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In the last lecture, I introduced inferentialist semantics.

In today's lecture, our aim is to extend this account to *quantifiers* and *modal operators*.

Arthur Prior (1914–1969)

> … possible worlds, in the sense of possible states of affairs are not really individuals (just as numbers are not really individuals). To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i.e. if something else were the case … We understand 'truth in states of affairs' because we understand 'necessarily'; not vice versa. [5]

If possibility and necessity are conceptually fundamental then (1) How *can* we grasp the concepts of possibility and necessity? (2) How is it that those concepts have a structure that makes possible worlds models appropriate for them?

But first, quantifiers and objects.

2.1 SAYING THINGS ABOUT THINGS

We can make claims *of* things. ¶ I can not only say that Clark Kent can fly. I can say *of* Clark Kent, that *he* can fly. ¶ Distinguish Fx (the *de re* judgement of x that it's F) from Ft (the *de dicto* judgement that t is F)

If I say of that man (who happens to *be* Clark, unbeknownst to me) that *he* can fly, then I am saying, *of Clark Kent* that he can fly.

SUBSTITUTION: Given any proof Π of A from $\mathcal{C}$, if we replace each free x in the proof by y, the result, Π^x_y is a proof of A^x_y from $\mathcal{C}^x_y$. ¶ The constraint means that these object variables are each *inferentially general*: no rule holds of one variable that does not hold of the others in this class of variables.

DETAILS: Any standard definition of substitution in first-order languages will suffice to make the details precise.

Should this substitution principle hold also for *terms*? ¶ There are reasons to hesitate. Our language might well contain singular terms that do not always take a value. ¶ There is no variable that takes $\frac{1}{0}$ as a value, because $\frac{1}{0}$ is not a number, or a thing. '$\frac{1}{0}$' is a non-denoting term.

REMEMBER: I am interested in logic as it applies to whatever we can think or say. We can make claims in languages with non-denoting terms.

In a language with terms that might not denote, but in which *variables* may be assumed to always take values—since Fx says *of the item* x that it's F—we need to mark the distinction between a term denoting and a term failing to denote. ¶ Represent term-judgements in positions, positively (s, t, etc., taking the terms to denote) and negatively ($\cancel{s}$, $\cancel{t}$, etc. rejecting terms as non-denoting), alongside declarative judgements. ¶ Then, we can define an existence predicate with an invertible rule. Since variables are assumed to always take values, we impose $\succ x$ as an axiom, for each variable x.

$$\frac{\mathcal{C} \succ t}{\mathcal{C} \succ E!t}\ E!Df$$

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We have a weaker principle for term substitution: a free variable may be replaced in a proof by a term, with the extra assumption that the term denotes. This rule is admissible:

$$\frac{\mathcal{C} \succ A}{t, \mathcal{C}^x_t \succ A^x_t}\ \mathit{Subst}^x_t$$

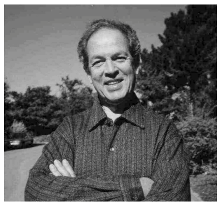
To verify this for a family of rules you must check that each inference rule is closed under substitution of terms for variables, possibly with the added extra premise to the effect that the term in variable denotes. Appeals to the axiom $\succ x$ are replaced by the assumption that t denotes (giving $t \succ t$) and the rest of the proof proceeds as before.

Given singular terms, the class of variables in that category, and substitution, we can now give invertible defining rules foir the quantifiers. ¶ The universal and existential quantifiers are simply defined:

$$\frac{\mathcal{C} \succ A}{\mathcal{C} \succ \forall x A}\ \forall Df \qquad \frac{\mathcal{C}, A \succ C}{\mathcal{C}, \exists x A \succ C}\ \exists Df$$

As usual, the variable x bound by the quantifier does not occur free in among the assumptions $\mathcal{C}$ or the conclusion C (in the existential quantifier rule).

Given our assumption that singular terms may fail to denote, we cannot necessarily conclude $\forall x A \succ A^x_t$. A universally quantified claim only applies to all *things*, and some terms may fail to denote. What does follow is $t, \forall x A \succ A^x_t$, by term substitution. Granted that the term t denotes, if everything has a given feature, t does. ¶ The resulting logic is a simple *free* logic, formulated by Sol Feferman [2, 7]. ¶ If we add the extra assumption that terms denote, this is no more and no less than classical first-order predicate logic.



Solomon Feferman (1928–2016)

I have left out details concerning the behaviour of predicates and function symbols. To get Feferman's negative free logic, we add rules of the form $Ft \succ t$ and $ft \succ t$, constraining predicates to apply to terms only when they denote, and functions having defined values only when their inputs are defined.

