## **Terms for Classical Sequents**

**Proof Invariants & Strong Normalisation** 

Greg Restall



GOTHENBURG LOGIC SEMINAR · 10 MAY 2016

Today's Plan

Background

**Preterms** 

**Derivations** 

Terms

Eliminating Cuts

Strong Normalisation

Further Work

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When is  $\pi_1$  the same proof as  $\pi_2$ ?

$$\frac{p \succ p}{p \succ p \lor q} \lor R \qquad \frac{p \land q}{p} \land E$$

$$p \land q \succ p \lor q \qquad \frac{p}{p} \lor I$$

$$\frac{\frac{p \succ p}{p \land q \succ p} \land L}{p \land q \succ p \lor q} \lor R$$

$$\frac{q \succ q}{q \succ p \lor q} \lor R$$
$$p \land q \succ p \lor q \land L$$

$$\frac{\frac{\mathsf{q}}{\mathsf{q}} \wedge \mathsf{E}}{\frac{\mathsf{p} \vee \mathsf{q}}{\mathsf{p} \vee \mathsf{q}}} \wedge \mathsf{E}$$

$$\frac{\frac{q \succ q}{q \succ p \lor q} \lor_{R}}{\frac{p \land q}{p \land q \succ p \lor q} \land_{L}} \qquad \frac{\frac{p \land q}{q}}{\frac{q}{p \lor q}} \lor_{R} \qquad \frac{\frac{q \succ q}{p \land q \succ q} \land_{L}}{\frac{p \land q \succ p \lor q}{p \land q \succ p \lor q}} \lor_{R}$$

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Girard, Lafont and Taylor: Proofs and Types, Chapter 2

Natural deduction is a slightly paradoxical system: it is limited to the intuitionistic case (in the classical case it has no particularly good properties) but it is only satisfactory for the  $(\land,\Rightarrow,\forall)$  fragment of the language: we shall defer consideration of  $\vee$  and  $\exists$  until chapter 10. Yet disjunction and existence are the two most *typically* intuitionistic connectors!

The basic idea of natural deduction is an asymmetry: a proof is a vaguely rea-like structure (this view is more a graphical illusion than a mathematical reality, but it is a pleasant illusion) with one or more hypotheses (possibly none) but a single conclusion. The deep symmetry of the calculus is shown by the introduction and elimination rules which match each other exactly. Observe, incidentally, that with a tree-like structure, one can always decide uniquely what was the last rule used, which is something we could not say if there were several conclusions.

To introduce a new invariant for classical propositional proofs and to show how they can be used.

Terms for Classical Sequents

## **BACKGROUND**

When is  $\pi_1$  the same proof as  $\pi_2$ ?

$$\frac{p \vee q \quad \frac{[p]^1}{q \vee p} \vee_I \quad \frac{[q]^1}{q \vee p} \vee_I}{\frac{q \vee p}{(q \vee p) \vee r} \vee_I} \quad \frac{\frac{[p]^1}{q \vee p} \vee_I \quad \frac{[q]^1}{q \vee p} \vee_I}{\frac{q \vee p}{(q \vee p) \vee r} \vee_I} \vee_I \quad \frac{[q]^1}{q \vee p} \vee_I}{\frac{q \vee p}{(q \vee p) \vee r} \vee_E} \vee_E$$

Are these different proofs, or different ways of presenting the same proof?

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#### Lambda Terms and Proofs

$$\frac{[x:p\supset (q\supset r)]\quad [z:p]}{xz:q\supset r}\supset E\quad \frac{[y:p\supset q]\quad [z:p]}{yz:q}\supset E$$

$$\frac{(xz)(yz):r}{\lambda z\,(xz)(yz):p\supset r}\supset I$$

$$\lambda y\lambda z\,(xz)(yz):(p\supset q)\supset (p\supset r)$$

$$\lambda x\lambda y\lambda z\,(xz)(yz):(p\supset (q\supset r))\supset ((p\supset q)\supset (p\supset r))$$

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#### Contraction and weakening are managed by variables

$$\frac{\frac{[x:p]}{\lambda y \, x \colon q \supset p} \supset I}{\lambda x \lambda y \, x \colon p \supset (q \supset p)} \supset I \qquad \frac{\frac{x \colon p \supset (p \supset q) \quad [y \colon p]}{x y \colon p \supset q} \supset E}{\frac{(xy)y \colon q}{\lambda y \, (xy)y \colon p \supset q} \supset I} \supset E$$

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## Sequents and Terms

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### $X \succ Y \qquad X \succ A, Y \qquad X, A \succ Y$

Where do you put the variables, and where do you put the terms?

