Defining Rules, Proofs and Counterexamples

Greg Restall

THE UNIVERSITY OF MELBOURNE

VII WORKSHOP ON PHILOSOPHICAL LOGIC · BUENOS AIRES · AUGUST 3, 2018

To present an account of *defining rules*, with the aim of explaining these rules they play a central role in *analytic proofs*.

Along the way, I'll explain how Kreisel's *squeezing argument* helps us understand the connection between an informal notion of validity and the notions formalised in our accounts of proofs and models, and the relationship between proof-theoretic and model-theoretic analyses of logical consequence.

Positions and Bounds

Definitions

What Proofs Are & What They Do

Counterexamples & Kreisel's Squeeze

POSITIONS AND BOUNDS

Assertions and *Denials*

$[X : Y]$

Assertions and denials are moves in a practice.

Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

I can *retract* an assertion or a denial.

Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

I can *retract* an assertion or a denial.

I can 'try on' assertion or denial hypothetically.

Assertions and denials are moves in a practice.

I can *deny* what you *assert*.

I can *retract* an assertion or a denial.

I can 'try on' assertion or denial hypothetically.

They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, *etc*.

Assertions and denials take a *stand* (*pro* or *con*) on something.

DENIAL clashes with assertion.
ASSERTION clashes with denial.

Yes/No Questions

Ask: p?

Ask: p?

YES

Ask: p?

YES

Assert p

Ask: p?

YES NO

Assert p

Ask: p?

YES	NO
Assert p	Deny p

Ask: p?

YES	NO
Assert p	Deny p

These two answers *clash*.

Strong and Weak Denial

Not all '*no*'s have the same force.

Not all ‘*no*’s have the same force.

Greg: Is Jen in the study?

Not all ‘*no*’s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Not all ‘*no*’s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She’s outside.

Not all '*no*'s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She's outside.

Greg: Jen is in the study.

Not all '*no*'s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She's outside.

Greg: Jen is in the study.

Lesley: No.

Strong and Weak Denial

Not all '*no*'s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She's outside.

Greg: Jen is in the study.

Lesley: No.

Lesley: She's either in the study or outside.

Not all '*no*'s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She's outside.

***Strong* denial**

Greg: Jen is in the study.

Lesley: No.

Lesley: She's either in the study or outside.

Not all '*no*'s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She's outside.

***Strong* denial**

Greg: Jen is in the study.

Lesley: No.

Lesley: She's either in the study or outside.

***Weak* denial**

Not all ‘*no*’s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She’s outside.

***Strong* denial**

Adds p to the common ground on the negative side

Greg: Jen is in the study.

Lesley: No.

Lesley: She’s either in the study or outside.

***Weak* denial**

Not all '*no*'s have the same force.

Greg: Is Jen in the study?

Lesley: No.

Lesley: She's outside.

***Strong* denial**

Adds p to the common ground on the negative side

Greg: Jen is in the study.

Lesley: No.

Lesley: She's either in the study or outside.

***Weak* denial**

Retracts p from the common ground on the positive side

Weak *Assertion*?

Perhaps p.

Perhaps p.

Retracts p from the common ground on the negative side?

Bounds for Positions

- IDENTITY: $[A : A]$ is out of bounds.

Bounds for Positions

- IDENTITY: $[A : A]$ is out of bounds.
- WEAKENING: If $[X : Y]$ is out of bounds, then $[X, A : Y]$ and $[X : A, Y]$ are also out of bounds.

Bounds for Positions

- IDENTITY: $[A : A]$ is out of bounds.
- WEAKENING: If $[X : Y]$ is out of bounds, then $[X, A : Y]$ and $[X : A, Y]$ are also out of bounds.
- CUT: If $[X, A : Y]$ and $[X : A, Y]$ are out of bounds, then so is $[X : Y]$.

Bounds for Positions

- IDENTITY: $[A : A]$ is out of bounds.
- WEAKENING: If $[X : Y]$ is out of bounds, then $[X, A : Y]$ and $[X : A, Y]$ are also out of bounds.
- CUT: If $[X, A : Y]$ and $[X : A, Y]$ are out of bounds, then so is $[X : Y]$.
- A position that is OUT OF BOUNDS is *overcommitted*.

On *Cut*

Suppose [X : Y] is not out of bounds.

Suppose [X : Y] is not out of bounds.

Suppose [X, A : Y] *is* out of bounds.

