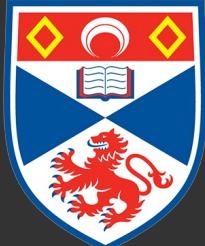


DEFINING QUANTIFIERS

GREG RESTALL



University of
St Andrews

TOPICS IN FREE LOGIC ⚡ MCMP ⚡ NOVEMBER 2024

1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO ...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

1. DEFINING RULES

2. QUANTIFIERS & GENERAITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

(How) DO SEQUENT RULES CHARACTERISE CONCEPTS?

$$\frac{X \succ A, Y}{X \succ A \text{ tonk } B, Y} \text{ tonk R}$$

$$\frac{X, B \succ Y}{X, A \text{ tonk } B \succ Y} \text{ tonk L}$$

Too STRONG!

$$\frac{p \succ p \text{ tonk } q \quad p \text{ tonk } q \succ q}{p \succ q}$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \text{ plink } B, Y} \text{ plink R}$$

$$\frac{X, A \succ Y \quad X, B \succ Y}{X, A \text{ plink } B \succ Y} \text{ plink L}$$

Too WEAK!

A plink B could be $A \wedge B$ or $A \vee B$ or anything in between!

$$\frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y} \rightarrow R$$

$$\frac{X \succ A, Y \quad X', B \succ Y'}{X, X', A \rightarrow B \succ Y, Y'} \rightarrow L$$

JUST RIGHT

A DEFINITION

x is equilateral iff x has three sides
& each side has equal length.

A DEFINITION

as an invertible rule

x has three sides

each side of x has equal length.

x is equilateral

DEFINING THE CONDITIONAL

$$\frac{x, A \vdash B, \gamma}{x \vdash A \rightarrow B, \gamma} \rightarrow \text{DF}$$

This is clearly uniquely defining (if \rightarrow_1 , $\not\vdash$, \rightarrow_2 are defined using the same rule, they are interchangeable)

$$\frac{x \vdash A \rightarrow_1 B, \gamma}{x, A \vdash B, \gamma} \rightarrow_1 \text{DF} \uparrow$$

$$\frac{x, A \vdash B, \gamma}{x \vdash A \rightarrow_2 B, \gamma} \rightarrow_2 \text{DF} \downarrow$$

$$\frac{}{A \rightarrow_2 B \vdash A \rightarrow_2 B} \text{Id}$$

$$\frac{A \rightarrow_2 B \vdash A \rightarrow_2 B}{A \rightarrow_2 B, A \vdash B} \rightarrow_2 \text{DF} \uparrow$$

$$\frac{A \rightarrow_2 B, A \vdash B}{A \rightarrow_2 B \vdash A \rightarrow_1 B} \rightarrow_1 \text{DF} \downarrow$$

$$\frac{x, A \rightarrow_1 B \vdash \gamma}{x, A \rightarrow_2 B \vdash \gamma} \text{Cut}$$

DEFINING THE CONDITIONAL

$$\frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y} \rightarrow DF$$

This is also conservatively extending — the usual left/right rules can be recovered from $\rightarrow DF$ using Cut & ID.

$$\frac{\frac{\frac{A \rightarrow B \succ A \rightarrow B}{A \rightarrow B \succ A \rightarrow B}^{Id}}{X_1 \succ A, Y_1 \quad A \rightarrow B, A \succ B} \rightarrow DF \uparrow}{\frac{X_1, A \rightarrow B \succ B, Y_1 \quad X_2, B \succ Y_2}{X_1, X_2, A \rightarrow B \succ Y_1, Y_2} \text{Cut}_B} \text{Cut}_A$$

DEFINING THE CONDITIONAL

$$\frac{x, A \vdash B, \gamma}{x \vdash A \rightarrow B, \gamma} \rightarrow \text{DF}$$

This is also conservatively extending — the usual left/right rules can be recovered from $\rightarrow \text{DF}$ using Cut & ID, and the principal cut-reduction step unwinds this definition.

