ISOMORPHISMS IN A CATEGORY OF PROPOSITIONS AND PROOFS

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I aim to show how a category of propositions and *classical* proofs can give rise to finely grained hyperintensional notions of sameness of content. One notion is *very* finely grained (distinguishing p and $p \wedge p$) others are is less finely grained. Another notion amounts to equivalence in R. B. Angell's logic of analytic containment [1].

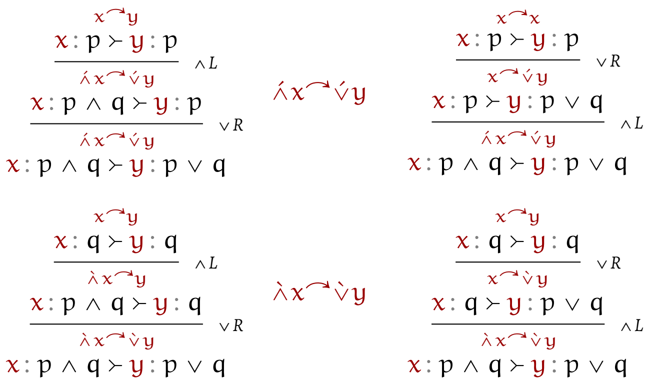
1 THE CATEGORY OF CLASSICAL PROOFS

Four different *derivations*, and two *proofs*.

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,{\wedge L}}{p \wedge q \succ p \vee q}\,{\vee R} \qquad \dfrac{\dfrac{p \wedge q}{p}}{p \vee q} \qquad \dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,{\vee R}}{p \wedge q \succ p \vee q}\,{\wedge L}$$

$$\dfrac{\dfrac{q \succ q}{p \wedge q \succ q}\,{\wedge L}}{p \wedge q \succ p \vee q}\,{\vee R} \qquad \dfrac{\dfrac{p \wedge q}{q}}{p \vee q} \qquad \dfrac{\dfrac{q \succ q}{q \succ p \vee q}\,{\vee R}}{p \wedge q \succ p \vee q}\,{\wedge L}$$

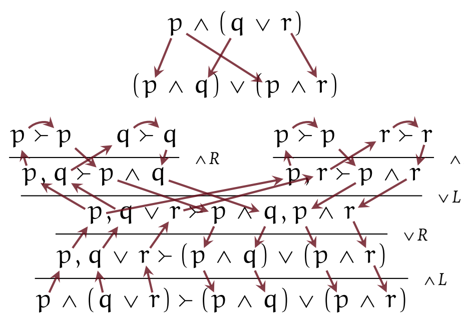
MOTIVATING IDEA: *Proof terms* are an *invariant* for derivations under rule permutation. δ_1 and δ_2 have the same *term* iff some permutation sends δ_1 to δ_2.



A *proof term* for $\Sigma \succ \Delta$ encodes the flow of information in a proof of $\Sigma \succ \Delta$. They can be represented as directed graphs on sequents [2].

$\acute{\wedge}x \frown \acute{\wedge}\acute{\vee}y \quad \acute{\wedge}x \frown \acute{\wedge}\grave{\vee}y \quad \acute{\vee}\grave{\wedge}x \frown \grave{\wedge}\acute{\vee}y \quad \grave{\vee}\grave{\wedge}x \frown \grave{\wedge}\grave{\vee}y$

$x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$



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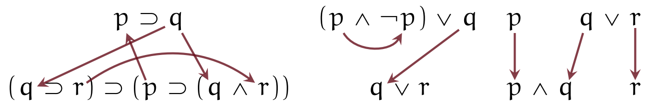
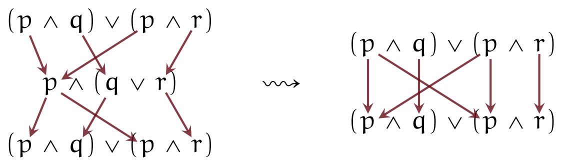
More examples:



Links wholly internal to a *premise* or a *conclusion* are called *cups* (⌣) and *caps* (⌢).

