Fixed Point Models for Theories of Properties and Classes

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THE UNIVERSITY OF MELBOURNE

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Our Target

Model Construction

Classifying Class Theories

Order and Continuity

Order Models

OUR TARGET

$$a \in \{x : \phi(x)\} \text{ iff } \phi(a)$$

$$a \mathrel{\varepsilon} \lambda x.\phi(x) \text{ iff } \phi(a)$$

Russell's Paradox

$$\{x : x \notin x\} \in \{x : x \notin x\} \text{ iff } \{x : x \notin x\} \notin \{x : x \notin x\}$$

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In general,

$$\{x : F(x \in x)\} \in \{x : F(x \in x)\} \text{ iff}$$
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$$\lambda x.(x \not\varepsilon x) \;\varepsilon\; \lambda x.(x \not\varepsilon x) \text{ iff } \lambda x.(x \not\varepsilon x) \not\varepsilon \lambda x.(x \not\varepsilon x)$$

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Extensionality

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(Extensionality will not play a significant role in what follows.)

MODEL CONSTRUCTION

Defining validity.

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Providing *counterexamples*,
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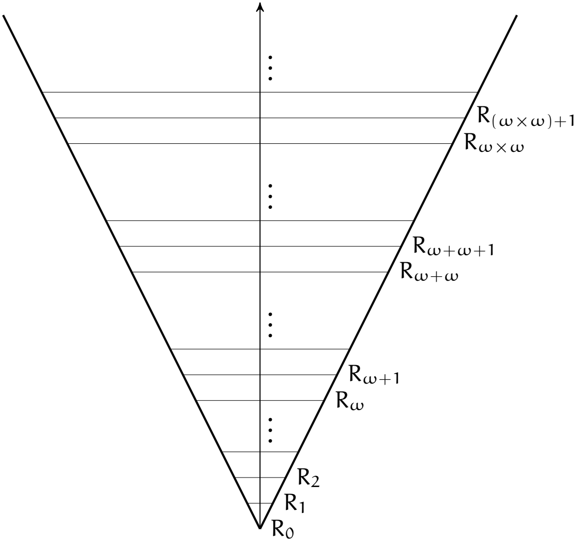
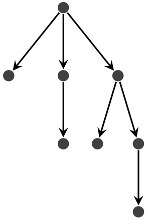
ZFC and its Cousins: The Iterative Conception of Set



ZFC and its Cousins: Anti-Foundation



$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

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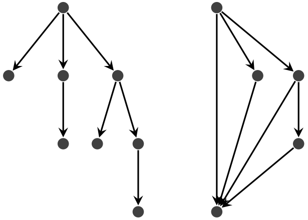

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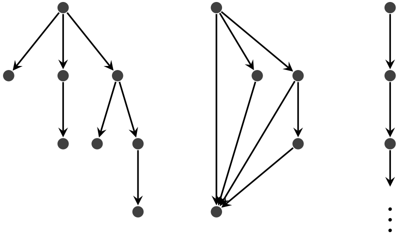

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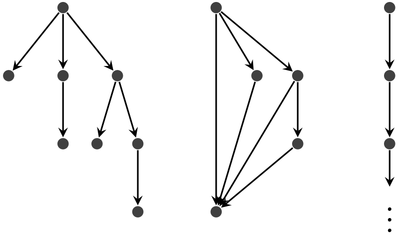

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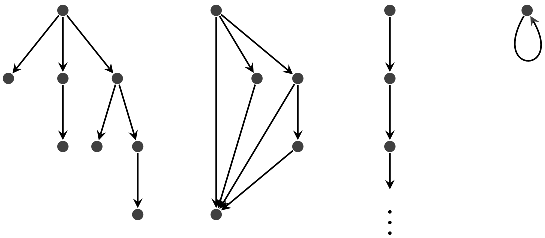

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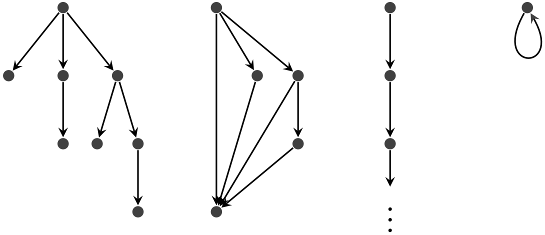