Suppose $[X : Y]$ is not out of bounds.

Suppose $[X, A : Y]$ *is* out of bounds.

Ask the question: A?

Suppose [X : Y] is not out of bounds.

Suppose [X, A : Y] *is* out of bounds.

Ask the question: A?

The answer *no* is forced,
as a *yes* answer is excluded
(given our other commitments).

Structural Rules

$$A \succ A \; Id \qquad \frac{X \succ A, Y \quad X', A \succ Y'}{X, X' \succ Y, Y'} \, Cut$$

$$\frac{X \succ Y}{X \succ A, Y} \, K \qquad \frac{X \succ Y}{X, A \succ Y} \, K$$

$$\frac{X \succ A, A, Y}{X \succ A, Y} \, W \qquad \frac{X, A, A \succ Y}{X, A \succ Y} \, W$$

The Power of Bounds: *Comparatives*

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity: $s >_F t, t >_F u \succ s >_F u$

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity: $s >_F t, t >_F u \succ s >_F u$

weak transitivity: $s \geqslant_F t, t \geqslant_F u \succ s \geqslant_F u$

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity: $s >_F t,\ t >_F u \succ s >_F u$

weak transitivity: $s \geqslant_F t,\ t \geqslant_F u \succ s \geqslant_F u$

strong irreflexivity: $s >_F s \succ$

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity: $s >_F t,\ t >_F u \succ s >_F u$

weak transitivity: $s \geqslant_F t,\ t \geqslant_F u \succ s \geqslant_F u$

strong irreflexivity: $s >_F s \succ$

weak reflexivity: $\succ s \geqslant_F s$

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity: $s >_F t, t >_F u \succ s >_F u$

weak transitivity: $s \geqslant_F t, t \geqslant_F u \succ s \geqslant_F u$

strong irreflexivity: $s >_F s \succ$

weak reflexivity: $\succ s \geqslant_F s$

contraries: $s >_F t, t \geqslant_F s \succ$

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity: $s >_F t, t >_F u \succ s >_F u$

weak transitivity: $s \geqslant_F t, t \geqslant_F u \succ s \geqslant_F u$

strong irreflexivity: $s >_F s \succ$

weak reflexivity: $\succ s \geqslant_F s$

contraries: $s >_F t, t \geqslant_F s \succ$

subcontraries: $\succ s >_F t, t \geqslant_F s$

The Power of Bounds: *Comparatives*

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity:	$s >_F t, t >_F u \succ s >_F u$
weak transitivity:	$s \geqslant_F t, t \geqslant_F u \succ s \geqslant_F u$
strong irreflexivity:	$s >_F s \succ$
weak reflexivity:	$\succ s \geqslant_F s$
contraries:	$s >_F t, t \geqslant_F s \succ$
subcontraries:	$\succ s >_F t, t \geqslant_F s$
strength:	$s >_F t \succ s \geqslant_F t$

The Power of Bounds: *Comparatives*

$$Fs \qquad s >_F t \qquad s \geqslant_F t$$

strong transitivity:	$s >_F t, t >_F u \succ s >_F u$
weak transitivity:	$s \geqslant_F t, t \geqslant_F u \succ s \geqslant_F u$
strong irreflexivity:	$s >_F s \succ$
weak reflexivity:	$\succ s \geqslant_F s$
contraries:	$s >_F t, t \geqslant_F s \succ$
subcontraries:	$\succ s >_F t, t \geqslant_F s$
strength:	$s >_F t \succ s \geqslant_F t$
preservation:	$Fs, t \geqslant_F s \succ Ft$

DEFINITIONS

By showing people how to *use* it.

Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

$$x \textit{ is a square} \quad =_{\mathrm{df}} \quad x \textit{ is a rectangle} \wedge \textit{ all sides of } x \textit{ are equal in length.}$$

Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

$$x \textit{ is a square} \quad =_{\mathrm{df}} \quad \begin{array}{l} x \textit{ is a rectangle } \land \\ \textit{all sides of } x \textit{ are equal in length.} \end{array}$$

Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

x *is a square* $\quad =_{\mathrm{df}} \quad$ x *is a rectangle* $\wedge$
all sides of x *are equal in length*.

Define a concept by showing how you can compose that concept out of more primitive concepts.

x *is a square* $=_{\text{df}}$ x *is a rectangle* $\wedge$ *all sides of* x *are equal in length*.

Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

$$x \textit{ is a square} \quad =_{\mathrm{df}} \quad x \textit{ is a rectangle} \land \textit{ all sides of } x \textit{ are equal in length}.$$

Concepts defined explicitly are *sharply delimited* (contingent on people accepting the definition).

Explicit Definition

Define a concept by showing how you can compose that concept out of more primitive concepts.

$$x \textit{ is a square} \quad =_{\mathrm{df}} \quad x \textit{ is a rectangle} \land \textit{all sides of } x \textit{ are equal in length}.$$

Concepts defined explicitly are *sharply delimited* (contingent on people accepting the definition).

Logical concepts are similarly sharply delimited, but they cannot *all* be given explicit definitions.

Definition through a *rule for use*

$[X, A \otimes B : Y]$ is out of bounds

if and only if

$[X, A, B : Y]$ is out of bounds

$[X, A \otimes B : Y]$ is out of bounds

if and only if

$[X, A, B : Y]$ is out of bounds

$$\frac{X, A, B \succ Y}{X, A \otimes B \succ Y}\,\wedge Df$$

What about when to *deny* a conjunction?

When do we have $X \succ A \otimes B, Y$?

What about when to *deny* a conjunction?

When do we have $X \succ A \otimes B, Y$?

$$
\dfrac{X \succ A, Y \qquad \dfrac{X' \succ B, Y' \qquad \dfrac{\dfrac{}{A \otimes B \succ A \otimes B}\,Id}{A, B \succ A \otimes B}\,\otimes Df}{X', A \succ A \otimes B, Y'}\,Cut}{X, X' \succ A \otimes B, Y, Y'}\,Cut
$$

What about when to *deny* a conjunction?

When do we have $X \succ A \otimes B, Y$?

$$\dfrac{X \succ A, Y \qquad \dfrac{X' \succ B, Y' \qquad \dfrac{\dfrac{}{A \otimes B \succ A \otimes B}\,Id}{A, B \succ A \otimes B}\,{\otimes}Df}{X', A \succ A \otimes B, Y'}\,Cut}{X, X' \succ A \otimes B, Y, Y'}\,Cut$$

So, we have

$$\dfrac{X \succ A, Y \qquad X' \succ B, Y'}{X, X' \succ A \otimes B, Y, Y'}\,{\otimes}R$$

What we've done

We have given norms governing $\otimes$ judgements
in terms of norms governing simpler judgements.

Definitions for other logical concepts

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\,\neg Df \qquad \frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y}\,{\rightarrow}Df \qquad \frac{X \succ A, B, Y}{X \succ A \oplus B, Y}\,\oplus Df$$

Definitions for other logical concepts

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg Df \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}\ {\to}Df \qquad \frac{X \succ A, B, Y}{X \succ A \oplus B, Y}\ \oplus Df$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \wedge B, Y}\ \wedge Df \qquad \frac{X, A \succ Y \quad X, B \succ Y}{X, A \vee B \succ Y}\ \vee Df$$

Definitions for other logical concepts

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg Df \qquad \frac{X, A \succ B, Y}{X \succ A \to B, Y}\ {\to} Df \qquad \frac{X \succ A, B, Y}{X \succ A \oplus B, Y}\ \oplus Df$$

$$\frac{X \succ A, Y \quad X \succ B, Y}{X \succ A \land B, Y}\ \land Df \qquad \frac{X, A \succ Y \quad X, B \succ Y}{X, A \lor B \succ Y}\ \lor Df$$

$$\frac{X \succ A|^x_n, Y}{X \succ (\forall x)A, Y}\ \forall Df \qquad \frac{X, A|^x_n \succ Y}{X, (\exists x)A \succ Y}\ \forall Df \qquad \frac{X, Fs \succ Ft, Y}{X \succ s = t, Y}\ {=}Df$$

(Where n and F are not present in X and Y.)

How do concepts defined in this way *work*?

