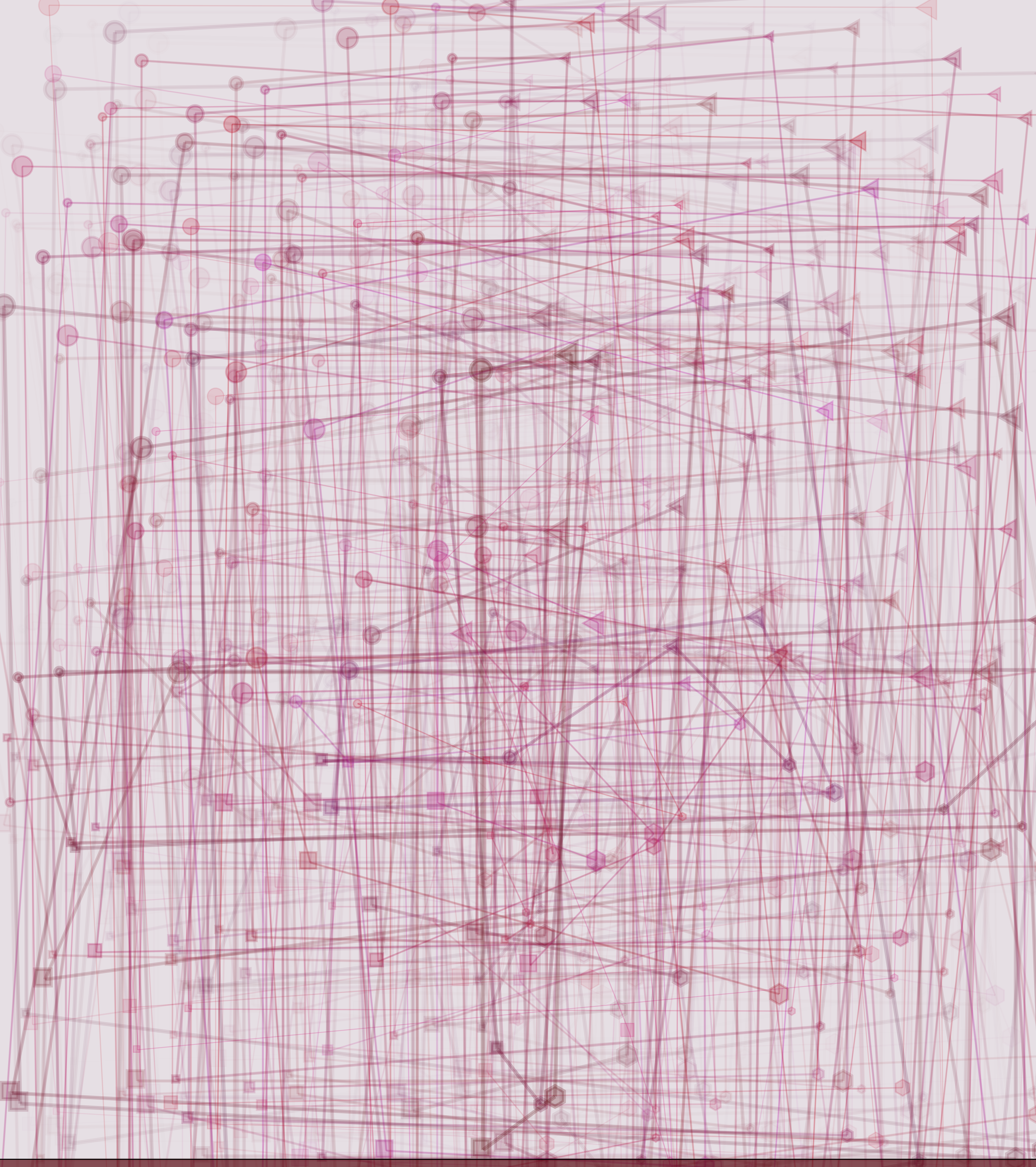

PROOF THEORY & PHILOSOPHY

Greg Restall

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ACKNOWLEDGEMENTS

This manuscript has been a very long time coming. Many people have contributed to my understanding of these topics over the last fifteen years. My debts are plentiful.

[Acknowledgements of specific people to come.]

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WHERE TO BEGIN

INTRODUCTION

This is a draft of a monograph on proof theory and philosophy. The focus will be a detailed examination of the different ways to understand *proof*, and how understanding the norms governing logical vocabulary can give us insight into questions in the philosophy of language, epistemology and metaphysics. Along the way, we will also take a few glances around to the other side of logical consequence, the kinds of *counterexamples* to be found when an deduction fails to be valid.

I should like to outline an image which is connected with the most profound intuitions which I always experience in the face of logistic. That image will perhaps shed more light on the true background of that discipline, at least in my case, than all discursive description could. Now, whenever I work even on the least significant logistic problem, for instance, when I search for the shortest axiom of the implicational propositional calculus I always have the impression that I am facing a powerful, most coherent and most resistant structure. I sense that structure as if it were a concrete, tangible object, made of the hardest metal, a hundred times stronger than steel and concrete. I cannot change anything in it; I do not create anything of my own will, but by strenuous work I discover in it ever new details and arrive at unshakable and eternal truths. Where is and what is that ideal structure? A believer would say that it is in God and is His thought. — Jan Łukasiewicz

The book is designed to serve a number of different purposes, and it can be used in a number of different ways. In writing the book I have two distinct aims in mind.

GENTLY INTRODUCING KEY IDEAS IN PROOF THEORY FOR PHILOSOPHERS: There are a number of very good introductions to proof theory: Bostock's *Intermediate Logic* [14], Tennant's *Natural Logic* [95], Troelstra and Schwichtenberg's *Basic Proof Theory* [97], and von Plato and Negri's *Structural Proof Theory* [61] are all excellent books, with their own virtues. However, they all introduce the core ideas of proof theory in what can only be described as a rather complicated fashion. The core *technical* results of proof theory (normalisation for natural deduction and cut elimination for sequent systems) are *relatively* simple ideas at their heart, but the expositions of these ideas in the available literature are quite difficult and detailed. This is through no fault of the existing literature. It is due to a choice. In each book, a proof system for the whole of classical or intuitionistic logic is introduced, and then, formal properties are demonstrated about such a system. Each proof system has different rules for each of the connectives, and this makes the proof-theoretical results such as normalisation and cut elimination case-ridden and lengthy. (The standard techniques are complicated inductions with different cases for each connective: the more connectives and rules, the more cases.)

In this book, the exposition will be rather different. Instead of taking a proof system as given and proving results about *it*, we will first look at the core ideas (normalisation for natural deduction, and cut elimination for sequent systems) and work with them in their simplest and purest manifestation. In Section 1.3 we will see a two-page normalisation proof. In Section 2.2 we will see a two-page cut-elimination proof. In each case, the aim is to understand the key concepts behind the central results. Then, we show how these results can be generalised to a much more abstract setting, in which they can be applied to a wide range of logical systems, and once we have established these general results, we apply them to specific systems of interest, including first order predicate logic, propositional modal and temporal logics, and quantified modal logics.

EXPLORING THE CONNECTIONS BETWEEN PROOF THEORY AND PHILOSOPHY: The central part of the book (Chapters 4 to 6) answer a central question in philosophical proof theory: When do inference rules define a logical concept? The first part of the book (Chapters 1 to 3) introduces the tools and techniques needed to both *understand* and to *address* the question. The central part of the book formulates the problem and offers a distinctive solution to it. A very particular kind of inference rule (a rule we will describe as a *defining rule*) defines a concept satisfying some very natural conditions—and there are good reasons to think of concepts satisfying these conditions as properly *logical* concepts. Then the remainder of the book (from Chapter 7) explores consequences and applications of these ideas for particular issues in logic, language, epistemology and metaphysics. Alone the way, we will explore the connections between proof theories and theories of meaning. What does this account of proof tell us about how we might *apply* the formal work of logical theorising? All accounts of meaning have something to say about the role of inference. For some, it is what things *mean* that tells you what inferences are appropriate. For others, it is what inferences are appropriate that helps constitute what particular words *mean*. For everyone, there is an intimate connection between inference and semantics.

The precise defintion is spelled out, along with its consequences, in Chapter 6.

I have in mind the distinction between *representationalist* and *inferentialist* theories of meaning. For a polemical and provocative account of the distinction, see Robert Brandom's *Articulating Reasons* [15].

The book includes marginal notes that expand on and comment on the central text. Feel free to read or ignore them as you wish, and to add your own comments. Each chapter (other than this one) contains definitions, examples, theorems, lemmas, and proofs. Each of these (other than the proofs) are numbered consecutively, first with the chapter number, and then with the number of the item within the chapter. Proofs end with a little box at the right margin, like this: ■

The manuscript is divided into three parts, each of which is divided into chapters. The first part, *Tools*, covers the basic concepts, arguments and results which we will use throughout the book. These chapters can be used as a gentle introduction to proof theory for anyone who is interested in the field, perhaps supplemented by (or supplementing) one or more of the texts mentioned earlier in this chapter. The second part, *The Core Argument*, introduces Prior's puzzle concerning inference rules and definitions, and presents and defends a distinct answer to that question. The answer takes the form of an *argument*, to the effect that a particular kind of rule—what I call a *defining rule*—can be used to introduce a logical concept into a discourse, and shows that this concept in an important sense both *free to add*, and *sharply delineated*. The third part, *Insights* then draws out the consequences of this argument to different kinds of logical concepts (the connectives, quantifiers, identity, and modal operators) and for different issues in the philosophy of language, epistemology, metaphysics and the philosophy of mathematics.

A slogan: A logical concept is one that can be introduced by means of a *defining rule*.

In addition to these three major parts, the book contains a small introduction designed to set the scene (this chapter) and a coda, which points forward to issues to be explored in the future.

Some chapters in the *Tools* section contain exercises to complete. Logic is never learned without hard work, so if you want to *learn* the material, work through the exercises: especially the basic and intermediate exercises, which should be taken as a guide to mastery of the techniques we discuss. The advanced exercises are more difficult, and should be dipped into as desired, in order to truly gain expertise in these tools and techniques. The *project* questions are examples of current research topics.

The book has an accompanying website: `http://consequently.org/writing/ptp`. From here you can look for an updated version of the book, leave comments, read the comments others have left, check for solutions to exercises and supply your own. Please visit the website and give your feedback. Visitors to the website have already helped me make this volume much better than it would have been were it written in isolation. It is a delight to work on logic within such a community, spread near and far.

MOTIVATION

Why? My first and overriding reason to be interested in proof theory is the beauty and simplicity of the subject. It is one of the central strands of the discipline of logic, along with its partner, model theory. Since the flowering of the field with the work of Gentzen, many beautiful definitions, techniques and results are to be found in this field, and they deserve a wider audience. In this book I aim to provide an introduction to proof theory that allows the reader with only a minimal background in logic to start with the flavour of the central results, and then understand techniques in their own right.

It is one thing to be interested in proof theory in its own right, or as a part of a broader interest in logic. It's another thing entirely to think that proof theory has a role in philosophy. Why would a *philosopher* be interested in the theory of proofs? Here are just three examples of concerns in philosophy where proof theory finds a place.

EXAMPLE 1: MEANING. Suppose you want to know when someone is using "or" in the same sense that you do. When does "or" in their vocabulary have the same significance as "or" in yours? One answer could be given in terms of *truth-conditions*. The significance of "or" can be given in a rule like this one:

Perhaps you've heard of the difference between 'inclusive' and 'exclusive' disjunction. And maybe you're worried that 'or' can be used in many ways, each meaning something different.

$$\ulcorner p \text{ or } q \urcorner \text{ is true if and only if } p \text{ is true or } q \text{ is true.}$$

Perhaps you have seen this information presented in a truth-table.

p	q	p or q
0	0	0
0	1	1
1	0	1
1	1	1

Clearly, this table can be used to distinguish between some uses of disjunctive vocabulary from others. We can use it to rule out *exclusive* disjunction. If we take $\ulcorner$p or q$\urcorner$ to be *false* when we take p and q to be both true, then we are using "or" in a manner that is at odds with the truth table.

However, what can we say of someone who is ignorant of the truth or falsity of p and of q? What does the truth table tell us about $\ulcorner$p or q$\urcorner$ in that case? It seems that the application of the truth table to our *practice* is less-than-straightforward.

It is for reasons like this that people have considered an alternate explanation of a logical connective such as "or." Perhaps we can say that someone is using "or" in the way that you do if you are disposed to make the following deductions to reason *to* a disjunction

$$\frac{p}{p \text{ or } q} \qquad \frac{q}{p \text{ or } q}$$

and to reason *from* a disjunction

$$\frac{p \text{ or } q \quad \begin{matrix}[p] \\ \vdots \\ r\end{matrix} \quad \begin{matrix}[q] \\ \vdots \\ r\end{matrix}}{r}$$

That is, you are prepared to infer *to* a disjunction on the basis of either disjunct; and you are prepared to reason by cases *from* a disjunction. Is there any more you need to do to fix the use of "or"? That is, if you and I both use "or" in a manner consonant with these rules, then is there any way that our usages can differ with respect to *meaning*?

Clearly, this is not the end of the story. Any proponent of a *proof*-first explanation of the meaning of a word such as "or" will need to say something about what it is to accept an inference rule, and what sorts of inference rules suffice to define a concept such as disjunction (or negation, or universal quantification, and so on). When does a definition work? What are the sorts of things that can be defined using inference rules? What are the sorts of rules that may be used to define these concepts? We will consider these issues in Chapter 6.

EXAMPLE 2: GENERALITY. It is a commonplace that it is impossible or very difficult to *prove* a nonexistence claim. After all, if there is *no* object with property F, then *every* object fails to have property F. How can we demonstrate that every object in the entire universe has some property? Surely we cannot survey each object in the universe one-by-one. Furthermore, even if we come to believe that object a has property F for each object a that happens to exist, it does not follow that we ought to believe that *every* object has that property. The universal judgement tells us more than the truth of each particular instance of that judgement, for given all of the objects $a_1, a_2, \ldots$, it certainly seems *possible* that a_1 has property F, that a_2 has property F and so on, without *everything* having property F since it seems possible that there might be some *new* object which does not *actually* exist. If you care to talk of 'facts' then we can express the matter by saying that the fact that everything is F cannot amount to just the fact that a_1 is F and the fact that a_2 is F, etc., it must also include the fact that $a_1, a_2, \ldots$ are all of the objects. There seems to be some irreducible *universality* in universal judgements.

If this was all that we could say about universality, then it would be very difficult to come to universal conclusions. However, we seem to manage to derive universal conclusions regularly. Consider mathematics: it is not difficult to prove that *every* whole number is either even or odd. We can do this without examining every number individually. Just how do we do this?

It is a fact that we *do* accomplish this, for we are able to come to universal conclusions as a matter of course. In the course of this book we will see how such a thing is possible. Our facility at reasoning with *quantifiers*, such as 'for every' and 'for some,' is intimately tied up with the structures of the claims we can make, and how the formation of judgements from *names* and *predicates* gives us a foothold which may be exploited in reasoning. When we understand the nature of proofs involving quantifiers, this will give us insight into how we can gain *general* information about our world.

It is one thing to know that $2 + 3 = 3 + 2$. It is quite another to conclude that for *every* pair of natural numbers n and m that $n+m = m+n$. Yet we do this sort of thing quite regularly.

example 3: modality. A third example is similar. Philosophical discussion is full of talk of *possibility* and *necessity*. What is the significance of this talk? What is its logical structure? One way to give an account of the logical structure of possibility and necessity talk is to analyse it in terms of possible worlds. To say that it is possible that Australia win the World Cup is to say that there is some possible world in which Australia wins the World Cup. Talk of possible worlds helps clarify the logical structure of possibility and necessity. It is possible that either Australia or New Zealand win the World Cup only if there's a possible world in which either Australia or New Zealand win the World Cup. In other words, either there's a possible world in which Australia wins, or a possible world in which New Zealand wins, and hence, it is either possible that Australia wins the World Cup or that New Zealand wins. We have reasoned from the possibility of a disjunction to the disjunction of the corresponding possibilities. Such an inference seems correct. Is talk of possible worlds required to explain this kind of derivation, or is there some other account of the logical structure of possibility and necessity? If we agree with Arthur Prior that we understand possible worlds be-

"… possible worlds, in the sense of possible states of affairs are not *really* individuals (just as numbers are not *really* individuals). To say that a state of affairs obtains is just to say that something is the case; to say that something is a possible state of affairs is just to say that something could be the case; and to say that something is the case 'in' a possible state of affairs is just to say that the thing in question would necessarily be the case if that state of affairs obtained, i. e. if something else were the case … We understand 'truth in states of affairs' because we understand 'necessarily'; not *vice versa*." — Arthur Prior, *Worlds, Times and Selves* [72].

cause we understand the concepts of possibility and necessity, then it's incumbent on us to give some explanation of how we come to understand those concepts—and how they come to have the structure that makes talk of possible worlds appropriate. I will argue in this book that when we attend to the structure of proofs involving modal notions, we will see how this use helps determine the concepts of necessity and possibility, and this *thereby* gives us an understanding the notion of a possible world. We don't first understand modal concepts by invoking possible worlds—we can invoke possible worlds when we first understand modal concepts, and the logic of modal concepts can be best understood when we understand what modal reasoning is *for* and how we do it.

EXAMPLE 4: A NEW ANGLE ON OLD IDEAS Lastly, one reason for studying proof theory is the perspective it brings on familiar themes. There is a venerable and well-trodden road between truth, models and logical consequence. Truth is well-understood, models (truth tables for propositional logic, or Tarski's models for first-order predicate logic, Kripke models for modal logic, or whatever else) are taken to be models of truth, and logical consequence is understood as the preservation of truth in all models. Then, some proof system is designed as a way to give a tractable account of that logical consequence relation. Nothing in this book will count as an argument *against* taking that road from truth, through logical consequence, to proof. However, we will travel that road in the other direction. By starting with proofs we will retrace those steps in reverse, to construct *models* from a prior understanding of proof, and then with an approach to *truth* once we have a notion of a model in hand. This is a very different way to chart the connection between proof theory and model theory. At the very least, tackling this terrain from that angle will allow us to take a different perspective on some familiar ground, and will give us the facility to offer new answers to some perennial questions about meaning, metaphysics and epistemology. Perhaps, when we see matters from this new perspective, the insights will be of lasting value.

However, the notion of truth is beset by paradox, and this should at least serve as a warning sign. Using the notion of truth as a starting point to define core features of logic may not provide the most stable foundation. It is at least worth exploring different approaches.

These are four examples of the kinds of issues that we will consider in the light of proof theory in the pages ahead. To broach these topics, we need to learn some proof theory, so let's dive in.

PART I

Tools

NATURAL DEDUCTION 1

We start with modest ambitions. In this section we focus on one way of understanding proof—natural deduction, in the style of Gentzen [30]—and we will consider just one kind of judgement: *conditionals*.

Gerhard Gentzen, German Logician: Born 1909, student of David Hilbert at Göttingen, died in 1945 in World War II. http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Gentzen.html

1.1 | CONDITIONALS

Conditional judgements have this shape

If . . . then . . .

where we can fill in both ". . ." with other judgements. Conditional judgements are a useful starting point for thinking about logic and proof, because conditionals play a central role in our thinking and reasoning, in reflection and in dialogue. If we move beyond judgements about what is the case to reflect on how our judgements hang together and stand with regard to one another, it is very natural to form *conditional* judgements. You may not want to claim that the Number 58 tram is about to arrive, but you may at be in a position to judge that *if the timetable is correct*, the Number 58 tram is about to arrive. This is a conditional judgement, with the antecedent "the timetable is correct," and consequent "the Number 58 tram is about to arrive."

"The aim of philosophy, abstractly formulated, is to understand how things in the broadest possible sense of the term hang together in the broadest possible sense of the term." — Wilfrid Sellars [87]

In the study of *formal* logic, we focus on the *form* or *structure* of judgements. One aspect of this involves being precise and attending to those structures and shapes in some detail. We will start this by defining a grammar for conditional judgements. Any grammar has to start somewhere, and we will start with labels for atomic judgements—those judgements which aren't themselves conditionals, but which can be used to build conditionals. We'll use the letters p, q and r for these atoms, and if they're not enough, we'll use numerical subscripts to make more—that way, we never run out.

$$p,\ q,\ r,\quad p_0,\ p_1,\ p_2,\ldots\quad q_0,\ q_1,\ q_2,\ldots\quad r_0, r_1,\ r_2,\ldots$$

Each of these formulas is an ATOM. Whenever we have two formulas A and B, whether A and B are ATOMs or not, we will say that $(A \to B)$ is also a formula. In other words, given two judgements, we can (at least, in theory) form the conditional judgement with the first as the antecedent and the second as consequent. Succinctly, this *grammar* can be represented as follows:

This is BNF, or "Backus Naur Form," first used in the specification of formal computer programming languages such as ALGOL. http://cui.unige.ch/db-research/Enseignement/analyseinfo/AboutBNF.html

$$\text{FORMULA} ::= \text{ATOM} \mid (\text{FORMULA} \to \text{FORMULA})$$

That is, a FORMULA is either an ATOM, or is found by placing an arrow (written like this '$\to$') between two FORMULAs, and surrounding the result with parentheses.

So, the next line contains four different formulas

$$p_3 \quad (q \to r) \quad ((p_1 \to (q_1 \to r_1)) \to (q_1 \to (p_1 \to r_1))) \quad (p \to (q \to (r \to (p_1 \to (q_1 \to r_1)))))$$

but these are not formulas:

$$t \qquad p \to q \to r \qquad p \to p$$

The first, t, fails to be a formula since is not in our set ATOM of atomic formulas (so it doesn't enter the collection of formulas by way of being an atom) and it does not contain an arrow (so it doesn't enter the collection through the clause for complex formulas). The second, $p \to q \to r$ does not enter the collection because it is short of a few parentheses. The only expressions that enter *our* language are those that bring a pair of parentheses along with every arrow: "$p \to q \to r$" has two arrows but no parentheses, so it does not qualify. You can see why it *should* be excluded because the expression is ambiguous. Does it express the conditional judgement to the effect that if p then if q then r, or is it the judgement that if it's true that if p then q, then it's also true that r? In other words, it is ambiguous between these two formulas:

You can do without parentheses if you use 'prefix' notation for the conditional: 'Cpq' instead of '$p \to q$'. The conditional are then $CpCqr$ and $CCpqr$. This is *Polish notation*.

$$(p \to (q \to r)) \quad ((p \to q) \to r)$$

We really need to distinguish these two judgements, so we make sure our formulas contain parentheses. Our last example of an offending non-formula, $p \to p$, does not offend nearly so much. It is not ambiguous. It merely offends against the letter of the law laid down, and not its spirit. I will feel free to use expressions such as "$p \to p$" or "$(p \to q) \to (q \to r)$" which are missing their outer parentheses, even though they are, strictly speaking, not FORMULAS.

If you like, you can think of them as including their outer parentheses very *faintly*, even more faintly than this: $((p \to q) \to (q \to r))$.

Given a formula containing at least one arrow, such as $(p \to q) \to (q \to r)$, it is important to be able to isolate its main connective (the last arrow introduced as it was constructed). In this case, it is the middle arrow. The formula to the left of the arrow (in this case $p \to q$) is said to be the *antecedent* of the conditional, and the formula to the right is the *consequent* (here, $q \to r$).

We can think of formulas generated in this way in at least two different ways. We can think of them as the sentences in a very simple language. This language is either something completely separate from our natural languages, or it is a fragment of a natural language, consisting only of atomic expressions and the expressions you can construct using a conditional construction like "if . . . then . . .".

On the other hand, you can think of formulas as not constituting a language in themselves, but as constructions used to display the *form* of expressions in a language. Both of these interpretations of this syntax are open to us, and everything in this chapter (and in much of the rest of the book) is written with both interpretations in mind. Formal languages can be used to describe the forms of different languages, and they can be thought to be languages in their own right.

The issue of interpreting the formal language raises another question: What is the relationship between languages (formal or informal) and the judgements expressed in those languages? This question is not unlike the question concerning the relationship between a name and the bearer of that name, or a term and the thing (if anything) denoted by that term. The numeral '2' is not to be identified with number 2, and the formula $p \to q$ (or a sentence with that shape) is not the same as the conditional judgement expressed by that formula. Talk of judgements is itself ambiguous between the act of judging (my act of judging that the Number 58 tram is coming soon is not the same act as your act of judging this), and the *content* of any such act. When it comes to interpreting and applying the formal language of logic, it is important to reflect on not only the languages that you and I might speak (or write, or use in computer programs, etc.) but also attend to the *content* expressed when we use such languages [98].

The term "$1 + 1$" to be identified with the numeral "2", though both denote the same number. One term contains the numeral "1" and the other doesn't.

»» ««

Often, we will want to talk quite generally about all formulas with a given shape. We do this very often, when it comes to logic, because we are interested in the forms of valid arguments. The structural or formal features of arguments apply generally, to more than just a particular argument. (If we know that an argument is valid in virtue of its possessing some particular form, then other arguments with that form are valid as well.) So, these formal or structural principles must apply *generally*. Our formal language goes some way to help us express this, but it will turn out that we will not want to talk merely about specific formulas in our language, such as $(p_3 \to q_7) \to r_{26}$. We will, instead, want to say things like

> A *modus ponens* inference is the inference from a conditional formula and the antecedent of that conditional, to its consequent.

This can get very complicated very quickly. It is not easy to understand

> *Given a conditional formula whose consequent is also a conditional, the conditional formula whose antecedent is the antecedent of the consequent of the original conditional, and whose consequent is a conditional whose antecedent is the antecedent of the original conditional and whose consequent is the consequent of the conditional inside the first conditional follows from the original conditional.*

Instead of that mouthful, we will use *variables* to talk generally about formulas in much the same way that mathematicians use variables to talk generally about numbers and other such things. We will use capital letters, such as

$$A, B, C, D, \ldots$$

as variables ranging over the FORMULAS. So, instead of the long paragraph above, it suffices to say

Number theory books don't often include lots of *numerals*. Instead, they're filled with *variables* like 'x' and 'y.' This isn't because these books aren't about numbers. They are, but they don't merely list *particular* facts about numbers. They talk about *general* features of numbers, and hence the use of variables.

From $A \to (B \to C)$ *you can infer* $B \to (A \to C)$.

which seems much more perspicuous and memorable. The letters A, B and C aren't any *particular* formulas. They each can stand in for any formula at all.

Now we have the raw formal materials to address the question of deduction using conditional judgements. How may we characterise proofs reasoning using conditionals? That is the topic of the next section.

1.2 | PROOFS FOR CONDITIONALS

Start with some of reasoning using conditional judgements. One example might be reasoning of this form:

> *Suppose* $A \to (B \to C)$. *Suppose* A. *It follows that* $B \to C$. *Suppose* B. *It follows that* C.

This kind of reasoning has two important features. We make *suppositions*. We also infer *from* these suppositions. From $A \to (B \to C)$ and A we inferred $B \to C$. From this new information, together with the supposition that B, we inferred a new conclusion, C.

One way to represent the structure of this piece of reasoning is in this *tree diagram* shown here

$$\dfrac{\dfrac{A \to (B \to C) \qquad A}{B \to C} \qquad B}{C}$$

The *leaves* of the tree are the formulas $A \to (B \to C)$, A and B. They are the assumptions upon which the deduction rests. The other formulas in the tree are *deduced* from formulas occurring above them in the tree. The formula $B \to C$ is written immediately below a line, above which are the formulas from which we deduced it. So, $B \to C$ didn't have to be supposed. It *follows from* the leaves $A \to (B \to C)$ and A. Then the *root* of the tree (the formula at the bottom), C, follows from that formula $B \to C$ and the other leaf B. The ordering of the formulas bears witness to the relationships of inference between those formulas in our process of reasoning.

The two steps in our example proof use the same kind of reasoning. The inference from a conditional, and from its antecedent to its consequent. This step is called *modus ponens*. It's easy to see that using *modus ponens* we always move from more complicated formulas to less complicated formulas. However, sometimes we wish to infer the conditional $A \to B$ on the basis of our information about A and about B. And it seems that sometimes this is legitimate. Suppose we want to know about the connection between A and C in a context in which we are happy to grant both $A \to (B \to C)$ and B. What kind of connection is there (if any) between A and C? It would seem that it would be appropriate to infer $A \to C$, since we can derive C if we are willing to grant

"*Modus ponens*" is short for "*modus ponendo ponens*," which means "the mode of affirming by affirming." You get to the affirmation of B by way of the affirmation of A (and the other premise, $A \to B$). It may be contrasted with *Modus tollendo tollens*, the mode of denying by denying: from $A \to B$ and *not* B to *not* A.

