identity, we need to keep track of which (simple or complex) predicates express features and which do not. We also need to keep track of which predicates express properties in the sense salient for serious actualism, and which do not. Is the feature/non-feature distinction for the epistemic logician the same as the property/non-property distinction for the actualist? There are different options to explore here, and different modal logics that arise, as we do so.

Of course, I do not expect the interventions here in this paper to *settle* any of the questions raised here, or even to significantly shift the debate between necessitists and contingentists. Nonetheless, I hope to have offered some new tools we can use for understanding, clarifying and refining some of the fundamental concepts that we find ourselves using as we think about our world and our place in it.

FORMAL APPENDIX

In this technical appendix, I collate the formal definitions of the *proof* system discussed here, the class of *models* that result, and I prove the soundness and completeness theorems connecting them. To keep this appendix relatively short, I will discuss the rules for $\neg$, $\vee$, $\Diamond$, the inner quantifier $\exists$, the outer quantifier $\exists^\Diamond$, and the "existence" predicate $\downarrow$, leaving the other connectives, necessity and the universal quantifiers aside.[32]

The proof rules and modelling conditions for each logical concept are independent of the other concepts: in particular, the results apply equally to a language with only the inner quantifier $\exists$, or only the outer quantifier $\exists^\Diamond$, and for a language containing both. At all times, our results will apply to any first-order language $\mathcal{L}$ whose logical concepts are some selection from among ($\neg$, $\vee$, $\exists$, $\exists^\Diamond$, $\Diamond$, $\downarrow$) given some countable family Var of first-order variables, and some stock of function symbols (including countably many zero-ary function symbols, i. e. constants, or names), and predicates of any desired arity. We allow for formulas to contain unbound variables, and in what follows, whenever we substitute a term inside some formula (or hypersequent), we require that the formula be free to substitute in that position, i.e., that no variables that were free in the term become bound under that substitution.

DEFINITION 1 [HYPERSEQUENTS]: A *hypersequent* in $\mathcal{L}$ is a nonempty finite multiset of pairs of finite multisets of formulas and terms from $\mathcal{L}$.

We call each pair of multisets of formulas in a given hypersequent $\mathcal{H}$ a *sequent* in $\mathcal{H}$. Order does not matter in hypersequents, either the ordering of sequents inside hypersequents, or the ordering of formulas in the left or the right of a sequent inside the hypersequent. As an example, notation $\Gamma, A \succ \Delta \mid \mathcal{H}$ represents a hypersequent in which there is one sequent $\Gamma, A \succ \Delta$, and the remaining sequents together form the multiset $\mathcal{H}$ of sequents.[33] In this first sequent, A occurs at least once on the left, and the remaining formulas on the left, if any—including any further occurrences of A—form the multiset Γ, while Δ is the multiset of formulas on the right. Here, any or all of Γ and Δ and $\mathcal{H}$ may be empty.

DEFINITION 2 [DERIVATIONS]: A *derivation* of a hypersequent $\mathcal{H}$ is a finite tree of hypersequents, in which each leaf is an *axiom*, each transition is an instance of a *rule*.

The axioms and rules of this proof system are given in Figure 1 (see page 35). The axioms are *identity* hypersequents of the form $\star \succ \star$, where $\star$ may be either a formula, or a term. Similarly, in the structural rules of left and right (internal) and external weakening and contraction, and (additive) cut, the active item may be either a formula or a term, so for concision, we represent either case, ambiguously, with a $\star$ in the table of rules. There are *term* rules governing the definedness of different kinds of singular terms in the vocabulary (variables take a value in at least

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[32] As usual, you could either treat them as defined concepts, given the usual definitions in terms of the available vocabulary, or you could give them the rules one would expect, generalising the given rules for the selected primitive vocabulary.

[33] $\mathcal{H}$ is another hypersequent in the case that it is nonempty.

$$\star \succ \star \ \mathit{Id} \qquad \begin{array}{c} \Gamma \succ \Delta \mid \mathcal{H} \\ \hline \Gamma, \star \succ \Delta \mid \mathcal{H} \end{array}\ \mathit{KL} \qquad \begin{array}{c} \Gamma \succ \Delta \mid \mathcal{H} \\ \hline \Gamma \succ \star, \Delta \mid \mathcal{H} \end{array}\ \mathit{KR} \qquad \begin{array}{c} \mathcal{H} \\ \hline \Gamma \succ \Delta \mid \mathcal{H} \end{array}\ \mathit{KE}$$

$$\begin{array}{c} \Gamma, \star, \star \succ \Delta \mid \mathcal{H} \\ \hline \Gamma, \star \succ \Delta \mid \mathcal{H} \end{array}\ \mathit{WL} \qquad \begin{array}{c} \Gamma \succ \star, \star, \Delta \mid \mathcal{H} \\ \hline \Gamma \succ \star, \Delta \mid \mathcal{H} \end{array}\ \mathit{WR} \qquad \begin{array}{c} \Gamma \succ \Delta \mid \Gamma \succ \Delta \mid \mathcal{H} \\ \hline \Gamma \succ \Delta \mid \mathcal{H} \end{array}\ \mathit{WE}$$

$$\begin{array}{c} \Gamma \succ \star, \Delta \mid \mathcal{H} \qquad \Gamma, \star \succ \Delta \mid \mathcal{H} \\ \hline \Gamma \succ \Delta \mid \mathcal{H} \end{array}\ \mathit{Cut} \qquad \begin{array}{c} \mathcal{H} \\ \hline \mathcal{H}[n := t] \end{array}\ \mathit{Spec}^{n}_{t}$$