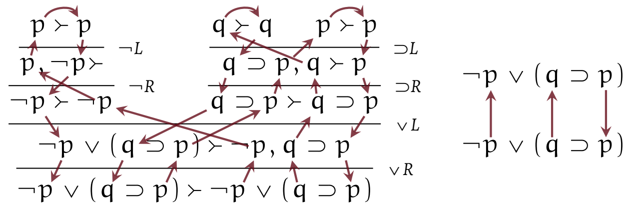
FACTS: Not every directed graph on occurrences of atoms in a sequent is a proof term. ¶ They *typecheck*. [An occurrence of p is linked only with an occurrence of p.] ¶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.] ¶ They must satisfy an "enough connections" condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

Cut is chaining of proof terms, composition of graphs.



Cut elimination is *confluent* and *terminating*. [So it can be understood as a kind of *evaluation*.] ¶ Cut elimination for proof terms is *local*. [So it is easily made parallel.]

$\mathfrak{C}$ is the *Category of Classical Proofs*. *OBJECTS*: Formulas — A, B, etc. *ARROWS*: Cut-Free Proof Terms — $\pi : A \succ B$. *COMPOSITION*: Composition of derivations with the elimination of *Cut* — If $\pi : A \succ B$ and $\tau : B \succ C$ then $\tau \circ \pi : A \succ C$. *IDENTITY*: Canonical identity proofs — $\mathrm{Id}(A) : A \succ A$.



The category $\mathfrak{C}$ is *symmetric monoidal* and *star autonomous*, but not *cartesian*, with structural *monoids* and *comonoids*, and is enriched in *SLat* (the category of semilattices). Being enriched in *SLat* means that proofs terms come ordered by $\subseteq$, and compose under $\cup$, and these interact sensibly with composition.

$$\begin{aligned} \pi \subseteq \pi' &\quad\Rightarrow\quad \pi \circ \tau \subseteq \pi' \circ \tau \\ \tau \subseteq \tau' &\quad\Rightarrow\quad \pi \circ \tau \subseteq \pi \circ \tau' \\ \pi \circ (\tau \cup \tau') &\quad=\quad (\pi \circ \tau) \cup (\pi \circ \tau') \\ (\pi \cup \pi') \circ \tau &\quad=\quad (\pi \circ \tau) \cup (\pi' \circ \tau) \end{aligned}$$

$\mathfrak{C}$ is just *classical* propositional logic, in a categorical setting. (The sequent calculus is playing no essential role here. You can define proof terms on other proof systems, e.g. *natural deduction*, *Hilbert proofs*, *tableaux*, *resolution*.)

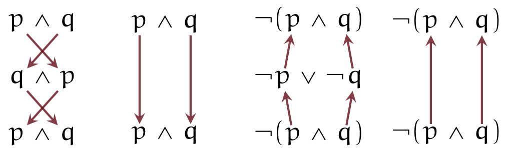
2 ISOMORPHISMS

$f : A \to B$ is an *isomorphism* in a category iff it has an *inverse* $g : B \to A$, where $g \circ f = id_A : A \to A$ and $f \circ g = id_B : B \to B$. (If g and g' are inverses, $g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g'$, so any inverse is unique. We can call it f^{-1}.)

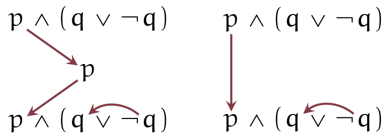
If A and B are isomorphic in a category $\mathcal{C}$, then what we can do with A (in $\mathcal{C}$) we can do with B, too.

If A and B are isomorphic in $\mathfrak{C}$, then they agree not only on *provability*, but also, on *proofs*. The distinctions drawn when you analyse how something is *proved* (from premises), are not far from what you want to understand when you ask how something is *made true*.

Isomorphisms in $\mathfrak{C}$: $p \wedge q \cong q \wedge p$; $\neg(p \wedge q) \cong \neg p \vee \neg q$



Non-isomorphisms in $\mathfrak{C}$: $p \wedge (q \vee \neg q) \ncong p$; $p \wedge p \ncong p$; $p \wedge (q \vee r) \ncong (p \wedge q) \vee (p \wedge r)$; $p \wedge (p \vee q) \ncong p \vee (p \wedge q)$



Occurrence Polarity Condition: If A is isomorphic to B in $\mathfrak{C}$ then each variable occurs positively [negatively] in A exactly as many times as it occurs positively [negatively] in B. (This condition is *necessary*, not *sufficient*: $p \wedge (p \vee q) \ncong p \vee (p \wedge q)$.)

