

Proof Identity, Invariants and Hyperintensionality

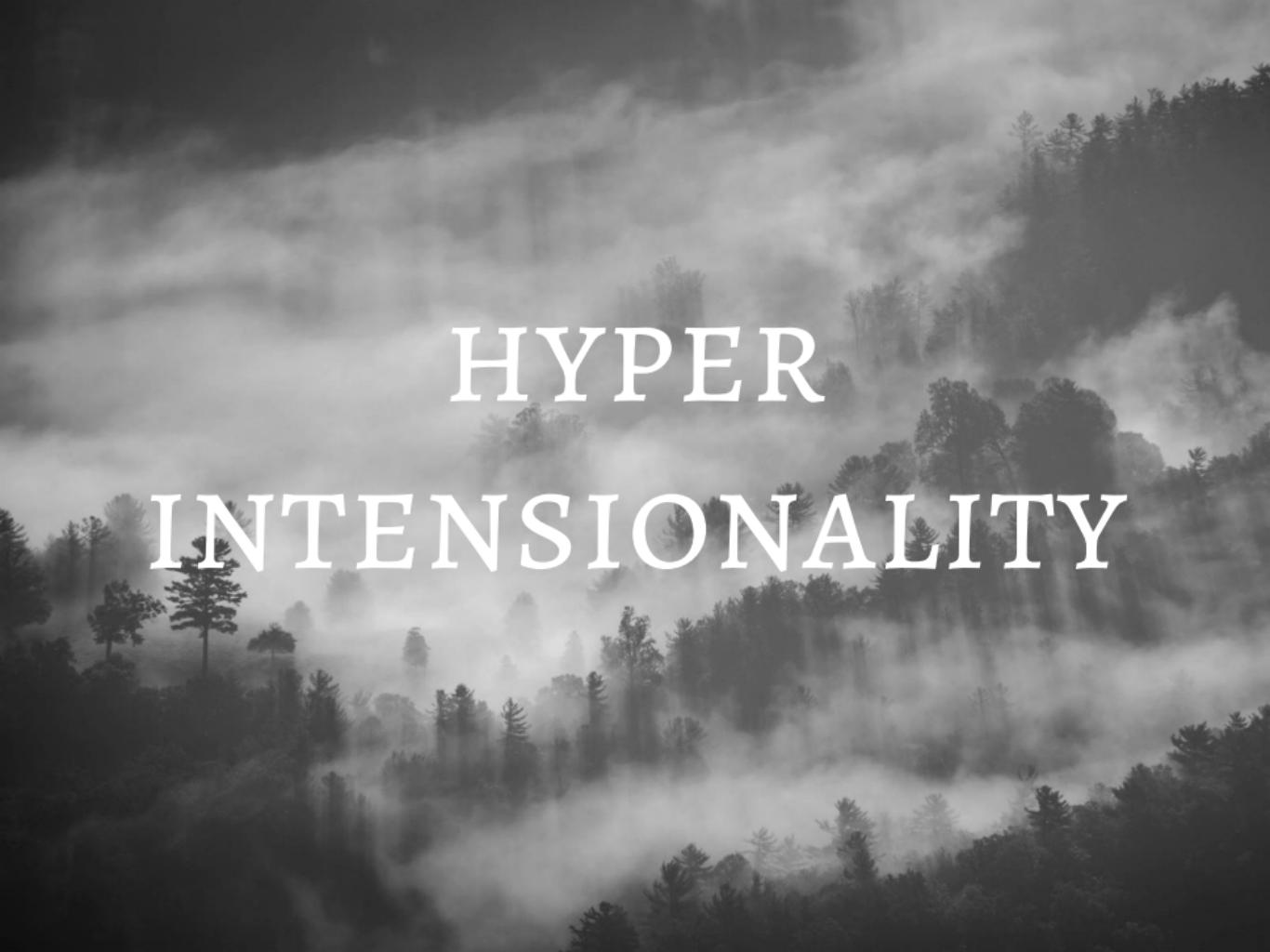
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THE UNIVERSITY OF
MELBOURNE

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Hyperintensionality
Models and Proofs
Truthmakers
Subject Matter
Proof Invariants
Where we've got, and where to from here



HYPER INTENSIONALITY

Extensionality, Intensionality and Hyperintensionality

- EXTENSIONAL concepts: compositional with respect to *truth values*, *referents*, etc. — \wedge , \vee , \neg , \forall , \exists , $=$.

