Greg Restall, *Arché, University of St Andrews* · 2025 Wendy Huang Lectures, National Taiwan University

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I have three aims for this set of lectures. ¶ To give an account of what is so distinctive about *logic*, insofar as logical notions have a grip on whatever can be *said* or *thought*. ¶ To clarify the connections between logic and *semantics*, the theory of meaning. ¶ To provide a philosophical motivation and non-technical introduction to the more technical work I have done in my forthcoming manuscript *Proof, Rules, and Meaning*.

`https://consequently.org/w/prm`

Today's topic is the fundamentals of *proof*, *rules*, and *meaning*. I aim to give an opinionated introduction to INFERENTIALIST SEMANTICS.

1.1 LOGIC & SEMANTICS / PROOFS & MODELS

What has *logic* to do with semantics? There are many possible answers:

- Logics give an account of the meanings of some concepts: the logical constants.
- Logical concepts (equivalence, entailment, contradiction, etc.) are useful when accounting for the significance of what is said.
- Logical tools provide frameworks for semantic theories. (In the 20th and 21st Centuries, truth-conditional semantic theories are chief among these.)

Kurt Gödel's *completeness theorem* is, by one measure, the greatest result in logic.

Kurt Gödel (1906–1978)

VALIDITY (PROOFS): $A \vdash B$ says that *there is a proof* from A to B.

VALIDITY (MODELS): $A \vDash B$ says that *there is no counterexample* to the argument from A to B. (That is, no *model* taking the premise to be *true* and the conclusion *false*.)

SOUNDNESS: If $A \vdash B$ then $A \vDash B$. (If *there is a proof* then *there is no counterexample*.)

COMPLETENESS: If $A \vDash B$ then $A \vdash B$. (If *there is no counterexample* then *there is a proof*.)

It would be surprising if models were significant for semantics and proofs were not.

However, many have taken model theory as properly *semantics* while relegating proof theory to mere *syntax*.

For a recent example, see Wolfgang Schwarz's recent review of Andrew Bacon's *Philosophical Introduction to Higher-order Logics*. `https://www.umsu.de/blog/2025/821`.

1.2 QUESTIONS & ANSWERS / NATURAL DEDUCTION

Start with *declaratives*: the expressions that we use to make *claims*. ¶ When I make a declarative claim or think a declarative thought *I take a stand on an issue*. ¶ This is an issue about which others, at least potentially, might *disagree*.

Though, with Nuel Belnap, we should remember that *declaratives are not enough* [2].

To make the claim that A, I take it, has the same upshot as answering *yes* to the question $A^?$. We'll call that speech act *asserting* A. ¶ To say *no* to $A^?$ is to *deny* A.

A proof is a specific species of the genus *answers to justification requests*. These are nearly as fundamental to the notion of making a *claim* as are polar questions. Whenever an assertion is made, a possible response for a hearer is to demur and ask for justification.

Nuel Belnap (1930–2024)

To *prove* A, given some background (in which certain given claims are taken for granted) is to *show* that A, in a certain rigorous gap-free sense. Some things follow from this thin characterisation:

'Gap-free' in *what* sense? Good question. This will be clarified as we go along.

First, proofs can be chained together. ¶ If we have a proof of A from $\mathcal{C}$, and a proof of B from $\mathcal{C}'$, A (a context, in which we take A for granted, among other things), then we can prove B from the $\mathcal{C}, \mathcal{C}'$ by proving B from $\mathcal{C}'$, A (meeting the request to justify B, but incurring the cost of appealing to A) and then using the proof from $\mathcal{C}$ to justify the use of A, on the basis of $\mathcal{C}'$.

π and π' are chained like this:

$$\begin{array}{c} \mathcal{C} \\ \vdots\,\pi \\ \mathcal{C}',\ A \\ \vdots\,\pi' \\ B \end{array}$$

Chaining proofs together is of little use unless we have some proofs to begin with. ¶ This is the other point at which the background context $\mathcal{C}$ can play a role. If the formula A is already taken for granted in $\mathcal{C}$, we need do nothing else to justify it *in that context*. It is taken as given.

We depict the proof of A in a context in which A is taken as given, like this:

$$A$$

Writing $\mathcal{C} \vdash A$ to represent the existence of a proof of A from the context $\mathcal{C}$, we have the following two *structural* features of proof:

- If A is in $\mathcal{C}$ then $\mathcal{C} \vdash A$
- If $\mathcal{C} \vdash A$ and $\mathcal{C}', A \vdash B$ then $\mathcal{C}, \mathcal{C}' \vdash B$.

