

# Generality & Existence III

## Predication & Identity

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To analyse the *quantifiers*

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(including their interactions with *modals*)

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(including their interactions with *modals*)  
using the tools of *proof theory*  
in order to better understand  
*quantification, existence and identity*.

Understanding the behaviour  
of the identity predicate.

## Today's Plan

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Sequents & Defining Rules

Identity & Indistinguishability

Defining Rules & Left/Right Rules

Identity & Predication

Free Logic & Identity

# SEQUENTS & DEFINING RULES



## Sequents

$$\Gamma \supset \Delta$$

Don't assert each element of  $\Gamma$   
and deny each element of  $\Delta$ .

*Identity:*  $A \succ A$

## Structural Rules

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*Weakening:*  $\frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta} \quad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}$

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*Cut:*  $\frac{\Gamma \succ A, \Delta \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}$

Structural rules govern declarative sentences *as such*.

With Left/Right rules?

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} [\wedge L]$$

$$\frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta} [\wedge R]$$

With Left/Right rules?

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} [\wedge L]$$

$$\frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta} [\wedge R]$$

$$\frac{\Gamma, B \succ \Delta}{\Gamma, A \text{ tonk } B \succ \Delta} [\text{tonk}L]$$

$$\frac{\Gamma \succ A, \Delta}{\Gamma \succ A \text{ tonk } B, \Delta} [\text{tonk}R]$$



What is involved in going from  $\mathcal{L}$  to  $\mathcal{L}'$ ?

Use  $\succ_{\mathcal{L}}$  to *define*  $\succ_{\mathcal{L}'}$ .

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*Desideratum #1:*  $\succ_{\mathcal{L}'}$  is conservative:  $(\succ_{\mathcal{L}'})|_{\mathcal{L}}$  is  $\succ_{\mathcal{L}}$ .

*Desideratum #2:* Concepts are defined *uniquely*.

## A Defining Rule

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} [\wedge^{Df}]$$

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Fully specifies norms governing conjunctions on the *left* in terms of simpler vocabulary.

*Identity* and *Cut* determine the behaviour of conjunctions on the *right*.

# From $[\wedge Df]$ to $[\wedge L/R]$

$$\begin{array}{c}
 \frac{\frac{\frac{\Gamma \succ B, \Delta}{\Gamma \succ A, \Delta} \quad \frac{\frac{\frac{\overline{A \wedge B \succ A \wedge B}}{A, B \succ A \wedge B} [Id]}{\Gamma, A \succ A \wedge B, \Delta} [\wedge Df]}{\Gamma, A \succ A \wedge B, \Delta} [Cut]}{\Gamma \succ A \wedge B, \Delta} [Cut]
 \end{array}$$

# From $[\wedge Df]$ to $[\wedge L/R]$

$$\begin{array}{c}
 \frac{\frac{\frac{}{A \wedge B \succ A \wedge B} [Id]}{\Gamma \succ B, \Delta \quad A, B \succ A \wedge B} [\wedge Df]}{\Gamma \succ A, \Delta \quad \Gamma, A \succ A \wedge B, \Delta} [Cut] \\
 \hline
 \Gamma \succ A \wedge B, \Delta \quad \Gamma \succ A, \Delta \quad \Gamma, A \succ A \wedge B, \Delta \quad \Gamma \succ B, \Delta \quad A, B \succ A \wedge B \quad \frac{}{A \wedge B \succ A \wedge B} [Id]
 \end{array}$$



# From $[\wedge Df]$ to $[\wedge L/R]$

$$\begin{array}{c}
 \frac{\frac{\frac{\Gamma \succ B, \Delta}{\Gamma, A \succ A \wedge B, \Delta} [\text{Cut}]}{\Gamma \succ A, \Delta} [\text{Cut}]}{\Gamma \succ A \wedge B, \Delta} [\text{Cut}]
 \end{array}$$

$\frac{\frac{\frac{\frac{\frac{\frac{\Gamma \succ B, \Delta}{\Gamma, A \succ A \wedge B, \Delta} [\text{Cut}]}{\Gamma \succ A, \Delta} [\text{Cut}]}{\Gamma \succ A \wedge B, \Delta} [\text{Cut}]}$

