Proof Terms as for Classical Derivations

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To introduce an *invariant*
for classical propositional proofs
to help address questions
about proof identity.

1. Why proof identity is important, but difficult.

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2. Proof terms for classical sequent derivations.

Project Outline

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2. Proof terms for classical sequent derivations.
3. Correctness for proof terms.

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6. Cut elimination for derivations.

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2. Proof terms for classical sequent derivations.
3. Correctness for proof terms.
4. Permutations for derivations.
5. Cut reduction for proof terms.
6. Cut elimination for derivations.

Background

Derivations and Terms

Permutations

Further Work


BACKGROUND

We can prove the same thing in different ways

On Proofs of THE IRRATIONALITY OF $\sqrt{2}$

By V. C. HARRIS
San Diego State College
San Diego, California

IN THIS paper there are given thirteen proofs that $\sqrt{2}$ is irrational. Indications are given as to whether the method employed extends to proving the irrationality of other square roots or of roots of higher order. In addition, a reference is provided to an incorrect proof recently published and its criticism.

Proof 1: **Terminal-digit proof.** Assume that $\sqrt{2}$ is rational, so that we can write $\sqrt{2}=a/b$ and $a^2=2b^2$, where a and b are

nal digit 0, which implies they are not relatively prime—a contradiction.

Proof 2: **First prime-divisor proof.** Assume $\sqrt{2} = a/b$ and $a^2 = 2b^2$, where a and b are relatively prime positive integers. We see that $b > 1$, since otherwise $b = 1$ and $\sqrt{2} = a$, an integer, which is incorrect. By division, $b^2 = (a/2)\cdot a$. Now (1) any positive integer > 1 is either a prime or a product of primes, and (2) if p is a prime dividing the product rs of two integers, then p divides r or p divides s. There is a prime p dividing b, since $b > 1$. Also, p divides $a/2$ or p divides a. In either case, p divides a, so that p divides both a and b—a contradiction to the assumption that

When is π_1 the *same proof* as π_2?

Can you see how *these* two proofs are different?

$$\dfrac{\dfrac{\dfrac{(p)^1 \qquad (p)^1}{p \wedge p}\,{\wedge}I}{p \supset (p \wedge p)}\,{\supset}I^1}{p \supset (p \supset (p \wedge p))}\,{\supset}I^2 \qquad\qquad \dfrac{\dfrac{\dfrac{(p)^1 \qquad (p)^2}{p \wedge p}\,{\wedge}I}{p \supset (p \wedge p)}\,{\supset}I^1}{p \supset (p \supset (p \wedge p))}\,{\supset}I^2$$

Annotate with λ-terms

$$\dfrac{\dfrac{\dfrac{(x : p)^1 \qquad (x : p)^1}{\langle x, x\rangle : p \wedge p}\wedge I}{\lambda x.\langle x, x\rangle : p \supset (p \wedge p)}\supset I^1}{\lambda y.\lambda x.\langle x, x\rangle : p \supset (p \supset (p \wedge p))}\supset I^2$$

$$\dfrac{\dfrac{\dfrac{(x : p)^1 \qquad (y : p)^2}{\langle x, y\rangle : p \wedge p}\wedge I}{\lambda x.\langle x, y\rangle : p \supset (p \wedge p)}\supset I^1}{\lambda y.\lambda x.\langle x, y\rangle : p \supset (p \supset (p \wedge p))}\supset I^2$$

Sequent Derivations can be Annotated, Too

$$\dfrac{\dfrac{\dfrac{x : p \succ x : p \qquad x : p \succ x : p}{x : p \succ \langle x, x\rangle : p \wedge p}\ {\wedge}R}{\succ \lambda x.\langle x, x\rangle : p \supset (p \wedge p)}\ {\supset}R}{\succ \lambda y.\lambda x.\langle x, x\rangle : p \supset (p \supset (p \wedge p))}\ {\supset}R$$

$$\dfrac{\dfrac{\dfrac{x : p \succ x : p \qquad y : p \succ y : p}{x : p, y : p \succ \langle x, y\rangle : p \wedge p}\ {\wedge}R}{y : p \succ \lambda x.\langle x, y\rangle : p \supset (p \wedge p)}\ {\supset}R}{\succ \lambda y.\lambda x.\langle x, y\rangle : p \supset (p \supset (p \wedge p))}\ {\supset}R$$

Different Sequent Derivations have the same term

$$\dfrac{\dfrac{x : p \succ x : p \qquad \dfrac{z : p \succ z : p}{y : p \wedge q \succ \mathrm{fst}\, y : p}\,{\wedge}L}{x : p, y : p \wedge q \succ \langle x, \mathrm{fst}\, y\rangle : p \wedge p}\,{\wedge}R}{\dfrac{y : p \wedge q \succ \lambda x.\langle x, \mathrm{fst}\, y\rangle : p \supset (p \wedge p)}{\succ \lambda y.\lambda x.\langle x, \mathrm{fst}\, y\rangle : (p \wedge q) \supset (p \supset (p \wedge p))}\,{\supset}R}\,{\supset}R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{x : p \succ x : p \qquad z : p \succ z : p}{x : p, z : p \succ \langle x, z\rangle : p \wedge p}\,{\wedge}R}{z : p \succ \lambda x.\langle x, z\rangle : p \supset (p \wedge p)}\,{\supset}R}{y : p \wedge q \succ \lambda x.\langle x, \mathrm{fst}\, y\rangle : p \supset (p \wedge p)}\,{\wedge}L}{\succ \lambda y.\lambda x.\langle x, \mathrm{fst}\, y\rangle : (p \wedge q) \supset (p \supset (p \wedge p))}\,{\supset}R$$

Different Sequent Derivations have the same term

$$
\dfrac{\dfrac{x : p \succ x : p \qquad \dfrac{z : p \succ z : p}{y : p \land q \succ \mathsf{fst}\, y : p}\ {\land}L}{x : p, y : p \land q \succ \langle x, \mathsf{fst}\, y\rangle : p \land p}\ {\land}R}{\dfrac{y : p \land q \succ \lambda x.\langle x, \mathsf{fst}\, y\rangle : p \supset (p \land p)}{\succ \lambda y.\lambda x.\langle x, \mathsf{fst}\, y\rangle : (p \land q) \supset (p \supset (p \land p))}\ {\supset}R}\ {\supset}R
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{x : p \succ x : p \qquad z : p \succ z : p}{x : p, z : p \succ \langle x, z\rangle : p \land p}\ {\land}R}{z : p \succ \lambda x.\langle x, z\rangle : p \supset (p \land p)}\ {\supset}R}{y : p \land q \succ \lambda x.\langle x, \mathsf{fst}\, y\rangle : p \supset (p \land p)}\ {\land}L}{\succ \lambda y.\lambda x.\langle x, \mathsf{fst}\, y\rangle : (p \land q) \supset (p \supset (p \land p))}\ {\supset}R
$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$
\dfrac{
\dfrac{\dfrac{p \succ p \qquad p \succ p}{p \vee p \succ p, p}\,{\vee L}}{p \vee p \succ p}\,W
\qquad
\dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \wedge p}\,{\wedge R}}{p \succ p \wedge p}\,W
}{p \vee p \succ p \wedge p}\,\mathit{Cut}
$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$
\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p \lor p \succ p, p}\;{\color{red}\lor L}}{p \lor p \succ p}\;W \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \land p}\;{\color{red}\land R}}{p \succ p \land p}\;W}{p \lor p \succ p \land p}\;\mathit{Cut}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \land p}\;\land R}{p \succ p \land p}\;W \qquad \dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \land p}\;\land R}{p \succ p \land p}\;W}{p \lor p \succ p \land p, p \land p}\;\lor L}{p \lor p \succ p \land p}\;W
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p, p}\;\lor L}{p \lor p \succ p}\;W \qquad \dfrac{\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p, p}\;\lor L}{p \lor p \succ p}\;W}{p \lor p, p \lor p \succ p \land p}\;\land R}{p \lor p \succ p \land p}\;W
$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p \lor p \succ p, p}\,{\lor L}}{p \lor p \succ p}\,W \qquad\qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \land p}\,{\land R}}{p \succ p \land p}\,W}{p \lor p \succ p \land p}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \land p}\,{\land R}}{p \succ p \land p}\,W \qquad \dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \land p}\,{\land R}}{p \succ p \land p}\,W}{p \lor p \succ p \land p, p \land p}\,{\lor L}}{p \lor p \succ p \land p}\,W \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p, p}\,{\lor L}}{p \lor p \succ p}\,W \qquad \dfrac{\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p, p}\,{\lor L}}{p \lor p \succ p}\,W}{p \lor p, p \lor p \succ p \land p}\,{\land R}}{p \lor p \succ p \land p}\,W$$

