Isomorphisms in a Category of Propositions and Proofs

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MELBOURNE LOGIC GROUP · 2 MARCH 2018

To show how a category of propositions and *classical* proofs can give rise to finely grained hyperintensional notions of sameness of content.

One notion is *very* finely grained (distinguishing p and $p \wedge p$) others are is less finely grained.

One of these notions amounts to equivalence in R. B. Angell's logic of analytic containment.

To apply distinctively
proof theoretical methods
to issues in philosophical logic.

Acknowledgements

Thanks to
Rohan French,
Dave Ripley, and
Shawn Standefer for
helpful conversations
on this material.

The Category of Classical Proofs

Isomorphisms

More Proofs from A to A

Matching & Logics of Analytic Containment

Matching as Isomorphism

THE CATEGORY OF CLASSICAL PROOFS

There can be different ways to prove the same thing

$$p \land q \succ p \lor q$$

Four different derivations,

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,{\wedge L}}{p \wedge q \succ p \vee q}\,{\vee R} \qquad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,{\vee R}}{p \wedge q \succ p \vee q}\,{\wedge L}$$

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,{\wedge L}}{p \wedge q \succ p \vee q}\,{\vee R} \qquad \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,{\vee R}}{p \wedge q \succ p \vee q}\,{\wedge L}$$

Four different derivations, two *proofs*

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,{\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\,{\scriptstyle \vee R} \quad \approx \quad \dfrac{\dfrac{p \wedge q}{p}}{p \vee q} \quad \approx \quad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,{\scriptstyle \vee R}}{p \wedge q \succ p \vee q}\,{\scriptstyle \wedge L}$$

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,{\scriptstyle \wedge L}}{p \wedge q \succ p \vee q}\,{\scriptstyle \vee R} \quad \approx \quad \dfrac{\dfrac{p \wedge q}{q}}{p \vee q} \quad \approx \quad \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,{\scriptstyle \vee R}}{p \wedge q \succ p \vee q}\,{\scriptstyle \wedge L}$$

Proof terms are an *invariant*
for derivations under rule permutation.

δ_1 and δ_2 have the same *term* iff
some permutation sends δ_1 to δ_2.

Four different derivations, two *proof terms*

$$\cfrac{\cfrac{\overset{x \curvearrowright y}{x : p \succ y : p}}{\overset{\acute{\wedge}x \curvearrowright \acute{\vee}y}{x : p \wedge q \succ y : p}}\;{\scriptstyle \wedge L}}{\overset{\acute{\wedge}x \curvearrowright \acute{\vee}y}{x : p \wedge q \succ y : p \vee q}}\;{\scriptstyle \vee R} \qquad \acute{\wedge}x \curvearrowright \acute{\vee}y \qquad \cfrac{\cfrac{\overset{x \curvearrowright x}{x : p \succ y : p}}{\overset{x \curvearrowright \acute{\vee}y}{x : p \succ y : p \vee q}}\;{\scriptstyle \vee R}}{\overset{\acute{\wedge}x \curvearrowright \acute{\vee}y}{x : p \wedge q \succ y : p \vee q}}\;{\scriptstyle \wedge L}$$

$$\cfrac{\cfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{\grave{\wedge}x \curvearrowright y}{x : p \wedge q \succ y : q}}\;{\scriptstyle \wedge L}}{\overset{\grave{\wedge}x \curvearrowright \grave{\vee}y}{x : p \wedge q \succ y : p \vee q}}\;{\scriptstyle \vee R} \qquad \grave{\wedge}x \curvearrowright \grave{\vee}y \qquad \cfrac{\cfrac{\overset{x \curvearrowright y}{x : q \succ y : q}}{\overset{x \curvearrowright \grave{\vee}y}{x : q \succ y : p \vee q}}\;{\scriptstyle \vee R}}{\overset{\grave{\wedge}x \curvearrowright \grave{\vee}y}{x : p \wedge q \succ y : p \vee q}}\;{\scriptstyle \wedge L}$$

Ingredients

λ terms • flow graphs • proof nets

Slogan

A *proof term* for $\Sigma \succ \Delta$
encodes the flow of information
in a proof of $\Sigma \succ \Delta$.