A as an assumption. In other words, we have the means to conclude C from A, using the other resources we have already granted. But what does the conditional judgement $A \to C$ say? That if A, then C. So we can make that explicit and conclude $A \to C$ from that reasoning. We can represent the structure of chain of reasoning in the following way:

$$\dfrac{\dfrac{\dfrac{A \to (B \to C) \quad [A]^{(1)}}{B \to C} \qquad B}{C}}{A \to C}\,[1]$$

This proof can be read as follows: At the step marked with [1], we make the inference to the *conditional* conclusion, on the basis of the reasoning up until that point. Since we can conclude C *using* A as an assumption, we can make the further conclusion $A \to C$. At this stage of the reasoning, A is no longer active as an assumption: we *discharge* it. It is still a leaf of the tree (there is no node of the tree above it), but it is no longer an active assumption in our reasoning. So, at this stage we bracket it, and annotate the brackets with a label, indicating the point in the demonstration at which the assumption is discharged. Our proof now has two assumptions, $A \to (B \to C)$ and B, and one conclusion, $A \to C$.

$$\dfrac{A \to B \qquad A}{B}\,{\scriptstyle \to E} \qquad\qquad \dfrac{\begin{matrix}[A]^{(i)} \\ \vdots \\ B\end{matrix}}{A \to B}\,{\scriptstyle \to I, i}$$

Figure 1.1: natural deduction rules for conditionals

We have motivated two rules for proofs with conditionals. These rules are displayed in Figure 1.1. The first rule, *modus ponens*, or *conditional elimination* [$\to E$] allows us to step from a conditional and its antecedent to the consequent of the conditional. We call the conditional premise $A \to B$ the *major* premise of the [$\to E$] inference, and the antecedent A the *minor* premise of that inference. When we apply the inference [$\to E$], we combine two proofs: the proof of $A \to B$ and the proof of A. The new proof has as assumptions any assumptions made in the proof of $A \to B$ and also any assumptions made in the proof of A. The conclusion is B.

The major premise in a connective rule features that connective.

The second rule, *conditional introduction* [$\to I$], allows us to use a proof from A to B as a proof of $A \to B$. The assumption of A is discharged in this step. The proof of $A \to B$ has as its assumptions all of the assumptions used in the proof of B except for the instances of A that we discharge in this step. Its conclusion is $A \to B$.

Now we come to the first formal definition, giving an account of what counts as a proof in this natural deduction system for the language of conditionals.

DEFINITION 1.1 [PROOFS FOR CONDITIONALS] A proof is a *tree* consisting of formulas, some of which may be *bracketed*. The formula at the root of a proof is said to be its CONCLUSION. The unbracketed formulas at the leaves of the tree are the PREMISES of the proof.

» Any FORMULA A is a proof, with premise A and conclusion A. The formula A is not bracketed.

» If π_l is a proof, with conclusion $A \to B$ and π_r is a proof, with conclusion A, then these proofs may be combined, into the following proof,

$$\dfrac{\begin{matrix}\vdots\, \pi_l \\ A \to B\end{matrix} \qquad \begin{matrix}\vdots\, \pi_r \\ A\end{matrix}}{B}{\scriptstyle \to E}$$

which has conclusion B, and which has premises consisting of the premises of π_l together with the premises of π_r.

How do you choose the number for the label (i) on the discharged formula? Find the largest number labelling a discharge in the proof π, and then choose the next one.

» If π is a proof with conclusion B, then the following tree

$$\dfrac{\begin{matrix}[A]^{(i)} \\ \vdots\, \pi \\ B\end{matrix}}{A \to B}{\scriptstyle \to I,i}$$

is a proof with conclusion $A \to B$. Its premises are the premises of the original proof π, except for the premise A which is now discharged. We indicate this discharge by *bracketing* it.

» Nothing else is a proof.

This is a recursive definition, in just the same manner as the recursive definition of the class FORMULA. We define atomic proofs (in this case, consisting of a single formula), and then show how new (larger) proofs can be built out of smaller proofs.

$$\dfrac{\dfrac{\dfrac{[B \to C]^{(2)} \qquad \dfrac{A \to B \quad [A]^{(1)}}{B}{\scriptstyle \to E}}{C}{\scriptstyle \to E}}{A \to C}{\scriptstyle \to I,1}}{(B \to C) \to (A \to C)}{\scriptstyle \to I,2}$$

SUFFIXING (DEDUCTION)

$$\dfrac{\dfrac{\dfrac{[A \to B]^{(1)} \quad [A]^{(2)}}{B}{\scriptstyle \to E}}{(A \to B) \to B}{\scriptstyle \to I,1}}{A \to ((A \to B) \to B)}{\scriptstyle \to I,2}$$

ASSERTION (FORMULA)

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \to B]^{(3)} \qquad \dfrac{[C \to A]^{(2)} \quad [C]^{(1)}}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}}{C \to B}{\scriptstyle \to I,1}}{(C \to A) \to (C \to B)}{\scriptstyle \to I,2}}{(A \to B) \to ((C \to A) \to (C \to B))}{\scriptstyle \to I,3}$$

PREFIXING (FORMULA)

Figure 1.2: THREE IMPLICATIONAL PROOFS

Figure 1.2 gives three proofs constructed using our rules. The first is a proof from $A \to B$ to $(B \to C) \to (A \to C)$. This is the inference of *suffixing*. (We "suffix" both A and B with $\to C$.) The other proofs conclude in formulas justified on the basis of *no* undischarged assumptions. It is worth your time to read through these proofs to make sure that you understand the way each proof is constructed. A good way to understand the shape of these proofs is to try writing them out from top-to-bottom, identifying the basic proofs you start with, and only adding the discharging brackets at the stage of the proof where the discharge occurs.

You can try a number of different strategies when making proofs for yourself without copying existing ones. For example, you might like to try your hand at constructing a proof to the conclusion that $B \to (A \to C)$ from the assumption $A \to (B \to C)$. Here are two strategies you could use to piece a proof together.

TOP–DOWN: You start with the assumptions and see what you can do with them. In this case, with $A \to (B \to C)$ you can, clearly, get $B \to C$, if you are prepared to assume A. And then, with the assumption of B we can deduce C. Now it is clear that we can get $B \to (A \to C)$ if we discharge our assumptions, A first, and then B.

BOTTOM–UP: Start with the conclusion, and find what you could use to prove it. Notice that to prove $B \to (A \to C)$ you could prove $A \to C$ using B as an assumption. Then to prove $A \to C$ you could prove C using A as an assumption. So, our goal is now to prove C using A, B and $A \to (B \to C)$ as assumptions. But this is an easy pair of applications of [$\to E$].

In either case, this is the proof we construct:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (B \to C) \quad [A]^{(1)}}{B \to C}{\scriptstyle \to E} \quad [B]^{(2)}}{C}{\scriptstyle \to E}}{A \to C}{\scriptstyle \to I,1}}{B \to (A \to C)}{\scriptstyle \to I,2}$$

» «

I have been intentionally unspecific when it comes how formulas are discharged formulas in proofs. In the examples in Figure 1.2 you will notice that at each step when a discharge occurs, one and only one formula is discharged. By this I do not mean that at each [$\to I$] step a formula A is discharged and a different formula B is not. I mean that in the proofs we have seen so far, at each [$\to I$] step, a single *instance* of the formula is discharged. Not all proofs are like this. Consider this proof from the assumption $A \to (A \to B)$ to the conclusion $A \to B$. At the final step of this proof, two instances of the assumption A are discharged at once.

$$\dfrac{\dfrac{\dfrac{A \to (A \to B) \quad [A]^{(1)}}{A \to B}{\scriptstyle \to E} \quad [A]^{(1)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I,1}$$

For this to count as a proof, we must read the rule [$\to I$] as licensing the discharge of *one or more instances* of a formula in the inference to the conditional. Once we think of the rule in this way, one further generalisation comes to mind: If we think of an $[\rightarrow I]$ move as discharging a *collection* of instances of our assumption, someone of a generalising spirit will ask if that collection can be empty. Can we discharge an assumption that *isn't there*? If we can, then *this* counts as a proof:

"Yesterday upon the stair, I met a man who wasn't there. He wasn't there again today. I wish that man would go away." — Hughes Mearns

$$\frac{A}{B \rightarrow A}\ {\scriptstyle \rightarrow I,1}$$

Here, we assume A, and then, we infer $B \rightarrow A$ discharging *all* of the active assumptions of B in the proof at this point. The collection of active assumptions of B is, of course, empty. No matter, they are all discharged, and we have our conclusion: $B \rightarrow A$.

You might think that this is silly: how can you discharge a nonexistent assumption? Nonetheless, discharging assumptions that are not there plays a role. To give you a taste of why, notice that the inference from A to $B \rightarrow A$ is *valid* if we read "$\rightarrow$" as the material conditional of standard two-valued classical propositional logic. In a pluralist spirit we will investigate different policies for discharging formulas.

For more in a "pluralist spirit" see my work with Jc Beall [7, 8, 79].

DEFINITION 1.2 [DISCHARGE POLICY] A DISCHARGE POLICY may either allow or disallow *duplicate* discharge (more than one instance of a formula at once) or *vacuous* discharge (*zero* instances of a formula in a discharge step). Here are the names for the four discharge policies:

I am not happy with the label "affine," but that's what the literature has given us. Does anyone have any better ideas for this? "Standard" is not "classical" because it suffices for intuitionistic logic in this context, not classical logic. It's not "intuitionistic" because "intuitionistic" is difficult to pronounce, and it is not *distinctively* intuitionistic. As we shall see later, it's the shape of proof and not the discharge policy that gives us intuitionistic implicational logic.

		VACUOUS	
		YES	NO
DUPLICATES	YES	*Standard*	*Relevant*
	NO	*"Affine"*	*Linear*

The "standard" discharge policy is to allow both vacuous and duplicate discharge.

There are reasons to explore each of the different policies. As I indicated above, you might think vacuous discharge does not make much sense. However, we can say more than that: it seems downright *mistaken* if we are to understand a judgement of the form $A \rightarrow B$ to record the claim that B may be inferred *from* A. If A is not used in the inference to B, then we hardly have reason to think that B follows from A in this sense. So, if you are after a conditional which is *relevant* in this way, you would be interested in discharge policies that ban vacuous discharge [1, 2, 75].

There are also reasons to ban duplicate discharge: Victor Pambuccian has found an interesting example of doing without duplicate discharge in early 20th Century geometry [63]. He traces cases where geometers took care to keep track of the number of times a postulate was used in a proof. So, they draw a distinction between $A \rightarrow (A \rightarrow B)$ and $A \rightarrow B$. The judgement that $A \rightarrow (A \rightarrow B)$ records the fact that B can be deduced from two uses of A. $A \rightarrow B$ records that B can be deduced from A used only once. More recently, work in *fuzzy logic* [10, 39, 56] motivates keeping track of the number of times premises are used. If a conditional $A \rightarrow B$ fails to be true to the degree that A is truer than B, then $A \rightarrow (A \rightarrow B)$ may be truer than $A \rightarrow B$.

Finally, for some [6, 57, 70, 76], Curry's Paradox motivates banning indiscriminate duplicate discharge. If we have a claim A which both implies $A \to B$ and is implied by it then we can reason as follows:

Consider the claim I'll call (α) — *If (α) is true, then I am a monkey's uncle.*

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^{(1)}}{A \to B}\,\dagger \quad [A]^{(1)}}{B}\,{\to}E}{A \to B}\,{\to}I,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A]^{(2)}}{A \to B}\,\dagger \quad [A]^{(2)}}{B}\,{\to}E}{A \to B}\,{\to}I,2}{A}\,\dagger}{B}\,{\to}E$$

Where we have used '†' to mark the steps where we have gone from A to $A \to B$ or back. Notice that this is a proof of B from *no premises at all*! So, if we have a claim A which is equivalent to $A \to B$, and if we allow vacuous discharge, then we can derive B.

DEFINITION 1.3 [KINDS OF PROOFS] A proof in which every discharge is *linear* is a *linear proof*. Similarly, a proof in which every discharge is *relevant* is a *relevant proof*, a proof in which every discharge is *affine* is an *affine proof*. If a proof has some duplicate discharge and some vacuous discharge, it is at least a *standard proof*.

Proofs underwrite *arguments*. If we have a proof from a collection X of assumptions to a conclusion A, then the argument $X \therefore A$ is *valid* by the light of the rules we have used. So, in this section, we will think of *arguments* as structures involving a collection of assumptions and a single conclusion. But what kind of thing is that collection X? It isn't a *set*, because the number of premises makes a difference: (The example here involves linear discharge policies. We will see later that even when we allow for duplicate discharge, there is a sense in which the number of occurrences of a formula in the premises might still matter.) There is a linear proof from $A \to (A \to B), A, A$ to B:

We will *generalise* the notion of an argument later, in a number of directions. But this notion of argument is suited to the kind of proof we are considering here.

$$\dfrac{\dfrac{A \to (A \to B) \quad A}{A \to B}\,{\to}E \qquad A}{B}\,{\to}E$$

We shall see later that there is *no* linear proof from $A \to (A \to B), A$ to B. (If we ban duplicate discharge, then the number of assumptions in a proof matters.) The *collection* appropriate for our analysis at this stage is what is called a *multiset*, because we want to pay attention to the number of times we make an assumption in an argument.

DEFINITION 1.4 [MULTISET] Given a class X of objects (such as the class FORMULA), a *multiset* M of objects from X is a special kind of collection of elements of X. For each x in X, there is a natural number $o_M(x)$, the number of OCCURRENCES of the object x in the multiset M. The number

$o_M(x)$ is sometimes said to be the DEGREE to which x is a member of M. The multiset M is FINITE if $o_M(x) > 0$ for only finitely many objects x. The multiset M is identical to the multiset M' if and only if $o_M(x) = o_{M'}(x)$ for every x in X.

Multisets may be presented in lists, in much the same way that sets can. For example, $[1, 2, 2]$ is the finite multiset containing 1 only once and 2 twice. $[1, 2, 2] = [2, 1, 2]$, but $[1, 2, 2] \neq [1, 1, 2]$. We shall only consider finite multisets of *formulas*, and not multisets that contain other multisets as members. This means that we can do without the brackets and write our multisets as lists. We will write "A, B, B, C" for the finite multiset containing B twice and A and C once. The empty multiset, to which everything is a member to degree zero, is $[\,]$.

If you like, you could define a multiset of formulas to *be* the occurrence function $o_M(x)$ function $o_M : \text{FORMULA} \to \omega$. Then $o_1 = o_2$ when $o_1(A) = o_2(A)$ for each formula A. $o(A)$ is the number of times A is in the multiset o.

DEFINITION 1.5 [COMPARING MULTISETS] When M and M' are multisets and $o_M(x) \leq o_{M'}(x)$ for each x in X, we say that M is a SUB-MULTISET of M', and M' is a SUPER-MULTISET of M.

The GROUND of the multiset M is the set of all objects that are members of M to a non-zero degree. So, for example, the ground of the multiset A, B, B, C is the set $\{A, B, C\}$.

We use finite multisets as a part of a discriminating analysis of proofs and arguments. (An even more discriminating analysis will consider premises to be structured in *lists*, according to which A, B differs from B, A. You can examine this in Exercise 24 on page 45.) We have no need to consider *infinite* multisets in this section, as multisets represent the premise collections in arguments, and it is quite natural to consider only arguments with finitely many premises, since proofs, as we have defined them feature only finitely many assumptions. So, we will consider arguments in the following way.

DEFINITION 1.6 [ARGUMENT] An ARGUMENT $X \therefore A$ is a structure consisting of a finite multiset X of formulas as its *premises*, and a single formula A as its *conclusion*. The premise multiset X may be empty. An argument $X \therefore A$ is *standardly valid* if and only if there is some proof with undischarged assumptions forming the multiset X, and with the conclusion A. It is *relevantly valid* if and only if there is a relevant proof from the multiset X of premises to A, and so on.

John Slaney has joked that the empty multiset $[\,]$ should be distinguished from the empty set $\emptyset$, since *nothing* is a member of $\emptyset$, but *everything* is a member of $[\,]$ *zero times*.

Here are some features of validity.

LEMMA 1.7 [VALIDITY FACTS] *Let v-validity be any of linear, relevant, affine or standard validity.*

1. $A \therefore A$ *is v-valid.*
2. $X, A \therefore B$ *is v-valid if and only if* $X \therefore A \to B$ *is v-valid.*
3. *If* $X, A \therefore B$ *and* $Y \therefore A$ *are both v-valid, so is* $X, Y \therefore B$.
4. *If* $X \therefore B$ *is affine or standardly valid, so is* $X, A \therefore B$.
5. *If* $X, A, A \therefore B$ *is relevantly or standardly valid, so is* $X, A \therefore B$.

Proof: (1) is given by the proof consisting of A as premise and conclusion.

For (2), take a proof π from X, A to B, and in a single step $\to I$, discharge the (single instance of) A to construct the proof of $A \to B$ from X. Conversely, if you have a proof from X to $A \to B$, add a (single) premise A and apply $\to E$ to derive B. In both cases here, if the original proofs satisfy a constraint (vacuous or multiple discharge) so do the new proofs.

For (3), take a proof from X, A to B, but replace the instance of assumption of A indicated in the premises, and replace this with the *proof* from Y to A. The result is a proof, from X, Y to B as desired. This proof satisfies the constraints satisfied by both of the original proofs.

For (4), if we have a proof π from X to B, we extend it as follows

$$\dfrac{\dfrac{\begin{matrix} X \\ \vdots\,\pi \\ B \end{matrix}}{A \to B}{\scriptstyle \to I} \qquad A}{B}{\scriptstyle \to E}$$

to construct a proof from to B involving the new premise A, as well as the original premises X. The $\to I$ step requires a vacuous discharge.

Finally (5): if we have a proof π from X, A, A to B (that is, a proof with X and *two* instances of A as premises to derive the conclusion B) we discharge the two instances of A to derive $A \to B$ and then reinstate a single instance of A to as a premise to derive B again.

$$\dfrac{\dfrac{\begin{matrix} X, [A, A]^{(i)} \\ \vdots\,\pi \\ B \end{matrix}}{A \to B}{\scriptstyle \to I, i} \qquad A}{B}{\scriptstyle \to E}$$

■

Now, having established these facts, we might focus all our attention on the distinction between those arguments that are valid and those that are not, to attend to facts about validity such as those we have just proved. That would be to ignore the distinctive features of proof theory. We care not only *that* an argument is proved, but *how* it is proved. For each of these facts about validity, we showed not only the bare existential fact (for example, if there is a proof from X, A to B, then there is a proof from X to $A \to B$) but the stronger and more specific fact (if there is a proof from X, A to B then from this proof we construct the proof from X to $A \to B$ in a uniform way). This is the power of proof theory. We focus on proofs, not only as a certificate for the validity of an argument, but as a structure worth attention in its own right.

» «

It is often a straightforward matter to show that an argument is valid. Find a proof from the premises to the conclusion, and you are done. It seems more difficult to show that an argument is not valid. According to the literal reading of this definition, if an argument is not valid there is no proof from the premises to the conclusion. So, the direct way to show that an argument is invalid is to show that it has no proof from the premises to the conclusion. There are infinitely many proofs. It would take forever to through all of the proofs and check that none of them are proofs from X to A in order to convince yourself that the argument from X to A is not valid. To show that the argument is not valid, that there is no proof from X to A, some subtlety is called for. We will end this section by looking at how we might summon up the skill we need.

One subtlety would be to change the terms of discussion entirely, and introduce a totally new concept. If you could show that all valid arguments have some special property – and one that is easy to detect when present and when absent – then you could show that an argument is invalid by showing it lacks that special property. How this might manage to work depends on the special property. We shall look at one of these properties in Chapter 3 when we show that all valid arguments *preserve truth* in all *models*. Then to show that an argument is invalid, you could provide a model in which truth is *not* preserved from the premises to the conclusion. If all valid arguments are truth-in-a-model-preserving, then such a model would count as a counterexample to the validity of your argument.

In this chapter, on the other hand, we will not go beyond the conceptual bounds of proof theory itself. We will find instead a way to show that an argument is invalid, using an analysis of the structure of proofs. The collection of *all* proofs is too large to survey. From premises X and conclusion A, the collection of *direct* proofs – those that go straight from X to A without any detours down byways or highways – should be more tractable. If we could show that there are not many *direct* proofs from a given collection of premises to a conclusion, then we might be able to exploit this fact to show that for a given set of premises and a conclusion there are *no* direct proofs from X to A. If, in addition, you were to how that any proof from a premise set to a conclusion could somehow be converted into a direct proof from the same premises to that conclusion, then you would have shown that there is no proof from X to A.

I *think* that the terminology 'normal' comes from Prawitz [68], though the idea comes from Gentzen.

Happily, this technique works. To show how it works we need to understand what it is for a proof to be have no detours. These proofs which head straight from the premises to the conclusion without detours are so important that they have their own name. They are called *normal*.

1.3 | NORMAL PROOFS

It is best to introduce normal proofs by contrasting them with *non*-normal proofs. Non-normal proofs are not difficult to find. Suppose you want to show that the following argument is valid

$$p \to q \therefore p \to ((q \to r) \to r)$$

You might note first that we have already seen an argument which takes us from $p \to q$ to $(q \to r) \to (p \to r)$. This is SUFFIXING. On page 8

$$
\begin{prooftree}
\AxiomC{$[q \to r]^{(2)}$}
\AxiomC{$p \to q$}
\AxiomC{$[p]^{(1)}$}
\RightLabel{$\to E$}
\BinaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,1$}
\UnaryInfC{$p \to r$}
\RightLabel{$\to I,2$}
\UnaryInfC{$(q \to r) \to (p \to r)$}
\end{prooftree}
$$

So, we have $p \to q \therefore (q \to r) \to (p \to r)$. But we also have the general principle *permuting* antecedents: $A \to (B \to C) \therefore B \to (A \to C)$.

$$
\begin{prooftree}
\AxiomC{$A \to (B \to C)$}
\AxiomC{$[A]^{(3)}$}
\RightLabel{$\to E$}
\BinaryInfC{$B \to C$}
\AxiomC{$[B]^{(4)}$}
\RightLabel{$\to E$}
\BinaryInfC{C}
\RightLabel{$\to I,3$}
\UnaryInfC{$A \to C$}
\RightLabel{$\to I,4$}
\UnaryInfC{$B \to (A \to C)$}
\end{prooftree}
$$

We can apply this in the case where $A = (q \to r)$, $B = p$ and $C = r$ to get $(q \to r) \to (p \to r) \therefore p \to ((q \to r) \to r)$. We then chain reasoning together, to get us from $p \to q$ to $p \to ((q \to r) \to r)$, which we wanted. But take a look at the whole proof:

$$
\begin{prooftree}
\AxiomC{$[q \to r]^{(2)}$}
\AxiomC{$p \to q$}
\AxiomC{$[p]^{(1)}$}
\RightLabel{$\to E$}
\BinaryInfC{q}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,1$}
\UnaryInfC{$p \to r$}
\RightLabel{$\to I,2$}
\UnaryInfC{$(q \to r) \to (p \to r)$}
\AxiomC{$[q \to r]^{(3)}$}
\RightLabel{$\to E$}
\BinaryInfC{$p \to r$}
\AxiomC{$[p]^{(4)}$}
\RightLabel{$\to E$}
\BinaryInfC{r}
\RightLabel{$\to I,3$}
\UnaryInfC{$(q \to r) \to r$}
\RightLabel{$\to I,4$}
\UnaryInfC{$p \to ((q \to r) \to r)$}
\end{prooftree}
$$

This proof is circuitous. It gets us from our premise $(p \to q)$ to our conclusion $(p \to ((q \to r) \to r))$, but it does it in a roundabout way. We break down the conditionals $p \to q$, $q \to r$ to construct $(q \to r) \to (p \to r)$ halfway through the proof, only to break that down again (deducing r on its own, for a second time) to build the required conclusion. This is most dramatic around the intermediate conclusion $p \to ((q \to r) \to r)$ which is built up *from* $p \to r$ only to be used to justify $p \to r$ at the next step. We may eliminate this redundancy by cutting out the intermediate formula $p \to ((q \to r) \to r)$ like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[q \to r]^{(3)} \qquad \dfrac{p \to q \quad [p]^{(1)}}{q}\,{\scriptstyle \to E}}{r}\,{\scriptstyle \to E}}{p \to r}\,{\scriptstyle \to I,1} \qquad [p]^{(4)}}{r}\,{\scriptstyle \to E}}{(q \to r) \to r}\,{\scriptstyle \to I,3}}{p \to ((q \to r) \to r)}\,{\scriptstyle \to I,4}$$

The resulting proof is a lot simpler already. But now the $p \to r$ is constructed from r only to be broken up immediately to return r. We can delete the redundant $p \to r$ in the same way.

$$\dfrac{\dfrac{\dfrac{[q \to r]^{(3)} \qquad \dfrac{p \to q \quad [p]^{(4)}}{q}\,{\scriptstyle \to E}}{r}\,{\scriptstyle \to E}}{(q \to r) \to r}\,{\scriptstyle \to I,3}}{p \to ((q \to r) \to r)}\,{\scriptstyle \to I,4}$$

This proof takes us directly from its premise to its conclusion, through no extraneous formulas. Every formula used in this proof is either found in the premise, or in the conclusion. This wasn't true in the original, roundabout proof. We say this new proof is *normal*, the original proof was not.

This is a general phenomenon. Take a proof ending in [$\to E$]: it goes from A to B by way of a sub-proof π_1, and then we discharge A to conclude $A \to B$. Imagine that at the very next step, we use a different proof – say π_2 – with conclusion A to deduce B by means of an implication elimination. This proof contains a redundant step. Instead of taking the detour through the formula $A \to B$, we could use the proof π_1 of B, but instead of taking A as an *assumption*, we could use the proof of A we have at hand, namely π_2. The before-and-after comparison is this:

$$\text{BEFORE:}\quad \dfrac{\dfrac{\begin{array}{c}[A]^{(i)}\\ \vdots\,\pi_1\\ B\end{array}}{A \to B}\,{\scriptstyle \to I,i} \qquad \begin{array}{c}\vdots\,\pi_2\\ A\end{array}}{B}\,{\scriptstyle \to E} \qquad\qquad \text{AFTER:}\quad \begin{array}{c}\vdots\,\pi_2\\ A\\ \vdots\,\pi_1\\ B\end{array}$$

The result is a proof of B from the same premises as our original proof. The premises are the premises of π_1 (other than the instances of A that were discharged in the other proof) together with the premises of π_2.

This new proof does not go through the formula $A \to B$, so it is, in a sense, simpler than the original.

Well … there are some subtleties with counting, as usual with proofs. If the discharge of A was vacuous, then we have nowhere to plug in the new proof π_2, so π_2, and its premises, don't appear in the final proof. On the other hand, if a number of duplicates of A were discharged, then the new proof will contain that many copies of π_2, and hence, that many copies of the premises of π_2.

Let's make this discussion more concrete, by considering an example where π_1 has two instances of A in the premise list. The original proof containing the introduction and then elimination of $A \to B$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (A \to B) \quad [A]^{(1)}}{A \to B}{\scriptstyle \to E} \quad [A]^{(1)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I,1} \qquad \dfrac{(A \to A) \to A \quad \dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2}}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

We can cut out the [$\to I$/$\to E$] pair (we call such pairs INDIRECT PAIRS) using the technique described above, we place a copy of the inference to A at *both* places that the A is discharged (with label 1). The result is this proof, which does not make that detour.

$$\dfrac{\dfrac{A \to (A \to B) \quad \dfrac{(A \to A) \to A \quad \dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2}}{A}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to E} \qquad \dfrac{(A \to A) \to A \quad \dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2}}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

which is a proof from the same premises ($A \to (A \to B)$ and $(A \to A) \to A$) to the same conclusion B, except for multiplicity. In this proof the premise $(A \to A) \to A$ is used twice instead of once. (Notice too that the label '2' is used twice. We could relabel one subproof to $A \to A$ to use a different label, but there is no ambiguity here because the two proofs to $A \to A$ do not overlap. Our convention for labelling is merely that at the time we get to an [$\to I$] label, the numerical tag is unique in the proof *above* that step.)

We have motivated the concept of normality. Here is the definition:

DEFINITION 1.8 [NORMAL PROOF] A proof is NORMAL if and only if the concluding formula $A \to B$ of an [$\to I$] step is also the major premise of an [$\to E$] step.

DEFINITION 1.9 [INDIRECT PAIR; DETOUR FORMULA] If a formula $A \to B$ introduced in an [$\to I$] step in a proof is also the major premise of an

[$\to E$] step in that proof, then we shall call this pair of inferences an INDIRECT PAIR and we will call the instance $A \to B$ in the middle of this indirect pair a DETOUR FORMULA in that proof.

So, a normal proof is one without any indirect pairs. It has no detour formulas.

Normality is not only important for proving that an argument is invalid by showing that it has no normal proofs. The claim that every valid argument has a normal proof could well be *vital*. If we think of the rules for conditionals as somehow *defining* the connective, then proving something by means of a roundabout [$\to I/\to E$] step that you *cannot* prove without it would seem to be illicit. If the conditional is *defined* by way of its rules then it seems that the things one can prove *from* a conditional ought to be merely the things one can prove from whatever it was you used to *introduce* the conditional. If we could prove more from a conditional $A \to B$ than one could prove on the basis on the information used to *introduce* the conditional, then we are conjuring new arguments out of thin air.

For this reason, many have thought that being able to convert non-normal proofs to normal proofs is not only desirable, it is critical if the proof system is to be properly logical. We will not continue in this philosophical vein here. We will take up this topic in a later section, after we understand the behaviour of normal proofs a little better. Let us return to the study of normal proofs.