An *excursus* on eliminating *Cut*: Contraction and Permutation

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p \lor p \succ p, p}\,{\lor L}}{p \lor p \succ p}\,W \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \land p}\,{\land R}}{p \succ p \land p}\,W}{p \lor p \succ p \land p}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \land p}\,{\land R}}{p \succ p \land p}\,W \qquad \dfrac{\dfrac{p \succ p \quad p \succ p}{p, p \succ p \land p}\,{\land R}}{p \succ p \land p}\,W}{p \lor p \succ p \land p, p \land p}\,{\lor L}}{p \lor p \succ p \land p}\,W \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p, p}\,{\lor L}}{p \lor p \succ p}\,W \qquad \dfrac{\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p, p}\,{\lor L}}{p \lor p \succ p}\,W}{p \lor p, p \lor p \succ p \land p}\,{\land R}}{p \lor p \succ p \land p}\,W$$

An *excursus* on eliminating *Cut*: Weakening and Blend

$$\dfrac{\dfrac{\begin{array}{c}\delta_1\\ \vdots\\ \Sigma_1 \succ \Delta_1\end{array}}{\Sigma_1 \succ C, \Delta_1}\,K \qquad \dfrac{\begin{array}{c}\delta_2\\ \vdots\\ \Sigma_2 \succ \Delta_2\end{array}}{\Sigma_2, C \succ \Delta_2}\,K}{\Sigma_{1,2} \succ \Delta_{1,2}}\,Cut$$

An *excursus* on eliminating *Cut*: Weakening and Blend

$$\dfrac{\dfrac{\begin{array}{c}\delta_1\\ \vdots\\ \Sigma_1 \succ \Delta_1\end{array}}{\Sigma_1 \succ C, \Delta_1}\,K \qquad \dfrac{\begin{array}{c}\delta_2\\ \vdots\\ \Sigma_2 \succ \Delta_2\end{array}}{\Sigma_2, C \succ \Delta_2}\,K}{\Sigma_{1,2} \succ \Delta_{1,2}}\,\mathit{Cut}$$

$$\dfrac{\Sigma_1 \succ \Delta_1 \qquad \Sigma_2 \succ \Delta_2}{\Sigma_{1,2} \succ \Delta_{1,2}}\,\mathit{Blend}$$


DERIVATIONS
AND TERMS

Symmetry in Classical Logic

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L \qquad\qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \vee \neg p}\ \vee R$$

Symmetry in Classical Logic

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ \neg L}{p \wedge \neg p \succ}\ \wedge L \qquad\qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\ \neg R}{\succ p \vee \neg p}\ \vee R$$

These sequents *aren't* $x_1 : A_1, \ldots, x_n : A_n \succ t(x_1, \ldots, x_n) : B$.

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L \qquad\qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R$$

These sequents *aren't* $x_1 : A_1, \ldots, x_n : A_n \succ t(x_1, \ldots, x_n) : B$.

Instead, the shape for terms will be

$$x_1 : A_1, \ldots, x_n : A_n \overset{\pi(x_1, \ldots, x_n)[y_1, \ldots, y_m]}{\succ} y_1 : B_1, \ldots, y_m : B_m$$

Example Derivations with Terms

$$\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{x \curvearrowright \grave{\neg} y}{x : p, y : \neg p \succ}}\,\neg L \qquad\qquad \dfrac{\overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{\grave{\neg} y \curvearrowright x}{\succ x : p, y : \neg p}}\,\neg R$$

$$\dfrac{x : p, y : \neg p \succ}{\overset{\acute{\wedge} z \curvearrowright \grave{\neg} \grave{\wedge} z}{z : p \wedge \neg p \succ}}\,\wedge L \qquad\qquad \dfrac{\succ x : p, y : \neg p}{\overset{\grave{\neg} \grave{\vee} z \curvearrowright \acute{\vee} z}{\succ z : p \vee \neg p}}\,\vee R$$

Labelled Formulas and Sequents

$$x : A, y : A, v : B \succ z : C, w : D$$

Labelled Formulas and Sequents

$$x : A, y : A, v : B \succ z : C, w : D$$

$$\frac{x : A, y : A \succ z : C, w : D \qquad v : B \succ z : C, w' : D}{x : A, u : A \vee B \succ z : C, w : D, w' : D}\ \vee L$$

Labelled Formulas and Sequents

$$x : A, y : A, v : B \succ z : C, w : D$$

$$\frac{x : A, y : A \succ z : C, w : D \qquad v : B \succ z : C, w' : D}{x : A, u : A \vee B \succ z : C, w : D, w' : D} \vee L$$

The Σ and Δ are *sets* of labelled formulas in $\Sigma \succ \Delta$.

Basic Axioms

$$\Sigma, \overset{x\curvearrowright}{x : \bot} \succ \Delta \qquad \Sigma, \overset{x \curvearrowright y}{x : p \succ y} : p, \Delta \qquad \Sigma \succ \overset{\curvearrowright y}{y : \top}, \Delta$$

Blending Sequents

$$\overset{\pi_i}{\Sigma_i \succ \Delta_i}$$

Blending Sequents

$$\begin{array}{c} \pi_i \\ \Sigma_i \succ \Delta_i \end{array}$$

$$\begin{array}{c} \bigcup_i \pi_i \\ \bigcup_i \Sigma_i \succ \bigcup_i \Delta_i \end{array}$$

Axioms

$$\Sigma, x : \bot \succ \Delta \qquad \Sigma, x : p \succ y : p, \Delta \qquad \Sigma \succ y : \top, \Delta$$

The *blend* of any finite collection of basic axioms is also an axiom.

$$\Sigma, x : \overset{x\curvearrowright}{\bot} \succ \Delta \qquad \Sigma, x : p \overset{x\curvearrowright y}{\succ} y : p, \Delta \qquad \Sigma \succ y : \overset{\curvearrowright y}{\top}, \Delta$$

The *blend* of any finite collection of basic axioms is also an axiom.

$$x : p, y : q, z : r \overset{x\curvearrowright x}{\wedge} \neg s \overset{\curvearrowright w}{\succ} x : p, y : q, w : \top$$

Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L \qquad \dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y : B, \Delta_2}}{\overset{\pi[\acute{\wedge} z] \;\; \pi'[\grave{\wedge} z]}{\Sigma_{1,2} \succ z : A \wedge B, \Delta_{1,2}}} \wedge R$$

Disjunction

$$\dfrac{\overset{\pi_1(x)}{\Sigma_1, x : A \succ \Delta_1} \qquad \overset{\pi_2(y)}{\Sigma_2, y : B \succ \Delta_2}}{\overset{\pi_1(\acute{\vee} z)\ \ \pi_2(\grave{\vee} z)}{\Sigma_{1,2}, z : A \vee B \succ \Delta_{1,2}}} \vee L \qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[\acute{\vee} z, \grave{\vee} z]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Conditional

$$\dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2(y)}{\Sigma_2, y : B \succ \Delta_2}}{\overset{\pi_1[\acute{\supset} z]\ \ \pi_2(\grave{\supset} z)}{\Sigma_{1,2}, z : A \supset B \succ \Delta_{1,2}}} \supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(\acute{\supset} z)[\grave{\supset} z]}{\Sigma \succ z : A \supset B, \Delta}} \supset R$$

Negation

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\underset{\Sigma, z : \neg A \succ \Delta}{\pi[\dot{\neg} z]}} \neg L \qquad \frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\underset{\Sigma \succ z : \neg A, \Delta}{\pi(\dot{\neg} z)}} \neg R$$

Cut

$$\frac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2(y)}{\Sigma_2, y : A \succ \Delta_2}}{\overset{\pi_1[\bullet]\ \ \pi_2(\bullet)}{\Sigma_{1,2} \succ \Delta_{1,2}}}\ \mathit{Cut}$$