# From $[\wedge Df]$ to $[\wedge L/R]$

$$\begin{array}{c}
 \frac{\frac{\frac{\Gamma \succ B, \Delta}{\Gamma, A \succ A \wedge B, \Delta} [\text{Cut}]}{\Gamma \succ A, \Delta} [\text{Cut}]}{\Gamma \succ A \wedge B, \Delta} [\text{Cut}]
 \end{array}$$

$\frac{\frac{\frac{\frac{\frac{\frac{\Gamma \succ B, \Delta}{\Gamma, A \succ A \wedge B, \Delta} [\text{Cut}]}{\Gamma \succ A, \Delta} [\text{Cut}]}{\Gamma \succ A \wedge B, \Delta} [\text{Cut}]}{\frac{\frac{\frac{\Gamma \succ B, \Delta}{\Gamma, A \succ A \wedge B, \Delta} [\text{Cut}]}{\Gamma, A \succ A \wedge B, \Delta} [\text{Cut}]} [\wedge Df]}{\frac{\Gamma \succ B, \Delta}{\Gamma, A \succ A \wedge B, \Delta} [\wedge Df]} [\text{Cut}]$

# From $[\wedge Df]$ to $[\wedge L/R]$

$$\begin{array}{c}
 \frac{\frac{\frac{}{A \wedge B \succ A \wedge B} [Id]}{A, B \succ A \wedge B} [\wedge Df]}{\Gamma \succ B, \Delta \quad A, B \succ A \wedge B} [Cut] \\
 \frac{\Gamma \succ A, \Delta \quad \Gamma, A \succ A \wedge B, \Delta}{\Gamma \succ A \wedge B, \Delta} [Cut]
 \end{array}$$

$$\frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta} [\wedge R]$$

## And Back

$$\frac{\frac{A \succ A \quad B \succ B}{A, B \succ A \wedge B} [\wedge R] \quad \Gamma, A \wedge B \succ \Delta}{\Gamma, A, B \succ \Delta} [Cut]$$

# Quantifier Rules

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall Df]$$

$$\frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists Df]$$

# Deductive Generality

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# Equivalence

$$\mathcal{L}[\wedge Df, Cut] = \mathcal{L}[\wedge L/R, Cut]$$

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This *generalises*:  $\wedge, \vee, \supset, \neg$  work in the same way.

# Equivalence

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This *generalises*:  $\wedge, \vee, \supset, \neg$  work in the same way.

I want to see how this works for *identity*.

# IDENTITY & INDIS- TINGUISHABILITY

# Identity and Harmony

## *Identity and harmony*

STEPHEN READ

### *1. Harmony*

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960–61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if ‘tonk’ has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its elimination-rules:

The introductions represent, as it were, the ‘definitions’ of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of ‘ $p$  tonk  $q$ ’ is given by Prior’s rule:

$$\frac{p}{p \text{ tonk } q} \text{ tonk-I}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{(p) \quad p \text{ tonk } q \quad r}{r} \text{ tonk-E}$$

that is, given ‘ $p$  tonk  $q$ ’, and a derivation of  $r$  from  $p$  (the ground for asserting ‘ $p$  tonk  $q$ ’), we can infer  $r$ , discharging the assumption  $p$ . We can state the rule more simply as follows:

$$q \text{ tonk } q$$

# Identity Axioms

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$$t = t$$

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$$s = t \supset t = s \qquad s = t \supset (t = u \supset s = u)$$

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$$s = t \supset t = s \quad s = t \supset (t = u \supset s = u)$$

$$s = t \supset (A(s) \equiv A(t))$$

# Identity Rules in Natural Deduction

$$\frac{\begin{array}{c} [Fs] \\ \vdots \\ Ft \end{array}}{s = t} [=I]$$



# Identity Rules in Natural Deduction

$$\frac{\begin{array}{c} [Fs] \\ \vdots \\ Ft \end{array}}{s = t} [=I]$$

$$\frac{s = t \quad A(s)}{A(t)} [=E]$$

## Defining Rule for Identity

$$\frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s = t, \Delta} [=Df]$$

# Generality in Predicate Position

$$\frac{\Gamma \succ \Delta}{\Gamma|_{A(x)}^{Fx} \succ \Delta|_{A(x)}^{Fx}} [Spec_{A(x)}^{Fx}]$$

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$$\frac{\Gamma \succ \Delta}{\Gamma|_{A(x)}^{Fx} \succ \Delta|_{A(x)}^{Fx}} [Spec_{A(x)}^{Fx}]$$

No norm holds of  $Fx$  that doesn't also hold of the sentence context  $A(x)$ .