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#### Example 1



## **PRETERMS**

#### Classical Sequent Derivations

$$\frac{p \succ p}{\stackrel{\succ}{\succ} p, \neg p} \neg_{R} \qquad \frac{p \succ p}{p, \neg p \succ} \neg_{L} \\ \stackrel{\succ}{\succ} p \lor \neg p} \lor_{R} \qquad \frac{p \succ p}{p, \neg p \succ} \land_{L}$$

$$\frac{p \succ p \quad \frac{q \succ q \quad r \succ r}{q \lor r \succ q, r} \lor_{L}}{p, q \lor r \succ p \land q, r} \land_{R}} \xrightarrow{p \land (q \lor r) \succ p \land q, r} \land_{L}} \land_{L}$$

$$\frac{p \land (q \lor r) \succ p \land q, r}{p \land (q \lor r) \succ (p \land q) \lor r} \lor_{R}$$

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#### Our Choice

$$\begin{matrix} \pi(x_1,\ldots,x_n)[y_1,\ldots,y_m] \\ x_1:A_1,\ldots,x_n:A_n\succ y_1:B_1,\ldots,y_m:B_m \end{matrix}$$

*Each* premise and conclusion is decorated with variables.

The sequent gets the term, showing how inputs & outputs are connected, with as much parallelism as possible.

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#### Example 2

$$\frac{x \cap x \qquad x \cap x \qquad x \cap x}{x : p \succ x : p \qquad x : p \succ x : p} \land R \qquad \frac{z \cap z}{z : p \succ z : p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times y : p \land p} \land R \qquad \frac{z \cap z}{F_{y} \cap z} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

$$\frac{x \cap F_{y} \times \cap S_{y} \qquad x \cap S_{y} \qquad x \cap p}{x \cap F_{y} \times x \cap S_{y} \qquad x \cap p} \land L$$

#### Variables and Cut Points

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► For each formula A,  $x_1^A$ ,  $x_2^A$ , . . . are VARIABLES of type A.

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- ► For each formula A,  $\bullet_1^A$ ,  $\bullet_2^A$ , ... are CUT POINTS of type A.
- We use  $x, y, z, u, v, w, \ldots$ ; •, \*, \*, \$, \beta as schematic letters for variables and cut points, ommitting type superscripts where possible.

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#### Nodes and Subnodes

- ► A variable x of type A and a cut point of type A are both A NODES.
- ► If  $\mathbf{n}$  is an A  $\wedge$  B node, then  $\mathbf{L}\mathbf{n}$  is an A node and  $\mathbf{R}\mathbf{n}$  is a B node.
- ▶ If n is an  $A \lor B$  node, then Fn is an A node and Sn is a B node.
- ► If n is an  $A \supset B$  node, then An is an A node and Cn is a B node.
- ► If n is a  $\neg$ A node, then Nn is an A node.
- ► For each complex node Ln, Rn, Fn, Sn, An, Cn and Nn, n is its IMMEDIATE subnode, and the subnodes of n are also subnodes of the original node.

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### Example Linkings

$$\boldsymbol{x}$$
 of type  $((\mathfrak{p}\supset\mathfrak{q})\supset\mathfrak{p})\supset\mathfrak{p}$ 

AAAx Cx

 $CAx^Cx$ 

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# **DERIVATIONS**

#### Annotating Derivations: Conjunction

$$\begin{array}{c|c} \pi(x,y) & \pi(x) \\ \hline \Sigma,x:A,y:B \succ \Delta \\ \hline \pi(Fz,Sz) & \Sigma,z:A \land B \succ \Delta \end{array} \land L \qquad \begin{array}{c|c} \pi(x) & \pi'(y) \\ \hline \Sigma \succ x:A,\Delta & \Sigma' \succ y:B,\Delta' \\ \hline \pi(Fz) & \pi'(Sz) \\ \hline \Sigma,\Sigma' \succ z:A \land B,\Delta,\Delta' \end{array}$$

#### Linkings, Inputs and Outputs

- ► A LINKING is a pair n m of nodes of the same type.
- ▶ In  $n^m$ , n is in input position, and m is in output position.
- ▶ Positions generalise to subnodes as follows:
  - ► If Ln, Rn, Fn, Sn or Cn are in input position, n is also in input position.
  - ► If Ln, Rn, Fn, Sn or Cn are in output position, n is also in output position.
  - L, R, F, S and C each preserve position.
  - ► If An or Nn is in input position, n is in output position.
  - ► If An or Nn is in output position, n is in input position.
  - A and N reverse position.
- ► The Inputs (Outputs) of a linking are the *variables* in Input (Output) position of that linking.