α Rules

$$\frac{\begin{array}{c}\pi(x, y)\\ \Sigma, x : A, y : B \succ \Delta\end{array}}{\begin{array}{c}\pi(\acute{\wedge} z, \grave{\wedge} z)\\ \Sigma, z : A \wedge B \succ \Delta\end{array}}\ {\scriptstyle \wedge L} \qquad \frac{\begin{array}{c}\pi[x, y]\\ \Sigma \succ x : A, y : B, \Delta\end{array}}{\begin{array}{c}\pi[\acute{\vee} z, \grave{\vee} z]\\ \Sigma \succ z : A \vee B, \Delta\end{array}}\ {\scriptstyle \vee R}$$

$$\frac{\begin{array}{c}\pi(x)[y]\\ \Sigma, x : A \succ y : B, \Delta\end{array}}{\begin{array}{c}\pi(\acute{\supset} z)[\grave{\supset} z]\\ \Sigma \succ z : A \supset B, \Delta\end{array}}\ {\scriptstyle \supset R} \qquad \frac{\begin{array}{c}\pi[x]\\ \Sigma \succ x : A, \Delta\end{array}}{\begin{array}{c}\pi[\grave{\neg} z]\\ \Sigma, z : \neg A \succ \Delta\end{array}}\ {\scriptstyle \neg L} \qquad \frac{\begin{array}{c}\pi(x)\\ \Sigma, x : A \succ \Delta\end{array}}{\begin{array}{c}\pi(\grave{\neg} z)\\ \Sigma \succ z : \neg A, \Delta\end{array}}\ {\scriptstyle \neg R}$$

$$\frac{\begin{array}{c}\pi\{a_1\}\{a_2\}\\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\end{array}}{\begin{array}{c}\pi\{\acute{\alpha} a\}\{\grave{\alpha} a\}\\ \mathfrak{S}\{a : \alpha\}\end{array}}$$

RULE	$\{a_1 : \alpha_1\}$	$\{a_2 : \alpha_2\}$	$\{a : \alpha\}$	$\acute{\alpha}$	$\grave{\alpha}$
$\wedge L$	input, A	input, B	input, $A \wedge B$	$\acute{\wedge}$	$\grave{\wedge}$
$\vee R$	output, A	output, B	output, $A \vee B$	$\acute{\vee}$	$\grave{\vee}$
$\supset R$	input, A	output, B	output, $A \supset B$	$\acute{\supset}$	$\grave{\supset}$
$\neg L$	output, A	output, A	input, $\neg A$	$\grave{\neg}$	$\grave{\neg}$
$\neg R$	input, A	input, A	output, $\neg A$	$\grave{\neg}$	$\grave{\neg}$

β Rules

$$\dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y : B, \Delta_2}}{\overset{\pi_1[\acute{\wedge} z] \;\; \pi_2[\grave{\wedge} z]}{\Sigma_{1,2} \succ z : A \wedge B, \Delta_{1,2}}} \wedge R \qquad \dfrac{\overset{\pi_1(x)}{\Sigma_1, x : A \succ \Delta_1} \qquad \overset{\pi_2(y)}{\Sigma_2, y : B \succ \Delta_2}}{\overset{\pi_1(\acute{\vee} z) \;\; \pi_2(\grave{\vee} z)}{\Sigma_{1,2}, z : A \vee B \succ \Delta_{1,2}}} \vee L$$

$$\dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2(y)}{\Sigma_2, y : B \succ \Delta_2}}{\overset{\pi_1[\acute{\supset} z] \;\; \pi_2(\grave{\supset} z)}{\Sigma_{1,2}, z : A \supset B \succ \Delta_{1,2}}} \supset L \qquad \dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2(y)}{\Sigma_2, y : A \succ \Delta_2}}{\overset{\pi_1[\bullet] \;\; \pi_2(\bullet)}{\Sigma_{1,2} \succ \Delta_{1,2}}} \mathit{Cut}$$

$$\dfrac{\overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1 : \beta_1\}} \qquad \overset{\pi_2\{b_2\}}{\mathfrak{S}_2\{b_2 : \beta_2\}}}{\overset{\pi_1\{\acute{\beta} b\} \;\; \pi_2\{\grave{\beta} b\}}{\mathfrak{S}_{1,2}\{b : \beta\}}}$$

β Rule Rubric

RULE	$\{b_1 : \beta_1\}$	$\{b_2 : \beta_2\}$	$\{b : \beta\}$	$\acute{\beta}$	$\grave{\beta}$
$\wedge R$	output, A	output, B	output, $A \wedge B$	$\acute{\wedge}$	$\grave{\wedge}$
$\vee L$	input, A	input, B	input, $A \vee B$	$\acute{\vee}$	$\grave{\vee}$
$\supset L$	output, A	input, B	input, $A \supset B$	$\acute{\supset}$	$\grave{\supset}$
Cut	output, A	input, A	—	$\bullet$	$\bullet$

Example 1—derivations with different terms

$$
\dfrac{
\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{\acute{\vee} y \curvearrowright x \quad \grave{\vee} y \curvearrowright x}{y : p \vee p \succ x : p}}\,\vee L
\qquad
\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z}{x : p \succ z : p \wedge p}}\,\wedge R
}{\overset{\acute{\vee} y \curvearrowright \bullet \quad \grave{\vee} y \curvearrowright \bullet \quad \bullet \curvearrowright \acute{\wedge} z \quad \bullet \curvearrowright \grave{\wedge} z}{y : p \vee p \succ y : p \wedge p}}\,\mathit{Cut}
$$

Example 1—derivations with different terms

$$
\dfrac{
\dfrac{
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
\qquad
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
}{
\begin{matrix} \acute{\vee} y \curvearrowright x \quad \grave{\vee} y \curvearrowright x \\ y : p \vee p \succ x : p \end{matrix}
}\,{\vee L}
\qquad
\dfrac{
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
\qquad
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
}{
\begin{matrix} x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z \\ x : p \succ z : p \wedge p \end{matrix}
}\,{\wedge R}
}{
\begin{matrix} \acute{\vee} y \curvearrowright \bullet \quad \grave{\vee} y \curvearrowright \bullet \quad \bullet \curvearrowright \acute{\wedge} z \quad \bullet \curvearrowright \grave{\wedge} z \\ y : p \vee p \succ y : p \wedge p \end{matrix}
}\,{\mathit{Cut}}
$$

$$
\dfrac{
\dfrac{
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
\qquad
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
}{
\begin{matrix} x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z \\ x : p \succ z : p \wedge p \end{matrix}
}\,{\wedge R}
\qquad
\dfrac{
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
\qquad
\begin{matrix} x \curvearrowright x \\ x : p \succ x : p \end{matrix}
}{
\begin{matrix} x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z \\ x : p \succ z : p \wedge p \end{matrix}
}\,{\wedge R}
}{
\begin{matrix} \acute{\vee} y \curvearrowright \acute{\wedge} z \quad \acute{\vee} y \curvearrowright \grave{\wedge} z \quad \grave{\vee} y \curvearrowright \acute{\wedge} z \quad \grave{\vee} y \curvearrowright \grave{\wedge} z \\ y : p \vee p \succ z : p \wedge p \end{matrix}
}\,{\vee L}
$$

Example 2—two *different* derivations with the *same* term

$$
\dfrac{
  \overset{x \curvearrowright x}{x : p \succ x : p}
  \qquad
  \dfrac{\overset{z \curvearrowright z}{z : p \succ z : p}}{\overset{\acute{\wedge} y \curvearrowright z}{y : p \wedge q \succ z : p}}\;\wedge L
}{
  \dfrac{\overset{x \curvearrowright \acute{\wedge} w \quad \acute{\wedge} y \curvearrowright \grave{\wedge} w}{x : p, y : p \wedge q \succ w : p \wedge p}}{
    \dfrac{\overset{\acute{\supset} v \curvearrowright \acute{\wedge} \grave{\supset} v \quad \acute{\wedge} y \curvearrowright \grave{\wedge} \grave{\supset} v}{y : p \wedge q \succ v : p \supset (p \wedge p)}}{
      \overset{\acute{\supset} \grave{\supset} u \curvearrowright \acute{\wedge} \grave{\supset} \grave{\supset} u \quad \acute{\wedge} \acute{\supset} u \curvearrowright \grave{\wedge} \grave{\supset} \grave{\supset} u}{\succ u : (p \wedge q) \supset (p \supset (p \wedge p))}
    }\;\supset R
  }\;\supset R
}\;\wedge R
$$

$$
\dfrac{
  \overset{x \curvearrowright x}{x : p \succ x : p}
  \qquad
  \overset{z \curvearrowright z}{z : p \succ z : p}
}{
  \dfrac{\overset{x \curvearrowright \acute{\wedge} w \quad z \curvearrowright \grave{\wedge} w}{x : p, z : p \succ w : p \wedge p}}{
    \dfrac{\overset{\acute{\supset} v \curvearrowright \acute{\wedge} \grave{\supset} v \quad z \curvearrowright \grave{\wedge} \grave{\supset} v}{z : p \succ v : p \supset (p \wedge p)}}{
      \dfrac{\overset{\acute{\supset} v \curvearrowright \acute{\wedge} \grave{\supset} v \quad \acute{\wedge} y \curvearrowright \grave{\wedge} \grave{\supset} v}{y : p \wedge q \succ v : p \supset (p \wedge p)}}{
        \overset{\acute{\supset} \grave{\supset} u \curvearrowright \acute{\wedge} \grave{\supset} \grave{\supset} u \quad \acute{\wedge} \acute{\supset} u \curvearrowright \grave{\wedge} \grave{\supset} \grave{\supset} u}{\succ u : (p \wedge q) \supset (p \supset (p \wedge p))}
      }\;\supset R
    }\;\wedge L
  }\;\supset R
}\;\wedge R
$$

