Defining Quantifiers

Greg Restall

University of St Andrews

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1. DEFINING RULES

2. QUANTIFIERS & GENERALITY

3. DEFINING RULES for FREE QUANTIFIERS

4. GOING WIDER, TOO...

5. WHAT is A QUANTIFIER?

on TERMS & VALUES

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(HOW) DO SEQUENT RULES CHARACTERISE CONCEPTS?

$$\frac{X \succ A, Y}{X \succ A \text{ tonk } B, Y}\ \text{tonk}R \qquad \frac{X, B \succ Y}{X, A \text{ tonk } B \succ Y}\ \text{tonk}L$$

TOO STRONG!

$$\frac{p \succ p \text{ tonk } q \quad p \text{ tonk } q \succ q}{p \succ q}$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \text{ plink } B, Y}\ \text{plink}R \qquad \frac{X, A \succ Y \quad X, B \succ Y}{X, A \text{ plink } B \succ Y}\ \text{plink}L$$

TOO WEAK!

A plink B could be $A \wedge B$ or $A \vee B$ or anything in between!

$$\frac{X, A \succ B, Y}{X \succ A \to B, Y}\ {\to}R \qquad \frac{X \succ A, Y \quad X', B \succ Y'}{X, X', A \to B \succ Y, Y'}\ {\to}L$$

JUST RIGHT

A DEFINITION

x is equilateral iff x has three sides

& each side has equal length.

A DEFINITION

as an invertible rule

$$\dfrac{x \text{ has three sides} \qquad\qquad x \text{ each side of } x \text{ has equal length.}}{x \text{ is equilateral}}$$

DEFINING THE CONDITIONAL

$$\begin{array}{c} X, A \succ B, Y \\ \hline\hline X \succ A \to B, Y \end{array} \;{\to}Df$$

This is clearly uniquely defining (if $\to_1$ & $\to_2$ are defined using the same rule, they are interchangeable)

$$\begin{array}{c} \begin{array}{c} X \succ A \to_1 B, Y \\ \hline X, A \succ B, Y \end{array} \;{\to_1}Df\uparrow \\ \hline X \succ A \to_2 B, Y \end{array} \;{\to_2}Df\downarrow$$

$$\begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c} \\ \hline A \to_2 B \succ A \to_2 B \end{array} \;Id \\ \hline A \to_2 B, A \succ B \end{array} \;{\to_2}Df\uparrow \\ \hline A \to_2 B \succ A \to_1 B \end{array} \;{\to_1}Df\downarrow \qquad X, A \to_1 B \succ Y \\ \hline X, A \to_2 B \succ Y \end{array} \;Cut$$

DEFINING THE CONDITIONAL

$$\dfrac{X, A \succ B, Y}{X \succ A \to B, Y}\ {\to}Df$$

This is also conservatively extending — the usual left/right rules can be recovered from $\to$Df using Cut & ID.

$$\dfrac{X_1 \succ A, Y_1 \qquad \dfrac{\dfrac{}{A \to B \succ A \to B}\ Id}{A \to B, A \succ B}\ {\to}Df{\uparrow}}{\dfrac{X_1, A \to B \succ B, Y_1 \qquad X_2, B \succ Y_2}{X_1, X_2, A \to B \succ Y_1, Y_2}\ Cut_B}\ Cut_A$$

DEFINING THE CONDITIONAL

$$\dfrac{X, A \succ B, Y}{X \succ A \to B, Y}\ \to Df$$

This is also *conservatively extending* — the usual left/right rules can be recovered from $\to Df$ using Cut & ID, and the principal cut-reduction step *unwinds* this definition.

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ X_3, A \succ B, Y_3\end{matrix}}{X_3 \succ A \to B \succ Y_3}\ \to Df\!\downarrow \qquad \dfrac{\dfrac{\begin{matrix}\vdots \\ X_1 \succ A, Y_1\end{matrix} \qquad \dfrac{\dfrac{}{A \to B \succ A \to B}\ Id}{A \to B, A \succ B}\ \to Df\!\uparrow}{X_1, A \to B \succ B, Y_1}\ Cut_A \qquad \begin{matrix}\vdots \\ X_2, B \succ Y_2\end{matrix}}{X_1, X_2, A \to B \succ Y_1, Y_2}\ Cut_B}{X_1, X_2, X_3 \succ Y_1, Y_2, Y_3}\ Cut_{A \to B}$$