Proof Terms

$$\acute{\wedge}x \curvearrowright \acute{\wedge}\,\acute{\vee}y \quad \acute{\wedge}x \curvearrowright \acute{\wedge}\,\grave{\vee}y \quad \acute{\vee}\,\grave{\wedge}x \curvearrowright \grave{\wedge}\,\acute{\vee}y \quad \grave{\vee}\,\grave{\wedge}x \curvearrowright \grave{\wedge}\,\grave{\vee}y$$

$$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$$

Proof Terms as Graphs on Sequents

$$\acute{\wedge}x \frown \acute{\wedge}\,\acute{\vee}y \quad \acute{\wedge}x \frown \acute{\wedge}\,\grave{\vee}y \quad \acute{\vee}\,\grave{\wedge}x \frown \grave{\wedge}\,\acute{\vee}y \quad \grave{\vee}\,\grave{\wedge}x \frown \grave{\wedge}\,\grave{\vee}y$$

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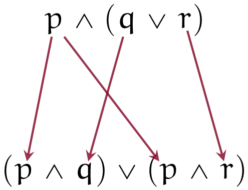

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,{\wedge R} \qquad \dfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,{\wedge R}}{p, q \vee r \succ p \wedge q, p \wedge r}\,{\vee L}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}\,{\vee R}}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,{\wedge L}$$

Finding a Proof Term from a Derivation

$$\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,{\wedge R} \qquad \dfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,{\wedge R}}{\dfrac{p, q \vee r \succ p \wedge q, p \wedge r}{\dfrac{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,{\wedge L}}\,{\vee R}}\,{\vee L}$$

$$p \wedge (q \vee r)$$

$$(p \wedge q) \vee (p \wedge r)$$

$$
\cfrac{\cfrac{\cfrac{\cfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,{\scriptstyle \wedge R} \qquad \cfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,{\scriptstyle \wedge R}}{p, q \vee r \succ p \wedge q, p \wedge r}\,{\scriptstyle \vee L}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}\,{\scriptstyle \vee R}}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,{\scriptstyle \wedge L}
$$

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Finding a Proof Term from a Derivation

$$
\dfrac{
\dfrac{
\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \land q}\,{\scriptstyle \land R}
\qquad
\dfrac{p \succ p \qquad r \succ r}{p, r \succ p \land r}\,{\scriptstyle \land R}
}{p, q \lor r \succ p \land q, p \land r}\,{\scriptstyle \lor L}
}{
\dfrac{p, q \lor r \succ (p \land q) \lor (p \land r)}{p \land (q \lor r) \succ (p \land q) \lor (p \land r)}\,{\scriptstyle \land L}
}\,{\scriptstyle \lor R}
$$

$$p \land (q \lor r)$$

$$(p \land q) \lor (p \land r)$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad q \succ q}{p, q \succ p \wedge q}\,{\wedge R} \qquad \dfrac{p \succ p \qquad r \succ r}{p, r \succ p \wedge r}\,{\wedge R}}{p, q \vee r \succ p \wedge q, p \wedge r}\,{\vee L}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}\,{\vee R}}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)}\,{\wedge L}
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$$

$$
p \wedge (q \vee r)
$$

$$
(p \wedge q) \vee (p \wedge r)
$$

More Flow Graphs

$$p \supset q$$

$$(q \supset r) \supset (p \supset (q \wedge r))$$

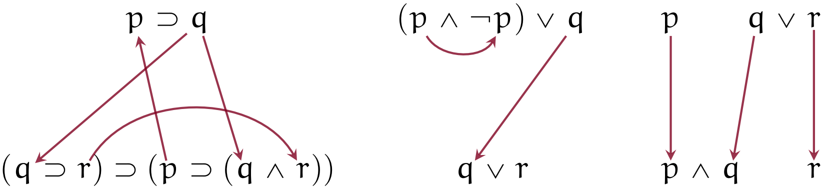
$$(p \wedge \neg p) \vee q$$

$$q \vee r$$

$$p \qquad q \vee r$$

$$p \wedge q \qquad r$$

More Flow Graphs



Links wholly internal to a *premise* or a *conclusion* are called *cups* ($\smile$) and *caps* ($\frown$).

Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

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- They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]

Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

- They *typecheck*. [An occurrence of p is linked only with an occurrence of p.]
- They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]
- They must satisfy an "enough connections" condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

Cut is chaining of proof terms

$$(p \land q) \lor (p \land r)$$

$$p \land (q \lor r)$$

Cut is chaining of proof terms

$$p \land (q \lor r)$$

$$(p \land q) \lor (p \land r)$$

Cut is chaining of proof terms

$$(p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \vee r)$$

$$(p \wedge q) \vee (p \wedge r)$$

The *cut formula* is no longer a premise or a conclusion in the proof term.

Eliminating Cuts is Local

$$(p \land q) \lor (p \land r)$$

$$p \land (q \lor r)$$

$$(p \land q) \lor (p \land r)$$

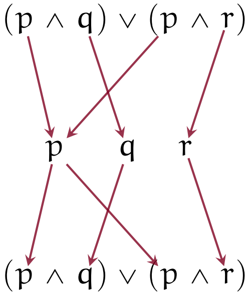
Eliminating Cuts is Local

$$(p \land q) \lor (p \land r)$$

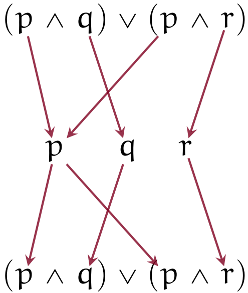
$$p \qquad q \lor r$$

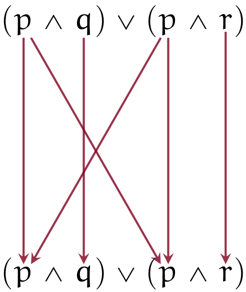
$$(p \land q) \lor (p \land r)$$

Eliminating Cuts is Local



Eliminating Cuts is Local





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Results

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- Cut elimination for proof terms is *local*.