Normal proofs are, intuitively at least, proofs without a kind of redundancy. It turns out that avoiding this kind of redundancy in a proof means that you must avoid another kind of redundancy too. A normal proof from X to A may use only a very restricted repertoire of formulas. It will contain only the *subformulas* of X and A.

DEFINITION 1.10 [SUBFORMULAS AND PARSE TREES] The PARSE TREE for an atom is that atom itself. The PARSE TREE for a conditional $A \to B$ is the tree containing $A \to B$ at the root, connected to the parse tree for A and the parse tree for B. The SUBFORMULAS of a formula A are those formulas found in A's parse tree. We let $sf(A)$ be the set of all subformulas of A, so $sf(p) = \{p\}$, and $sf(A \to B) = \{A \to B\} \cup sf(A) \cup sf(B)$. To generalise, when X is a multiset of formulas, we will write $sf(X)$ for the set of subformulas of each formula in X.

Here is the parse tree for $(p \to q) \to ((q \to r) \to p)$:

$$\dfrac{\dfrac{p \quad q}{p \to q} \qquad \dfrac{\dfrac{q \quad r}{q \to r} \quad p}{(q \to r) \to p}}{(p \to q) \to ((q \to r) \to p)}$$

So, $sf((p \to q) \to ((q \to r) \to p)) = \{(p \to q) \to ((q \to r) \to p), p \to q, p, q, (q \to r) \to p, q \to r, r\}$.

We may prove the following theorem.

THEOREM 1.11 [THE SUBFORMULA THEOREM] *If π is a normal proof from the premises X to the conclusion A, then π contains only formulas in* $\mathrm{sf}(X, A)$.

Notice that this is *not* the case for non-normal proofs. Consider the following circuitous proof from A to A.

$$\dfrac{\dfrac{[A]^{(1)}}{A \to A}{\scriptstyle \to I,1} \quad A}{A}{\scriptstyle \to E}$$

Here $A \to A$ is in the proof, but it is not a subformula of the premise (A) or the conclusion (also A).

The subformula property for normal proofs goes some way to reassure us that a normal proof is *direct*. A normal proof from X to A cannot stray so far away from the premises and the conclusion so as to incorporate material outside X and A. This fact goes some way to defend the notion that validity is *analytic* in a strong sense. The validity of an argument is grounded a proof where the consitituents of that proof are found by *analysing* the premises and the conclusion into their constituents. Here is how the subformula theorem is proved.

Proof: We look carefully at how proofs are constructed. If π is a normal proof, then it is constructed in exactly the same way as all proofs are, but the fact that the proof is normal gives us some useful information. By the definition of proofs, π either is a lone assumption, or π ends in an application of [$\to I$], or it ends in an application of [$\to E$]. Assumptions are the basic building blocks of proofs. We will show that assumption-only proofs have the subformula property, and then, also show on the assumption that the proofs we have on had have the subformula property, then the normal proofs we construct from them also have the property. Then it will follow that all normal proofs have the subformula property, because all of the normal proofs can be generated in this way.

Notice that the subproofs of normal proofs are normal. If a subproof of a proof contains an indirect pair, then so does the larger proof.

ASSUMPTION: A sole assumption, considered as a proof, satisfies the subformula property. The assumption A is the only constituent of the proof and it is both a premise and the conclusion.

INTRODUCTION: In the case of [$\to I$], π is constructed from another normal proof π' from X to B, with the new step added on (and with the discharge of a number – possibly zero – of assumptions). π is a proof from X' to $A \to B$, where X' is X with the deletion of some number of instances of A. Since π' is normal, we may assume that every formula in π' is in $\mathrm{sf}(X, B)$. Notice that $\mathrm{sf}(X', A \to B)$ contains every element of $\mathrm{sf}(X, B)$, since X differs only from X' by the deletion of some instances of A. So, every formula in π (namely, those formulas in π', together with $A \to B$) is in $\mathrm{sf}(X', A \to B)$ as desired.

ELIMINATION: In the case of $[\to E]$, π is constructed out of *two* normal proofs: one (call it π_1) to the conclusion of a conditional $A \to B$ from premises X, and the other (call it π_2) to the conclusion of the antecedent of that conditional A from premises Y. Both π_1 and π_2 are normal, so we may assume that each formula in π_1 is in $sf(X, A \to B)$ and each formula in π_2 is in $sf(Y, A)$. We wish to show that every formula in π is in $sf(X, Y, B)$. This seems difficult ($A \to B$ is in the proof—where can it be found inside X, Y or B?), but we also have some more information: π_1 cannot end in the *introduction* of the conditional $A \to B$. So, π_1 is either the assumption $A \to B$ itself (in which case $Y = A \to B$, and clearly in this case each formula in π *is* in $sf(X, A \to B, B)$) or π_1 ends in a $[\to E]$ step. But if π_1 ends in an $[\to E]$ step, the major premise of that inference is a formula of the form $C \to (A \to B)$. So π_1 contains the formula $C \to (A \to B)$, so *whatever* list Y is, $C \to (A \to B) \in sf(Y, A)$, and so, $A \to B \in sf(Y)$. In this case too, every formula in π is in $sf(X, Y, B)$, as desired.

This completes the proof of our theorem. Every normal proof is constructed from assumptions by introduction and elimination steps in this way. The subformula property is preserved through each step of the construction. ■

Normal proofs are handy to work with. Even though an argument might have very many proofs, it will have many fewer normal proofs. We can exploit this fact when searching for proofs.

EXAMPLE 1.12 [NO NORMAL PROOFS] There is no normal proof from p to q. There is no normal relevant proof from $p \to r$ to $p \to (q \to r)$.

Proof: Normal proofs from p to q (if there are any) contain only formulas in $sf(p, q)$: that is, they contain only p and q. That means they contain no $[\to I]$ or $[\to E]$ steps, since they contain no conditionals at all. It follows that any such proof must consist solely of an assumption. As a result, the proof cannot have a premise p that differs from the conclusion q. There is no normal proof from p to q.

For the second example, if there is a normal proof of $p \to (q \to r)$, from $p \to r$, it must end in an $\to I$ step, from a normal (relevant) proof from $p \to r$ and p to $q \to r$. Similarly, this proof must also end in an $\to I$ step, from a normal (relevant) proof from $p \to r$, p and q to r. Now, what normal relevant proofs can be found from $p \to r$, p and q to r? There are none. Any such proof would have to use q as a premise somewhere, but since it is normal, it contains only subformulas of $p \to r$, p, q and r—namely those formulas themselves. There is no formula involving q other than q itself on that list, so there is nowhere for q to go. It cannot be used, so it will not be a premise in the proof. There is no normal relevant proof from the premises $p \to r$, p and q to the conclusion r. ■

These facts are interesting enough. It would be more productive, however, to show that there is no proof at all from p to q, and no relevant proof from $p \to r$ to $p \to (q \to r)$. We can do this if we have some way of showing that if we have a proof for some argument, we have a normal proof for that argument.

So, we now work our way towards the following theorem:

THEOREM 1.13 [NORMALISATION THEOREM] *A proof π from X to A reduces in some number of steps to a normal proof π' from X' to A.*

If π is linear, so is π', and $X = X'$. If π is affine, so is π', and X' is a sub-multiset of X. If π is relevant, then so is π', and X' covers the same ground as X, and is a super-multiset of X. If π is standard, then so is π', and X' covers no more ground than X.

$[1,2,2,3]$ covers the same ground as–and is a super-multiset of–$[1,2,3]$. And $[2,2,3,3]$ covers no more ground than $[1,2,3]$.

Notice how the premise multiset of the normal proof is related to the premise multiset of the original proof. If we allow duplicate discharge, then the premise multiset may contain formulas to a greater degree than in the original proof, but the normal proof will not contain any premises that weren't in the original proof. If we allow vacuous discharge, then the normal proof might contain fewer premises than the original proof.

The normalisation theorem mentions the notion of *reduction*, so let us first define it.

DEFINITION 1.14 [REDUCTION] A proof π *reduces* to π' (shorthand: $\pi \rightsquigarrow \pi'$) if some indirect pair in π is eliminated, to result in π'.

$$\dfrac{\dfrac{\begin{matrix}[A]^{(i)}\\ \vdots\ \pi_1\\ B\end{matrix}}{A \to B}{\scriptstyle \to I,i} \qquad \begin{matrix}\vdots\ \pi_2\\ A\end{matrix}}{\begin{matrix}B\\ \vdots\\ C\end{matrix}}{\scriptstyle \to E} \qquad \rightsquigarrow \qquad \begin{matrix}\vdots\ \pi_2\\ A\\ \vdots\ \pi_1\\ B\\ \vdots\\ C\end{matrix}$$

If there is no π' such that $\pi \rightsquigarrow \pi'$, then π' is normal. If $\pi_0 \rightsquigarrow \pi_2 \rightsquigarrow \cdots \rightsquigarrow \pi_n$ we write "$\pi_0 \rightsquigarrow_* \pi_n$" and we say that π_0 reduces to π_n in a number of steps. We aim to show that for any proof π, there is some normal π^* such that $\pi \rightsquigarrow_* \pi^*$.

We allow that $\pi \rightsquigarrow_* \pi$. A proof 'reduces' to itself in *zero* steps.

The only difficult part in proving the normalisation theorem is showing that the process reduction can terminate in a normal proof. In the case where we do not allow duplicate discharge, there is no difficulty at all.

Proof [Theorem 1.13: *linear and affine cases*]: If π is a linear proof, or is an affine proof, then whenever you pick an indirect pair and normalise it, the result is a shorter proof. At most one copy of the proof π_2 for A is inserted into the proof π_1. (Perhaps no substitution is made in the case of an affine proof, if a vacuous discharge was made.) Proofs have some finite size, so this process cannot go on indefinitely. Keep deleting indirect pairs until there are no pairs left to delete. The result is a normal proof to the conclusion A. The premises X remain undisturbed, except in the affine case, where we may have lost premises along the way. (An assumption from π_2 might disappear if we did not need to make the substitution.) In this case, the premise multiset X' from the normal proof is a *sub*-multiset of X, as desired. ∎

If we allow duplicate discharge, however, we cannot be sure that in normalising we go from a larger to a smaller proof. The example on page 17 goes from a proof with 11 formulas to another proof with 11 formulas. In some cases a reduction step can take us from a smaller proof to a properly larger proof. Sometimes, the result is *much* larger. So size alone is no guarantee that the process terminates.

To gain some understanding of the general process of transforming a non-normal proof into a normal one, we must find some other measure that decreases as normalisation progresses. If this measure has a least value then we can be sure that the process will stop. The appropriate measure in this case will not be too difficult to find. Let's look at a part of the process of normalisation: the complexity of the formula that is normalised.

Well, the process stops if the measures are ordered appropriately—so that there's no *infinitely descending chain*.

DEFINITION 1.15 [COMPLEXITY] A formula's *complexity* is the number of connectives in that formula. In this case, it is the number of instances of '$\to$' in the formula.

The crucial features of complexity are that each formula has a finite complexity, and that the proper subformulas of a formula each have a lower complexity than the original formula. This means that complexity is a good measure for an induction, like the size of a proof.

Now, suppose we have a proof containing just one indirect pair, introducing and eliminating $A \to B$, and suppose that otherwise, π_1 (the proof of B from A) and π_2 (the proof of A) are normal.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\begin{matrix}[A]^{(i)}\\ \vdots\,\pi_1\\ B\end{matrix}}{A \to B}{\scriptstyle \to I,i} \qquad \begin{matrix}\vdots\,\pi_2\\ A\end{matrix}}{B}{\scriptstyle \to E} \qquad\qquad \text{AFTER:}\quad \begin{matrix}\vdots\,\pi_2\\ A\\ \vdots\,\pi_1\\ B\end{matrix}$$

This the new proof need not be necessarily normal, even though π_1 and π_2 are. The new proof is non-normal if π_2 ends in the introduction of A and π_1 starts off with the elimination of A. Notice, however, that the non-normality of the new proof is, somehow, *smaller*. There is no non-normality with respect to $A \to B$ or any other formula as complex as that. The potential non-normality is with respect to a subformula A. This result would still hold if the proofs π_1 and π_2 weren't normal themselves, but when they might have [$\to I/\to E$] pairs for formulas less complex than $A \to B$. If $A \to B$ is the most complex detour formula in the original proof, then the new proof has a *smaller* most complex detour formula.

DEFINITION 1.16 [NON-NORMALITY] The *non-normality measure* of a proof is a sequence $\langle c_1, c_2, \ldots, c_n \rangle$ of numbers such that c_i is the number of indirect pairs of formulas of complexity i. The sequence for a proof stops at the last non-zero value. Sequences are ordered with their last number as most significant. That is, $\langle c_1, \ldots, c_n \rangle > \langle d_1, \ldots, d_m \rangle$ if and only if $n > m$, or if $n = m$, when $c_n > d_n$, or if $c_n = d_n$, when $\langle c_1, \ldots, c_{n-1} \rangle > \langle d_1, \ldots, d_{n-1} \rangle$.

Non-normality measures satisfy the finite descending chain condition. Starting at any particular measure, you cannot find an infinite descending chain of measures below it. Of course, there are infinitely many measures smaller than $\langle 0, 1 \rangle$ (in this case, $\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \ldots$). However, to form a *descending sequence* from $\langle 0, 1 \rangle$ you must choose one of these as your next measure. Say you choose $\langle 500 \rangle$. From that, you have only finitely many (500, in this case) steps until $\langle \rangle$ and the sequence stops. This generalises. From the sequence $\langle c_1, \ldots, c_n \rangle$, you lower c_n until it gets to zero. Then you look at the index for $n - 1$, which might have grown enormously. Nonetheless, it is some finite number, and now you must reduce this value. And so on, until you reach the last quantity, and from there, the empty sequence $\langle \rangle$. Here is an example sequence using this ordering $\langle 3, 2, 30 \rangle > \langle 2, 8, 23 \rangle > \langle 1, 47, 15 \rangle > \langle 138, 478 \rangle > \cdots > \langle 1, 3088 \rangle > \langle 314159 \rangle > \cdots > \langle 1 \rangle > \langle \rangle$.

LEMMA 1.17 [NON-NORMALITY REDUCTION] *Any a proof with an indirect pair reduces in one step to some proof with a lower measure of non-normality.*

Proof: Choose a detour formula in π of greatest complexity (say n), such that its proof contains no other detour formulas of complexity n. Normalise that proof. The result is a proof π' with fewer detour formulas of complexity n (and perhaps many more of $n - 1$, etc.). So, it has a lower non-normality measure. ■

Now we have a proof of our normalisation theorem.

Proof [of Theorem 1.13: *for the relevant and standard cases*]: Start with π, a proof that isn't normal, and use Lemma 1.17 to choose a proof π' with a lower measure of non-normality. If π' is normal, we're done. If it isn't, continue the process. There is no infinite descending chain of non-normality measures, so this process will stop at some point, and the result is a normal proof. ■

Every proof may be transformed into a normal proof. If there is a linear proof from X to A then there is a normal linear proof from X to A. Linear proofs are satisfying and strict in this manner. If we allow vacuous discharge or duplicate discharge, matters are not so straightforward. For example, there is a non-normal standard proof from p, q to p:

$$\dfrac{\dfrac{p}{q \to p}{\scriptstyle \to I,1} \qquad q}{p}{\scriptstyle \to E}$$

but there is no normal standard proof from exactly these premises to the same conclusion, since any normal proof from atomic premises to an atomic conclusion must be an assumption alone. We have a normal proof from p to p (it is very short!), but there is no normal proof from p to p that involves q as an extra premise.

Similarly, there is a relevant proof from $p \to (p \to q), p$ to q, but it is non-normal.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \to (p \to q) \qquad [p]^{(1)}}{p \to q}\,{\scriptstyle \to E} \qquad [p]^{(1)}}{q}\,{\scriptstyle \to E}}{p \to q}\,{\scriptstyle \to I,1} \qquad p}{q}\,{\scriptstyle \to E}$$

There is no normal relevant proof from $p \to (p \to q), p$ to q. Any normal relevant proof from $p \to (p \to q)$ and p to q must use [$\to E$] to deduce $p \to q$, and then the only other possible move is either [$\to I$] (in which case we return to $p \to (p \to q)$ none the wiser) or we perform another [$\to E$] with another assumption p to deduce q, and we are done. Alas, we have claimed two undischarged assumptions of p. In the non-linear cases, the transformation from a non-normal to a normal proof does damage to the number of times a premise is used.

1.4 | STRONG NORMALISATION AND TERMS

This passage is the hardest part of Chapter 1. Feel free to skip over the proofs of theorems in this section, until page 37 on first reading.

It is very tempting to view normalisation as a process of reducing a proof down to its essence, of unwinding detours, and making explicit the essential logical connections made in the proof between the premises and the conclusion. The result of normalising a proof π from X to A shows the connections made from X to A in that proof π, without the need to bring in the extraneous information in any detours that may have been used in π. Another analogy is that the complex non-normal proof is *evaluated* into its normal form, in the same way that a numerical term like $5 + (2 \times (7 + 3))$ is evaluated into *its* normal form, the numeral 25.

If this is the case, then the process of normalisation should give us two distinct "answers" for the underlying structure of the one proof. Can two different reduction sequences for a single proof result in *different* normal proofs? To investigate this, we need to pay attention to the different processes of reduction we can take when reducing a proof. To do that, we'll introduce a new notion of reduction:

DEFINITION 1.18 [PARALLEL REDUCTION] A proof π *parallel reduces* to π' if some number of indirect pairs in π are eliminated in parallel. We write "$\pi \mathrel{\substack{\rightsquigarrow\\\rightsquigarrow}} \pi'$."

For example, consider the proof with the following two detour formulas marked:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (A \to B) \quad [A]^{(1)}}{A \to B}{\scriptstyle \to E} \quad [A]^{(1)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I,1} \qquad \dfrac{\dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2} \quad A}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

To process them we can take them in any order. Eliminating the $A \to B$, we have

$$\dfrac{\dfrac{A \to (A \to B) \qquad \dfrac{\dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2} \quad A}{A}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to E} \qquad \dfrac{\dfrac{[A]^{(2)}}{A \to A}{\scriptstyle \to I,2} \quad A}{A}{\scriptstyle \to E}}{B}{\scriptstyle \to E}$$

which now has two copies of the $A \to A$ to be reduced. However, these copies do not overlap in scope (they cannot, as they are duplicated in the place of assumptions discharged in an eliminated $\to I$ rule) so they can be processed together. The result is the proof

$$\dfrac{\dfrac{A \to (A \to B) \quad A}{A \to B}{\scriptstyle \to E} \qquad A}{B}{\scriptstyle \to E}$$

You can check that if you had processed the formulas to be eliminated in the other order, the result would have been the same.

LEMMA 1.19 [DIAMOND PROPERTY FOR $\mathrel{\overset{\rightsquigarrow}{\rightsquigarrow}}$] *If $\pi \mathrel{\overset{\rightsquigarrow}{\rightsquigarrow}} \pi_1$ and $\pi \mathrel{\overset{\rightsquigarrow}{\rightsquigarrow}} \pi_2$ then there is some proof π' where $\pi_1 \mathrel{\overset{\rightsquigarrow}{\rightsquigarrow}} \pi'$ and $\pi_2 \mathrel{\overset{\rightsquigarrow}{\rightsquigarrow}} \pi'$.*

Can you see why this is called the diamond property?

Proof: Take the detour formulas in the proof π that are eliminated in either the move to π_1 or the move to π_2. 'Colour' them in π, and transform the proof to π_1. Some of the coloured formulas may remain. Do the same in the move from π to π_2. The result are two proofs π_1 and π_2 in which some formulas may be coloured. The proof π' is found by parallel reducing either collection of formulas in π_1 or π_2. ■

They may have multiplied, if they occurred in a proof part duplicated in the reduction step. But some may have vanished, too, if they were in a part of the proof that disappeared during reduction.

THEOREM 1.20 [ONLY ONE NORMAL FORM] *Given any proof π, if $\pi \rightsquigarrow_* \pi'$ then if $\pi \rightsquigarrow_* \pi''$, it must be that $\pi' = \pi''$. That is, any sequence of reduction steps from π that terminates in a normal form must terminate in a unique normal form.*

Proof: Suppose that $\pi \rightsquigarrow_* \pi'$, and $\pi \rightsquigarrow_* \pi''$. It follows that we have two reduction sequences

$$\pi \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi'_1 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi'_2 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \cdots \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi'_n \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi'$$

$$\pi \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi''_1 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi''_2 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \cdots \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi''_m \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi''$$

By the diamond property, we have a $\pi_{1,1}$ where $\pi'_1 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi_{1,1}$ and $\pi''_1 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi_{1,1}$. Then $\pi''_1 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi_{1,1}$ and $\pi''_1 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi''_2$ so by the diamond property there is some $\pi_{2,1}$ where $\pi''_2 \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi_{2,1}$ and $\pi_{1,1} \mathrel{\substack{\rightsquigarrow\\ \rightsquigarrow}} \pi_{2,1}$. Continue in this vein, guided by the picture below:

$$\begin{array}{ccccccccc}
\pi & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi'_1 & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi'_2 & \substack{\rightsquigarrow\\ \rightsquigarrow} & \cdots & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi'_n \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\pi''_1 & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{1,1} & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{1,2} & \substack{\rightsquigarrow\\ \rightsquigarrow} & \cdots & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{1,n} \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\pi''_2 & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{2,1} & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{2,2} & \substack{\rightsquigarrow\\ \rightsquigarrow} & \cdots & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{2,n} \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\wr\wr & & \wr\wr & & \wr\wr & & & & \wr\wr \\
\pi''_m & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{m,1} & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi_{m,2} & \substack{\rightsquigarrow\\ \rightsquigarrow} & \cdots & \substack{\rightsquigarrow\\ \rightsquigarrow} & \pi^*
\end{array}$$

to find the desired proof π^*. So, if π'_n and π''_n are *normal* they must be identical. ■

So, sequences of reductions from π cannot terminate in two different proofs. A normal form for a proof is unique.

This result goes a lot of the way towards justifying the idea that normalisation corresponds to evaluating the underlying essence of a proof. The normal form is well defined an unique. But this leaves us with a remaining question. We have seen that for each proof π there is some process to evaluate its normal form π^*, and further, the proof of the previous theorem shows us that *any* finite sequence of reductions from π can be extended to eventually reach π^*. Does it follow that any process of reductions from π terminates in its normal form π^*? That is: are our proofs *strongly normalising*?

DEFINITION 1.21 [STRONGLY NORMALISING] A proof π is strongly normalising (under a reduction relation $\rightsquigarrow$) if and only if there is no infinite reduction sequence starting from π.

This does not follow from weak normalisation (there is some reduction to a normal form) and the diamond property, which gives us unique normal form theorem. This is straightforward to see, because a "reduction" process which allows us to run reduction steps backwards as well as forwards if the proof is not already normal, would still allow for weak normalisation, would still have the diamond property and would have a unique normal form. But it would not be strongly normalising. (We could go on forever reducing one detour only to put it back, forever.) Is there any guarantee that our reduction process process will always terminate?

A naive approach would be to define some measure on proofs which always reduces under any reduction step. This seems hopeless, because anything like the measure we have already defined can increase, rather than decrease, under reductions. (Take a proof with a small detour formula $A \to B$ where the assumption A is discharged a number of times in the proof of the major premise $A \to B$, and in which there are larger detour formulas in the proof of the minor premise A. This proof is duplicated in the reduction, and the measure of the new proof could rise significantly, as we have eliminated a small detour formula at the cost of many large detour formulas.)

We will prove that every proof is strongly normalising under the relation $\rightsquigarrow$ of deleting detour formulas. To assist in talking about this, we need to make a few more definitions. First, the *reduction tree*.

DEFINITION 1.22 [REDUCTION TREE] The reduction tree (under $\rightsquigarrow$) of a proof π is the tree whose branches are the reduction sequences on the relation $\rightsquigarrow$. So, from the root π we reach any proof accessible in one $\rightsquigarrow$ step from π. From each π' where $\pi \rightsquigarrow \pi'$, we branch similarly. Each node has only finitely many successors as there are only finitely many detour formulas in a proof. For each proof π, $\nu(\pi)$ is the size of its reduction tree.

LEMMA 1.23 [THE SIZE OF REDUCTION TREES] *A strongly normalising proof has a finite reduction tree. It follows that not only is every reduction path finite, but there is a longest reduction path.*

Proof: This is a corollary of König's Lemma, which states that every tree in which the number of immediate descendants of a node is finite (it is finitely *branching*), and in which every branch is finitely long, is itself *finite*. Since the reduction tree for a strongly normalising proof is finitely branching, and each branch has a finite length, it follows that any strongly normalising proof not only has only finite reduction paths, it also has a *longest* reduction path. ■

It's true that every finitely branching tree with finite branches is finite. But is it *obvious* that it's true?

Now to prove that every proof is strongly normalising. To do this, we define a new property that proofs can have: of being **red**. It will turn out that all **red** proofs are strongly normalising. It will also turn out that all proofs are **red**.

DEFINITION 1.24 [**red** PROOFS] We define a new predicate '**red**' applying to proofs in the following way.

» A proof of an atomic formula is **red** if and only if it is strongly normalising.

The term '**red**' should bring to mind 'reducible.' This formulation of strong normalisation is originally due to William Tait [94]. I am following the presentation of Jean-Yves Girard [33, 35].

» A proof π of an implication formula $A \rightarrow B$ is **red** if and only if whenever π' is a **red** proof of A, then the proof

$$\dfrac{\begin{matrix}\vdots\,\pi \\ A \rightarrow B\end{matrix} \quad \begin{matrix}\vdots\,\pi' \\ A\end{matrix}}{B}$$

is a **red** proof of type B.

We will have cause to talk often of the proof found by extending a proof π of $A \rightarrow B$ and a proof π' of A to form the proof of B by adding an $\rightarrow E$ step. We will write '$(\pi\,\pi')$' to denote this proof. If you like, you can think of it as the application of the proof π to the proof π'.

Now, our aim will be twofold: to show that every **red** proof is strongly normalising, and to show that every proof is **red**. We start by proving the following crucial lemma:

LEMMA 1.25 [PROPERTIES OF **red** PROOFS] *For any proof π, the following three conditions hold:*

c1 *If π is* **red** *then π is strongly normalisable.*

c2 *If π is* **red** *and π reduces to π' in one step, then π' is* **red** *too.*

c3 *If π is a proof not ending in $\rightarrow I$, and whenever we eliminate one indirect pair in π we have a* **red** *proof, then π is* **red** *too.*

Proof: We prove this result by induction on the formula proved by π. We start with proofs of atomic formulas.

c1 Any **red** proof of an atomic formula is strongly normalising, by the definition of '**red**'.

c2 If π is strongly normalising, then so is any proof to which π reduces.

c3 π does not end in $\rightarrow I$ as it is a proof of an atomic formula. If whenever $\pi \Rightarrow_1 \pi'$ and π' is **red**, since π' is a proof of an atomic formula, it is strongly normalising. Since *any* reduction path through π must travel through one such proof π', each such path through π terminates. So, π is **red**.

Now we prove the results for a proof π of $A \rightarrow B$, under the assumption that c1, c2 and c3 they hold for proofs of A and proofs of B. We can then conclude that they hold of *all* proofs, by induction on the complexity of the formula proved.

c1 If π is a **red** proof of $A \rightarrow B$, consider the proof

$$\sigma : \quad \dfrac{\begin{matrix}\vdots\,\pi \\ A \rightarrow B\end{matrix} \quad A}{B}$$

The assumption A is a normal proof of its conclusion A not ending in $\to I$, so C3 applies and it is **red**. So, by the definition of **red** proofs of implication formulas, σ is a **red** proof of B. Condition C1 tells us that **red** proofs of B are strongly normalising, so any reduction sequence for σ must terminate. It follows that any reduction sequence for π must terminate too, since if we had a non-terminating reduction sequence for π, we could apply the same reductions to the proof σ. But since σ is strongly normalising, this cannot happen. It follows that π is strongly normalising too.

C2 Suppose that π reduces in one step to a proof π'. Given that π is **red**, we wish to show that π' is **red** too. Since π' is a proof of $A \to B$, we want to show that for any **red** proof π'' of A, the proof $(\pi' \, \pi'')$ is **red**. But this proof is **red** since the **red** proof $(\pi \, \pi'')$ reduces to $(\pi' \, \pi'')$ in one step (by reducing π to π'), and C2 applies to proofs of B.

C3 Suppose that π does not end in $[\to I]$, and suppose that all of the proofs reached from π in one step are **red**. Let σ be a **red** proof of A. We wish to show that the proof $(\pi \, \sigma)$ is **red**. By C1 for the formula A, we know that σ is strongly normalising. So, we may reason by induction on the length of the longest reduction path for σ. If σ is normal (with path of length 0), then $(\pi \, \sigma)$ reduces in one step only to $(\pi' \, \sigma)$, with π' one step from π. But π' is **red** so $(\pi' \, \sigma)$ is too.