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#### Preterms

- ► DEFINITION: A PRETERM is a finite set of linkings.
- The inputs of a preterm are the inputs of its linkings.
- Its OUTPUTS are the outputs of its linkings.

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#### Annotating Derivations: Identity

$$\Sigma, \mathbf{x} : \mathbf{A} \succ \mathbf{y} : \mathbf{A}, \Delta$$

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#### Excursus on Weakening and Variables

$$\frac{\frac{[x:p]}{\lambda y x: q \supset p} \supset^{I}}{\lambda x \lambda y x: p \supset (q \supset p)} \supset^{I}$$

$$\begin{array}{c|c} \pi(x,y) & \pi(x) \\ \hline \Sigma, x: A, y: B \succ \Delta \\ \hline \pi(Fz, Sz) & \wedge L & can be & \hline \begin{array}{c} \pi(x) \\ \Sigma, x: A \succ \Delta \\ \hline \pi(Fz) & \\ \Sigma, z: A \wedge B \succ \Delta & \Sigma, z: A \wedge B \succ \Delta \end{array} \end{array} \wedge L$$

In a premise  $\pi(x, y)$  the indicated x and y display all of the x and y inputs to the proof term.

There might be none.

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## Annotating Derivations: Negation

$$\frac{\sum \times x : A, \Delta}{\pi[Nz]} \neg L \qquad \frac{\pi(x)}{\sum, x : A \times \Delta} \neg R$$

$$\sum, z : \neg A \times \Delta \qquad \qquad \Sigma \times z : \neg A, \Delta$$

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## Annotating Derivations: Disjunction

$$\begin{array}{c|c} \frac{\pi(x)}{\Sigma, x : A \succ \Delta} & \frac{\pi'(y)}{\Sigma', y : B \succ \Delta'} \vee_{L} & \frac{\pi(x, y)}{\Sigma \succ x : A, y : B, \Delta} \vee_{R} \\ \hline \Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta' & \frac{\pi(Lz, Rz)}{\Sigma \succ z : A \vee B, \Delta} \end{array}$$

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#### Annotating Derivations: Conditional

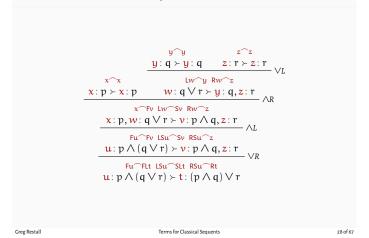
$$\frac{\sum \boldsymbol{\times} \boldsymbol{x} : \boldsymbol{A}, \boldsymbol{\Delta} \qquad \boldsymbol{\Sigma}', \boldsymbol{y} : \boldsymbol{B} \boldsymbol{\succ} \boldsymbol{\Delta}'}{\boldsymbol{\pi}(Az)} \supset L \qquad \frac{\boldsymbol{\pi}(x)[y]}{\boldsymbol{\pi}(Az)[Cz]} \supset R$$

$$\boldsymbol{\Sigma}, \boldsymbol{\Sigma}', \boldsymbol{z} : \boldsymbol{A} \supset \boldsymbol{B} \boldsymbol{\succ} \boldsymbol{\Delta}, \boldsymbol{\Delta}' \qquad \qquad \boldsymbol{\Sigma}', \boldsymbol{x} : \boldsymbol{A} \boldsymbol{\succ} \boldsymbol{y} : \boldsymbol{B}, \boldsymbol{\Delta}$$

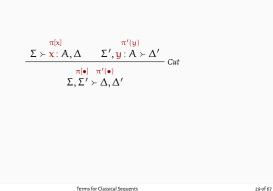
$$\boldsymbol{\Sigma}, \boldsymbol{\Sigma}', \boldsymbol{z} : \boldsymbol{A} \supset \boldsymbol{B} \boldsymbol{\succ} \boldsymbol{\Delta}, \boldsymbol{\Delta}' \qquad \qquad \boldsymbol{\Sigma} \boldsymbol{\Sigma} \boldsymbol{\succ} \boldsymbol{z} : \boldsymbol{A} \supset \boldsymbol{B}, \boldsymbol{\Delta}$$

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#### Example Annotation



#### Annotating Derivations: Cut



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#### Identify Terms up to $\alpha$ equivalence

If  $\pi$  can be transformed into  $\pi'$  by relabelling cut points we treat them as identical (they are  $\alpha$  equivalent).