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- ▶ HYPERINTENSIONAL: more finely grained still: *e.g. proves that, makes true, because.*
 - Distinguish between logically equivalent propositions. $p \vee \neg p$ and \top ; $p \vee (p \wedge q)$ and p , etc.
- ▶ I want to explore the difference between some *model theoretic* and *proof theoretic* approaches to hyperintensionality.

The background image shows a wide-angle view of a mountainous region. In the foreground, a river flows through a valley, its path winding through a dense forest of green trees. The mountains in the background are rugged and partially covered in snow or ice. The sky is a clear blue with scattered white clouds.

MODELS AND PROOFS

Two Traditions in Logic and Semantics

MODEL THEORY (1) Define *models* and give recursive conditions explaining when a claim is true in a model.

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Two More Tools for Understanding Aboutness

MODEL THEORY

Situations as Truthmakers

PROOF THEORY

Proof Invariants

TRUTHMAKERS

Case 1: $p \vee \neg p$

$p \vee \neg p$

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$$p \vee \neg p$$

$s \Vdash p \vee \neg p$ iff $s \Vdash p$ or $s \Vdash \neg p$

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$$p \vee \neg p$$

$s \Vdash p \vee \neg p$ iff $s \Vdash p$ or $s \Vdash \neg p$

Many situations are silent on whether p or not.

(*Making $p \vee \neg p$ true* is different from *making $q \vee \neg q$ true*.)

Case 2: $p \vee (p \wedge q)$

$$p \vee (p \wedge q)$$

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$$p \vee (p \wedge q)$$

$s \Vdash p \vee (p \wedge q)$ iff $s \Vdash p$ or $s \Vdash p \wedge q$,
iff $s \Vdash p$ or ($s \Vdash p$ and $s \Vdash q$),
iff $s \Vdash p$.

Case 2: $p \vee (p \wedge q)$

$$p \vee (p \wedge q)$$

$s \Vdash p \vee (p \wedge q)$ iff $s \Vdash p$ or $s \Vdash p \wedge q$,
iff $s \Vdash p$ or ($s \Vdash p$ and $s \Vdash q$),
iff $s \Vdash p$.

Making $p \vee (p \wedge q)$ true just is making p true.

Case 3: $(p \vee \neg p) \vee (q \vee \neg q)$ and $(p \vee \neg p) \wedge (q \vee \neg q)$

$$(p \vee \neg p) \vee (q \vee \neg q)$$

Either make p true, or make p false,
or make q true, or make q false.

Case 3: $(p \vee \neg p) \vee (q \vee \neg q)$ and $(p \vee \neg p) \wedge (q \vee \neg q)$

$$(p \vee \neg p) \vee (q \vee \neg q)$$

Either make p true, or make p false,
or make q true, or make q false.

$$(p \vee \neg p) \wedge (q \vee \neg q)$$

Either make p and q true, or make p true and q false,
or make p false and q true, or make p and q false.

A photograph showing a vast, dense grid of green theater chairs, all facing towards the left. The chairs are arranged in numerous rows, creating a pattern of repeating curved shapes. Some chairs have small red tags attached to their backrests.

SUBJECT MATTER

Positive Subject Matter

“The [positive] subject matter of S
is the relation m such that
worlds are m -dissimilar iff
 S is differently true in them.”

Positive Subject Matter

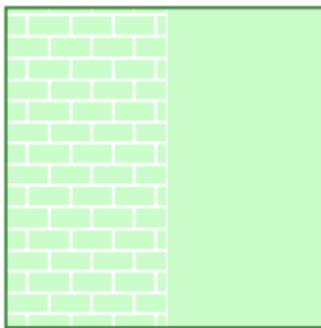
“The [positive] subject matter of S
is the relation m such that
worlds are m -dissimilar iff
 S is differently true in them.”

This is: how S is true

Picturing Positive Subject Matter



Picturing Positive Subject Matter



Negative Subject Matter

The negative subject matter of S
is the relation m' such that
worlds are m' -dissimilar iff
 S is differently false in them.”