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We represent *showing* that a context $\mathcal{C}$ is defective by expanding our notion of proof:

$$\dfrac{A \qquad \cancel{A}}{\sharp}$$

More generally, if I have a proof from a starting context $\mathcal{C}$ to A, I can extend this into a proof that leads us from $\mathcal{C}, \cancel{A}$ to $\sharp$. We can use the rule to extend a proof in this way:

$$\dfrac{\begin{matrix}\mathcal{C}\\ \vdots\,\pi \\ A\end{matrix} \qquad \cancel{A}}{\sharp}\;\lightning$$

Conversely, If I *try* to deny A, and it turns out that this is ruled out, then the context implicitly settles the question $A^?$. This grounds the following rule:

$$\dfrac{\begin{matrix}\mathcal{C}\;[\cancel{A}]^i\\ \vdots\,\pi \\ \sharp\end{matrix}}{A}\;\downarrow^i$$

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The contexts $\mathcal{C}$ in which claims are ruled in and others are ruled out form the different POSITIONS we are able to formulate using the conceptual resources of the language at hand. ¶ We will say that a position $\mathcal{C}$ is OUT OF BOUNDS if there is a proof from $\mathcal{C}$ to $\sharp$. ¶ A position is AVAILABLE if it is not out of bounds.

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Here are the basic inference rules governing conjunction, from Gentzen's pioneering work on natural deduction [4], popularised and systematised by Dag Prawitz [6].

Gerhard Gentzen (1909–1945)

$$\dfrac{A \qquad B}{A \wedge B}\;\wedge I \qquad \dfrac{A \wedge B}{A}\;\wedge E \qquad \dfrac{A \wedge B}{B}\;\wedge E$$

Similar rules can be given for the conditional, and for negation.

$$\dfrac{\begin{matrix}[A]^i\\ \vdots \\ B\end{matrix}}{A \to B}\;\to I^i \qquad \dfrac{A \to B \qquad A}{B}\;\to E \qquad \dfrac{\begin{matrix}[A]^i\\ \vdots \\ \sharp\end{matrix}}{\neg A}\;\neg I^i \qquad \dfrac{\neg A \qquad A}{\sharp}\;\neg E$$

We can combine these rules, to formulate longer proofs. ¶ The first is a proof from $\neg\neg p$ to p. The second, from $\neg(p \to q)$ to $p \wedge \neg q$.

In the second proof, note that the rightmost $\to I$ inference is *vacuous*: zero instances of the assumption p are discharged in the step from q to $p \to q$. Similarly, the first $\downarrow$ inference is vacuous, in that the discharged denial q occurs zero times in the context. This proof system is unapologetically *irrelevant*.

$$\dfrac{\dfrac{\neg\neg p \qquad \dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\;\lightning}{\neg p}\;\neg I^1}{\sharp}\;\neg E}{p}\;\downarrow^2$$

$$\dfrac{\dfrac{\dfrac{\neg(p \to q) \qquad \dfrac{\dfrac{\dfrac{[p]^1 \qquad [\cancel{p}]^2}{\sharp}\;\lightning}{q}\;\downarrow}{p \to q}\;\to I^1}{\sharp}\;\neg E}{p}\;\downarrow^2 \qquad \dfrac{\dfrac{\neg(p \to q) \qquad \dfrac{[q]^3}{p \to q}\;\to I}{\sharp}\;\neg E}{\neg q}\;\neg I^3}{p \wedge \neg q}\;\wedge I$$

These rules suffice for classical propositional logic [9], given the bilateral setting in which the context can contain both positive judgements (A) and negative ones ($\cancel{A}$).

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It is compelling to think of the introduction and elimination rules for logical concepts as *definitions* of the concepts they govern. ¶ Can we take this thought *seriously*?

Taking certain inference rules to be definitional in this sense provides one possible response to the Tortoise's behavior in Lewis Carroll's classic paper [3, 10].