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- Cut elimination for proof terms is *local*.

  [So it is easily made parallel.]

Cuts with Caps and Cups

$$p \land (p \supset q)$$

$$q \land (r \lor \neg r)$$

$$\neg p \lor (p \land q)$$

Cuts with Caps and Cups

$$p \wedge (p \supset q)$$

$$q \wedge (r \vee \neg r)$$

$$\neg p \vee (p \wedge q)$$

$$p \wedge (p \supset q)$$

$$\neg p \vee (p \wedge q)$$

$\mathfrak{C}$ is the Category of Classical Proofs

OBJECTS Formulas — A, B, etc.

ARROWS Cut-Free Proof Terms — $\pi : A \succ B$.

COMPOSITION Composition of derivations with the elimination of *Cut* — If $\pi : A \succ B$ and $\tau : B \succ C$ then $\tau \circ \pi : A \succ C$.

IDENTITY Canonical identity proofs — $Id(A) : A \succ A$.

Identity Proofs

$$
\dfrac{
\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R
\qquad
\dfrac{\dfrac{p \succ p \quad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R
}{
\dfrac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R
}\,\vee L
$$

$$\neg p \vee (p \supset p)$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\,\vee L}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R$$

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$$\begin{tikzcd} \neg p \vee (p \supset p) \\ \neg p \vee (p \supset p) \arrow[u] \end{tikzcd}$$

Identity Proofs

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,\supset L}{p \supset p \succ p \supset p}\,\supset R}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\,\vee L}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,\vee R$$

$$\begin{array}{c} \neg p \vee (p \supset p) \\ \Big\uparrow \\ \neg p \vee (p \supset p) \end{array}$$

$$
\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\ {\scriptstyle \neg L}}{\neg p \succ \neg p}\ {\scriptstyle \neg R} \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\ {\scriptstyle \supset L}}{p \supset p \succ p \supset p}\ {\scriptstyle \supset R}}{\dfrac{\neg p \lor (p \supset p) \succ \neg p, p \supset p}{\neg p \lor (p \supset p) \succ \neg p \lor (p \supset p)}\ {\scriptstyle \lor R}}\ {\scriptstyle \lor L}
$$

$$
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$$

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$$

Identity Proofs

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,{\supset}L}{p \supset p \succ p \supset p}\,{\supset}R}{\neg p \vee (p \supset p) \succ \neg p, p \supset p}\,{\vee}L}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}\,{\vee}R
$$

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Identity Proofs

$$\dfrac{\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{\neg p \succ \neg p}\,\neg R \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \supset p, p \succ p}\,{\supset}L}{p \supset p \succ p \supset p}\,{\supset}R}{\dfrac{\neg p \lor (p \supset p) \succ \neg p, p \supset p}{\neg p \lor (p \supset p) \succ \neg p \lor (p \supset p)}\,{\lor}R}\,{\lor}L$$

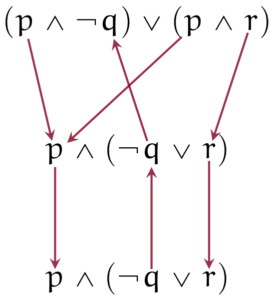
$$\begin{array}{c} \neg p \lor (p \supset p) \\ \\ \neg p \lor (p \supset p) \end{array}$$

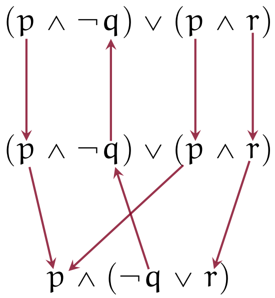
Identity Proofs

In the identity proof from A to A,

- A *positive* occurrence of an atom in the premise linked *to* its mate in the conclusion.
- A *negative* occurrence of an atom in the premise is linked *from* its mate in the conclusion.
- There are no other links.

Identity and Composition in $\mathfrak{C}$





Identity and Composition in $\mathfrak{C}$

$$(p \wedge \neg q) \vee (p \wedge r) \qquad\qquad (p \wedge \neg q) \vee (p \wedge r)$$

$$(p \wedge \neg q) \vee (p \wedge r)$$

$$p \wedge (\neg q \vee r) \qquad\qquad p \wedge (\neg q \vee r)$$

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The Category $\mathfrak{C}$ …

- … is *symmetric monoidal* and *star autonomous*
- but not *Cartesian*,
- with structural *monoids* and *comonoids*,
- and is enriched in *SLat* (the category of semilattices).