Negative Subject Matter

The negative subject matter of S
is the relation m' such that
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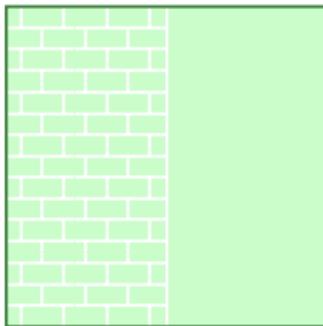
This is: how S is false

The Subject Matter of S

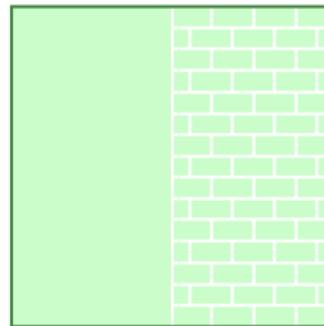
{how S is true, how S is false}

Aboutness, Chapter 2

Picturing Subject Matter

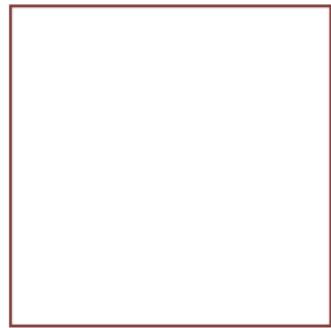


how S is true



how S is false

The subject matter of $p \vee \neg p$



how $p \vee \neg p$ is true

how $p \vee \neg p$ is false

The subject matter of $p \vee \neg p$

p

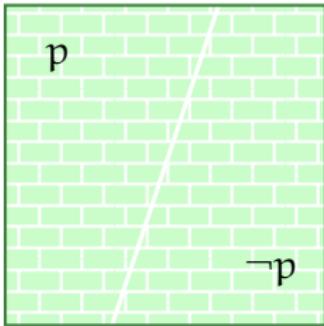
$\neg p$



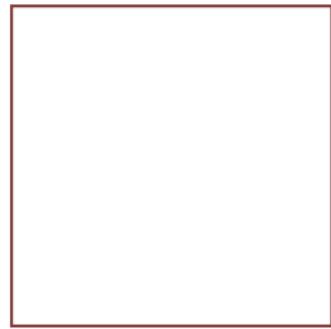
how $p \vee \neg p$ is true

how $p \vee \neg p$ is false

The subject matter of $p \vee \neg p$

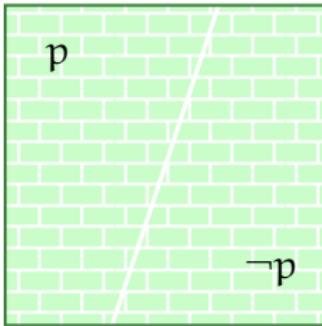


how $p \vee \neg p$ is true

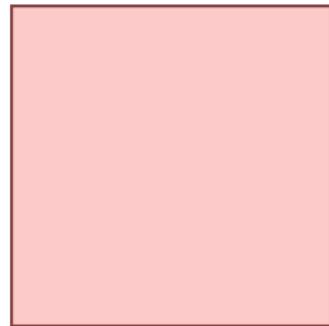


how $p \vee \neg p$ is false

The subject matter of $p \vee \neg p$



how $p \vee \neg p$ is true



how $p \vee \neg p$ is false

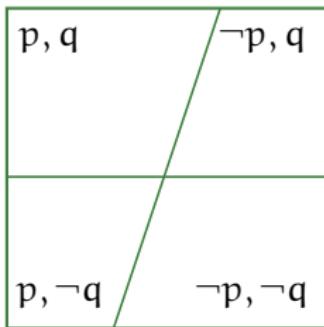
The subject matter of $p \vee (p \wedge q)$



how $p \vee (p \wedge q)$ is true

how $p \vee (p \wedge q)$ is false

The subject matter of $p \vee (p \wedge q)$

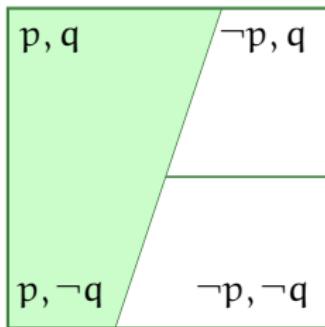


how $p \vee (p \wedge q)$ is true



how $p \vee (p \wedge q)$ is false

The subject matter of $p \vee (p \wedge q)$

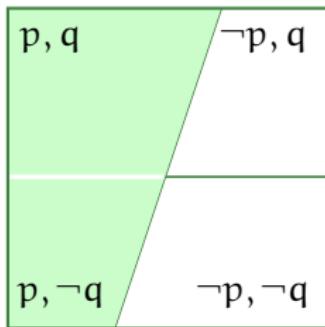


how $p \vee (p \wedge q)$ is true



how $p \vee (p \wedge q)$ is false

The subject matter of $p \vee (p \wedge q)$

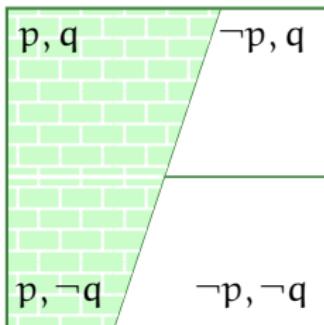


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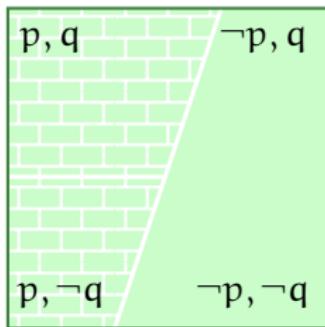