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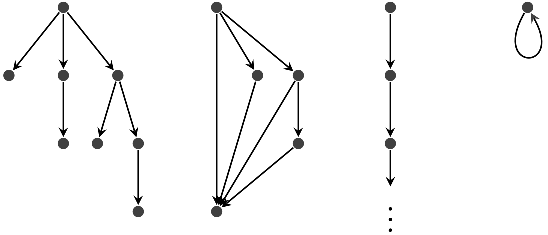

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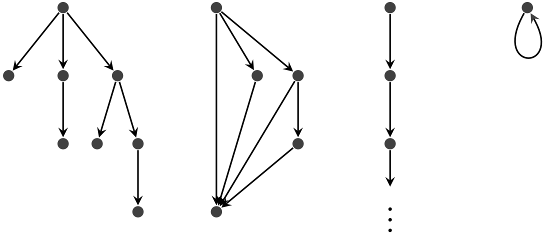

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These models are good for (1) *relating* ZFC to AFA,
(2) motivating a choice of the anti-foundation axiom, and
(3) explaining what the theory could be *about*.

If x is a *variable* and M is a *term*, $\lambda x.M$ is a *term*.

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$$(\lambda x.M)N = M[x := N].$$

Models of the Untyped λ Calculus

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$$\mathbf{D} \qquad\qquad \mathbf{D} \to \mathbf{D}$$

$$D \cong D \to D$$

You bump up against *Cantor's Theorem*.

$$D \cong [D \to D]$$

$[D \to E]$: the *order preserving functions* from $(D, \sqsubseteq)$ to $(E, \sqsubseteq)$.

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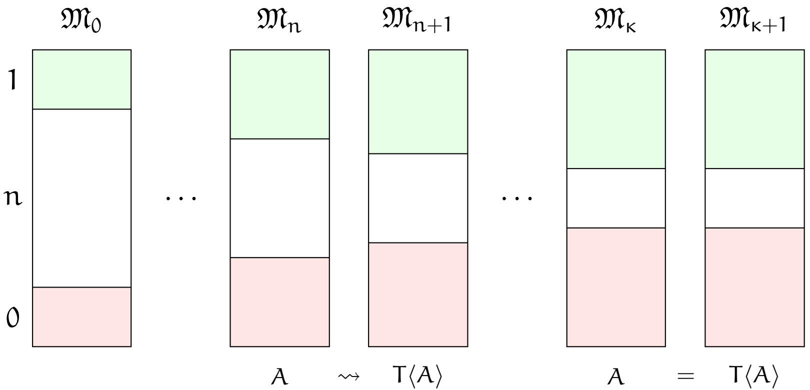
Embed D_i into $[D_i \to D_i] = D_{i+1}$
(Use the constant functions.)

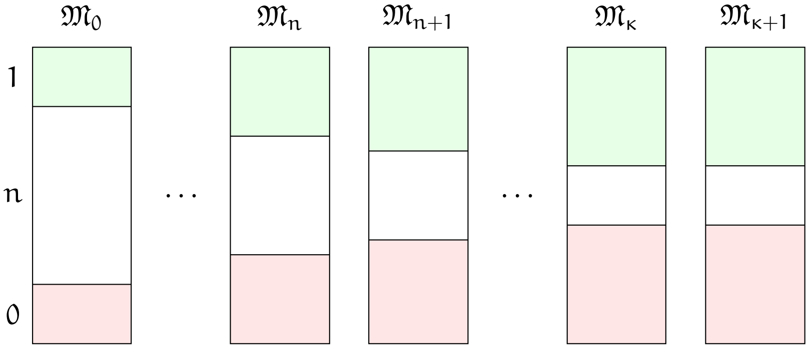
Let D_∞ be the limit: $D_\infty \widehat{\cong} [D_\infty \to D_\infty]$.

This is a model of the untyped λ calculus.

