Terms for Classical Sequents

Proof Invariants & Strong Normalisation

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My Aim

To introduce a new *invariant*
for classical propositional proofs
and to show how they can be used.

Today's Plan

Background
Preterms
Derivations
Terms
Eliminating Cuts
Strong Normalisation
Further Work

BACKGROUND

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\lor R}{p \land q \succ p \lor q}\land L \qquad \dfrac{\dfrac{p \land q}{p}\land E}{p \lor q}\lor I \qquad \dfrac{\dfrac{p \succ p}{p \land q \succ p}\land L}{p \land q \succ p \lor q}\lor R$$

$$\dfrac{\dfrac{q \succ q}{q \succ p \lor q}\lor R}{p \land q \succ p \lor q}\land L \qquad \dfrac{\dfrac{p \land q}{q}\land E}{p \lor q}\lor I \qquad \dfrac{\dfrac{q \succ q}{p \land q \succ q}\land L}{p \land q \succ p \lor q}\lor R$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \lor q \quad \dfrac{[p]^1}{q \lor p}\lor I \quad \dfrac{[q]^1}{q \lor p}\lor I}{q \lor p}\lor E^1}{(q \lor p) \lor r}\lor I \qquad \dfrac{p \lor q \quad \dfrac{\dfrac{[p]^1}{q \lor p}\lor I}{(q \lor p) \lor r}\lor I \quad \dfrac{\dfrac{[q]^1}{q \lor p}\lor I}{(q \lor p) \lor r}\lor I}{(q \lor p) \lor r}\lor E^1$$

Are these *different proofs*, or *different ways of presenting the same proof*?

Girard, Lafont and Taylor: *Proofs and Types*, Chapter 2

Natural deduction is a slightly paradoxical system: it is limited to the intuitionistic case (in the classical case it has no particularly good properties) but it is only satisfactory for the $(\land, \Rightarrow, \forall)$ fragment of the language: we shall defer consideration of $\lor$ and $\exists$ until chapter 10. Yet disjunction and existence are the two most *typically* intuitionistic connectors!

The basic idea of natural deduction is an asymmetry: a proof is a vaguely tree-like structure (this view is more a graphical illusion than a mathematical reality, but it is a pleasant illusion) with one or more hypotheses (possibly none) but a single conclusion. The deep symmetry of the calculus is shown by the *introduction* and *elimination* rules which match each other exactly. Observe, incidentally, that with a tree-like structure, one can always decide uniquely what was the *last* rule used, which is something we could not say if there were several conclusions.

Lambda Terms and *Proofs*

$$\dfrac{\dfrac{\dfrac{\dfrac{[x : p \supset (q \supset r)] \quad [z : p]}{xz : q \supset r}\supset E \quad \dfrac{[y : p \supset q] \quad [z : p]}{yz : q}\supset E}{(xz)(yz) : r}\supset E}{\lambda z\,(xz)(yz) : p \supset r}\supset I}{\dfrac{\lambda y \lambda z\,(xz)(yz) : (p \supset q) \supset (p \supset r)}{\lambda x \lambda y \lambda z\,(xz)(yz) : (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))}\supset I}\supset I$$

Contraction and *weakening* are managed by variables

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\,{\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\,{\supset}I \qquad \dfrac{\dfrac{\dfrac{x : p \supset (p \supset q) \quad [y : p]}{xy : p \supset q}\,{\supset}E \quad [y : p]}{(xy)y : q}\,{\supset}E}{\lambda y\,(xy)y : p \supset q}\,{\supset}I$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,\lor R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,\land L$$

$$\dfrac{\dfrac{\dfrac{p \succ p \quad \dfrac{q \succ q \quad r \succ r}{q \lor r \succ q, r}\,\lor L}{p, q \lor r \succ p \land q, r}\,\land R}{p \land (q \lor r) \succ p \land q, r}\,\land L}{p \land (q \lor r) \succ (p \land q) \lor r}\,\lor R$$

Sequents and Terms

$$X \succ Y \qquad X \succ A, Y \qquad X, A \succ Y$$

Where do you put the *variables*,
and where do you put the *terms*?

Our Choice

$$\overset{\pi(x_1,\ldots,x_n)[y_1,\ldots,y_m]}{x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m}$$

Each premise and conclusion is decorated with variables.

The *sequent* gets the term, showing how inputs & outputs are connected, with as much parallelism as possible.

Example 1

$$\dfrac{\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \quad \dfrac{\overset{y\frown y}{y : q \succ y : q} \quad \overset{z\frown z}{z : r \succ z : r}}{\overset{Lw\frown y \;\; Rw\frown z}{w : q \lor r \succ y : q, z : r}}\,\lor L}{\overset{x\frown Fv \;\; Lw\frown Sv \;\; Rw\frown z}{x : p, w : q \lor r \succ v : p \land q, z : r}}\,\land R}{\overset{Fu\frown Fv \;\; LSu\frown Sv \;\; RSu\frown z}{u : p \land (q \lor r) \succ v : p \land q, z : r}}\,\land L}{\overset{Fu\frown FLt \;\; LSu\frown SLt \;\; RSu\frown Rt}{u : p \land (q \lor r) \succ t : (p \land q) \lor r}}\,\lor R$$

Example 2

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \quad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fy \;\; x\frown Sy}{x : p \succ y : p \land p}}\,\land R \quad \dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fw\frown z}{w : p \land p \succ z : p}}\,\land L}{\overset{x\frown F\bullet \;\; x\frown S\bullet \;\; F\bullet\frown z}{x : p \succ z : p}}\,Cut$$

PRETERMS

Variables and Cut Points

- For each formula A, $x_1^A, x_2^A, \ldots$ are VARIABLES of type A.
- For each formula A, $\bullet_1^A, \bullet_2^A, \ldots$ are CUT POINTS of type A.