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The subject matter of $p \vee (p \wedge q)$

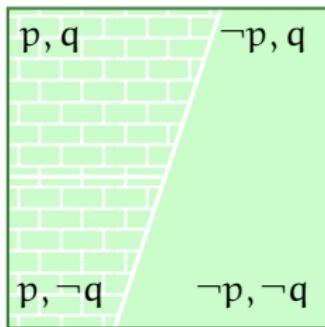


how $p \vee (p \wedge q)$ is true

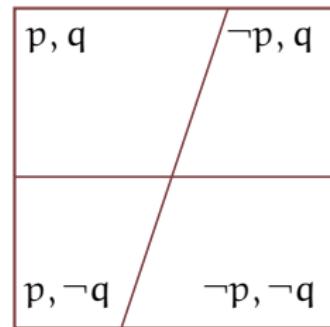


how $p \vee (p \wedge q)$ is false

The subject matter of $p \vee (p \wedge q)$

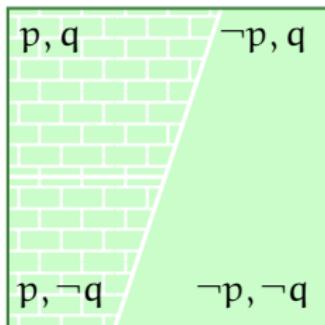


how $p \vee (p \wedge q)$ is true

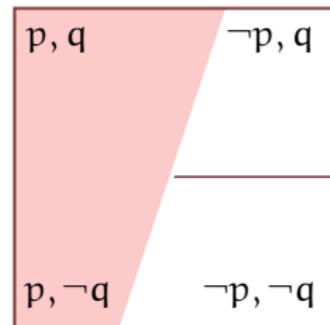


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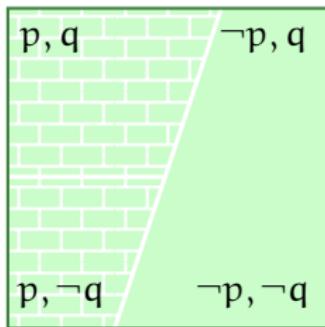


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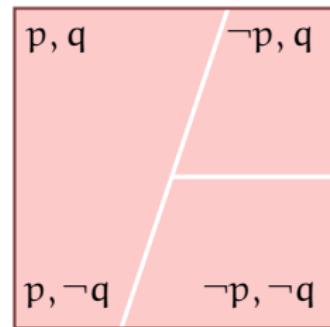


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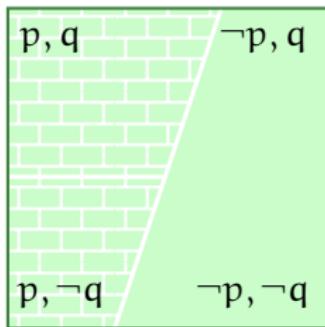


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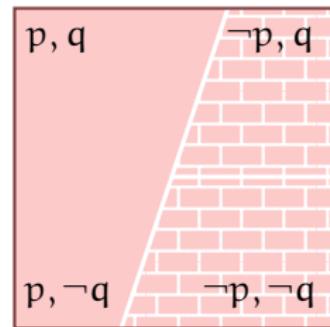