$$
\dfrac{\dfrac{\begin{matrix}\vdots\\ X_1 \succ A, Y_1\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ X_3, A \succ B, Y_3\end{matrix}}{X_3 \succ A \to B \succ Y_3}\,{\to}\mathrm{Df}{\downarrow} \qquad \dfrac{\dfrac{}{A \to B \succ A \to B}\,\mathrm{Id}}{A \to B, A \succ B}\,{\to}\mathrm{Df}{\uparrow}}{X_3, A \succ B, Y_3}\,\mathrm{Cut}_{A\to B}}{X_1, X_3 \succ B, Y_3}\,\mathrm{Cut}_A \qquad \begin{matrix}\vdots\\ X_2, B \succ Y_2\end{matrix}}{X_1, X_2, X_3 \succ Y_1, Y_2, Y_3}\,\mathrm{Cut}_B
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\begin{matrix}\vdots\\ X_1 \succ A, Y_1\end{matrix} \qquad \begin{matrix}\vdots\\ X_3, A \succ B, Y_3\end{matrix}}{X_1, X_3 \succ B, Y_3}\,\mathrm{Cut}_A \qquad \begin{matrix}\vdots\\ X_2, B \succ Y_2\end{matrix}}{X_1, X_2, X_3 \succ Y_1, Y_2, Y_3}\,\mathrm{Cut}_B
$$

THIS IS TOTALLY GENERAL

(it works like this for any invertible defining rule, in the presence or absence of contraction & weakening, in different sequent structures.)

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge Df \qquad \frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\ \otimes Df \qquad \frac{X, A \succ Y \quad X \succ B, Y}{X \succ A \sqsupset B, Y}\ \sqsupset Df$$

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Defining Rules for Quantifiers

$$\frac{X \succ A(n), Y}{X \succ \forall x A(x), Y}\ \forall Df \qquad \frac{X, A(n) \succ Y}{X, \exists x A(x) \succ Y}\ \exists Df$$

The term n must be absent in the lower sequent.

n is an eigenvariable (more on what this means, soon).

RECOVERING THE USUAL LEFT/RIGHT RULES

$$\dfrac{X \succ A(n), Y}{X \succ \forall_x A(x), Y}\,\forall Df$$

$\forall R$

$$\dfrac{X \succ A(n), Y}{X \succ \forall_x A(x), Y}\,\forall Df\!\downarrow$$

$\forall L$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall_x A(x) \succ \forall_x A(x)}\,Id}{\forall_x A(x) \succ A(n)}\,\forall Df\!\uparrow}{\forall_x A(x) \succ A(t)}\,Spec^n_t \qquad X, A(t) \succ Y}{X, \forall_x A(x) \succ Y}\,Cut$$

(t can be any term at all, not necessarily an eigenvariable)

THE SPECIALISATION^n_t RULE

$$\frac{X \succ Y}{[X \succ Y]^n_t}\ \mathrm{Spec}^n_t$$

If a sequent obtains concerning some eigenvariable n, it also obtains for the term t.

Eigenvariables are inferentially general among terms.

Spec^n_t is admissible in Gentzen's sequent calculus.

EQUIVALENT FORMULATIONS

$$\begin{tikzcd}[column sep=huge]
\begin{array}{l}\mathrm{Id}\\ \mathrm{Cut}\\ \forall\mathrm{Df}\\ \mathrm{Spec}\end{array} \arrow[r, bend left, "\text{define } \forall L \text{ by } \forall\mathrm{Df}, \mathrm{Spec}, \ldots"] & \begin{array}{l}\mathrm{Id}\\ \mathrm{Cut}\\ \forall L/\forall R\\ \mathrm{Spec}\end{array} \arrow[l, bend left, "\text{define } \forall\mathrm{Df} \text{ by } \forall L, \mathrm{Spec}, \ldots"] \arrow[r, bend left, "\text{Spec elimination}"] & \begin{array}{l}\mathrm{Id}\\ \mathrm{Cut}\\ \forall L/\forall R\end{array} \arrow[l, bend left, "\text{inclusion}"] \arrow[r, bend left, "\text{Cut elimination}"] & \begin{array}{l}\mathrm{Id}\\ \forall L/\forall R\end{array} \arrow[l, bend left, "\text{inclusion}"]
\end{tikzcd}$$

[See my "Generality & Existence 1", RSL (2019) for details.]