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CONSERVATIVE EXTENSION, the argument proceeds in the same way as for the propositional constants. (By translation into a left/right rule sequent calculus and eliminating *Cut*. Substitution plays a critical role.) ¶ For UNIQUENESS, we can pass from $\forall_1 x A$ to $\forall_2 x A$ through the original language, as before.

What about multi-sorted first-order logics, with different quantifiers? ¶ Uniqueness requires coordination on the class of variables and terms used to define the quantifiers.

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You might think that this is a substitutional account of the quantifiers [1, 4], according to which a quantified expression $\forall x F x$ [$\exists x F x$] is true if and only if every [some] substitution instance of Fx is true. ¶ This is a mistake, and not only because we have allowed for non-denoting terms. Even if we imposed the condition that every term denotes, the truth of every instance Ft, Ft', Fx, Fy, etc., would not be enough to guarantee the truth of $\forall x F x$. ¶ There is no way to prove the conclusion $\forall x F x$ from the premises Ft, Ft', Fx, Fy, etc., no matter *how* many terms of the language are included. So, whatever $\forall x F x$ says, it must say something more than what is said by the instances taken together.

Take a model in which there are at least two objects, one of which has feature F and another of which does not, and let's interpret our language so that every term denotes the F-object. Furthermore, assign the value of each variable, *that* object, too. Here, the premises are satisfied, but the conclusion $\forall x F x$ is not.

2.2 SUPPOSING THINGS WERE OTHERWISE

When we *modalise*—considering matters concerning what is possible and what is necessary—we consider not only what *is* the case, but what *might be*. We do not just assert and deny and ask questions. We *suppose*. ¶ Not all acts of supposition merely add the supposed item to the current context. ¶ In modal reasoning we can understand 'supposing things were otherwise' and 'considering alternative possibilities' in a straightforward way. We apply our thought and talk a richer structure, which might look like this:

$$[_{@}\, p, q, \cancel{r} \mid p, s, \cancel{r} \mid r, \cancel{p}, \cancel{q}\,]$$

Such a richer position represents the commitment of someone who has granted p and q and denied r (that is their '*actual*' commitment, as marked by the '@') and who has admitted as *possible* two other alternative outcomes: p and s without r or r without p and q. ¶ To say that something is necessary is to preserve that commitment across alternatives. So, a position like this is defective:

There is also a characteristically *epistemic* kind of supposition, arising when we attempt to rationally manage disagreement. You and I might disagree over p, and I might say: suppose you're right, and that it's actually the case that p . . . Here, I do not suppose p counterfactually in the subjunctive sense important for planning. Here, I consider p as part of an alternative account of *how things actually are* [3, 6]. We will not consider this epistemic supposition and the associated modality here, but the techniques under consideration apply equally well to epistemic modality.

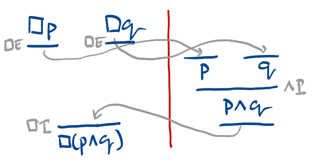
$$[\cdots \mid \Box A \cdots \mid \cancel{A} \cdots \mid \cdots]$$

To grant (somewhere) that A is *necessary* and to deny A (elsewhere) is to contradict yourself. ¶ Similarly, this position is also out of bounds:

At least, to grant $\Box A$ in some zone and to deny A in another is to contradict yourself if the notion of necessity in question encompasses the kinds of alternatives represented by the zones in our position.

$$[\cdots \mid \cancel{\Diamond A} \cdots \mid A \cdots \mid \cdots]$$

This motivates modal reasoning, which could be represented like this:



This reasoning presumes that the semantic rules for conjunction apply in *other* zones, as much as they do in our 'home' zone. That is the constraint that we will impose throughout. We take our definitions to apply to the language as we use it *across supposition boundaries*. (How *else* could we employ our concepts in subjunctive reasoning?)

This graphical representation is hard to typeset. Instead, we can distinguish different zones with tags. Here, 'a' labels commitments in the starting zone, while 'b' labels those in the other zone. ¶ The defining rules for the modal operators arise out of the richer

$$\dfrac{\dfrac{\dfrac{\Box p \cdot a}{p \cdot b}\,\Box E \qquad \dfrac{\Box q \cdot a}{q \cdot b}\,\Box E}{p \wedge q \cdot b}\,\wedge I}{\Box(p \wedge q) \cdot a}\,\Box I$$

structure of the zones in our positions, rather than quantifying over *objects* to which we might be able to *refer*.

$$\frac{\mathcal{C} \succ A \cdot i}{\mathcal{C} \succ \Box A \cdot j}\ \Box Df \qquad \frac{\mathcal{C}, A \cdot i \succ C}{\mathcal{C}, \Diamond A \cdot j \succ C}\ \Diamond Df$$

Here, the zone tag i in the premise is absent from the assumptions $\mathcal{C}$ (and the conclusion C in the $\Diamond$ rule). ❡ Our demonstration concerning A is *arbitrary*, no matter what else we have supposed in that counterfactual zone.