A is *isomorphic to* B iff A and B are equivalent in the following calculus:

$$\begin{array}{cc} A \wedge B \leftrightarrow B \wedge A, & A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C. \\ A \vee B \leftrightarrow B \vee A, & A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C. \\ \neg(A \vee B) \leftrightarrow \neg A \vee \neg B, & \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B. \\ \neg\neg A \leftrightarrow A. & A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B). \end{array}$$

(This allows for a *negation normal form*, but not DNF or CNF.)

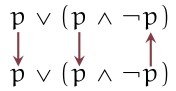
Proof Sketch (Došen and Petrić, 2012 [3]).

If $A \leftrightarrow B$ holds in the calculus, A and B are isomorphic. ¶ A is isomorphic to B iff there are *diversified* A' and B' where A' and B' are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution σ. ¶ A is isomorphic to B iff their negation normal forms are isomorphic. (If A is diversified, so is its negation normal form.) ¶ If A and B are diversified, isomorphic, and in negation normal form, if $l \wedge m$ is a conjunction in A (l and m, literals), a substitution argument (substituting $\top$ and $\bot$ for the *other* atoms) shows that l and m must be conjunctively joined in B, too. The same goes for $l \vee m$. ¶ Replace $l \wedge m$ by a new atom in both A and B, and repeat. ¶ This shows how to reconstruct a proof of equivalence for A and B in the syntactic calculus for isomorphic formulas.

Isomorphism is a very tight constraint: If A and B are isomorphic, they can play *essentially* the same role in proof. ¶ Replacing A by B in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*. ¶ Not even A and $A \wedge A$ are equivalent in *this* sense. ¶ Yet, A and $A \wedge A$ seem to have identical *subject matter* (insofar as we understand that notion). ¶ Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

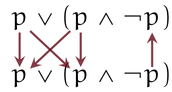
3 MORE PROOFS FROM A TO A

$Id(p \vee (p \wedge \neg p))$



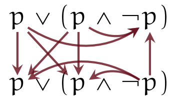
In $Id(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion. *Different occurrences of atoms in A are treated differently.*

$Mx(p \vee (p \wedge \neg p))$



In $Hz(A)$, each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion. *We treat occurrences of an atom in A—with the same polarity—equally.*

$Hz(p \vee (p \wedge \neg p))$



In $Mx(A)$, each syntactically possible linking is included. *We treat all occurrences of an atom in A equally.*

Note: $Hz(A)$ is $Mx(A)$ with the caps and cups removed.

Let's look at relations like isomorphism, but which erase distinctions, up to *Hz* or *Mx*.

Let's say that A and B *Hz*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$. We write "$\approx_{Hz}$" for the *Hz*-matching relation, and we write "$\pi, \pi' : A \approx_{Hz} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Hz*-match between A and B.

Let's say that A and B *Mx*-MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$. We write "$\approx_{Mx}$" for the *Mx*-matching relation, and we write "$\pi, \pi' : A \approx_{Mx} B$" to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Mx*-match between A and B.

Isomorphism $\subseteq$ *Hz-Matching*: If $\pi : A \succ B$ and $\pi^{-1} : B \succ A$, then consider $\pi' = Hz(B) \circ \pi \circ Hz(A)$ and $\tau' = Hz(A) \circ \pi^{-1} \circ Hz(B)$. These satisfy the *Hz*-matching criteria, $\tau' \circ \pi' = Hz(A)$ and $\pi' \circ \tau' = Hz(B)$.

Hz-Matching $\subseteq$ *Mx-Matching*: If $\pi, \pi' : A \approx_{Hz} B$, then consider $\tau = Mx(B) \circ \pi \circ Mx(A)$ and $\tau' = Mx(A) \circ \pi' \circ Mx(B)$. These satisfy the *Mx*-matching criteria, $\tau' \circ \pi' = Mx(A)$ and $\pi' \circ \tau' = Mx(B)$.

Mx-Matching $\subseteq$ *Logical Equivalence*: If $A \approx_{Mx} B$ then there are proofs $\pi : A \succ B$ and $\tau : B \succ A$.

Matching Relations are Equivalences: *REFLEXIVE* $Hz(A), Hz(A) : A \approx_{Hz} A$. $Mx(A), Mx(A) : A \approx_{Mx} A$. ¶ *SYMMETRIC* If $\pi, \pi' : A \approx_{Hz} B$, then $\pi', \pi : B \approx_{Hz} A$. If $\pi, \pi' : A \approx_{Mx} B$, then $\pi', \pi : B \approx_{Mx} A$. ¶ *TRANSITIVE* If $\pi, \pi' : A \approx_{Hz} B$ and $\tau, \tau' : B \approx_{Hz} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Hz} C$. If $\pi, \pi' : A \approx_{Mx} B$ and $\tau, \tau' : B \approx_{Mx} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Mx} C$.

