

Proofs with Star & Perp

pluralism & proofs for different logics

Greg Restall



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10 DECEMBER 2021

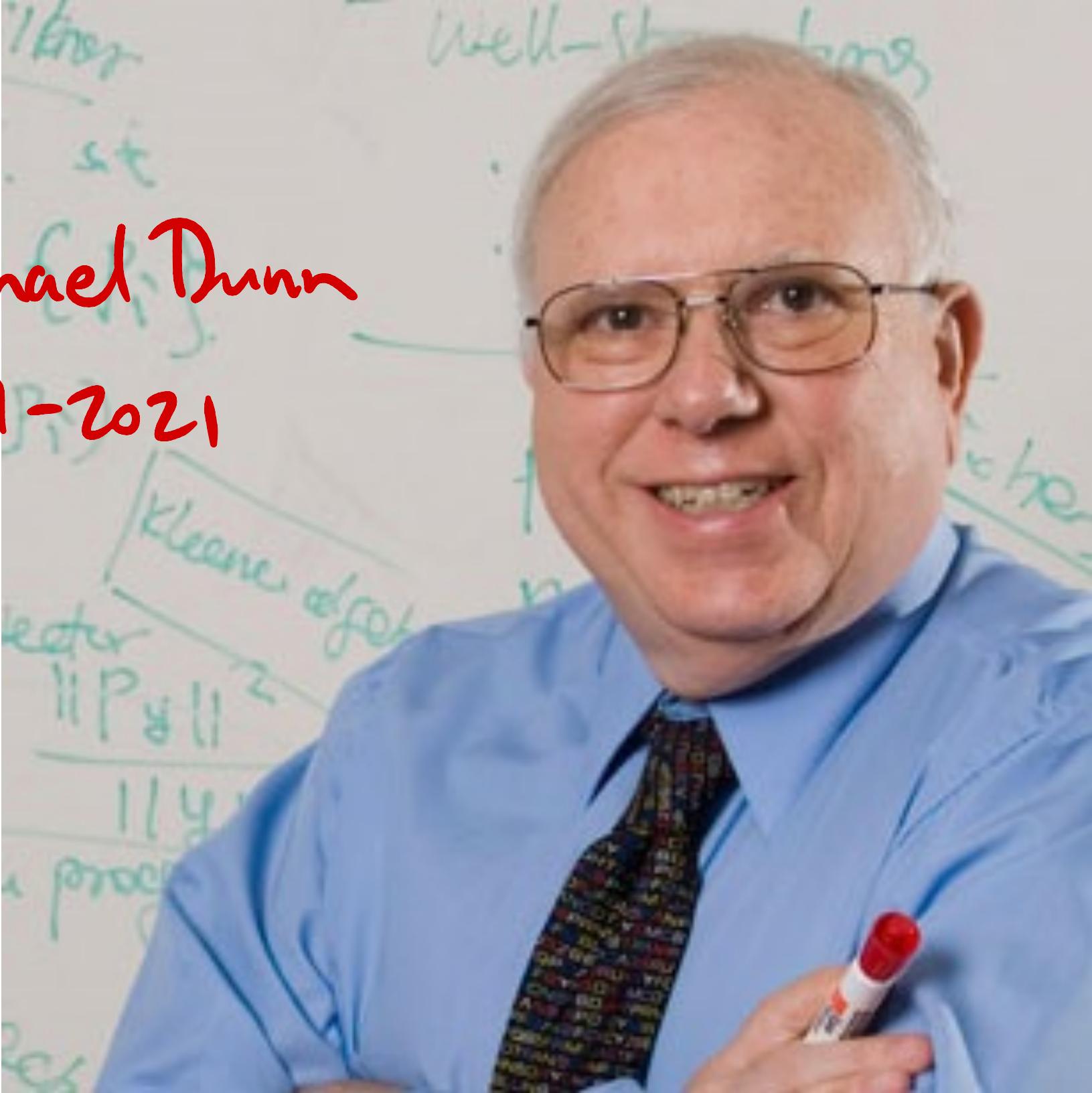
My Goal

To analyse a pluralist approach
to classical, intuitionist & paraconsistent
logics from a properly proof-first perspective.

[This continues the work of my paper,
"Pluralism & Proofs" Erkenntnis 2014.]

J. Michael Dunn

1941–2021



Star and Perp: Two Treatments of Negation¹

J. Michael Dunn
Departments of Philosophy and Computer Science

1. Star and Perp. In the literature on non-classical logics there are two common treatments of negation, illustrated by the following semantic clauses:

- (\neg^*) $\chi \models \neg\varphi$ iff $\chi^* \not\models \varphi$
- ($\neg\perp$) $\chi \models \neg\varphi$ iff $\forall\alpha(\alpha \models \varphi \text{ implies } \alpha \perp \chi)$.

The first uses a unary operation * ("star") on some underlying set of "states" ("worlds," "situations," "set-ups," "cases," whatever). The second definition uses a binary relation \perp ("perp") on such an underlying set.² It is the purpose of this paper to show that there is a close connection between these two apparently different treatments.