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\begin{aligned}
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\pi \circ (\tau \cup \tau') \quad &= \quad (\pi \circ \tau) \cup (\pi \circ \tau')
\end{aligned}
$$

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(\pi \cup \pi') \circ \tau &\quad=\quad (\pi \circ \tau) \cup (\pi' \circ \tau)
\end{aligned}
$$

$\mathfrak{C}$ is just *classical* propositional logic,
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in a categorical setting.

(The sequent calculus plays no essential role here.
You can define proof terms on other proof systems,
e.g. *natural deduction, Hilbert proofs, tableaux, resolution*.)

ISOMORPHISMS

$f : A \to B$ is an *isomorphism* in a category iff it has an *inverse* $g : B \to A$, where $g \circ f = id_A : A \to A$ and $f \circ g = id_B : B \to B$.

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it has an *inverse* $g : B \to A$, where
$g \circ f = id_A : A \to A$ and $f \circ g = id_B : B \to B$.

If g and g' are both inverses, we have
$g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g'$,
so any inverse is unique. We can call it f^{-1}.

Why Isomorphisms?

If A and B are isomorphic in a category $\mathcal{C}$,
then what we can do with A (in $\mathcal{C}$)
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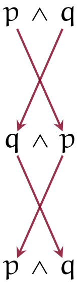
If A and B are isomorphic in $\mathfrak{C}$, then
they agree not only on *provability*,
but also, on *proofs*.

The distinctions drawn when you analyse how something is *proved* (from premises), are not far from what you want to understand when you ask how something is *made true*.

Isomorphisms in $\mathfrak{C}$

$$\mathfrak{p} \wedge \mathfrak{q} \cong \mathfrak{q} \wedge \mathfrak{p}$$

$$p \wedge q \cong q \wedge p$$



Isomorphisms in $\mathfrak{C}$

$$p \wedge q \cong q \wedge p$$

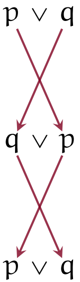
$$\begin{tikzcd}
p \arrow[drr] & \wedge & q \arrow[dll] \\
q \arrow[drr] & \wedge & p \arrow[dll] \\
p & \wedge & q
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
p \arrow[d] & \wedge & q \arrow[d] \\
p & \wedge & q
\end{tikzcd}$$

Isomorphisms in $\mathfrak{C}$

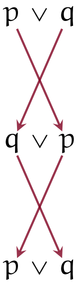
$$p \lor q \cong q \lor p$$

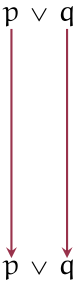
Isomorphisms in $\mathfrak{C}$

$$p \vee q \cong q \vee p$$



$$p \vee q \cong q \vee p$$





Isomorphisms in $\mathfrak{C}$

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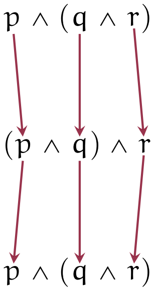
$$p \wedge (q \wedge r)$$

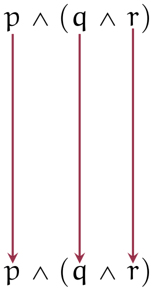
$$(p \wedge q) \wedge r$$

$$p \wedge (q \wedge r)$$

Isomorphisms in $\mathfrak{C}$

$$p \wedge (q \wedge r) \cong (p \wedge q) \wedge r$$





$$\neg(p \wedge q) \cong \neg p \vee \neg q$$

Isomorphisms in $\mathfrak{C}$

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$$\begin{tikzcd}
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p \arrow[d] \\
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\end{tikzcd}$$

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Non-isomorphisms in $\mathfrak{C}$

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$$p$$

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p

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$$\begin{tikzcd} p \wedge (q \vee \neg q) \arrow[d] \\ p \arrow[d] \\ p \wedge (q \vee \neg q) \end{tikzcd} \qquad\qquad \begin{tikzcd} p \wedge (q \vee \neg q) \arrow[dd] \\ \\ p \wedge (q \vee \neg q) \end{tikzcd}$$

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$$\begin{array}{ccc}
p \wedge (q \vee \neg q) & \qquad\qquad & p \wedge (q \vee \neg q) \\
p & & \\
p \wedge (q \vee \neg q) & & p \wedge (q \vee \neg q)
\end{array}$$

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$$\begin{tikzcd} & p \arrow[dl] \arrow[dr] & \\ p & \wedge & p \end{tikzcd}$$

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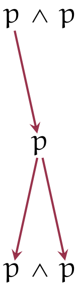

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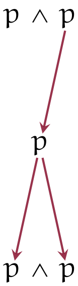

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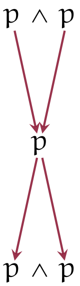

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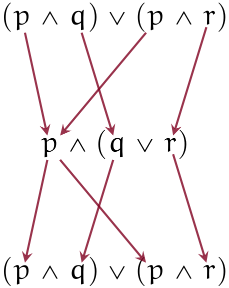