$$\begin{array}{c} x \succ\ \mid \mathcal{H} \\ \hline \mathcal{H} \end{array}\ \mathit{VarL} \qquad \begin{array}{c} t_i, \Gamma \succ \Delta \mid \mathcal{H} \\ \hline Ft_1 \cdots t_n, \Gamma \succ \Delta \mid \mathcal{H} \end{array}\ \mathit{FL} \qquad \begin{array}{c} t_i, \Gamma \succ \Delta \mid \mathcal{H} \\ \hline ft_1 \cdots t_n, \Gamma \succ \Delta \mid \mathcal{H} \end{array}\ \mathit{fL}$$

$$\begin{array}{c} \Gamma \succ A, B, \Delta \mid \mathcal{H} \\ \hline\hline \Gamma \succ A \vee B, \Delta \mid \mathcal{H} \end{array}\ \vee\mathit{Df} \qquad \begin{array}{c} \Gamma, A \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma \succ \neg A, \Delta \mid \mathcal{H} \end{array}\ \neg\mathit{Df} \qquad \begin{array}{c} A \succ\ \mid \Gamma \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, \Diamond A \succ \Delta \mid \mathcal{H} \end{array}\ \Diamond\mathit{Df}$$

$$\begin{array}{c} \Gamma, t \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, t\!\downarrow\ \succ \Delta \mid \mathcal{H} \end{array}\ \downarrow\!\mathit{Df} \qquad \begin{array}{c} \Gamma, n, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, \exists x A(x) \succ \Delta \mid \mathcal{H} \end{array}\ \exists\mathit{Df} \qquad \begin{array}{c} n \succ\ \mid \Gamma, A(n) \succ \Delta \mid \mathcal{H} \\ \hline\hline \Gamma, \exists^{\Diamond} x A(x) \succ \Delta \mid \mathcal{H} \end{array}\ \exists^{\Diamond}\mathit{Df}$$

Figure 1: *Structural* rules, *term* rules, and *defining* rules

one zone; complex terms are defined only at zones at which their constituents are defined, and similarly, atomic predications of terms are true only at zones at which those terms are defined), as motivated by serious actualist considerations.

When constructing proofs and derivations, it is most natural to work with multisets rather than *sets* of formulas, because we wish to keep track of the *use* of a given formula in justifying another.[34] When it comes to models cutting things so finely is often more trouble than it is worth.[35] When our attention turns to hypersequents that are *not* derivable, we pass from hypersequents to sets, and to mark this difference, we use different terminology.

DEFINITION 3 [POSITIONS, HYPERPOSITIONS, EXTENSION AND COVERAGE]: A *position* in $\mathcal{L}$ is a pair of sets of formulas and terms from $\mathcal{L}$. (Positions need not be composed of *finite* sets.) The position $[X' : Y']$ *extends* $[X : Y]$ when $X \subseteq X'$ and $Y \subseteq Y'$. ¶ A *hyperposition* $\mathcal{P}$ in $\mathcal{L}$ is a non-empty set of positions in $\mathcal{L}$ where no position in $\mathcal{P}$ extends any other position in $\mathcal{P}$. (This is the *nonredundancy* condition on hyperpositions.) ¶ The hyperposition $\mathcal{P}'$ *extends* $\mathcal{P}$ (written $\mathcal{P} \sqsubseteq \mathcal{P}'$) when for each $[X : Y]$ in $\mathcal{P}$ there is some $[X' : Y']$ in $\mathcal{P}'$ extending $[X : Y]$. ¶ Analogously, the hyper*sequent* $\mathcal{H}$ is *covered by* the hyperposition $\mathcal{P}$ when for each $\Gamma \succ \Delta$ in $\mathcal{H}$ there is some position $[X : Y]$ in $\mathcal{P}$ where each member of the multiset Γ (or Δ) is in the set X (or Y) respectively.

So, for example $[p : r \mid q : s]$ is extended by $[p, q : r \mid q : s]$ which is extended by $[p, q : r, s]$. The set $[p : q : r \mid p, q : r, s]$ is not a hyperposition because it violates the nonredundancy condition. We impose the nonredundancy condition for two reasons: first, if position $[X : Y]$ is extended by $[X' : Y']$, then making the assertions and denials recorded in $[X' : Y']$ (in some zone of a discourse) counts as making those assertions and denials recorded in $[X : Y]$ too, so as far as keeping track of the commitments in dialogue goes, nonredundant positions suffice. Second, if we restrict our attention to nonredundant hyperpositions, the following lemma holds:

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[34] *One* appeal to A might be used to justify B where a different use of A might refute C.

[35] At least for models where the underlying logic is classical or constructive, and not so substructural so as to reject the contraction rule.

LEMMA 1 [EXTENSION IS A PARTIAL ORDER]: *If* $\mathcal{P} \sqsubseteq \mathcal{P}'$ *and* $\mathcal{P}' \sqsubseteq \mathcal{P}$ *then* $\mathcal{P} = \mathcal{P}'$.