The definition (\neg^*) is perhaps most famous from the Routley-Meyer semantics for relevance logic (see e.g., Routley and Routley (1972), Routley and Meyer (1973)), though its mathematical essence can be traced to the Bialynicki-Birula and Rasiowa (1957) representation of De Morgan lattices (cf. Dunn (1966, 1967, 1986)).³ The definition ($\neg\perp$) is perhaps most famous from the Goldblatt (1974) semantics for orthologic, though its most familiar current use is in the Girard (1987) semantics for linear logic. It too has a more ancient history, going back to Birkhoff (1941) in his example of a Galois connection as determined by a "polarity," defined using an arbitrary binary relation. This in turn generalizes the orthogonality operator on closed subspaces of a Hilbert space. K. Došen (1986) should also be recommended for a treatment of various negations in the neighborhood of the intuitionistic one, but with semantics done in the perp style.⁴

My main interest in the relationship between the two treatments of negation is motivated by the fact that the perp definition is the one that falls right out of the general "gaggle theoretic" considerations of Dunn (1990) about how to define semantical conditions for n -placed logical operators using $n + 1$ -placed accessibility relations,⁵ and yet it is often convenient to understand the De



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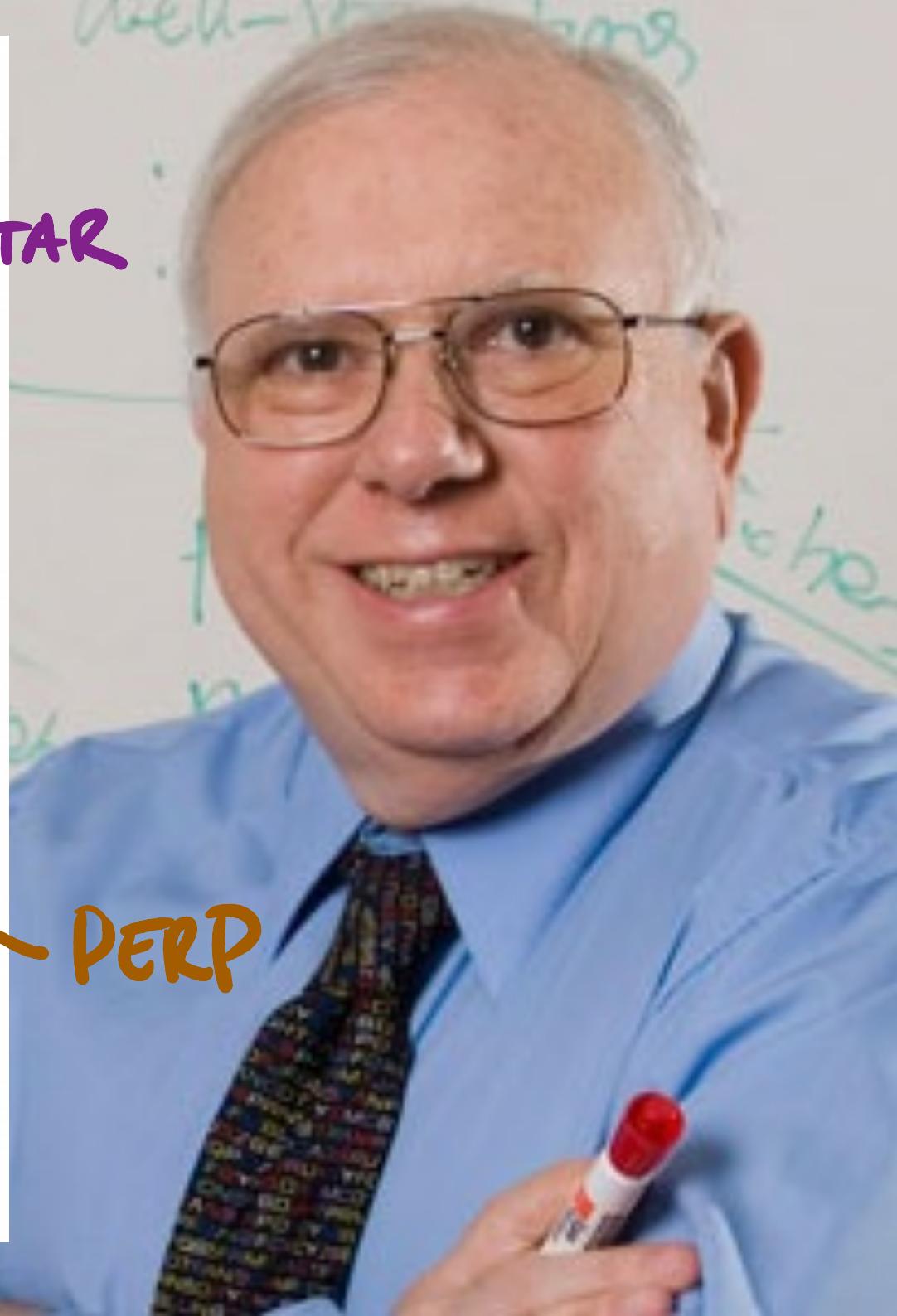
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PERP