$$\begin{array}{c}
 \frac{\vdots}{x_3, A \vdash B, \gamma_3} \rightarrow \text{DF} \downarrow \\
 \hline
 x_3 \vdash A \rightarrow B \vdash \gamma_3
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\vdots}{A \rightarrow B \vdash A \rightarrow B} \text{Id} \\
 \hline
 \frac{x_1 \vdash A, \gamma_1 \quad A \rightarrow B, A \vdash B}{x_1, A \rightarrow B \vdash B, \gamma_1} \rightarrow \text{DF} \uparrow
 \end{array}
 \quad
 \begin{array}{c}
 \frac{\vdots}{x_2, B \vdash \gamma_2} \text{Cut}_3 \\
 \hline
 x_1, x_2, A \rightarrow B \vdash \gamma_1, \gamma_2
 \end{array}$$

Cut_{A→B}

$$\hline
 x_1, x_2, x_3 \vdash \gamma_1, \gamma_2, \gamma_3$$

$$\frac{\vdots}{\frac{\frac{x_3, A \succ B, y_3}{x_3 \succ A \rightarrow B \succ y_3} \rightarrow Df \downarrow \quad \frac{A \rightarrow B \succ A \rightarrow B}{A \rightarrow B, A \succ B} \stackrel{Id}{\longrightarrow} \rightarrow Df \uparrow}{x_3, A \succ B, y_3 \quad A \rightarrow B, A \succ B} \text{Cut}_{A \rightarrow B}}
 {x_1, x_3 \succ B, y_3 \quad x_2, B \succ y_2} \text{Cut}_A \quad \vdots \quad \text{Cut}_B$$

$x_1, x_2, x_3 \succ y_1, y_2, y_3$

⋮ ⋮

$$\frac{x_1 \succ A, y_1 \quad x_3, A \succ B, y_3}{x_1, x_3 \succ B, y_3} \text{Cut}_A \quad \frac{x_2, B \succ y_2}{x_2, B \succ y_2} \text{Cut}_B$$

$x_1, x_2, x_3 \succ y_1, y_2, y_3$

THIS IS TOTALLY GENERAL

(it works like this for any invertible defining rule, in the presence or absence of contraction & weakening, in different sequent structures.)

$$\frac{x \succ A, Y \quad x \succ B, Y}{x \succ A \wedge B, Y} \text{ AND}$$

$$\frac{x, A, B \vdash Y}{x, A \otimes B \vdash Y} \text{ OTF}$$

$$\frac{x, A \succ Y \quad x \succ B, Y}{x \succ A \supset B, Y} \text{ ODF}$$

1. DEFINING RULES

2. QUANTIFIERS & GENERA^RITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO ...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

DEFINING RULES for QUANTIFIERS

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y} \forall Df$$

$$\frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y} \exists Df$$

The term n must be absent in the lower sequent.

n is an eigenvariable (more on what this means, soon).

RECOVERING THE USUAL LEFT/RIGHT RULES

$$\frac{\forall R}{X \succ A(n), Y} \frac{X \succ A(n), Y}{X \succ \forall x A(x), Y} \forall D F \downarrow$$

$$\frac{\forall D F}{X \succ A(n), Y} \frac{X \succ A(n), Y}{X \succ \forall x A(x), Y} \forall D F$$

$$\frac{\forall L}{X \succ \forall x A(x), Y} \frac{}{\frac{\frac{\forall x A(x) \succ \forall x A(x)}{\forall x A(x) \succ A(n)} \frac{\forall x A(x) \succ A(n)}{\frac{\forall x A(x) \succ A(t)}{X, A(t) \succ Y}} \text{Id}}{\forall x A(x) \succ A(t)}} \forall D F \uparrow \text{Spec}_t^n$$

$$\frac{}{X, \forall x A(x) \succ Y} \text{Cut}$$

(t can be any term at all, not necessarily an eigenvariable)

