Geometric Models for Relevant Logics

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THE UNIVERSITY OF MELBOURNE

MELBOURNE LOGIC GROUP ⋆ 20 MARCH 2020 ⋆ *ONLINE TALK & ZOOM TEST*

To present some thoughts on *geometric models* for relevant logics, written for a festschrift for Alasdair Urquhart …

My ***Aim***

To present some thoughts on *geometric models* for relevant logics, written for a festschrift for Alasdair Urquhart …

… and to test giving a talk using *Zoom* before I use all this for my classes.

Alasdair Urquhart,
Banff (2007)

Alasdair Urquhart,
& Robert K. Meyer
(2007)

THE JOURNAL OF SYMBOLIC LOGIC
Volume 37, Number 1, March 1972

SEMANTICS FOR RELEVANT LOGICS

ALASDAIR URQUHART

§1. Introduction. In what follows there is presented a unified semantic treatment of certain "paradox-free" systems of entailment, including Church's weak theory of implication (Church [7]) and logics akin to the systems E and R of Anderson and Belnap (Anderson [3], Belnap [6]).[1] We shall refer to these systems generally as *relevant logics*.

The leading idea of the semantics is that just as in modal logic validity may be defined in terms of certain valuations on a binary relational structure so in relevant logics validity may be defined in terms of certain valuations on a semilattice—interpreted informally as the semilattice of possible pieces of information. Completeness theorems can be given relative to these semantics for the implicational fragments of relevant logics. The semantical viewpoint affords some insights into the structure of the systems—in particular light is thrown upon admissible modes of negation and on the assumptions underlying rejection of the "paradoxes of material implication".

THE JOURNAL OF SYMBOLIC LOGIC
Volume 49, Number 4, Dec. 1984

THE UNDECIDABILITY OF ENTAILMENT AND RELEVANT IMPLICATION

ALASDAIR URQUHART

§1. Introduction. In this paper we show that the propositional logics E of entailment, R of relevant implication and T of ticket entailment are undecidable. The decision problem is also shown to be unsolvable in an extensive class of related logics. The main tool used in establishing these results is an adaptation of the von Neumann coordinatization theorem for modular lattices.

Interest in the decision problem for these systems dates from the late 1950s. The earliest result was obtained by Anderson and Belnap who proved that the first degree fragment of all these logics is decidable. Kripke [11] proved that the pure implicational fragments $R_{\rightarrow}$ and $E_{\rightarrow}$ of R and E are decidable. His methods were extended by Belnap and Wallace to the implication-negation fragments of these systems [3]; Kripke's methods also extend easily to include the implication-

Failure of Interpolation in Relevant Logics (1993)

ALASDAIR URQUHART

FAILURE OF INTERPOLATION IN RELEVANT LOGICS

ABSTRACT. Craig's interpolation theorem fails for the propositional logics *E* of entailment, *R* of relevant implication and *T* of ticket entailment, as well as in a large class of related logics. This result is proved by a geometrical construction, using the fact that a non-Arguesian projective plane cannot be imbedded in a three-dimensional projective space. The same construction shows failure of the amalgamation property in many varieties of distributive lattice-ordered monoids.

Journal of Philosophical Logic (22) 449–479, 1993

My ***Plan***

Models for Relevant Logics

Collection Frames

Points at Infinity

Functional Geometric Set Frames

MODELS FOR RELEVANT LOGICS

THE JOURNAL OF SYMBOLIC LOGIC
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Relevant logics

$$A \to B$$

When is $A \to B$ *true at a world?*

A natural thought:

$x \Vdash A \to B$ iff at some appropriately chosen worlds y,
if $y \Vdash A$, then we have $y \Vdash B$ too.

The Problem of $p \to (q \to q)$

Check the antecedent A,
and the consequent B,
at *different places*.

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at *different places*.

$x \Vdash A \to B$ iff for each y
if $y \Vdash A$, then $x \sqcup y \Vdash B$.

$p \to (q \to q)$ *is no longer a problem*

Provided that $\sqcup$ is not *cumulative*.

These aren't "worlds"

Urquhart calls them *pieces of information*.

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I'll call them *points*.

Urquhart calls them *pieces of information*.

I'll call them *points*.

For Urquhart, $\sqcup$ is *commutative*, *associative* and *idempotent*.