Transforming Systems of Rules

$$*Df + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut + Id$$

Transforming Systems of Rules

$$*Df + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut + Id$$

$$*Df \leftrightarrow *L/R$$

Transforming Systems of Rules

$$*Df + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut + Id$$

$$*Df \leftrightarrow *L/R$$

$$*L/R + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut$$

Transforming Systems of Rules

$$*Df + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut + Id$$

$*Df \leftrightarrow *L/R$

$$*L/R + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut$$

Id Elimination

Transforming Systems of Rules

$$*Df + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut + Id$$

$*Df \leftrightarrow *L/R$

$$*L/R + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut$$

Id Elimination

$$*L/R + Cut \quad \Leftrightarrow \quad *L/R$$

Transforming Systems of Rules

$$*Df + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut + Id$$

$*Df \leftrightarrow *L/R$

$$*L/R + Cut + Id \quad \Leftrightarrow \quad *L/R + Cut$$

Id Elimination

$$*L/R + Cut \quad \Leftrightarrow \quad *L/R$$

Cut Elimination

Concepts defined in this way...

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)

Concepts defined in this way…

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)
- Are *conservatively extending*. (Adding a logical concept to your vocabulary in this way doesn't constrain the bounds in the original language.)

Concepts defined in this way...

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)
- Are *conservatively extending*. (Adding a logical concept to your vocabulary in this way doesn't constrain the bounds in the original language.)
- *Play useful dialogical roles*. (You can do things with these concepts that you cannot do without. Denying a conjunction does something different to denying the conjuncts.)

Concepts defined in this way...

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)
- Are *conservatively extending*. (Adding a logical concept to your vocabulary in this way doesn't constrain the bounds in the original language.)
- *Play useful dialogical roles*. (You can do things with these concepts that you cannot do without. Denying a conjunction does something different to denying the conjuncts.)
- Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)

Concepts defined in this way...

- Are *uniquely defined*. (If you and I use the same rule, we define the same concept.)
- Are *conservatively extending*. (Adding a logical concept to your vocabulary in this way doesn't constrain the bounds in the original language.)
- *Play useful dialogical roles*. (You can do things with these concepts that you cannot do without. Denying a conjunction does something different to denying the conjuncts.)
- Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)
- In Brandom's terms, they *make explicit* some of what was implicit in the practice of assertion and denial.

WHAT PROOFS ARE & WHAT THEY DO

A Tiny Proof

If it's Thursday, I'm in Melbourne.

It's Thursday.

Therefore, I'm in Melbourne.

If it's Thursday, I'm in Melbourne.

It's Thursday.

Therefore, I'm in Melbourne.

$$\dfrac{\dfrac{}{A \to B \succ A \to B}\ Id}{A \to B, A \succ B}\ {\to}Df$$

A Tiny Proof

If it's Thursday, I'm in Melbourne.

It's Thursday.

Therefore, I'm in Melbourne.

$$\dfrac{\overline{A \to B \succ A \to B}\ \mathit{Id}}{A \to B, A \succ B}\ {\to}\mathit{Df}$$

$[\mathit{It's\ Thursday} \to \mathit{I'm\ in\ Melbourne}, \mathit{It's\ Thursday} : \mathit{I'm\ in\ Melbourne}]$

(This is out of bounds.)

The *Undeniable*

Take a context in which I've asserted
it's Thursday $\to$ *I'm in Melbourne*
and I've asserted *it's Thursday*,
then *I'm in Melbourne* is *undeniable*.

The *Undeniable*

Take a context in which I've asserted
it's Thursday $\rightarrow$ *I'm in Melbourne*
and I've asserted *it's Thursday*,
then *I'm in Melbourne* is *undeniable*.

Adding the *assertion* makes explicit
what was *implicit* before that assertion.

The *Undeniable*

Take a context in which I've asserted
it's Thursday $\to$ *I'm in Melbourne*
and I've asserted *it's Thursday*,
then *I'm in Melbourne* is *undeniable*.

Adding the *assertion* makes explicit
what was *implicit* before that assertion.

The *stance* (*pro* or *con*)
on *I'm in Melbourne* was already made.

A *proof* for $X \succ Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.

A *proof* for $X \succ Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.

In this sense, proofs are *analytic*.

They apply, given the definitions, independently of the positions taken by those giving the proof.

A proof of $A, B \succ C, D$ can be seen as a *proof* of C from $[A, B : D]$,

A proof of $A, B \succ C, D$ can be seen
as a *proof* of C from $[A, B : D]$,
and a *refutation* of A from $[B : C, D]$,
and *more*.

COUNTEREXAMPLES & KREISEL'S SQUEEZE

Enlarging Positions

$$\frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\,Cut$$

$$\frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\,Cut$$

If $X \succ Y$ is not derivable
then one of $X, A \succ Y$ and $X \succ A \succ Y$
is also not derivable.