THE SPECIATISATIONⁿ_t RULE

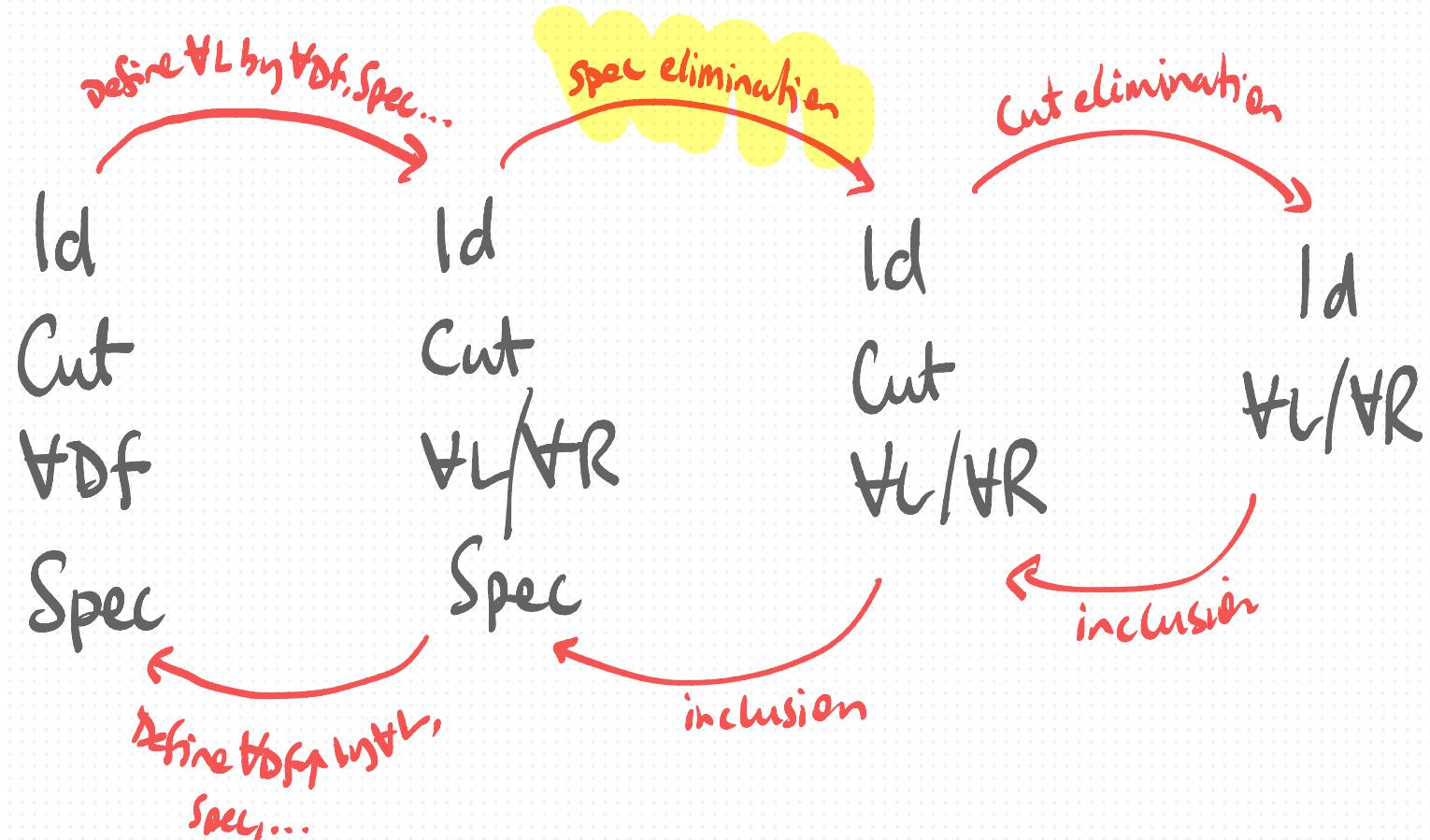
$$\frac{X \succ Y}{[X \succ Y]^n_t} \text{ Spec}_t^n$$

If a segment obtains concerning some eigenvariable n , it also obtains for the term t .

Eigenvariables are inferentially general among terms.

Spec_t^n is admissible in Gentzen's segment calculus.

EQUIVALENT FORMULATIONS



[See my "Generality & Existence 1", RSL (2019) for details.]

SPEC ELIMINATION

$$\frac{\delta : \begin{array}{c} X \succ Y \\ \hline [X \succ Y]^n_t \end{array}}{\text{Spec}_t^n} \rightsquigarrow \delta_t^{\hat{n}} : [X \succ Y]_t^n$$

It's easy to see that each of the rules in Gentzen's system are closed under specialisation — ie there is nothing inferentially special in an eigenvariable as conclusion.

