DAY 4

MODAL LOGIC

TODAY'S PLAN

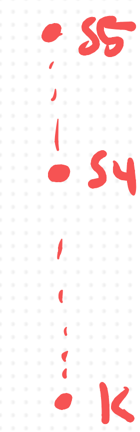
TRADITIONAL PROOFS FOR NECESSITY / POSSIBILITY

PROOFS / DERIVATIONS WITH LABELS

HYPERSEQUENTS

LIMIT POSITIONS & COMPLETENESS

TWO SIMPLE MODAL LOGICS



* S5 $\langle W, \Vdash \rangle$

$w \Vdash A \wedge B$ iff $w \Vdash A$ & $w \Vdash B$

$w \Vdash \neg A$ iff $w \nVdash A$

$w \Vdash \Box A$ iff $\forall v \in W, v \Vdash A$

$w \Vdash \Diamond A$ iff $\exists v \in W, v \Vdash A$

* S4 $\langle W, R, \Vdash \rangle$

$\wedge, \neg$ as before,

R reflexive & transitive

$w \Vdash \Box A$ iff $\forall v \in W (wRv \Rightarrow v \Vdash A)$

$w \Vdash \Diamond A$ iff $\exists v \in W (wRv \;\&\; v \Vdash A)$

(Can also model S5 by making R symmetric, too...)

$\Box A \vdash_{S4} A$ $\qquad$ $\Box A \vdash_{S4} \Box\Box A$ $\qquad$ $\Diamond A \nvdash_{S4} \Box \Diamond A$

$\vdash_{S5}$

MODAL NATURAL DEDUCTION

$$\dfrac{\begin{array}{c}\vdots\\ \Box A\end{array}}{A}\ \Box E \qquad\qquad \dfrac{\begin{array}{c}X\\ \vdots\\ A\end{array}}{\Box A}\ \Box I^{*}$$

MODAL NATURAL DEDUCTION

$$\dfrac{\begin{matrix}\vdots\\ \Box A\end{matrix}}{A}\ \Box E \qquad\qquad \dfrac{\begin{matrix}X\\ \vdots\\ A\end{matrix}}{\Box A}\ \Box I^{*}$$

* CONDITIONS APPLY:

X must only contain formulas starting with $\Box$.

(NECESSITIVES)

EXAMPLE PROOFS

- ✓ $\Box p \vdash p$
- ✓ $\Box p \vdash \Box\Box p$
- ✓ $\Box p, \Box q \vdash \Box(p \wedge q)$
- ✗ $\Box p \wedge \Box q \vdash \Box(p \wedge q)$

$$\dfrac{\dfrac{\dfrac{\Box p}{p}\;\Box E}{\Box p}\;\Box I}{\Box\Box p}\;\Box I \qquad\qquad \dfrac{\Box p}{\Box\Box p}\;\Box I$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box p \wedge \Box q}{\Box p}}{p}\;\Box E \qquad \dfrac{\dfrac{\Box p \wedge \Box q}{\Box q}}{q}\;\Box E}{p \wedge q}\;\wedge I}{\Box(p \wedge q)}\;\Box I \;\times$$

WHY THESE RULES ARE SOUND for S4

Show by induction on the construction of Π (for $X \vdash A$) that in any S4 model $\langle W, R, \Vdash \rangle$, for each $w \in W$, if each member of X is true at w then so is A.

Cases to Check: $\dfrac{\Box A}{A}\,\Box E$ $\qquad$ $\left\{\begin{array}{c} X \\ \vdots \\ A \end{array}\right.$ $\dfrac{A}{\Box A}\,\Box I^{*}$

Spose $X \vdash A$ is valid. $\forall w$ (If $w \Vdash B$ for each $\underline{\underline{B \in X}}$ then $w \Vdash A$)

$B \in X$ so $B = \Box C$ for some C.

If $w \Vdash \Box C$ true, & Rwv, then $v \Vdash \Box C$ true too,

since $w \Vdash \Box\Box C$ true.

$v \Vdash A$ $\longrightarrow$ $w \Vdash \Box A$.

WHY THESE RULES ARE S4 COMPLETE

- **THEFT:** Prove each of the Hilbert axioms in Rineke's system for S4 & apply her completeness proof.

- **HARD WORK:** Mimic that proof directly.