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#### Example Annotation, with Cut

$$\frac{x \cdot x}{x \cdot p \cdot x \cdot p} \quad x \cdot p \cdot x \cdot p}{\underbrace{y \cdot p \cdot p \cdot x \cdot p}_{Ly \cap \bullet} \quad x \cdot p}_{Ly \cap \bullet} \quad x \cdot p \cdot x \cdot p}_{X \cdot p \cdot x \cdot p} \quad x \cdot p \cdot x \cdot p}_{X \cdot p \cdot x \cdot p} \quad AR$$

$$\frac{y \cdot p \lor p \cdot x \cdot p}{Ly \cap \bullet} \quad x \cdot p \cdot x \cdot p}_{Ly \cap \bullet} \quad x \cdot p \cdot p \cdot p}_{Cut}$$

$$\frac{x \cdot x}{x \cdot p \cdot x \cdot p} \quad x \cdot p \cdot x \cdot p}_{X \cdot p \cdot x \cdot p} \quad AR$$

$$\frac{x \cdot p \cdot x \cdot p}{x \cdot p \cdot x \cdot p} \quad x \cdot p \cdot x \cdot p}_{Cut}$$

$$\frac{x \cdot x}{x \cdot p \cdot x \cdot p} \quad x \cdot p \cdot x \cdot p}_{Cut}$$

$$\frac{x \cdot p \cdot x \cdot p}{x \cdot p \cdot x \cdot p} \quad x \cdot p \cdot x \cdot p}_{Cut}$$

When is  $\pi_1$  the same proof as  $\pi_2$  (revisited)?

$$\frac{z \cdot z}{z \cdot p \times z \cdot p} \vee_{R} \qquad \frac{p \wedge q}{p} \wedge_{E} \qquad \frac{z \cdot z}{Fx \cdot z} \vee_{R}$$

$$\frac{z \cdot p \times y \cdot p \vee q}{Fx \cdot Ly} \wedge_{L} \qquad \frac{p \wedge q}{p \vee q} \vee_{I} \qquad \frac{z \cdot p \times z \cdot p}{Fx \cdot z} \vee_{R}$$

$$\frac{x \cdot p \wedge q \times y \cdot p \vee q}{x \cdot p \wedge q \times y \cdot p \vee q} \wedge_{L} \qquad \frac{x \cdot p \wedge q \times z \cdot p}{x \cdot p \wedge q \times y \cdot p \vee q} \wedge_{L}$$

$$\frac{w \cdot w}{w \cdot q \times w \cdot q} \vee_{R} \qquad \frac{p \wedge q}{q} \wedge_{E} \qquad \frac{w \cdot q \times w \cdot q}{Sx \cdot w} \vee_{R}$$

$$\frac{w \cdot q \times w \cdot q}{x \cdot p \wedge q \times y \cdot p \vee q} \wedge_{L} \qquad \frac{w \cdot q \times w \cdot q}{x \cdot p \wedge q \times w \cdot q} \wedge_{L}$$

$$x \cdot p \wedge q \times y \cdot p \vee q$$

$$x \cdot p \wedge q \times y \cdot p \vee q$$
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#### When is $\pi_1$ the same proof as $\pi_2$ (revisited)?

$$\frac{p \vee q \quad \frac{[p]^1}{q \vee p} \vee I \quad \frac{[q]^1}{q \vee p} \vee I}{\frac{q \vee p}{(q \vee p) \vee r} \vee I} \vee E^1$$

$$\frac{x \cdot p \times x \cdot p}{x \cdot Rz} \vee R \qquad \frac{y \cdot y}{y \cdot q \times y \cdot q} \vee R$$

$$\frac{x \cdot p \times z \cdot q \vee p}{x \cdot Rz} \qquad y \cdot q \times z \cdot q \vee p$$

$$\frac{Lw \cdot Rz \quad Rw \cdot Lz}{w \cdot p \vee q \times z \cdot q \vee p} \vee R$$

$$\frac{w \cdot p \vee q \times z \cdot q \vee p}{Lw \cdot RLu \quad Rw \cdot LLu} \vee R \qquad VR$$

$$w \cdot p \vee q \times u \cdot (q \vee p) \vee r$$

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#### When is $\pi_1$ the same proof as $\pi_2$ (revisited)?

