TWO NEGATIONS ARE MORE THAN ONE

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Abstract: In models for paraconsistent logics, the semantic values of sentences and their negations are less tightly connected than in classical logic. In "American Plan" logics for negation, truth and falsity are, to some degree, independent. The truth of $\sim p$ is given by the falsity of p, and the falsity of $\sim p$ is given by the truth of p. Since truth and falsity are only loosely connected, p and $\sim p$ can both hold, or both fail to hold. In "Australian Plan" logics for negation, negation is treated rather like a modal operator, where the truth of $\sim p$ in a situation amounts to p failing *in certain other situations*. Since those situations can be different from this one, p and $\sim p$ might both hold here, or might both fail here.

So much is well known in the semantics for paraconsistent logics, and for first degree entailment and logics like it, it is relatively easy to translate between the American Plan and the Australian Plan. It seems that the choice between them seems to be a matter of taste, or of preference for one kind of semantic treatment or another. This paper explores some of the differences between the American Plan and the Australian Plan by exploring the tools they have for modelling a language in which we have *two* negations.

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I owe my start in logic and in philosophy to Graham Priest. From our first meetings, when he took me on, as an enthusiastic and naïve undergraduate Science student finding his way into philosophical logic from mathematics in 1989, through PhD supervision in the early 1990s, and being a colleague and friend in the years since, my debt to Graham has been profound. My earliest work with Graham was on paraconsistent logics, and in particular, the first year of my PhD was spent trying to work through the details of Australian Plan and American Plan semantics for substructural logics [13–15].

*Thanks to Jc Beall, Rohan French, Lloyd Humberstone, Dave Ripley and Shawn Standefer for discussions on these topics, and especially to Graham Priest, whose encouragement, example, guidance and challenge have given me more – over nearly 30 years! – than I can express in words. Whereof one cannot speak, . . . ¶ This research is supported by the Australian Research Council, through Grant DP150103801, and Elephant9 and Reine Fiske's album, *Atlantis*. ¶ A version of this paper is available at http://consequently.org/writing/two-negations/.

It is a delight to return to some of these topics with the hindsight and experience of nearly 30 years, learning from Graham, being challenged and inspired by him, arguing with him, and wrangling over these and many other issues. Thanks, Graham!

1 NEGATION ON THE AMERICAN PLAN

In a paraconsistent logic, the truth of the negation $\sim p$ doesn't necessarily *rule out* the truth of the thing negated, p. Truth and falsity can overlap. In many paraconsistent logics, models allow not only truth value "gluts" like this, we also allow the dual case, of truth value gaps. The most natural, straightforward and elegant logic built on such a plan is now known as FDE, the logic of *first degree entailment*.

It can be defined and understood in a number of different ways, but for our purposes it suits to introduce it as the generalisation of classical two-valued logic according to which evaluations are no longer functions assigning each sentence of a language a truth value from $\{0, 1\}$, but *relations* to those truth values. Relaxing the constraint that evaluations be Boolean functions means that sentences can be *neither* true nor false (the evaluation fails to relate the sentence to either 0 or 1) or *both* true and false (the evaluation relates the sentence to both truth values).[1]

DEFINITION 1 A FDE-model for a propositional language consists of a relation ρ defined as follows: For each atomic sentence p, we posit by fiat whether it is true ($p\rho 1$) and whether it is false ($p\rho 0$). We then extend the relation ρ to a language including conjunction, disjunction, and negation, as follows:

$$
\begin{array}{rl}
(A \land B)\rho 1 \text{ iff } A\rho 1 \text{ and } B\rho 1 & (A \land B)\rho 0 \text{ iff } A\rho 0 \text{ or } B\rho 0 \\
(A \lor B)\rho 1 \text{ iff } A\rho 1 \text{ or } B\rho 1 & (A \lor B)\rho 0 \text{ iff } A\rho 0 \text{ and } B\rho 0 \\
\sim A\rho 1 \text{ iff } A\rho 0 & \sim A\rho 0 \text{ iff } A\rho 1
\end{array}
$$