Negation on the Australian Plan

Francesco Berto^{1,2} · Greg Restall³

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Abstract

We present and defend the Australian Plan semantics for negation. This is a comprehensive account, suitable for a variety of different logics. It is based on two ideas. The first is that negation is an exclusion-expressing device: we utter negations to express incompatibilities. The second is that, because incompatibility is modal, negation is a modal operator as well. It can, then, be modelled as a quantifier over points in frames, restricted by accessibility relations representing compatibilities and incompatibilities between such points. We defuse a number of objections to this Plan, raised by supporters of the American Plan for negation, in which negation is handled via a many-valued semantics. We show that the Australian Plan has substantial advantages over the American Plan.

Keywords Negation · Compatibility semantics · Kripke semantics · Non-classical logics · Many-valued logics · Modal logics

1. Natural Deduction

2. Proofs & speech acts in context

3. Negation & \perp

4. Plurality

1. Natural Deduction

2. Proofs & speech acts in context

3. Negation & \perp

4. Plurality

$$\neg p \wedge \neg q \succ \neg(p \vee q)$$

$$\frac{\begin{array}{c} \neg p \wedge \neg q \\ \hline \neg p \quad [\neg p] \end{array} \wedge E \quad \begin{array}{c} \neg p \wedge \neg q \\ \hline \neg q \quad [\neg q] \end{array} \neg E}{\begin{array}{c} \neg p \quad \neg q \\ \hline \bot \quad \bot \end{array} \vee E^{2,3}}$$

$$\frac{\bot}{\neg(p \vee q) \quad \neg \Gamma^3}$$

Proofs from X to A meet a **justification request** for A from a position in which the members of X are granted.

$$\frac{A \quad B}{A \wedge B} \wedge I$$

$$\frac{A \wedge B}{A} \wedge E$$

$$\frac{A \wedge B}{B} \wedge E$$

$$\frac{A}{A \vee B} \vee I$$

$$\frac{B}{A \vee B} \vee I$$

$$\frac{\begin{matrix} [A]^i & [B]^j \\ \vdots & \vdots \\ A \vee B & C & C \end{matrix}}{C} \vee E^{ij}$$

$$\frac{\vdash \vdots \quad \vdash \vdots}{\neg A \qquad A} \neg E$$

$$\frac{\vdash \vdots \quad \vdash \vdots}{\perp \qquad \neg \perp} \neg I$$

$$\frac{\vdash \vdots \quad \vdash \vdots}{A \qquad \perp E} \perp E$$

$$\begin{array}{ccc}
 A & \cancel{\not\vdash} & [A] \\
 \hline
 \bot & \uparrow & \vdots \\
 & & \bot \\
 & & \hline
 & & A \downarrow
 \end{array}$$

This is a mild bilaterism.

Proofs from a background where claims are ruled in
& others ruled out.

$X:Y \vdash A$, or equivalently, $X \succ A; Y$

$$\frac{1}{A} \text{LE}$$

(LE turns out
to just be the
vacuous case
of \downarrow , which
unifies two
different kinds
of irrelevance in
one phenomenon.)

$$\frac{1}{A} \downarrow$$

$$\frac{\frac{\frac{\neg(p \wedge q)}{p \wedge q} \frac{[p]^2 [q]'}{\neg p} \text{N}}{\neg p} \text{E}}{\neg q} \text{I}^1 \\
 \frac{\frac{\perp}{\neg q} \text{I}^1}{\neg p \vee \neg q} \text{V I} \\
 \frac{\frac{\frac{\perp}{\neg p} \text{I}^2}{\neg p} \text{V I}}{\neg p \vee \neg q} \frac{[\neg p \vee \neg q]^3}{\neg p \vee \neg q} \text{I}^3$$

$$\neg(p \wedge q) \succ \neg p \vee \neg q$$

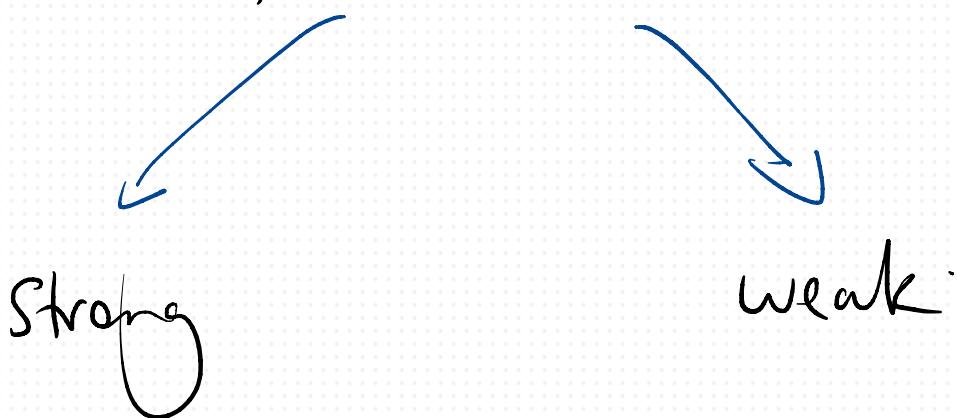
$$\frac{\frac{[\neg P]'}{[\neg P]^2} \uparrow}{\perp} \neg I' \\
 \frac{\neg \neg P \quad \neg P}{\perp} \neg E \downarrow^2 \\
 \frac{\perp}{P}$$