# DEFINING RULES & LEFT/RIGHT RULES

# From [=Df] to [=L]

$$\begin{array}{c}
 \frac{s = t \succ s = t}{s = t, Fs \succ Ft} [=Df] \\
 \frac{\Gamma \succ A(s), \Delta \quad \frac{s = t, A(s) \succ A(t)}{[Spec_{A(x)}^{Fx}]} }{s = t, \Gamma \succ A(t), \Delta} [Cut] \\
 \frac{s = t, \Gamma \succ A(t), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [Cut]
 \end{array}$$

# From [=Df] to [=L]

$$\begin{array}{c}
 \frac{s = t \succ s = t}{s = t, Fs \succ Ft} [=Df] \\
 \frac{\Gamma \succ A(s), \Delta \quad \frac{s = t, A(s) \succ A(t)}{[Spec_{A(x)}^{Fx}]} }{s = t, \Gamma \succ A(t), \Delta} [Cut] \\
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 \end{array}$$

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$$\begin{array}{c}
 \dfrac{s = t \succ s = t}{s = t, Fs \succ Ft} [=Df] \\
 \dfrac{\Gamma \succ A(s), \Delta \quad \dfrac{s = t, Fs \succ Ft}{s = t, A(s) \succ A(t)} [Spec_{A(x)}^{Fx}]}{s = t, \Gamma \succ A(t), \Delta} [Cut] \\
 \dfrac{s = t, \Gamma \succ A(t), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [Cut]
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 \frac{s = t, \Gamma \succ A(t), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [Cut]
 \end{array}$$

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$[=L]$

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$

[=L]

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This is *valid*, but *ugly*.

$[=L]$

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$

This is *valid*, but *ugly*.

Proof search?

$[=L]$

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$

This is *valid*, but *ugly*.

Proof search?

Subformula property?



# Backtracking a little

$$\begin{array}{c}
 \frac{s = t \succ s = t}{s = t, Fs \succ Ft} [=Df] \\
 \frac{\Gamma \succ A(s), \Delta \quad \frac{s = t, A(s) \succ A(t)}{s = t, A(s) \succ A(t)} [=Spec_{A(x)}^{Fx}]}{s = t, \Gamma \succ A(t), \Delta} [Cut] \\
 \frac{s = t, \Gamma \succ A(t), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [Cut]
 \end{array}$$

# Backtracking a little

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 \frac{s = t, \Gamma \succ A(t), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [Cut]
 \end{array}$$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L']$$

# $[=L']$ is Enough to recover $[=Df]$

$$\begin{array}{c}
 \frac{\Gamma \succ s = t, \Delta}{\Gamma, Fs \succ s = t, Ft, \Delta} [K] \qquad \frac{\frac{\frac{}{Ft \succ Ft} [Id]}{s = t, Fs \succ Ft} [=L']}{\Gamma, s = t, Fs \succ Ft, \Delta} [K] \\
 \hline
 \Gamma, Fs \succ Ft, \Delta \quad [Cut]
 \end{array}$$

$[=L']$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L']$$

$[=L']$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L']$$

This is *better*...

$$[=L']$$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L']$$

This is *better*...

But it is still *strange*.

$$[=L']$$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L']$$

This is *better*...

But it is still *strange*.

It operates at *two places* in the concluding sequent.

This puts *compositionality* in question.

# IDENTITY & PREDICATION



## Decomposing [=L']: conjunctions

$$\frac{\frac{\frac{\Gamma \succ A(s) \wedge B(s), \Delta}{\Gamma \succ A(s), \Delta} [\wedge E]}{s = t, \Gamma \succ A(t), \Delta} [=L']}{s = t, \Gamma \succ A(t) \wedge B(t), \Delta} [\wedge R]$$
$$\frac{\frac{\frac{\Gamma \succ A(s) \wedge B(s), \Delta}{\Gamma \succ B(s), \Delta} [\wedge E]}{s = t, \Gamma \succ B(t), \Delta} [=L']}{s = t, \Gamma \succ A(t) \wedge B(t), \Delta} [\wedge R]$$

(Where the  $[\wedge E]$  is given by a *Cut* on  $A(t) \wedge B(t) \succ A(t)$ , or  $A(t) \wedge B(t) \succ B(t)$ .)