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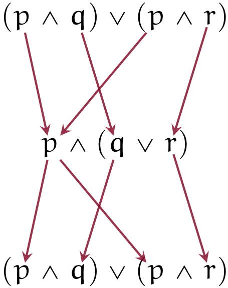

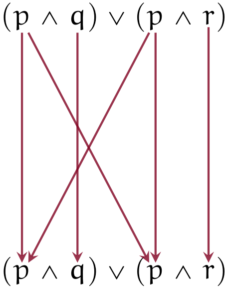

Non-isomorphisms in $\mathfrak{C}$

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Occurrence Polarity Condition

If A is isomorphic to B in $\mathfrak{C}$
then each variable occurs positively [negatively] in A
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If A is isomorphic to B in $\mathfrak{C}$
then each variable occurs positively [negatively] in A
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(This condition is *necessary*, not *sufficient*: $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.)

A is *isomorphic to* B
iff A and B are equivalent
in the following calculus:

$$A \wedge B \leftrightarrow B \wedge A, \qquad A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C.$$

$$A \vee B \leftrightarrow B \vee A, \qquad A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C.$$

$$\neg(A \vee B) \leftrightarrow \neg A \vee \neg B, \qquad \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B.$$

$$\neg\neg A \leftrightarrow A. \qquad A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B).$$

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This allows for a *negation normal form*, but not DNF or CNF.

Proof Sketch (Došen and Petrić, 2012)

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- If A and B are diversified, isomorphic, and in negation normal form, if $l \land m$ is a conjunction in A (l and m, literals), a substitution argument (substituting $\top$ and $\bot$ for the *other* atoms) shows that l and m must be conjunctively joined in B, too. The same goes for $l \lor m$.

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- Replace $l \wedge m$ by a new atom in both A and B, and repeat.
- This shows how to reconstruct a proof of equivalence for A and B in the syntactic calculus for isomorphic formulas.

This is a *very* tight constraint

- If A and B are isomorphic, they can play *essentially* the same role in proof.

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- Replacing A by B in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*.

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- Not even A and $A \wedge A$ are equivalent in *this* sense.
- Yet, A and $A \wedge A$ seem to have identical *subject matter* (insofar as we understand that notion).
- Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

MORE PROOFS FROM A TO A

$Id(A), Hz(A), Mx(A)$

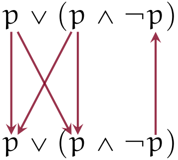
$Id(p \lor (p \land \neg p))$

$$\begin{tikzcd}
p \arrow[d] & \lor & (p \arrow[d] & \land & \neg p) \\
p & \lor & (p & \land & \neg p) \arrow[u]
\end{tikzcd}$$

In $Id(A)$, each occurrence of an atom in the premise
is linked with every its corresponding occurrence of in the conclusion.
Different occurrences of atoms in A are treated differently.

$Id(A), Hz(A), Mx(A)$

$Hz(p \vee (p \wedge \neg p))$



In $Id(A)$, each occurrence of an atom in the premise
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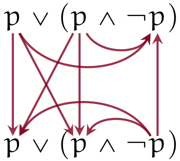
Different occurrences of atoms in A are treated differently.

In $Hz(A)$, each positive [negative] occurrence of an atom in the premise
is linked with every positive [negative] occurrence in the conclusion.
We treat occurrences of an atom in A—with the same polarity—equally.

$Id(A), Hz(A), Mx(A)$

$Mx(p \vee (p \wedge \neg p))$



In $Id(A)$, each occurrence of an atom in the premise
is linked with every its corresponding occurrence of in the conclusion.
Different occurrences of atoms in A are treated differently.

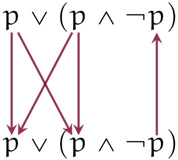
In $Hz(A)$, each positive [negative] occurrence of an atom in the premise
is linked with every positive [negative] occurrence in the conclusion.
We treat occurrences of an atom in A—with the same polarity—equally.

In $Mx(A)$, each syntactically possible linking is included.
We treat all occurrences of an atom in A equally.

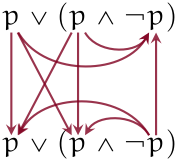
$Hz(A)$, $Mx(A)$, Caps and Cups

Note: $Hz(A)$ is $Mx(A)$ with the caps and cups removed.

$Hz(p \vee (p \wedge \neg p))$



$Mx(p \vee (p \wedge \neg p))$



Let's look at relations like isomorphism,
but which erase distinctions, up to *Hz* or *Mx*.

Let's say that A and B *Hz*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$.

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We write "$\approx_{Hz}$" for the *Hz*-matching relation, and we write "$\pi, \pi' : A \approx_{Hz} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Hz*-match between A and B.

Let's say that A and B *Mx*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$.