(HdF↑ violates this constraint — hence the requirement to impose Spec.)

DEFINING RULES

- ... specify concepts that are available (conservative) & determinate (unique) over a basic structural context.
- & for quantifiers, these rules use prior notions of substitution & inferential generality.

1. DEFINING RULES

2. QUANTIFIERS & GENERA^TY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO ...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

FREE LOGIC, DEFINED & UNDEFINED TERMS

n is a number.

n/m is not necessarily a number.

n/m is not defined when $m=0$.
(does not exist)

' n/m ' is a term, whatever values n & m take.

$$\forall x(\neg \exists y(y = n/x) \leftrightarrow x=0)$$

THE BASIC STRUCTURAL CONTEXT

RULING TERMS IN & RULING THEM OUT

$s, t, A, B \succ C, D, s', t'$

[defined true false undefined]

ACCEPT REJECT

$$\frac{t \cdot \text{Id}}{t, X \succ Y, t}$$

$$\frac{X \succ Y, t \quad t, X' \succ Y'}{X, X' \succ Y, Y'} \text{-Cut}$$

DEFINING DEFINEDNESS

$$\frac{X \succ Y, t}{X \succ Y, t \downarrow} \downarrow_{DF}$$

& equivalently, ...

$$\frac{X \succ Y, t}{X \succ Y, t \downarrow} \downarrow_R$$

$$\frac{X, t \succ Y}{X, t \downarrow \succ Y} \downarrow_L$$

POSSIBLE CONDITIONS ON PREDICATES & FUNCTION SYMBOLS

$$\frac{t_i, X \vdash Y}{Ft, \dots t_n, X \vdash Y} \text{ FL}$$

$$\frac{t_i, X \vdash Y}{ft, \dots t_n, X \vdash Y} \text{ fL}$$

(If we grant these, then nonexistence is not a predicate in this sense, since we have $\neg \frac{1}{0} \downarrow$, while $\frac{1}{0}$ is not defined.)

DEFINING RULES FOR \forall/\exists with existential commitment

$$\frac{n, X \vdash Y, A(n)}{X \vdash Y, \forall x A(x)} \forall DF$$

$$\frac{n, A(n), X \vdash Y}{\exists x A(x), X \vdash Y} \exists DF$$

$$\frac{\frac{\frac{}{\forall x A(x) \vdash \forall x A(x)}}{\forall x A(x) \vdash \forall x A(x)} Id}{n, \forall x A(x) \vdash A(n)} \forall DF \uparrow$$

$$\frac{n, \forall x A(x) \vdash A(n)}{t, \forall x A(x) \vdash A(t)} Spec^n_t$$

$$X', A(t) \vdash Y' \quad Cut$$

$$\frac{X \vdash Y, t \quad t, X', \forall x A(x) \vdash Y'}{X, X', \forall x A(x) \vdash Y, Y'} t Cut$$