Spec Elimination

$$\dfrac{\begin{array}{c}\delta \\ \vdots \\ X \succ Y\end{array}}{[X \succ Y]^n_t}\,\mathrm{Spec}^n_t \quad \rightsquigarrow \quad \begin{array}{c}\delta^n_t \\ \vdots \\ [X \succ Y]^n_t\end{array}$$

It's easy to see that each of the rules in Gentzen's system are closed under specialisation — ie there is nothing inferentially special in an eigenvariable as conclusion.

($\forall Df\uparrow$ violates this constraint — hence the requirement to impose Spec.)

DEFINING RULES

... specify concepts that are available (conservative) & determinate (unique) over a basic structural context

.... & for quantifiers, these rules use prior notions of substitution & inferential generality.

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FREE LOGIC, DEFINED & UNDEFINED TERMS

n is a number.

n/m is not necessarily a number.

n/m is not defined when $m=0$.
(does not exist)

'n/m' is a term, whatever values n & m take.

$$\forall x(\neg\exists y(y = n/x) \longleftrightarrow x=0)$$

THE BASIC STRUCTURAL CONTEXT

RULING TERMS IN & RULING THEM OUT

$$s, t, A, B \succ C, D, s', t'$$

[defined true false undefined]

for a counterexample to the sequent

ACCEPT

REJECT

$$\frac{}{t, X \succ Y, t}\ t\text{-Id}$$

$$\frac{X \succ Y, t \qquad t, X' \succ Y'}{X, X' \succ Y, Y'}\ t\text{-Cut}$$

DEFINING DEFINEDNESS

$$\frac{X \succ Y, t}{X \succ Y, t{\downarrow}}\ {\downarrow}Df$$

& equivalently, ...

$$\frac{X \succ Y, t}{X \succ Y, t{\downarrow}}\ {\downarrow}R \qquad\qquad \frac{X, t \succ Y}{X, t{\downarrow} \succ Y}\ {\downarrow}L$$

POSSIBLE CONDITIONS ON PREDICATES & FUNCTION SYMBOLS

$$\frac{t_i, X \succ Y}{Ft_1 \ldots t_n, X \succ Y}\,FL \qquad\qquad \frac{t_i, X \succ Y}{ft_1 \ldots t_n, X \succ Y}\,fL$$

(If we grant these, then nonexistence is not a predicate in this sense, since we have $\neg \frac{1}{0}\!\downarrow$, while $\frac{1}{0}$ is not defined.)

DEFINING RULES FOR $\forall / \exists$ with existential commitment

$$\begin{array}{c} n, X \succ Y, A(n) \\ \hline\hline X \succ Y, \forall_x A(x) \end{array}\ \forall Df \qquad\qquad \begin{array}{c} n, A(n), X \succ Y \\ \hline\hline \exists_x A(x), X \succ Y \end{array}\ \exists Df$$