Example 2—two *different* derivations with the *same* term

$$
\dfrac{
\overset{x \curvearrowright x}{x : p \succ x : p}
\qquad
\dfrac{
\dfrac{
\overset{x \curvearrowright \acute{\wedge} w \quad \acute{\wedge} y \curvearrowright \grave{\wedge} w}{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \dfrac{\overset{z \curvearrowright z}{z : p \succ z : p}}{\overset{\acute{\wedge} y \curvearrowright z}{y : p \wedge q \succ z : p}} \; {\scriptstyle \wedge L}}{x : p, y : p \wedge q \succ w : p \wedge p} \; {\scriptstyle \wedge R}}
}{
\overset{\acute{\supset} v \curvearrowright \acute{\wedge} \grave{\supset} v \quad \acute{\wedge} y \curvearrowright \grave{\wedge} \grave{\supset} v}{y : p \wedge q \succ v : p \supset (p \wedge p)}
} \; {\scriptstyle \supset R}
}{
\overset{\acute{\supset} \grave{\supset} u \curvearrowright \acute{\wedge} \grave{\supset} \grave{\supset} u \quad \acute{\wedge} \acute{\supset} u \curvearrowright \grave{\wedge} \grave{\supset} \grave{\supset} u}{\succ u : (p \wedge q) \supset (p \supset (p \wedge p))}
} \; {\scriptstyle \supset R}
}{}
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{z \curvearrowright z}{z : p \succ z : p}}{\overset{x \curvearrowright \acute{\wedge} w \quad z \curvearrowright \grave{\wedge} w}{x : p, z : p \succ w : p \wedge p}} \; {\scriptstyle \wedge R}
}{
\overset{\acute{\supset} v \curvearrowright \acute{\wedge} \grave{\supset} v \quad z \curvearrowright \grave{\wedge} \grave{\supset} v}{z : p \succ v : p \supset (p \wedge p)}
} \; {\scriptstyle \supset R}
}{
\overset{\acute{\supset} v \curvearrowright \acute{\wedge} \grave{\supset} v \quad \acute{\wedge} y \curvearrowright \grave{\wedge} \grave{\supset} v}{y : p \wedge q \succ v : p \supset (p \wedge p)}
} \; {\scriptstyle \wedge L}
}{
\overset{\acute{\supset} \grave{\supset} u \curvearrowright \acute{\wedge} \grave{\supset} \grave{\supset} u \quad \acute{\wedge} \acute{\supset} u \curvearrowright \grave{\wedge} \grave{\supset} \grave{\supset} u}{\succ u : (p \wedge q) \supset (p \supset (p \wedge p))}
} \; {\scriptstyle \supset R}
$$

Example 3—two *different* derivations with the *same* term

$$\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z}{x : p \succ z : p \wedge p}}\,{\wedge R} \qquad \dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z}{x : p \succ z : p \wedge p}}\,{\wedge R}}{\overset{\acute{\vee} y \curvearrowright \acute{\wedge} z \quad \acute{\vee} y \curvearrowright \grave{\wedge} z \quad \grave{\vee} y \curvearrowright \acute{\wedge} z \quad \grave{\vee} y \curvearrowright \grave{\wedge} z}{y : p \vee p \succ z : p \wedge p}}\,{\vee L}$$

$$\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{\acute{\vee} y \curvearrowright x \quad \grave{\vee} y \curvearrowright x}{y : p \vee p \succ x : p}}\,{\vee L} \qquad \dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{\acute{\vee} y \curvearrowright x \quad \grave{\vee} y \curvearrowright x}{y : p \vee p \succ x : p}}\,{\vee L}}{\overset{\acute{\vee} y \curvearrowright \acute{\wedge} z \quad \acute{\vee} y \curvearrowright \grave{\wedge} z \quad \grave{\vee} y \curvearrowright \acute{\wedge} z \quad \grave{\vee} y \curvearrowright \grave{\wedge} z}{y : p \vee p \succ z : p \wedge p}}\,{\wedge R}$$

Example 3—two *different* derivations with the *same* term

$$\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z}{x : p \succ z : p \wedge p}}{\scriptstyle \wedge R} \qquad \dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{x \curvearrowright \acute{\wedge} z \quad x \curvearrowright \grave{\wedge} z}{x : p \succ z : p \wedge p}}{\scriptstyle \wedge R}}{\overset{\acute{\vee} y \curvearrowright \acute{\wedge} z \quad \acute{\vee} y \curvearrowright \grave{\wedge} z \quad \grave{\vee} y \curvearrowright \acute{\wedge} z \quad \grave{\vee} y \curvearrowright \grave{\wedge} z}{y : p \vee p \succ z : p \wedge p}}{\scriptstyle \vee L}$$

$$\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{\acute{\vee} y \curvearrowright x \quad \grave{\vee} y \curvearrowright x}{y : p \vee p \succ x : p}}{\scriptstyle \vee L} \qquad \dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{x \curvearrowright x}{x : p \succ x : p}}{\overset{\acute{\vee} y \curvearrowright x \quad \grave{\vee} y \curvearrowright x}{y : p \vee p \succ x : p}}{\scriptstyle \vee L}}{\overset{\acute{\vee} y \curvearrowright \acute{\wedge} z \quad \acute{\vee} y \curvearrowright \grave{\wedge} z \quad \grave{\vee} y \curvearrowright \acute{\wedge} z \quad \grave{\vee} y \curvearrowright \grave{\wedge} z}{y : p \vee p \succ z : p \wedge p}}{\scriptstyle \wedge R}$$

For the single premise rules of *weakening* and *contraction*

$$\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma, x : A \succ \Delta}}\,KL \qquad \dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}}\,KR \qquad \dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : A \succ \Delta}}{\overset{\pi(x,x)}{\Sigma, x : A \succ \Delta}}\,WL \qquad \dfrac{\overset{\pi[x,y]}{\Sigma \succ x : A, y : A, \Delta}}{\overset{\pi[x,x]}{\Sigma \succ x : A, \Delta}}\,WR$$

if there is a derivation δ of the *premise* of the rule, of height $h(\delta)$, then there is a derivation δ' of the *conclusion* of the rule, of the same height, $h(\delta)$.

Height Preserving Admissibility—Blend

For the two premise rule of *blend*,

$$\dfrac{\overset{\pi_1}{\Sigma_1 \succ \Delta_1} \qquad \overset{\pi_2}{\Sigma_2 \succ \Delta_2}}{\overset{\pi_1\ \pi_2}{\Sigma_{1,2} \succ \Delta_{1,2}}}\ \textit{Blend}$$

if there is a derivation δ_1 of the premise $\Gamma_1 \succ \Delta_1$ (with term π) and a derivation δ_2 of the premise $\Gamma_2 \succ \Delta_2$ (with term π') then there is a derivation of height less than or equal to $h(\delta) + h(\delta')$ of the blend $\Gamma_{1,2} \succ \Delta_{1,2}$ with term $\pi_1\ \pi_2$.