$$\frac{\frac{[p]^{1}}{q \vee p} \vee I}{(q \vee p) \vee r} \vee I \frac{\frac{[q]^{1}}{q \vee p} \vee I}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x}{(q \vee p) \vee r} \vee I \frac{y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x}{(q \vee p) \vee r} \vee I \frac{y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x}{(q \vee p) \vee r} \vee I \frac{y \wedge y \wedge y}{(q \vee p) \vee r} \vee I \frac{y \wedge y \wedge y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x \wedge y \wedge y \wedge y}{(q \vee p) \vee r} \vee I \frac{y \wedge y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x \wedge y \wedge y}{(q \vee p) \vee r} \vee I \frac{y \wedge y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x \wedge y \wedge y}{(q \vee p) \vee r} \vee I \frac{y \wedge y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x \wedge y \wedge y}{(q \vee p) \vee r} \vee I \frac{y \wedge y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x \wedge y \wedge y}{(q \vee p) \vee r} \vee I \frac{y \wedge y}{(q \vee p) \vee r} \vee I$$

$$\frac{x \wedge x \wedge y \wedge y}{(q \vee p) \vee r} \vee I \frac{y \wedge y}{(q \vee p) \vee r} \vee I$$

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#### Sequentialisable Preterms

DEFINITION: A preterm is SEQUENTIALISABLE iff it is the conclusion of some derivation.

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## **TERMS**

#### Nonsequentialisable Preterms

$$\begin{array}{ccc} & Lx ^Fy & Rx ^Sy \\ \mathbf{x} \colon \mathbf{p} \bigvee \mathbf{q} \succ \mathbf{y} \colon \mathbf{p} \bigwedge \mathbf{q} \end{array}$$

This is connected, but it is not connected *enough*.

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#### Switching Example

$$x: p \lor q \succ y: p \land q$$

$$x: p \lor - \succ y: p \land -$$

$$x: p \lor - \succ y: - \land q$$

$$\begin{array}{ccc} & \text{Jx} & \text{Fy} & \text{Rx} & \text{Sy} \\ x : - & y : p \land - \end{array}$$

$$x: - \bigvee q \succ y: - \bigwedge q$$

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#### Switchings

- The switchings of a preterm π are found by selecting for each pair of subterms Ln and Rn in input position; Fn and Sn in output position, An in output position and Cn in input position; or the cut point (in both input and output position), one item of the pair to keep, and the other to DELETE.
- A LINKING in a switching of a preterm π survives if and only if neither side of the link involves a deletion.
- ► A preterm is SPANNED if every switching has at least one surviving linking.

#### Example

Fu FLt LSu SLt RSu Rt

This has two pairs for switching:

LSu/RSu in input position. FLt/SLt in output position.