1.3 RULES & DEFINITIONS / INFERENCE & MEANING

As Arthur Prior showed in 1960 [7], if we could simply specify introduction and elimination rules for a putative concept and declare them to be *definitions*, we could prove anything we like. ¶ Prior introduced tonk with two rules:

Arthur Prior (1914–1969)

$$\dfrac{A}{A \text{ tonk } B}\;\text{tonk}I \qquad \dfrac{A \text{ tonk } B}{B}\;\text{tonk}E$$

These rules trivialise. From any premise A, you can prove any conclusion B.

$$\dfrac{\dfrac{A}{A \text{ tonk } B}\;\text{tonk}I}{B}\;\text{tonk}E$$

A definition shows us how extend a starting language $\mathcal{L}_1$ with a new item of vocabulary, to form an extended language, $\mathcal{L}_2$, in which we can use the new item in some determinate way. ¶ An *abbreviative* definition is a paradigm case.

In the language of *proofs*, introducing a newly defined expression E in terms of previously introduced vocabulary D, can be seen as adopting an *invertible* rule. ¶ Abbreviative definitions have two useful features. ¶ They are CONSERVATIVE: positions in the original language ($\mathcal{L}_1$) that were available beforehand remain available afterward from the point of view of $\mathcal{L}_2$.

$$\frac{D}{\overline{\overline{E}}}\,{\scriptstyle EDf}$$

Abbreviative definitions are also UNIQUELY DETERMINED: The newly introduced expression E is *fixed*, in the sense that if we happened to add the term *twice* (E_1 and E_2, say), using definitions of the same form, then the two defined terms are equivalent, in the sense that we can infer one from the other, in any context.

CONSERVATIVITY and UNIQUENESS are useful features. A *conservative* definition is safe. ¶ A *uniquely defining* rule means that the concept is as completely delineated as the prior vocabulary. It introduces no new 'give' into the system that wasn't already there.

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The inference rules for conjunction, the conditional, and negation, given above are *not* abbreviative definitions. They are also not necessarily conservative, unless we are careful to specify the commitments in our starting language. ¶ Using the conjunction rules alone, we can construct a proof from p, q to p. If our original language $\mathcal{L}_1$ did not tell us that there was a proof from p, q to p, then the addition of these conjunction rules would be non-conservative.

$$\dfrac{\dfrac{p \quad q}{p \wedge q}\,{\scriptstyle \wedge I}}{p}\,{\scriptstyle \wedge E}$$

Our assumptions ensure that there is a proof from p, q to p. That was one reason for being explicit about the structural assumptions governing proof *as such*. Other choices are open to us [9].

The introduction and elimination rules for the conditional do not look anything like *abbreviative definitions*. Making the contexts explicit, we have:

$$\frac{\mathcal{C}, A \succ B}{\mathcal{C} \succ A \to B}\,{\scriptstyle \to I} \qquad \frac{\mathcal{C} \succ A \to B \qquad \mathcal{C}' \succ A}{\mathcal{C}, \mathcal{C}' \succ B}\,{\scriptstyle \to E}$$

$$\frac{\begin{array}{c}[A]^i\\ \vdots \\ B\end{array}}{A \to B}\,{\scriptstyle \to I^i} \qquad \frac{A \to B \qquad A}{B}\,{\scriptstyle \to E}$$

If, in the *elimination* rule we restrict our attention to the case in which A alone is assumed, then the two rules are in fact two directions of the one transition:

$$\frac{\mathcal{C}, A \succ B}{\overline{\overline{\mathcal{C} \succ A \to B}}}\,{\scriptstyle \to Df}$$

This is a two-way rule, and one that can be treated in just the same way as an abbreviative definition. ¶ This rule is clearly *uniquely defining*. ¶ That the rule is *conservative* over $\mathcal{L}_1$ requires a rather more work to demonstrate, and the details depend a little on how the starting language $\mathcal{L}_1$ is formulated.

The details of how this works are spelled out in Chapter 6 of *Proof, Rules, and Meaning*.

$$\frac{\mathcal{C}, A \succ \sharp}{\overline{\overline{\mathcal{C} \succ \neg A}}}\,{\scriptstyle \neg Df} \qquad \frac{\mathcal{C} \succ A \qquad \mathcal{C} \succ B}{\overline{\overline{\mathcal{C} \succ A \wedge B}}}\,{\scriptstyle \wedge Df} \qquad \frac{\mathcal{C}, A \succ C \qquad \mathcal{C}, B \succ C}{\overline{\overline{\mathcal{C}, A \vee B \succ C}}}\,{\scriptstyle \vee Df}$$

We can see these rules as showing how to *add* these concepts to our vocabulary, demarcating exactly how to evaluate claims involving the newly defined concepts.