The only deviation from classical propositional logic is that we allow for truth value gaps (ρ may fail to relate a given formula to a truth value) or gluts (ρ may relate a given formula to both truth values). Indeed, the possibilities of gaps and of gluts are, in a sense, *separable* or *modular*. It is quite straightforward to show that if a given interpretation ρ is a *partial function* on the basic vocabulary of a language – if it never over-assigns values to the extension of any predicate in that language – then it remains so over every sentence in that language. Sentences can be assigned gaps and not gluts. Similarly, if an interpretation is *decisive* over the basic vocabulary of some language – it never under-assigns values to the extensions of any predicate in that language – then

[1]The idea of a relational evaluation goes back to work by J. Michael Dunn, in the 1970s [4], and has been made use of in some influential papers by Nuel Belnap [1, 2]. My presentation follows Graham Priest's treatment in *An Introduction to Non-Classical Logics* [11, 12].

it remains so over every sentence of that language. These sentences can be assigned gluts and not gaps. If an evaluation is *sharp* (if it allows for neither gaps nor gluts in the interpretation of any predicate), then it remains so over the whole language.

We can use FDE evaluations to analyse truth and consequence in the language of first order logic. One important notion goes like this:

DEFINITION 2 An interpretation ρ is said to be a *counterexample* to the sequent $X \succ Y$ if and only if ρ relates each member of X to 1 while it relates no member of Y to 1.[2] In other words, an interpretation provides a counterexample to a sequent if it shows some way that the sequent fails to preserve truth. Given some set $\mathcal{M}$ of evaluations, a sequent is said to be $\mathcal{M}$-valid if it has no counterexamples in the set $\mathcal{M}$. We reserve the term 'FDE-valid' for those sequents which have no counterexamples at all.

All this is very well known in the literature on non-classical logics – see, for example, Priest's *An Introduction to Non-Classical Logic* [12, Chapter 8] for more detail on the behaviour of these logical systems. The FDE-valid sequents include all of distributive lattice logic, with a de Morgan negation. For example, sequents such as these

$$A \vee (A \wedge B) \succ A \qquad A \succ A \wedge (A \vee B)$$

$$A \wedge (B \vee C) \succ (A \wedge B) \vee C$$

$$\sim(A \wedge B) \succ \sim A \vee \sim B \qquad \sim(A \vee B) \succ \sim A \wedge \sim B$$

$$\sim A \vee \sim B \succ \sim(A \wedge B) \qquad \sim A \wedge \sim B \succ \sim(A \vee B)$$

$$A \succ \sim\sim A \qquad \sim\sim A \succ A$$

are FDE-valid, while $A, \sim A \succ B$ and $B \succ A, \sim A$ are not. (Any relational evaluation ρ where $A\rho 1$ and $A\rho 0$, while $B\not\rho 1$ is a counterexample to $A, \sim A \succ B$, and when $B\rho 1$ and $A\not\rho 1$ and $A\not\rho 0$, ρ is a counterexample to $B \succ A, \sim A$.) In addition, FDE-validity defined in this way satisfies the usual structural rules of identity, weakening (on the left and the right), and *Cut*:

$$A \succ A \qquad \frac{X \succ Y}{X, A \succ Y} \qquad \frac{X \succ Y}{X \succ A, Y} \qquad \frac{X \succ A, Y \qquad X, A \succ Y}{X \succ Y}$$

Relational evaluations provide a very natural model for FDE. They show it to be an elementary generalisation of classical logic, allowing for gaps between truth values and over-assignment of those values. The interpretation of the connectives remains as classical as in two-valued logic, except for the generalisation to allow for gaps and gluts

[2] For this paper it suffices to take sequents to be pairs $X \succ Y$ of sets of formulas. We allow these sets to be infinite. I follow Humberstone [5] in using '$\succ$' as a sequent separator, as a sequent with a counterexample remains a sequent. I will reserve the assertion sign '$\vdash$' for other uses.

between the two semantic values. This is semantics on the "American Plan." It is interesting, substantial, and philosophically salient. This understanding of negation is at the heart of Graham Priest's view of semantics [9–12].

This semantics raises a question concerning the status of negation, and the interaction between truth and falsity. While truth and falsity are completely on a par when it comes to relational evaluations, they are not on a par when it comes to the definition of logical consequence. If we were to define logical consequence on the positive fragment of FDE – to restrict the language to $\wedge$ and $\vee$ – we would not need to concern ourselves with falsity. We could just use ρ to keep track of whether a formula is related to truth or not. Falsity, as far as validity is concerned, is a fifth wheel in the positive vocabulary language. It does enters into the semantics of logical consequence in order to give us the truth conditions for negation, but it does no other work. So, the question that is raised is this: what is so special about *negation* that means that we enrich our semantic statuses to keep track of this extra kind of information (negative information) as well as the information we already needed to track for evaluating validity?