Fu FLt LSú SLt RSu Rt

Fu^FLt LSú^SŁť RSu^Rt

Fu Ftt LSu SLt RSt Rt

Fu FLt LSu Stt RSu Rt

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Terms	Theorem: Sequentialisable Preterms are Terms
DEFINITION: A preterm $\pi$ is a TERM when it is SPANNED.	By induction on the derivation sequentialising $\pi$ .
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Sequentialisable Preterms are Terms: Identity	Sequentialisable Preterms are Terms: Conjunction
$\Sigma, x: A \succ y: A, \Delta$	$\frac{\sum, x : A, y : B \succ \Delta}{\sum, z : A \land B \succ \Delta} \land L \qquad \frac{\sum x : A, \Delta \qquad \sum' x' : y : B, \Delta'}{\sum x : A, \Delta \qquad \sum' x' : z : A \land B} \land R$
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Sequentialisable Preterms are Terms: Negation	Sequentialisable Preterms are Terms: Disjunction
$ \frac{\sum \times \mathbf{x}: \mathbf{A}, \Delta}{\pi_{1}(Nz)} \neg L \qquad \frac{\sum \mathbf{x}: \mathbf{x} \times \mathbf{A} \times \Delta}{\pi_{1}(Nz)} \neg R $ $ \sum \mathbf{x}: \mathbf{A} \times \Delta \qquad \frac{\mathbf{x}(Nz)}{\pi_{1}(Nz)} \nabla R $ $ \sum \mathbf{x}: \mathbf{x} \times \mathbf{A} \times \Delta \qquad \sum \mathbf{x} \times \mathbf{x} \times \mathbf{A} \times \Delta $	$\frac{\sum_{, \mathbf{x}: \mathbf{A} \succeq \Delta} \sum_{z', \mathbf{y}: \mathbf{B} \succeq \Delta'}^{\pi'(\mathbf{y})}}{\sum_{, \mathbf{x}', \mathbf{z}: \mathbf{A} \lor \mathbf{B} \succeq \Delta, \Delta'} \vee_{\mathbf{L}} \frac{\sum_{x \in \mathbf{X}: \mathbf{A}, \mathbf{y}: \mathbf{B}, \Delta}^{\pi[\mathbf{x}, \mathbf{y}]}}{\sum_{x \in \mathbf{X}: \mathbf{A}, \mathbf{y}: \mathbf{B}, \Delta}} \vee_{\mathbf{R}}$
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Sequentialisable Preterms are Terms: Conditional	Sequentialisable Preterms are Terms: Cut
$\frac{\sum \succ x : A, \Delta \qquad \sum', y : B \succ \Delta'}{\sum \prod_{\substack{\pi[Az] \\ \Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L} \xrightarrow{\sum \prod_{\substack{\pi(x)[y] \\ \pi(Az)[Cz] \\ \Sigma, \Sigma', z : A \supset B, \Delta}} \supset R$	$\frac{\sum x: A, \Delta \qquad \sum', y: A \succ \Delta'}{\sum x: A, \Delta \qquad \sum', y: A \succ \Delta'} Cut$ $\sum, \sum' \succ \Delta, \Delta'$

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By induction on the number of pairs for switching in  $\pi$ . Except ...

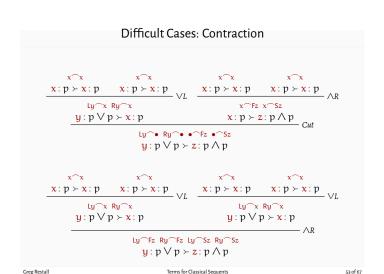


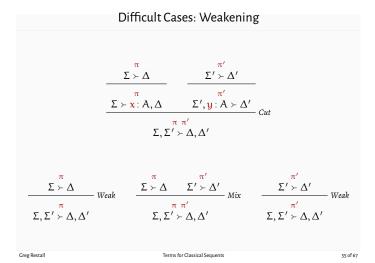
Terms for Classical Sequents

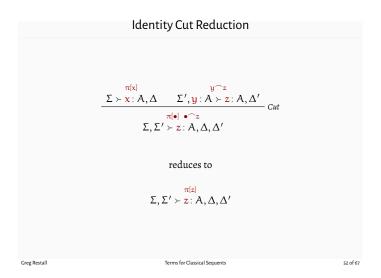
## **ELIMINATING CUTS**

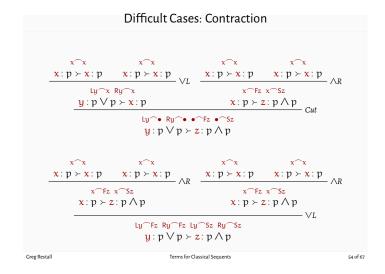
## **Conjunction Cut Reduction** $\begin{array}{ccc} \pi[\mathsf{F}\bullet] & \pi'[\mathsf{S}\bullet] & \pi''(\mathsf{F}\bullet,\mathsf{S}\bullet) \\ \Sigma,\Sigma',\Sigma'' \succ \Delta,\Delta',\Delta'' \end{array}$ reduces to $\pi''(\mathfrak{u},\mathfrak{v})$ $\Sigma' \succeq \underline{y} : B, \Delta \qquad \Sigma'', \underline{u} : A, \underline{v} : B \succeq \Delta''$ Cut $\Sigma', \Sigma'', \mathbf{u} : A \succ \Delta', \Delta''$ $\pi[*]$ $\pi'[\star]$ $\pi''(*,\star)$ $\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''$ Terms for Classical Sequents

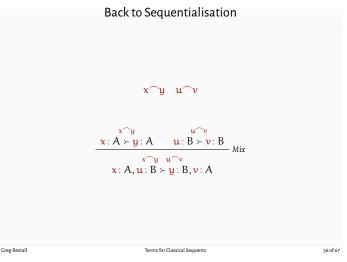
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#### Terms with Switchings

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By induction on the number of switched pairs.