how $p \vee (p \wedge q)$ is false

The subject matter of $p \vee (p \wedge q)$



how $p \vee (p \wedge q)$ is true



how $p \vee (p \wedge q)$ is false

$$(p \vee \neg p) \vee (q \vee \neg q)$$

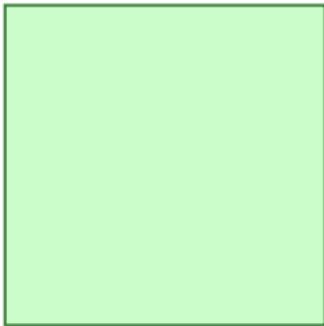


how $(p \vee \neg p) \vee (q \vee \neg q)$ is true



how $(p \vee \neg p) \vee (q \vee \neg q)$ is false

$$(p \vee \neg p) \vee (q \vee \neg q)$$

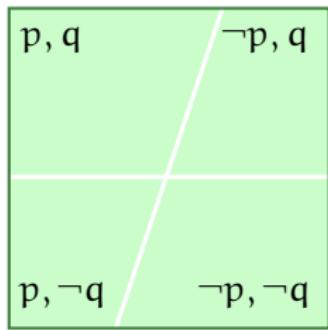


how $(p \vee \neg p) \vee (q \vee \neg q)$ is true



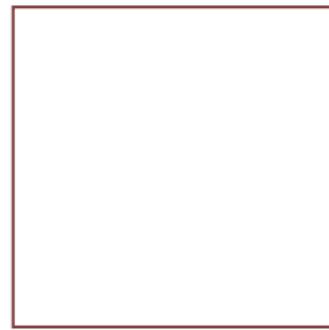
how $(p \vee \neg p) \vee (q \vee \neg q)$ is false

$$(p \vee \neg p) \vee (q \vee \neg q)$$

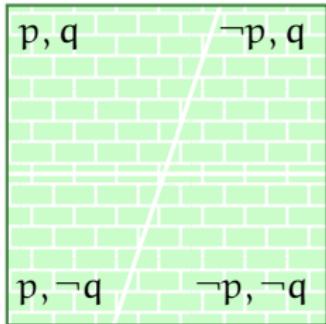


how $(p \vee \neg p) \vee (q \vee \neg q)$ is true

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$$(p \vee \neg p) \vee (q \vee \neg q)$$

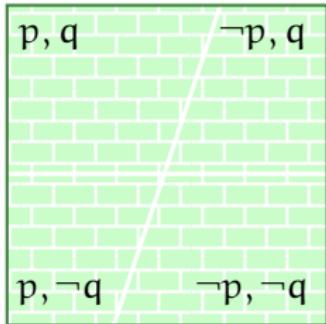


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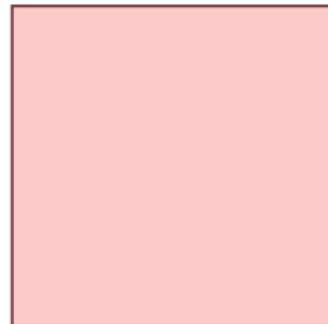


$$(p \vee \neg p) \vee (q \vee \neg q)$$



how $(p \vee \neg p) \vee (q \vee \neg q)$ is true

how $(p \vee \neg p) \vee (q \vee \neg q)$ is false



$$(p \vee \neg p) \wedge (q \vee \neg q)$$



how $(p \vee \neg p) \wedge (q \vee \neg q)$ is true



how $(p \vee \neg p) \wedge (q \vee \neg q)$ is false

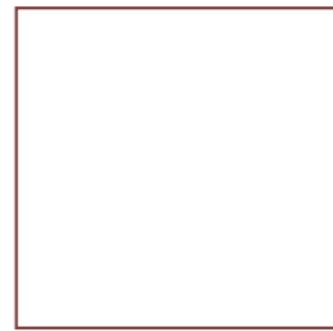
$$(p \vee \neg p) \wedge (q \vee \neg q)$$

p, q

$\neg p, q$

$p, \neg q$

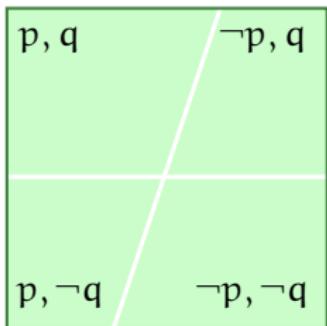
$\neg p, \neg q$



how $(p \vee \neg p) \wedge (q \vee \neg q)$ is true

how $(p \vee \neg p) \wedge (q \vee \neg q)$ is false

$$(p \vee \neg p) \wedge (q \vee \neg q)$$

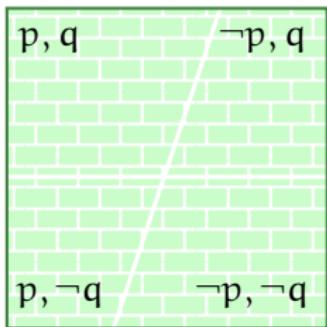


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$$(p \vee \neg p) \wedge (q \vee \neg q)$$