Proof: Take any position $[X : Y]$ in $\mathcal{P}$. We show that this position is also in $\mathcal{P}'$. First, since $\mathcal{P}'$ extends $\mathcal{P}$, there is some $[X' : Y']$ in $\mathcal{P}'$ extending $[X : Y]$. Then, since $\mathcal{P}$ extends $\mathcal{P}'$ there is some $[X'' : Y'']$ in $\mathcal{P}$ extending $[X' : Y']$, and *a fortiori*, $[X : Y]$. Since $\mathcal{P}$ is nonredundant, $[X : Y]$ is the only position in $\mathcal{P}$ extending $[X : Y]$, so $[X : Y] = [X' : Y'] = [X'' : Y'']$, and hence, $[X : Y]$ is also in $\mathcal{P}'$. This reasoning also works in reverse, so all positions in $\mathcal{P}$ are also in $\mathcal{P}'$, so by extensionality, $\mathcal{P} = \mathcal{P}'$. ■

Our interest in hyperpositions arises from our desire to understand what cannot be derived. A hyperposition is said to be *available* if that particular combination of assertions and denials, partitioned into different zones, is not ruled out by the meaning rules. To make availability precise, we must clarify the connection between the possibly *infinite* hyperpositions, and hypersequents, which are by design, finite.

DEFINITION 4 [HYPERPOSITIONS, HYPERSEQUENTS AND AVAILABILITY]: A hyperposition $\mathcal{P}$ is *available* when no hypersequent $\mathcal{H}$ covered by $\mathcal{P}$ is derivable.

So, for example, the position $[n, \Diamond p : q \mid q : p, n]$ is not available because it covers the derivable sequent $\Diamond p \succ \ \mid\ \succ p$.

An available hyperposition represents how things might be thought to *be*, modally speaking Of course, I could assent to the claim that something is possible, without entertaining any scenario in which that possibility take place; I might take a disjunction to hold without taking either disjunct to hold, and I might take there to be an item that satisfies condition $A(x)$ without being able to identify any such thing. However, if I grant a possibility statement, a disjunction or an existentially quantified claim, I am inviting for my claim to be *spelled out* in one of these ways. A hyperposition is *fully refined* if each of the claims it explicitly makes is spelled out in just this sort of way:

DEFINITION 5 [FULL REFINEMENT]: A hyperposition $\mathcal{P}$ is *fully refined* if and only if the following conditions hold:

- If $[ft_1 \cdots t_n, X : Y] \in \mathcal{P}$ then $t_1, \ldots, t_n \in X$.
- If $[Ft_1 \cdots t_n, X : Y] \in \mathcal{P}$ then $t_1, \ldots, t_n \in X$.
- For each variable x occurring free in $\mathcal{P}$, for some $[X : Y] \in \mathcal{P}$, $x \in X$.
- If $[A \vee B, X : Y] \in \mathcal{P}$ then either $A \in X$ or $B \in X$.
- If $[X : A \vee B, Y] \in \mathcal{P}$ then $A, B \in Y$.
- If $[\neg A, X : Y] \in \mathcal{P}$ then $A \in Y$.
- If $[X : \neg A, Y] \in \mathcal{P}$ then $A \in X$.
- If $[\Diamond A, X : Y] \in \mathcal{P}$ then for some $[X' : Y'] \in \mathcal{P}$, $A \in X'$.
- If $[X : \Diamond A, Y] \in \mathcal{P}$ then for every $[X' : Y'] \in \mathcal{P}$, $A \in Y'$.
- If $[X, t{\downarrow} : Y] \in \mathcal{P}$ then $t \in X$.
- If $[X : t{\downarrow}, Y] \in \mathcal{P}$ then $t \in Y$.
- If $[X, \exists x A(x) : Y] \in \mathcal{P}$ then for some term t, both $t, A(t) \in X$.
- If $[X : \exists x A(x), Y] \in \mathcal{P}$ then for every term t, either or $A(t) \in Y$ or $t \in Y$.
- If $[X, \exists^{\Diamond} x A(x) : Y] \in \mathcal{P}$ then for some term t and some $[X' : Y'] \in \mathcal{P}$, $t \in X'$, and $A(t) \in X$.
- If $[X : \exists^{\Diamond} x A(x) : Y] \in \mathcal{P}$ then for every term t, either $A(t) \in Y$, or for every $[X' : Y'] \in \mathcal{P}$, $t \in Y'$.

Fully refined positions are specific about the claims they make. As we will see, they are a mere hop, skip and a jump away from *models* for our logic. So, the following lemma will be the key in proving completeness, but it is interesting for its own sake, as well:

LEMMA 2 [HYPERPOSITION REFINEMENT]: *For any available hyperposition $\mathcal{P}$ in language $\mathcal{L}$, and any extension $\mathcal{L}'$ of $\mathcal{L}$ with denumerably many more constants, there is some fully refined and available hyperposition $\mathcal{P}^*$ in $\mathcal{L}'$ extending $\mathcal{P}$.*

Proof: This is a standard construction, familiar from tableaux calculi [10, 67]. We work in $\mathcal{L}'$, starting with our original hyperposition $\mathcal{P}$. We select a formula in a position inside $\mathcal{P}$ to which one of the conditions apply, and we insert the new items as required by the refinement conditions, sure at least one of the possible results is indeed an available position. For the $\Diamond$-*left* condition (dictating what is required when $\Diamond A$ occurs in the left of a position), we add a fresh position in which A occurs by itself in the left. For the $\exists$- and $\exists^\Diamond$-*left* conditions, we add a fresh constant from the supply added in $\mathcal{L}'$ as witnesses. A fully refined hyperposition is the limit of this procedure, under the extension ordering, when every formula has been processed. The only exception to this structure is the *variable* condition, which requires that any variable occurring unbound in a formula in $\mathcal{P}$ must occur positively in some position. For this, we appeal simply to the term rule *VarL*, which delivers this result immediately: if a given hyperposition is available, it is safe to add a fresh zone at which any variable at all (whether present or absent in $\mathcal{P}$), at no cost to the position.[36]

The rules of the proof calculus ensure that at each stage, we have an available position. I will illustrate this with enough examples to show how the result is proved.