To put this question another way: if our language were equipped with *two* kinds of negation instead of one, would this mean that we extend our evaluations to incorporate two different kinds of falsity? After all, that is the strategy of the American Plan.

This is a live question, because the negation of first degree entailment, while a logical constant in the everyday sense, is not quite a logical *constant* in the same sense as the conjunction and disjunction of FDE. It is straightforward to see that conjunction and disjunction in FDE satisfy the following *defining rules*:

$$\frac{X, A, B \succ Y}{\overline{X, A \wedge B \succ Y}} \qquad \frac{X \succ A, B, Y}{\overline{X \succ A \vee B, Y}}$$

and that if any *other* connective satisfied one of the defining rule for conjunction, it would be logically equivalent to conjunction, and similarly for disjunction.[3] The same cannot be said for negation. There are no rules satisfied negation in FDE-interpretations that force all FDE-negations to be equivalent. Negation, in FDE, is more like a $\Box$ satisfying the conditions of the modal logic S5. There are bimodal logics with *two* S5 non-equivalent necessities, and there are "bi-negation" logics with two non-equivalent FDE-negations.

This should not be surprising, for it is possible already to extend any FDE relational evaluation to model so-called "Boolean negation", by adding the clauses:

[3] Suppose & also satisfied the defining rule for conjunction. By identity, $A \mathbin{\&} B \succ A \mathbin{\&} B$ holds, so by the defining rule for &, we have $A, B \succ A \mathbin{\&} B$. But by the defining rule applied to $\wedge$, we have $A \wedge B \succ A \mathbin{\&} B$. Similarly, we can derive $A \mathbin{\&} B \succ A \wedge B$, and then there is a short appeal through *Cut* to show that any derivable sequent in which $A \wedge B$ is used as a premise or a conclusion holds with $A \mathbin{\&} B$ in its place, and *vice versa*. And the same holds for disjunction, too. Nothing like this holds for negation.

$$-A\rho 1 \text{ iff } A\not\rho 1 \qquad -A\rho 0 \text{ iff } A\not\rho 0$$

and this is another FDE-negation. Now, to be sure, Boolean negation is not quite fully in the spirit of FDE, because it is the degenerate kind of negation with neither gaps nor gluts. We have a new kind of falsity (*un*truth) which stands to "$-$" as 0 stands to "$\sim$". Classical Boolean extensions of FDE are examples of logics with two non-equivalent FDE-negations.[4] It is possible, of course, for there to be *other* kinds of negations that satisfy the FDE conditions. One possible example might be found using the notion of polarity reversal for graded predicates. Consider the difference between being *not* tall (and the boundary between tall and *not* tall, being vague, may perhaps permit gaps or overlap), and the difference between being tall and the *polar opposite* of tall – *short*. In a language where predicates have not only *complements* but *opposites*, there is scope for something that also satisfies the FDE conditions (for example, the opposite of the opposite takes you back to the original predicate), but care must be used in filling this out to the wider vocabulary. (A case could me made that the polar opposite of "tall and reserved" should be "short and outgoing" and not "short our outgoing", so polarity reversal generalised to a sentential operator is not necessarily a de Morgan negation.) Regardless of how the details are filled out, there is a case to at least explore the behaviour of languages in which there are different notions of negation and opposition that satisfy the FDE constraints, because there is nothing in the logic that rules this out.

Nothing in what follows hangs on any particular choice of an interpretation for a language with two negations. My chief aim here is to understand the behaviour of negation in FDE by examining what happens when the language is extended to two negations. The aim is to not only shed light on FDE, relational evaluations and the American plan, but to also illuminate the distinctive features of that *other* semantics for FDE and negation, the *Australian* plan.