$$X, X', \forall x A(x) \vdash Y, Y'$$

DEFINING Rules for \forall/\exists with existential commitment

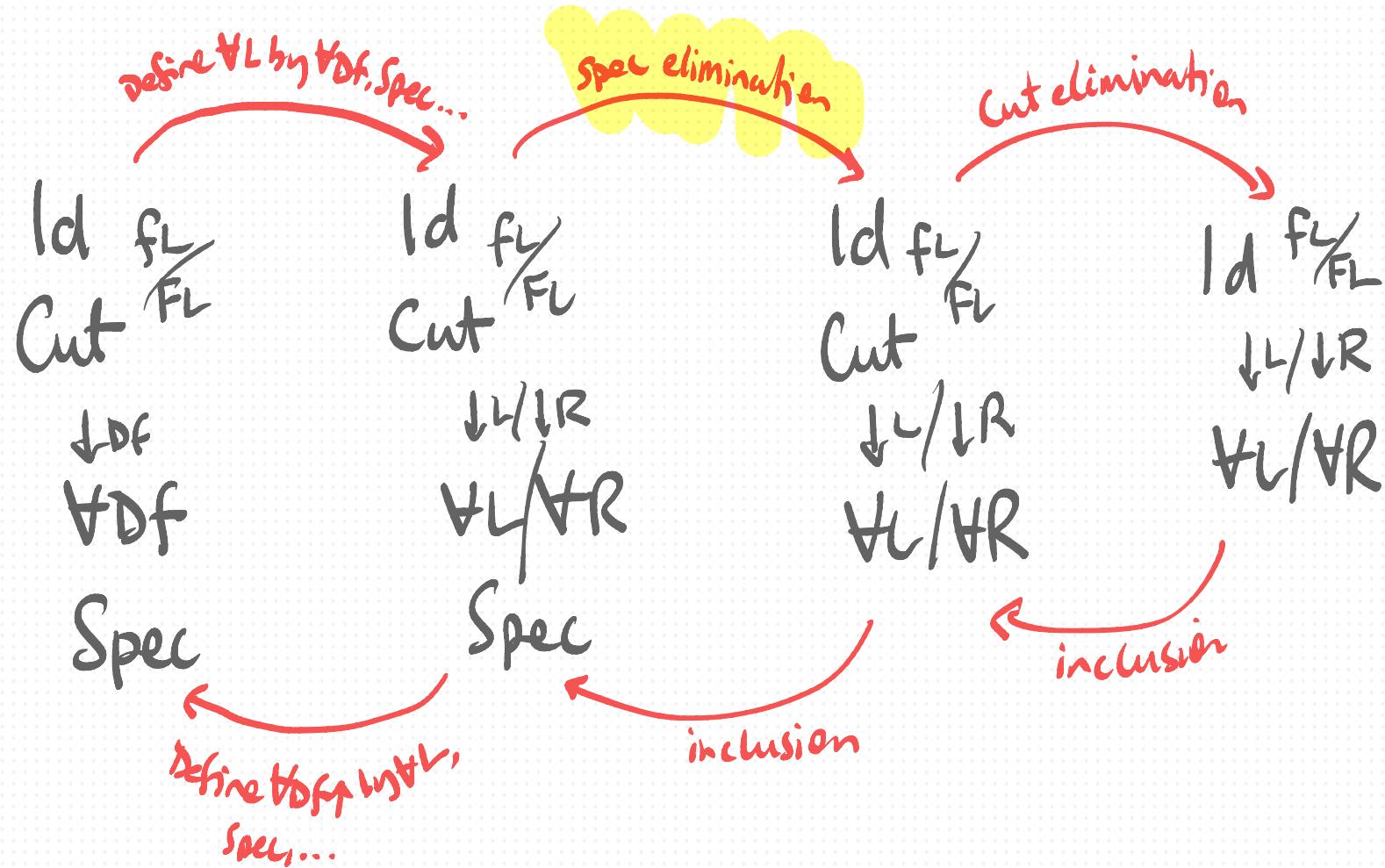
$$\frac{n, X \vdash Y, A(n)}{X \vdash Y, \forall x A(x)} \forall DF$$

$$\frac{n, A(n), X \vdash Y}{\exists x A(x), X \vdash Y} \exists DF$$

$$\frac{\frac{\frac{\frac{\frac{\exists x A(x) \vdash \exists x A(x)}{\exists x A(x) \vdash \exists x A(x)}}{\frac{n, A(n) \vdash \exists x A(x)}{n, A(n) \vdash \exists x A(x)}}}{t, A(t) \vdash \exists x A(x)}}{\text{Spec}_t^n} \text{ Id} \quad \text{DF} \uparrow}{\text{Cut}}$$

$$\frac{X \vdash Y, t \quad t, X' \vdash \forall x A(x), Y'}{X, X', \forall x A(x) \vdash Y, Y'} t Cut$$

THE EQUIVALENCES STILL HOLD...



1. DEFINING RULES

2. QUANTIFIERS & GENERAITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

BUT WE ARE FREE TO DEFINE MORE...

$$\frac{n, X \succ Y, A(n)}{X \succ Y, \forall x A(x)} \text{ VDF}$$

$$\frac{X \succ Y, A(n)}{X \succ Y, \Pi x A(x)} \text{ PIDF}$$

With Existential Commitment

without Existential Commitment

$$\begin{array}{c} X \succ Y, \forall x A(x) \\ \hline n, X \succ Y, A(n) \\ \hline m, X \succ Y, A(n) \\ \hline X \succ Y, m \downarrow \rightarrow A(n) \\ \hline X \succ Y, \Pi x (x \downarrow \rightarrow A(n)) \end{array}$$