$$\frac{[\neg P]'}{\neg P \vee \neg P} \text{VI} \\
 \frac{\perp}{\neg \neg P} \uparrow \\
 \frac{\neg P}{\neg P \vee \neg P} \text{VI} \\
 \frac{\perp}{\neg P \vee \neg P} \downarrow^2$$

To reason constructively, you could avoid the classical rules.

But... Why?

Two forms of denial.



Two forms of denial.

Strong

weak

It is natural
to understand this
in terms of a
scope distinction.

inner

outer

$$(\neg A) \omega$$

$$\neg (A \omega)$$

$$\begin{array}{c} \neg A \omega \quad A \omega \\ \hline \perp \end{array}$$

$$\begin{array}{c} A \omega \quad \cancel{A \omega} \\ \hline \perp \end{array}$$

$$\begin{array}{c} [A \omega] \\ \vdots \\ \perp \\ \hline \neg A \omega \end{array}$$

$$\frac{\neg A \omega \quad A \omega}{\perp}$$

✓

$$\frac{\cancel{A \omega} \quad \cancel{A \omega}}{\perp}$$

✓

$$\begin{array}{c} [A \omega] \\ \vdots \\ \underline{\perp} \\ \hline \neg A \omega \end{array}$$

$$\frac{\neg A \omega \quad A \omega}{\perp}$$

✓

$$\frac{A \omega \quad \cancel{A \omega}}{\perp}$$

✓

$$\begin{array}{c} [A \omega] \\ \vdots \\ \frac{}{\perp} \\ \hline \neg A \omega \end{array}$$

✗

Not in the
presence of
negative
assumptions.

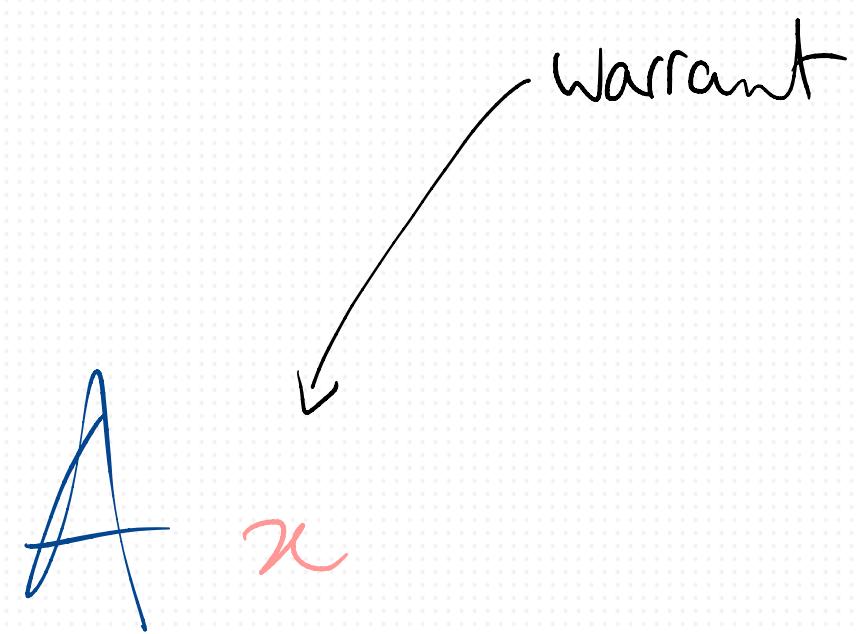
Let's step back ...