On the other hand, suppose σ is not yet normal, but the result holds for all σ' with shorter reduction paths than σ. So, suppose τ reduces to $(\pi \, \sigma')$ with σ' one step from σ. σ' is **red** by the induction hypothesis C2 for A, and σ' has a shorter reduction path, so the induction hypothesis for σ' tells us that $(\pi \, \sigma')$ is **red**.

There is no other possibility for reduction as π does not end in $\to I$, so reductions must occur wholly in π or wholly in σ, and not in the last step of $(\pi \, \sigma)$.

This completes the proof by induction. The conditions C1, C2 and C3 hold of every proof. ■

Now we prove one more crucial lemma.

LEMMA 1.26 [**red** PROOFS ENDING IN $[\to I]$] *If for each* **red** *proof σ of A, the proof*

$$\pi(\sigma) : \quad \begin{array}{c} \vdots\ \sigma \\ A \\ \vdots\ \pi \\ B \end{array}$$

is **red**, *then so is the proof*

$$\tau : \quad \dfrac{\begin{array}{c} [A] \\ \vdots\ \pi \\ B \end{array}}{A \to B}\,{\to}I$$

Proof: We show that the $(\tau\,\sigma)$ is **red** whenever σ is **red**. This will suffice to show that the proof τ is **red**, by the definition of the predicate '**red**' for proofs of $A \to B$. We will show that every proof resulting from $(\tau\,\sigma)$ in one step is **red**, and we will reason by induction on the sum of the sizes of the reduction trees of π and σ. There are three cases:

» $(\tau\,\sigma) \rightsquigarrow \pi(\sigma)$. In this case, $\pi(\sigma)$ is **red** by the hypothesis of the proof.

» $(\tau\,\sigma) \rightsquigarrow (\tau'\,\sigma)$. In this case the sum of the size of the reduction trees of τ' and σ is smaller, and we may appeal to the induction hypothesis.

» $(\tau\,\sigma) \rightsquigarrow (\tau\,\sigma')$. In this case the sum of the size of the reduction trees is τ and σ' smaller, and we may appeal to the induction hypothesis. ∎

We are set to prove our major theorem:

THEOREM 1.27 [ALL PROOFS ARE **red**] *Every proof π is* **red**.

To do this, we'll approach it by induction, as follows:

LEMMA 1.28 [**red** PROOFS BY INDUCTION] *For each proof π with assumptions $A_1, \ldots, A_n$, and for any* **red** *proofs $\sigma_1, \ldots, \sigma_n$ of the formulas $A_1, \ldots, A_n$ respectively, the proof $\pi(\sigma_1, \ldots, \sigma_n)$ in which each assumption A_i is replaced by the proof σ_i is* **red**.

Proof: We prove this by induction on the construction of the proof.

» If π is an assumption A_1, the claim is a tautology (if σ_1 is **red**, then σ_1 is **red**).

» If π ends in [$\to$*E*], and is $(\pi_1\,\pi_2)$, then by the induction hypothesis $\pi_1(\sigma_1, \ldots, \sigma_n)$ and $\pi_2(\sigma_1, \ldots, \sigma_n)$ are **red**. Since $\pi_1(\sigma_1, \ldots, \sigma_n)$ has type $A \to B$ the definition of **red**ness tells us that when ever it is applied to a **red** proof the result is also **red**. Therefore, the proof $(\pi_1(\sigma_1, \ldots, \sigma_n)\,\pi_2(\sigma_1, \ldots, \sigma_n))$ is **red**, but this proof is simply $\pi(\sigma_1, \ldots, \sigma_n)$.

» If π ends in an application of [$\to$*I*], then this case is dealt with by Lemma 1.26: if π is a proof of $A \to B$ ending in $\to$*E*, then we may assume that π', the proof of B from A inside π is **red**, so by Lemma 1.26, the result π is **red** too.

It follows that *every* proof is **red**. ∎

It follows also that every proof is strongly normalising, since all **red** proofs are strongly normalising.

» «

It is very tempting to think of proofs as *processes* or *functions* that convert the information presented in the premises into the information in the conclusion. This is doubly tempting when you look at the notation for implication. In $\to E$ we apply something which converts A to B (a function from A to B?) to something which delivers you A (from premises) into something which delivers you B. In $\to I$ if we can produce B (when supplied with A, at least in the presence of other resources—the other premises) then we can (in the context of the other resources at least) convert As into Bs at will.

Let's make this talk a little more precise, by making *explicit* this kind of *function*-talk. It will give us a new vocabulary to talk of proofs.

We start with simple notation to talk about functions. The idea is straightforward. Consider numbers, and addition. If you have a number, you can add 2 to it, and the result is another number. If you like, if x is a number then

$$x + 2$$

is another number. Now, suppose we don't want to talk about a particular number, like $5+2$ or $7+2$ or $x+2$ for any choice of x, but we want to talk about the *operation* or of adding two. There is a sense in which just writing "$x + 2$" should be enough to tell someone what we mean. It is relatively clear that we are treating the "x" as a marker for the input of the function, and "$x + 2$" is the output. The *function* is the output as it varies for different values of the input. Sometimes leaving the variables there is not so useful. Consider the subtraction

$$x - y$$

You can think of this as the function that takes the input value x and takes away y. Or you can think of it as the function that takes the input value y and subtracts it from x. or you can think of it as the function that takes two input values x and y, and takes the second away from the first. Which do we mean? When we apply this function to the input value 5, what is the result? For this reason, we have a way of making explicit the different distinctions: it is the λ-notation, due to Alonzo Church [18]. The function that takes the input value x and returns $x + 2$ is denoted

$$\lambda x.(x + 2)$$

The function taking the input value y and subtracts it from x is

$$\lambda y.(x - y)$$

The function that takes *two* inputs and subtracts the second from the first is

$$\lambda x.\lambda y.(x - y)$$

Notice how this function works. If you feed it the input 5, you get the output $\lambda y.(5 - y)$. We can write *application* of a function to its input by way of juxtaposition. The result is that

$$(\lambda x.\lambda y.(x - y)\, 5)$$

evaluates to the result $\lambda y.(5 - y)$. This is the function that subtracts y from 5. When you feed *this* function the input 2 (i.e., you evaluate $(\lambda y.(5 - y)\ 2)$) the result is $5 - 2$ — in other words, 3. So, functions can have other functions as outputs.

Now, suppose you have a function f that takes two inputs y and z, and we wish to consider what happens when you apply f to a pair where the first value is the repeated as the second value. (If f is $\lambda x.\lambda y.(x - y)$ and the input value is a number, then the result should be 0.) We can do this by applying f to the value x twice, to get $((f\ x)\ x)$. But this is not a function, it is the result of applying f to x and x. If you consider this as a function of x you get

$$\lambda x.((f\ x)\ x)$$

This is the function that takes x and feeds it *twice* into f. But just as functions can create other functions as *outputs*, there is no reason not to make functions take other functions as *inputs*. The process here was competely general — we knew nothing specific about f — so the function

$$\lambda y.\lambda x.((y\ x)\ x)$$

takes an input y, and returns the function $\lambda x.((y\ x)\ x)$. This function takes an input x, and then applies y to x and then applies the result to x again. When you feed it a function, it returns the *diagonal* of that function.

Draw the function as a table of values for each pair of inputs, and you will see why this is called the '*diagonal*.'

Now, sometimes this construction does not work. Suppose we feed our diagonal function $\lambda y.\lambda x.((y\ x)\ x)$ an input that is not a function, or that is a function that does not expect two inputs? (That is, it is not a function that returns another function.) In that case, we may not get a sensible output. One response is to bite the bullet and say that everything is a function, and that we can apply anything to anything else. We won't take that approach here, as something becomes very interesting if we consider what happens if we consider variables (the x and y in the expression $\lambda y.\lambda x.((y\ x)\ x)$) to be *typed*. We could consider y to only take inputs which are functions of the right kind. That is, y is a function that expects values of some kind (let's say, of type A), and when given a value, returns a function. In fact, the function it returns has to be a function that expects values of the very same kind (also type A). The *result* is an object (perhaps a function) of some kind or other (say, type B). In other words, we can say that the variable y takes values of type $A \to (A \to B)$. Then we expect the variable x to take values of type A. We'll write these facts as follows:

This is the *untyped* λ-calculus.

$$y : A \to (A \to B) \qquad x : A$$

Now, we may put these two things together, to say derive the type of the result of applying the function y to the input value x.

$$\frac{y : A \to (A \to B) \quad x : A}{(y\ x) : A \to B}$$

Applying the result to x again, we get

$$\dfrac{\dfrac{y : A \to (A \to B) \quad x : A}{(y\,x) : A \to B} \qquad x : A}{((y\,x)\,x) : B}$$

Then when we abstract away the particular choice of the input value x, we have this

$$\dfrac{\dfrac{\dfrac{y : A \to (A \to B) \quad [x : A]}{(y\,x) : A \to B} \qquad [x : A]}{((y\,x)\,x) : B}}{\lambda x.((y\,x)\,x) : A \to B}$$

and abstracting away the choice of y, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{[y : A \to (A \to B)] \quad [x : A]}{(y\,x) : A \to B} \qquad [x : A]}{((y\,x)\,x) : B}}{\lambda x.((y\,x)\,x) : A \to B}}{\lambda y.\lambda x.((y\,x)\,x) : (A \to (A \to B)) \to (A \to B)}$$

so the diagonal function $\lambda y.\lambda x.((y\,x)\,x)$ has type $(A \to (A \to B)) \to (A \to B)$. It takes functions of type $A \to (A \to B)$ as input and returns an output of type $A \to B$.

Does that process look like something you have already seen?

We may use these λ-terms to represent proofs. Here are the definitions. We will first think of formulas as *types*.

$$\textsc{type} ::= \textsc{atom} \mid (\textsc{type} \to \textsc{type})$$

Then, given the class of types, we can construct terms for each type.

DEFINITION 1.29 [TYPED SIMPLE λ-TERMS] The class of typed simple λ-terms is defined as follows:

- For each type A, there is an infinite supply of variables x^A, y^A, z^A, w^A, x_1^A, x_2^A, etc.
- If M is a term of type $A \to B$ and N is a term of type A, then $(M\,N)$ is a term of type B.
- If M is a term of type B then $\lambda x^A.M$ is a term of type $A \to B$.

These formation rules for types may be represented in ways familiar to those of us who care for proofs. See Figure 1.3.

Sometimes we write variables without superscripts, and leave the typing of the variable understood from the context. It is simpler to write $\lambda y.\lambda x.((y\,x)\,x)$ instead of $\lambda y^{A\to(A\to B)}.\lambda x^A((y^{A\to(A\to B)}\,x^A)\,x^A)$.

$$\dfrac{M : A \to B \quad N : A}{(M\,N) : B}{\scriptstyle \to E} \qquad \dfrac{\begin{array}{c}[x : A]^{(i)} \\ \vdots \\ M : B\end{array}}{\lambda x.M : A \to B}{\scriptstyle \to I, i}$$

Figure 1.3: RULES FOR λ-TERMS

Not everything that *looks* like a typed λ-term actually is. Consider the term

$$\lambda x.(x\,x)$$

There is no such simple typed λ-term. Were there such a term, then x would have to both have type $A \to B$ and type A. But as things stand now, a variable can have only one type. Not every λ-term is a *typed* λ-term.

Now, it is clear that typed λ-terms stand in some interesting relationship to proofs. From any typed λ-term we can reconstruct a unique proof.Take $\lambda x.\lambda y.(y\,x)$, where y has type $p \to q$ and x has type p. We can rewrite the unique formation pedigree of the term as a tree.

$$\dfrac{\dfrac{\dfrac{[y : p \to q] \quad [x : p]}{(y\,x) : q}}{\lambda y.(y\,x) : (p \to q) \to q}}{\lambda x.\lambda y.(y\,x) : p \to ((p \to q) \to q)}$$

and once we erase the terms, we have a proof of $p \to ((p \to q) \to q)$. The term is a compact, linear representation of the proof which is presented as a tree.

The mapping from terms to proofs is many-to-one. Each typed term constructs a single proof, but there are many different terms for the one proof. Consider the proofs

$$\dfrac{p \to q \quad p}{q} \qquad \dfrac{p \to (q \to r) \quad p}{(q \to r)}$$

we can label them as follows

$$\dfrac{x : p \to q \quad y : p}{(xy) : q} \qquad \dfrac{z : p \to (q \to r) \quad y : p}{(zy) : q \to r}$$

we could combine them into the proof

$$\dfrac{\dfrac{z : p \to (q \to r) \quad y : p}{(zy) : q \to r} \qquad \dfrac{x : p \to q \quad y : p}{(xy) : q}}{(zy)(xy) : r}$$

but if we wished to discharge just *one* of the instances of p, we would have to have chosen a different term for one of the two subproofs. We could have chosen the variable w for the first p, and used the following term:

$$\dfrac{\dfrac{\dfrac{z : p \to (q \to r) \quad [w : p]}{(zw) : q \to r} \quad \dfrac{x : p \to q \quad y : p}{(xy) : q}}{(zw)(xy) : r}}{\lambda w.(zw)(xy) : p \to r}$$

So, the choice of variables allows us a great deal of choice in the construction of a term for a proof. The choice of variables both does *not* matter (who cares if we replace x^A by y^A) and *does* matter (when it comes to discharge an assumption, the formulas discharged are exactly those labelled by the particular free variable bound by λ at that stage).

DEFINITION 1.30 [FROM TERMS TO PROOFS AND BACK] For every typed term M (of type A), we find $\text{PROOF}(M)$ (of the formula A) as follows:

» $\text{PROOF}(x^A)$ is the identity proof A.
» If $\text{PROOF}(M^{A \to B})$ is the proof π_1 of $A \to B$ and $\text{PROOF}(N^A)$ is the proof π_2 of A, then extend them with one [$\to$*E*] step into the proof $\text{PROOF}(MN^B)$ of B.
» If $\text{PROOF}(M^B)$ is a proof π of B and x^A is a variable of type A, then construct the proof $\text{PROOF}((\lambda x.M)^{A \to B})$ of type $A \to B$ as follows: Extend the proof π by discharging each premise in π of type A labelled with the variable x^A.

Conversely, for any proof π, we find the set $\text{TERMS}(\pi)$ as follows:

» $\text{TERMS}(A)$ is the set of variables of type A. (Note that the term is an unbound variable, whose type is the only assumption in the proof.)
» If π_l is a proof of $A \to B$, and M (of type $A \to B$) is a member of $\text{TERMS}(\pi_l)$, and N (of type A) is a member of $\text{TERMS}(\pi_r)$, then (MN) (which is of type B) is a member of $\text{TERMS}(\pi)$, where π is the proof found by extending π_l and π_r by the [$\to$*E*] step. (Note that if the unbound variables in M have types corresponding to the assumptions in π_l and those in N have types corresponding to the assumptions in π_r, then the unbound variables in (MN) have types corresponding to the variables in π.)
» Suppose π is a proof of B, and we extend π into the proof π' by discharging some set (possibly empty) of instances of the formula A, to derive $A \to B$ using [$\to$*I*]. Then M is a member of $\text{TERMS}(\pi)$ for which a variable x labels *all* and *only* those assumptions A that are discharged in this [$\to$*I*] step, then $\lambda x.M$ is a member of $\text{TERMS}(\pi')$. (Notice that the free variables in $\lambda x.M$ correspond to the remaining active assumptions in π'.)

THEOREM 1.31 [TERMS ARE PROOFS ARE TERMS] *If* $M \in$ TERMS(π) *then* $\pi =$ PROOF(M). *Conversely,* $M' \in$ TERMS(PROOF(M)) *if and only if* M' *is a relabelling of* M.

Proof: For the first part, we proceed by induction on the proof π. If π is an atomic proof, then since TERMS(A) is the set of variables of type A, and since PROOF(x^A) is the identity proof A, we have the base case of the induction. If π is composed of two proofs, π_l of $A \to B$, and π_r of A, joined by an $[{\to}E]$ step, then M is in TERMS(π) if and only if $M = (N_1 N_2)$ where $N_1 \in$ TERMS(π_l) and $N_2 \in$ TERMS(π_r). But by the induction hypothesis, if $N_1 \in$ TERMS(π_l) and $N_2 \in$ TERMS(π_r), then $\pi_l =$ PROOF(N_1) and $\pi_r =$ PROOF(N_2), and as a result, $\pi =$ PROOF(M), as desired.

Finally, if π is a proof of B, extended to the proof π' of $A \to B$ by discharging some (possibly empty) set of instances of A, then if M is in TERMS(π) if and only if $M = \lambda x.N$, $N \in$ TERMS(π'), and x labels those (and only those) instances of A discharged in π. By the induction hypothesis, $\pi' =$ PROOF(N). It follows that $\pi =$ PROOF$(\lambda x.N)$, since x labels all and only the formulas discharged in the step from π' to π.

For the second part of the proof, if $M' \in$ TERMS(PROOF(M)), then if M is a variable, PROOF(M) is an identity proof of some formula A, and TERMS(PROOF(M)) is a variable with type A, so the base case of our hypothesis is proved. Suppose the hypothesis holds for terms simpler than our term M. If M is an application term $(N_1 N_2)$, then PROOF$(N_1 N_2)$ ends in $[{\to}E]$, and the two subproofs are PROOF(N_1) and (N_2) respectively. By hypothesis, TERM(PROOF(N_1)) is some relabelling of N_1 and TERM(PROOF(N_2)) is some relabelling of N_2, so TERM(PROOF$(N_1 N_2)$) may only be relabelling of $(N_1 N_2)$ as well. Similarly, if M is an abstraction term $\lambda x.N$, then PROOF$(\lambda x.N)$ ends in $[{\to}I]$ to prove some conditional $A \to B$, and PROOF(N) is a proof of B, in which some (possibly empty) collection of instances of A are about to be discharged. By hypothesis, TERM(PROOF(N)) is a relabelling of N, so TERM(PROOF$(\lambda x.N)$) can only be a relabelling of $\lambda x.N$. ∎

The following theorem shows that the λ-terms of different kinds of proofs have different features.

THEOREM 1.32 [DISCHARGE CONDITIONS AND TERMS] M *is a linear* λ*-term (a term of some linear proof) iff each* λ *expression in* M *binds exactly one variable.* M *is a relevant* λ*-term (a term of a relevant proof) iff each* λ *expression in* M *binds at least one variable.* M *is a an affine* λ*-term (a term of some affine proof) iff each* λ *expression binds at most one variable.*

Proof: Check the definition of PROOF(M). If M satisfies the conditions on variable binding, PROOF(M) satisfies the corresponding discharge conditions. Conversely, if π satisfies a discharge condition, the terms in TERM(π) are the corresponding kinds of λ-term. ∎

The most interesting connection between proofs and λ-terms is not simply this pair of mappings. It is the connection between *normalisation* and *evaluation*. We have seen how the application of a function, like $\lambda x.((y\,x)\,x)$ to an input like M is found by removing the lambda binder, and substituting the term M for each variable x that was bound by the binder. In this case, we get $((y\,M)\,M)$.

DEFINITION 1.33 [β REDUCTION] The term $\lambda x.M\,N$ is said to directly β-reduce to the term $M[x := N]$ found by substituting the term N for each free occurrence of x in M.

Furthermore, M β-reduces in one step to M' if and only if some subterm N inside M immediately β-reduces to N' and $M' = M[N := N']$. A term M is said to β-reduce to M^* if there is some chain $M = M_1, \cdots M_n = M^*$ where each M_i β-reduces in one step to M_{i+1}.

Consider what this means for *proofs*. The term $(\lambda x.M\,N)$ immediately β-reduces to $M[x := N]$. Representing this transformation as a proof, we have

$$\dfrac{\dfrac{\begin{matrix}[x:A]\\ \vdots\,\pi_l\\ M:B\end{matrix}}{\lambda x.M : A \to B} \quad \begin{matrix}\vdots\,\pi_r\\ N:A\end{matrix}}{(\lambda x.M\,N) : B} \quad \Longrightarrow_\beta \quad \begin{matrix}\vdots\,\pi_r\\ N:A\\ \vdots\,\pi_l\\ M[x:=N]:B\end{matrix}$$

and β-reduction corresponds to normalisation. This fact leads immediately to the following theorem.

THEOREM 1.34 [NORMALISATION AND β-REDUCTION] *A proof* PROOF(N) *is normal if and only if the term* N *does not* β*-reduce to another term. If* N β*-reduces to* N' *then a normalisation process sends* PROOF(N) *to* PROOF(N').

This natural reading of normalisation as function application, and the easy way that we think of $(\lambda x.M\,N)$ as *being identical to* $M[x := N]$ leads some to make the following claim:

> If π and π' *normalise* to the same proof,
> then π and π' are *really* the same proof.

We will discuss proposals for the identity of proofs in a later section.

1.5 | HISTORY

Gentzen's technique for natural deduction is not the only way to represent this kind of reasoning, with introduction and elimination rules for connectives. Independently of Gentzen, the Polish logician, Stanisław Jaśkowski constructed a closely related, but different system for presenting proofs in a natural deduction style. In Jaśkowski's system, a proof is a *structured list* of formulas. Each formula in the list is either a *supposition*, or it follows from earlier formulas in the list by means of the rule of *modus ponens* (conditional elimination), or it is proved by *conditionalisation*. To prove something by conditionalisation you first make a supposition of the antecedent: at this point you start a *box*. The contents of a box constitute a proof, so if you want to use a formula from outside the box, you may *repeat* a formula into the inside. A conditionalisation step allows you to exit the box, discharging the supposition you made upon entry. Boxes can be nested, as follows:

1.	$A \to (A \to B)$	Supposition
2.	A	Supposition
3.	$A \to (A \to B)$	1, Repeat
4.	$A \to B$	2, 3, Modus Ponens
5.	B	2, 4, Modus Ponens
6.	$A \to B$	2–5, Conditionalisation
7.	$(A \to (A \to B)) \to (A \to B)$	1–6, Conditionalisation

This nesting of boxes, and repeating or reiteration of formulas to enter boxes, is the distinctive feature of Jaśkowski's system. Notice that we could prove the formula $(A \to (A \to B)) \to (A \to B)$ without using a duplicate discharge. The formula A is used twice as a minor premise in a Modus Ponens inference (on line 4, and on line 5), and it is then discharged at line 6. In a Gentzen proof of the same formula, the assumption A would have to be made twice.

Jaśkowski proofs also straightforwardly incorporate the effects of a vacuous discharge in a Gentzen proof. We can prove $A \to (B \to A)$ using the rules as they stand, without making any special plea for a vacuous discharge:

1.	A	Supposition
2.	B	Supposition
3.	A	1, Repeat
4.	$B \to A$	2–3, Conditionalisation
5.	$A \to (B \to A)$	1–4, Conditionalisation

The formula B is supposed, and it is not *used* in the proof that follows. The formula A on line 4 occurs *after* the formula B on line 3, in the subproof, but it is harder to see that it is inferred *from* that$B$. Conditionalisation, in Jaśkowski's system, colludes with reiteration to allow the effect of vacuous discharge. It appears that the "fine control" over inferential connections between formulas in proofs in a Gentzen proof is somewhat obscured in the *linearisation* of a Jaśkowski proof. The fact that one formula occurs *after* another says nothing about how that formula is inferentially connected to its forbear.

Jaśkowski's account of proof was modified in presentation by Frederic Fitch (boxes become assumption *lines* to the left, and hence become somewhat simpler to draw and to typeset). Fitch's natural deduction ststem gained quite some popularity in undergraduate education in logic in the 1960s and following decades in the United States [29]. Edward Lemmon's text *Beginning Logic* [49] served a similar purpose in British logic education. Lemmon's account of natural deduction is similar to this, except that it does without the need to reiterate by *breaking the box*.

1	(1)	$A \to (A \to B)$	Assumption
2	(2)	A	Assumption
1,2	(3)	$A \to B$	1, 2, Modus Ponens
1,2	(4)	B	2,3, Modus Ponens
1	(5)	$A \to B$	2, 4, Conditionalisation
	(6)	B	1, 5, Conditionalisation

Now, line numbers are joined by *assumption numbers*: each formula is tagged with the line number of each assumption upon which that formula depends. The rules for the conditional are straightforward: If $A \to B$ depends on the assumptions X and A depends on the assumptions Y, then you can derive B, depending on the assumptions X, Y. (You should ask yourself if X, Y is the *set* union of the *sets* X and Y, or the *multiset* union of the *multisets* X and Y. For Lemmon, the assumption collections are *sets*.) For conditionalisation, if B depends on X, A, then you can derive $A \to B$ on the basis of X alone. As you can see, vacuous discharge is harder to motivate, as the rules stand now. If we attempt to use the strategy of the Jaśkowski proof, we are soon stuck:

1	(1)	A	Assumption
2	(2)	B	Assumption
$\vdots$	(3)	$\vdots$	

There is no way to attach the assumption number "2" to the formula A. The linear presentation is now explicitly *detached* from the inferential connections between formulas by way of the assumption numbers. Now the assumption numbers tell you all you need to know about the provenance of formulas. In Lemmon's own system, you *can* prove the formula $A \to (B \to A)$ but only, as it happens, by taking a detour through conjunction or some other connective.

1	(1)	A	Assumption
2	(2)	B	Assumption
1,2	(3)	$A \wedge B$	1,2, Conjunction intro
1,2	(4)	A	3, Conjunction elim
1	(5)	$B \to A$	2,4, Conditionalisation
	(6)	$A \to (B \to A)$	1,5, Conditionalisation

This seems quite unsatisfactory, as it breaks the normalisation property. (The formula $A \to (B \to A)$ is proved only by a non-normal proof—in this case, a proof in which a conjunction is introduced and then immediately eliminated.) Normalisation can be restored to Lemmon's system, but at the cost of the introduction of a new rule, the rule of *weakening*, which says that if A depends on assumptions X, then we can infer A depending on assumptions X together with another formula.

For more information on the history of natural deduction, consult Jeffrey Pelletier's article [65].

Notice that the lines in a Lemmon proof don't just contain *formulas* (or formulas tagged a line number and information about how the formula was deduced). They are *pairs*, consisting of a formula, and the formulas upon which the formula depends. In a Gentzen proof this information is implicit in the structure of the proof. (The formulas upon which a formula depends in a Gentzen proof are the leaves in the tree above that formula that are undischarged at the moment that this formula is derived.) This feature of Lemmon's system was not original to him. The idea of making completely explicit the assumptions upon which a formula depends had also occurred to Gentzen, and this insight is our topic for the next section.

» «

Linear, relevant and affine implication have a long history. Relevant implication bust on the scene through the work of Alan Anderson and Nuel Belnap in the 1960s and 1970s [1, 2], though it had precursors in the work of the Russian logician, I. E. Orlov in the 1920s [21, 62]. The idea of a proof in which conditionals could only be introduced if the assumption for discharge was genuinely *used* is indeed one of the motivations for relevant implication in the Anderson–Belnap tradition. However, *other* motivating concerns played a role in the development of relevant logics. For other work on relevant logic, the work of Dunn [23, 25], Routley and Meyer [80], Read [75] and Mares [52] are all useful. Linear logic arose much more centrally out of proof-theoretical concerns in the work of the proof-theorist Jean-Yves Girard in the 1980s [34, 35]. A helpful introduction to linear logic is the text of Troelstra [96]. Affine logic is introduced in the tradition of linear logic as a variant on linear implication. Affine implication is quite close, however to the implication in Łukasiewicz's infinitely valued logic—which is slightly stronger, but shares the property of rejecting all *contraction*-related principles [77]. These logics are all *substructural* logics [22, 64, 78]

The definition of normality is due to Prawitz [68], though glimpses of the idea are present in Gentzen's original work [30].

The λ-calculus is due to Alonzo Church [18], and the study of λ-calculi has found many different applications in logic, computer science, type theory and related fields [4, 41, 86]. The correspondence between formulas/proofs and types/terms is known as the Curry–Howard correspondence [19, 42].

1.6 | EXERCISES

I am not altogether confident about the division of the exercises into "basic," "intermediate," and "advanced." I'd appreciate your feedback on whether some exercises are too easy or too difficult for their categories.

Working through these exercises will help you understand the material. As with all logic exercises, if you want to deepen your understanding of these techniques, you should attempt the exercises until they are no longer difficult. So, attempt each of the different kinds of *basic* exercises, until you know you can do them. Then move on to the *intermediate* exercises, and so on. (The *project* exercises are not the kind of thing that can be completed in one sitting.)

BASIC EXERCISES

Q1 Which of the following formulas have proofs with no premises?

1 : $p \to (p \to p)$
2 : $p \to (q \to q)$
3 : $((p \to p) \to p) \to p$
4 : $((p \to q) \to p) \to p$
5 : $((q \to q) \to p) \to p$
6 : $((p \to q) \to q) \to p$
7 : $p \to (q \to (q \to p))$
8 : $(p \to q) \to (p \to (p \to q))$
9 : $((q \to p) \to p) \to ((p \to q) \to q)$
10 : $(p \to q) \to ((q \to p) \to (p \to p))$
11 : $(p \to q) \to ((q \to p) \to (p \to q))$
12 : $(p \to q) \to ((p \to (q \to r)) \to (p \to r))$
13 : $(q \to p) \to ((p \to q) \to ((q \to p) \to (p \to q)))$
14 : $((p \to p) \to p) \to ((p \to p) \to ((p \to p) \to p))$
15 : $(p_1 \to p_2) \to ((q \to (p_2 \to r)) \to (q \to (p_1 \to r)))$

Formula 4 is *Peirce's Law*. It is a two-valued classical logic tautology.