Truth Theories: Kripke, Woodruff, Gilmore, Brady



Class Theories



$\phi(a) \rightsquigarrow a \in \{x : \phi(x)\}$ $\qquad$ $\phi(a) = a \in \{x : \phi(x)\}$

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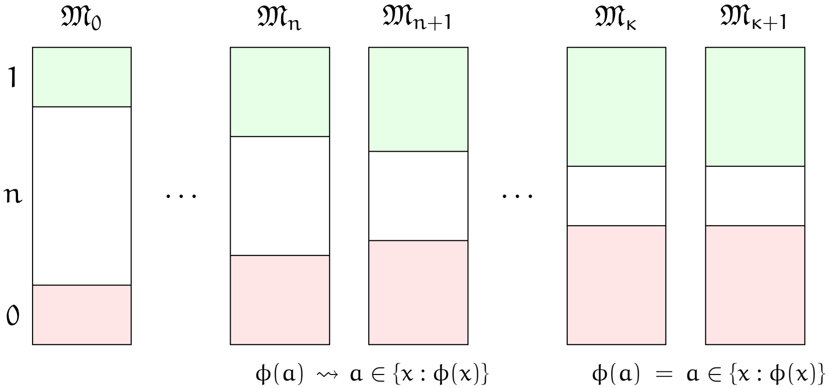

This is *not* like the other model constructions: the domain is constant—the terms $\{x : \phi(x)\}$.

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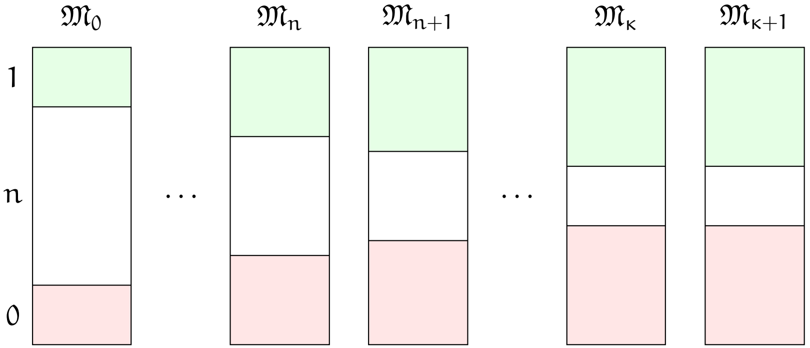

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This is *not* like the other model constructions:
the domain is constant—the terms $\{x : \phi(x)\}$.

This shows what the theory is *about* in only a very weak sense.

CLASSIFYING CLASS THEORIES

Gaps or *Gluts*?

Underlying Logic: Negation

Gaps or *Gluts*?

Paraconsistent or *Paracomplete*?

Underlying Logic: The Conditional

Do we have a conditional in the language?

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And if so, what is it like?

These decisions are not *that* important.

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The logic must allow for *fixed points*.

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For *any* sentence context $F(-)$, we need to allow for some p to be *equivalent to* $F(p)$.
If $c =_{df} \{x : F(x \in x)\}$, then $c \in c$ *iff* $F(c \in c)$

$$D$$

- D: the *ordinary* domain.

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$$D \qquad \Omega$$

- D: the *ordinary* domain.
- Ω: truth values.

$$\color{red}{C} \qquad\qquad D \to \Omega$$

- D: the *ordinary* domain.
- Ω: truth values.
- C: the classes

$$\mathsf{C} \qquad (\mathsf{C} \cup \mathsf{D}) \to \Omega$$

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$$C \cong (C \cup D) \to \Omega$$

- D: the *ordinary* domain.
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- C: the classes

$$C \stackrel{\sim}{=} [(C \cup D) \to \Omega]$$

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We won't focus on extensionality here.

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But we'll *identify* classes by their extensions as much as possible.

Sharpening our Target

$$C \cong [C \cup D \to \Omega]$$

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$\phi(x)$ gives a function $[C \cup D \to \Omega]$.
So we can find a class C to *match*.

$a \in \{x : \phi(x)\}$ has the *same* value in Ω as $\phi(a)$.

ORDER AND CONTINUITY

Underlying Logic: Preservation

$$\sqsubseteq$$

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Ω is ordered by $\sqsubseteq$.

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(If $x \sqsubseteq x'$ and $y \sqsubseteq y'$ then $x \sharp y \sqsubseteq x' \sharp y'$, etc.)

Preservation on candidates for Ω

$$\begin{tikzcd} 0 & & 1 \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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In $Ł_3$, $* \to *$ is 1; but $1 \to 0$ is 0

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In RM_3, $1 \to *$ is 0; but $1 \to 1$ is 1

Preservation on candidates for Ω

$$
\begin{tikzcd}
0 & & 1 \\
& * \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \star & \\
0 \arrow[ur] & & 1 \arrow[ul] \\
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\end{tikzcd}
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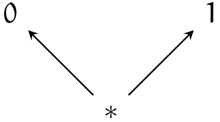

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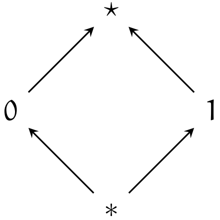

FDE, but no robust conditionals.