$p \to p$ *fails somewhere, not necessarily everywhere*

If we have a point 0 where $0 \sqcup x = x$,
then $0 \Vdash A \to A$ for each A.

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We call 0 a *normal* point.

Disjunction: The Fly in the Ointment

$$x \Vdash A \lor B \text{ iff } x \Vdash A \text{ or } x \Vdash B$$

$x \Vdash A \lor B$ iff $x \Vdash A$ or $x \Vdash B$

Suppose $x \Vdash p \to (q \lor r)$ and $y \Vdash p$.

$x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$

Suppose $x \Vdash p \to (q \vee r)$ and $y \Vdash p$.

Then $x \sqcup y \Vdash q \vee r$, but *which one*?

THE SEMANTICS OF ENTAILMENT

Richard ROUTLEY
Australian National University

Robert K. MEYER
University of Toronto, Indiana University

Once upon a time, modal logics "had no semantics". Bearing a real world G, a set of worlds ***K***, and a relation R of relative possibility between worlds, Saul Kripke beheld this situation and saw that it was formally explicable, and made model structures. It came to pass that soon everyone was making model structures, and some were deontic, and some were temporal, and some were epistemic, according to the conditions on the binary relation R.

None of the model structures that Kripke made, nor that Hintikka made, nor that Thomason made, nor that their co-workers and colleagues made, were, however, relevant. This caused great sadness in the city of Pittsburgh, where dwelt the captains of American Industry. The logic industry was there represented by Anderson, Belnap & Sons, discoverers of entailment and scourge of material impliers, strict impliers, and of all that to which their falsehoods and contradictions led. Yea, every year or so Anderson & Belnap turned out a new logic, and they did call it E, or R, or $E_{\bar{I}}$, or P – W, and they beheld each such logic, and they were called relevant. And these logics were looked upon with favor by many, for they captureth the intuitions, but by many more

Alasdair Urquhart,
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The Fix: Generalise!

Replace $x \sqcup y = z$ by the *ternary relation* $Rxyz$.

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There may be more than one
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$x \Vdash A \to B$ iff for each y, z where $Rxyz$, if $y \Vdash A$ then $z \Vdash B$.

$x \Vdash A \lor B$ iff $x \Vdash A$ or $x \Vdash B$

$x \Vdash A \land B$ iff $x \Vdash A$ and $x \Vdash B$.

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- $N \subseteq P$
- $\sqsubseteq \subseteq P \times P$
- $R \subseteq P \times P \times P$

The Twofold Cost: Complexity

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3. R is downward preserved in the its two positions and upward preserved in the third, i.e. if $Rx'y'z$ and $x \sqsubseteq x'$, $y \sqsubseteq y'$, $z \sqsubseteq z'$, then $Rxyz'$.

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4. $y \sqsubseteq y'$ iff $(\exists x)(Nx \wedge Rxyy')$.

*Then, to model nice logics, like **R**, you need more*

$$(\exists u)(Rxyu \wedge Ruzw) \quad \Leftrightarrow \quad (\exists v)(Ryzv \wedge Rxvw)$$

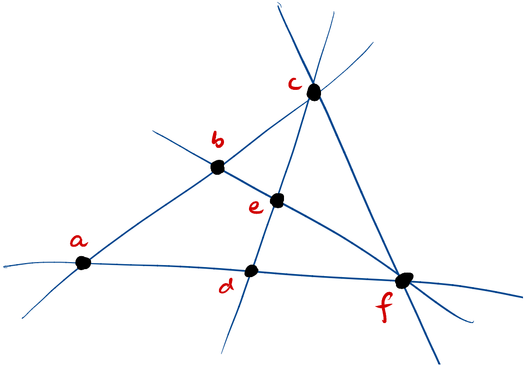
$$Rxyz \quad \Leftrightarrow \quad Ryxz$$

$$Rxxx$$

The Twofold Cost: Complexity

Ternary relations are not so easy to think about, by themselves.

Ternary relations occur naturally in *geometry*.