Enlarging Positions

$$\frac{X \succ A, Y \quad X, A \succ Y}{X \succ Y}\ \mathit{Cut}$$

If $X \succ Y$ is not derivable
then one of $X, A \succ Y$ and $X \succ A \succ Y$
is also not derivable.

If $[X : Y]$ is available, then
so is either $[X, A : Y]$ or $[X : A, Y]$

If $[X : Y]$ is available,
we can extend it into
a *partition* $[X' : Y']$
of the entire language.

If $[X : Y]$ is available,
we can extend it into
a *partition* $[X' : Y']$
of the entire language.

$U \succ V$ is not derivable
for any finite $U \subseteq X'$ and $V \subseteq Y'$.

Adding Witnesses

If $(\exists x)A$ is added on the left, we also add a *witness* $A|^x_n$, where n is fresh

If $(\exists x)A$ is added on the left, we also add a *witness* $A|^x_n$, where n is fresh

$$\frac{X, A|^x_n, (\exists x)A \succ Y}{X, (\exists x)A \succ Y}\ \exists Df,W$$

If $(\exists x)A$ is added on the left, we also add a *witness* $A|^x_n$, where n is fresh and similarly when $(\forall x)A$ is added on the right.

$$\frac{X, A|^x_n, (\exists x)A \succ Y}{X, (\exists x)A \succ Y}\ \exists Df,W \qquad \frac{X \succ (\forall x)A, A|^x_n, Y}{X \succ (\forall x)A, Y}\ \forall Df,W$$

Witnessed Limit Positions give rise to Models

Witnessed Limit Positions give rise to Models

$$A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y',$$

$$A \wedge B \in X' \text{ iff } A \in X' \text{ and } B \in X'.$$

$$A \vee B \in X' \text{ iff } A \in X' \text{ or } B \in X'.$$

$$A \to B \in X' \text{ iff } A \in Y' \text{ or } B \in X'.$$

$$(\forall x)A \in X' \text{ iff } A|^x_n \in X' \text{ for each name } n.$$

$$(\exists x)A \in X' \text{ iff } A|^x_n \in X' \text{ for some name } n.$$

Witnessed Limit Positions give rise to Models

$$A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y',$$

$$A \wedge B \in X' \text{ iff } A \in X' \text{ and } B \in X'.$$

$$A \vee B \in X' \text{ iff } A \in X' \text{ or } B \in X'.$$

$$A \to B \in X' \text{ iff } A \in Y' \text{ or } B \in X'.$$

$$(\forall x)A \in X' \text{ iff } A|^x_n \in X' \text{ for each name } n.$$

$$(\exists x)A \in X' \text{ iff } A|^x_n \in X' \text{ for some name } n.$$

This is a *model*, where the *true formulas* are in X' and the *false formulas* are in Y', and whose *domain* is the set of names.

Witnessed Limit Positions give rise to Models

$$A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y',$$

$$A \wedge B \in X' \text{ iff } A \in X' \text{ and } B \in X'.$$

$$A \vee B \in X' \text{ iff } A \in X' \text{ or } B \in X'.$$

$$A \to B \in X' \text{ iff } A \in Y' \text{ or } B \in X'.$$

$$(\forall x)A \in X' \text{ iff } A|^x_n \in X' \text{ for each name } n.$$

$$(\exists x)A \in X' \text{ iff } A|^x_n \in X' \text{ for some name } n.$$

This is a *model*, where the *true formulas* are in X' and the *false formulas* are in Y', and whose *domain* is the set of names.

(Things are *little* more delicate when the language contains the identity predicate.)

$X \succ Y$ is derivable
iff there is no *model*
in which each member of X is true
and each member of Y is false.

$X \succ Y$ is *informally* valid

Kreisel's *Squeeze*

$X \succ Y$ has a derivation

$$\Downarrow$$

$X \succ Y$ is *informally* valid

$$X \succ Y \text{ has a derivation}$$

$$\Downarrow$$

$$X \succ Y \text{ is } \textit{informally} \text{ valid}$$

$$\Downarrow$$

$$X \succ Y \text{ has no countermodel}$$

$X \succ Y$ has a derivation

$\Downarrow$

$X \succ Y$ is *informally* valid

$\Downarrow$

$X \succ Y$ has no countermodel

$\Downarrow$

$X \succ Y$ has a derivation.