Consider the disjunction clauses: for a disjunction in the right of a position, it is always safe to add both disjuncts to the right of that position, since by the following derivation

$$\cfrac{\cfrac{\Gamma \succ A \lor B, A, B, \Delta \mid \mathcal{H}}{\Gamma \succ A \lor B, A \lor B, \Delta \mid \mathcal{H}}\,{\lor}Df}{\Gamma \succ A \lor B, \Delta \mid \mathcal{H}}\,W$$

if the position with $A \lor B, A, B$ on the left is not available, then so is the corresponding position in which $A \lor B$ (without the A and the B) is also not available. For a disjunction on the *left* the derivation must use *Cut* and *Id*.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \lor B \succ A \lor B}\,Id}{\Gamma, A \lor B \succ A \lor B, \Delta \mid \mathcal{H}}\,K}{\Gamma, A \lor B \succ A, B, \Delta \mid \mathcal{H}}\,{\lor}Df \quad \dfrac{\Gamma, A, A \lor B \succ \Delta \mid \mathcal{H}}{\Gamma, A, A \lor B \succ B, \Delta \mid \mathcal{H}}\,K}{\Gamma, A \lor B \succ B, \Delta \mid \mathcal{H}}\,Cut \qquad \Gamma, B, A \lor B \succ \Delta \mid \mathcal{H}}{\Gamma, A \lor B \succ \Delta \mid \mathcal{H}}\,Cut$$

So, if a hyperposition with $A \lor B$ on the left in some position is available, then either the result of adding A to the left of that position, or the result of adding B to the left of that position must be available. If both are unavailable, then this reasoning shows that the original position must be unavailable, too.

We can reason in the same way for the $\Diamond$ conditions. First, for the left, since we have this derivation:

$$\cfrac{\cfrac{A \succ \;\mid\; \Diamond A, \Gamma \succ \Delta \mid \mathcal{H}}{\Diamond A, \Diamond A, \Gamma \succ \Delta \mid \mathcal{H}}\,{\Diamond}Df}{\Diamond A, \Gamma \succ \Delta \mid \mathcal{H}}\,W$$

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[36] The *VarL* rule is what ensures that free variables in the calculus always take values at some zone or other, as is expected by their behaviour in the quantifier rules. In this calculus, variables are not inferentially general among singular terms, as this rule holds for *variables*, but does not for arbitrary terms, which may fail to denote at all zones.

it follows that any available hyperposition with $\Diamond A$ in the left component of some position may be extended to an available hyperposition with some position with A on the left. Conversely, for the right rule, the following derivation shows that any available hyperposition with $\Diamond A$ on the right of one position may be safely extended to include A on the right of any given position.[37]

$$\dfrac{\Gamma \succ \Diamond A, \Delta \mid \Gamma' \succ A, \Delta' \mid \mathcal{H} \qquad \dfrac{\dfrac{\dfrac{}{\Diamond A \succ \Diamond A}\,\mathit{Id}}{A \succ \mid \succ \Diamond A}\,\Diamond\mathit{Df}}{\Gamma', A \succ \Delta' \mid \Gamma \succ \Diamond A, \Delta \mid \mathcal{H}}\,K}{\Gamma \succ \Diamond A, \Delta \mid \Gamma' \succ \Delta' \mid \mathcal{H}}\,\mathit{Cut}$$

Finally, we consider the $\exists^\Diamond$ conditions, leaving the remaining conditions as a straightforward exercise. First, we show that if we have any available hyperposition in which $\exists^\Diamond x A(x)$ occurs on the left in a position may be safely extended with a fresh name n, where n is on the left of a new position, while $A(n)$ is on the left in the original position.

$$\dfrac{\dfrac{n \succ \mid A(n), \exists^\Diamond x A(x), \Gamma \succ \Delta \mid \mathcal{H}}{\exists^\Diamond x A(x), \exists^\Diamond x A(x), \Gamma \succ \Delta \mid \mathcal{H}}\,\exists^\Diamond\mathit{Df}}{\exists^\Diamond x A(x), \Gamma \succ \Delta \mid \mathcal{H}}\,W$$

For $\exists^\Diamond x A(x)$ in the right of an available hyperposition, we must show that for any term t, we may either safely extend the hyperposition with $A(t)$ on the right of this same position, or we can safely extend it so that the term t is on the right of each position. Since we have the derivation in Figure 2 (for any selected sequent $\Gamma' \succ \Delta'$ in the original hypersequent), we are assured that one of these two possibilities is open to us.

This completes the proof. ■

The *atoms* in a fully refined hyperposition determine the status of each complex formula, in the manner discussed above (see p. 3). As such, they will be key to facilitating the connection between the logic characterised by this proof system and the class of models.