## Decomposing [=L']: conjunctions

$$\frac{\frac{\frac{\Gamma \succ A(s) \wedge B(s), \Delta}{\Gamma \succ A(s), \Delta} [\wedge E]}{s = t, \Gamma \succ A(t), \Delta} [=L']}{s = t, \Gamma \succ A(t) \wedge B(t), \Delta} [\wedge R]$$
$$\frac{\frac{\frac{\Gamma \succ A(s) \wedge B(s), \Delta}{\Gamma \succ B(s), \Delta} [\wedge E]}{s = t, \Gamma \succ B(t), \Delta} [=L']}{s = t, \Gamma \succ A(t) \wedge B(t), \Delta} [\wedge R]$$

(Where the  $[\wedge E]$  is given by a *Cut* on  $A(t) \wedge B(t) \succ A(t)$ , or  $A(t) \wedge B(t) \succ B(t)$ .)

$[=L']$  on *conjunctions* is given by  $[=L']$  on its *conjuncts*.

## Decomposing [=L']: disjunctions

$$\frac{\frac{\frac{\Gamma \succ A(s) \vee B(s), \Delta}{\Gamma \succ A(s), B(s), \Delta} [\vee Df]}{s = t, \Gamma \succ A(t), B(s), \Delta} [=L']}{s = t, s = t, \Gamma \succ A(t), B(t), \Delta} [=L'] \frac{}{s = t, \Gamma \succ A(t) \vee B(t), \Delta} [W] \frac{}{s = t, \Gamma \succ A(t) \vee B(t), \Delta} [\vee Df]$$

## Decomposing [=L']: disjunctions

$$\frac{\frac{\frac{\Gamma \succ A(s) \vee B(s), \Delta}{\Gamma \succ A(s), B(s), \Delta} [\vee Df]}{s = t, \Gamma \succ A(t), B(s), \Delta} [=L']}{s = t, s = t, \Gamma \succ A(t), B(t), \Delta} [=L'] \quad \frac{}{s = t, \Gamma \succ A(t) \vee B(t), \Delta} [W] \quad \frac{}{s = t, \Gamma \succ A(t) \vee B(t), \Delta} [\vee Df]$$

[=L'] on *conjunctions* is given by [=L'] on its *conjuncts*.

## Decomposing [=L']: universal quantifiers

$$\frac{\frac{\frac{\Gamma \succ (\forall x)A(x, s), \Delta}{\Gamma \succ A(n, s), \Delta} [\forall Df]}{s = t, \Gamma \succ A(n, t), \Delta} [=L']}{s = t, \Gamma \succ (\forall x)A(x, t), \Delta} [\forall Df]$$

[=L'] on a *universally quantified statement* is given by [=L'] on an *instance*.

## Decomposing [=L']: existential quantifiers

$$\begin{array}{c}
 \frac{}{A(n, s) \succ A(n, s)} [Id] \\
 \frac{}{s = t, A(n, s) \succ A(n, t)} [=L'] \\
 \frac{}{s = t, A(n, s) \succ (\exists x)A(x, t)} [\exists R] \\
 \frac{}{s = t, (\exists x)A(x, s) \succ (\exists x)A(x, t)} [\exists Df] \quad \Gamma \succ (\exists x)A(x, s), \Delta \\
 \hline
 s = t, \Gamma \succ (\exists x)A(x, t), \Delta \quad [Cut]
 \end{array}$$

[=L'] on an *existentially quantified statement* is given by [=L'] on an *instance*.

## But for *negation*...

$$\frac{\Gamma \succ \neg A(s), \Delta}{\Gamma, A(s) \succ \Delta} [\neg Df]$$
$$\frac{\Gamma, A(s) \succ \Delta}{s = t, A(t), \Gamma \succ \Delta} [=L' \text{ on the wrong side!}]$$
$$\frac{s = t, A(t), \Gamma \succ \Delta}{s = t, \Gamma \succ \neg A(t), \Delta} [\neg Df]$$

## Different Identity Rules

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$



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$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L_r^f]$$

## Different Identity Rules

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L_r^f] \quad \frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=R]$$