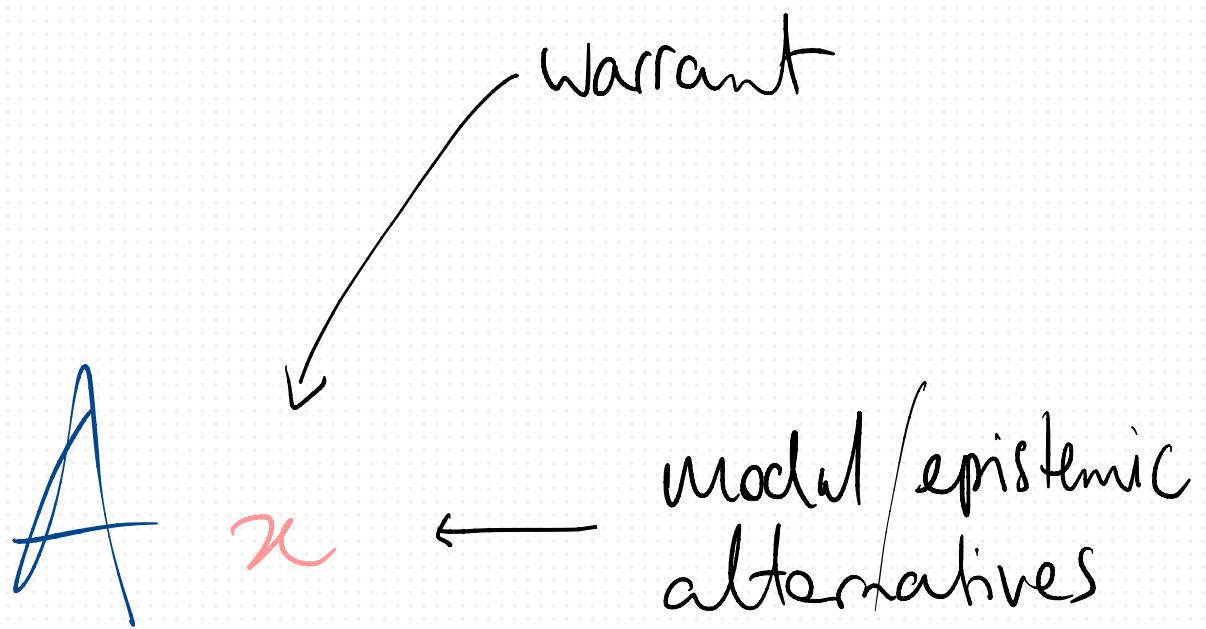
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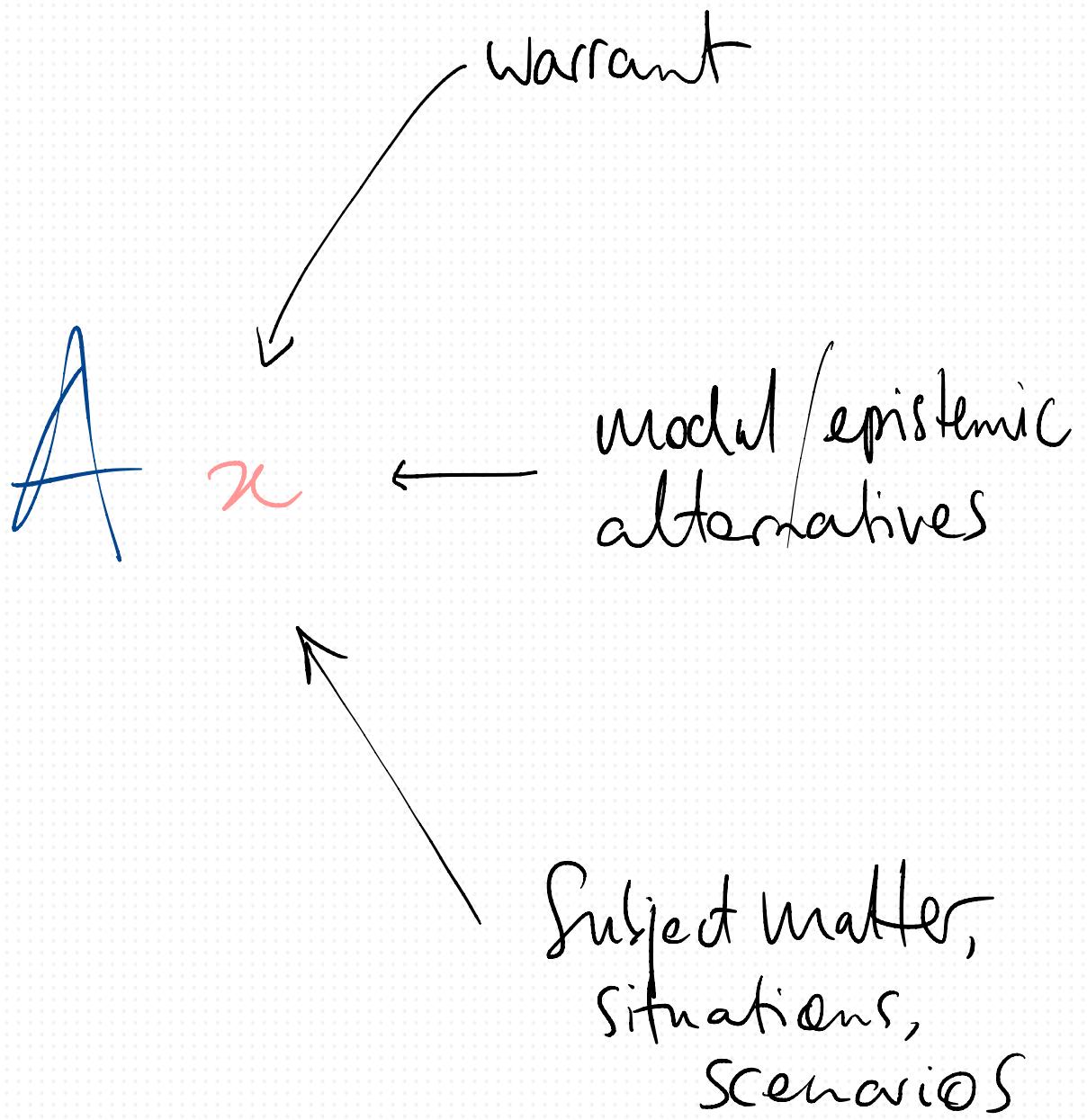
2. Proofs & speech acts in context