S5 PROOFS?

Change the side condition for $\Box I$.

$$\cfrac{\begin{matrix} X \\ \vdots \\ A \end{matrix}}{\Box A}\,\Box I^{*}$$

* CONDITIONS APPLY:

X must only contain formulas starting with $\Box$ or $\Diamond$ (MODAL formulas)

(SOUNDNESS & COMPLETENESS is straightforward.)

THESE RULES ARE UGLY

$$\dfrac{\begin{matrix}\Pi_1\\ q \wedge \Box p\end{matrix}}{\Box p}\,\wedge E \qquad\qquad \dfrac{\begin{matrix}\Pi_2\\ \Box p\end{matrix}}{\Box\Box p}\,\Box I$$

Π_1 is a proof to $\Box p$

Π_2 is a proof from $\Box p$

But you cannot (directly) compose them into one proof!

BUT INDIRECTLY?

... SURE THING! HERE'S ONE WAY

$$\dfrac{\dfrac{\dfrac{[\Box p]}{\Box\Box p}\,\Box I}{\Box p \to \Box\Box p}\,{\to}I \qquad \dfrac{q \wedge \Box p}{\Box p}\,{\wedge}E}{\Box\Box p}\,{\to}E$$

But notice — the detour cannot be simplified.

THERE MUST BE A BETTER WAY

* S5 is just monadic predicate logic with one variable.

$$t(p) = Px$$

$$t(A \wedge B) = t(A) \wedge t(B)$$

$$t(\neg A) = \neg t(A)$$

$$t(\Box A) = \forall x\, t(A)$$

$$t(\Diamond A) = \exists x\, t(A)$$

So $t(\Diamond p \to \Box \Diamond p) = \exists x Px \to \forall x \exists x Px$

AND WE CAN DO NICE PROOFS FOR THAT

$$\dfrac{\begin{array}{c}\vdots\\ \forall x A(x)\end{array}}{A(t)}\ \forall E$$

$$\dfrac{\begin{array}{c}X\\ \vdots\\ A(n)\end{array}}{\forall x A(x)}\ \forall I^{*}$$

* n absent from X & from $\forall x A(x)$

HANG ON... IS THIS ANY BETTER?

$$\dfrac{q \wedge \Box p}{\Box p}\,\wedge E \quad \& \quad \dfrac{\dfrac{\dfrac{\Box p}{p}\,\Box E}{p \vee q}\,\vee I}{\Box(p \vee q)}\,\Box I^{*}$$

don't compose.

What about these? — they don't compose either?!

$$\dfrac{Qx \wedge \forall x Px}{\forall x Px}\,\forall E \quad \& \quad \dfrac{\dfrac{\dfrac{\forall x Px}{Px}\,\forall E}{Px \vee Qx}\,\vee I}{\forall x (Px \vee Qx)}\,\forall I^{*}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Qx \wedge \forall x Px}{\forall x Px}\,\forall E}{Px}\,\forall E}{Px \vee Qx}\,\vee I}{\forall x (Px \vee Qx)}\,\forall I^{*}$$

Not a proof, since the $\forall I$ condition is now violated

$$\dfrac{\dfrac{\dfrac{\dfrac{Qn \wedge \forall x\, Px}{\forall x\, Px}\;\forall E}{Pn}\;\forall E}{Pn \vee Qn}\;\vee I}{\forall x\,(Pn \vee Qn)}\;\forall I^{*}$$

Not a proof, since the ∀I condition is now violated

But there is an easy fix!

Use a different eigenvariable inside the sealed-off subproof

$$\dfrac{\dfrac{\dfrac{\dfrac{Qn \wedge \forall x\, Px}{\forall x\, Px}\;\forall E}{Pm}\;\forall E}{Pm \vee Qm}\;\vee I}{\forall x\,(Pn \vee Qn)}\;\forall I^{*}$$

Isn't this really the same proof, notated differently?