(1) From Derivability to Informal Validity

- To say that $X \succ Y$ is *informally valid* means that is a clash involved in asserting each member of X and denying each member of Y.

(1) From Derivability to Informal Validity

- To say that $X \succ Y$ is *informally valid* means that is a clash involved in asserting each member of X and denying each member of Y.
- Axiomatic sequents ($A \succ A$) are informally valid in this sense.

(1) From Derivability to Informal Validity

- To say that $X \succ Y$ is *informally valid* means that is a clash involved in asserting each member of X and denying each member of Y.
- Axiomatic sequents ($A \succ A$) are informally valid in this sense.
- Structural rules preserve informal validity.

(1) From Derivability to Informal Validity

- To say that $X \succ Y$ is *informally valid* means that is a clash involved in asserting each member of X and denying each member of Y.
- Axiomatic sequents ($A \succ A$) are informally valid in this sense.
- Structural rules preserve informal validity.
- Defining rules *define* the connectives/quantifiers.

(2) From Informal Validity to Absence of Countermodel

(2) From Countermodel to Informal *In*validity

(2) From Countermodel to Informal *In*validity

- Refine our notion of informal validity: *Literals* (Fa, Gbc, etc.) are informally logically independent. We *ignore* logical connections between literals—we fix on informal validity *in virtue of first order logical form*.

(2) From Countermodel to Informal *In*validity

- Refine our notion of informal validity: *Literals* (Fa, Gbc, etc.) are informally logically independent. We *ignore* logical connections between literals—we fix on informal validity *in virtue of first order logical form*.
- Given a witnessed partition position $[X : Y]$ (i.e., given a model), there is no informal clash (in virtue of logical form) involved in asserting any of the literals in X and denying any in Y.

(2) From Countermodel to Informal *In*validity

- Refine our notion of informal validity: *Literals* (Fa, Gbc, etc.) are informally logically independent. We *ignore* logical connections between literals—we fix on informal validity *in virtue of first order logical form*.
- Given a witnessed partition position $[X : Y]$ (i.e., given a model), there is no informal clash (in virtue of logical form) involved in asserting any of the literals in X and denying any in Y.
- So, there is no clash involved in asserting *any* formulas in X and denying any formulas in Y, by appeal to the defining rules.

(2) From Countermodel to Informal *In*validity

- Refine our notion of informal validity: *Literals* (Fa, Gbc, etc.) are informally logically independent. We *ignore* logical connections between literals—we fix on informal validity *in virtue of first order logical form*.

- Given a witnessed partition position $[X : Y]$ (i.e., given a model), there is no informal clash (in virtue of logical form) involved in asserting any of the literals in X and denying any in Y.

- So, there is no clash involved in asserting *any* formulas in X and denying any formulas in Y, by appeal to the defining rules. (This is an induction on the depth of the structure of the formulas. The defining rules reduce clashes involving formulas into clashes involving subformulas.)

(2) From Countermodel to Informal *In*validity

- Refine our notion of informal validity: *Literals* (Fa, Gbc, etc.) are informally logically independent. We *ignore* logical connections between literals—we fix on informal validity *in virtue of first order logical form*.
- Given a witnessed partition position $[X : Y]$ (i.e., given a model), there is no informal clash (in virtue of logical form) involved in asserting any of the literals in X and denying any in Y.
- So, there is no clash involved in asserting *any* formulas in X and denying any formulas in Y, by appeal to the defining rules. (This is an induction on the depth of the structure of the formulas. The defining rules reduce clashes involving formulas into clashes involving subformulas.)
- So, a countermodel for a sequent shows *how* there is no clash involved in asserting each member of X and denying each member of Y.

(3) From Absence of Countermodel to Derivability

That's the *Completeness Theorem*.

Kreisel's *Squeeze*

$X \succ Y$ has a derivation

$$\Downarrow$$

$X \succ Y$ is *informally* valid

$$\Downarrow$$

$X \succ Y$ has no countermodel

$$\Downarrow$$

$X \succ Y$ has a derivation.

Informal validity (in virtue of first order logical form), for the language given by the defining rules, is *first order classical logic*, as given by the sequent calculus and Tarski's models.

THANK YOU!

https://consequently.org/presentation/2018/defining-rules-proofs-and-counterexamples-ba-logic-vii/

@consequently on Twitter