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Modal Reasoning

$$\frac{\Box p^1}{p^2} \text{ DE} \quad \frac{\Box q^1}{q^2} \text{ DE}$$
$$\frac{p^2 \quad q^2}{\wedge I}$$

$$\frac{p \wedge q^2}{\Box(p \wedge q)^1} \text{ DI}$$

$$\frac{\Box A \ i}{A \ j} \text{ DE}$$

$$\vdots \\ \frac{A \ i}{\Box A \ j} \text{ DI}$$

(here, i is appropriately arbitrary — e.g. not present in the assumptions)

We can understand the 'world' labels **thickly** (in terms of a prior understanding of worlds), or **thinly**, as discourse markers.

Let's grant that p is necessary. Now, suppose things had gone differently. Then (since p is necessary) regardless, we have p .

1

Let's grant that p is necessary. Now, suppose things had gone differently. Then (since p is necessary) regardless, we have p .

2

$$\frac{[P]^1 \quad [\cancel{P}]^2}{\perp} \uparrow$$

$$\frac{\neg\neg P \quad \neg P}{\perp} \neg E$$

$$\frac{\perp}{P} \downarrow^2$$

Classical proof
from $\neg\neg p$ to p .

$$\begin{array}{c}
 \frac{[P]^1 [P]^2}{\perp} \uparrow \\
 \frac{\perp}{\neg I^1} \\
 \frac{\neg\neg P \quad \neg P}{\perp} \neg E \\
 \frac{\perp}{P} \downarrow^2
 \end{array}$$

Classical proof
from $\neg\neg p$ to p .

$$\begin{array}{c}
 \frac{[P\omega]^1 [P\omega]^2}{\perp} \uparrow \\
 \frac{\perp}{\neg I^1} \\
 \frac{\neg\neg P\omega \quad \neg P\omega}{\perp} \neg E \\
 \frac{\perp}{P\omega} \downarrow^2
 \end{array}$$

Breaks down
here.

The $\neg I$ step is classically acceptable, but not so, constructively.

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NEGATION in Contexts

... more generally

$\neg A_x \quad \forall y$

NEGATION in Contexts

... more generally

$$\frac{\neg A_x \quad \neg y}{\neg E}$$

$x \perp y$

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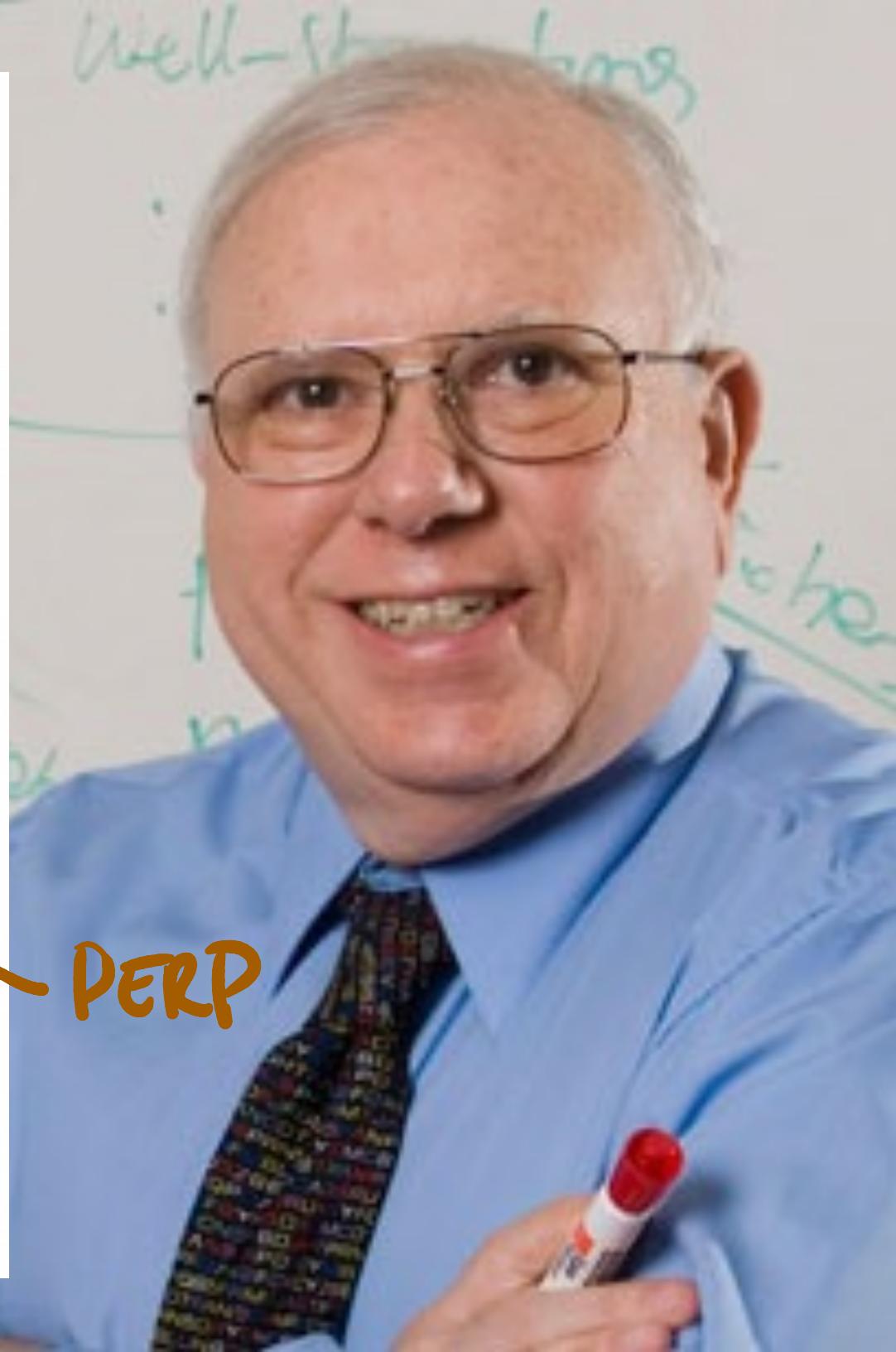
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NEGATION in Contexts