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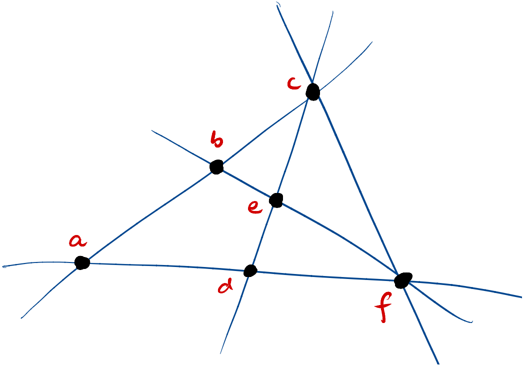

Collinearity: $Cabc$ but not $Cabe$.

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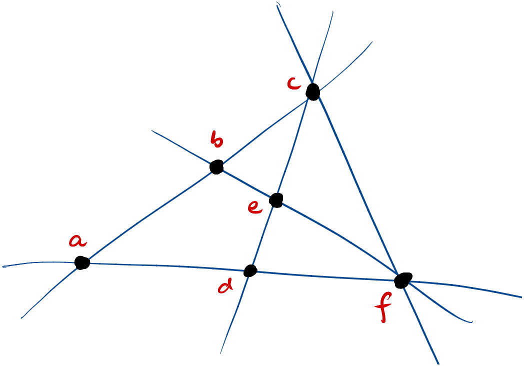

Collinearity: $Cabc$ but not $Cabe$.

Betweenness/Surrounding: $Sacb$ but not $Sabc$.

The properly *geometric* properties are those preserved under *translation*, *rotation*, *scaling*, etc.

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Collinearity, *Betweenness*,

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Inside/Outside, *Angle*,

Relative Size, *Shape*, etc.

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Collinearity, *Betweenness*,

Intersection, *Parallels*,

Inside/Outside, *Angle*,

Relative Size, *Shape*, etc.

Not: *Absolute Size*, *Absolute Position*.

Ordering: There is no *natural* ordering $\sqsubseteq$ (other than $=$) in *most* geometries.

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If R is *collinearity* or *betweenness*, then there is *no* candidate normal point in any $\mathbb{R}^n$ (or in any affine or projective space).

Add one!

Add one!

This follows in the steps of a venerable tradition in mathematics, of adding *ideal points*. In this case, a *point at infinity*.

COLLECTION FRAMES

$$\mathsf{N}\,\overline{z} \qquad \underline{x} \mathrel{\underline{\sqsubseteq}} \overline{z} \qquad \mathsf{R}\,\underline{x}\,\underline{y}\,\overline{z}$$

$$\mathsf{N}\,\overline{z} \qquad \underline{x} \sqsubseteq \overline{z} \qquad \mathsf{R}\,\underline{x}\underline{y}\overline{z}$$

- ▷ The position of an *underlined* variable is closed *downwards* along $\sqsubseteq$.

The Behaviour of N, $\sqsubseteq$ *and* R

$$\mathsf{N}\,\overline{z} \qquad \underline{x} \mathrel{\underline{\sqsubseteq}} \overline{z} \qquad \mathsf{R}\,\underline{x}\underline{y}\overline{z}$$

- ▷ The position of an <u>*underlined*</u> variable is closed *downwards* along $\sqsubseteq$.
- ▷ The position of an $\overline{\textit{overlined}}$ variable is closed *upwards* along $\sqsubseteq$.

The Behaviour of N, $\sqsubseteq$ *and* R

$$N\ \overline{z} \qquad \underline{x} \sqsubseteq \overline{z} \qquad \underline{x}\,\underline{y}\ R\ \overline{z}$$

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$R\,z \qquad x\,R\,z \qquad xy\,R\,z$

$$X \; R \; z$$

X is a finite *collection* of elements of P; z is in P.

What kind of finite collection?

Leaf-Labelled Trees   *Lists*   *Multisets*   *Sets*   *more …*

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$$Rxyz \Leftrightarrow Ryxz$$

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$$Rxxx$$

What kind of finite collection?

Leaf-Labelled Trees *Lists* *Multisets* ***Sets*** *more . . .*

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Set Relations

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R generalises $\sqsubseteq$.

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R generalises $\sqsubseteq$.

So, it should satisfy analogues of *reflexivity* and *transitivity*.