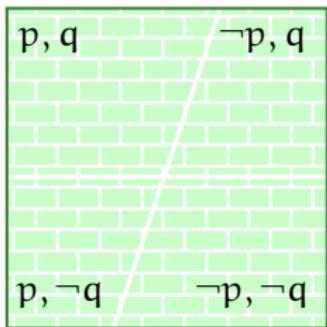


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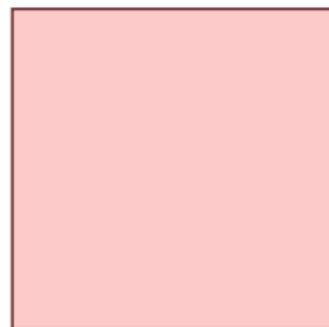


$$(p \vee \neg p) \wedge (q \vee \neg q)$$



how $(p \vee \neg p) \wedge (q \vee \neg q)$ is true

how $(p \vee \neg p) \wedge (q \vee \neg q)$ is false





PROOF INVARIANTS

When is π_1 the same proof as π_2 ?

$$\frac{p \wedge q}{\frac{p}{p \vee q}} \wedge E$$

$$\frac{p \succ p}{\frac{p \succ p \vee q}{p \wedge q \succ p \vee q}} \vee R$$

When is π_1 the same proof as π_2 ?

$$\frac{p \wedge q}{\frac{p}{p \vee q}} \wedge E$$

$$\frac{p \succ p}{\frac{p \succ p \vee q}{p \wedge q \succ p \vee q}} \vee R$$

When is π_1 the same proof as π_2 ?

$$\frac{p \wedge q}{\frac{p}{p \vee q}} \wedge E$$

$$\frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

When is π_1 the same proof as π_2 ?

$$\frac{p \wedge q}{\frac{p}{p \vee q}} \wedge E$$

$$\frac{p \succ p}{\frac{p \succ p \vee q}{p \wedge q \succ p \vee q}} \vee R \quad \wedge L$$

When is π_1 the same proof as π_2 ?

$$\frac{\begin{array}{c} p \wedge q \\ \hline p \end{array}}{p \vee q} \wedge E$$

$$\frac{\begin{array}{c} p \succ p \\ \hline p \succ p \vee q \end{array}}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{\begin{array}{c} p \wedge q \\ \hline q \end{array}}{p \vee q} \wedge E$$

$$\frac{\begin{array}{c} q \succ q \\ \hline q \succ p \vee q \end{array}}{p \wedge q \succ p \vee q} \wedge L$$

When is π_1 the same proof as π_2 ?

$$\frac{p \wedge q}{\frac{p}{p \vee q} \vee I} \wedge E$$

$$\frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{p \wedge q}{\frac{q}{p \vee q} \vee I} \wedge E$$

$$\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

When is π_1 the same proof as π_2 ?

$$\frac{p \wedge q}{\frac{p}{p \vee q} \vee I} \wedge E$$

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$$\frac{p \wedge q}{\frac{q}{p \vee q} \vee I} \wedge E$$

$$\frac{\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L}{p \wedge q \succ p \vee q} \wedge L$$

When is π_1 the same proof as π_2 ?

$$\frac{p \wedge q}{\frac{p}{p \vee q} \vee I} \wedge E$$

$$\frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{p \wedge q}{\frac{q}{p \vee q} \vee I} \wedge E$$

$$\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

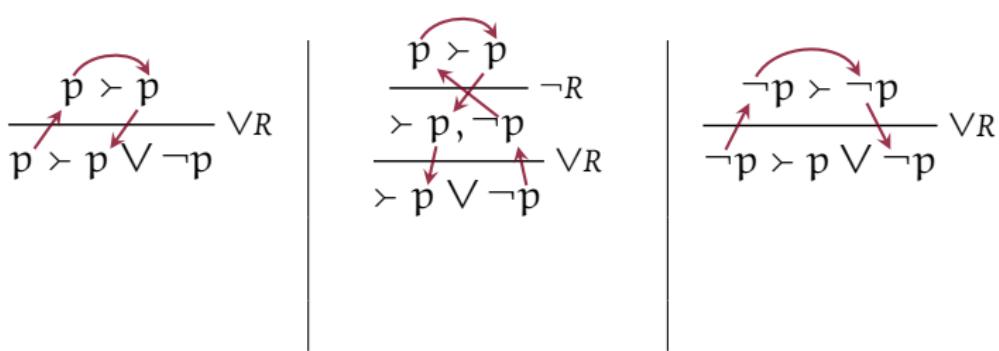
Proofs, proofinvariants and truth conditions

SLOGAN 1: A proof of C from P shows how C can obtain in P-circumstances.