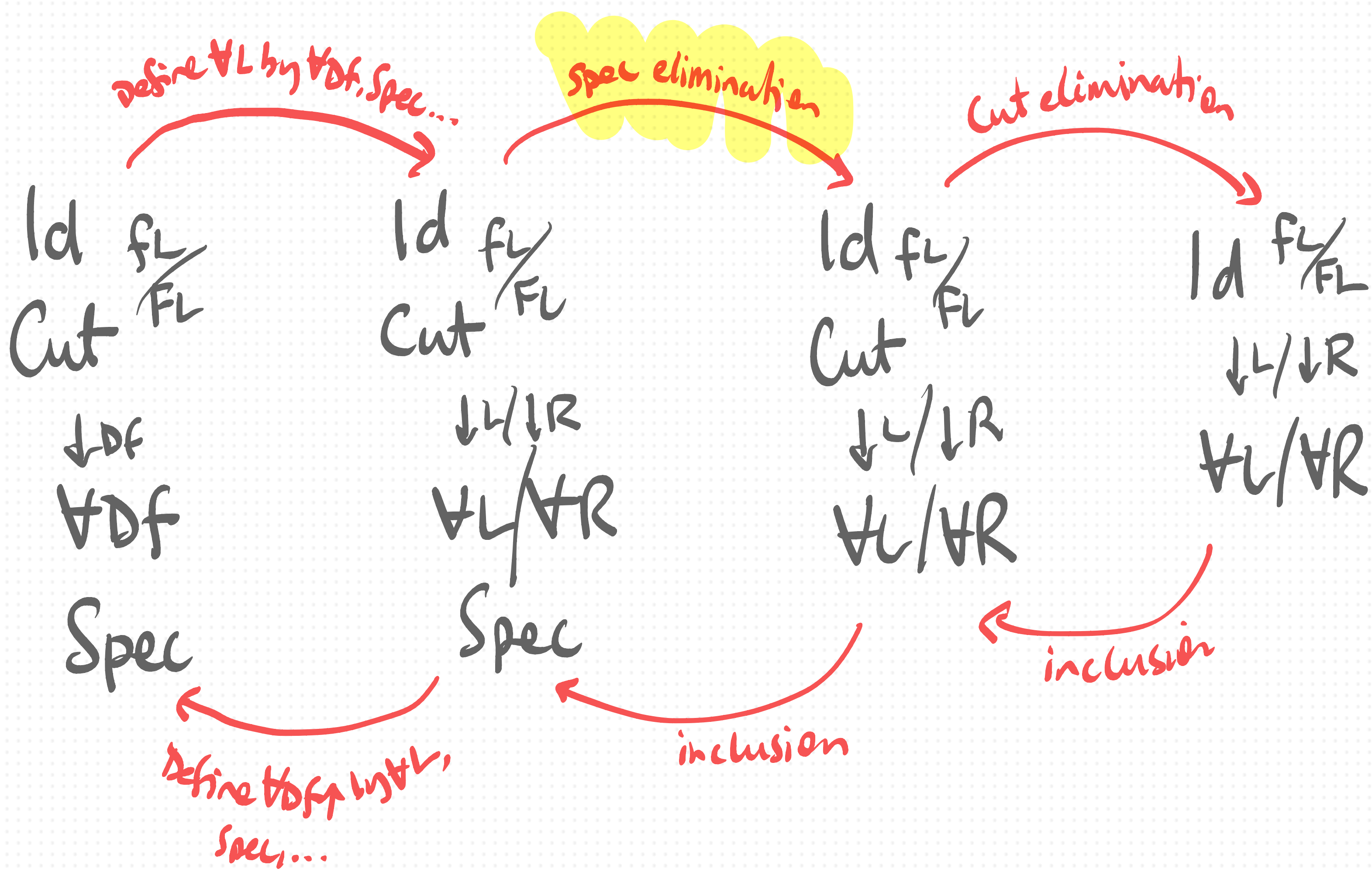
$$\dfrac{X \succ Y, t \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall_x A(x) \succ \forall_x A(x)}\,Id}{n, \forall_x A(x) \succ A(n)}\,\forall Df\uparrow}{t, \forall_x A(x) \succ A(t)}\,Spec^n_t \qquad X', A(t) \succ Y'}{t, X', \forall_x A(x) \succ Y'}\,Cut}{X, X', \forall_x A(x) \succ Y, Y'}\,t\,Cut$$

Defining Rules for $\forall / \exists$ with existential commitment

$$\frac{n, X \succ Y, A(n)}{\overline{\overline{\;X \succ Y, \forall x A(x)\;}}}\;\forall Df \qquad\qquad \frac{n, A(n), X \succ Y}{\overline{\overline{\;\exists x A(x), X \succ Y\;}}}\;\exists Df$$

$$\dfrac{X \succ Y, t \qquad \dfrac{X' \succ A(t), Y' \qquad \dfrac{\dfrac{\dfrac{}{\exists x A(x) \succ \exists x A(x)}\;Id}{n, A(n) \succ \exists x A(x)}\;Df\uparrow}{t, A(t) \succ \exists x A(x)}\;Spec^{n}_{t}}{t, X' \succ \forall x A(x), Y'}\;Cut}{X, X', \forall x A(x) \succ Y, Y'}\;t\,Cut$$

THE EQUIVALENCES STILL HOLD...



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BUT WE ARE FREE TO DEFINE MORE...

$$\begin{array}{c} n, X \succ Y, A(n) \\ \hline\hline X \succ Y, \forall x A(x) \end{array}\ \forall Df \qquad\qquad \begin{array}{c} X \succ Y, A(n) \\ \hline\hline X \succ Y, \Pi x A(x) \end{array}\ \Pi Df$$

with Existential Commitment   without Existential Commitment

$$\begin{array}{c} X \succ Y, \forall x A(x) \\ \hline\hline n, X \succ Y, A(n) \\ \hline\hline n{\downarrow}, X \succ Y, A(n) \\ \hline\hline X \succ Y, n{\downarrow} \to A(n) \\ \hline\hline X \succ Y, \Pi x (x{\downarrow} \to A(n)) \end{array} \begin{array}{l} \\ \forall Df \\ \\ {\downarrow} Df \\ \\ \to Df \\ \\ \Pi Df \\ \\ \end{array}$$