An *actuality* operator may be defined with a rule defining $@A \cdot a$ to have the same effect as $A \cdot @$. Granting $@A$ in any zone of the discourse has the same effect as granting A in the *actual* zone. This, in combination with the rules for $\Box$ and $\Diamond$, is a simple family of rules for the standard logic s5@.

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These rules are CONSERVATIVELY EXTENDING and UNIQUELY DEFINING, for the same reasons as for the quantifiers, except that this reasoning depends on supposition, rather than *substitution*. ❡ The uniqueness result requires coordination on what counts as the relevant kind of *modal supposition*.

2.3 BUT WHAT IF *THIS* THING HADN'T EXISTED?

Now let's combine the quantifier rules and the modal rules. ❡ We can prove the infamous BARCAN FORMULAS, named after Ruth Barcan Marcus. ❡ If everything is necessarily F, then it is necessary that everything is F.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \Box Fx \cdot a]^1}{\Box Fx \cdot a}\ \forall E}{Fx \cdot b}\ \Box E}{\forall x Fx \cdot b}\ \forall I}{\Box \forall x Fx \cdot a}\ \Box I}{\forall x \Box Fx \to \Box \forall x Fx \cdot a}\ {\to} I^1$$

There are reasons to resist the Barcan formulas. If everything that *happens* to exist is necessarily an F, it follows only that it is necessary that everything (that happens to have existed) is also an F only if there could be no *other* things than those things that happen to exist. ❡ Our quantifier rules, as given in Section 2.1, make no distinction between what exists and what doesn't.

Ruth Barcan Marcus (1921–2012)

This assumption should be rejected if you take it to be coherent to suppose, of a given item, that *it* might not have existed. ❡ *Being defined* and *having a value* come apart. When we take a given item x (say *me*) and suppose x *had not existed*, then the variable x is *is defined*, but we are considering a circumstance where in which the value of that variable does not exist. ❡ We modify the variable rule. We require only that their values be present in *some* zone. ❡ With $\Diamond Var$ in place, the quantifier rules come in two forms: one is existentially committing, and the other, *possibilist*.

$$\frac{x \cdot i, \mathcal{C} \succ C}{\mathcal{C} \succ C}\ \Diamond Var$$

Here, the zone i must not be used elsewhere in $\mathcal{C}$ and C.

$$\frac{\mathcal{C}, x \cdot j \succ A \cdot j}{\mathcal{C} \succ \forall x A \cdot j}\ \forall Df \quad \frac{\mathcal{C}, x \cdot j, A \cdot j \succ C}{\mathcal{C}, \exists x A \cdot j \succ C}\ \exists Df \quad \frac{\mathcal{C} \succ A \cdot j}{\mathcal{C} \succ \forall^{\Diamond} x A \cdot j}\ \forall^{\Diamond} Df \quad \frac{\mathcal{C}, A \cdot j \succ C}{\mathcal{C}, \exists^{\Diamond} x A \cdot j \succ C}\ \exists^{\Diamond} Df$$

In these rules we have the proviso that the variable x bound by the quantifier does not occur free in among the assumptions $\mathcal{C}$ or the conclusion C.

With the existentially committing rules for $\forall$ and $\exists$, the Barcan proofs fail, without the principle to the effect that something existing *there* also exists *here*. We start like this … but stop there. ❡ One way to verify that we *cannot* derive $\Box \forall x Fx$ from $\forall x \Box Fx$, using these rules, is to systematically reason about what can be derived. ❡ Instead, we will consider what this perspective can tell us about *models*, and use this to give counterexamples to invalid reasoning.

$$\dfrac{\dfrac{x \cdot a \qquad \forall x \Box Fx \cdot a}{\Box Fx \cdot a}\ \forall E}{Fx \cdot b}\ \Box E$$

2.4 PUSHING THINGS TO THE LIMIT

Limit positions provide another way to think of traditional two-valued models. ❡ Given a propositional language $\mathcal{L}$, limit positions settle all $\mathcal{L}$-issues. ❡ Quantifiers and modal operators complicate this picture.