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We write "$\approx_{Mx}$" for the *Mx*-matching relation, and we write "$\pi, \pi' : A \approx_{Mx} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Mx*-match between A and B.

If $\pi : A \succ B$ and $\pi^{-1} : B \succ A$, then
consider $\pi' = Hz(B) \circ \pi \circ Hz(A)$
and $\tau' = Hz(A) \circ \pi^{-1} \circ Hz(B)$.

These satisfy the *Hz*-matching criteria,
$\tau' \circ \pi' = Hz(A)$ and $\pi' \circ \tau' = Hz(B)$.

Proof

$$
\begin{aligned}
Hz(A) &= Id(A) \circ Id(A) \circ Hz(A) \\
&\subseteq Hz(A) \circ Id(A) \circ Hz(A) \\
&= Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \\
&= Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \\
&\subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \\
&= (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \\
&= \tau' \circ \pi' \\
&\subseteq Hz(A)
\end{aligned}
$$

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Hz(A) &= Id(A) \circ Id(A) \circ Hz(A) \\
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&= Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \\
&\subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \\
&= (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \\
&= \tau' \circ \pi' \\
&\subseteq Hz(A)
\end{aligned}
$$

...and similarly, $Hz(B) \subseteq \pi' \circ \tau' \subseteq Hz(B)$

If $\pi, \pi' : A \approx_{Hz} B$, then
consider $\tau = Mx(B) \circ \pi \circ Mx(A)$
and $\tau' = Mx(A) \circ \pi' \circ Mx(B)$.

These satisfy the *Mx*-matching criteria,
$\tau' \circ \pi' = Mx(A)$ and $\pi' \circ \tau' = Mx(B)$.

Proof

$$
\begin{aligned}
Mx(A) &= Id(A) \circ Id(A) \circ Mx(A) \\
&\subseteq Mx(A) \circ Hz(A) \circ Mx(A) \\
&= Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \\
&= Mx(A) \circ (\pi' \circ Id(B) \circ Id(B) \circ \pi) \circ Mx(A) \\
&\subseteq Mx(A) \circ (\pi' \circ Mx(B) \circ Mx(B) \circ \pi) \circ Mx(A) \\
&= (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \\
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\end{aligned}$$

...and similarly, $Mx(B) \subseteq \pi' \circ \tau' \subseteq Mx(B)$

Mx-Matching $\subseteq$ Logical Equivalence

If $A \approx_{Mx} B$ then there are proofs $\pi : A \succ B$ and $\tau : B \succ A$.

Matching Relations are Equivalence Relations

REFLEXIVE $Hz(A), Hz(A) : A \approx_{\mathrm{Hz}} A.$

$Mx(A), Mx(A) : A \approx_{\mathrm{Mx}} A.$

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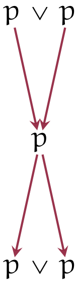
TRANSITIVE If $\pi, \pi' : A \approx_{\mathrm{Hz}} B$ and $\tau, \tau' : B \approx_{\mathrm{Hz}} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{\mathrm{Hz}} C.$

If $\pi, \pi' : A \approx_{\mathrm{Mx}} B$ and $\tau, \tau' : B \approx_{\mathrm{Mx}} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{\mathrm{Mx}} C.$

More Matchings

$$p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$$



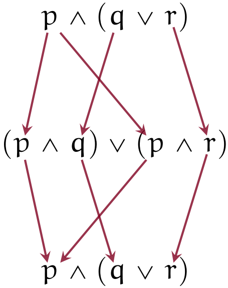
More Matchings

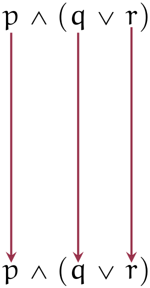
$$p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$$

$$\begin{tikzcd}
& p \arrow[dl] \arrow[dr] & \\
p \arrow[dr] & \vee & p \arrow[dl] \\
& p &
\end{tikzcd}
\qquad\qquad\qquad
\begin{tikzcd}
p \arrow[dd] \\
\\
p
\end{tikzcd}$$

More Matchings

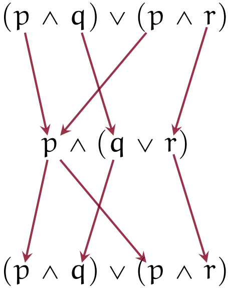
$$p \wedge (q \vee r) \approx_{\mathrm{Hz}} (p \wedge q) \vee (p \wedge r)$$

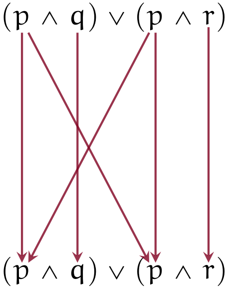




More Matchings

$$p \wedge (q \vee r) \approx_{Hz} (p \wedge q) \vee (p \wedge r)$$





FACT: If an atom p occurs positively in A but not in B, then A and B do not Mx-match.