... more generally

$$\frac{\neg A_x \quad \exists y}{\neg E}$$

$x \perp y$

What might a matching
 $\neg I$ rule look like?

Invert $\neg E$!

$$\frac{\neg A[x] \quad A[y]}{x \perp y} \neg E$$

$$\frac{\begin{array}{c} [A[y]] \\ \Pi \\ \hline x \perp y \end{array}}{\neg A[x]} \neg I$$

where y is free in Π .
(ie $\Pi[y:=z]$ is a proof
of $x \perp z$ from $A \not\models \$$
the same other assumptions
as used in Π .)

Normalisierung

$$\frac{[\underset{\text{A}y}{\overline{\pi}}]}{\neg A x} \xrightarrow{\gamma I} \frac{\underset{\text{A}z}{\overline{\pi'}}}{\neg A z} \xrightarrow{\gamma E} \underset{x \perp z}{\sim} \frac{\pi'}{\pi[y := z]}$$

$$\frac{\frac{\frac{[\neg p x]^1 \frac{[p \wedge q y]^3}{py}^{\wedge E}}{x \perp y}^{\neg E}}{\frac{[\neg q x]^2 \frac{[p \wedge q y]^3}{q y}^{\wedge E}}{x \perp y}^{\neg E}}^{\vee E^{1,2}}}{x \perp y}^{\neg I^3}$$

$P \rightarrow \gamma\gamma P$??

$$\frac{[\neg P]^1 \quad P}{\neg E}$$
$$\frac{\perp}{\neg I^1}$$
$$\neg\neg P$$

$\vdash P \rightarrow \neg \neg P$??

$$\frac{[\neg P]^\dagger \quad P}{\neg E}$$
$$\frac{\perp}{\neg \neg P} \neg I^\dagger$$

$$\frac{[\neg P y]^\dagger \quad P x}{\neg E}$$
$$\frac{y \perp x}{x \perp y} \quad ???$$
$$\frac{x \perp y}{\neg \neg P x} \neg I^\dagger$$

$P \rightarrow \neg\neg P$??

$$\frac{[\neg P]^\dagger \quad P}{\perp} \neg E$$
$$\frac{\perp}{\neg\neg P} \neg I^\dagger$$

$$\frac{[\neg P y]^\dagger \quad P x}{\perp} \neg E$$

$$\frac{y \perp x}{x \perp y} \perp\text{-Symmetry}$$

$$\frac{x \perp y}{\neg\neg P x} \neg I^\dagger$$

$P \rightarrow \neg\neg P$??

$$\frac{[\neg P]^\dagger \quad P}{\perp} \neg E$$
$$\frac{\perp}{\neg\neg P} \neg I'$$

$$\frac{[\neg P y]^\dagger \quad P x}{\perp} \neg E$$
$$\frac{y \perp x}{x \perp y} \neg I' \quad \text{Invert}$$
$$\frac{x \perp y}{\neg\neg P x} \neg I'$$

P, $\neg P \vdash \perp$??