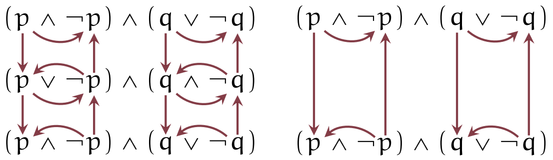
Matchings: $p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$; $p \wedge (q \vee r) \approx_{Hz} (p \wedge q) \vee (p \wedge r)$.

Mx-Matching $\subset$ *Logical Equivalence*: If an atom p occurs positively [negatively] in A but not in B, then A and B do not *Mx*-match.

PROOF: $Mx(A) : A \succ A$ contains the link from [to] that occurrence of p in the premise A to [from] its corresponding occurrence in the conclusion A. ¶ No proof from A to B contains a link from [to] that occurrence to anything in B (since there is no positive [negative] occurrence in B at all). ¶ So, in the composition proof from A to A, there is no link from [to] the premise occurrence to [from] the conclusion occurrence. No proof from A to B and back can recreate $Mx(A)$.

COROLLARY: $p \not\approx_{Mx} p \wedge (q \vee \neg q)$; $p \wedge \neg p \not\approx_{Mx} q \wedge \neg q$.

Hz-matching $\subset$ *Mx-matching*: $(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$.



However, $(p \wedge \neg p) \wedge (q \vee \neg q) \not\approx_{Hz} (p \vee \neg p) \wedge (q \wedge \neg q)$. So:

Isomorphism $\subset$ *Hz-Matching* $\subset$ *Mx-Matching* $\subset$ *Logical Equivalence*

4 MATCHING & LOGICS OF ANALYTIC CONTAINMENT

Angell's Logic of Analytic Containment: [AC1] $A \leftrightarrow \neg\neg A$ [AC2] $A \leftrightarrow (A \wedge A)$ [AC3] $(A \wedge B) \leftrightarrow (B \wedge A)$ [AC4] $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$ [AC5] $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$ [R1] $A \leftrightarrow B, C(A) \Rightarrow C(B)$

Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$. You can define $A \rightarrow B$ as $A \leftrightarrow (A \wedge B)$.

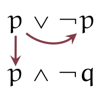
The first degree fragment of *Parry's* Logic of Analytic Containment is found by adding $(A \vee (B \wedge \neg B)) \rightarrow A$ to Angell's Logic. (Parry's logic still satisfies this relevance constraint: $A \rightarrow B$ is provable only when the atoms in B are present in A.)

First Degree Entailment (FDE) is found by adding $A \rightarrow (A \vee B)$ to Angell's Logic. ¶ FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that $p \vee \neg p$, and $q \wedge \neg q$ are both non-trivial, and ineliminable. ¶ A simple translation encodes FDE inside classical logic. Choose, for each atom p, a fresh atom p', its *shadow*. For each FDE formula A, its translation is the formula A' found by replacing the negative occurrences of atoms p in A by their shadows. An argument is FDE valid iff its translation is classically valid.

DEFINITION: $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from A to B. ¶ That is, it contains a link from {positive atoms in A, negative atoms in B} to matching {positive atoms in B, negative atoms in A}.

FACT: $Mx(A, B)$ is a proof iff there is some proof from A to B. (And if so, it is the maximal such proof.)

$Mx(p \vee \neg p, p \wedge \neg q)$ is not a proof:



LEMMA: If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

PROOF: If $\pi, \pi' : A \approx_{Mx} B$, then $\pi \subseteq Mx(A, B)$ and $\pi' \subseteq Mx(B, A)$, so $Mx(A, B)$ and $Mx(B, A)$ are both proofs. ¶ Since $\pi' \circ \pi = Mx(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$, and similarly, $Mx(B) = Mx(A, B) \circ Mx(A)$, so $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

FACT: If A is classically equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Mx*-match—and conversely.