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PROOF: $Mx(A) : A \succ A$ contains the link from that occurrence of p in the premise A to its corresponding occurrence in the conclusion A.

Mx-Matching $\subset$ Logical Equivalence

FACT: If an atom p occurs positively in A but not in B, then A and B do not *Mx*-match.

PROOF: $Mx(A) : A \succ A$ contains the link from that occurrence of p in the premise A to its corresponding occurrence in the conclusion A.

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$$p \wedge \neg p \not\approx_{Mx} q \wedge \neg q.$$

Hz-matching $\subset$ *Mx*-matching

$$(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$$

$$(p \wedge \neg p) \wedge (q \vee \neg q)$$

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$$\begin{array}{c}
(p \wedge \neg p) \wedge (q \vee \neg q) \\[2em]
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(p \wedge \neg p) \wedge (q \vee \neg q)
\end{array}
\qquad\qquad
\begin{array}{c}
(p \wedge \neg p) \wedge (q \vee \neg q) \\[5em]
(p \wedge \neg p) \wedge (q \vee \neg q)
\end{array}$$

$$\textit{Isomorphism} \subset \textit{Hz-Matching} \subset \textit{Mx-Matching} \subset \textit{Logical Equivalence}$$

So what *are* the *matching* relations?

MATCHING & LOGICS OF ANALYTIC CONTAINMENT

Angell's Logic of Analytic Containment

AC1 $A \leftrightarrow \neg\neg A$

AC2 $A \leftrightarrow (A \wedge A)$

AC3 $(A \wedge B) \leftrightarrow (B \wedge A)$

AC4 $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$

AC5 $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$

R1 $A \leftrightarrow B, C(A) \Rightarrow C(B)$

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Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$.

You can define $A \rightarrow B$ as $A \leftrightarrow (A \wedge B)$.

Famously, $A \rightarrow (A \vee B)$ is not derivable in Angell's logic.
We cannot prove $A \leftrightarrow (A \wedge (A \vee B))$.

Extensions of Angell's Logic

- The first degree fragment of *Parry's* Logic of Analytic Containment is found by adding $(A \lor (B \land \neg B)) \to A$ to Angell's Logic.
  - Parry's logic still satisfies this relevance constraint: $A \to B$ is provable only when the atoms in B are present in A.
- *First Degree Entailment* (FDE) is found by adding $A \to (A \lor B)$ to Angell's Logic.
  - FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that $p \lor \neg p$, and $q \land \neg q$ are both non-trivial, and ineliminable.
  - A simple translation encodes FDE inside classical logic. Choose, for each atom p, a fresh atom p', its *shadow*. For each FDE formula A, its translation is the formula A' found by replacing the negative occurrences of atoms p in A by their shadows. An argument is FDE valid iff its translation is classically valid.

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FACT: $Mx(A, B)$ is a proof iff there is some proof from A to B.

$Mx(A, B)$ examples

$Mx(p, q)$

$$\begin{matrix} p \\ \\ q \end{matrix}$$

$Mx(A, B)$ examples

$Mx(p, q)$

$$\begin{array}{c} p \\ \\ q \end{array}$$

No links.

$Mx(p, q)$

$$p$$

$$q$$

No links.

$Mx(p \lor \neg p, p \land \neg q)$

$$p \lor \neg p$$

$$p \land \neg q$$

$Mx(A, B)$ examples

$Mx(p, q)$

$$p$$

$$q$$

No links.

$Mx(p \vee \neg p, p \wedge \neg q)$

$$p \vee \neg p$$

$$p \wedge \neg q$$

Not a proof.

$Mx(A, B)$ and matching

LEMMA: If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$

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PROOF: If $\pi, \pi' : A \approx_{Mx} B$, then $\pi \subseteq Mx(A, B)$ and $\pi' \subseteq Mx(B, A)$, so $Mx(A, B)$ and $Mx(B, A)$ are both proofs.

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Since $\pi' \circ \pi = Mx(A)$, we have
$Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$,
and similarly, $Mx(B) = Mx(A, B) \circ Mx(A)$,
so $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

FACT: If A is classically logically equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Mx*-match — and conversely.

Proof

If A is logically equivalent to B, then $Mx(A, B)$ and $Mx(B, A)$ are both proofs.

It suffices to show that $Mx(B, A) \circ Mx(A, B) = Mx(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Mx(A, B)$ composed with a link in $Mx(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Mx(A, B)$ and $Mx(B, A)$.

Conversely, if $A \approx_{Mx} B$, we have already seen that A and B must be equivalent, and no atom occurs positively [negatively] in A but not B.

This is *not* Equivalence in Parry's Logic

FACT: A is equivalent to B in Parry's logic of analytic containment iff A is classically equivalent to B and the atoms present in A are the atoms present in B.

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$$(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$$

But this pair satisfies Parry's variable sharing criteria.