Reflxivity

$$\{x\}\ R\ x$$

$X \; R \; x$

$$X \mathrel{R} x \qquad \{x\} \cup Y \mathrel{R} y$$

$$X\,R\,x \qquad \{x\} \cup Y\,R\,y \qquad\qquad X \cup Y\,R\,y$$

$$(X \mathrel{R} x \wedge \{x\} \cup Y \mathrel{R} y) \Rightarrow X \cup Y \mathrel{R} y$$

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$X \cup Y \mathrel{R} y$

$$(X\ R\ x \land \{x\} \cup Y\ R\ y) \Rightarrow X \cup Y\ R\ y$$

$X \cup Y\ R\ y$

$X\ R\ x$

Generalised Transitivity

$$(X\ R\ x \wedge \{x\} \cup Y\ R\ y) \Rightarrow X \cup Y\ R\ y$$

$X \cup Y\ R\ y$ $\quad$ $X\ R\ x$ $\quad$ $\{x\} \cup Y\ R\ y$

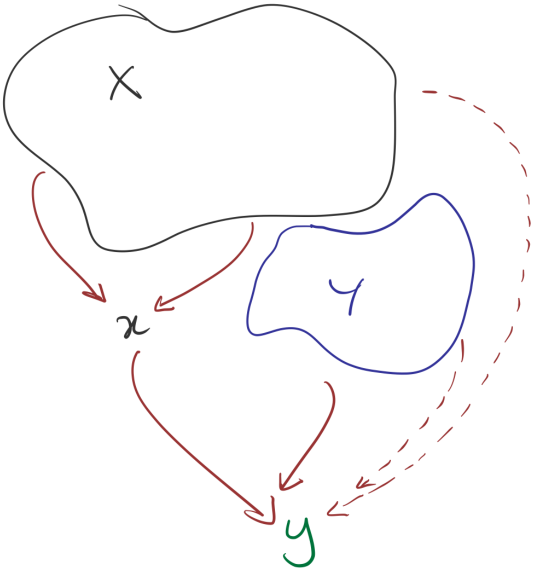
Generalised Transitivity

$$(X\ R\ x \land \{x\} \cup Y\ R\ y) \Rightarrow X \cup Y\ R\ y$$

$$X \cup Y\ R\ y \;\Rightarrow\; (\exists x)(X\ R\ x \;\land\; \{x\} \cup Y\ R\ y)$$

$$(\exists x)(X \mathrel{R} x \land \{x\} \cup Y \mathrel{R} y) \Leftrightarrow X \cup Y \mathrel{R} y$$

Left to Right



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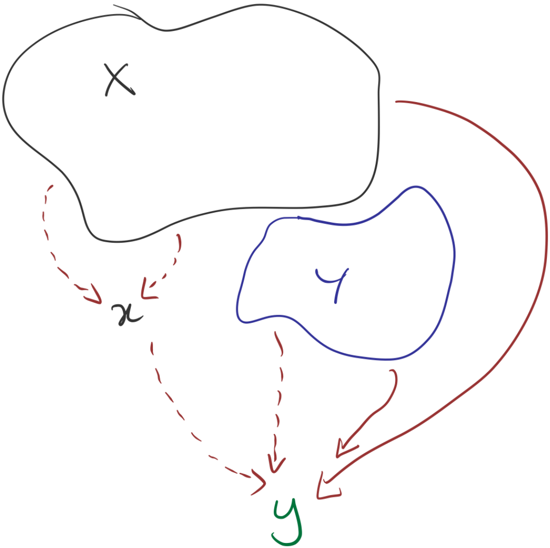

Compositional Set Relations

$R \subseteq \mathcal{P}^{\mathsf{fin}}(P) \times P$ is *compositional* iff for each $X, Y \in \mathcal{P}^{\mathsf{fin}}(P)$ and $y \in P$

- $\{y\}\ R\ y$
- $(\exists x)(X\ R\ x \land \{x\} \cup Y\ R\ y) \iff X \cup Y\ R\ y$

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3. R is downward preserved in the its two positions and upward preserved in the third.
4. $y \sqsubseteq y'$ iff $(\exists x)(Nx \wedge Rxyy')$.
5. $Rxyz \Leftrightarrow Rxyz$
6. $(\exists v)(Rxyv \wedge Rvzw) \Leftrightarrow (\exists u)(Ryzu \wedge Rxuw)$
7. $Rxxx$

$$\langle P, R \rangle$$

- P: a non-empty set
- $R \subseteq \mathcal{P}^{\mathsf{fin}}(P) \times P$

1. R is *compositional*. That is, $\{x\}\ R\ x$ and $(\exists x)(X\ R\ x \land \{x\} \cup Y\ R\ y) \Leftrightarrow X \cup Y\ R\ y$