DEFINITION 6 [VDQS5 MODEL]: A *variable domain quantified S5 model* is a structure $\mathfrak{M}$ consisting of (*a*) A non-empty set W of *worlds*; (*b*) For each world, W, a set D_w, the *domain at* W; (Given the family D_w of domains for each $w \in W$, the *global* domain D_* is $\bigcup_{w \in W} D_w$); (*c*) An *interpretation* $[\![\cdot]\!]$ assigning values to each predicate and function symbol as follows: (c_1) For each n-ary function symbol f, $[\![f]\!]$ is a partial function $D_*^n \rightharpoonup D_*$, such that if $[\![f]\!](a_1, \ldots, a_n) \in D_w$, then $a_1, \ldots, a_n \in D_w$. (c_2) For each n-ary predicate F, $[\![F]\!]$ is a function $W \longrightarrow (D_*^n \longrightarrow \{0, 1\})$ such that if $[\![F]\!](w)(a_1, \ldots, a_n) = 1$ then $a_1, \ldots, a_n \in D_w$.

Models provide truth conditions in terms of more general *satisfaction* conditions, using assignments of values to variables.

DEFINITION 7 [ASSIGNMENTS OF VALUES]: An assignment α of values to variables, given a language $\mathcal{L}$ and model $\langle W, \{D_w : w \in W\}, [\![\cdot]\!]\rangle$ is a function from the set Var of variables to D_*.

As usual, we say that the assignment α' is an x-variant of α if α' assigns the same values to all variables as α does, except possibly the variable x.

Given an assignment of values to variables and the interpretation of terms in a given model $\mathfrak{M}$, we can give values to each term:

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[37] It is an easy exercise to simplify this derivation for the special case in which the selected position is the *same* position as that containing $\Diamond A$.

$$\dfrac{\dfrac{\dfrac{\Gamma \succ \exists^{\Diamond}xA(x),\Delta \mid \Gamma' \succ t,\Delta' \mid \mathcal{H}}{\Gamma, A(t) \succ \exists^{\Diamond}xA(x),\Delta \mid \Gamma' \succ t,\Delta' \mid \mathcal{H}}\,\mathit{K} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\exists^{\Diamond}xA(x) \succ \exists^{\Diamond}xA(x)}\,\mathit{Id}}{n \succ \;\mid\; A(n) \succ \exists^{\Diamond}xA(x)}\,\exists^{\Diamond}\mathit{Df}}{t \succ \;\mid\; A(t) \succ \exists^{\Diamond}xA(x)}\,\mathit{Spec}}{\Gamma, A(t) \succ \exists^{\Diamond}xA(x),\Delta \mid \Gamma', t \succ \Delta' \mid \mathcal{H}}\,\mathit{K}}{\Gamma, A(t) \succ \exists^{\Diamond}xA(x),\Delta \mid \Gamma' \succ \Delta' \mid \mathcal{H}}\,\mathit{Cut} \qquad \Gamma \succ \exists^{\Diamond}xA(x), A(t),\Delta \mid \Gamma' \succ \Delta' \mid \mathcal{H}}{\Gamma \succ \exists^{\Diamond}xA(x),\Delta \mid \Gamma' \succ \Delta' \mid \mathcal{H}}\,\mathit{Cut}$$

Figure 2: Derivation for the Wide Existential Quantifier on the Left, used in the proof for Lemma 2

DEFINITION 8 [INTERPRETATION OF SINGULAR TERMS]: For any model $\mathfrak{M}$ for $\mathcal{L}$ we define the interpretation partial function $[\![\cdot]\!]_\alpha$, for all singular terms (including those with free variables) by induction on the structure of a term:

- $[\![x]\!]_\alpha = \alpha(x)$, for each variable x.
- $[\![f(t_1, \ldots, t_n)]\!]_\alpha = [\![f]\!]([\![t_1]\!]_\alpha, \ldots, [\![t_n]\!]_\alpha)$ when each $[\![t_i]\!]_\alpha$ is defined, and when $[\![f]\!]$ is defined on those values; and is undefined otherwise[38]

Each singular term is a rigid designator in the sense that a term t's denotation $[\![t]\!]_\alpha$ does not depend on a choice of world. The object denoted by a term t may *exist* in some worlds and not in others, and the predicates satisfied by that object may well vary from world-to-world, but in these models, but denotation is not world relative.

DEFINITION 9 [SATISFACTION CONDITIONS IN VDQS5 MODELS]: For any model $\mathfrak{M} = \langle W, \{D_w : w \in W\}, [\![\cdot]\!]\rangle$, a world w in W and an assignment α of values to the variables, we determine the satisfaction relation $\models$ on terms[39] and formulas as follows:

- $\mathfrak{M}, w, \alpha \models t$ iff $[\![t]\!]_\alpha \in D_w$.
- $\mathfrak{M}, \alpha, w \models Ft_1 \cdots t_n$ iff $[\![F]\!](w)([\![t_1]\!]_w, \ldots, [\![t_n]\!]_w) = 1$
- $\mathfrak{M}, w, \alpha \models t{\downarrow}$ iff $\mathfrak{M}, w, \alpha \models t$.
- $\mathfrak{M}, w, \alpha \models \neg A$ iff $\mathfrak{M}, w, \alpha \not\models A$.
- $\mathfrak{M}, w, \alpha \models A \vee B$ iff $\mathfrak{M}, w, \alpha \models A$ or $\mathfrak{M}, \alpha, w \models B$.
- $\mathfrak{M}, w, \alpha \models \Diamond A$ iff $\mathfrak{M}, w', \alpha \models A$ for some $w' \in W$.
- $\mathfrak{M}, w, \alpha \models \exists x A$ iff $\mathfrak{M}, w, \alpha' \models A$ for some x-variant α' of α, where $\alpha'(x) \in D_w$.
- $\mathfrak{M}, w, \alpha \models \exists^\Diamond x A$ iff $\mathfrak{M}, w, \alpha' \models A$ for some x-variant α' of α.