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$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L_r^f] \quad \frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=R]$$

$$\frac{\Gamma \succ Fs, \Delta}{s = t, \Gamma \succ Ft, \Delta} [=L_r^p]$$

## Different Identity Rules

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$$\frac{\Gamma \succ Fs, \Delta}{s = t, \Gamma \succ Ft, \Delta} [=L_r^p] \quad \frac{Fs, \Gamma \succ \Delta}{s = t, Ft, \Gamma \succ \Delta} [=L_l^p]$$

## Different Identity Rules

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L_r^f]$$

$$\frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=R]$$

$$\frac{\Gamma \succ Fs, \Delta}{s = t, \Gamma \succ Ft, \Delta} [=L_r^p]$$

$$\frac{Fs, \Gamma \succ \Delta}{s = t, Ft, \Gamma \succ \Delta} [=L_l^p]$$

$$\frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=Df]$$

## Different Identity Rules

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L_r^f] \quad \frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=R]$$

$$\frac{\Gamma \succ Fs, \Delta}{s = t, \Gamma \succ Ft, \Delta} [=L_r^p] \quad \frac{Fs, \Gamma \succ \Delta}{s = t, Ft, \Gamma \succ \Delta} [=L_l^p]$$

$$\frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=Df] \quad \frac{\Gamma \succ \Delta}{\Gamma|_{A(x)}^{Fx} \succ \Delta|_{A(x)}^{Fx}} [Spec_{A(x)}^{Fx}]$$

# Equivalences

$$\frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=Df] \quad \frac{\Gamma \succ \Delta}{\Gamma|_{A(x)}^{Fx} \succ \Delta|_{A(x)}^{Fx}} [Spec_{A(x)}^{Fx}]$$

$$\mathcal{L}[=Df, Spec, Cut]$$

# Equivalences

$$\frac{\Gamma \succ A(s), \Delta \quad \Gamma, A(t) \succ \Delta}{s = t, \Gamma \succ \Delta} [=L]$$

$$\mathcal{L}[=Df, Spec, Cut] = \mathcal{L}[=L/R, Cut]$$



# Equivalences

$$\frac{\Gamma \succ A(s), \Delta}{s = t, \Gamma \succ A(t), \Delta} [=L_r^f]$$

$$\begin{aligned}\mathcal{L}[=Df, Spec, Cut] &= \mathcal{L}[=L/R, Cut] \\ &= \mathcal{L}[=L_r^f/R, Cut]\end{aligned}$$

# Equivalences

$$\frac{\Gamma \succ Fs, \Delta}{s = t, \Gamma \succ Ft, \Delta} [=L_r^p] \quad \frac{Fs, \Gamma \succ \Delta}{s = t, Ft, \Gamma \succ \Delta} [=L_l^p] \quad \frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=R]$$

$$\begin{aligned} \mathcal{L}[=Df, Spec, Cut] &= \mathcal{L}[=L/R, Cut] \\ &= \mathcal{L}[=L_r^f/R, Cut] \\ &= \mathcal{L}[=L_r^p/L_l^p/R, Cut] \end{aligned}$$

# Equivalences

$$\frac{\Gamma \succ Fs, \Delta}{s = t, \Gamma \succ Ft, \Delta} [=L_r^p] \quad \frac{Fs, \Gamma \succ \Delta}{s = t, Ft, \Gamma \succ \Delta} [=L_l^p] \quad \frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=R]$$

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# Equivalences

$$\frac{\Gamma \succ Fs, \Delta}{s = t, \Gamma \succ Ft, \Delta} [=L_r^p] \quad \frac{Fs, \Gamma \succ \Delta}{s = t, Ft, \Gamma \succ \Delta} [=L_l^p] \quad \frac{\Gamma, Fa \succ Fb, \Delta}{\Gamma \succ a = b, \Delta} [=R]$$

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Each system gives you classical first-order predicate logic with identity.

## Non-Symmetric 'Identity'

$$\frac{\Gamma \succ Fs, \Delta}{s \text{ is } t, \Gamma \succ Ft, \Delta} [isL^p_r] \qquad \frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s \text{ is } t, \Delta} [isR]$$

## Non-Symmetric 'Identity'

$$\frac{\Gamma \succ Fs, \Delta}{s \text{ is } t, \Gamma \succ Ft, \Delta} [isL^p] \qquad \frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s \text{ is } t, \Delta} [isR]$$

There are models of this system in which  $s \text{ is } t \not\vdash t \text{ is } s$ .