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K3 or LP, but not Ł3 or RM3

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In RM3, $1 \rightarrow *$ is 0; but $1 \rightarrow 1$ is 1

$$\begin{tikzcd} & \star & \\ 0 \arrow[ur] & & 1 \arrow[ul] \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}$$

FDE, but no robust conditionals.

Similar behaviour here.

Many other choices for Ω are possible.

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Even $\{0, 1\}$ can be ordered: $0 \sqsubseteq 1$. Then $\wedge, \vee, 0, 1$ are order preserving, but $\neg$ and $\supset$ are *not* order preserving.

3: our choice of Ω

$$
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(I *really* don't care if you think of $*$ as *true*, or as *untrue*.)

ORDER MODELS

Given an order algebra Ω, and a domain D of urelements

$\langle C, \sqsubseteq, \Uparrow, \Downarrow \rangle$ is a $\langle D, \Omega \rangle$ *order model* iff

Defining Order Models

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– Write '$\Uparrow(c)$' as '$c_{\Uparrow}$' and '$\Downarrow(f)$' as '$f_{\Downarrow}$.' So $c_{\Uparrow\Downarrow} = c$ and $f_{\Downarrow\Uparrow} = f$.

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- If $b \in C \cup D$ and $c \in C$, then $c_{\Uparrow}(b)$ tells you whether b is in c.

Membership is order preserving

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$x_{\Uparrow}(y') \sqsubseteq x'_{\Uparrow}(y')$	—	by the definition of $\sqsubseteq$ for functions.

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- (Connectives and quantifiers are order preserving functions on 3 or $[C \cup D \to 3]$.)

Extending the Language with Terms

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$$[\![\{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} = (\lambda v.[\![\phi(x)]\!]_{\mathfrak{M},\alpha[x:=v]})_{\Downarrow}$$

Strong Comprehension

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$$[\![t \in \{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} \quad = \quad [\![\{x : \phi(x)\}]\!]_{\alpha_{\Uparrow}}([\![t]\!]_{\alpha})$$

(I've dropped reference to $\mathfrak{M}$ as it is constant throughout.)

Strong Comprehension

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\begin{aligned}
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&= (\lambda v.[\![\phi(x)]\!]_{\alpha[x:=v]})_{\Downarrow\Uparrow}([\![t]\!]_\alpha) \\
&= (\lambda v.[\![\phi(x)]\!]_{\alpha[x:=v]})([\![t]\!]_\alpha) \\
&= [\![\phi(x)]\!]_{\alpha[x:=[\![t]\!]_\alpha]}
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Logical Constants

$0 \qquad\qquad 1$

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$$\Lambda = \{x : 0\}$$

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$x \in \Lambda$ is always false.

Λ, V and $\rlap{V}\Lambda$

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Λ, V and $\mathbb{X}$

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$x \in V$ is always true.

$$\mathbb{X} = \{x : *\}$$

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Ordering the Classes

$$\begin{tikzcd} 0 & & 1 \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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\begin{tikzcd}[column sep=small]
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& * \arrow[ul] \arrow[ur] & & & & \wedge\!\!\!\!\vee \arrow[ul] \arrow[ur] &
\end{tikzcd}
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In fact, $[\![\mathbb{X}]\!] \sqsubseteq c$ for every class $c \in C$.

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In fact, $[\![\mathbb{X}]\!] \sqsubseteq c$ for every class $c \in C$.

From now, we'll use '$\emptyset$', 'V' and '$\mathbb{X}$' as both the *class terms* in the language, and as their denotations, names for objects in C.

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for each object b in $C \cup D$
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Λ and V are sharp.

$\mathbb{X}$ is *not* sharp.

If $c_{\Uparrow}(b) = 1$ and $c_{\Uparrow}(b') = 0$, then $c_{\Uparrow}(\mathfrak{X}) = *$.