2 NEGATION ON THE AUSTRALIAN PLAN

First degree entailment, like all good logics, can be understood in more than one way. Another prespective on FDE is given when you follow the *Australian Plan*, and in particular, utilising the Routley star due to Richard and Valerie Routley [16]. On the Australian Plan [6], negation is understood in the manner of a modal operator. The fact that the extension of a proposition is independent from its anti-extension – or better, from the extension of its negation – is no more surprising than that there is independence between the extension of p and the extension of its necessitation $\Box p$. There are some statements p where p and $\Box p$ are both true. There are others where p is true and $\Box p$ is false. In the same way, the truth of p does not fix – on the Australian Plan – the truth

[4] I first learned of this sort of combination of negations in Meyer and Routley's work on Classical Relevant Logics [7, 8].

of $\sim$p. Rather than taking falsity to be an independent and separate component of the semantic value of a statement from its truth, falsity is understood in terms of truth *elsewhere*.

DEFINITION 3 A Routley frame is a set P of points endowed with a function $*$ where for each $x \in P$, $x^{**} = x$. A *Routley frame* $\langle P, * \rangle$ may be endowed with an interpretation $\Vdash$ for atomic sentences, determining whether $x \Vdash p$ or not at each point $x \in P$, and this is extended to the language of propositional logic by setting

$$x \Vdash A \land B \text{ iff } x \Vdash A \text{ and } x \Vdash B$$

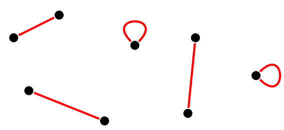
$$x \Vdash A \lor B \text{ iff } x \Vdash A \text{ or } x \Vdash B$$

$$x \Vdash \sim A \text{ iff } x^* \nVdash A$$

and $\langle P, *, \Vdash \rangle$, so defined, is a *Routley model.*

We say that a sequent $X \succ Y$ holds on a model if and only there is no point $x \in P$ which serves as a counterexample to the sequent – i.e., where $x \Vdash A$ for each $A \in X$ and $x \nVdash B$ for each $B \in Y$.

A single Routley star $*$ on a frame P of points does not impose much structure on P. A operator determines a diffferent partition of P into unordered pairs and singletons like so: $\{\{x, x^*\} : x \in P\}$. You can represent this in a diagram by connecting a point x and its star mate x^* with a link, like this:



A graph of this structure is graph is special. Each node participates in one and only one link. They are very simple structures, with little complexity.

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These models are rather unlike relational evaluations. However, it is straightforward to show that any sequents that hold in relational evaluations are exactly the sequents that hold in all Routley models.

FACT 1 *For each relational evaluation, there is a Routley model with a point satisfying exactly the formulas true at that evaluation. Conversely, for any point in a Routley model, there is a relational evaluation satisfying exactly the formulas true at that point.*

Proof: Given each relational evaluation ρ, take the Routley model $\langle\{\rho,\rho^*\},*,\Vdash\rangle$, where ρ^* is the relational evaluation that sets $p\rho^*1$ iff $p\not\rho 0$, and $p\rho 0$ iff $p\not\rho 1$. Define $*$ in the obvious way (so, $\rho^{**}=\rho$) and set for atomic formulas p, $\rho\Vdash p$ iff $p\rho 1$, and $\rho*\Vdash p$ iff $p\rho^*1$, i.e., $p\not\rho 0$. It is an easy inductive argument to show that $\rho\Vdash A$ iff $A\rho 1$, and $\rho^*\Vdash A$ iff $A\not\rho 0$, and so, this is our Routley model that mimics the relational evaluation ρ.

Conversely, given a Routley model $\langle P,*,\Vdash\rangle$, and a point $x\in P$, define ρ by setting $p\rho 1$ iff $x\Vdash p$, and $p\rho 0$ iff $x^*\not\Vdash p$. This is a relational evaluation, and again, it is a straightforward inductive verification to show that $A\rho 1$ iff $x\Vdash A$, and $A\rho 0$ iff $x^*\not\Vdash A$, i.e., iff $x\Vdash {\sim}A$. ■

FACT 2 *It is a consequence of this that* FDE *is not only the logic of arbitrary Routley models, it is also the logic of two point Routley models. No more than two points are needed to model* FDE.

So, we have two very different kinds of semantics for the one logic. What is the difference between the American plan and the Australian plan? This is a real question, with philosophical bite. [5] These approaches are equivalent, in the sense that they model the same logic, but they do it in very different ways. These tools have vary different affordances, as we will see when we attempt to take them beyond the very simple domain of modelling FDE.

We will not need to take them very *far* beyond that domain to address our question. We will see that these semantics are very different when we address the question of how they model *two* negations. Here, the difference between the American and Australian plans will come into sharper relief.