For each formula that can be proved, find a proof that complies with the strictest discharge policy possible.

Q2 Annotate your proofs from Exercise 1 with λ-terms. Find a most general λ-term for each provable formula.

Q3 Construct a proof from $q \to r$ to $(q \to (p \to p)) \to (q \to r)$ using vacuous discharge. Then construct a proof of $q \to (p \to p)$ (also using vacuous discharge). Combine the two proofs, using [$\to$*E*] to deduce $q \to r$. Normalise the proof you find. Then annotate each proof with λ-terms, and explain the β reductions of the terms corresponding to the normalisation.

Then construct a proof from $(p \to r) \to ((p \to r) \to q))$ to $(p \to r) \to q$ using duplicate discharge. Then construct a proof from $p \to (q \to r)$ and $p \to q$ to $p \to r$ (also using duplicate discharge). Combine the two proofs, using [$\to$*E*] to deduce q. Normalise the proof you find. Then annotate each proof with λ-terms, and explain the β reductions of the terms corresponding to the normalisation.

Q4 Find types and proofs for each of the following terms.

1 : $\lambda x.\lambda y.x$
2 : $\lambda x.\lambda y.\lambda z.((xz)(yz))$
3 : $\lambda x.\lambda y.\lambda z.(x(yz))$
4 : $\lambda x.\lambda y.(yx)$
5 : $\lambda x.\lambda y.((yx)x)$

Which of the proofs are linear, which are relevant and which are affine?

Q5 Show that there is no normal relevant proof of these formulas.

1 : $p \to (q \to p)$
2 : $(p \to q) \to (p \to (r \to q))$
3 : $p \to (p \to p)$

Q6 Show that there is no normal affine proof of these formulas.

1 : $(p \to q) \to ((p \to (q \to r)) \to (p \to r))$

$2 : (p \to (p \to q)) \to (p \to q)$

Q7 Show that there is no normal proof of these formulas.

$1 : ((p \to q) \to p) \to p$
$2 : ((p \to q) \to q) \to ((q \to p) \to p)$

Q8 Find a formula that can has both a relevant proof and an affine proof, but no linear proof.

INTERMEDIATE EXERCISES

Q9 Consider the following "truth tables."

GD3

$\to$	t	n	f
t	t	n	f
n	t	t	f
f	t	t	t

Ł3

$\to$	t	n	f
t	t	n	n
n	t	t	f
f	t	t	t

RM3

$\to$	t	n	f
t	t	f	f
n	t	n	f
f	t	t	t

A GD3 tautology is a formula that receives the value t in every GD3 valuation. An Ł3 tautology is a formula that receives the value t in every Ł3 valuation. Show that every formula with a standard proof is a GD3 tautology. Show that every formula with an affine proof is an Ł3 tautology.

Q10 Consider proofs that have paired steps of the form [$\to E/\to I$]. That is, a conditional is eliminated only to be introduced again. The proof has a sub-proof of the form of this proof fragment:

$$\dfrac{\dfrac{A \to B \quad [A]^{(i)}}{B}{\scriptstyle \to E}}{A \to B}{\scriptstyle \to I,i}$$

These proofs contain redundancies too, but they may well be normal. Call a proof with a pair like this CIRCUITOUS. Show that all circuitous proofs may be transformed into non-circuitous proofs with the same premises and conclusion.

Q11 In Exercise 5 you showed that there is no normal relevant proof of $p \to (p \to p)$. By normalisation, it follows that there is no relevant proof (normal or not) of $p \to (p \to p)$. Use this fact to explain why it is more natural to consider relevant arguments with *multisets* of premises and not just *sets* of premises. (HINT: is the argument from p, p to p relevantly valid?)

Q12 You might think that "if … then …" is a slender foundation upon which to build an account of logical consequence. Remarkably, there is rather a lot that you can do with implication alone, as these next questions ask you to explore.

First, define $A \hat{\vee} B$ as follows: $A \hat{\vee} B ::= (A \to B) \to B$. In what way is "$\hat{\vee}$" like *disjunction*? What usual features of disjunction are not had by $\hat{\vee}$?

(Pay attention to the behaviour of $\hat{\vee}$ with respect to different discharge policies for implication.)

Q13 Provide introduction and elimination rules for $\hat{\vee}$ that do not involve the conditional connective $\rightarrow$.

Q14 Now consider *negation*. Given an ATOM p, define the p-negation $\neg_p A$ to be $A \rightarrow p$. In what way is "$\neg_p$" like negation? What usual features of negation are not had by $\neg_p$ defined in this way? (Pay attention to the behaviour of $\neg$ with respect to different discharge policies for implication.)

Q15 Provide introduction and elimination rules for $\neg_p$ that do not involve the conditional connective $\rightarrow$.

Q16 You have probably noticed that the inference from $\neg_p\neg_p A$ to A is not, in general, valid. Define a *new* language CFORMULA inside FORMULA as follows:

$$\text{CFORMULA} ::= \neg_p\neg_p\text{ATOM} \mid (\text{CFORMULA} \rightarrow \text{CFORMULA})$$

Show that $\neg_p\neg_p A \mathrel{\therefore} A$ and $A \mathrel{\therefore} \neg_p\neg_p A$ are valid when A is a CFORMULA.

Q17 Now define $A\dot{\wedge}B$ to be $\neg_p(A \rightarrow \neg_p B)$, and $A\dot{\vee}B$ to be $\neg_p A \rightarrow B$. In what way are $A\dot{\wedge}B$ and $A\dot{\vee}B$ like conjunction and disjunction of A and B respectively? (Consider the difference between when A and B are FORMULAS and when they are CFORMULAS.)

Q18 Show that if there is a normal relevant proof of $A \rightarrow B$ then there is an ATOM occurring in both A and B.

Q19 Show that if we have two conditional connectives $\rightarrow_1$ and $\rightarrow_2$ defined using different discharge policies, then the conditionals collapse, in the sense that we can construct proofs from $A \rightarrow_1 B$ to $A \rightarrow_2 B$ and *vice versa*.

Q20 Explain the significance of the result of Exercise 19.

Q21 Add rules the obvious introduction rules for a *conjunction* connective $\otimes$ as follows:

$$\frac{A \quad B}{A \otimes B}\,{\scriptstyle \otimes I}$$

Show that if we have the following two $\otimes E$ rules:

$$\frac{A \otimes B}{A}\,{\scriptstyle \otimes I_1} \qquad \frac{A \otimes B}{B}\,{\scriptstyle \otimes I_2}$$

we may simulate the behaviour of vacuous discharge. Show, then, that we may normalise proofs involving these rules (by showing how to eliminate all indirect pairs, including $\otimes I/\otimes E$ pairs).

ADVANCED EXERCISES

Q22 Another demonstration of the subformula property for normal proofs uses the notion of a *track* in a proof.

DEFINITION 1.35 [TRACK] A sequence $A_0, \ldots, A_n$ of formula instances in the proof π is a *track* of length $n+1$ in the proof π if and only if

- A_0 is a *leaf* in the proof tree.
- Each A_{i+1} is immediately below A_i.
- For each $i < n$, A_i is not a minor premise of an application of [$\to E$].

A track whose terminus A_n is the conclusion of the proof π is said to be a TRACK OF ORDER 0. If we have a track t whose terminus A_n is the minor premise of an application of [$\to E$] whose conclusion is in a track of order n, we say that t is a TRACK OF ORDER $n+1$.

The following annotated proof gives an example of tracks.

$$\dfrac{\dfrac{\heartsuit[B \to C]^{(2)} \qquad \dfrac{\dfrac{\spadesuit\, A \to ((D \to D) \to B) \qquad \diamondsuit[A]^{(2)}}{\spadesuit\, (D \to D) \to B}{\scriptstyle \to E} \qquad \dfrac{\clubsuit[D]^{(1)}}{\clubsuit\, D \to D}{\scriptstyle \to I,1}}{\spadesuit\, B}{\scriptstyle \to E}}{\dfrac{\heartsuit\, C}{\heartsuit\, A \to C}{\scriptstyle \to I,2}}{\scriptstyle \to E}}{\heartsuit\, (B \to C) \to (A \to C)}{\scriptstyle \to I,3}$$

(Don't let the fact that this proof has one track of each order 0, 1, 2 and 3 make you think that proofs can't have more than one track of the same order. Look at this example —

$$\dfrac{\dfrac{A \to (B \to C) \quad A}{B \to C} \qquad B}{C}$$

— it has two tracks of order 1.) The formulas labelled with $\heartsuit$ form one track, starting with $B \to C$ and ending at the conclusion of the proof. Since this track ends at the conclusion of the proof, it is a track of order 0. The track consisting of $\spadesuit$ formulas starts at $A \to ((D \to D) \to B)$ and ends at B. It is a track of order 1, since its final formula is the minor premise in the [$\to E$] whose conclusion is C, in the $\heartsuit$ track of order 0. Similarly, the $\diamondsuit$ track is order 2 and the $\clubsuit$ track has order 3.

For this exercise, prove the following lemma by induction on the construction of a proof.

LEMMA 1.36 *In every proof, every formula is in one and only one track, and each track has one and only one order.*

Then prove this lemma.

LEMMA 1.37 *Let $t : A_0, \ldots, A_n$ be a track in a normal proof. Then*

a) *The rules applied within the track consist of a sequence (possibly empty) of [$\to$E] steps and then a sequence (possibly empty) of [$\to$I] steps.*

b) *Every formula A_i in t is a subformula of A_0 or of A_n.*

Now prove the subformula theorem, using these lemmas.

Q23 Consider the result of Exercise 19. Show how you might define a natural deduction system containing (say) both a linear and a standard conditional, in which there is *no* collapse. That is, construct a system of natural deduction proofs in which there are two conditional connectives: $\rightarrow_l$ for linear conditionals, and $\rightarrow_s$ for standard conditionals, such that whenever an argument is valid for a linear conditional, it is (in some appropraite sense) valid in the system you design (when $\rightarrow$ is translated as $\rightarrow_l$) and whenever an argument is valid for a standard conditional, it is (in some appropriate sense) valid in the system you design (when $\rightarrow$ is translated as $\rightarrow_s$). What mixed inferences (those using both $\rightarrow_l$ and $\rightarrow_s$) are valid in your system?

Q24 Suppose we have a new discharge policy that is "stricter than linear." The *ordered* discharge policy allows you to discharge only the *rightmost* assumption at any one time. It is best paired with a strict version of $\rightarrow E$ according to which the major premise ($A \rightarrow B$) is on the left, and the minor premise (A) is on the right. What is the resulting logic *like*? Does it have the normalisation property?

Q25 Take the logic of Exercise 24, and extend it with *another* connective $\leftarrow$, with the rule $\leftarrow E$ in which the major premise ($B \leftarrow A$) is on the *right*, and the minor premise (A) is on the *left*, and $\leftarrow I$, in which the *leftmost* assumption is discharged. Examine the connections between $\rightarrow$ and $\leftarrow$. Does normalisation work for *these* proofs? This is *Lambek's* logic for syntactic types [46, 47, 59, 60].

Q26 Show that there is a way to be even *stricter* than the discharge policy of Exercise 24. What is the *strictest* discharge policy for $\rightarrow I$, that will result in a system which normalises, provided that $\rightarrow E$ (in which the major premise is leftmost) is the only other rule for implication.

Q27 Consider the introduction rule for $\otimes$ given in Exercise 21. Construct an appropriate *elimination* rule for fusion which does not allow the simulation of vacuous (or duplicate) discharge, and for which proofs normalise.

Q28 Identify two proofs where one can be reduced to the other by way of the elimination of *circuitous* steps (see Exercise 10). Characterise the identities this provides among λ-terms. Can this kind if identification be maintained along with β-reduction?

PROJECT

Q29 Thoroughly and systematically explain and evaluate the considerations for choosing one discharge policy over another. This will involve looking at the different *uses* to which one might put a system of natural deduction, and then, relative to a use, what one might say in favour of a different policy.

SEQUENT CALCULUS 2

In this chapter we will look at a different way of thinking about deduction: Gentzen's *sequent calculus*. The core idea is straightforward. We want to know what follows from what, so we will keep a track of facts of consequence: facts we will record in the following form:

$$A \vdash B$$

One can read "$A \vdash B$" in a number of ways. You can say that B follows from A, or that A entails B, that the argument from A to B is valid, or, saliently, that there is a proof from A to B. The symbol used here is sometimes called the TURNSTILE.

"Scorning a turnstile wheel at her reverend helm, she sported there a tiller; and that tiller was in one mass, curiously carved from the long narrow lower jaw of her hereditary foe. The helmsman who steered by that tiller in a tempest, felt like the Tartar, when he holds back his fiery steed by clutching its jaw. A noble craft, but somehow a most melancholy! All noble things are touched with that." — Herman Melville, *Moby Dick*.

Once we have this notion of consequence, we can ask ourselves what properties consequence has. There are many different ways you could answer this question. The focus of this section will be a particular technique, originally due to Gerhard Gentzen. We can think of consequence—relative to a particular *language*—like this: when we want to know about the relation of consequence, we first consider each different kind of formula in the language. To make the discussion concrete, let's consider a very simple language: the language of propositional logic with only two connectives, *conjunction* $\land$ and *disjunction* $\lor$. That is, we will now look at formulas expressed in a language with the following grammar:

$$\text{FORMULA} ::= \text{ATOM} \mid (\text{FORMULA} \land \text{FORMULA}) \mid (\text{FORMULA} \lor \text{FORMULA})$$

To characterise consequence relations, we need to characterise how consequence works on the *atoms* of the language, and then how the addition of $\land$ and $\lor$ expands the repertoire of facts about consequence. To do this, we need to know when $A \vdash B$ when A is a conjunction, or when A is a disjunction, and when B is a conjunction, or when B is a disjunction. In other words, for each connective, we need to know when it is appropriate to deduce something *from* a formula featuring that connective, and when it is appropriate to deduce a formula featuring that connective. Another way of putting it is that we wish to know how a connective behave on the left of the turnstile, and how it behaves on the right.

In a sequent system, we will have rules concerning statements about consequence—and these statements are the *sequents* at the heart of the system. Because we can make false claims as well as true ones, we will use the following *bent* turnstile for the general case of a sequent

I follow Lloyd Humberstone, who, as far as I am aware, introduced this convention for sequents [43]. Gentzen used the arrow, which we have already used for the object-language conditional.

$$A \succ B$$

and we reserve the straight turnstile $A \vdash B$ for when we wish to explicitly claim that the sequent $A \succ B$ is *derivable*. In what follows, $p \succ p \land q$ is a perfectly good sequent, though it will not be a derivable one (for $p \wedge q$ does not follow from p), so we will not have $p \vdash p \wedge q$.

This is a *formal* account of consequence. We look only at the form of propositions and not their content. For atomic propositions (those with no internal form) there is nothing upon which we could pin a claim to consequence. Thus we never have $p \vdash q$ where p and q are different atoms, while $p \vdash p$ for all atoms p.

The answers for our language seem straightforward. For atomic formulas, p and q, $p \succ q$ is derivable only if p and q are the *same* atom: so we have $p \vdash p$ for each atom p. For conjunction, we can say that if $A \succ B$ and $A \succ C$ are derivable, then so is $A \succ B \wedge C$. That's how we can infer *to* a conjunction. Inferring *from* a conjunction is also straightforward. We can say that $A \wedge B \succ C$ when $A \succ C$, or when $B \succ C$. For disjunction, we can reason similarly. We can say $A \vee B \succ C$ when $A \succ C$ and $B \succ C$. We can say $A \succ B \vee C$ when $A \succ B$, or when $A \succ C$. This is *inclusive* disjunction, not exclusive disjunction.

You can think of these definitions as adding new material (in this case, conjunction and disjunction) to a pre-existing language. Think of the inferential repertoire of the basic language as settled (in our discussion this is *very* basic, just the atoms), and the connective rules are "definitional" extensions of the basic language. These thoughts are the raw materials for the development of an account of logical consequence.

2.1 | DERIVATIONS

Like natural deduction proofs, derivations involving sequents are trees. The structure is as before:

$$\dfrac{\dfrac{\bullet \quad \bullet}{\bullet} \qquad \dfrac{\dfrac{\bullet \quad \bullet}{\bullet}}{\bullet}}{\bullet}$$

Where each position on the tree follows from those above it. In a tree, the *order* of the branches does not matter. These are two different ways to present the *same* tree:

$$\frac{A \quad B}{C} \qquad \frac{B \quad A}{C}$$

In this case, the tree structure is at the one and the same time *simpler* and *more complicated* than the tree structure of natural deduction proofs. They are simpler, in that there is no discharge. They are more complicated, in that trees are not trees of formulas. They are trees consisting of *sequents*. As a result, we will call these structures DERIVATIONS instead of PROOFS. The distinction is simple. For us, a proof is a structure in which the *formulas* are connected by inferential relations in a tree-like structure. A proof will go *from* some formulas *to* other formulas, *via* yet other formulas. Our structures involving sequents are quite different. The last sequent in a tree (the *endsequent*) is itself a statement of consequence, with its own antecedent and consequent (or premise and conclusion, if you prefer.) The tree *derivation* shows you why (or perhaps how) you can infer from the antecedent to the consequent. The rules for constructing sequent derivations are found in Figure 2.1.

I say "tree-like" since we will see different structures in later chapters.

$$p \succ p \; {\scriptstyle Id}$$

$$\dfrac{L \succ C \quad C \succ R}{L \succ R}\,{\scriptstyle Cut}$$

$$\dfrac{A \succ R}{A \wedge B \succ R}\,{\scriptstyle \wedge L_1} \qquad \dfrac{A \succ R}{B \wedge A \succ R}\,{\scriptstyle \wedge L_2} \qquad \dfrac{L \succ A \quad L \succ B}{L \succ A \wedge B}\,{\scriptstyle \wedge R}$$

$$\dfrac{A \succ R \quad B \succ R}{A \vee B \succ R}\,{\scriptstyle \vee L} \qquad \dfrac{L \succ A}{L \succ A \vee B}\,{\scriptstyle \vee R_1} \qquad \dfrac{L \succ A}{L \succ B \vee A}\,{\scriptstyle \vee R_2}$$

Figure 2.1: a simple sequent system

definition 2.1 [simple sequent derivation] If the leaves of a tree are instances of the *Id* rule, and if its transitions from node to node are instances of the other rules in Figure 2.1, then the tree is said to be a simple sequent derivation.

We must read these rules completely literally. Do not presume any properties of conjunction or disjunction other than those that can be demonstrated on the basis of the rules. We will take these rules as *constituting* the behaviour of the connectives $\wedge$ and $\vee$.

example 2.2 [example sequent derivations] In this section, we will look at a few sequent derivations, demonstrating some simple properties of conjunction, disjunction, and the consequence relation.

The first derivations show some commutative and associative properties of conjunction and disjunction. Here is the conjunction case, with derivations to the effect that $p \wedge q \succ q \wedge p$, and that $p \wedge (q \wedge r) \succ (p \wedge q) \wedge r$.

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,{\scriptstyle \wedge L_2} \quad \dfrac{p \succ p}{p \wedge q \succ p}\,{\scriptstyle \wedge L_1}}{p \wedge q \succ q \wedge p}\,{\scriptstyle \wedge R}$$

$$\dfrac{\dfrac{\dfrac{p \succ p}{p \wedge (q \wedge r) \succ p}\,{\scriptstyle \wedge L_1} \quad \dfrac{\dfrac{q \succ q}{q \wedge r \succ q}\,{\scriptstyle \wedge L_1}}{p \wedge (q \wedge r) \succ q}\,{\scriptstyle \wedge L_2}}{p \wedge (q \wedge r) \succ p \wedge q}\,{\scriptstyle \wedge R} \quad \dfrac{\dfrac{r \succ r}{q \wedge r \succ r}\,{\scriptstyle \wedge L_2}}{p \wedge (q \wedge r) \succ r}\,{\scriptstyle \wedge L_2}}{p \wedge (q \wedge r) \succ (p \wedge q) \wedge r}\,{\scriptstyle \wedge R}$$

Here are the cases for disjunction. The first derivation is for the commutativity of disjunction, and the second is for associativity. (It is important to notice that these are not derivations of the commutativity or associativity of conjunction or disjunction in *general*. They only show the commutativity and associativity of conjunction and disjunction of *atomic* formulas. These are not derivations of $A \wedge B \succ B \wedge A$ (for example) since $A \succ A$ is not an axiom if A is a complex formula. We will see more on this in the next section.)

$$
\dfrac{\dfrac{p \succ p}{p \succ q \lor p}{\scriptstyle\lor R_1} \quad \dfrac{q \succ q}{q \succ p \lor q}{\scriptstyle\lor R_2}}{p \lor q \succ q \lor p}{\scriptstyle\lor L}
\qquad
\dfrac{\dfrac{\dfrac{p \succ p}{p \succ p \lor (q \lor r)}{\scriptstyle\lor R_1} \quad \dfrac{\dfrac{q \succ q}{q \succ q \lor r}{\scriptstyle\lor R_1}}{q \succ p \lor (q \lor r)}{\scriptstyle\lor R_2}}{p \lor q \succ p \lor (q \lor r)}{\scriptstyle\lor L} \quad \dfrac{\dfrac{}{r \ldots}}{r \succ p \ldots}}{(p \lor q) \lor r \succ p \lor (q \lor r)}
$$

You can see that the disjunction derivations have the same structure as those for conjunction. You can convert any derivation into another (its *dual*) by swapping conjunction and disjunction, and swapping the left-hand side of the sequent with the right-hand side. Here are some more examples of duality between derivations. The first is the dual of the second, and the third is the dual of the fourth.

Exercise 15 on page 64 asks you to make this duality precise.

$$
\dfrac{p \succ p \quad p \succ p}{p \lor p \succ p}{\scriptstyle\lor L}
\qquad
\dfrac{p \succ p \quad p \succ p}{p \succ p \land p}{\scriptstyle\land R}
\qquad
\dfrac{p \succ p \quad \dfrac{p \succ p}{p \land q \succ p}{\scriptstyle\land L_1}}{p \lor (p \land q) \succ p}{\scriptstyle\lor L}
\qquad
\dfrac{p \succ p \quad \dfrac{p \succ p}{p \succ p \lor q}{\scriptstyle\lor R}}{p \succ p \land (p \lor q)}{\scriptstyle\land R}
$$

You can use derivations you have at hand, like these, as components of other derivations. One way to do this is to use the *Cut* rule.

$$
\dfrac{\dfrac{p \succ p \quad \dfrac{p \succ p}{p \land q \succ p}{\scriptstyle\land L_1}}{p \lor (p \land q) \succ p}{\scriptstyle\lor L} \quad \dfrac{p \succ p \quad \dfrac{p \succ p}{p \succ p \lor q}{\scriptstyle\lor R}}{p \succ p \land (p \lor q)}{\scriptstyle\land R}}{p \lor (p \land q) \succ p \land (p \lor q)}{\scriptstyle Cut}
$$

Notice, too, that each of these derivations we've seen so far move from less complex formulas at the top to more complex formulas, at the bottom. Reading from bottom to top, you can see the formulas decomposing into their constituent parts. This isn't the case for all sequent derivations. Derivations that use the *Cut* rule can include new (more complex) material in the process of deduction. Here is an example:

$$
\dfrac{\dfrac{\dfrac{p \succ p}{p \succ q \lor p}{\scriptstyle\lor R_1} \quad \dfrac{q \succ q}{q \succ q \lor p}{\scriptstyle\lor R_2}}{p \lor q \succ q \lor p}{\scriptstyle\lor L} \quad \dfrac{\dfrac{q \succ q}{q \succ p \lor q}{\scriptstyle\lor R_1} \quad \dfrac{p \succ p}{p \succ p \lor q}{\scriptstyle\lor R_2}}{q \lor p \succ p \lor q}{\scriptstyle\lor L}}{p \lor q \succ p \lor q}{\scriptstyle Cut}
$$

We call the concluding sequent of a derivation the "ENDSEQUENT."

This derivation is a complicated way to deduce $p \lor q \succ p \lor q$, and it includes $q \lor p$, which is not a subformula of any formula in the final sequent of the derivation. Reading from bottom to top, the *Cut* step can introduce new formulas into the derivation.

2.2 | IDENTITY & CUT CAN BE ELIMINATED

The two distinctive rules in our proof system are *Id* and *Cut*. These rules are not about any particular kind of formula—they are *structural*, governing the behaviour of derivations, no matter *what* the nature of the formulas flanking the turnstiles. In this section we will look at the distinctive behaviour of *Id* and of *Cut*. We start with *Id*.

IDENTITY

This derivation of $p \vee q \succ p \vee q$ is *a* derivation of an identity (a sequent of the form $A \succ A$). There is a more systematic way to show that $p \vee q \succ p \vee q$, and any identity sequent. Here is a derivation of the sequent without *Cut*, and its dual, for conjunction.

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\vee R_1 \qquad \dfrac{q \succ q}{q \succ p \vee q}\vee R_2}{p \vee q \succ p \vee q}\vee L \qquad \dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\wedge L_1 \qquad \dfrac{q \succ q}{p \wedge q \succ q}\wedge L_2}{p \wedge q \succ p \wedge q}\wedge R$$

We can piece together these little derivations in order to derive any sequent of the form $A \succ A$. For example, here is the start of derivation of $p \wedge (q \vee (r_1 \wedge r_2)) \succ p \wedge (q \vee (r_1 \wedge r_2))$.

$$\dfrac{\dfrac{p \succ p}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p}\wedge L_1 \qquad \dfrac{q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)}{p \wedge (q \vee (r_1 \wedge r_2)) \succ q \vee (r_1 \wedge r_2)}\wedge L_2}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p \wedge (q \vee (r_1 \wedge r_2))}\wedge R$$

It's not a complete derivation yet, as one leaf $q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)$ is not an axiom. However, we can add the derivation for it.

$$\dfrac{\dfrac{p \succ p}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p} \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q \succ q \vee (r_1 \wedge r_2)}\vee R_1 \qquad \dfrac{\dfrac{\dfrac{r_1 \succ r_1}{r_1 \wedge r_2 \succ r_1}\wedge L_1 \qquad \dfrac{r_2 \succ r_2}{r_1 \wedge r_2 \succ r_2}\wedge L_2}{r_1 \wedge r_2 \succ r_1 \wedge r_2}\wedge R}{r_1 \wedge r_2 \succ q \vee (r_1 \wedge r_2)}\vee R_2}{q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)}\vee L}{p \wedge (q \vee (r_1 \wedge r_2)) \succ q \vee (r_1 \wedge r_2)}}{p \wedge (q \vee (r_1 \wedge r_2)) \succ p \wedge (q \vee (r_1 \wedge r_2))}$$

The derivation of $q \vee (r_1 \wedge r_2) \succ q \vee (r_1 \wedge r_2)$ itself contains a smaller identity derivation, for $r_1 \wedge r_2 \succ r_1 \wedge r_2$. The derivation displayed here uses shading to indicate the way the derivations are nested together. This result is general, and it is worth a theorem of its own.

THEOREM 2.3 [IDENTITY DERIVATIONS] *For each formula* A*, the sequent* $A \succ A$ *has a derivation. A derivation for* $A \succ A$ *may be systematically constructed from the identity derivations for the subformulas of* A*.*

Proof: We define Id_A, the IDENTITY DERIVATION FOR A by induction on the construction of A, as follows. Id_p is the axiom $p \succ p$. For complex formulas, we have

$$Id_{A\lor B}: \quad \cfrac{\cfrac{Id_A}{A \succ A\lor B}\,{\lor R_1} \qquad \cfrac{Id_B}{B\succ A\lor B}\,{\lor R_2}}{A\lor B\succ A\lor B}\,{\lor L} \qquad Id_{A\land B}: \quad \cfrac{\cfrac{Id_A}{A\land B \succ A}\,{\land L_1} \qquad \cfrac{Id_B}{A\land B\succ B}\,{\land}}{A\land B\succ A\land B}\,{\land}$$

We say that $A \succ A$ is DERIVABLE in the sequent system. If we think of *Id* as a degenerate *rule* (a rule with no premise), then its generalisation, Id_A, is a *derivable rule*.

It might seem *crazy* to have a proof of identity, like $A \succ A$ where A is a complex formula. Why don't we take Id_A as an axiom? There are a few different reasons we might like to consider for taking Id_A as derivable instead of one of the primitive axioms of the system.

THE SYSTEM IS SIMPLE: In an axiomatic theory, it is always preferable to minimise the number of primitive assumptions. Here, it's clear that Id_A is derivable, so there is no need for it to be an axiom. A system with fewer axioms is preferable to one with more, for the reason that we have reduced derivations to a smaller set of primitive notions.

These are part of a general story, to be explored throughout this book, of what it is to be a logical constant. These sorts of considerations have a long history [38].