As is usual, we will say that if $\mathfrak{M}, w, \alpha \models A$ then A is *true* at world w under assignment α, in model $\mathfrak{M}$, and similarly, if $\mathfrak{M}, w, \alpha \not\models A$ then A is *false*. For *terms*, we say that t *exists* at w (under α) or *doesn't exist*, depending on whether $\mathfrak{M}, w, \alpha \models t$ obtains or not. If $\star$ is either a term or a formula, we will use the generic terms *holding* and *failing* to cover both distinctions.

One crucial feature of models like these is that the only contribution of a singular term (whether a name, a variable, or a complex function term) to the satisfaction of a formula is given by way of its semantic value: we can substitute one term by another (with the same semantic value) at no change to satisfaction. We appeal to this fact at a surprising number of places in this Appendix, so it is worth calling out with another lemma.

LEMMA 3 [SEMANTIC VALUE LEMMA]: *For each term* t *and name* n, *If* $[\![t]\!]_\alpha = [\![n]\!]$, *then for each formula* A, $\mathfrak{M}, w, \alpha \models A$ *iff* $\mathfrak{M}, w, \alpha \models A[n := t]$. ¶ *Similarly, for each variable* x *(not free in* t*) if* $[\![t]\!]_\alpha = \alpha(x)$, *then for each formula* A, $\mathfrak{M}, w, \alpha \models A[n := x]$ *iff* $\mathfrak{M}, w, \alpha \models A[n := t]$.

Proof: This proof is a straightforward induction on the truth conditions for the formula A. The assumption that $[\![t]\!]_\alpha = [\![n]\!]$ (or that $[\![t]\!]_\alpha = \alpha(x)$) ensures that the result holds when A is a predication, and then this fact suffices to complete the proof, noting that t is required to be free for n in A. ■

These models are sound and complete for our hypersequent proof system, not just in the sense that each provable formula is true at every world of every model (under any assignment), but in the strong sense that no derivable hypersequent has a counterexample in any model.

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[38] Note that n may be zero, so this clause applies to zero-ary function symbols, i.e. constants, and this definition allows for such constants to be undefined by $[\![\cdot]\!]$.

[39] Since our hypersequents include terms as well as formulas, extending the satisfaction relation to terms makes the presentation simpler.

DEFINITION 10 [COUNTEREXAMPLES, VALIDITY AND VERIFYING POSITIONS]: A model $\mathfrak{M}$ (and assignment α) is a COUNTEREXAMPLE to a hypersequent $\mathcal{H}$ iff for each sequent $\Gamma \succ \Delta$ in $\mathcal{H}$ there is some world w in $\mathfrak{M}$ where for each $\star \in \Gamma$, $\star$ holds at w under α, and for each $* \in \Delta$, $*$ fails at w under α. ¶ We say that a hypersequent $\mathcal{H}$ *holds* on the model $\mathfrak{M}$ iff $\mathfrak{M}$ is not a counterexample to $\mathcal{H}$ under any assignment. ¶ A hypersequent is VDQS5 VALID if and only if it holds on every model. ¶ We think of hyperpositions as dual to hypersequents, so given a hyperposition $\mathcal{P}$ composed of positions $[X_i : Y_i]$ we say that the model $\mathfrak{M}$ (and assignment α) VERIFIES $\mathcal{P}$ iff for each position $[X_i : Y_i]$ there is some world w in $\mathfrak{M}$ where (under α) each member of X_i holds at w and each member of Y_i fails.

THEOREM 4 [SOUNDNESS FOR VDQS5 MODELS]: *No derivable hypersequent has a counterexample in any* VDQS5 *model. That is, if the hypersequent* $\mathcal{H}$ *is derivable, it is* VDQS5 *valid.*

Proof: The proof is an induction on the derivation of the hypersequent $\mathcal{H}$. Any counterexample to an axiomatic identity sequent is impossible by definition, so the base case is proved. For each rule, it suffices to show that from a counterexample to the *conclusion* of a rule, we may find a counterexample to at least one premise of that rule. This holds immediately for the structural rules of weakening and contraction: any counterexample to the conclusion simply *is* a counterexample to the premise. ¶ Similarly, for *Cut*, any counterexample to the conclusion is a counterexample to one premise or other: take the world w and assignment α at which the sequent $\Gamma \succ \Delta$, the site of the cut (formula or term) is evaluated. At this world, if the item $\star$ fails at w (under α), then this model is a counterexample to the first premise of the inference, and if it holds, it is a counterexample to the second premise. ¶ For *Spec*$^{n}_{t}$, the final structural rule, any counterexample $\mathfrak{M}, \alpha$ to a hypersequent $\mathcal{H}[n := t]$ can be transformed into a counterexample $\mathfrak{M}', \alpha$ to $\mathcal{H}$, by setting $\mathfrak{M}'$ to be the same as the model $\mathfrak{M}$, except that $\llbracket n \rrbracket$ is set to be $\llbracket t \rrbracket_\alpha$. This is a counterexample to $\mathcal{H}$, given Lemma 3.