3 TWO KINDS OF FALSITY

Given two negations and the American Plan, it is obvious how to *start* answering the question. We now have three semantic statuses: We have truth, falsity_1 and falsity_2. We will represent these as 1, 0 and F. That seems fair enough. We can extend the clauses for disjunction and conjunction:

$$(A\land B)\rho 1 \text{ iff } A\rho 1 \text{ and } B\rho 1$$

$$(A\land B)\rho 0 \text{ iff } A\rho 0 \text{ or } B\rho 0 \qquad (A\land B)\rho F \text{ iff } A\rho F \text{ or } B\rho F$$

$$(A\lor B)\rho 1 \text{ iff } A\rho 1 \text{ or } B\rho 1$$

$$(A\lor B)\rho 0 \text{ iff } A\rho 0 \text{ and } B\rho 0 \qquad (A\lor B)\rho F \text{ iff } A\rho F \text{ and } B\rho F$$

[5] De and Omori, for example, take it that the Australian Plan is philosophically defective, and the American Plan is the appropriate way to understand negation [3].

For our two negations, we know that we should *at least* enforce these conditions:

$$(\sim_1 A)\rho 1 \text{ iff } A\rho 0 \qquad (\sim_1 A)\rho 0 \text{ iff } A\rho 1$$

$$(\sim_2 A)\rho 1 \text{ iff } A\rho F \qquad (\sim_2 A)\rho F \text{ iff } A\rho 1$$

But what should we say about when $\sim_1 A$ is false$_2$, or when $\sim_2 A$ is false$_1$? There are many options, including adding semantic complexity to the values we already have at hand. If we aren't going to go beyond the three semantic statuses, however, the natural answer is this:

$$(\sim_1 A)\rho F \text{ iff } A\rho 1 \qquad (\sim_2 A)\rho 0 \text{ iff } A\rho 1$$

A negation's being *false* (in any sense of false) is, in some sense, a *positive* notion, and 1 is our only positive semantic status. In the American Plan, each semantic status of a complex formula is determined by the holding of a semantic status (positively) by the constituent formulas. We don't define truth or falsity in terms of the *failure* of the semantic status to subformulas.[6] Given the raw materials of 1, 0 and F, we don't have much to work with. This clause is the natural one unless we wish to extend the modelling conditions further with other semantic statuses again such as keeping separate track not only of the two different kinds of falsity, but also the 1-falsity of 2-falsity (as a different kind of truth?), and more. But that way lies further complexity, and as far as I can see, makes the semantics look much less like relational evaluations and the purity of the American Plan.

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So, what do our clauses for $\sim_1$ and $\sim_2$ tell us about the semantics for negation? Keeping our interpretation for validity as before, we know that both negations are FDE negations. The $\langle \wedge, \vee, \sim_1 \rangle$ and $\langle \wedge, \vee, \sim_2 \rangle$ fragments of the language are FDE, just as we would expect. Where things get interesting is the interaction between $\sim_1$ and $\sim_2$. One way to get a sense of just how interesting and different the logic is can be found by compiling "truth tables" for the negations. Abusing notation a little, we can think of our evaluations as determining an eight valued logic, with the values

$$\emptyset \quad \{1\} \quad \{0\} \quad \{F\} \quad \{1,0\} \quad \{1,F\} \quad \{0,F\} \quad \{1,0,F\}$$

where a formula is *assigned* the value S by a relational evaluation ρ where it is related to the values in S – and only those values – by ρ. Abusing notation a little more, we'll drop

[6]This gives relational evaluations their significant power in terms of preservation and heridity results. If $\rho \subseteq \rho'$, in that ρ' assigns all the values that ρ does, and perhaps more, then ρ' makes true (and false) all the formulas made true (and false) by ρ. This result requires the positivity of the clauses for complex formulas. To break this is to lose one of the significant benefits of relational evaluations.