THE SYSTEM IS SYSTEMATIC: In the system without Id_A as an axiom, when we consider a sequent like $L \succ R$ in order to know whether it it is derived (in the absence of *Cut*, at least), we can ask two separate questions. We can consider L. If it is complex perhaps $L \succ R$ is derivable by means of a left rule like [$\land$L] or [$\lor$L]. On the other hand, if R is complex, then perhaps the sequent is derivable by means of a right rule, like [$\land$R] or [$\lor$R]. If both are primitive, then $L \succ R$ is derivable by identity only. And that is it! You check the left, check the right, and there's no other possibility. There is no other condition under which the sequent is derivable. In the presence of Id_A, one would have to check if $L = R$ as well as the other conditions.

THE SYSTEM PROVIDES A CONSTRAINT: In the absence of a general identity axiom, the burden on deriving identity is passed over to the connective rules. Allowing derivations of identity statements is a hurdle over which a connective rule might be able to jump, or over which it might *fail*. As we shall see later, this is provides a constraint we can use to sort out "good" definitions from "bad" ones. Given that the left and right rules for conjunction and disjunction tell you how the connectives are to be introduced, it would seem that the rules are defective (or at the very least, *incomplete*) if they don't allow the derivation of each instance of *Id*. We will make much more of this when we consider other connectives. However, before we make more of the philosophical motivations and implications of this constraint, we will add another possible constraint on connective rules, this time to do with the other rule in our system, *Cut*.

CUT

Some of the nice properties of a sequent system are as a matter of fact, the nice features of derivations that are constructed without the *Cut* rule. Derivations constructed without *Cut* satisfy the subformula property.

THEOREM 2.4 [SUBFORMULA PROPERTY] *If δ is a sequent derivation not containing Cut, then the formulas in δ are all subformulas of the formulas in the endsequent of δ.*

Proof: You can see this merely by looking at the rules. Each rule except for *Cut* has the subformula property. ■

Notice how much simpler this proof is than the proof of Theorem 1.11.

A derivation is said to be CUT-FREE if it does not contain an instance of the *Cut* rule. Doing without *Cut* is good for some things, and bad for others. In the system of proof we're studying in this section, sequents have *very many* more proofs with *Cut* than without it.

EXAMPLE 2.5 [DERIVATIONS WITH OR WITHOUT CUT] $p \succ p \vee q$ has only one *Cut*-free derivation, it has infinitely many derivations using *Cut*.

You can see that there is only one *Cut*-free derivation with $p \succ p \vee q$ as the endsequent. The only possible last inference in such a derivation is [$\vee$R], and the only possible premise for that inference is $p \succ p$. This completes that proof.

On the other hand, there are very many different last inferences in a derivation featuring *Cut*. The most trivial example is the derivation:

$$\dfrac{p \succ p \qquad \dfrac{p \succ p}{p \succ p \vee q}\,{\scriptstyle \vee R_1}}{p \succ p \vee q}\,{\scriptstyle Cut}$$

which contains the *Cut*-free derivation of $p \succ p \vee q$ inside it. We can nest the cuts with the identity sequent $p \succ p$ as deeply as we like.

$$\dfrac{p \succ p \qquad \dfrac{p \succ p \qquad \dfrac{p \succ p}{p \succ p \vee q}\,{\scriptstyle \vee R_1}}{p \succ p \vee q}\,{\scriptstyle Cut}}{p \succ p \vee q}\,{\scriptstyle Cut} \qquad \dfrac{p \succ p \qquad \dfrac{p \succ p \qquad \dfrac{p \succ p \qquad \dfrac{p \succ p}{p \succ p \vee q}\,{\scriptstyle \vee R_1}}{p \succ p \vee q}\,{\scriptstyle Cut}}{p \succ p \vee q}\,{\scriptstyle Cut}}{p \succ p \vee q}\,{\scriptstyle Cut} \qquad \cdots$$

However, we can construct quite different derivations of our sequent, and we involve different material in the derivation. For any formula A you wish to choose, we could implicate A (an "innocent bystander") in the derivation as follows:

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee (q \wedge A)}\,{\scriptstyle \vee R_1} \qquad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,{\scriptstyle \vee R_1} \qquad \dfrac{\dfrac{q \succ q}{q \wedge A \succ q}\,{\scriptstyle \wedge L_1}}{q \wedge A \succ p \vee q}\,{\scriptstyle \vee R_2}}{p \vee (q \wedge A) \succ p \vee q}\,{\scriptstyle \vee L}}{p \succ p \vee q}\,{\scriptstyle Cut}$$

Well, it's doing *work*, in that $p \vee (q \wedge A)$ is, for many choices for A, genuinely intermediate between p and $p \vee q$. However, A is doing the kind of work that could be done by *any* formula. Choosing different values for A makes no difference to the shape of the derivation. A is doing the kind of work that doesn't require special qualifications.

In this derivation the *Cut* formula $p \vee (q \wedge A)$ is doing no genuine work. It is merely repeating the left formula p or the right formula q.

So, using *Cut* makes the search for derivations rather difficult. There are very many more *possible* derivations of a sequent, and many more actual derivations. The search space is much more constrained if we are looking for *Cut*-free derivations instead. Constructing derivations, on the other hand, is easier if we are permitted to use *Cut*. We have very many more options for constructing a derivation, since we are able to pass through formulas "intermediate" between the desired antecedent and consequent.

Do we *need* to use *Cut*? Is there anything derivable with *Cut* that cannot be derived without? Take a derivation involving *Cut*, like this:

$$
\dfrac{
  \dfrac{
    \dfrac{p \succ p}{p \wedge (q \wedge r) \succ p}\,{\scriptstyle \wedge L_1}
    \qquad
    \dfrac{\dfrac{q \succ q}{q \wedge r \succ q}\,{\scriptstyle \wedge L_1}}{p \wedge (q \wedge r) \succ q}\,{\scriptstyle \wedge L_2}
  }{p \wedge (q \wedge r) \succ p \wedge q}\,{\scriptstyle \wedge R}
  \qquad
  \dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,{\scriptstyle \wedge L_1}}{p \wedge q \succ q \vee r}\,{\scriptstyle \vee R_1}
}{p \wedge (q \wedge r) \succ q \vee r}\,{\scriptstyle Cut}
$$

The systematic technique I am using will be revealed in detail very soon.

This sequent $p \wedge (q \wedge r) \succ q \vee r$ did not have to be derived using *Cut*. We can *eliminate* the *Cut*-step from the derivation in a systematic way by showing that whenever we use a *Cut* in a derivation we could have either done without it, or used it *earlier*. For example in the last inference here, we did not need to leave the *Cut* until the last step. We could have *Cut* on the sequent $p \wedge q \succ q$ and left the inference to $q \vee r$ until later:

$$
\dfrac{
\dfrac{
  \dfrac{
    \dfrac{p \succ p}{p \wedge (q \wedge r) \succ p}\,{\scriptstyle \wedge L_1}
    \qquad
    \dfrac{\dfrac{q \succ q}{q \wedge r \succ q}\,{\scriptstyle \wedge L_1}}{p \wedge (q \wedge r) \succ q}\,{\scriptstyle \wedge L_2}
  }{p \wedge (q \wedge r) \succ p \wedge q}\,{\scriptstyle \wedge R}
  \qquad
  \dfrac{q \succ q}{p \wedge q \succ q}\,{\scriptstyle \wedge L_1}
}{p \wedge (q \wedge r) \succ q}\,{\scriptstyle Cut}
}{p \wedge (q \wedge r) \succ q \vee r}\,{\scriptstyle \vee R_1}
$$

The similarity with non-normal proofs as discussed in the previous section is *not* an accident.

Now the *Cut* takes place on the conjunction $p \wedge q$, which is introduced immediately before the application of the *Cut*. Notice that in this case we use the *Cut* to get us to $p \wedge (q \wedge r) \succ$, which is one of the sequents already seen in the derivation! This derivation repeats itself. (Do not be deceived, however. It is not a *general* phenomenon among proofs involving *Cut* that they repeat themselves. The original proof did not repeat any sequents except for the axiom $q \succ q$.)

No, the interesting feature of this new proof is that before the *Cut*, the *Cut* formula is introduced on the right in the derivation of left sequent $p \wedge (q \wedge r) \succ p \wedge q$, and it is introduced on the left in the derivation of the right sequent $p \wedge q \succ q$.

Notice that in general, if we have a *Cut* applied to a conjunction which is introduced on both sides of the step, we have a shorter route to $L \succ R$. We can sidestep the move through $A \land B$ to *Cut* on the formula A, since we have $L \succ A$ and $A \succ R$.

$$\dfrac{\dfrac{L \succ A \quad L \succ B}{L \succ A \land B}\,{\scriptstyle \land R} \qquad \dfrac{A \succ R}{A \land B \succ R}\,{\scriptstyle \land L_1}}{L \succ R}\,{\scriptstyle Cut}$$

In our example we do the same: We *Cut* with $p \land (q \land r) \succ q$ on the left and $q \succ q$ on the right, to get the first proof below in which the *Cut* moves *further* up the derivation. Clearly, however, this *Cut* is redundant, as cutting on an identity sequent does nothing. We could eliminate that step, without cost.

$$\dfrac{\dfrac{\dfrac{\dfrac{q \succ q}{q \land r \succ q}\,{\scriptstyle \land L_1}}{p \land (q \land r) \succ q}\,{\scriptstyle \land L_2} \qquad q \succ q}{p \land (q \land r) \succ q}\,{\scriptstyle Cut}}{p \land (q \land r) \succ q \lor r}\,{\scriptstyle \lor R_1} \qquad \dfrac{\dfrac{\dfrac{q \succ q}{q \land r \succ q}\,{\scriptstyle \land L_1}}{p \land (q \land r) \succ q}\,{\scriptstyle \land L_2}}{p \land (q \land r) \succ q \lor r}\,{\scriptstyle \lor R_1}$$

We have a *Cut*-free derivation of our concluding sequent.

As I hinted before, this technique is a general one. We may use exactly the same method to convert *any* derivation using *Cut* into a derivation without it. To do this, we will make explicit a number of the concepts we saw in this example.

DEFINITION 2.6 [ACTIVE AND PASSIVE FORMULAS] The formulas L and R in each inference in Figure 2.1 are said to be *passive* in the inference (they "do nothing" in the step from top to bottom), while the other formulas are *active*.

A formula is active in a step in a derivation if that formula is either introduced or eliminated. The active formulas in the connective rules are the *principal* formula (the conjunction or disjunction introduced, below the line) or the *constituents* from which the principal formula is constructed. The active formulas in a *Cut* step are the two instances of the *Cut*-formula, present above the line, but absent below the line.

DEFINITION 2.7 [DEPTH OF AN INFERENCE] The DEPTH of an inference in a derivation δ is the number of nodes in the sub-derivation of δ in which that inference is the last step, minus one. In other words, it is the number of sequents above the conclusion of that inference.

Now we can proceed to present the technique for eliminating *Cut*s from a derivation. First we show that *Cut*s may be moved upward. Then we show that this process will terminate in a *Cut*-free derivation. This first lemma is the bulk of the procedure for eliminating *Cut*s from derivations.

LEMMA 2.8 [CUT-DEPTH REDUCTION] *Given a derivation* δ *of* $A \succ C$, *whose final inference is Cut, but which is otherwise Cut-free, and in which that inference has a depth of* n, *we can transform* δ *another derivation* δ' *of* $A \succ C$ *which is Cut-free, or in which each Cut step has a depth less than* n.

Proof: Our derivation δ contains two subderivations: δ_l ending in $A \succ B$ and δ_r ending in $B \succ C$. These subderivations are *Cut*-free.

$$\dfrac{\begin{matrix}\vdots\,\delta_l \\ A \succ B\end{matrix} \qquad \begin{matrix}\vdots\,\delta_r \\ B \succ C\end{matrix}}{A \succ C}$$

To find our new derivation, we look at the two instances of the *Cut*-formula B and its roles in the final inference in δ_l and in δ_r. We have the following two cases: either B is passive in one or other of these inferences, or it is not.

CASE 1: THE CUT-FORMULA IS PASSIVE IN EITHER INFERENCE Suppose that the formula B is *passive* in the last inference in δ_l or passive in the last inference in δ_r. For example, if δ_l ends in $\wedge L_1$, then we may push the *Cut* above it like this:

The $\wedge L_2$ case is the same, except for the choice of A_2 instead of A_1.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\delta_l' \\ A_1 \succ B\end{matrix}}{A_1 \wedge A_2 \succ B}{\scriptstyle\wedge L_1} \qquad \begin{matrix}\vdots\,\delta_r \\ B \succ C\end{matrix}}{A_1 \wedge A_2 \succ C}{\scriptstyle Cut} \qquad\qquad \text{AFTER:}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\delta_l' \\ A_1 \succ B\end{matrix} \qquad \begin{matrix}\vdots\,\delta_r \\ B \succ C\end{matrix}}{A_1 \succ C}{\scriptstyle Cut}}{A_1 \wedge A_2 \succ C}{\scriptstyle \wedge L_1}$$

The resulting derivation has a *Cut*-depth lower by one. If, on the other hand, δ_l ends in $\vee L$, we may push the *Cut* above that $\vee L$ step. The result is a derivation in which we have duplicated the *Cut*, but we have reduced the *Cut*-depth more significantly, as the effect of δ_l is split between the two cuts.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\delta_l^1 \\ A_1 \succ B\end{matrix} \qquad \begin{matrix}\vdots\,\delta_l^2 \\ A_2 \succ B\end{matrix}}{A_1 \vee A_2 \succ B}{\scriptstyle\vee L} \qquad \begin{matrix}\vdots\,\delta_r \\ B \succ C\end{matrix}}{A_1 \vee A_2 \succ C}{\scriptstyle Cut} \qquad \text{AFTER:}\quad \dfrac{\dfrac{\begin{matrix}\vdots\,\delta_l^1 \\ A_1 \succ B\end{matrix} \quad \begin{matrix}\vdots\,\delta_r \\ B \succ C\end{matrix}}{A_1 \succ C}{\scriptstyle Cut} \qquad \dfrac{\begin{matrix}\vdots\,\delta_l^2 \\ A_2 \succ B\end{matrix} \quad \begin{matrix}\vdots\,\delta_r \\ B \succ C\end{matrix}}{A_2 \succ C}{\scriptstyle Cut}}{A_1 \vee A_2 \succ C}{\scriptstyle\vee L}$$

The other two ways in which the *Cut* formula could be passive are when δ_2 ends in $\vee R$ or $\wedge R$. The technique for these is identical to the examples we have seen. The *Cut* passes over $\vee R$ trivially, and it passes over $\wedge R$ by splitting into two cuts. In every instance, the depth is reduced.

CASE 2: THE CUT-FORMULA IS ACTIVE In the remaining case, the *Cut*-formula formula B may be assumed to be active in the last inference in both δ_l and in δ_r, because we have dealt with the case in which it is passive in either inference. What we do now depends on the form of the formula B. In each case, the structure of the formula B determines the final rule in both δ_l and δ_r.

CASE 2A: THE *Cut*-FORMULA IS ATOMIC If the *Cut*-formula is an atom, then the only inference in which an atomic formula is active in the conclusion is *Id*. In this case, the *Cut* is redundant.

$$\text{BEFORE:}\quad \dfrac{p \succ p \quad p \succ p}{p \succ p}\,{\scriptstyle Cut} \qquad \text{AFTER:}\quad p \succ p$$

CASE 2B: THE *Cut*-FORMULA IS A CONJUNCTION If the *Cut*-formula is a conjunction $B_1 \land B_2$, then the only inferences in which a conjunction is active in the conclusion are $\land R$ and $\land L$. Let us suppose that in the inference $\land L$, we have inferred the sequent $B_1 \land B_2 \succ C$ from the premise sequent $B_1 \succ C$. In this case, it is clear that we could have *Cut* on B_1 instead of the conjunction $B_1 \land B_2$, and the *Cut* is shallower.

The choice for $\land L_2$ instead of $\land L_1$ involves choosing B_2 instead of B_1.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\dfrac{\vdots\,\delta_l^1}{A \succ B_1} \quad \dfrac{\vdots\,\delta_l^2}{A \succ B_2}}{A \succ B_1 \land B_2}\,{\scriptstyle \land R} \qquad \dfrac{\dfrac{\vdots\,\delta_r'}{B_1 \succ C}}{B_1 \land B_2 \succ C}\,{\scriptstyle \land L_1}}{A \succ C}\,{\scriptstyle Cut} \qquad \text{AFTER:}\quad \dfrac{\dfrac{\vdots\,\delta_l^1}{A \succ B_1} \quad \dfrac{\vdots\,\delta_r'}{B_1 \succ C}}{A \succ C}\,{\scriptstyle Cut}$$

CASE 2C: THE *Cut*-FORMULA IS A DISJUNCTION The case for disjunction is similar. If the *Cut*-formula is a disjunction $B_1 \lor B_2$, then the only inferences in which a conjunction is active in the conclusion are $\lor R$ and $\lor L$. Let's suppose that in $\lor R$ the disjunction $B_1 \lor B_2$ is introduced in an inference from B_1. In this case, it is clear that we could have *Cut* on B_1 instead of the disjunction $B_1 \lor B_2$, with a shallower *Cut*.

$$\text{BEFORE:}\quad \dfrac{\dfrac{\dfrac{\vdots\,\delta_l'}{A \succ B_1}}{A \succ B_1 \lor B_2}\,{\scriptstyle \lor R_1} \qquad \dfrac{\dfrac{\vdots\,\delta_r^1}{B_1 \succ C} \quad \dfrac{\vdots\,\delta_r^2}{B_2 \succ C}}{B_1 \lor B_2 \succ C}\,{\scriptstyle \lor L}}{A \succ C}\,{\scriptstyle Cut} \qquad \text{AFTER:}\quad \dfrac{\dfrac{\vdots\,\delta_l'}{A \succ B_1} \quad \dfrac{\vdots\,\delta_r^1}{B_1 \succ C}}{A \succ C}\,{\scriptstyle Cut}$$

In every case, then, we have traded in a derivation for a derivation either without *Cut* or with a shallower cut. ■

The process of reducing *Cut*-depth cannot continue indefinitely, since the starting *Cut*-depth of any derivation is finite. At some point we find a derivation of our sequent $A \succ C$ with a *Cut*-depth of *zero*: We find a derivation of $A \succ C$ without a cut. That is,

THEOREM 2.9 [CUT ELIMINATION] *If a sequent is derivable with Cut, it is derivable without Cut.*

Proof: Given a derivation of a sequent $A \succ C$, take a *Cut* with no *Cut*s above it. This *Cut* has some depth, say n. Use the lemma to find a derivation with lower *Cut*-depth. Continue until there is no *Cut* remaining in this part of the derivation. (The depth of each *Cut* decreases, so this process cannot continue indefinitely.) Keep selecting cuts in the original derivation and eliminate them one-by-one. Since there are only finitely many cuts, this process terminates. The result is a *Cut*-free derivation. ■

This result has a number of fruitful consequences, which we will consider in the end of this chapter, but before that, we will extend our result to a richer language, putting together what natural deduction and the sequent calculus.

2.3 | COMPLEX SEQUENTS

Definition of intuitionist sequents (sketching their correspondence with proofs).

Definition of classical sequents and proofs.

Definition of the Belnap cut conditions and the general cut elimination argument for the range of systems discussed so far.

2.4 | CONSEQUENCES OF CUT ELIMINATION

COROLLARY 2.10 [DECIDABILITY FOR SIMPLE SEQUENTS] *There is an algorithm for determining whether or not a simple sequent* $A \succ B$ *is valid.*

Proof: To determine whether or not $A \succ B$ has a simple sequent derivation, look for the finitely many different sequents from which this sequent may be derived. Repeat the process until you find atomic sequents. Atomic sequents of the form $p \succ p$ are derivable, and those of the form $p \succ q$ are not. ∎

Here is an example:

EXAMPLE 2.11 [DISTRIBUTION IS NOT DERIVABLE] The sequent $p \land (q \lor r) \succ (p \land q) \lor r$ is not derivable.

Proof: Any *Cut*-free derivation of $p \land (q \lor r) \succ (p \land q) \lor r$ must end in either a $\land L$ step or a $\lor R$ step. Consider the wo cases:

CASE 1: THE DERIVATION ENDS WITH $\land$L: Then we infer our sequent from either $p \succ (p \land q) \lor r$, or from $q \lor r \succ (p \land q) \lor r$. Neither of these are derivable. As you can see, $p \succ (p \land q) \lor r$ is derivable only, using $\lor R$ from either $p \succ p \land q$ or from $p \succ r$. The latter is not derivable (it is not an axiom, and it cannot be inferred from *anywhere*) and the former is derivable only when $p \succ q$ is — and it isn't. Similarly, $q \lor r \succ (p \land q) \lor r$ is derivable only when $q \succ (p \land q) \lor r$ is derivable, and this is only derivable when either $q \succ p \land q$ or when $q \succ r$ are derivable, and as before, neither of *these* are derivable either.

CASE 2: THE DERIVATION ENDS WITH $\lor$R: Then we infer our sequent from either $p \land (q \lor r) \succ p \land q$ or from $p \land (q \lor r) \succ r$. By analogous reasoning, (more precisely, by *dual* reasoning) neither of these sequents are derivable. So, $p \land (q \lor r) \succ (p \land q) \lor r$ has no *Cut*-free derivation, and by Theorem 2.9 it has no derivation at all. ∎

Searching for derivations in this naïve manner is not as efficient as we can be: we don't need to search for *all* possible derivations of a sequent if we know about some of the special properties of the rules of the system. For example, consider the sequent $A \lor B \succ C \land D$ (where A, B, C and D are possibly complex statements). This is derivable in two ways (a) from $A \succ C \land D$ and $B \succ C \land D$ by $\lor L$ or (b) from $A \lor B \succ C$ and $A \lor B \succ D$ by $\land R$. Instead of searching *both* of these possibilites, we may notice that *either* choice would be enough to search for a derivation, since the rules $\lor L$ and $\land R$ 'lose no information' in an important sense.

DEFINITION 2.12 [INVERTIBILITY] A sequent rule of the form

$$\frac{S_1 \quad \cdots \quad S_n}{S}$$

is *invertible* if and only if whenever the sequent S is derivable, so are the sequents $S_1, \ldots, S_n$.

THEOREM 2.13 [INVERTIBLE SEQUENT RULES] *The rules $\lor L$ and $\land R$ are invertible, but the rules $\lor R$ and $\land L$ are not.*

Proof: Consider $\lor L$. If $A \lor B \succ C$ is derivable, then since we have a derivation of $A \succ A \lor B$ (by $\lor R$), a use of *Cut* shows us that $A \succ C$ is derivable. Similarly, since we have a derivation of $B \succ A \lor B$, the sequent $B \succ C$ is derivable too. So, from the conclusion $A \lor B \succ C$ of a $\lor L$ inference, we may derive the premises. The case for $\land R$ is completely analogous.

For $\land L$, on the other hand, we have a derivation of $p \land q \succ p$, but no derivation of the premise $q \succ p$, so this rule is not invertible. Similarly, $p \succ q \lor p$ is derivable, but $p \succ q$ is not. ■

It follows that when searching for a derivation of a sequent, instead of searching for *all* of the ways that a sequent may be derived, if it may be derived from an invertible rule we can look to the premises of that rule *immediately*, and consider those, without pausing to check the other sequents from which our target sequent is constructed.

EXAMPLE 2.14 [DERIVATION SEARCH] The sequent $(p \land q) \lor (q \land r) \succ (p \lor r) \land p$ is not derivable. By the invertibility of $\lor L$, it *is* derivable only if (a) $p \land q \succ (p \lor r) \land p$ and (b) $q \land r \succ (p \lor r) \land p$ are both derivable. Using the invertibility of $\land R$, the sequent (b) this is derivable only if (b_1) $q \land r \succ p \lor r$ and (b_2) $q \land r \succ p$ are both derivable. But (b_2) is not derivable because $q \succ p$ and $r \succ p$ are underivable.

The elimination of *Cut* is useful for more than just limiting the search for derivations. The fact that any derivable sequent has a *Cut*-free derivation has other consequences. One consequence is the fact of *interpolation*.

COROLLARY 2.15 [INTERPOLATION FOR SIMPLE SEQUENTS] *If $A \succ B$ is derivable in the simple sequent system, then there is a formula C containing only atoms present in both A and B such that $A \succ C$ and $C \succ B$ are derivable.*

This result tells us that if the sequent $A \succ B$ is derivable then that consequence "factors through" a statement in the vocabulary shared between A and B. This means that the consequence $A \succ B$ not only relies only upon the material in A and B and nothing *else* (that is due to the availability of a *Cut*-free derivation) but also in some sense the derivation 'factors through' the material in common between A and B. The result is a straightforward consequence of the *Cut*-elimination theorem. A *Cut*-free derivation of $A \succ B$ provides us with an interpolant.

Proof: We prove this by induction on the construction of the derivation of $A \succ B$. We keep track of the interpolant with these rules:

$$p \succ_p p \; {\scriptstyle Id}$$

$$\frac{A \succ_C R}{A \land B \succ_C R}\wedge L_1 \qquad \frac{A \succ_C R}{B \land A \succ_C R}\wedge L_2 \qquad \frac{L \succ_{C_1} A \qquad L \succ_{C_2} B}{L \succ_{C_1 \land C_2} A \land B}\wedge R$$

$$\frac{A \succ_{C_1} R \qquad B \succ_{C_2} R}{A \lor B \succ_{C_1 \lor C_2} R}\vee L \qquad \frac{L \succ_C A}{L \succ_C A \lor B}\vee R_1 \qquad \frac{L \succ_C A}{L \succ_C B \lor A}\vee R_2$$

We show by induction on the length of the derivation that if we have a derivation of $L \succ_C R$ then $L \succ C$ and $C \succ R$ and the atoms in C present in both L and in R. These properties are satisfied by the atomic sequent $p \succ_p p$, and it is straightforward to verify them for each of the rules. ■

EXAMPLE 2.16 [A DERIVATION WITH AN INTERPOLANT] Take the sequent $p \land (q \lor (r_1 \land r_2)) \succ (q \lor r_1) \land (p \lor r_2))$. We may annotate a *Cut*-free derivation of it as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{q \succ_q q}{q \succ_q q \lor r}\vee R \qquad \dfrac{r_1 \succ_{r_1} r_1}{r_1 \land r_2 \succ_{r_1} r_1}\wedge L}{q \lor (r_1 \land r_2) \succ_{q \lor r_1} q \lor r_1}\vee L}{p \land (q \lor (r_1 \land r_2)) \succ_{q \lor r_1} q \lor r_1}\wedge L \qquad \dfrac{\dfrac{p \succ p}{p \succ_p p \lor r_2}\vee R}{p \land (q \lor (r_1 \land r_2)) \succ_p p \lor r_2}\wedge L}{p \land (q \lor (r_1 \land r_2)) \succ_{(q \lor r_1) \land p} (q \lor r_1) \land (p \lor r_2)}\wedge R$$

Notice that the interpolant $(q \lor r_1) \land p$ does not contain r_2, even though r_2 is present in both the antecedent and the consequent of the sequent. This tells us that r_2 is doing no 'work' in this derivation. Since we have

$$p \land (q \lor (r_1 \land \boxed{r_2})) \succ (q \lor r_1) \land p, \quad (q \lor r_1) \land p \succ (q \lor r_1) \land (p \lor \boxed{r_2})$$

We can replace the r_2 in either derivation with another statement – say r_3 – preserving the structure of each derivation. We get the more general fact:

$$p \land (q \lor (r_1 \land \boxed{r_2})) \succ (q \lor r_1) \land (p \lor \boxed{r_3})$$

More consequences of *Cut*-elimination and the admissibility of the identity rules Id_A will be considered as the book goes on. Exercises 8–14 ask you to consider different possible connective rules, some of which will admit of *Cut*-elimination and *Id*-admissibility when added, and others of which which will not. In Chapter 4 we will look at reasons why this might help us demarcate definitions of a kind of *properly logical* concept from those which are not logical in that sense.

2.5 | HISTORY

The idea of taking the essence of conjunction and disjunction to be expressed in these sequent rules is to take conjunction and disjunction to form what is known as a *lattice*. A lattice is on *ordered* structure in which we have for every pair of objects a *greatest lower bound* and a *least upper bound*. A *greatest lower bound* of x and y is something below both x and y but which is greatest among such things. A *least upper bound* of x and y is something above both x and y but which is the least among such things. Among statements, taking $\succ$ to be the ordering, $A \land B$ is the greatest lower bound of A and of B (since $A \land B \succ A$ and $A \land B \succ B$, and if $C \succ A$ and $C \succ B$ then $C \succ A \land B$) and $A \lor B$ is their least upper bound (for dual reasons).

Here, $\succ$ is the ordering. If $A \succ B$, think of A as occuring 'below' B in the ordering from stronger to weaker.

Well, you need to *squint*, and take A and A' to be *the same* if $A \succ A'$ and $A' \succ A$ to make $A \land B$ *the unique* greatest lower bound. If it helps, don't think of the *sentence* A but the *proposition*, where two logically equivalent sentences express the same proposition.