LABELLED NATURAL DEDUCTION for S5

$$\dfrac{A\ w \qquad B\ w}{A \wedge B\ w}\ \wedge I \qquad \dfrac{A \wedge B\ w}{A\ w}\ \wedge E \qquad \dfrac{A \wedge B\ w}{B\ w}\ \wedge E$$

$$\dfrac{\neg A\ w \qquad A\ w}{\#}\ \neg E \qquad \dfrac{\begin{array}{c}[A\ w]^i \\ \vdots \\ \#\end{array}}{\neg A\ w}\ \neg I^i \qquad \dfrac{A\ w \qquad \cancel{A\ w}}{\#}\ \uparrow \qquad \dfrac{\begin{array}{c}[\cancel{A}\ w]^i \\ \vdots \\ \#\end{array}}{A\ w}\ \downarrow^i$$

$$\dfrac{\Box A\ w}{A\ v}\ \Box E \qquad \dfrac{\begin{array}{c}X \\ \vdots \\ A\ w\end{array}}{\Box A\ v}\ \Box I^*$$

* w is absent from X

EXAMPLE PROOFS

$\Box p \wedge \Box q \vdash \Box(p \wedge q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box p \wedge \Box q \ \ w}{\Box p \ \ w}{\scriptstyle \wedge E}}{p \ \ v}{\scriptstyle \Box E} \qquad \dfrac{\dfrac{\Box p \wedge \Box q \ \ w}{\Box q \ \ w}{\scriptstyle \wedge E}}{q \ \ v}{\scriptstyle \Box E}}{p \wedge q \ \ v}{\scriptstyle \wedge I}}{\Box(p \wedge q) \ \ w}{\scriptstyle \Box I}$$

$\neg \Box p \vdash \Box \neg \Box p$

$$\dfrac{\dfrac{\neg \Box p \ \ w \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\Box p \ \ v]^1}{p \ \ v}{\scriptstyle \Box E}}{\Box p \ \ w}{\scriptstyle \Box I} \qquad \cancel{[\Box p \ \ w]}^2}{\#}{\scriptstyle \uparrow}}{\neg \Box p \ \ v}{\scriptstyle \neg I^1}}{\Box \neg \Box p \ \ w} \qquad \cancel{[\Box \neg \Box p \ \ w]}^3}{\#}{\scriptstyle \uparrow}}{\Box p \ \ w}{\scriptstyle \downarrow^2}}{}}{\#}{\scriptstyle \neg E}}{\Box \neg \Box p \ \ w}{\scriptstyle \downarrow^3}$$

S4 is relatively straightforward...

$$\frac{\Box A\, w \qquad Rwv}{A\, v}\ \Box E$$

$$\frac{\begin{matrix} X[Rvw]^i \\ \vdots \\ A\, w \end{matrix}}{\Box A\, v}\ \Box I^{i\,*}$$

* w is absent from X

$$\frac{}{Rxx}\ \text{Refl}$$

$$\frac{Rxy \qquad Ryz}{Rxz}\ \text{Trans}$$

Example: $\Box p \vdash \Box\Box p$

$$\dfrac{\dfrac{\dfrac{\Box p\, w \qquad \dfrac{[Rwv]^2 \quad [Rvu]^1}{Rwu}}{p\, u}\ \Box E}{\Box p\, v}\ \Box I^1}{\Box\Box p\, w}\ \Box I^2$$

SEQUENTS?

$$\frac{X, Rwv \vdash Av, Y}{X \vdash \Box Aw, Y}\ \Box R^{*}$$

(v absent from conclusion)

$$\frac{X, Rww \vdash Y}{X \vdash Y}\ \mathit{Refl}$$

$$\frac{X, Av \vdash Y}{X, \Box Aw, Rwv \vdash Y}\ \Box L$$

$$\frac{X, Rwu \vdash Y}{X, Rwv, Ru \vdash Y}\ \mathit{Trans}$$

SEQUENTS?

$$\frac{X, Rwv \vdash Av, Y}{X \vdash \Box Aw, Y}\ \Box R^{*}$$

(v absent from conclusion)

$$\frac{X, Rww \vdash Y}{X \vdash Y}\ \text{Refl}$$

$$\frac{X, Av \vdash Y}{X, \Box Aw, Rwv \vdash Y}\ \Box L$$

$$\frac{X, Rwu \vdash Y}{X, Rwv, Rvu \vdash Y}\ \text{Trans}$$

Example Proof!

$$\cfrac{\cfrac{\cfrac{\cfrac{pu \vdash pu}{\Box pw, Rwu \vdash pu}\ \Box L}{\Box pw, Rwv, Rvu \vdash pu}\ \text{Trans}}{\Box pw, Rwv \vdash \Box pv}\ \Box R}{\Box pw \vdash \Box\Box pw}\ \Box R$$

SEQUENTS?

$$\dfrac{X \succ A v, Y}{X \succ \Box A w, Y}\ \Box R^*$$

(v absent from conclusion)

$$\dfrac{X, A v \succ Y}{X, \Box A w \succ Y}\ \Box L$$

AND FOR S5 you can erase the relation!