PERMUTATIONS

$\wedge L/\neg R$ Permutation

$$\dfrac{\dfrac{\overset{\pi(x,y,z)}{\Sigma, x: A, y: B, z: C \succ \Delta}}{\overset{\pi(\acute{\wedge}v, \grave{\wedge}v, z)}{\Sigma, v: A \wedge B, z: C \succ \Delta}}\wedge L}{\overset{\pi(\acute{\wedge}v, \grave{\wedge}v, \dot{\neg}w)}{\Sigma, v: A \wedge B \succ w: \neg C, \Delta}}\neg R \qquad \dfrac{\dfrac{\overset{\pi(x,y,z)}{\Sigma, x: A, y: B, z: C \succ \Delta}}{\overset{\pi(x, y, \dot{\neg}w)}{\Sigma, x: A, y: B \succ w: \neg C, \Delta}}\neg R}{\overset{\pi(\acute{\wedge}v, \grave{\wedge}v, \dot{\neg}w)}{\Sigma, v: A \wedge B \succ w: \neg C, \Delta}}\wedge L$$

$$
\cfrac{\cfrac{\overset{\pi(x,y,z)}{\Sigma, x : A, y : B, z : C \succ \Delta}}{\overset{\pi(\acute{\wedge}v,\ \grave{\wedge}v,\ z)}{\Sigma, v : A \wedge B, z : C \succ \Delta}}\ \wedge L}{\overset{\pi(\acute{\wedge}v,\ \grave{\wedge}v,\ \dot{\neg}w)}{\Sigma, v : A \wedge B \succ w : \neg C, \Delta}}\ \neg R
\qquad\qquad
\cfrac{\cfrac{\overset{\pi(x,y,z)}{\Sigma, x : A, y : B, z : C \succ \Delta}}{\overset{\pi(x,\ y,\ \dot{\neg}w)}{\Sigma, x : A, y : B \succ w : \neg C, \Delta}}\ \neg R}{\overset{\pi(\acute{\wedge}v,\ \grave{\wedge}v,\ \dot{\neg}w)}{\Sigma, v : A \wedge B \succ w : \neg C, \Delta}}\ \wedge L
$$

α/α Permutations

$$\dfrac{\dfrac{\overset{\pi\{a_1\}\{a_2\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a_1' : \alpha_1'\}\{a_2' : \alpha_2'\}}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a : \alpha\}\{a_1' : \alpha_1'\}\{a_2' : \alpha_2'\}}}\,\alpha}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}'a'\}\{\grave{\alpha}'a'\}}{\mathfrak{S}\{a : \alpha\}\{a' : \alpha'\}}}\,\alpha'$$

α/α Permutations

$$\dfrac{\dfrac{\overset{\pi\{a_1\}\{a_2\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a_1' : \alpha_1'\}\{a_2' : \alpha_2'\}}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a : \alpha\}\{a_1' : \alpha_1'\}\{a_2' : \alpha_2'\}}}\,\alpha}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}'a'\}\{\grave{\alpha}'a'\}}{\mathfrak{S}\{a : \alpha\}\{a' : \alpha'\}}}\,\alpha'$$

$$\dfrac{\dfrac{\overset{\pi\{a_1\}\{a_2\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a_1' : \alpha_1'\}\{a_2' : \alpha_2'\}}}{\overset{\pi\{a_1\}\{a_2\}\{\acute{\alpha}'a'\}\{\grave{\alpha}'a'\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a' : \alpha'\}}}\,\alpha'}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}'a'\}\{\grave{\alpha}'a'\}}{\mathfrak{S}\{a : \alpha\}\{a' : \alpha'\}}}\,\alpha$$

$$
\dfrac{
  \dfrac{
    \overset{\pi_1(z)[x]}{\Sigma_1, z : C \succ x : A, \Delta_1}
    \qquad
    \overset{\pi_2(z)[y]}{\Sigma_2, z : C \succ y : B, \Delta_2}
  }{
    \overset{\pi_1(z)[\acute{\wedge} v] \quad \pi_2(z)[\grave{\wedge} v]}{\Sigma_{1,2}, z : C \succ v : A \wedge B, \Delta_{1,2}}
  }\ {\wedge R}
}{
  \overset{\pi_1(\dot{\neg} w)[\acute{\wedge} v] \quad \pi_2(\dot{\neg} w)[\grave{\wedge} v]}{\Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}
}\ {\neg R}
$$

$\wedge R/\neg R$ Permutations

$$\dfrac{\dfrac{\overset{\pi_1(z)[x]}{\Sigma_1, z : C \succ x : A, \Delta_1} \qquad \overset{\pi_2(z)[y]}{\Sigma_2, z : C \succ y : B, \Delta_2}}{\overset{\pi_1(z)[\acute{\wedge}v] \quad \pi_2(z)[\grave{\wedge}v]}{\Sigma_{1,2}, z : C \succ v : A \wedge B, \Delta_{1,2}}}\,{\wedge R}}{\overset{\pi_1(\grave{\neg}w)[\acute{\wedge}v] \quad \pi_2(\grave{\neg}w)[\grave{\wedge}v]}{\Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}}\,{\neg R}$$

$$\dfrac{\dfrac{\overset{\pi_1(z)[x]}{\Sigma_1, z : C \succ x : A, \Delta_1}}{\overset{\pi_1(\grave{\neg}w)[x]}{\Sigma_1 \succ w : \neg C, x : A, \Delta_1}}\,{\neg R} \qquad \dfrac{\overset{\pi_2(z)[y]}{\Sigma_2, z : C \succ y : B, \Delta_2}}{\overset{\pi_2(\grave{\neg}w)[y]}{\Sigma_2 \succ w : \neg C, y : B, \Delta_2}}\,{\neg R}}{\overset{\pi_1(\grave{\neg}w)[\acute{\wedge}v] \quad \pi_2(\grave{\neg}w)[\grave{\wedge}v]}{\Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}}}\,{\wedge R}$$

α/β Permutations

$$\dfrac{\dfrac{\overset{\pi_1\{a_1\}\{a_2\}\{b_1\}}{\mathfrak{S}_1\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b_1:\beta_1\}}}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{b_1\}}{\mathfrak{S}_1\{a:\alpha\}\{b_1:\beta_1\}}}\,\alpha \qquad \dfrac{\overset{\pi_2\{a_1\}\{a_2\}\{b_2\}}{\mathfrak{S}_2\{a_1:\alpha_1\}\{a_2:\alpha_1\}\{b_2:\beta_2\}}}{\overset{\pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{b_2\}}{\mathfrak{S}_2\{a:\alpha\}\{b_2:\beta_2\}}}\,\alpha}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\beta}b\}\quad \pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{a:\alpha\}\{b:\beta\}}}\,\beta$$

α/β Permutations

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\{a_1\}\{a_2\}\{b_1\}\\ \mathfrak{S}_1\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b_1:\beta_1\}\end{array}}{\begin{array}{c}\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{b_1\}\\ \mathfrak{S}_1\{a:\alpha\}\{b_1:\beta_1\}\end{array}}\,\alpha \qquad \dfrac{\begin{array}{c}\pi_2\{a_1\}\{a_2\}\{b_2\}\\ \mathfrak{S}_2\{a_1:\alpha_1\}\{a_2:\alpha_1\}\{b_2:\beta_2\}\end{array}}{\begin{array}{c}\pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{b_2\}\\ \mathfrak{S}_2\{a:\alpha\}\{b_2:\beta_2\}\end{array}}\,\alpha}{\begin{array}{c}\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\beta}b\} \quad \pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\grave{\beta}b\}\\ \mathfrak{S}_{1,2}\{a:\alpha\}\{b:\beta\}\end{array}}\,\beta$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\{a_1\}\{a_2\}\{b_1\}\\ \mathfrak{S}_1\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b_1:\beta_1\}\end{array} \qquad \begin{array}{c}\pi_2\{a_1\}\{a_2\}\{b_2\}\\ \mathfrak{S}_2\{a_1:\alpha_1\}\{a_2:\alpha_1\}\{b_2:\beta_2\}\end{array}}{\begin{array}{c}\pi_1\{a_1\}\{a_2\}\{\acute{\beta}b\} \quad \pi_2\{a_1\}\{a_2\}\{\grave{\beta}b\}\\ \mathfrak{S}_{1,2}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b:\beta\}\end{array}}\,\beta}{\begin{array}{c}\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\beta}b\} \quad \pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\grave{\beta}b\}\\ \mathfrak{S}_{1,2}\{a:\alpha\}\{b:\beta\}\end{array}}\,\alpha$$

$$
\cfrac{
\cfrac{
\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1 : \beta_1\}\{b_1' : \beta_1'\}}
\qquad
\overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2 : \beta_2\}\{b_1' : \beta_1'\}}
}{
\overset{\pi_1\{\acute{\beta}\,b\}\{b_1'\}\quad \pi_2\{\grave{\beta}\,b\}\{b_1'\}}{\mathfrak{S}_{1,2}\{b : \beta\}\{b_1' : \beta_1'\}}
}\;\beta
\qquad
\cfrac{
\overset{\pi_3\{b_1\}\{b_2'\}}{\mathfrak{S}_3\{b_1 : \beta_1\}\{b_2' : \beta_2'\}}
\qquad
\overset{\pi_4\{b_2\}\{b_2'\}}{\mathfrak{S}_4\{b_2 : \beta_2\}\{b_2' : \beta_2'\}}
}{
\overset{\pi_3\{\acute{\beta}\,b\}\{b_2'\}\quad \pi_2\{\grave{\beta}\,b\}\{b_2'\}}{\mathfrak{S}_{3,4}\{b : \beta\}\{b_2' : \beta_2'\}}
}\;\beta
}{
\overset{\pi_1\{\acute{\beta}\,b\}\{\acute{\beta}'\,b'\}\quad \pi_2\{\grave{\beta}\,b\}\{\acute{\beta}'\,b'\}\quad \pi_3\{\acute{\beta}\,b\}\{\grave{\beta}'\,b'\}\quad \pi_4\{\grave{\beta}\,b\}\{\grave{\beta}'\,b'\}}{\mathfrak{S}_{1-4}\{b : \beta\}\{b' : \beta'\}}
}\;\beta'
$$