Lattices are wonderful structures, which may be applied in many different ways, not only to logic, but in many other domains as well. Davey and Priestley's *Introduction to Lattices and Order* [20] is an excellent way into the literature on lattices. The concept of a lattice dates to the late 19th Century in the work of Charles S. Peirce and Ernst Schröder, who independently generalised Boole's algebra of propositional logic. Richard Dedekind's work on 'ideals' in algebraic number theory was an independent mathematical motivation for the concept. Work in the area found a focus in the groundbreaking series of papers by Garrett Birkhoff, culminating in a the book *Lattice Theory* [11]. For more of the history, and for a comprehensive state of play for lattice theory and its many applications, George Grätzer's 1978 *General Lattice Theory* [36], and especially its 2003 Second Edition [37] is a good port of call.

We will not study much algebra in this book. However, algebraic technqiues find a very natural home in the study of logical systems. Helena Rasiowa's 1974 *An Algebraic Approach to Non-Classical Logics* [74] was the first look at lattices and other structures as models of a wide range of different systems. For a good guide to why this technique is important, and what it can do, you cannot go past J. Michael Dunn and Gary Hardegree's *Algebraic Methods in Philosophical Logic* [24].

Well, we won't study algebra *explicitly*. Algebraic considerations and sensibilities underly *much* of what will go on. But that will almost always stay under the surface.

The idea of studying derivations consisting of sequents, rather than proofs from premises to conclusions, is entirely due to Gentzen, in his groundbreaking work in proof theory. His motivation was to extend his results on normalisation from what we called the standard natural deduction system to classical logic as well as intuitionistic logic [30, 31]. To do this, it was fruitful to step back from proofs from premises X to a conclusion

But for more connectives than just the conditional.

Gentzen didn't use the turnstile. His notation was '$\Gamma \rightarrow \mathfrak{A}$'. We use the arrow for a conditional, and the turnstile for a sequent separator.

A to consider statements of the form '$X \succ A$,' making explicit at each step on which premises X the conclusion A depends. Then as we will see in the next chapter, normalisation 'corresponds' in some sense to the elimination of *Cut*s in a derivation. One of Gentzen's great insights was that sequents could be generalised to the form $X \succ Y$ to provide a uniform treatment of traditional Boolean classical logic. We will make much of this connection in the next chapter.

Gentzen's own logic wasn't lattice logic, but traditional classical logic (in which the distribution of conjunction over disjunction—that is, $A \wedge (B \vee C) \succ (A \wedge B) \vee (A \wedge C)$—is valid) and intuitionistic logic. I have chosen to start with simple sequents for lattice logic for two reasons. First, it makes the procedure for the elimination of *Cut*s much more simple. There are fewer cases to consider and the essential shape of the argument is laid bare with fewer inessential details. Second, once we see the technique applied again and again, it will hopefully reinforce the thought that it is very general indeed. Sequents were introduced as a way of looking at an underlying proof structure. As a pluralist, I take it that there is more than one sort of underlying proof structure to examine, and so, sequents may take more than one sort of shape. Much work has been done recently on *why* Gentzen chose the rules he did for his sequent calculi. I have found papers by Jan von Plato [66, 67] most helpful. Gentzen's papers are available in his collected works [32], and a biography of Gentzen, whose life was cut short in the Second World War, has recently been written [54, 55].

2.6 | EXERCISES

BASIC EXERCISES

Q1 Find a derivation for $p \succ p \wedge (p \vee q)$ and a derivation for $p \vee (p \wedge q) \succ p$. Then find a *Cut*-free derivation for $p \vee (p \wedge q) \succ p \wedge (p \vee q)$ and compare it with the derivaion you get by joining the two original derivations with a *Cut*.

Q2 Show that there is no *Cut*-free derivation of the following sequents

$$\begin{array}{rl} 1: & p \vee (q \wedge r) \succ p \wedge (q \vee r) \\ 2: & p \wedge (q \vee r) \succ (p \wedge q) \vee r \\ 3: & p \wedge (q \vee (p \wedge r)) \succ (p \wedge q) \vee (p \wedge r) \end{array}$$

Q3 Suppose that there is a derivation of $A \succ B$. Let $C(A)$ be a formula containing A as a subformula, and let $C(B)$ be that formula with the subformula A replaced by B. Show that there is a derivation of $C(A) \succ C(B)$. Furthermore, show that a derivation of $C(A) \succ C(B)$ may be systematically constructed from the derivation of $A \succ B$ together with the context $C(-)$ (the shape of the formula $C(A)$ with a 'hole' in the place of the subformula A).

Q4 Find a derivation of $p \wedge (q \wedge r) \succ (p \wedge q) \wedge r$. Find a derivation of $(p \wedge q) \wedge r \succ p \wedge (q \wedge r)$. Put these two derivations together, with a *Cut*, to show that $p \land (q \land r) \succ p \land (q \land r)$. Then eliminate the cuts from this derivation. What do you get?

Q5 Do the same thing with derivations of $p \succ (p \land q) \lor p$ and $(p \land q) \lor p \succ p$. What is the result when you eliminate this cut?

Q6 Show that (1) $A \succ B \land C$ is derivable if and only if $A \succ B$ and $A \succ C$ is derivable, and that (2) $A \lor B \succ C$ is derivable if and only if $A \succ C$ and $B \succ C$ are derivable. Finally, (3) when is $A \lor B \succ C \land D$ derivable, in terms of the derivability relations between A, B, C and D.

Q7 Under what conditions do we have a derivation of $A \succ B$ when A contains only propositional atoms and *disjunctions* and B contains only propositional atoms and *conjunctions*.

Q8 Expand the system with the following rules for the propositional *constants* $\bot$ and $\top$.

$$A \succ \top \;\; [\top R] \qquad \bot \succ A \;\; [\bot L]$$

Show that *Cut* is eliminable from the new system. (You can think of $\bot$ and $\top$ as zero-place connectives. In fact, there is a sense in which $\top$ is a zero-place *conjunction* and $\bot$ is a zero-place *disjunction*. Can you see why?)

Q9 Show that simple sequents including $\top$ and $\bot$ are decidable, following Corollary 2.10 and the results of the previous question.

Q10 Show that every formula composed of just $\top$, $\bot$, $\land$ and $\lor$ is *equivalent* to either $\top$ or $\bot$. (What does this result remind you of?)

Q11 Prove the interpolation theorem (Corollary 2.15) for derivations involving $\land$, $\lor$, $\top$ and $\bot$.

Q12 Expand the system with rules for a propositional connective with the following rules:

$$\frac{A \succ R}{A \text{ tonk } B \succ R}\,\text{tonk } L \qquad \frac{L \succ B}{L \succ A \text{ tonk } B}\,\text{tonk } R$$

What new things can you derive using tonk? Can you derive A tonk $B \succ A$ tonk B? Is *Cut* eliminable for formulas involving tonk?

See Arthur Prior's "The Runabout Inference-Ticket" [71] for tonk's first appearance in print.

Q13 Expand the system with rules for a propositional connective with the following rules:

$$\frac{A \succ R}{A \text{ honk } B \succ R}\,\text{honk } L \qquad \frac{L \succ A \quad L \succ B}{L \succ A \text{ honk } B}\,\text{honk } R$$

What new things can you derive using honk? Can you derive A honk $B \succ A$ honk B? Is *Cut* eliminable for formulas involving honk?

Q14 Expand the system with rules for a propositional connective with the following rules:

$$\frac{A \succ R \quad B \succ R}{A \text{ plonk } B \succ R}\,\text{plonk } L \qquad \frac{L \succ B}{L \succ A \text{ plonk } B}\,\text{plonk } R$$

What new things can you derive using plonk? Can you derive A plonk $B \succ A$ plonk B? Is *Cut* eliminable for formulas involving plonk?

INTERMEDIATE EXERCISES

Q15 Give a formal, recursive definition of the *dual* of a sequent, and the *dual* of a derivation, in such a way that the dual of the sequent $p_1 \wedge (q_1 \vee r_1) \succ (p_2 \vee q_2) \wedge r_2$ is the sequent $(p_2 \wedge q_2) \vee r_2 \succ p_1 \vee (q_1 \wedge r_1)$. And then use this definition to prove the following theorem.

THEOREM 2.17 [DUALITY FOR DERIVATIONS] *A sequent* $A \succ B$ *is derivable if and only if its dual* $(A \succ B)^d$ *is derivable. Furthermore, the dual of the derivation of* $A \succ B$ *is a derivation of the dual of* $A \succ B$.

Q16 Even though the distribution sequent $p \wedge (q \vee r) \succ (p \wedge q) \vee r$ is not derivable (Example 2.11), some sequents of the form $A \wedge (B \vee C) \succ (A \wedge B) \vee C$ *are* derivable. Give an independent characterisation of the triples $\langle A, B, C \rangle$ such that $A \wedge (B \vee C) \succ (A \wedge B) \vee C$ is derivable.

Q17 Prove the invertibility result of Theorem 2.13 without appealing to the *Cut* rule or to *Cut*-elimination. (HINT: if a sequent $A \vee B \succ C$ has a derivation δ, consider the instances of $A \vee B$ 'leading to' the instance of $A \vee B$ in the conclusion. How does $A \vee B$ appear first in the derivation? Can you change the derivation in such a way as to make it derive $A \succ C$? Or to derive $B \succ C$ instead? Prove this, and a similar result for $\wedge L$.)

ADVANCED EXERCISES

Q18 Define a notion of reduction for simple sequent derivations parallel to the definition of reduction of natural deduction proofs in Chapter 1. Show that it is strongly normalising and that each derivation reduces to a unique *Cut*-free derivation.

Q19 Define *terms* corresponding to simple sequent derivations, in an analogy to the way that λ-terms correspond to natural deduction proofs for conditional formulas. For example, we may annotate each derivation with *terms* in the following way:

$$p \succ_x p \;\; {\scriptstyle Id} \qquad \frac{L \succ_f A \quad A \succ_g R}{L \succ_{f \circ g} R}\,{\scriptstyle Cut}$$

$$\frac{A \succ R}{A \wedge B \succ_{l[f]} R}\,{\scriptstyle \wedge L_1} \qquad \frac{B \succ R}{A \wedge B \succ_{r[f]} R}\,{\scriptstyle \wedge L_2} \qquad \frac{L \succ_f A \quad L \succ_g B}{L \succ_{f \| g} A \wedge B}\,{\scriptstyle \wedge R}$$

where x is an atomic term (of type p), f and g are terms, $l[\,]$ and $r[\,]$ are one-place term constructors and $\|$ is a two-place term constructor (of a kind of parallel composition), and $\circ$ is a two-place term constructor (of *serial composition*). Define similar term constructors for the disjunction rules.

Then reducing a *Cut* will correspond to simplifying terms by eliminating serial composition. A *Cut* in which $A \wedge B$ is active will take the following form of reduction:

$$(f \| g) \circ l[h] \text{ reduces to } f \circ h \qquad (f \| g) \circ r[h] \text{ reduces to } g \circ h$$

Fill out all the other reduction rules for every other kind of step in the *Cut*-elimination argument.

Do these terms correspond to anything like computation? Do they have any other interpretation?

PROJECTS

Q20 Provide sequent formulations for logics intermediate between simple sequent logic and the logic of *distributive lattices* (in which $p \land (q \lor r) \succ (p \land q) \lor r$). Characterise *which* logics intermediate between lattice logic (the logic of simple sequents) and distributive lattice logic *have* sequent presentations, and which do not. (This requires making explicit what counts as a *logic* and what counts as a sequent presentation of a logic.)

FROM PROOFS TO MODELS 3

3.1 POSITIONS

[This goes back to Aristotle and the method of counterexamples for invalid syllogisms, I think!]

3.2 SOUNDNESS AND COMPLETENESS

3.3 ANOTHER ARGUMENT FOR THE ADMISSIBILITY OF CUT

3.4 FROM RULES TO TRUTH CONDITIONS

[Consider Raatikainen's point, recovering Carnap's old argument that the rules of inference aren't enough to give us the truth conditions [73]. Explain how *these* rules are strong enough, and why they're genuinely more than the traditional rules.] [Is this the right point to put a Kreiselesque squeezing argument? I cannot say too much for the intuitive notion of logical consequence or its connection to proof theory here, but there is, perhaps, something to say about the argument here. [88, 89]]

3.5 BEYOND SIMPLE SEQUENTS

Sketch what one might do for Quantifiers, and for Modal Operators.

3.6 HISTORY

PART II

The Core Argument

4 TONK

POSITIONS 5

5.1 | ASSERTION AND DENIAL

[In this chapter, I present the justification of the structural rules in terms of the norms governing assertion and denial, and the individuation of assertions and denials.]

[Here is where I indicate what is involved in kinds of norms of assertion, including the literature on norms of assertion [16, 45, 53]. The traditional norms of assertion (Knowledge norm, belief norm, truth norm, etc.) are norms for correctness of individual assertions. The coherence of a position is a different thing entirely, related to those norms, to be sure, but not the same. This is the section where I need to explain what is going on here.]

5.2 | POSITIONS AND THEIR STRUCTURE

Positions: formally speaking, they are pairs $[X : Y]$ of sets of formulas. What are these formulas modelling? Positions in a discourse, keeping track of assertions and denials, commitments undertaken in a dialogue. So, sentences are a suitable proxy for assertions and denials insofar as they represent those commitments, and in many cases they do. [When do they not? When we interpret the language differently. I might say "this is a bank" and you might say "this isn't a bank" where we do not disagree, because "bank" in your mouth means *riverbank* and in mine, it means *financial institution*. We might *agree* with each other, asserting "this is a bank" under one reading, and denying "this is a bank" under the other. In this case, the sentence is not a good proxy for the commitment, but it would be acceptable if we disambiguated, replacing "bank" by unamibguous words, "bank$_1$" and "bank$_2$."

The sentences may still be a bad proxy for the commitments, because the demonstrative term "this" may also vary in significance. My assertion "this is a bank" and your denial would not amount to a disagreement if my "this" picked out something different from yours. In this case, it isn't that we are speaking a different language, or using different rules to interpret the term "this," but rather, that the term is interpreted differently, due to a differing context of use. The same holds for indexicals. If I say "I'm tired" and you say "I'm not tired", or if I say "it's 3 o'clock now" and one hour later you say "it's not 3 o'clock now", we do not disagree, despite the surface disagreement between the sentences used. A comprehensive theory of positions involving assertions and denials will address these issues, but for now, we mention them to put them aside. For now, treating suitably disambiguated sentences as proxies for the commitments undertaken in assertion and denials will do us well. So, we have the following definition.

DEFINITION 5.1 [POSITIONS] Given a language $\mathcal{L}$, a POSITION $[X : Y]$ IN $\mathcal{L}$ is a pair of sets of sentences from $\mathcal{L}$.

This definition is meant quite literally. Positions are individuated by the sets of assertions and denials made in them. Positions take no account of the order in which assertions or denials are made, nor the order in which they are made. A position is a kind of *score*. Different games can have the same score.

There are norms governing positions. Some of these are norms governing individual acts of assertion (or denial). Belief, Knowledge, Accuracy. Much can be said for these. For each norm there is a clear sense in which an assertion fails if it fails to fall under that norm, and another clear sense in which it succeeds if it falls under the norm. Perhaps one norm is, in some sense, prior to the others, perhaps not. The same could be said for denial and parallel norms [spell these out].

Some of the norms governing assertion and denial are not norms governing individual acts of assertion and denial—they are norms that can be understood as governing positions. Consider a position in which the very same claim is asserted and denied. The position

$$[A : A]$$

is, in some important sense, defective. The very same item is ruled *in* and ruled *out*. It is confused concerning A, and it is confused concerning A in a way that the prior positions $[A : \ ]$ and $[\ : A]$ are not. There is a sense in which the position $[A : A]$ does not succeed in taking a stand on A, while the more modest positions $[A : \ ]$ and $[\ : A]$ do take a stand.

How is this defect in $[A : A]$ to be understood? One way to do so is to appeal to norms governing individual acts of assertion and denial. Do any of the norms we have seen explain the particular defect of the position $[A : A]$? In some sense, they can offer part of an explanation. Individual norms concerning assertion and denial can all (with varying degrees of success) predict that this position is defective. According to the *success* norm, $[A : A]$ must fail because either the *assertion* fails to be correct, or the *denial* fails to be correct. Invariably, one of the speech acts will fail, so $[A : A]$ is guaranteed to fail. Of course, on the success criterion for assertion and denial, the defect of $[A : A]$ is shared by one of the more modest positions, $[\ : A]$ and $[A : \ ]$, so the success criterion cannot, by itself, give an account of the special defect of the inconsistent position $[A : A]$. However, we could go further. The defect of $[A : A]$ isn't just that it contingently happens to fail, but that it does so no matter how things turn out concerning A. $[A : A]$ is guaranteed to fail while $[\ : A]$ and $[A : \ ]$ may not be.

The same might be said for the knowledge norm as well as the success norm. If I know that A then my knowledge cannot rule out A. However, I might be in a position to know A and (had things gone differently) my knowledge might be enough to rule A out. The same cannot be said for the belief norm, if belief states allow for confusion. If someone can (in a confused state) believe A and also disbelieve it, then $[A : A]$ might be blameless regarding the belief norm.

How are we to understand these norms governing positions as a whole? There is clearly something defective in the position $[A : A]$ that need not be shared by $[A : \ ]$ and $[\ : A]$ and the correctness norm (and perhaps the knowledge norm) can go some way to help articulate that defect—at least with the help of the generalising modality "no matter how things go"—how is this modalisation to be understood? Is this a metaphysical modality, grounded in the different possibilities of his things could have turned out had things gone differently? Or is it an epistemic matter? Or is the generality to be understood in terms of analyticity? If you are to understand the distinct failing of the position $[A : A]$ in terms of some modalised version of the correctness norm, you should specify which generalisation is doing the work. If the target is to have anything to do logic and proof, the generalisation should extend beyond bare metaphysical necessity, unless we are to bite the bullet and take necessities (like $2 + 2 = 4$) to be derivable from anything and everything whatsoever. Our target is to match derivation and positions, in such a way that $[X : Y]$ is out of bounds if and only if there is a derivation of the sequent $X \succ Y$. [How could there be rational disagreement over necessities? All necessities would reduce to logicality, and that's just wrong.]

Instead of making this choice, On the other hand, One alternative is to hold this norm as primitive, constituted by the dual relationship between assertion and denial. (One aim of denial is to be opposed to assertion; one aim of assertion is to be opposed to denial. This is why the positive norms for single assertions are so straightforward to generalise to single denials.) It follows that any position of the form $[A : A]$ is self-undermining, or *constituitively* out of bounds, in a way that need not be explained in terms of any external modality. That is how we will proceed.

Consider my assertion of p and your denial. Taking two opposing positions on it. No shared position there to be found without more work. (Disambiguating, one or both falling back. Suppose I assert $p \land q$ and you deny it. I might fall back to me asserting p, and you agreeing $p \land \neg q$.) (Could I both believe and disbelieve p? Perhaps, if I am confused. Could p be both true and untrue? No. Could I know both? No. But the problem with)

Justify the Structural Rules, and their behaviour in terms of positions and norms.

5.3 | HYPOTHETICAL POSITIONS

Here is a puzzle: On this view what are we to say about this sequent?

$$p \succ \text{I assert } p$$

Surely there is a clash between asserting p and denying that you assert p. Is the argument from p to the claim that you assert p valid? Surely, since p could be true without being asserted, we have reason to resist

Thanks to Hannes Leitgeb, Lloyd Humberstone and Nico Silins for independently raising this issue with me.

the inference, and as a result, we have reason to deny our analysis of the connection between logical consequence and norms governing assertion and denial. Or do we?

Consider what is going on in the case. A position in which I assert p and deny that I assert p is certainly *mistaken*. I cannot correctly assert p and also correctly deny that I assert p. However, is it *inconsistent*? Is there a *clash* of the required sense, between the two speech acts? Notice that *you* can assert p and deny that I have asserted p. So, there is reason to think that there is no such clash, and that the position, while mistaken, is nonetheless, consistent and within bounds. Consider what happens when the assertion p is hypothetical. Suppose that we are not, in fact, committed to p, but are considering what would follow from p, were it to be the case. Given that dialectical scenario, we could happily reason as follows: suppose that p is the case. Does it follow that I assert p? No, because I may well be ignorant of p, or think it too impolite to assert, or I could lie. We are very happy to grant scenarios in which p is the case but I do not assert p. The strong norms of deductive logic are norms that govern deduction, and deductive consequence applies not only to reasoning from premises taken to be true, but also, reasoning from premises which are merely supposed, and this is enough to defeat the worry that our norms cut too coarsely, and license as valid inferences about *assertion* or *denial* that are not themselves valid.

We consider different norms governing kinds of supposition in Chapter 9 when we consider modal logics.

5.4 | POSITIONS, MODELS AND SEMANTICS

Question from Timothy Williamson: What does the sequent semantics offer as a *semantics*? It's one thing to have a theory of logical consistency given by these bounds on positions, but does this say anything about the semantics of atoms? What can you say about that in this picture?

The same story holds with regard to model theory. There is nothing in Tarski-style model theory which distinguishes one model from another. Another principle must be given to select out those models which are *good* models of how things are, and those that aren't. There are different choices to be made here, but in the case of an interpreted propositional language, it's not as difficult as it is in a quantified language.

What can we do in the proof theory?

Two options. 1. Directly. Assuming truth and falsity. Given an interpreted language, a position $\Gamma \succ \Delta$ *completely accurate* if every element of Γ is true and every element of Δ is false. A position $\Gamma \succ \Delta$ is *atomically accurate* iff every atom in Γ is true and every atom in Δ is false.

Atomically accurate positions can fail to be accurate.

Fact: any fully refined atomically accurate position is completely accurate—assuming the boolean interpretation of the connectives.

Proof: straightforward. See Chapter 4 [[Add Reference]]

This is fine enough, and it works as an answer to the question, given the prior notion of truth and falsity, and the fit between these notions and the connectives in the traditional Boolean way. So another explanation closer to the spirit of this account of positions would be nice.

Option 2. To *adopt* the position $\Gamma \succ \Delta$ is to assert Γ and deny Δ—and this is what it is to take all of Γ to be true and all of Δ to be false. [If I adopt the position $\Gamma \succ \Delta$, I take these positions to be *available* and other positions to be *excluded*, and a closure of my position (in the salient sense) to be *forced*. The commitments shared in all limit available positions are those that are *forced*. These are some of the norms consequent on the bounds norms we already have, and this tells us something about how abstract positions in a formal proof theory are to be understood in an interpreted language and what it is to take things to be true. [spell out what it is, then, to take p to be true, for an atom p, and why it is that either p is true or $\neg$p is true, given this approach. Fill out what components of the truth conditions remain here.]

5.5 | GROUDING THE NORMS

Where do these norms come from? How do they arise? Consider the argument in Esfeld's "Inferentialism and the Normativity Trilemma", to the effect that the following three claims are jointly inconsistent [27, page 13].

1. Normative statements are true or false. Regarding a certain normative statement as true does not imply that it is true, not even if a whole community takes the statement in question to be true (*cognitivism*)'
2. There are no normative entities in the world that make normative statements true (*naturalism*).
3. It is not possible to deduce normative statements from descriptive statements (*naturalistic fallacy*).

How does an inferentialist like me respond?

5.6 | OTHER SPEECH ACTS

6 DEFINING RULES

The first part of the book leads up to a single argument, applying the techniques of proof theory to the question of whether or not a rule succeeds in defining a logical concept. This central argument takes up this chapter. The argument is an extension and a refinement of Nuel Belnap's response [9] to Prior's problem of the runabout inference ticket [71].

The central argument shows that given an appropriate background context of deducibility (a family of position structures and structural rules), with a language made up of a set of concepts governed by *defining rules*, then the addition of any new concept also governed by a defining rule is both *conservative* and *uniquely defining*. Given the background context structural rules arising out of norms governing assertion and denial, then the rules defining the classical concepts of conjunction, disjunction, negation and the material conditional are, indeed, conservative and uniquely defining. Given the syntactic distinction between singular terms and predicates, then the universal and existential quantifiers are also conservative and uniquely defining. And more can be shown to be conservative and uniquely defining given a richer structure in the background context of deducibility, as we will see in the chapters ahead.

This result is relatively simple to state, but it requires two different kinds of elaboration. *First,* the crucial concepts (the appropriate context of deducibility, and defining rules) must be precisely made out, general enough that these ideas can be applied to a wide range of logical systems, but not so general so as to be hopelessly abstract and unintelligible. We will cover this in Sections 6.1 and 6.2. *Second,* there is the elaboration of the proof itself. This will take up Sections 6.3 and 6.4. The proof falls into two parts. First, we show that any system given an appropriate background context of deducibility and a system of defining rules can be converted into system consisting not of defining rules but of *Left* and *Right* rules in Gentzen's sense, using the *Cut* rule. Then, in the second part, we show that derivations in this system using the *Cut* rule can be reformulated without the use of that rule, and that the resulting derivations have some version of the subformula property—which then shows that the addition of new defining rules is *conservative,* because no sequent not involving new vocabulary must make use of that vocabulary in its derivation.

That is the Hauptsatz, the central result of the book. Once we've completed that result, there is much to be done to draw out its significance. We'll begin our descent from the mountain in this chapter in the final two sections, in Section 6.5 with an explanation of exactly how this answers Prior's original question concerning tonk, and in Section 6.6 with a look forward to the different ways this result might apply, to the kinds of concepts (connectives, modal operators, quantifiers, identity) which might count as *logical*, given this result.

6.1 POSITIONS AND STRUCTURAL RULES

6.2 DEFINING RULES DEFINED

The sequent rules for conjunction are more of a piece with an explanation of the meaning of "hello." — I owe this lovely example to Dave Ripley.

[Background on defining rules. Their importance is discussed by Dana Scott [84, 85] in the 1970s, and the work of Giovanni Sambin and his colleagues on Basic Logic [5, 28, 82, 83].]

$$\begin{array}{c}\mathfrak{S}\{A\}_{\star_l}\{B\}_{\star_r}\\ \hline\hline \mathfrak{S}\{A\star B\}_{\star_c}\end{array}\ {\scriptstyle \star Df}$$

[Sequents and positions.]

6.3 DEFINING RULES AND LEFT/RIGHT RULES

Here's how we get from defining rules to L/R rules in a modular way.

$$\dfrac{\mathfrak{S}_1\{A\}_{\overline{\star_l}} \qquad \dfrac{\mathfrak{S}_2\{B\}_{\overline{\star_r}} \qquad \dfrac{\mathfrak{I}\{A\star B\}_{\star_c}\{A\star B\}_{\overline{\star_c}}}{\mathfrak{I}\{A\}_{\star_l}\{B\}_{\star_r}\{A\star B\}_{\overline{\star_c}}}{\scriptstyle \star Df}}{\mathfrak{S}_1\{A\}_{\star_l}\{A\star B\}_{\overline{\star_c}}}{\scriptstyle Cut}}{\mathfrak{S}_{1,2}\{A\star B\}_{\overline{\star_c}}}{\scriptstyle Cut}$$

So, the defining rule motivates (with *Cut* and *Id*), the following rule:

$$\dfrac{\mathfrak{S}_1\{A\}_{\overline{\star_l}} \qquad \mathfrak{S}_2\{B\}_{\overline{\star_r}}}{\mathfrak{S}_{1,2}\{A\star B\}_{\overline{\star_c}}}$$

Now consider the eliminability of matching principle constituents. If we have the following two $\star$ intro inferences, with a Cut:

$$\dfrac{\dfrac{\mathfrak{S}_1\{A\}_{\overline{\star_l}} \qquad \mathfrak{S}_2\{B\}_{\overline{\star_r}}}{\mathfrak{S}_{1,2}\{A\star B\}_{\overline{\star_c}}} \qquad \dfrac{\mathfrak{S}_3\{A\}_{\star_l}\{B\}_{\star_r}}{\mathfrak{S}_3\{A\star B\}_{\star_c}}{\scriptstyle \star Df}}{\mathfrak{S}_{1-3}}{\scriptstyle Cut}$$

This *Cut* can be eliminated, and transformed into cuts on the subformulas, A and B.

$$\dfrac{\mathfrak{S}_1\{A\}_{\overline{\star_l}} \qquad \dfrac{\mathfrak{S}_2\{B\}_{\overline{\star_r}} \qquad \mathfrak{S}_3\{A\}_{\star_l}\{B\}_{\star_r}}{\mathfrak{S}_{2,3}\{A\}_{\star_l}}{\scriptstyle Cut}}{\mathfrak{S}_{1-3}}{\scriptstyle Cut}$$

6.4 ELIMINATING CUT

6.5 ANSWERING PRIOR'S QUESTION

6.6 THE SCOPE OF LOGICALITY

One can see … main technical limitations in current proof-theory: The lack in *modularity*: in general, neighbouring problems can be attacked by neighbouring methods; but it is only exceptionally that one of the problems will be a corollary of the other … Most of the time, a completely new proof will be necessary (but without any new idea). This renders work in the domain quite long and tedious. For instance, if we prove a cut-elimination theorem for a certain system of rules, and then consider a new system including just a new pair of rules, then

¶ BELNAP'S RESPONSE: A rule (or collection of rules) defines a logical concept relative to a background context of deducibility if and only if that addition satisfies two criteria. (*a*) the EXISTENCE criterion: the addition is conservative, in the sense of adding no new consequences in the old vocabulary; and (*b*) the UNIQUENESS criterion: the rules are well-characterised, in the sense that if they were used *twice* to introduce two versions of the one concept, these two versions would agree.