$$\frac{\neg P x \quad P x}{\neg E}$$

$$\frac{x \perp x}{\perp} \text{ Consistency}$$

P, $\neg P \vdash \perp$??

$$\frac{\neg P \ x \quad P \ x}{\neg E}$$

$$\frac{x \perp x \quad \text{Cons } x}{\text{Cons.}} \quad \perp$$

$\neg\neg P \vdash P ??$

This is more difficult to
prove using the \perp rules.

$$\frac{\frac{\frac{\frac{[\neg P]}' [\neg\neg P]^2}{\perp} \uparrow}{\frac{\neg P}{\neg\neg P}} \neg I^1}{\frac{\perp}{P}} \downarrow^2}{\neg\neg P}$$

$\neg\neg P \vdash P ??$

It is easy if we use * instead.

$$\frac{\neg A x^* \quad A x}{\perp} \neg E^*$$

$$\begin{array}{c} [A x] \\ \vdots \\ \perp \\ \hline \neg P^* \\ \hline \neg A x^* \end{array}$$

$$\frac{A x}{A x^{**}} **+$$

$$\frac{A x^{**}}{A x} ** -$$

$$\frac{\frac{[\mathbf{P}x]^1 [\mathbf{P}x]^2}{\perp}}{\frac{\gamma \mathbf{P}x}{\gamma \mathbf{P}x^{**}}^{**+} \quad \frac{\perp}{\gamma \mathbf{P}x^*} \gamma I^1} \uparrow$$

$$\frac{\perp}{\mathbf{P}x} \downarrow^2$$

There is a lot we can do
with tagged proofs of
this form, and classical,
intuitionistic, & paraconsistent/
relevant logics are well-
suited to this framework.

1. Natural Deduction
2. Proofs & speech acts in context
3. Negation & \perp
4. Plurality

Option 1

Unique best match between
everyday deductive reasoning
& a proof formalism.

(eg: untagged proofs, tagged proofs, with
denial, without, etc...)

Option 1

Unique best match between
everyday deductive reasoning
& a proof formalism,

1a \in One unique best set
of rules, relative to
that formalism.

1b \in a plurality of
sets of rules possible,
formalising different
norms of deductive
proof.

Option 1

Unique best match between
everyday deductive reasoning
& a proof formalism,

~~1a~~ \in One unique best set
of rules, relative to
that formalism.

NOT REALLY PLURALIST,
OR ONLY VERY WEAKLY SO.

~~1b~~ \in a plurality of
sets of rules possible,
formalising different
norms of deductive
proof.

Option 1

Unique best match between
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sets of rules possible
formalising different
norms of deductive
proof.

THIS OPTION IS PLURALIST

EXAMPLE 1: UNTAGGED PROOFS

$$\frac{\frac{[P] [P]^{\perp} \downarrow}{\perp} \neg I^1}{\perp} \neg E$$

$$\frac{\neg \neg P \quad \neg P}{\perp} \neg E$$

$$\frac{\perp}{P} \downarrow^2$$

EXAMPLE 2: TAGGED PROOFS

$$\frac{\neg P x \quad P x}{x \perp x} \neg E \quad \text{Cons I}$$

$$\frac{x \perp x \quad \text{Cons } x}{\perp} \text{ConsE}$$

The $\neg I$ step is classically acceptable, but not so, constructively.

Consistency holds constructively, but not so in a paraconsistent logic.

You can model all the way up to classical logic from the \perp foundation with two extra primitives

$$\frac{\text{Cons } x \quad x \perp x}{\perp} \text{ Cons E}$$

$$\frac{\text{Cons } x}{\text{Cons I}}$$

$$\frac{\text{Comp } x \quad \cancel{A}x \quad A y}{x \perp y} \text{ Comp E}$$

$$\frac{\text{Comp } x}{\text{Comp I}}$$

$$\frac{[\mathbf{P}]' [\mathbf{P}]^2}{\perp} \uparrow$$

$$\frac{\perp}{\perp} \quad \gamma I'$$

$$\frac{\gamma \gamma P \quad \gamma P}{\gamma E}$$

$$\frac{\perp}{P} \downarrow^2$$

$$\frac{\text{Comp} \times \cancel{[\mathbf{P}]^2} [\mathbf{P}]^2}{x \perp y} \frac{[\mathbf{P}]^2}{\gamma E} \text{CompE}$$

$$\frac{x \perp y}{\gamma C'} \text{ConsI}$$

$$\frac{x \perp x}{\text{ConsX}} \text{Const}$$

$$\frac{\perp}{P x} \downarrow^2$$

Is one of these better or more deeply
logical than the other?

Option 1

Unique 'best' match between
everyday deductive reasoning
& a proof formalism.

Option 2

There is a plurality of levels of
formal analysis of everyday
deductive reasoning, each of which
gets at different norms of deductive
proof.

Options 1b & 2 both have virtues.

Both are kinds of pluralisms absent
proof of deductive logic.

This framework gives us different ways
to explore logical pluralism.