SLOGAN 2: Different *proofinvariants* give different truth conditions.

$$\frac{\frac{p \wedge q}{p} \wedge E}{p \vee q} \vee I \qquad \frac{\frac{p \wedge q}{q} \wedge E}{p \vee q} \vee I$$

Case 1: $p \vee \neg p$



Case 1: $p \vee \neg p$

$$\frac{p \succ p}{p \succ p \vee \neg p} \text{ VR}$$

$$\frac{\frac{\frac{p \succ p}{\cancel{\succ p, \neg p}} \neg R}{\succ p \vee \neg p} \text{ VR}}{\succ p \vee \neg p}$$

$$\frac{\frac{\frac{\neg p \succ \neg p}{\cancel{\neg p \succ p \vee \neg p}} \neg R}{\neg p \succ p \vee \neg p} \text{ VR}}{\neg p \succ p \vee \neg p}$$

Holds when p

Case 1: $p \vee \neg p$

$$\frac{p \succ p}{p \succ p \vee \neg p} \text{ VR}$$

Holds when p

$$\frac{\begin{array}{c} p \succ p \\ \hline \succ p, \neg p \end{array}}{\succ p \vee \neg p} \text{ VR}$$

$$\frac{\neg p \succ \neg p}{\neg p \succ p \vee \neg p} \text{ VR}$$

Holds when $\neg p$

Case 1: $p \vee \neg p$

$$\frac{p \succ p}{p \succ p \vee \neg p} \vee R$$

Holds when p

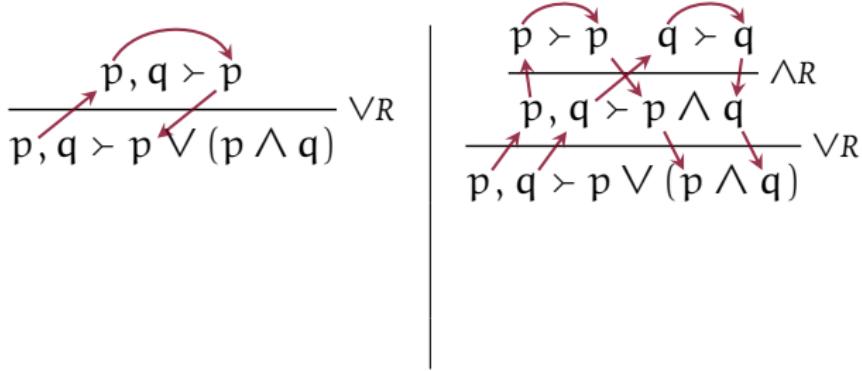
$$\frac{\begin{array}{c} p \succ p \\ \succ p, \neg p \end{array}}{\succ p \vee \neg p} \vee R$$

Holds independently

$$\frac{\neg p \succ \neg p}{\neg p \succ p \vee \neg p} \vee R$$

Holds when $\neg p$

Case 2: $p \vee (p \wedge q)$



Case 2: $p \vee (p \wedge q)$

$$\frac{p, q \succ p}{p, q \succ p \vee (p \wedge q)} \vee R$$

$$\frac{\begin{array}{c} p \succ p \\ \hline p, q \succ p \wedge q \end{array}}{\begin{array}{c} p \succ p \wedge q \\ \hline p, q \succ p \vee (p \wedge q) \end{array}} \wedge R$$

Holds when p

Case 2: $p \vee (p \wedge q)$

$$\frac{p, q \succ p}{p, q \succ p \vee (p \wedge q)} \vee R$$

$$\frac{\begin{array}{c} p \succ p \\ q \succ q \end{array}}{p, q \succ p \wedge q} \wedge R$$
$$\frac{}{p, q \succ p \vee (p \wedge q)} \vee R$$

Holds when p

Case 2: $p \vee (p \wedge q)$

$$\frac{\frac{p, q \succ p}{p, q \succ p \vee (p \wedge q)} \vee R \quad \frac{\frac{p \succ p \quad q \succ q}{p, q \succ p \wedge q} \wedge R}{p, q \succ p \vee (p \wedge q)}}{p, q \succ p \vee (p \wedge q)} \vee R$$

Holds when p

Case 2: $p \vee (p \wedge q)$

$$\frac{p, q \succ p}{p, q \succ p \vee (p \wedge q)} \vee R$$

Holds when p

$$\frac{\begin{array}{c} p \succ p \\ q \succ q \end{array}}{\frac{p, q \succ p \wedge q}{p, q \succ p \vee (p \wedge q)}} \wedge R \quad \vee R$$