the set braces and think of the values as 0, 1F, 10F, etc. The truth tables for $\sim_1$, $\sim_2$, $\sim_1\sim_1$ and $\sim_2\sim_1$ are then:

p	$\sim_1 p$	$\sim_2 p$	$\sim_1\sim_1 p$	$\sim_2\sim_1 p$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	0F	0F	1	1
0	1	$\emptyset$	0F	0F
F	$\emptyset$	1	$\emptyset$	$\emptyset$
10	10F	0F	10F	10F
1F	0F	10F	1	1
0F	1	1	0F	0F
10F	10F	10F	10F	10F

and a number of features stand out immediately. As in FDE, there are two fixed points for negation – for *both* negations here – the empty value $\emptyset$ and the total value 10F. That is to be expected. The table shows clearly that the sequents $p \succ \sim_1\sim_1 p$ and $\sim_1\sim_1 p \succ p$ are valid (and similarly for $\sim_2$). The formulas p and $\sim_1\sim_1 p$ are designated in exactly the same rows. However, unlike in simple relational evaluations, p and $\sim_1\sim_1 p$ no longer receive the same semantic evaluations. When p is false$_1$ (only), $\sim_1\sim_1 p$ is not only false$_1$, it is also false$_2$. This double negation of p has gained a semantic status not had by p. But when p is false$_2$ only, its double negation $\sim_1\sim_1 p$ has value $\emptyset$. In this case, its double negation has *lost* a semantic status. In these bifurcated relational semantic valuations, A and $\sim_1\sim_1 A$ agree as far as 1 and 0, but they can disagree about F. A and $\sim_1\sim_1 A$ are no longer semantically equivalent in the strong sense: $\sim_2 A$ might fail (to be true) where $\sim_2\sim_1\sim_1 A$ might succeed (i.e. be true). The equivalence of A and $\sim_1\sim_1 A$ in relational evaluations is fragile. It can be disturbed by the addition of a new semantic status.

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This illustrates one well-understood issue with the American Plan. The bifurcated semantics, with separate conditions for truth and falsity, makes giving semantic clauses for *everything* else more complicated. The case of two "negations" is one simple illustration of this fact. It is one thing to give truth conditions for something (say, conditionals [15], or in this case, another negative operator, another negation). Giving these truth conditions leaves open the thorny question of their *falsity* conditions. There is nothing in the *logic* that tells us, on the American Plan, how these falsity conditions are to relate to the truth conditions, and furthermore, there are ways to assign these conditions in such a way as to break the substitutivity of logical equivalents.

The Australian Plan has none of those difficulties. The extra work done to give 'modal' truth conditions for negation means that once truth conditions – at each point of the model structure – are given for a concept, the interaction between that concept and negation is already fixed. This is true, also, for the interaction between one negation and another, as we will see in the next section.

4 TWO ROUTLEY STARS

On the Australian Plan, the prospect of having two FDE negations is immediate and affords no difficulty in giving truth conditions. The definition of the appropriate structure stares us in the face.

DEFINITION 4 A *two-star frame* is a structure $\langle P, \ast, \star\rangle$ with two Routley stars. We endow a two star frame with an interpretation $\Vdash$ in the usual way, adding clauses for two negations as follows:

$$x \Vdash {\sim_1} A \text{ iff } x^{\ast} \nVdash A \qquad x \Vdash {\sim_2} A \text{ iff } x^{\star} \nVdash A$$

and the definition of sequents holding on a two-star model are exactly the same as with Routley models.

There is no further question concerning how to evaluate ${\sim_2}{\sim_1}p$. The model tells us that ${\sim_2}{\sim_1}p$ holds at x if and only if p holds at $x^{\ast\star}$. Of course, where that *is* depends on the model, and on how the star operators, $\ast$ and $\star$, interact.

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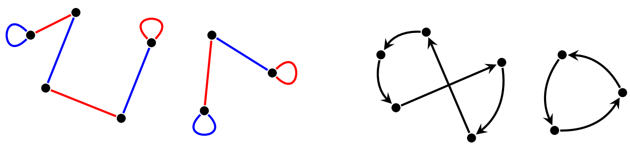
As to how $\ast$ and $\star$ interact, there is significant scope for complexity. Here is one way to get some sense of just how much complexity there is:

FACT 3 *Given any two star frame* $\langle P, \ast, \star\rangle$*, the relation* RB *given by setting* $xRBy$ *iff* $x^{\ast\star} = y$ *is a permutation on* P.