## Non-Symmetric 'Identity'

$$\frac{\Gamma \succ Fs, \Delta}{s \text{ is } t, \Gamma \succ Ft, \Delta} [isL_r^p] \qquad \frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s \text{ is } t, \Delta} [isR]$$

There are models of this system in which  $s \text{ is } t \not\vdash t \text{ is } s$ .

DOMAIN: *Animal* < *Mammal* < *Human*.

ATOMIC PREDICATES: closed upward under <.

*Spec*: holds for atomic predicates, closed under  $\wedge, \vee, \forall, \exists$  but not  $\neg$  or  $\supset$ .

# FREE LOGIC & IDENTITY



# Free Quantification

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists Df]$$

# Free Quantification

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists Df]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t \downarrow \succ \Delta} [\downarrow Df]$$

# Free Quantification

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists Df]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t \downarrow \succ \Delta} [\downarrow Df]$$

$$(\forall x)Fx \not\succ Ft$$

# Free Quantification

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists Df]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t \downarrow \succ \Delta} [\downarrow Df]$$

$$(\forall x)Fx \not\vdash Ft \quad A(t) \not\vdash (\exists x)A(x)$$

# Free Quantification

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists Df]$$

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$$(\forall x)Fx \not\succ Ft \quad A(t) \not\succ (\exists x)A(x) \quad (\forall x)Fx, t \succ Ft$$

# Free Quantification

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} [\exists Df]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t \downarrow \succ \Delta} [\downarrow Df]$$

$$(\forall x)Fx \not\succ Ft \quad A(t) \not\succ (\exists x)A(x) \quad (\forall x)Fx, t \succ Ft \quad A(t), t \downarrow \succ (\exists x)A(t)$$

# Is Predication Existentially Committing?

$$\frac{t_i, \Gamma \succ \Delta}{\exists t_1 \cdots t_n, \Gamma \succ \Delta} [FL]$$

# Which Identity Rule?

$$\text{NON-COMMITAL: } \frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s =_n t, \Delta} [=_n Df]$$



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$$\text{COMMITTAL: } \frac{\Gamma \succ s, \Delta \quad \Gamma \succ t, \Delta \quad \Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s =_c t, \Delta} [=_c Df]$$

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$$\text{NON-COMMITAL: } \frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s =_n t, \Delta} [=_n Df]$$

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$$\succ t =_n t \quad \not\succ t =_c t$$

## Which Identity Rule?

$$\text{NON-COMMITAL: } \frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s =_n t, \Delta} [=_n Df]$$

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$$\succ t =_n t$$

$$\not\succ t =_c t$$

$$s =_c t \succ t \downarrow$$

$$s =_c t \succ s \downarrow$$

# Non-committal identity clashes with committing predication

$$\frac{\frac{\frac{s \succ s, Ft}{Fs \succ s, Ft} [FL]}{Fs \succ s \downarrow, Ft} [\downarrow Df]}{\neg s \downarrow, Fs \succ Ft} [\neg Df] \\ \frac{\neg s \downarrow, Fs \succ Ft}{\neg s \downarrow \succ s =_n t} [=_n Df]$$

# Non-committal identity clashes with committing predication

$$\frac{\frac{\frac{s \succ s, Ft}{Fs \succ s, Ft} [FL]}{Fs \succ s \downarrow, Ft} [\downarrow Df]}{\neg s \downarrow, Fs \succ Ft} [\neg Df] \\ \frac{\neg s \downarrow, Fs \succ Ft}{\neg s \downarrow \succ s =_n t} [=_n Df]$$

MORAL: For non-committal identity, allow  $F$  to be *negative* as well as *positive* (e.g., *nonexistence*) so  $FL$  might fail for this predicate.

# The Generality of Predication Matters

$$\frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s = t, \Delta} [=Df]$$

What can be substituted for the F here makes a *real* difference.

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$$\frac{\Gamma, Fs \succ Ft, \Delta}{\Gamma \succ s = t, \Delta} [=Df]$$

What can be substituted for the F here makes a *real* difference.

I'll consider this more on this in the next talk,  
when I consider the interaction with modality.



# THANK YOU!

[http://consequently.org/presentation/2015/  
generality-and-existence-3](http://consequently.org/presentation/2015/general-ity-and-existence-3)

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