Example Proof!

$$\dfrac{\dfrac{\dfrac{p\,u \succ p\,u}{\Box p\,w \succ p\,u}\ \Box L}{\Box p\,w \succ \Box p\,v}\ \Box R}{\Box p\,w \succ \Box\Box p\,w}\ \Box R$$

DO WE NEED THE LABELS?

Instead of using labels, simply collect together the formulas using the same label in different zones.

THE RESULT IS A HYPERSEQUENT

$$A x, C y \vdash B x, D y, E z$$

$$\Updownarrow$$

$$A \vdash B \mid C \vdash D \mid \vdash E$$

SIMPLE HYPERSEQUENTS for S5

$$\frac{X \succ A_v, Y}{X \succ \Box A_w, Y}\;\Box R^{*}$$

$$\frac{\mathcal{H} \mid X \succ Y \mid \;\succ A}{\mathcal{H} \mid X \succ \Box A, Y}\;\Box R$$

$$\frac{X, A_v \succ Y}{X, \Box A_w \succ Y}\;\Box L$$

$$\frac{\mathcal{H} \mid U \succ V \mid X, A \succ Y}{\mathcal{H} \mid U, \Box A \succ V \mid X \succ Y}\;\Box L$$

$$\dfrac{\dfrac{\dfrac{p_u \succ p_u}{\Box p_w \succ p_u}\;\Box L}{\Box p_w \succ \Box p_v}\;\Box R}{\Box p_w \succ \Box\Box p_w}\;\Box R$$

$$\dfrac{\dfrac{\dfrac{\succ \mid p \succ p}{\Box p \succ \mid\; \succ p}\;\Box L}{\Box p \succ \mid\; \succ \Box p}\;\Box R}{\Box p \succ \Box\Box p}\;\Box R$$

TREE HYPERSEQUENTS

$$\frac{\mathcal{H} \mid X \vdash Y \mid \;\vdash A}{\mathcal{H} \mid X \vdash \Box A, Y}\,\Box R \qquad\qquad \frac{\mathcal{H}(X \vdash Y \quad \vdash A)}{\mathcal{H}(X \vdash \Box A, Y)}\,\Box R$$

$$\frac{\mathcal{H} \mid U \vdash V \mid X, A \vdash Y}{\mathcal{H} \mid U, \Box A \vdash V \mid X \vdash Y}\,\Box L \qquad\qquad \frac{\mathcal{H}(U \vdash V \quad X, A \vdash Y)}{\mathcal{H}(U, \Box A \vdash V \quad X \vdash Y)}\,\Box L$$

DERIVE $\Box p \wedge \Box q \vdash \Box(p \wedge q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box p, \Box q \vdash \quad p \vdash p}{\Box p, \Box q \vdash \quad \vdash p}\,\Box L}{\Box p \wedge \Box q \vdash \quad \vdash p}\,\wedge L \qquad \dfrac{\dfrac{\Box p, \Box q \vdash \quad q \vdash q}{\Box p, \Box q \vdash \quad \vdash q}\,\Box L}{\Box p \wedge \Box q \vdash \quad \vdash q}\,\wedge L}{\Box p \wedge \Box q \vdash \quad \vdash p \wedge q}\,\wedge R}{\Box p \wedge \Box q \vdash \Box(p \wedge q)}\,\Box L$$

COMPLETENESS

A position is an underivable Hypersequent

we will write them $[X:Y] \mid [U:V] \mid \cdots$ for S5

or $[X:Y] \; [U:V] \; [W:Z]$ in general

Let's say a position is SATURATED when every zone is a partition of $\mathcal{L}$ and if whenever $\Box A$ is in the right of a zone, A is in the right of some zone [targeted from the first].

SATURATED positions are Kripke models!