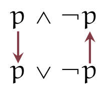
PROOF: If A is logically equivalent to B, then $Mx(A, B)$ and $Mx(B, A)$ are both proofs. ¶ It suffices to show that $Mx(B, A) \circ Mx(A, B) = Mx(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Mx(A, B)$ composed with a link in $Mx(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Mx(A, B)$ and $Mx(B, A)$. ¶ Conversely, if $A \approx_{Mx} B$, we have already seen that A and B must be equivalent, and no atom occurs positively [negatively] in A but not B.

This is *not* Equivalence in Parry's Logic. A is equivalent to B in Parry's logic of analytic containment iff A is classically equivalent to B and the atoms present in A are the atoms present in B. ¶ $(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$, but this pair satisfies Parry's variable sharing criteria.

QUESTION: Does the equivalence relation of *Mx*-matching occur elsewhere in the literature?

DEFINITION: $Hz(A, B)$ is the set of all possible linkings which could occur in any proof from A to B, excluding caps and cups. ¶ That is, it contains a link from positive atoms in A to corresponding positive atoms in B and negative atoms in A to corresponding negative atoms in B.

$Hz(p \wedge \neg p, q \vee \neg q)$ contains no links. $Hz(p \wedge \neg p, p \vee \neg p)$ is a proof, but not the maximal one:



FACT: $Hz(A, B)$ is a proof iff A entails B in FDE.

PROOF: From FDE-validity to *Hz*-proof: straightforward induction on an FDE-axiomatisation. ¶ From the *Hz*-proof $Hz(A, B)$ to FDE-validity: Notice that no negative occurrences of atoms in A or B are linked to any positive occurrences of atoms in A or B. So, there is another *Hz*-proof $Hz(A', B')$ for the FDE translations for A and B.

LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$.

PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, π and π' are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$, so $Hz(A, B)$ and $Hz(B, A)$ are both proofs. ¶ Since $\pi' \circ \pi = Hz(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$, and similarly, $Hz(B) = Hz(A, B) \circ Hz(A)$, so $Hz(A, B), Hz(B, A) : A \approx_{Mx} B$.

FACT: If A is FDE-equivalent to A, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Hz*-match — and conversely.

PROOF: If A is FDE-equivalent to B, then $Hz(A, B)$ and $Hz(B, A)$ are both proofs. ¶ It suffices to show that $Hz(B, A) \circ Hz(A, B) = Hz(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Hz(A, B)$ composed with a link in $Hz(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Hz(A, B)$ and $Hz(B, A)$. ¶ Conversely, if $A \approx_{Hz} B$, we have already seen that A and B must be FDE-equivalent, and no atom occurs positively [negatively] in A but not B.

FACT: (Ferguson 2016 [4]; Fine [5]) A is equivalent to B in Angell's logic of analytic containment iff A is FDE equivalent to B, and any atom occurs positively [negatively] in A iff it occurs positively [negatively] in B.

So, *Hz-matching* $\equiv$ *Angellic Equivalence*.

5 MATCHING AS ISOMORPHISM

$Hz(A)$ and $Mx(A)$ are Idempotents: $Hz(A) \circ Hz(A) = Hz(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.

For any category $\mathcal{C}$, if i_A is an idempotent for each object A, we can form a new category $\mathcal{C}_i$ with the same objects as $\mathcal{C}$, and with arrows $i_B \circ f \circ i_A : A \to B$. ❡ In this new category, the idempotents i_A are the new identity arrows. ❡ So, $\mathfrak{C}_{Hz}$ and $\mathfrak{C}_{Mx}$ are both categories — like $\mathfrak{C}$, but less discriminating, with fewer arrows.

Hz-matching is *isomorphism* in $\mathfrak{C}_{Hz}$.

Mx-matching is *isomorphism* in $\mathfrak{C}_{Mx}$.

$\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$ are nontrivial, nonetheless.

$$\begin{array}{ccc} p & \wedge & q \\ \downarrow & & \\ p & \vee & q \end{array} \qquad \begin{array}{ccc} p & \wedge & q \\ \downarrow & & \downarrow \\ p & \vee & q \end{array} \qquad \begin{array}{ccc} p & \wedge & q \\ & & \downarrow \\ p & \vee & q \end{array}$$

These are each different proofs in $\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$.

6 CONCLUSION

- Extending these results to include the units $\top$ and $\bot$ are not difficult. (They were left out only to ease the presentation).
- Proof theoretical resources *for classical logic* provide tools for fine-grained hyperintensional distinctions.
- One question is how to relate these results to *models* of logics of content.
- Another next step is these results to first order logic is an open question.

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