Holds when p and q

Case 3a: $(p \vee \neg p) \vee (q \vee \neg q)$

$$\frac{\frac{\frac{p \succ p}{p \succ p \vee \neg p} \vee R}{p \succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R \quad \frac{\neg p \succ \neg p}{\neg p \succ p \vee \neg p} \vee R}{\neg p \succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R$$
$$\frac{\frac{\frac{p \succ p}{\succ p, \neg p} \neg R}{\succ p \vee \neg p} \vee R}{\succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R$$

Case 3a: $(p \vee \neg p) \vee (q \vee \neg q)$

$$\frac{\frac{\frac{p \succ p}{p \succ p \vee \neg p} \vee R}{p \succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R \quad \frac{\neg p \succ \neg p}{\neg p \succ p \vee \neg p} \vee R}{\neg p \succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R$$
$$\frac{\frac{\frac{p \succ p}{\succ p, \neg p} \neg R}{\succ p \vee \neg p} \vee R}{\succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R}{\succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R$$

These conditions *don't involve q*.

Case 3a: $(p \vee \neg p) \vee (q \vee \neg q)$

$$\frac{\frac{\frac{q \succ q}{q \succ q \vee \neg q} \vee R}{q \succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R}{q \succ (p \vee \neg p) \vee (q \vee \neg q)}$$
$$\frac{\frac{\neg q \succ \neg q}{\neg q \succ q \vee \neg q} \vee R}{\neg q \succ (p \vee \neg p) \vee (q \vee \neg q)}$$
$$\frac{\frac{\frac{q \succ q}{\succ q, \neg q} \neg R}{\succ q \vee \neg q} \vee R}{\succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R}{\succ (p \vee \neg p) \vee (q \vee \neg q)}$$

Case 3a: $(p \vee \neg p) \vee (q \vee \neg q)$

$$\frac{\frac{\frac{q \succ q}{q \succ q \vee \neg q} \vee R}{q \succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R \quad \frac{\neg q \succ \neg q}{\neg q \succ q \vee \neg q} \vee R}{\neg q \succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R$$
$$\frac{\frac{\frac{q \succ q}{\succ q, \neg q} \neg R}{\succ q \vee \neg q} \vee R}{\succ (p \vee \neg p) \vee (q \vee \neg q)} \vee R$$

These conditions *don't involve p*.

Case 3b: $(p \vee \neg p) \wedge (q \vee \neg q)$

$$\frac{q \succ q}{q \succ q \vee \neg q} \vee R \quad \frac{\neg p \succ \neg p}{\neg p \succ p \vee \neg p} \vee R$$
$$\frac{}{\neg p, q \succ (p \vee \neg p) \wedge (q \vee \neg q)} \wedge R$$
$$\frac{p \succ p}{\succ p, \neg p} \neg R \quad \frac{q \succ q}{\succ q, \neg q} \neg R$$
$$\frac{\succ p \vee \neg p}{\succ q \vee \neg q} \vee R$$
$$\frac{\succ p \vee \neg p \quad \succ q \vee \neg q}{\succ (p \vee \neg p) \wedge (q \vee \neg q)} \wedge R$$


These conditions *always involve both p and q.*

A scenic view of Bryce Canyon National Park, featuring a vast landscape of red rock hoodoos and a winding trail. The sky is clear and blue. The text "WHERE WE'VE GOT, AND WHERE TO FROM HERE" is overlaid in large, white, serif capital letters.

WHERE WE'VE GOT,
AND WHERE TO
FROM HERE

Scorecard

SUBJECT MATTER	SITUATIONS	INVARIANTS
$p \vee \neg p / \top$	<i>different</i>	<i>different</i>
$p \vee (p \wedge q) / p$	<i>different</i>	<i>same</i>
$(p \vee \neg p) \vee (q \vee \neg q) /$ $(p \vee \neg p) \wedge (q \vee \neg q)$	<i>same</i>	<i>different</i>

Places to Go

- ▶ In what can proofs and proof invariants do interesting work in articulating *subject matter*?
- ▶ Can invariants be used to define a notion of *logical subtraction*? (See Stephen Yablo's *Aboutness* for details.)
- ▶ Extend proof invariants beyond propositional logic.

THANK YOU!

[http://consequently.org/presentation/2017/
proof-identity-aboutness-and-meaning](http://consequently.org/presentation/2017/proof-identity-aboutness-and-meaning)

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