VDF ↓DF →DF PIDF

BUT WE ARE FREE TO DEFINE MORE...

$$\frac{n, A(n), X \vdash Y}{\exists x A(x), X \vdash Y} \exists Df$$

$$\frac{A(n), X \vdash Y}{\Sigma_x A(x), X \vdash Y} \Sigma Df$$

$$\frac{\sum_x \neg x \downarrow \vdash \sum_x \neg x \downarrow}{\neg n \downarrow \vdash \sum_x \neg x \downarrow} \Sigma Df \uparrow$$
$$\frac{\neg n \downarrow \vdash \sum_x \neg x \downarrow}{\neg \frac{1}{0} \downarrow \vdash \sum_x \neg x \downarrow} Spec^n_{\frac{1}{0}}$$

Are outer quantifiers just
as acceptable as inner quantifiers?

Do they involve the same sort of
ontological {
ideological } commitment?
theoretical }

Not necessarily
There are differences...

1. DEFINING RULES

2. QUANTIFIERS & GENERAITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO ...

5. WHAT is A QUANTIFIER?
on TERMS & VALUES

LOOKING CLOSER...

$$\frac{n, X \succ Y, A(n)}{X \succ Y, \forall x A(x)}$$

VDF

$$\frac{X \succ Y, A(n)}{X \succ Y, \exists x A(x)}$$

TDF

Diagram: A red curly brace groups the two top equations. A red arrow points from the brace down to the handwritten note below.

this n is { an eigenvariable
an inferentially general singular term.

WHAT IS A VARIABLE?

What is the meaning of 'n' in $\forall x A(x)$?

"Everything is A..." "For every value n can take..."

What is the meaning of 'n' in $A(n)$?

- A variable of some sort?
- A pronoun? a demonstrative expression?
- A singular term of some kind?

Should these be connected?

IN FAVOUR OF CONNECTION... CONNECTION

$$\frac{x, X \vdash Y, A(x)}{X \vdash Y, \forall x A(x)} \forall\text{Df}$$

$$\frac{x, A(x), X \vdash Y}{\exists x A(x), X \vdash Y} \exists\text{Df}$$

The defining rules become compositional, just like the other defining rules.

BUT THEN...

$$\frac{x, X \vdash Y, A(x)}{X \vdash Y, \forall x A(x)}$$

\forall DF

$$\frac{x, A(x), X \vdash Y}{\exists x A(x), X \vdash Y}$$

\exists DF

THIS SEEMS REDUNDANT!

What would it mean for the variable x
to be undefined in this context?

Isn't using a variable tantamount to
treating it as having some value or other?

(E.g. Feferman takes $x \downarrow$ to be a theorem.)

ALTERNATE FORMULATION

- Variables are always treated as defined

$$\frac{}{X \succ Y, x}$$

- They are no longer inferentially general
but are general among the defined terms.

$$\frac{X \succ Y}{t, [X \succ Y]_t} \text{Spec}_t^x$$

- Variables are used for quantification,
not general singular terms.

$$\frac{X \succ Y, A(x)}{X \succ Y, \forall x A(x)} \text{VDF}_x$$

EQUivalence

Derivations in the two systems are inter-translatable

Id, Cut

FL fL

Specⁿ_t

Connective Df...

$\forall Df_n \exists Df_n$

x, y... only ever bound.

Eigenvariables n occur free.

Eigenvariable with existential commitment

Id, Cut

fL fL

Spec^x_t $x \downarrow$

Connective Df...

$\forall Df_x \exists Df_x$

x, y... occur free

No eigenvariables

Variable



THE UPSHOT

This formulation provides a context in which inner quantification is definable, using the resources available, but outer quantification (defined - the obvious way) uses extra resources.

This is not to say that outer quantification is meaningless, of course!

LESSONS & FURTHER QUESTIONS

- Quantification involves a number of connected issues
SUBSTITUTION, GENERALITY, DEFINEDNESS, VARIABLES/VALUES, ETC.
- Choosing a structural context for deduction means taking sides on these issues.
- Quantification into sentence position? predicate position?
 - do similar issues apply there?
- Model calculi?
 - What of singular terms that take values in some worlds and not others?