BUT WE ARE FREE TO DEFINE MORE...

$$\frac{n, A(n), X \vdash Y}{\exists_x A(x), X \vdash Y}\ \exists\mathrm{Df} \qquad\qquad \frac{A(n), X \vdash Y}{\Sigma_x A(x), X \vdash Y}\ \Sigma\mathrm{Df}$$

$$\dfrac{\dfrac{\Sigma_x \neg x{\downarrow} \vdash \Sigma_x \neg x{\downarrow}}{\neg n{\downarrow} \vdash \Sigma_x \neg x{\downarrow}}\ \Sigma\mathrm{Df}{\uparrow}}{\neg \tfrac{1}{0}{\downarrow} \vdash \Sigma_x \neg x{\downarrow}}\ \mathrm{Spec}^{n}_{1/0}$$

Are outer quantifiers just as acceptable as inner quantifiers?

Do they involve the same sort of

$$\left\{\begin{array}{l}\text{ontological}\\\text{ideological}\\\text{theoretical}\end{array}\right\}$$ commitment?

Not necessarily

There are differences...

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LOOKING CLOSER...

$$\frac{n, X \succ Y, A(n)}{X \succ Y, \forall x A(x)}\ \forall Df \qquad \frac{X \succ Y, A(n)}{X \succ Y, \Pi x A(x)}\ \Pi Df$$

this n is $\begin{cases} \text{an eigenvariable} \\ \text{an inferentially general singular term.} \end{cases}$

WHAT IS A VARIABLE?

What is the meaning of 'x' in $\mathcal{Q}x\, A(x)$?

"Everything is A..." "For every value x can take..."

What is the meaning of 'n' in $A(n)$?

- A variable of some sort?
- A pronoun? a demonstrative expression?
- A singular term of some kind?

Should these be connected?

IN FAVOUR OF CONNECTION...

$$\frac{x, X \succ Y, A(x)}{X \succ Y, \forall_x A(x)}\ \forall Df \qquad\qquad \frac{x, A(x), X \succ Y}{\exists_x A(x), X \succ Y}\ \exists Df$$

The defining rules become compositional, just like the other defining rules.

BUT THEN...

$$\frac{x, X \vdash Y, A(x)}{X \vdash Y, \forall x A(x)} \; \forall Df \qquad\qquad \frac{x, A(x), X \vdash Y}{\exists x A(x), X \vdash Y} \; \exists Df$$

THIS SEEMS REDUNDANT!

What would it mean for the variable x to be undefined in this context?

Isn't using a variable tantamount to treating it as having some value or other?

(E.g. Feferman takes $x\downarrow$ to be a theorem.)

ALTERNATE FORMULATION

- Variables are always treated as defined

$$\frac{}{X \succ Y, x}\; x{\downarrow}$$

- They are no longer inferentially general but are general among the defined terms.

$$\frac{X \succ Y}{t, [X \succ Y]^x_t}\; \mathrm{Spec}^x_t$$

- Variables are used for quantification, not general singular terms.

$$\frac{X \succ Y, A(x)}{X \succ Y, \forall x A(x)}\; \forall Df_x$$

EQUIVALENCE

Derivations in the two systems are inter-translatable

Id, Cut

Fl fl

Spec^{n}_{t}

Connective Df...

$\forall\mathrm{Df}_{n}$ $\exists\mathrm{Df}_{n}$

x, y.. only ever bound.

Eigenvariables n occur free.

Id, Cut

Fl fl

Spec^{x}_{t} $x\downarrow$

Connective Df...

$\forall\mathrm{Df}_{x}$ $\exists\mathrm{Df}_{x}$

x, y... occur free

No eigenvariables

Eigenvariable with existential commitment $\rightleftarrows$ Variable

THE UPSHOT

This formulation provides a context in which inner quantification is definable, using the resources available, but outer quantification (defined in the obvious way) uses extra resources.

This is not to say that outer quantification is meaningless, of course!

LESSONS & FURTHER QUESTIONS

- Quantification involves a number of connected issues
  SUBSTITUTION, GENERALITY, DEFINEDNESS, VARIABLES/VALUES, ETC.
- Choosing a structural context for deduction means taking sides on these issues.
- Quantification into sentence position? predicate position?
  — do similar issues apply there?
- Modal calculi?
  — What of singular terms that take values in some worlds and not others?

Thanks!