Almost No Classes are *Sharp*

$$\text{If } c_{\Uparrow}(b) = 1 \text{ and } c_{\Uparrow}(b') = 0, \text{ then } c_{\Uparrow}(\mathbb{X}) = *.$$

$$\mathbb{X} \sqsubseteq b, \text{ so } c_{\Uparrow}(\mathbb{X}) \sqsubseteq c_{\Uparrow}(b) = 1.$$

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$\mathbb{X} \sqsubseteq b$, so $c_{\Uparrow}(\mathbb{X}) \sqsubseteq c_{\Uparrow}(b) = 1$.

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Almost No Classes are *Sharp*

If $c_{\Uparrow}(b) = 1$ and $c_{\Uparrow}(b') = 0$, then $c_{\Uparrow}(\mathfrak{X}) = *$.

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It follows that $c_{\Uparrow}(\mathfrak{X}) = *$

Once a class *includes* something
and *excludes* something,
it is *indecisive* about $\mathbb{X}$.

There is no *classical recapture* through crisp classes

Once a class *includes* something
and *excludes* something,
it is *indecisive* about $\mathbb{X}$.

It follows that there are no *crisp singletons*:
objects $\{a\}$ for which $[\![a \in \{x\}]\!] = 1$
and $[\![b \in \{x\}]\!] = 0$ *for all other* b.

Singletons and Anti-Signetons: $\{t\}$ and $\}t\{$

- $[\![\{t\}]\!]_\alpha$: (the class representative of) the function that
  - assigns 1 to x iff $[\![t]\!]_\alpha \sqsubseteq x$,
  - and 0 to x iff there is no z where $x \sqsubseteq z$ and $[\![t]\!]_\alpha \sqsubseteq z$,
  - and $*$ otherwise.
- $[\![\}t\{]\!]_\alpha$: (the class representative of) the function that
  - assigns 0 to x iff $[\![t]\!]_\alpha \sqsubseteq x$, and
  - and 1 to x if there is no z where $x \sqsubseteq z$ and $[\![t]\!]_\alpha \sqsubseteq z$,
  - and $*$ otherwise.

From Here

- Study *pure* order models (where D is empty),

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- Study *pure* order models (where D is empty),

  ... and *impure* order models.
- Find perspicuous ways to *construct* order models.
- Relate these constructions to other known model constructions.
- *Axiomatise* the logic of order models.
- Examine different *motivations* of order models.

$$
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\Lambda & & V \\
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\end{tikzcd}
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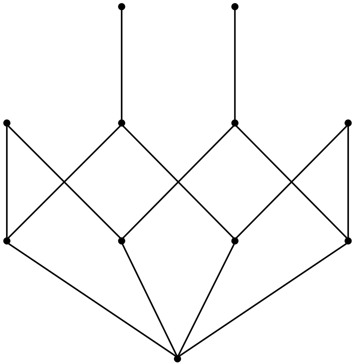
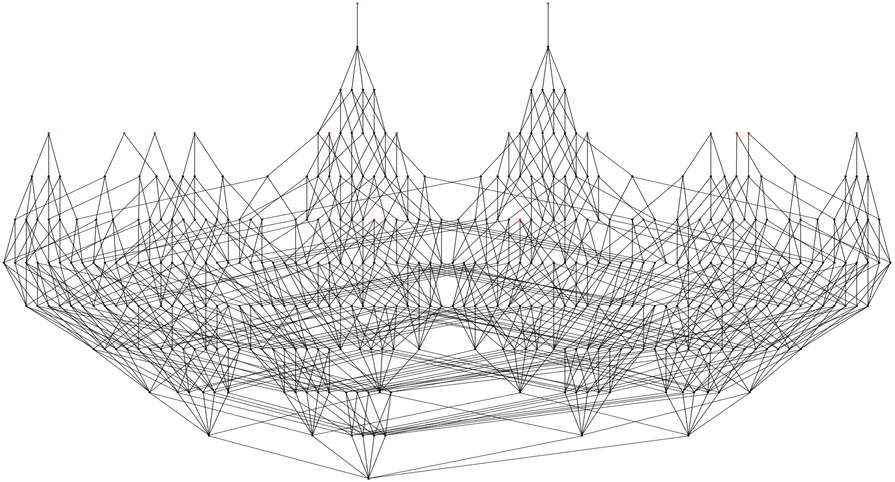
Model Construction: D_2 — $[D_1 \to \Omega]$



Model Construction: $D_3 — [D_2 \to \Omega]$



THANK YOU!

http://consequently.org/presentation/2016/fixed-point-models-fnclmp-2016

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