The introduction and elimination rules for *tonk* are not defining rules, and cannot be transformed into such a form, since they do not even conservatively extend a basic language containing only atomic vocabulary.

If we selected either tonk*I* or tonk*E* to be strengthened into an invertible rule, then the result would be safe. However, the first choice would render A tonk B a needlessly complicated way to say A, and the second, a needlessly complicated way to say B.

Viewing the logical concepts as introduced by defining rules, we have a ready answer to how it is that proofs can be gap-free.

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Treating these concepts as given by defining rules doesn't mean that everyday appeals to 'and', 'if', 'or', and 'not' must be given by these rules, or that the everyday concepts agree with the sharply defined concepts ∧, →, ∨ and, ¬.

In what sense do defining rules act as *semantics* for the connectives they govern? In one sense—as giving rules for how to interpret, and to use these concepts—this is completely straightforward. The defining rules are rules for *use* of the governed concepts.

1.4 POSITIONS & LIMITS / MODELS & TRUTH

Given any available position $\mathcal{C}$, we can say that A is *true-in-$\mathcal{C}$* if $\mathcal{C} \vdash A$, and that A is *false-in-$\mathcal{C}$* if $\mathcal{C}, A \vdash \sharp$. We then have:

- $A \wedge B$ is true-in-$\mathcal{C}$ iff A is true-in-$\mathcal{C}$ and B is true-in-$\mathcal{C}$.
- $A \wedge B$ is false-in-$\mathcal{C}$ *if* A is false-in-$\mathcal{C}$ or B is false-in-$\mathcal{C}$.
- $A \vee B$ is true-in-$\mathcal{C}$ *if* A is true-in-$\mathcal{C}$ or B is true-in-$\mathcal{C}$.
- $A \vee B$ is false-in-$\mathcal{C}$ iff A is false-in-$\mathcal{C}$ and B is false-in-$\mathcal{C}$.
- $A \to B$ is true-in-$\mathcal{C}$ *if* A is false-in-$\mathcal{C}$ or B is true-in-$\mathcal{C}$.
- $A \to B$ is false-in-$\mathcal{C}$ iff A is true-in-$\mathcal{C}$ and B is false-in-$\mathcal{C}$.
- $\neg A$ is true-in-$\mathcal{C}$ iff A is false-in-$\mathcal{C}$.
- $\neg A$ is false-in-$\mathcal{C}$ iff A is true-in-$\mathcal{C}$.

Note the *if* in the disjunctive clauses.

Finite available positions are essentially incomplete. ¶ Any available finite position $\mathcal{C}$ may be extended, systematically, into a *limit* position $\mathcal{C}^*$, settling all issues, one way or another [8]. ¶ In any limit position $\mathcal{C}^*$, the truth conditions given above hold, except that the disjunctive conditions hold as *biconditionals*, rather than conditionals.

A limit position for $\mathcal{L}$ determines a two-valued boolean valuation on $\mathcal{L}$. ¶ Connectives are defined in terms of inference rules. Proofs are primary, and models are secondary.

There is more to semantics than truth-in-a-model—we would like to give an account of *truth* [5], and how it relates to truth-in-a-model. What can we say about about truth *per se*? Here, we must step from talking *about* the language we are modelling, to actually *using* it. ¶ Here is how to specify the limit position $\mathcal{C}^*$ which counts as a model of truth. For each sentence A in the language $\mathcal{L}$, if A, then you add A to the context, affirmatively. Otherwise, you add its denial, $\not{A}$.

This is the point at which the sentence A is *used*, and not merely mentioned.

If $\mathcal{L}$ contains context-sensitive expressions, here, we assume that the context of use for asserting A when checking for the position is the same context of use *in* the position so evaluated. Matters are more subtle if we wish to include positions containing judgements with different contexts of use, and context-sensitive expressions.

There is more to say about the relationship between inference rules, in general, and truth-conditional theories of meaning, but this is enough for now.

FOR NEXT TIME

ONE QUESTION: A common criticism about inferentalism, is that the semantic relations are language-internal, and are disconnected from *reality*. This criticism is understandable, but mistaken. In the next lecture I will show how an inferentialist semantics can take account of language–world relations.

ONE CHALLENGE: I will also given an account of how inferentialist conditions can shed light on the meanings of modal concepts.

ONE INSIGHT: The answers provided for both issues will involve saying more about the background context governing judgements *as such*. Once that is clarified, the details of the definitions of the quantifiers and modal operators—and their interactions—will be relatively straightforward.

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