Open Question

Does the equivalence relation of Mx-matching occur elsewhere in the literature?

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That is, it contains a link from positive atoms in A to corresponding positive atoms in B and negative atoms in A to corresponding negative atoms in B.

$Hz(A, B)$ examples

$Hz(p \wedge \neg p, q \vee \neg q)$

$Hz(A, B)$ examples

$$p \wedge \neg p$$

$Hz(p \wedge \neg p, q \vee \neg q)$

$$q \vee \neg q$$

$Hz(A, B)$ examples

$Hz(p \wedge \neg p, q \vee \neg q)$

$$p \wedge \neg p$$

$$q \vee \neg q$$

No links!

$Hz(A, B)$ examples

$Hz(p \land \neg p, q \lor \neg q)$

$$p \land \neg p$$

No links!

$$q \lor \neg q$$

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$Hz(A, B)$ examples

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No links!

$Hz(p \land \neg p, p \lor \neg p)$

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$Hz(A, B)$ examples

$Hz(p \land \neg p, q \lor \neg q)$

$$\begin{array}{c} p \land \neg p \\ \\ q \lor \neg q \end{array}$$

No links!

$Hz(p \land \neg p, p \lor \neg p)$

$$\begin{tikzcd} p \arrow[d] & \land & \neg p \\ p & \lor & \neg p \arrow[u] \end{tikzcd}$$

A proof, but not the maximal one.

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- From FDE-validity to *Hz*-proof: straightforward induction on an FDE-axiomatisation.
- From the *Hz*-proof $Hz(A, B)$ to FDE-validity. Notice that no negative occurrences of atoms in A or B are linked to any positive occurrences of atoms in A or B. So, there is another *Hz*-proof $Hz(A', B')$ for the FDE translations for A and B.

LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$

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PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, π and π' are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$, so $Hz(A, B)$ and $Hz(B, A)$ are both proofs.

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$Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$,
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FACT: If A is FDE-equivalent to A,
and all atoms occurring positively [negatively] in A
also occur positively [negatively] in B,
and *vice versa*, then A and B *Hz*-match
— and conversely.

Proof

If A is FDE-equivalent to B, then $Hz(A, B)$ and $Hz(B, A)$ are both proofs.

It suffices to show that $Hz(B, A) \circ Hz(A, B) = Hz(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Hz(A, B)$ composed with a link in $Hz(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Hz(A, B)$ and $Hz(B, A)$.

Conversely, if $A \approx_{Hz} B$, we have already seen that A and B must be FDE-equivalent, and no atom occurs positively [negatively] in A but not B.

Hz-matching ≡ Angellic Equivalence

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So, *Hz*-matching *is* equivalence in Angell's Logic of Analytic Containment.

MATCHING AS ISOMORPHISM

$Hz(A)$ and $Mx(A)$ are Idempotents

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- For any category $\mathcal{C}$, if i_A is an idempotent for each object A, we can form a new category $\mathcal{C}_i$ with the same objects as $\mathcal{C}$, and with arrows $i_B \circ f \circ i_A : A \to B$.

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- In this new category, the idempotents i_A are the new identity arrows.

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- In this new category, the idempotents i_A are the new identity arrows.
- So, $\mathfrak{C}_{Hz}$ and $\mathfrak{C}_{Mx}$ are both categories — like $\mathfrak{C}$, but less discriminating, with fewer arrows.

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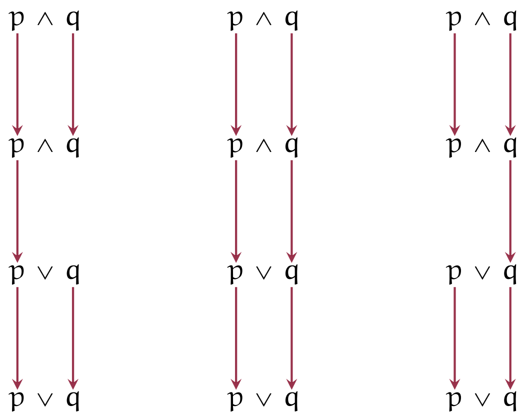
$\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$ are nontrivial, nonetheless



These are each different proofs in $\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$.

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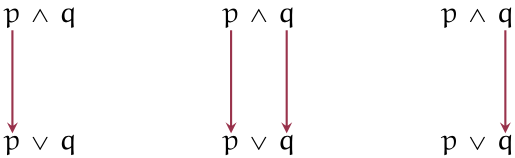

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- Proof theoretical resources *for classical logic* provide tools for fine-grained hyperintensional distinctions.
- Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation).
- Relate these results to *models* of logics of content.
- Extend these results to first order logic, and beyond!

THANK YOU!

Thank you!

SLIDES: `http://consequently.org/presentation/`

FEEDBACK: `@consequently` on *Twitter*,

or *email* at `restall@unimelb.edu.au`