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$$Nx \qquad x \sqsubseteq y \qquad Rxyz$$

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$$Nx \qquad x \sqsubseteq y \qquad Rxyz$$

$$\{\,\}Rx \qquad \{x\}Ry \qquad \{x,y\}Rz$$

POINTS AT INFINITY

$$\langle P, R\rangle$$

- P: a non-empty set
- $R \subseteq \mathcal{P}^{\mathsf{fin}*}(P) \times P$
— $\mathcal{P}^{\mathsf{fin}*}(P)$ is the set of *non-empty* finite subsets of P.

1. R is compositional. That is, $\{x\}\ R\ x$ and $(\exists x)(X\ R\ x \land \{x\} \cup Y\ R\ y) \Leftrightarrow X \cup Y\ R\ y$

- $\{x, y\}Sv$

Surrounding

- $\{x, y\}Sv$
- $\{x, y, z\}Sv$

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- $\{x, y\}Sv$
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The result is an *elegant* geometric metaphor
for concept combination and containment.

Adding a Point at Infinity

If $R \subseteq \mathcal{P}^{\mathsf{fin}*}(P) \times P$, we want to add a point ∞, extending R to a relation R' where

$$\{\,\}R'\infty$$

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We want $R' \subseteq \mathcal{P}^{\mathsf{fin}}(P \cup \{\infty\}) \times P \cup \{\infty\}$

Two Natural Options

$$XR^{+}z \ \text{ iff } \ (X\backslash\infty)Rz \text{ or } z = \infty$$

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($XR^{+}\infty$ always: so $x \sqsubseteq^{+} \infty$ for all x.)

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($XR^{+}\infty$ always: so $x \sqsubseteq^{+} \infty$ for all x.)

$$XR^{\times}z \text{ iff } \begin{cases} (X\backslash\infty)Rz, & X\backslash\infty \neq \{\} \\ z = \infty, & X\backslash\infty = \{\} \end{cases}$$

($XR^{\times}\infty$ only when $X = \{\}$ or $\{\infty\}$: so $x \sqsubseteq^{\times} \infty$ iff $x = \infty$.)

$$XR^{+}z \ \text{ iff } \ (X\backslash\infty)Rz \text{ or } z = \infty$$

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($XR^{\times}\infty$ only when $X = \{\}$ or $\{\infty\}$: so $x \sqsubseteq^{\times} \infty$ iff $x = \infty$.)

Both R^{+} and $R^{\times}$ are compositional if R is.

FUNCTIONAL GEOMETRIC SET FRAMES

Different Desiderata for Geometric Relations

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- R is PRESERVED: If XRy then $\tau(X)R\tau(x)$ for *geometric transformations* τ

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- R is REGIONAL: If $b(X) = b(Y)$ then XRz iff YRz.
- R is PRESERVED: If XRy then $\tau(X)R\tau(x)$ for *geometric transformations* τ
- R is FUNCTIONAL: There is a *unique* x where XRx.

There are exactly two relations satisfying the desiderata on $\mathbb{R}$

INCLUSIVE PRESERVED REGIONAL FUNCTIONAL

There are exactly two relations satisfying the desiderata on $\mathbb{R}$

INCLUSIVE	PRESERVED	REGIONAL	FUNCTIONAL

$$\mathsf{min} \qquad \mathsf{max}$$

There are no relations satisfying the desiderata on $\mathbb{R}^n$ $(n \geqslant 2)$

$\mathbb{R}$ is inside $\mathbb{R}^n$.

$\mathbb{R}$ is inside $\mathbb{R}^n$.

In $\mathbb{R}^n$ $(n \geqslant 2)$ you can rotate an interval 180°,
sending min onto max (and *vice versa*),
so neither is preserved under this rotation.

What does this all mean?

We've seen how we can *use* and *extend* Urquhart's insights into the connection between geometries and models of relevant logics, to provide new ways to *build* and *understand* those models.

THANK YOU!

QUESTIONS, COMMENTS?