Proof: If $xRBy$ and $xRBz$ then $y = z$, since $y = x^{\ast\star} = z$. The converse relation RB^{-1} can be found by setting $xRB^{-1}y$ iff $x^{\star\ast} = y$, since if $x^{\ast\star} = y$ then $x^{\ast\star\ast} = y^{\star}$ so $x^{\ast} = y^{\star}$, so $x = x^{\ast\ast} = y^{\star\star}$. ∎

Here is an example of a two-star frame and its underlying RB permutation:



On any two-star frame, RB is a (single alternative) binary accessibility relation, and we have

$$\begin{aligned} x \Vdash {\sim_2}{\sim_1}A &\text{ iff for } all\ y \text{ where } xRBy,\ y \Vdash A, \\ &\text{ iff for } some\ y \text{ where } xRBy,\ y \Vdash A. \end{aligned}$$

Let's abbreviate $\sim_2\sim_1 A$ by $\Box A$ (it could equally be $\Diamond A$, of course). This is the modal operator of single alternative modal logic, by the permutation RB on our set P of points. The converse modality $\Box^{-1}$, modelled by RB^{-1}, is given by setting $\Box^{-1}A = \sim_1\sim_2 A$. Clearly $\Box\Box^{-1}A$ is equivalent to $\Box\Box^{-1}A$ is equivalent to A on all of our two-star frames.

So, we have an interesting logical structure on our frames, and a recognisable normal modal logic – the logic of permutations – is embedded in two-star frames. In fact, we can say more. The structure of two Routley stars is enough to generate *any* permutation.

FACT 4 *Given any permutation σ on a set P, there is some two star frame $\langle P, *, \star\rangle$ such that the* RB *relation on that frame is the permutation σ.*

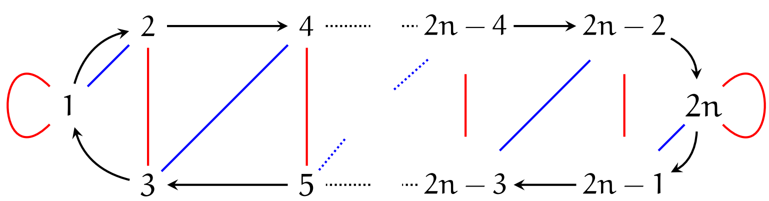
Proof: The permutation σ on P decomposes into its cycles in the following way: we say that $x, y \in P$ are in the same cycle if and only if some finite number (including zero) of applications of σ or its converse sends x to y. It is easy to see this is an equivalence relation on P, by design. Each cycle has some finite non-zero size (is an n-cycle for some n), or is infinite, in which case it has of order type $\omega^* + \omega$. The action of the permutation on P is determined by the structure of its cycles.

So, for any cycle, use the following rubric to define $*$ and $\star$ in order to generate such a cycle out of the action of two Routley stars:

For a cycle of even length, on $\{1, \ldots, 2n\}$ (where $n \geqslant 1$), set

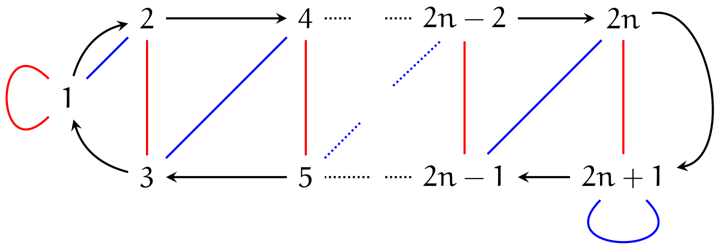
$$1^* = 1, (2k)^* = 2k + 1\ (k = 1, \ldots, n-1), 2n^* = n$$
$$(2l + 1)^\star = 2l + 2\ (l = 0, \ldots, n-1)$$

The structure looks like this:



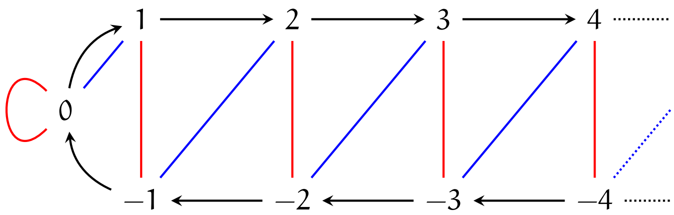
For a cycle of odd length, on $\{1, \ldots, 2n + 1\}$ ($n \geqslant 0$), assign

$$1^* = 1, (2k)^* = 2k + 1\ (k = 1, \ldots, n)$$
$$(2l + 1)^\star = 2l + 2\ (l = 0, \ldots, n-1), (2n + 1)^\star = 2n + 1$$



And finally, for an infinite $\omega^* + \omega$ sequence on $\mathbb{Z}$ assign

$$n^{*} = -n \text{ for each } n \qquad n^{\star} = 1 - n \text{ for each } n$$



In this way, we can assign $*$ and $\star$ on each cycle of the permutation σ, such that the RB permutation generated by them agrees with σ. Each permutation is given by some two star model. ∎

So, we have a rich family of structures. The logic of such structures can vary significantly. Here is one example of how the logic of a structure varies with the kinds of cycles present in its RB permutation.