I wholeheartedly endorse Belnap's proposal, but I aim to give an *argument* as to how these criteria are grounded in the normative pragmatics of assertion and denial. Here is how I extend this response into an argument.

(1) The background context of deductibility involves norms governing acceptable patterns of assertion and denial (understood as distinctive kinds of *acts* undertaken by agents) and patterns of acceptance and rejection (understood as distinctive kinds of *states* which may be ascribed to agents). The core norms are connect assertion and denial, or accepting and rejecting.

> IDENTITY: Accepting A precludes rejecting A, no matter what else is accepted and rejected. Similarly, asserting A precludes denying A, no matter what else is asserted or denied

This is not to be understood as saying that asserting A makes denying A impossible: one can be inconsistent. However, being inconsistent in this sense is a mistake, and not merely a mistake of fact. This constraint is formally recorded in a sequent system with the identity rule.

$$X, A \succ A, Y \quad [\text{ID}]$$

Similarly, we have the rule for CUT. As a rule in a formal system, we have:

$$\frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\,[\text{CUT}]$$

This can be read in the contrapositive:

> CUT: If asserting X does not preclude denying Y, but asserting X does preclude denying A, Y then asserting X, A does not preclude denying Y.

That is, if asserting X and denying Y is acceptable (at least, not precluded) then either adding the denial of A is not precluded or adding the assertion of A is not precluded. In other words, if (relative to the assertion of X and the denial of Y) A is undeniable, then it is not precluded to assert it. (Think of the ease with which we move from undeniability to assertion.)

(2) It is *constituitive* of the acts of assertion and denial, and the states of accepting and rejecting that they are governed by these norms. If we treat some pattern of behaviour as not governed by those norms, we are

not treating it as assertion and denial (or accepting and rejecting). If I take some pair of speech acts concerning the content A to not preclude each other, then I am not taking them to be denial and assertion, but to be some other acts.

(3) We can treat a proposed family of rules for concepts (say, rules for negation; rules for conjunction, quantifiers, modalities, number terms, etc.) as *defining* those concepts—by showing how patterns of assertions and denials involving those new concepts are to be treated, in terms of patterns of assertions and denials already so constrained.

(4) Belnap's constraint on definitions, then, amounts to showing that the new rules provide for the admissibility of [CUT] and of [ID]. At least, the details here are to be further worked out.

(5) For if [CUT] is no longer admissible in the wider space of acts allowed by the introduction of the new rules, then the acts so constrained are no longer constrained in a way governed by [CUT] and [ID] and thus, the acts are no assertion and denial (or accepting and rejecting).

(1) – (5) is the central argument. If it works, it provides a template against which we may judge different concepts for logicality. As it stands, we can see that the rules for classical logic meet the test, and as a result, we have an answer to Hacking's three questions about the scope of logic [38, p. 304].

> Why should rules for "pure logic" have the subformula property and be conservative? There are several kinds of answers. They correspond to criteria of adequacy like the following. (A) The demarcation should give the "right" logicist class of logical constants and theorems. That is, it should include the traditional (and consistent) core of what logicists said was logic and should exclude what they denied to be logic. (B) Since the demarcation is couched in terms of how logical constants are characterized, it should provide the semantics for the constants called "logical." (C) It should explicate why logic is important to the analytic program. Although (A) and (B) are important, (C) is essential. A demarcation of logic that leaves the analytic program unintelligible is of little philosophical interest (unless the point is to show that the analytic program *is* unintelligible).

¶ The central part of the book will present the argument, and defending it against objections to it. Part 2 will be the smallest part of the book. As it is the central argument, it deserves to be highlighted in its own part. It is the focus. Part 1 leads to the argument in Part 2, and Part 3 is a range of consequences for a number of topics.

PART III

Insights

MEANING AND PROOF 7

The foregoing chapters have said a lot about language and rules for use, norms governing assertion and denial, and the behaviour of particular concepts, such as the logical connectives and operators and modalities. In this chapter we will attend to some of the upshot of this work for issues in the philosophy of language and epistemology. We'll start with considering connections between this work and more traditional concerns in the philosophy of language.

7.1 | PROOF AND MEANING

Logic has long borne a close relationship with the philosophy of language. Since the pioneering work of Frege and Russell at the start of the twentieth century, through Tarski's groundbreaking work on truth, and the application of techniques from modal logic in semantics through the later part of the twentieth century, there has been a fruitful interplay between logic and philosophy of language. In large part, the traffic has been through *model theory* rather than proof theory. To provide a semantics for natural language, the standard tools of model theory give us the *reference* of expressions or means to construct their truth conditions, understood formally as the conditions under which a sentence is true in a model, or true at a world in a model, and it is clear that this begins to address some of the issues important to those interested in the meanings of expression in natural languages. How does proof theory compare on this score? Of what use is proof theory to theories of meaning?

PROOFS AND MEANINGS

To answer this, it will help to consider two distinct senses of a "theory of meaning." Following Jeff Speaks [92], it is helpful to distinguish two senses of the phrase. First, a *semantic theory* which assigns semantic contents to expressions of a language, and second a *foundational theory of meaning* which explains how and why expressions in a language have the meanings that they do. [Explain presemantics too???]

[Slot in Soames [91] here, too, for whom a proposition is an event type—the proposition that p is the event type of a particular kind. The proposition Fa is the event type of predicating F of a.]

[Slot in Lycan [50] on truth conditional theory of meanings.]

Language—a norm-governed communicative practice. Many things to be said about the norms, but we will focus on the declaratives, even though declaratives are not enough. But we could have norms governing not only assertion and denial (under suppositions) but also stronger things. Assertion/Denial as taking a stand. To put something out there for others to agree with or disagree with (or demur from). Different norms governing assertion (and denial) have been proposed. (Truth, Belief, Knowledge, of the form that an assertion is something that aims to express that property. [[Check the Wilson/Cappelen Assertion volume on the characterisation.) What is important for our task is not to take a stand on the nature of assertion or its aims (realistically, an assertion can be criticised on all three grounds in different ways), but rather, to characterise a particular kind of norm governing assertion, not the narrow bounds of *good play*, but the wider bounds of *legal* play. [[Why think there is any such constraint? Examine the relationship between assertion and denial. If I assert p and you deny it, then there is no (legal) position incorporating both of our acts. That's the point. They don't unify. (That's what's up in the example of the people without "no" in Price's "Why 'Not'?" we want to explain how these are genuinely different—in the sense of *divergent* positions.) That's what grounds the sense of no legal play. Once that's settled, we can go on and attempt to use this as a lever for the definition of vocabulary items.]]

[[This is where the argument about positions and norms really needs to go, and I need to be careful about it.]]

[[Then the matter of rules.]]

[[Then semantic values.]]

Second: we have the connection between truth conditions and positions that we have marked out in earlier chapters. We can use this, and explain the detail of what that means for the referent of an expression of a different syntactic category. notice that it will be open-ended depending on what is needed for the kind of explanation that is required, and the kind of background context of deducibility in play. This seems to be the right sort of thing to ask.

Third: hyperintensionality is straightforward in this setting.

Fourth: the direction of explanation can be given from the norms to the models rather than vice versa, and we get a standard and helpful structure, according to which not only does language use seem to be a kind of thing that we could engage in, but it gives us an explanation of why it is that the models do their explanatory work.

ACHILLES AND THE TORTOISE

And here is where we address Achilles and the Tortoise come in.

[Bring in Lewis Carroll and the Achilles/Tortoise question [17, 93] and connect the issue of question answering and the norms of assertion and denial here, giving the answer to what is odd in the Carroll *modus ponens* question.

- Set up the dialogue.
- Explain the nature of the problem. (There is a kind of mistake or failure on the behaviour of the tortoise, but what sort of mistake is it?)
- Explain what kind of theory this is a problem for.

- Explain attempted resolutions.
  - HAMBLIN/MACKENZIE [40, 51]: bureaucratic "rules of order" for discussion. Don't talk to people with a different logic. It's just *primitive* that $p, p \to q \therefore q$ is a valid rule, and is not withdrawable. GOOD: explaining what dialogue rules are broken; giving a semirealistic but real example of how a norm is broken. BAD: arbitrariness of rules of logic, or no explanation of why one might want these rules rather than those
  - SMILEY [90]: GOOD: Achilles doesn't grasp the difference between accepting a rule and asserting or commitment to a hypothetical proposition. INCOMPLETE: A general account of the connection and why one might accept or reject a rule and its application to coordination problems.
  - BLACKBURN [12]: GOOD: discussion of the will and imperatives or prescriptives. BAD: assumes the problem is to explain how logical deduction should move belief.
  - MIŠČEVIĆ [58]: GOOD: connects up with Bonjour and the argument for Rationalism, and nicely defeats the Bonjour argument.

7.2 | PROOF AND NECESSITY

7.3 | PROOF AND WARRANT

Engage with Prawitz here: [69]

GROUNDS AND PROOF TERMS

POSSESSING A PROOF

COMBINING GROUNDS AND THE PREFACE PARADOX

[Explain the mechanics of a proof from $X \succ Y$ shows us why the position $[X : Y]$ is out of bounds, and what that means for warrant preservation.]

[Then explain how there is a real difference, even when it comes to propositional logic which is decidable, between the possession of a proof for $X \succ Y$, which can be used to reduce the position to absurdity, and the mere fact of validity when the agent in question does not possess a proof. We use Prawitz's understanding of the proof as something to *do*, the possession of which is a genuine cognitive achievement. So, if it turns out that $X \succ Y$ has some derivation then an agent who has asserted X and denied Y has made a mistake, but there is another sense in which the the absence of such a proof allows for a kind of (weakly) permissible ignorance.]

[Flesh this out formally, with gestures at what is involved in *showing*.]

[This brings us up to the P vs NP problem, http://www.scottaaronson.com/blog/?p=459, for which something as simple as 3SAT, for example, is NP complete, while any proof is checkable in polynomial time.]

Bring in Dag Prawitz on grounds and warrants, etc. The crucial issue is transitions like the following. We want to be able to say, with Dag, that if $X \succ A$ is derivable, then anyone justified in asserting X, if they make the inference from X to A they are thereby also justified in asserting A. So, to multiple-conclusionise that step, we need to read things in general as follows. If $X \succ A, Y$ is derivable then anyone justified in asserting X and denying Y, if they make the inference from X/Y to A, they are thereby also justified in asserting A, *and* that if $X, A \succ Y$ derivable, then anyone justified in asserting X and denying Y, if they make the inference from A to Y/X, they are thereby also justified in denying A. The trick, of course, is in the structural rule of contraction. We wish to move from $X \succ A, A, Y$ to $X \succ A, Y$. We generally wish to bring out a connection between the two different ways of reading $X, A \succ B, Y$: that anyone justified in asserting X, A and denying Y, who uses the inference from X, A to B, Y is thereby justified in asserting B, and dually, anyone justified in asserting X and denying B, Y who uses the inference from X, A to B, Y is thereby justified in denying A. How can I make all these matters clear? This is necessary if we are to make an account of the connection between deduction and justification.

It would be even nicer to have a sense of 'ground' (positive and negative) for which we can construct these sorts of justifications in terms of primitive grounds, in such a way as to bring out the distinctive classicality of the approach without being overly ridiculous in terms of omniscience (or logical omniscience). That is the nice goal.

7.4 | PARADOX AND NEGATION

7.5 | THE EPISTEMOLOGY OF LOGIC

Russell on Metaphysical Analyticity and the Epistemology of Logic [81].

… FOR QUANTIFIERS AND OBJECTS 8

THE SEMANTICS OF QUANTIFICATION

Consider the difference between three different kinds of interpretation of the quantifiers [13, 26, 48].

- OBJECTUAL: Truth conditional, and referential. $(\forall x)A(x)$ is *true* (in a model $\mathfrak{M}$) if and only if $A(x)$ is true *of* d in the model $\mathfrak{M}$ for each object d in the domain of $\mathfrak{M}$. Or more generally, $(\forall x)A(x)$ is *satisfied by* the assignment α of values to the variables in the model $\mathfrak{M}$ iff $A(x)$ is satisfied by the assignment $\alpha[x := d]$ for each object d in the domain. Relies either on (a) and unrealistic expansion of the vocabulary or (b) a mysterious or difficult satisfaction relation. It gives the "right results"—I have no dispute concerning first order logic (within limits, depending on what you say about the semantics of empty names), but in general, it's hard to see what the *intended* domain of objects is. (Is it a set? A class? Does it contain absoultely everything? Including the model itself? (this makes the theory of sets very difficult.)) The objectual interpretation fixes on one model as the intended model. There is no need to fix on that in order to get the benefits of first order logic and the local use of models as modelling restricted parts of reality.

- SUBSTITUTIONAL: Motivated by anaphora. Fails for "unnamed objects". Extension to possible addition of names is better, but requires an account of possible names. Gives the wrong results under language (or domain) expansion.

- INFERENTIAL: requires inferentially inert names (pronouns), and gives "right results." Has different goals to the truth conditional objectual interpretation.

Give the reasons for preferring an inferential explanation, though this keeps what is good in the objectual and the substitutional interpretation. The important feature in an inferential account is the kind of generality that's involved.

8.1 | GENERALITY AND QUANTIFIER RULES

How does normalisation or cut elimination work in the principle case for a quantifier? It trades on a kind of substitution. A term has to be inferentially (deductively, with respect to the bounds) general in order to play the appropriate role in the rule. To prove $A(b)$ where b is *arbitrary* or *general* is enough to prove $(\forall x)A(x)$, because that is enough to prove

$A(t)$ for any term t at all. That is the core of what is required in generality. Let's spell out what is involved in making this work, and see what it means for our account of defining rules and logical concepts.

The logical concepts we have seen so far are propositional connectives. They *combine* propositions to make new propositions, which may be asserted and denied, supposed, questioned, etc. Quantifiers do more. They exploit some of the internal structure of sentences. In the language of first order logic, there is a relationship between the following four sentences

$$Fa \quad Fb \quad (\forall x)Fx \quad (\exists y)Fy$$

despite the fact that none of the sentences are components of any of the others, and neither is there a sentence shared between them. They share components (the predicate F) but this component is not itself an assertible or a deniable. A predicate is something different. Similarly, the sentences Fa and Fb contain *singular terms*: a and b. It is these singular terms which allow us to define quantifiers.

So, let's suppose that our language has a designated class of *singular terms* (we'll use $m, n, \ldots, s, t, \ldots$ for singular terms of various kinds[1]), and a class of *variables* (we'll use $x, y, z, \ldots$ for variables) for the quantifiers $(\forall x), (\forall y), \ldots$ and $(\exists x), (\exists y), \ldots$ In general, if a language allows for sentences to include variables unbound by any quantifier, then the variables will be among the class of singular terms. We will not make that assumption here, and for clarity, we will take it that all variables occuring in sentences must be bound by quantifiers, though nothing important hangs on this.

For each sentence A in our language, we may single out some instances of a singular term occuring in A by enclosing that term parentheses. $A(n)$ is a sentence with some number of instances of n singled out.[2] Given $A(n)$, the formula $(\forall x)A(x)$ is found by replacing those designated instances of n by the variable x and binding the formula with the quantifier $(\forall x)$. So, for example, if $A(n)$ is the formula

$$(Lm\underline{n} \supset L\underline{n}m) \wedge Lnm'$$

with the designated instances of n indicated by underlining, then the corresponding formula $(\exists x)A(n)$ is

$$(\exists x)((Lmx \supset Lxm) \wedge Lnm')$$

This means that we can replace some number instances of a singular term in a formula by variables, and bind them with a quantifier in order to construct a new formula.[3] This leads to the natural question; What is the connection between $A(n)$ and $(\forall x)A(x)$? Or between $A(n)$ and

[1]A mnemonic: 'n' for 'name' and 't' for 'term'. The distinction between names and terms is not important now, but will become important soon.

[2]Yes, that number can be zero.

[3]We allow for languages in which variables themselves count as singular terms, and languages in which they never occur in formulas unbound, and are not themselves proper singular terms.

$(\exists x)A(x)$? The classical behaviour of the quantifiers suggests the following pair of defining rules:

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\ \forall Df \qquad \frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\ \exists Df$$

(where n is not present in the bottom sequent of both rules). These rules are satisfied by the quantifiers in classical first order predicate logic. For example, $(\forall x)A(x)$ is false in a model if and only if we can assign a value to name n, for which $A(n)$ is false in that model (provided that n is free to interpret however we wish—as it is in first order predicate *logic*, where there is no restriction on the interpretation of names). On the other hand $(\exists x)A(x)$ is true in a model if and only if we can assign some value for n such that $A(n)$ is true in that model.

These rules can be motivated in terms of assertion and denial, too. Denying $(\forall x)A(x)$ (in the context of asserting Γ and denying Δ) involves a clash if and only if denying $A(n)$ would involve a clash, where n is a name free of any commitments. Asserting $(\exists x)A(x)$ (in the context of asserting Γ and denying Δ) involves a clash if and only if asserting $A(n)$ would involve a clash, where n is a new name.

For these rules to work in the intended way, the name n has to be appropriate. Not every singular term in every language can do the job. Here is an example. Consider RA the finite set of axioms of Robinson's Arithmetic.[4] In the context of classical predicate logic, we can derive $RA \succ 0 \neq 3088$. There is a clash involved in asserting the axioms of RA and in denying that 0 is unequal to 3088. However, the term 3088 does not appear in the axioms of RA. Nonetheless, it would be a mistake to generalise using [$\forall Df$] to deduce $PA \succ (\forall x)(0 \neq x)$. Although 3088 does not appear in the axioms of RA, it is not logically independent of them. In the syntax of RA, 3088 is a *function* term,[5] which means that it is not free to be interpreted arbitrarily, given the commitments made in the other assertions and denials we have made—in this case, in RA.

In general, in a given language $\mathcal{L}$ with a consequence relation $\succ$, we will say that a term α in a category $\mathfrak{A}$ is DEDUCTIVELY GENERAL iff for each sequent $\Gamma \succ \Delta$ that can be derived, so can $\Gamma[\alpha := \beta] \succ \Delta[\alpha := \beta]$ where α is globally replaced in that sequent by another term β in the category $\mathfrak{A}$.[6] In first order classical predicate logic, *function* terms are not deductively general singular terms, but primitive names are. If we consider the conseqence relation of *Peano Arithmetic*, defined by setting $\Gamma \succ_{PA} \Delta$ iff $PA, \Gamma \succ \Delta$ over the language of predicate logic with 0, successor, addition, multiplication and identity, where PA is an axiomatisation of PA, then the constant term 0 is not deductively general, since we

[4] In fact, we don't need all of the axioms in RA. The single axiom $(\forall x)(0 \neq x')$ stating that 0 is not the successor of any number will do.

[5] Literally, it is the term '0' with the successor function applied to it 3088 times.

[6] In the rest of this paper, the only syntactic category we will consider is the category of singular terms. However, in other contexts, we will consider other syntactic categories, such as the category of predicates, and the category of sentences.

have $(\exists x)(0 \neq x')\succ_{PA}$ (it's inconsistent with the axioms of PA for 0 be the successor of some number), but we do not have $(\exists x)(0' \neq x')\succ_{PA}$.

Our proposed defining rules [$\forall$*Df*] and [$\exists$*Df*] make sense only when we impose the restriction that the terms n appearing in the defining rule are deductively general. However, there is a tension between this condition and the form of the defining rules themselves. Consider a toy example, in which we have a language with a primitive predicate F, a stock of names n, m,..., and a single one-place function symbol g, and it is interpreted in the usual classical fashion. In this language, n and m are names, while $g(n)$, $g(m)$, $g(g(n))$, etc., are terms but not names. How can we derive $(\forall x)Fx \succ Fg(n)$ when we extend the language using [$\forall$*Df*]? We can certainly derive $(\forall x)Fx \succ Fn$ for each name n, as follows:

$$\dfrac{\dfrac{}{(\forall x)Fx \succ (\forall x)Fx}\,\textit{Id}}{(\forall x)Fx \succ Fn}\,\forall\textit{Df}$$

However, there is no way to apply the rule [$\forall$*Df*] to generate the conclusion $Fg(n)$, since $g(n)$ is not a name—it is not deductively general. If we were to require that in the extended language, the names remained deductively general, we would require that the consequence relation satisfy the condition of *Specification*:

$$\dfrac{\Gamma \succ \Delta}{\Gamma[n := t] \succ \Delta[n := t]}\,\textit{Spec}^{n}_{t}$$

which permits a gobal replacement of the deductively general term n by the term t of the same category, which may be less general.[7] In other words, the specification rule ensures that the terms n indeed are deductively general. If we could appeal to the specification rule in a derivation, we could conclude $(\forall x)Fx \succ Fg(n)$ in the following way:

$$\dfrac{\dfrac{\dfrac{}{(\forall x)Fx \succ (\forall x)Fx}\,\textit{Id}}{(\forall x)Fx \succ Fn}\,\forall\textit{Df}}{(\forall x)Fx \succ Fg(n)}\,\textit{Spec}^{n}_{g(n)}$$

Specification differs from the other sequent rules we have seen. It is not a local rule in a derivation introducing a single formula, but a global rule modifying the entire sequent.[8] In the classical sequent calculus it is an *admissible* rule, rather than a primitive rule, because any step in the derivation in which ends in a sequent involving a general term could be

[7]The first paper in which a rule of this form is explicitly considered in a Gentzen system is Arnon Avron's 1993 paper "Gentzen-type systems, resolution and tableaux" [3].

[8]So, it is not easily understood as corresponding to a natural deduction rule in which a proof is modified either at a premise or a conclusion position, but is rather understood as a global transformation of a proof. We transform a proof involving the deductively general term n into a proof involving the more specific term t.

converted into a step in which that term is replaced by a more specific one. The rules [$\forall Df$] and [$\exists Df$] do not have this feature, when read from bottom to top.

[End inclusion to edit.]

This is enough to ground classical predicate logic, and this is good. (We'll see later what it this means concerning models.)

But perhaps this is not the kind of generality we want in *all* and *some*. Perhaps the language contains non-denoting terms. These terms come in a number of kinds:

Fictional characters.

Failed reference chains.

Complex mathematical terms are only partially defined, and which give rise to reference failure.

Similar examples from empirical sciences seem possible [check Shawn's recommendation about Mark Wilson and Bob Batterman.]

Error theorists about X say that there are no entities of kind X. If others, who aren't error theorists attempt to name entities of kind X, then the natural understaning of what is going on, names in their vocabulary are, according to the error theorist, empty. So, for example, the fictionalist about mathematics takes there to be no mathematical entities. On a natural view, they take 0, 1, 2, … to not denote. Mathematical claims, such as "$1 + 2 = 3$" are, literally speaking, false. **[[Check Field's *Science without Numbers* to confirm that this is how he characterises things]]** The same goes for other terminology.

So, we take it that there are terms which don't denote. There is no Zeus. Perhaps Homer does not exist. Certainly $1/0$ doesn't exist, and neither does $\lim_{x\to\infty} a_n$ when a is a diverging sequence. In other words, $\neg(1/0\downarrow)$ and $\neg(\lim_{x\to\infty} f(x)\downarrow)$ when f diverges.

How do non-denoting terms interact with predication and the quantifiers?

Let's start with predication. It's not an option, for us, to take it that Fa is truth valueless when a fails to denote, at least, not when 'truth valueless' is given its standard meaning. For us, given that Fa can be asserted and it can be denied, $Fa \vee \neg Fa$ is undeniable. So, in this very small sense, options which take us to deny Fa and deny $\neg Fa$, are off the table. Of course, the option remains to find a *stronger* negation operator, $-$ say, for which both Fa and $-Fa$ can be denied, so $Fa \vee -Fa$ can also be denied.[9]

[9] An example might be simple, at least in the case of predication: $-Fa$ is $a\downarrow \wedge \neg Fa$. (It's a bit more effort to define $-$ as a sentential operator.) If you take it that basic predication is existence entailing, then this seems to make some kind of sense. An outer negation (that defined by $\neg$, simply denies Fa, where it can deny it on the grounds of reference failure for a), and inner negation says *of* a that it's not F, which is modelled here by the conjunction $a\downarrow \wedge \neg Fa$. Perhaps generalising this kind of strong negation to an operator on arbitrary sentences will give room to preserve what is worth preserving in the idea that Fa is truth valueless when 'a' fails to denote, despite the semantics not allowing for truth value gaps in the stronger sense. Fa is false when a fails to denote, and $\neg Fa$ is true. But Fa is not *strongly* false in the sense of its strong negation being true.

[material from Generality and Existence papers on specification and generality. Reflect on this and decide what to include with rewriting, and what to totally restructure.]

[End of inclusion.]

[Then do the same thing for free logic.]

8.2 | APPLYING THE ARGUMENT

8.3 | POSITIONS AND MODELS FOR LOGIC WITH QUANTIFIERS

[Consider what I said in the section on truth and positions in chapter 6, and what this means, then, for what this means for a *domain*. What it means to take something to *exist*.]

8.4 | ABSOLUTE GENERALITY

Here is how we have absolute generality.

We can agree on the universal quantifier by coordinating on the defining rules for that quantifier. When we do this, we agree on the meaning of the quantifier in every way that matters for us: your use and my use are inferentially equivalent.We do not have to agree on what exists in order to coordinate in this way.

This results in a kind of *absolute generality*. The semantics of the universal quantifier thus understood gives us what everyone requires of absolute generality.

(I need to check what people have actually said about this. How to specify that??)

It is not syntactically restricted by explicit restrictors. Given anything that might be left out of the domain of quantification, a singular term can capture it. (This also seems a little subtle and needs careful treatment, both in the boundaries of meaningful singular term application, and in the flexibility of pronouns. What do I say about fictional discourse and open ended expansion of mathematical vocabulary, etc?)

Whatever people have give as a requirement of absolute generality (apart from something specified in terms of Tarski style models) we meet it.

This semantics does *not* go by way of initially identifying a domain and defining the semantics in terms of that domain.

This is one argument *against* absolute generality: meaningful quantification requires an interpretation, interpretation requires a domain, that domain cannot be a member of itself (or, more generally, that domain must leave some things out). — We do not follow this argument at the second step: interpretation does not require a domain in that sense.

The completeness proof delivers *small models*. This proof is always relative to a particular language, even with the use of free variables or pronouns. The result will not be an absolutely general model, the "standard model". Because in *that* sense, there is no such thing. [Can I make out a strong case for this?]

Under conceptual expansion, we'd get a *larger* model.

What shall I say about class models?

8.5 | THE EXAMPLE OF ARITHMETIC

8.6 | REALISM AND ANTI-REALISM

[Check Alex Miller SEP on Realism to see what kind of realism I'm committed to or not]

... FOR MODALITY AND WORLDS 9

9.1 | HYPERSEQUENTS AND 2D HYPERSEQUENTS

9.2 | VERIFYING THAT THINGS WORK

9.3 | SOLVING PRIOR'S OTHER PROBLEM

9.4 | MIXING AND MATCHING WITH QUANTIFIERS

9.5 | MORE ON REALISM AND ANTI-REALISM

What is the status of the worlds (or the *possibilia*) that are generated in the model theory? This is a subtle point. Add a pointer to Ruth Kempson on representationalism in semantics as a guide to where the discussion might go [44].

[So, Greg, suppose I agree with you about the modal logic: we grok the modal concepts by using them in just the way you describe. Excellent.

What does this tell us about modal *reality*? I can feel fine that we now mean the same thing by box. But have I learned anything about modal reality? I'd like to learn something significant about what is really really true, modally speaking. How does what you have done help with that?

I think this is a good question, and in a sense, I haven't given a complete answer. These rules is compltely consistent with fatalism, and with plenitudinous views of modal reality. Just like first order logic is consistent with monism, and with there being very very many objects.

But further, it tells us where to look if we want to know *more*. How do you prove non-monism, that there is more than one thing? well, one way to do that is by proving it from Fa and $\neg Fb$ for some choices of F, a and b.

Same here in the modal case, except we don't have names to specify different zones: need to say something about what primitives we have for modal vocabulary...Do we have nominals? Not clearly, but perhaps things can be introduced *locally*.]

Inspired by a comment from Al Wilson, after my philosophy seminar. September 15, 2011.

Run the same argument as in Chapter 9, and in Chapter 6, concerning what these positions are. What is it for me to rule a term in or rule it out in a zone? What is it to rule a sentence in or rule it out in a zone? What is it to include a zone in a position? Explain that this is what it is to adopt a position, and therefore what is involved in taking it to be *true*. This will answer Nick Smith's question [Taiwan, October 2016] concerning the status of the model constructed. They're not necessarily syntactic devices, but they're not necessarily objects to which you're committed, in the fully fledged sense. I think this gives a nice answer to what's going on in the necessitist and contingentist debate.

How to continue

What about truth?

What else is missing?

The open endedness of extension

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