β/β Permutations

$$
\dfrac{
\dfrac{
\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1 : \beta_1\}\{b_1' : \beta_1'\}}
\quad
\overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2 : \beta_2\}\{b_1' : \beta_1'\}}
}{
\overset{\pi_1\{\acute{\beta}\, b\}\{b_1'\}\ \ \pi_2\{\grave{\beta}\, b\}\{b_1'\}}{\mathfrak{S}_{1,2}\{b : \beta\}\{b_1' : \beta_1'\}}
}\,\beta
\qquad
\dfrac{
\overset{\pi_3\{b_1\}\{b_2'\}}{\mathfrak{S}_3\{b_1 : \beta_1\}\{b_2' : \beta_2'\}}
\quad
\overset{\pi_4\{b_2\}\{b_2'\}}{\mathfrak{S}_4\{b_2 : \beta_2\}\{b_2' : \beta_2'\}}
}{
\overset{\pi_3\{\acute{\beta}\, b\}\{b_2'\}\ \ \pi_2\{\grave{\beta}\, b\}\{b_2'\}}{\mathfrak{S}_{3,4}\{b : \beta\}\{b_2' : \beta_2'\}}
}\,\beta
}{
\overset{\pi_1\{\acute{\beta}\, b\}\{\acute{\beta}'\, b'\}\ \ \pi_2\{\grave{\beta}\, b\}\{\acute{\beta}'\, b'\}\ \ \pi_3\{\acute{\beta}\, b\}\{\grave{\beta}'\, b'\}\ \ \pi_4\{\grave{\beta}\, b\}\{\grave{\beta}'\, b'\}}{\mathfrak{S}_{1-4}\{b : \beta\}\{b' : \beta'\}}
}\,\beta'
$$

$$
\dfrac{
\dfrac{
\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1 : \beta_1\}\{b_1' : \beta_1'\}}
\quad
\overset{\pi_3\{b_1\}\{b_2'\}}{\mathfrak{S}_3\{b_1 : \beta_1\}\{b_2' : \beta_2'\}}
}{
\overset{\pi_1\{b_1\}\{\acute{\beta}'\, b'\}\ \ \pi_3\{b_1\}\{\grave{\beta}'\, b'\}}{\mathfrak{S}_{1,3}\{b_1 : \beta_1\}\{b' : \beta'\}}
}\,\beta'
\qquad
\dfrac{
\overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2 : \beta_2\}\{b_1' : \beta_1'\}}
\quad
\overset{\pi_4\{b_2\}\{b_2'\}}{\mathfrak{S}_4\{b_2 : \beta_2\}\{b_2' : \beta_2'\}}
}{
\overset{\pi_3\{b_2\}\{\acute{\beta}'\, b'\}\ \ \pi_2\{b_2\}\{\grave{\beta}'\, b'\}}{\mathfrak{S}_{2,4}\{b_2 : \beta_2\}\{b' : \beta'\}}
}\,\beta'
}{
\overset{\pi_1\{\acute{\beta}\, b\}\{\acute{\beta}'\, b'\}\ \ \pi_2\{\grave{\beta}\, b\}\{\acute{\beta}'\, b'\}\ \ \pi_3\{\acute{\beta}\, b\}\{\grave{\beta}'\, b'\}\ \ \pi_4\{\grave{\beta}\, b\}\{\grave{\beta}'\, b'\}}{\mathfrak{S}_{1-4}\{b : \beta\}\{b' : \beta'\}}
}\,\beta
$$

Duplicating Rule Instances

$$\dfrac{\dfrac{\begin{array}{c}\pi_1(z)[x]\\ \Sigma_1, z : C \succ x : A, \Delta_1\end{array}}{\begin{array}{c}\pi_1(\dot{\neg}w)[x]\\ \Sigma_1 \succ w : \neg C, x : A, \Delta_1\end{array}}\,{\scriptstyle \neg R} \qquad \dfrac{\begin{array}{c}\pi_2(z)[y]\\ \Sigma_2, z : C \succ y : B, \Delta_2\end{array}}{\begin{array}{c}\pi_2(\dot{\neg}w)[y]\\ \Sigma_2 \succ w : \neg C, y : B, \Delta_2\end{array}}\,{\scriptstyle \neg R}}{\begin{array}{c}\pi_1(\dot{\neg}w)[\acute{\wedge}v] \quad \pi_2(\dot{\neg}w)[\grave{\wedge}v]\\ \Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}\end{array}}\,{\scriptstyle \wedge R}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1(z)[x]\\ \Sigma_1, z : C \succ x : A, \Delta_1\end{array}}{\begin{array}{c}\pi_1(\dot{\neg}w)[x]\\ \Sigma_1 \succ w : \neg C, x : A, \Delta_1\end{array}}\,{\scriptstyle \neg R} \qquad \begin{array}{c}\pi_2(z)[y]\\ \Sigma_2, z : C \succ y : B, \Delta_2\end{array}}{\begin{array}{c}\pi_1(\dot{\neg}w)[\acute{\wedge}v] \quad \pi_2(z)[\grave{\wedge}v]\\ \Sigma_{1,2}, z : C \succ w : \neg C, v : A \wedge B, \Delta_{1,2}\end{array}}\,{\scriptstyle \wedge R}}{\begin{array}{c}\pi_1(\dot{\neg}w)[\acute{\wedge}v] \quad \pi_2(\dot{\neg}w)[y]\\ \Sigma_{1,2} \succ w : \neg C, v : A \wedge B, \Delta_{1,2}\end{array}}\,{\scriptstyle \neg R}$$

$\wedge L / \wedge L$ Contraction/Expansion

$$\dfrac{\dfrac{\overset{\pi(x, y, x', y')}{\Sigma, x : A, y : B, x' : A, y' : B \succ \Delta}}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z, x', y')}{\Sigma, z : A \wedge B, x' : A, y' : B \succ \Delta}} \wedge L}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z, \acute{\wedge} z, \grave{\wedge} z)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L$$

$\wedge L / \wedge L$ Contraction/Expansion

$$\dfrac{\dfrac{\overset{\pi(x,y,x',y')}{\Sigma, x : A, y : B, x' : A, y' : B \succ \Delta}}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z, x', y')}{\Sigma, z : A \wedge B, x' : A, y' : B \succ \Delta}} \wedge L}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z, \acute{\wedge} z, \grave{\wedge} z)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L$$

$$\dfrac{\overset{\pi(x,y,x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(\acute{\wedge} z, \grave{\wedge} z, \acute{\wedge} z, \grave{\wedge} z)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L$$

α/α Contraction/Expansion

$$
\dfrac{\dfrac{\overset{\pi\{a_1\}\{a_2\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a_1' : \alpha_1'\}\{a_2' : \alpha_2'\}}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a : \alpha\}\{a_1' : \alpha_1'\}\{a_2' : \alpha_2'\}}}\ \alpha}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a : \alpha\}}}\ \alpha
$$

α/α Contraction/Expansion

$$
\cfrac{\cfrac{\overset{\pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\}}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{a'_1\}\{a'_2\}}{\mathfrak{S}\{a : \alpha\}\{a'_1 : \alpha'_1\}\{a'_2 : \alpha'_2\}}}\ \alpha}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a : \alpha\}}}\ \alpha
$$

$$
\cfrac{\overset{\pi\{a_1\}\{a_2\}\{a_1\}\{a_2\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a : \alpha\}}}\ \alpha
$$

$\wedge R/\wedge R$ Contraction/Expansion?