Now for the *term* rules: for F*L* and f*L*, we note that $Ft_1 \cdots t_n$ is true at w under α only when $\llbracket t_1 \rrbracket_\alpha, \ldots, \llbracket t_n \rrbracket_\alpha$ are defined and in D_w, and similarly $\llbracket ft_1 \cdots t_n \rrbracket_\alpha$ is defined and in D_w only when $\llbracket t_1 \rrbracket_\alpha, \ldots, \llbracket t_n \rrbracket_\alpha$ are defined and in D_w, so any counterexample to the conclusion is, by construction, a counterexample to the premise. ¶ For *VarL*, we note that for any model and assignment α that serves as a counterexample to $\mathcal{H}$, and any variable x, the assignment α gives x a value in D_*, which means the value is in D_w for some world w. The sequent $x \succ$ has a counterexample at w, since x takes a value at is present at that world, so our model (and assignment) serves as a counterexample to premise hypersequent, too.

Finally, for the defining rules, we note that the $\vee$, $\neg$, $\Diamond$ and $\downarrow$ rules are immediate: any counterexample to the conclusion simply is a counterexample to the premise, and vice versa. ¶ For $\exists$*Df* if we have a model in which there is a world at which n is defined and $A(n)$ is true (under the assignment α) then under this assignment, $\exists x A(x)$ is true, since $A(x)$ holds when x is assigned the value $\llbracket n \rrbracket$ (by Lemma 3). So, our model that is a counterexample to the premise sequent is a counterexample to the conclusion. Conversely, if $\exists x A(x)$ holds at a world w in our model $\mathfrak{M}$, under α, then it follows that there is some x-variant α' of α where $\alpha'(x) \in D_w$ and $A(x)$ holds at w under α'. Since n is a fresh name (as is required for this rule) then under model $\mathfrak{M}'$ which is like $\mathfrak{M}$ except we assign $\llbracket n \rrbracket = \alpha'(x)$, then in $\mathfrak{M}'$, $A(n)$ holds (under α') by Lemma 3, and n is defined at w. The other formulas in Γ, Δ and $\mathcal{H}$ do not contain n, so their values are unchanged under $\mathfrak{M}'$, and so, $\mathfrak{M}'$ (and α') provide a counterexample to the premise sequent. ¶ The reasoning for $\exists^{\Diamond}$*Df* follows in the same way, and so, we can declare the soundness theorem proved. ■

Now we turn to the *completeness theorem*. We aim to show that *only* the derivable sequents are VDQS5 valid. As usual, the proof construction establishes the contrapositive, that for any underivable hypersequent, there is some VDSQ5 model that serves as a counterexample. Here, most of our work has already been done, in Lemma 2. We know that for any underivable hypersequent in a given language $\mathcal{L}$, we have a fully refined hyperposition, covering the hypersequent, in a language extending $\mathcal{L}$ with a countable stock of extra names. We use such a hyperposition to construct a model, and we then show that this model provides a counterexample to our original hypersequent.

DEFINITION 11 [THE MODEL $\mathfrak{M}_{\mathcal{P}^*}$ FOR A REFINED HYPERPOSITION $\mathcal{P}^*$]: For any fully refined hyperposition $\mathcal{P}^*$, we define the model $\mathfrak{M}_{\mathcal{P}^*}$ as follows: the *worlds* are the *positions* in $\mathcal{P}^*$. ¶ The domain at the world $[X : Y]$ is the set of *terms* in X. So, the global domain is the set of terms occurring in the left at *some* position inside $\mathcal{P}^*$. ¶ We interpret the n-ary function symbol f by setting $[\![f]\!](t_1, \dots, t_n) = ft_1 \cdots t_n$, if the term $ft_1 \cdots t_n$ is in the global domain, and allowing the value of the partial function $[\![f]\!]$ at $(t_1, \dots, t_n)$ to be undefined, otherwise. ¶ We interpret the n-ary predicate F by setting $[\![F]\!](t_1, \dots, t_n)([X : Y]) = 1$ if $Ft_1 \cdots t_n \in X$, and 0 otherwise.

LEMMA 5 [CORRECTNESS OF $\mathfrak{M}_{\mathcal{P}^*}$]: *The structure $\mathfrak{M}_{\mathcal{P}^*}$ defined in terms of the fully refined hyperposition $\mathcal{P}^*$ is indeed a model, and it verifies the hyperposition $\mathcal{P}^*$.*

Proof: To confirm that $\mathfrak{M}_{\mathcal{P}^*}$ is a model, we must check that the conditions for function symbols and predicates hold. For function symbols, we require that if $[\![f]\!](t_1, \dots, t_n) \in D_w$ then $t_1, \dots, t_n \in D_w$. That is, we need that for each position $[X : Y]$ in $\mathcal{P}^*$, if $ft_1 \cdots t_n \in X$ then $t_1, \dots, t_n \in X$ too. This is a condition of full refinement, so it holds of $\mathcal{P}^*$. ¶ Similarly for predicate symbols, we require that if $[\![F]\!](w)(t_1, \dots, t_n) = 1$ then $t_1, \dots, t_n \in D_w$. That is, we need that for each $[X : Y]$ in $\mathcal{P}^*$, if $Ft_1 \cdots t_n \in X$ then $t_1, \dots, t_n \in X$ too. This is a condition of full refinement, so it also holds of $\mathcal{P}^*$. So, our structure $\mathfrak{M}_{\mathcal{P}^*}$ is indeed a model.