FACT 5 *The following sequents hold:*

$$\Box^{n_1}p \wedge \cdots \wedge \Box^{n_k}p \succ p \qquad p \succ \Box^{n_1}p \vee \cdots \vee \Box^{n_k}p$$

on any frame if and only if its RB permutations consists of cycles where each cycle has some length l where $l \mid n_i$ for some i.

Proof: Suppose $\Box^{n_1}p \wedge \cdots \wedge \Box^{n_k}p$ holds at a point x, where x is on some l cycle where $l \mid n_i$. Since $\Box^{n_i}p$ holds at x, and x is n_i RB-steps away from itself, p holds at x too. Suppose that p holds at a point x, where x is on some l cycle where $l \mid n_i$. The only point n_i steps away from x is x itself, so $\Box^{n_i}p$ holds at x also, and so does $\Box^{n_1}p \vee \cdots \vee \Box^{n_k}p$.

Conversely, suppose the frame has a component which is *not* an l cycle for any $l \mid n_i$. That is, it has a component which is an $\omega^* + \omega$ chain, or a cycle of some length k where $k \not\mid n_i$ for each i. In any case, choose a point x in this component, and let p be true at each of the points n_i steps from x for each i. This does not include x itself. So, $\Box^{n_1}p \wedge \cdots \wedge \Box^{n_k}p$ holds at x but p doesn't. For the other sequent, do the reverse: p holds at x but an none of the points n_i steps from x. This is a counterexample to the other sequent. ∎

So, for example, $\Box^4 p \wedge \Box^5 p \succ p$ holds in frames consisting of 4-cycles and 5-cycles. This sequent *also* holds in frames consisting of 2-cycles, 4-cycles and 5-cycles, because 2 divides 4.

These differences make a difference for the logic of negation. Frames consisting solely of 73-cycles are such that iterated $\sim_2\sim_1$ double negations of length 73 collapse into nothingness, *but not before that*. Can sense be made of that?

So, constructions like these show us that we have infinitely many different logics, corresponding to different classes of frames, each with subtly different negation interaction conditions.

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The variations among the logics of two-star frames does not end there. We have seen just a little of what you can do with different RB permutations. However, there is more to a frame than this. (You cannot, in general, recover the two star frame $\langle P, *, \star\rangle$ from its RB permutation. For example, these two different two star frames on $\{a, b\}$, one in which the stars $*$ and $\star$ keep a and b fixed, and the other, in which both stars swap a and b, have the same RB permutation, the identity function on $\{a, b\}$.) Exactly what other rich variety can be found in the class of two-star frames is left for further research.

5 ON SEMANTICS FOR NEGATIONS

So, while the Australian Plan and the American Plan are equivalent when it comes to the semantics of negation by itself, they differ significantly when we add negation to something else – even if that something else is just another negation. There are complexities on both sides. Following the American Plan, and extending relational evaluations to include another semantic status beyond truth and (the original) falsity, leads to questions about how this new status and falsity are to interact – questions that are not easy to answer. Furthermore, when you answer them, you may find that desirable features which you formerly had (such as the substitutivity of logical equivalents) no longer hold. However, there is little risk of semantic hyperinflation, of rich stuctures of possible propositional values generated by your semantic primitives.

On the Australian Plan, the situation is reversed. There is no need to ask difficult questions about how different semantic statuses are to interact. Once the modelling condition for each negation is set up, the interactions between those negations are fixed. Giving the truth conditions for each negation in each point in the model structure *fixes* the semantic values – and hence the interactions with other connectives, including the other negation. However, the potential for interactions between the devices used to interpret each negation means that we move beyond a simple four-valued or eight-valued logic into the complexity and variety of many different algebras of potential semantic values, many of them infinite and intricate.

Once we move beyond the simple language of first degree entailment itself, we discover that the American Plan and the Australian Plan are very different. We are only beginning to understand the rich possibilities for modelling non-classical logics.

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