$$
\dfrac{
\dfrac{
\overset{\color{red}\pi_1[x,x']}{\Sigma_1 \succ {\color{red}x} : A, {\color{red}x'} : A, \Delta_1}
\quad
\overset{\color{red}\pi_2[x,y']}{\Sigma_2 \succ {\color{red}x} : A, {\color{red}y'} : B, \Delta_2}
}{
\overset{\color{red}\pi_1[x,\,\acute{\wedge}z] \quad \pi_2[x,\,\grave{\wedge}z]}{\Sigma_{1,2} \succ {\color{red}x} : A, {\color{red}z} : A \wedge B, \Delta_{1,2}}
}
\quad
\dfrac{
\overset{\color{red}\pi_3[y,x']}{\Sigma_3 \succ {\color{red}y} : B, {\color{red}x'} : A, \Delta_3}
\quad
\overset{\color{red}\pi_4[y,y']}{\Sigma_4 \succ {\color{red}y} : B, {\color{red}y'} : B, \Delta_4}
}{
\overset{\color{red}\pi_3[y,\,\acute{\wedge}z] \quad \pi_4[y,\,\grave{\wedge}z]}{\Sigma_{3,4} \succ {\color{red}y} : B, {\color{red}z} : A \wedge B, \Delta_{3,4}}
}
}{
\overset{\color{red}\pi_1[\acute{\wedge}z,\,\acute{\wedge}z] \quad \pi_2[\acute{\wedge}z,\,\grave{\wedge}z \quad \pi_3[\grave{\wedge}z,\,\acute{\wedge}z] \quad \pi_4[\acute{\wedge}z,\,\grave{\wedge}z]}{\Sigma_{1-4} \succ {\color{red}z} : A \wedge B, \Delta_{1-4}}
}
$$

$$\frac{\dfrac{\overset{\pi_1[x,x']}{\Sigma_1 \succ x : A, x' : A, \Delta_1} \quad \overset{\pi_2[x,y']}{\Sigma_2 \succ x : A, y' : B, \Delta_2}}{\overset{\pi_1[x,\acute{\lambda} z] \quad \pi_2[x,\grave{\lambda} z]}{\Sigma_{1,2} \succ x : A, z : A \wedge B, \Delta_{1,2}}} \quad \dfrac{\overset{\pi_3[y,x']}{\Sigma_3 \succ y : B, x' : A, \Delta_3} \quad \overset{\pi_4[y,y']}{\Sigma_4 \succ y : B, y' : B, \Delta_4}}{\overset{\pi_3[y,\acute{\lambda} z] \quad \pi_4[y,\grave{\lambda} z]}{\Sigma_{3,4} \succ y : B, z : A \wedge B, \Delta_{3,4}}}}{\overset{\pi_1[\acute{\lambda} z,\acute{\lambda} z] \quad \pi_2[\acute{\lambda} z,\grave{\lambda} z \quad \pi_3[\grave{\lambda} z,\acute{\lambda} z] \quad \pi_4[\acute{\lambda} z,\grave{\lambda} z]}{\Sigma_{1-4} \succ z : A \wedge B, \Delta_{1-4}}}$$

Collapsing this into one step removes the π_2 and π_3 connections.

$$\frac{\overset{\pi_1[x,x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_4[y,y]}{\Sigma_4 \succ y : B, \Delta_4}}{\overset{\pi_1[\acute{\lambda} z,\acute{\lambda} z] \quad \pi_4[\acute{\lambda} z,\grave{\lambda} z]}{\Sigma_{1,4} \succ z : A \wedge B, \Delta_{1,4}}}$$

We can contract stacked occurrences of the same β rule into this triangle.

$$
\dfrac{
\dfrac{\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1 : \beta_1\}\{b_1' : \beta_1\}} \quad \overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2 : \beta_2\}\{b_1' : \beta_1\}}}{\overset{\pi_1\{\acute{\beta}\,b\}\{b_1'\} \quad \pi_2\{\grave{\beta}\,b\}\{b_1'\}}{\mathfrak{S}_{1,2}\{b : \beta\}\{b_1' : \beta_1\}}}\,\beta
\qquad
\dfrac{\overset{\pi_3\{b_1\}\{b_2'\}}{\mathfrak{S}_3\{b_1 : \beta_1\}\{b_2' : \beta_2\}} \quad \overset{\pi_4\{b_2\}\{b_2'\}}{\mathfrak{S}_4\{b_2 : \beta_2\}\{b_2' : \beta_2\}}}{\overset{\pi_3\{\acute{\beta}\,b\}\{b_2'\} \quad \pi_2\{\grave{\beta}\,b\}\{b_2'\}}{\mathfrak{S}_{3,4}\{b : \beta\}\{b_2' : \beta_2\}}}\,\beta
}{
\overset{\pi_1\{\acute{\beta}\,b\}\{\acute{\beta}\,b\} \quad \pi_2\{\grave{\beta}\,b\}\{\acute{\beta}\,b\} \quad \pi_3\{\acute{\beta}\,b\}\{\grave{\beta}\,b\} \quad \pi_4\{\grave{\beta}\,b\}\{\grave{\beta}\,b\}}{\mathfrak{S}_{1-4}\{b : \beta\}}
}\,\beta
$$

There's one more permutation…

I'll introduce that later.

(It's more complicated, and I'm not satisfied that I've got the best formulation.)

Soundness of permutations

If $\delta_1 \approx \delta_2$ then $\pi(\delta_1) = \pi(\delta_2)$

Completeness of permutations

If $\pi(\delta_1) = \pi(\delta_2)$ then $\delta_1 \approx \delta_2$

Strategy for Completeness

- Start with δ_1 and δ_2 where $\pi(\delta_1) = \pi(\delta_2)$

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  - Permute the *other* instances of that rule application in δ_1 to the bottom, where they collapse into one instance (α)—or a triangle (β).

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  - Do the same, with the corresponding instances in δ_2.
  - Chop off those last rules in both derivations.
  - The premises are the same, with the same proof terms [*Caveat*].
- Rinse and repeat.

In a derivation δ with term π, annotate steps with the *node* in π introduced.

$$
\dfrac{\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \dfrac{\overset{z \curvearrowright z}{z : p \succ z : p}}{\overset{\acute{\wedge} y \curvearrowright z}{y : p \wedge q \succ z : p}}\;\acute{\supset} u}{\overset{x \curvearrowright \acute{\wedge} w \quad \acute{\wedge} y \curvearrowright \grave{\wedge} w}{x : p, y : p \wedge q \succ w : p \wedge p}}\;\grave{\supset}\grave{\supset} u}{\overset{\acute{\supset} v \curvearrowright \acute{\wedge}\grave{\supset} v \quad \acute{\wedge} y \curvearrowright \grave{\wedge}\grave{\supset} v}{y : p \wedge q \succ v : p \supset (p \wedge p)}}\;\grave{\supset} u}{\overset{\acute{\supset}\grave{\supset} u \curvearrowright \acute{\wedge}\grave{\supset}\grave{\supset} u \quad \acute{\wedge}\acute{\supset} u \curvearrowright \grave{\wedge}\grave{\supset}\grave{\supset} u}{\succ u : (p \wedge q) \supset (p \supset (p \wedge p))}}\;u
\qquad
\dfrac{\dfrac{\dfrac{\overset{x \curvearrowright x}{x : p \succ x : p} \qquad \overset{z \curvearrowright z}{z : p \succ z : p}}{\overset{x \curvearrowright \acute{\wedge} w \quad z \curvearrowright \grave{\wedge} w}{x : p, z : p \succ w : p \wedge p}}\;\grave{\supset}\grave{\supset} u}{\overset{\acute{\supset} v \curvearrowright \acute{\wedge}\grave{\supset} v \quad z \curvearrowright \grave{\wedge}\grave{\supset} v}{z : p \succ v : p \supset (p \wedge p)}}\;\grave{\supset} u}{\dfrac{\overset{\acute{\supset} v \curvearrowright \acute{\wedge}\grave{\supset} v \quad \acute{\wedge} y \curvearrowright \grave{\wedge}\grave{\supset} v}{y : p \wedge q \succ v : p \supset (p \wedge p)}}{\overset{\acute{\supset}\grave{\supset} u \curvearrowright \acute{\wedge}\grave{\supset}\grave{\supset} u \quad \acute{\wedge}\acute{\supset} u \curvearrowright \grave{\wedge}\grave{\supset}\grave{\supset} u}{\succ u : (p \wedge q) \supset (p \supset (p \wedge p))}}\;u}\;\acute{\supset} u
$$