To confirm that $\mathfrak{M}_{\mathcal{P}^*}$ verifies the hyperposition $\mathcal{P}^*$ we show that for each position $[X : Y]$ in $\mathcal{P}^*$, the members of X *hold* at the world $[X : Y]$ while the members of Y all *fail*, under the *identity* assignment of values to the variables. We proceed by induction on the structure, starting with terms, and then moving to formulas. ¶ For terms: by the definition of $\mathfrak{M}_{\mathcal{P}^*}$, the term t holds at $[X : Y]$ iff $t \in X$. If $t \in Y$ then t cannot be in X (since $\mathcal{P}^*$ is available) so t fails at $[X : Y]$, as desired. ¶ Again, by the definition of $\mathfrak{M}_{\mathcal{P}^*}$, the atomic formula $Ft_1 \cdots t_n$ holds at $[X : Y]$ iff it is a member of X. If $Ft_1 \cdots t_n \in Y$, then it cannot be in X (since $\mathcal{P}^*$ is available), so it fails at $[X : Y]$. ¶ For negations, if $\neg A$ is in the left component X then by full refinement, A is in Y, and by the induction hypothesis, A fails at $[X : Y]$, so by the modelling condition, $\neg A$ holds at $[X : Y]$. On the other hand, if $\neg A$ is in Y, then by full refinement, $A \in X$, and so, A holds at $[X : Y]$, ensuring that $\neg A$ fails at $[X : Y]$ as desired. ¶ For disjunctions: if $A \vee B \in X$, then by full refinement, $A \in X$ or $B \in X$, which means either A or B holds at $[X : Y]$, and hence $A \vee B$ holds at $[X : Y]$. On the other hand, if $A \vee B \in Y$, then by full refinement, $A, B \in Y$, which means that both A and B fail at $[X : Y]$, ensuring that $A \vee B$ fails at $[X : Y]$ as desired. ¶ If $\Diamond A \in X$, then by full refinement, there is some position $[X' : Y'] \in \mathcal{P}^*$ where $A \in X'$, and so, by hypothesis, some world in $\mathfrak{M}_{\mathcal{P}^*}$ at which A holds. So, $\Diamond A$ holds at $[X : Y]$. On the other hand, if $\Diamond A \in Y$, then by full refinement, $A \in Y'$ for each position $[X' : Y']$, and so, by hypothesis, A fails at each world in $\mathfrak{M}_{\mathcal{P}^*}$. So, $\Diamond A$ fails at $[X : Y]$ as desired. ¶ If $\exists x A(x) \in X$, then by full refinement, there is some term t where $t \in X$, and $A(t) \in X$. So, by hypothesis, t holds at $[X : Y]$ and is in the domain at $[X : Y]$. Since $A(t)$ holds at $[X : Y]$, by Lemma 3, $A(x)$ holds there under the x-variant assignment that sends x to t. So, $\exists x A(x)$ holds at $[X : Y]$ under the original (identity) assignment. On the other hand, if $\exists x A(x) \in Y$, then for any term t, by full refinement, either $A(t) \in Y$ or $t \in Y$. So, under any possible x-variant assignment where x takes a value in the domain of $[X : Y]$ (again, by Lemma 3) $A(x) \in Y$, meaning that $\exists x A(x)$ fails at $[X : Y]$ as desired. ¶ If $\exists^{\Diamond} x A(x) \in X$, then by full refinement, there is some term t and some position $[X' : Y']$ in $\mathcal{P}^*$ where $t \in X'$, and $A(t) \in X$. Since $t \in X'$, t is in the global domain, and since $A(t)$ holds at $[X : Y]$, by Lemma 3, $A(x)$ holds there under the x-variant assignment that sends x to t. So, $\exists^{\Diamond} x A(x)$ holds at $[X : Y]$ under the original (identity) assignment. On the other hand, if $\exists^{\Diamond} x A(x) \in Y$, then for any term t, by full refinement, either $A(t) \in Y$ or t is not in the global domain. So, under any possible x-variant assignment (again, by Lemma 3) $A(x) \in Y$, meaning that $\exists x^{\Diamond} A(x)$ fails at $[X : Y]$ as desired. ■

This proof was the bulk of the work for showing completeness. Now for the payoff:

THEOREM 6 [COMPLETENESS FOR VDQS5 MODELS]: *If $\mathcal{H}$ is valid in every VDQS5 model, it is derivable.*

Proof: If $\mathcal{H}$ is *not* derivable, then consider the corresponding position fully refined position $\mathcal{P}^*$ extending $\mathcal{H}$ (which exists, by Lemma 2). The model $\mathfrak{M}_{\mathcal{P}^*}$ verifies $\mathcal{P}^*$, and so, serves as a counterexample to the original hypersequent $\mathcal{H}$. So, any underivable hypersequent has a counterexample, as desired. ■

In this appendix, we provided an independently motivated hypersequent proof system to represent natural reasoning for the connectives, quantifiers and modal operators, combined in a way to respect the contingentist's use of those concepts. The proof system incorporates rules for inner quantifiers (which, for the contingentist, have existential import) and outer quantifiers (which, for the contingentist, allow for *possibilist* quantification). The defining rules for the logical connectives, operators and quantifiers are modular, in that each can be independently adopted or avoided, independently of the others. The *models* corresponding to this proof system are standard, so the fact that we have soundness and completeness theorems connecting the proof system and the models, means that the *inference* rules governing modal and quantificational concepts can be used to give us an independent account of why the concepts that we use in our own thought and talk might correspond to the models that have proved to be so useful.

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