Take a switched pair at the *adjacent to variables* or *cut points* (peel away unswitched steps if there aren't any).

$$\frac{\sum_{\boldsymbol{\lambda}}^{\pi[\boldsymbol{x}](-)} \qquad \pi[-](\boldsymbol{y})}{\sum_{\boldsymbol{\lambda}} \boldsymbol{x} : \boldsymbol{A}, \boldsymbol{\Delta} \qquad \sum_{\boldsymbol{\lambda}}^{\prime}, \boldsymbol{y} : \boldsymbol{B} \succ \boldsymbol{\Delta}^{\prime}} \supset L}{\sum_{\boldsymbol{\lambda}} \sum_{\boldsymbol{\lambda}}^{\pi[\boldsymbol{Az}](Lz)}} \supset L$$

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#### **Cut Reductions**

Given a term  $\pi(\bullet)[\bullet]$  and a cut-point  $\bullet$ , the  $\bullet$ -REDUCTION of  $\pi$  is found by:

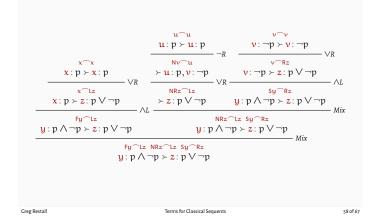
- ► *atomic*: replace each pair n and m by n m.
- conjunction: for each F•/S•, add new cut points \* and \*. For any n add l(n) for each link l(•) with n as input. For any n add l[n] for each link l[•] with n as output.

$$Sz \cap F \bullet Fz \cap S \bullet F \bullet \cap Sx \quad S \bullet \cap Fx \quad Ny \cap \bullet \bullet \cap v$$
 $Sz \cap \star Fz \cap \star \star \cap Sx \quad \star \cap Fx \quad Fy \cap Sx \quad SNy \cap Fx \quad Ny \cap v \quad Sz \cap Fv \quad Fz \cap Sv$ 

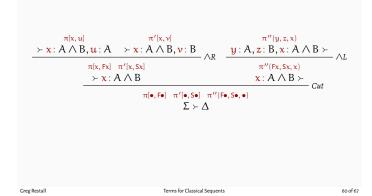
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# STRONG NORMALISATION

#### Terms with No Switchings: Example



#### Back to Eliminating Cuts: Cuts can be Complicated



#### **Cut Reductions**

Given a term  $\pi(\bullet)[\bullet]$  and a cut-point  $\bullet$ , the  $\bullet$ -REDUCTION of  $\pi$  is found by:

- ► *atomic*: replace each pair n and m by n m.
- conjunction: for each F•/S•, add new cut points \* and \*. For any n add l(n) for each link l(•) with n as input. For any n add l[n] for each link l[•] with n as output.
- negation: for each N•, add a new cut point \*. For any n add l(n) for each link l(•) with n as input. For any n add l(n) for each link l[•] with n as output.
- disjunction: for each Le/Re, add new cut points \* and \*. For any n add l(n) for each link l(•) with n as input. For any n add l[n] for each link l[•] with n as output.
- conditional: for each A•/C•, add new cut points \* and \*. For any n add l(n) for each link l(•) with n as input. For any n add l[n] for each link l[•] with n as output.

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#### Any reduction for $\pi$ terminates in a unique\* term $\pi^*$

- ► There is *some* terminating reduction process.
- ► Proof reduction is confluent.
- If  $\pi \leadsto_{\bullet} \pi'$  and  $\pi \leadsto_{\star} \pi''$  then there is a  $\pi'''$  where  $\pi' \leadsto_{\star} \pi'''$  and  $\pi'' \leadsto_{\bullet} \pi'''$ . (This is where  $\alpha$  equivalence is required.)

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## FURTHER WORK

# THANK YOU!

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#### To Do List

- ► Are these genuine *invariants*? (Can we show that if two derivations have the same term, some set of permutations permute one to the other?)
- ► Apply these terms to other kinds of proofs (Fitch, Lemmon, tableaux, Hilbert, resolution...)
- ► Categories (The class of single input, single output terms with composition by defined by Cut + reduction is a category. What are its properties?)
- ► Apply terms to theories of warrants.
- ► Extend beyond propositional logic.

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