This allows identification of steps inside the *same* derivation

$$\dfrac{\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p}\qquad\overset{x\curvearrowright x}{x:p\succ x:p}}{\overset{x\curvearrowright\acute{\wedge}z\quad x\curvearrowright\grave{\wedge}z}{x:p\succ z:p\wedge p}}z\qquad\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p}\qquad\overset{x\curvearrowright x}{x:p\succ x:p}}{\overset{x\curvearrowright\acute{\wedge}z\quad x\curvearrowright\grave{\wedge}z}{x:p\succ z:p\wedge p}}z}{\overset{\acute{\vee}y\curvearrowright\acute{\wedge}z\quad\acute{\vee}y\curvearrowright\grave{\wedge}z\quad\grave{\vee}y\curvearrowright\acute{\wedge}z\quad\grave{\vee}y\curvearrowright\grave{\wedge}z}{y:p\vee p\succ z:p\wedge p}}y$$

$$\dfrac{\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p}\qquad\overset{x\curvearrowright x}{x:p\succ x:p}}{\overset{\acute{\vee}y\curvearrowright x\quad\grave{\vee}y\curvearrowright x}{y:p\vee p\succ x:p}}y\qquad\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p}\qquad\overset{x\curvearrowright x}{x:p\succ x:p}}{\overset{\acute{\vee}y\curvearrowright x\quad\grave{\vee}y\curvearrowright x}{y:p\vee p\succ x:p}}y}{\overset{\acute{\vee}y\curvearrowright\acute{\wedge}z\quad\acute{\vee}y\curvearrowright\grave{\wedge}z\quad\grave{\vee}y\curvearrowright\acute{\wedge}z\quad\grave{\vee}y\curvearrowright\grave{\wedge}z}{y:p\vee p\succ z:p\wedge p}}z$$

Deleting an α rule

$$\dfrac{\begin{array}{c}\pi\{a_1\}\{a_2\}\\ \mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\end{array}}{\begin{array}{c}\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\\ \mathfrak{S}\{a : \alpha\}\end{array}}$$

Deleting a β rule — where do you split?

$$\dfrac{\overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1 : \beta_1\}} \qquad \overset{\pi_2\{b_2\}}{\mathfrak{S}_2\{b_2 : \beta_2\}}}{\overset{\pi_1\{\acute{\beta} b\} \quad \pi_2\{\grave{\beta} b\}}{\mathfrak{S}_{1,2}\{b : \beta\}}}$$

Deleting a β rule — where do you split?

$$\dfrac{\overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1 : \beta_1\}} \qquad \overset{\pi_2\{b_2\}}{\mathfrak{S}_2\{b_2 : \beta_2\}}}{\overset{\pi_1\{\acute{\beta}b\} \quad \pi_2\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{b : \beta\}}}$$

For example . . .

$$\dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y : B, \Delta_2}}{\overset{\pi_1[\acute{\wedge}z] \quad \pi_2[\grave{\wedge}z]}{\Sigma_{1,2} \succ z : A \wedge B, \Delta_{1,2}}} \qquad \dfrac{\overset{\pi_3[x]}{\Sigma_3 \succ x : A, \Delta_3} \qquad \overset{\pi_4[y]}{\Sigma_4 \succ y : B, \Delta_4}}{\overset{\pi_3[\acute{\wedge}z] \quad \pi_4[\grave{\wedge}z]}{\Sigma_{3,4} \succ z : A \wedge B, \Delta_{3,4}}}$$

Be Lazy

$$\frac{\overset{\pi_1[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y : B, \Delta_2}}{\overset{\pi_1[\acute{\wedge} z] \;\; \pi_2[\grave{\wedge} z]}{\Sigma_{1,2} \succ z : A \wedge B, \Delta_{1,2}}} \qquad \frac{\overset{\pi[x, -]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi[-, y]}{\Sigma \succ y : B, \Delta}}{\overset{\pi[\acute{\wedge} z, \grave{\wedge} z]}{\Sigma \succ z : A \wedge B, \Delta}}$$

Be Lazy

$$\dfrac{\begin{array}{c}\pi_1[x] \\ \Sigma_1 \succ x : A, \Delta_1\end{array} \qquad \begin{array}{c}\pi_2[y] \\ \Sigma_2 \succ y : B, \Delta_2\end{array}}{\begin{array}{c}\pi_1[\acute{\wedge} z] \;\; \pi_2[\grave{\wedge} z] \\ \Sigma_{1,2} \succ z : A \wedge B, \Delta_{1,2}\end{array}} \qquad \dfrac{\begin{array}{c}\pi[x, -] \\ \Sigma \succ x : A, \Delta\end{array} \qquad \begin{array}{c}\pi[-, y] \\ \Sigma \succ y : B, \Delta\end{array}}{\begin{array}{c}\pi[\acute{\wedge} z, \grave{\wedge} z] \\ \Sigma \succ z : A \wedge B, \Delta\end{array}}$$

$$\dfrac{\begin{array}{c}\pi_1[x] \;\; \pi_2[-] \\ \Sigma \succ x : A, \Delta\end{array} \qquad \begin{array}{c}\pi_1[-] \;\; \pi_2[y] \\ \Sigma \succ y : B, \Delta\end{array}}{\begin{array}{c}\pi_1[\acute{\wedge} z] \;\; \pi_2[\grave{\wedge} z] \\ \Sigma \succ z : A \wedge B, \Delta\end{array}}$$

The last permutation—horizontal copying of *proof fragments*

$$\frac{\overset{\pi_1}{\Sigma_1 \succ \Delta_1} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y : A, \Delta_2}}{\overset{\pi}{\Sigma \succ \Delta}} \qquad\qquad \frac{\overset{\pi_1 \;\; \pi_2[-]}{\Sigma_{1,2} \succ \Delta_{1,2}} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y : A, \Delta_2}}{\overset{\pi}{\Sigma \succ \Delta}}$$

Example of weakening in a proof fragment

$$\dfrac{\overset{y \curvearrowright y}{y : A \succ y : A} \qquad\qquad \overset{z \curvearrowright z}{z : B \succ z : B}}{\overset{\acute{\vee}x \curvearrowright y \quad \grave{\vee}x \curvearrowright z}{x : A \vee B \succ y : A, z : B}} \; {\vee}L$$

Example of weakening in a proof fragment

$$\dfrac{\overset{y \curvearrowright y}{y : A \succ y : A} \qquad\qquad \overset{z \curvearrowright z}{z : B \succ z : B}}{\overset{\acute{\vee}x \curvearrowright y \quad \grave{\vee}x \curvearrowright z}{x : A \vee B \succ y : A, z : B}}\ \vee L$$

Delete y to get $\grave{\vee}x \curvearrowright z$, for the (invalid) sequent $x : A \vee B \succ z : B$.

$$\dfrac{\overset{y \curvearrowright y}{y : A \succ y : A} \qquad\qquad \overset{z \curvearrowright z}{z : B \succ z : B}}{\overset{\acute{\vee} x \curvearrowright y \quad \grave{\vee} x \curvearrowright z}{x : A \vee B \succ y : A, z : B}} \; \vee L$$

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$$\Sigma \overset{\pi}{\succ} \Delta$$

$$\dfrac{\overset{y \curvearrowright y}{y : A \succ y : A} \qquad \overset{z \curvearrowright z}{z : B \succ z : B}}{\overset{\acute{\vee} x \curvearrowright y \;\; \grave{\vee} x \curvearrowright z}{x : A \vee B \succ y : A, z : B}}\,{\vee}L$$

Delete y to get $\grave{\vee} x \curvearrowright z$, for the (invalid) sequent $x : A \vee B \succ z : B$.

$$\Sigma \overset{\pi}{\succ} \Delta \qquad \text{becomes} \qquad \dfrac{\overset{\pi}{\Sigma, y : A \succ \Delta} \qquad \overset{z \curvearrowright z}{z : B \succ z : B}}{\overset{\pi \;\; \grave{\vee} x \curvearrowright z}{\Sigma, x : A \vee B \succ z : B, \Delta}}\,{\vee}L$$


FURTHER WORK

To Do List

- Apply these terms to other kinds of proofs (Fitch, Lemmon, tableaux, Hilbert, resolution…)
- *Categories* (The class of *single input*, *single output* terms with composition by defined by *Cut* + *reduction* is a category. What are its properties?)
- Extend beyond propositional logic.

THANK YOU!

http://consequently.org/presentation/2017/proof-terms-invariants-amsterdam

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