Consider $\mathcal{L}^{F}_{\exists}$: with one predicate F, no connectives, a family of variables and singular terms, and the existential quantifier, governed by $\exists Df$. ❡ We can consistently affirm $\exists x Fx$ and deny Ft for every term t. A position denying every formula of the form Ft and affirming $\exists x Fx$ is available, and can be extended into a partition of $\mathcal{L}^{F}_{\exists}$. ❡ This issue is unsettled: have affirmed $\exists x Fx$, but can ask *which* item has feature F.

This is a fact concerning the inference rules for $\mathcal{L}^{F}_{\exists}$. There is no way to derive a contradiction from $\exists x Fx$ and any number of denials of the form $\cancel{Ft}$.

Call a position $\exists$-WITNESSED if whenever it includes $\exists x A$ it includes A^{x}_{y} for some y.

The defining rule for $\exists$ ensures that any available finite position including $\exists xA$, is extended by an available position that contains A^x_y for some fresh variable y. ¶ So, any available (small) position may be extended to a $\exists$-*witnessed* limit position.

The finiteness condition ensures that we can choose a variable that is, as yet, unused by the position. This can be weakened in natural ways, but those details are not necessary for our purposes.

What goes for $\exists$ goes also for $\forall$. I leave those details for you to consider.

The modal operators raise parallel issues. Given a position in which I affirm $\Diamond A$ (in zone i), this raises the question: *How* could A have been the case? To appropriately *settle* the issue, our position also involve a zone in which A is affirmed.

If I say it's possible that $2+2=5$, you're within your rights to ask me *how*, and if you suppose that $2+2=5$ and reduce that (counterfactual) supposition to absurdity, using agreed premises and principles, you've ruled out my claim that it's possible.

Call positions which contain, for each $\Diamond A \cdot i$, some $A \cdot j$, $\Diamond$-WITNESSED.

That this constraint can be met arises from the defining rule $\Diamond Df$. Any available position containing $\Diamond A \cdot i$ may be extended to contain $A \cdot j$ for a fresh zone j.

What goes for $\Diamond$ goes also for $\Box$. I leave those details for you to consider.

Witnessed limit positions in a language $\mathcal{L}$ settle every $\mathcal{L}$-issue in this extended sense.

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Witnessed limit positions describe Kripke models. ¶ Each *zone* describes a possible world. The terms ruled *in* at a zone are the inner domain of objects *existing* at that world, while the remaining terms that are ruled in at *some* world are the outer domain. ¶ The truth conditions for a varying domain quantified S5 (with actuality) are satisfied.

For the technical details, see my draft "Modal Logic and Contingent Existence (Generality and Existence 2)" [8].

Start with this position, affirming $\Diamond\exists xFx$ and denying $\exists x\Diamond Fx$:

$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}]$

For the witnessing condition for $\Diamond$ we must add a zone affirming $\exists xFx$. We have:

$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx} \mid \exists xFx]$

We must add a witness for $\exists xFx$, which is also present at that zone:

$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx} \mid x, \exists xFx, Fx]$

Since Fx is affirmed in the second zone, $\Diamond Fx$ is undeniable in each zone (given $\Diamond Df$), so as we fill out the position, $\Diamond Fx$ must be settled affirmatively in each zone:

$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}, \Diamond Fx \mid x, \exists xFx, Fx, \Diamond Fx]$

$\exists x\Diamond Fx$ is *denied* but $\Diamond Fx$ is affirmed in the first zone, so x must fail to be present there:

$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}, \Diamond Fx, \cancel{x} \mid x, \exists xFx, Fx, \Diamond Fx]$

If the predicate F expresses an existence entailing property then Fx must also be rejected in our starting zone. We end up with ($\mathfrak{P}$), describing a tiny model, settling every issue concerning x and F. ¶ This solves Prior's modal semantics puzzle. Possible worlds models model modal vocabulary when that vocabulary is governed by defining rules, since witnessed limit positions describe these models.

$[_{@}\Diamond\exists xFx, \cancel{\exists x\Diamond Fx}, \Diamond Fx, \cancel{Fx}, \cancel{x} \mid x, \exists xFx, Fx, \Diamond Fx]$

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ISSUE 1: What about the *ontological* significance of possibilia and of possible worlds?

ISSUE 2: The models given by constructing witnessed limit positions are a *scale models*. They need not be one-to-one models of the universe.

FOR NEXT TIME

We will take this perspective on models for modal vocabulary, and see how this can give a new angle on the value of truth-conditional semantics for natural languages.

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