– We use $x, y, z, u, v, w, \ldots; \bullet, \star, *, \sharp, \flat$ as schematic letters for variables and cut points, ommitting type superscripts where possible.

Nodes and Subnodes

- A variable x of type A and a cut point $\bullet$ of type A are both A NODES.
- If n is an $A \wedge B$ node, then Ln is an A node and Rn is a B node.
- If n is an $A \vee B$ node, then Fn is an A node and Sn is a B node.
- If n is an $A \supset B$ node, then An is an A node and Cn is a B node.
- If n is a $\neg A$ node, then Nn is an A node.
- For each complex node Ln, Rn, Fn, Sn, An, Cn and Nn, n is its IMMEDIATE subnode, and the subnodes of n are also subnodes of the original node.

Linkings, Inputs and Outputs

- A LINKING is a pair $n \frown m$ of nodes of the same type.
- In $n \frown m$, n is in INPUT POSITION, and m is in OUTPUT POSITION.
- Positions generalise to subnodes as follows:
  - If Ln, Rn, Fn, Sn or Cn are in *input position*, n is also in *input position*.
  - If Ln, Rn, Fn, Sn or Cn are in *output position*, n is also in *output position*.
  - – L, R, F, S and C each *preserve position*.
  - If An or Nn is in *input position*, n is in *output position*.
  - If An or Nn is in *output position*, n is in *input position*.
  - – A and N *reverse position*.
- The INPUTS (OUTPUTS) of a linking are the *variables* in INPUT (OUTPUT) position of that linking.

Example Linkings

x of type $((p \supset q) \supset p) \supset p$

$AAAx \frown Cx$

$CAx \frown Cx$

Preterms

- DEFINITION: A PRETERM is a finite set of linkings.
- – The INPUTS of a preterm are the inputs of its linkings.
- – Its OUTPUTS are the outputs of its linkings.

DERIVATIONS

Annotating Derivations: Identity

$$\overset{x \frown y}{\Sigma, x : A \succ y : A, \Delta}$$

Annotating Derivations: Conjunction

$$\frac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\underset{}{\overset{\pi(Fz, Sz)}{\Sigma, z : A \wedge B \succ \Delta}}} \wedge L \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \quad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}} \wedge R$$

Excursus on *Weakening* and *Variables*

$$\cfrac{\cfrac{[x : p]}{\lambda y\, x : q \supset p} \supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)} \supset I$$

$$\frac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz, Sz)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L \quad \text{can be} \quad \frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L$$

In a premise $\pi(x, y)$ the indicated x and y display all of the x and y inputs to the proof term.

There might be *none*.

Annotating Derivations: Negation

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\overset{\pi[Nz]}{\Sigma, z : \neg A \succ \Delta}}\,\neg L \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Nz)}{\Sigma \succ z : \neg A, \Delta}}\,\neg R$$

Annotating Derivations: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \lor B \succ \Delta, \Delta'}}\,\lor L \qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \lor B, \Delta}}\,\lor R$$

Annotating Derivations: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az] \quad \pi'(Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}}\,\supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}}\,\supset R$$

Example Annotation

$$\dfrac{\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \dfrac{\overset{y\frown y}{y : q \succ y : q} \qquad \overset{z\frown z}{z : r \succ z : r}}{\overset{Lw\frown y \quad Rw\frown z}{w : q \lor r \succ y : q, z : r}}\,\lor L}{\overset{x\frown Fv \quad Lw\frown Sv \quad Rw\frown z}{x : p, w : q \lor r \succ v : p \land q, z : r}}\,\land R}{\overset{Fu\frown Fv \quad LSu\frown Sv \quad RSu\frown z}{u : p \land (q \lor r) \succ v : p \land q, z : r}}\,\land L}{\overset{Fu\frown FLt \quad LSu\frown SLt \quad RSu\frown Rt}{u : p \land (q \lor r) \succ t : (p \land q) \lor r}}\,\lor R$$

Annotating Derivations: Cut

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet] \quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\,\mathit{Cut}$$

Identify Terms up to α equivalence

If π can be transformed into π' by relabelling cut points we treat them as identical (they are α equivalent).

Example Annotation, with *Cut*

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,\lor L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,\land R}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}}\,\mathit{Cut}$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \lor q}}\,\lor R}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\,\land L \qquad \dfrac{\dfrac{p \land q}{p}\,\land E}{p \lor q}\,\lor I \qquad \dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \land q \succ z : p}}\,\lor R}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\,\land L$$

$$\dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{w\frown Ry}{w : q \succ y : p \lor q}}\,\lor R}{\overset{Sx\frown Ry}{x : p \land q \succ y : p \lor q}}\,\land L \qquad \dfrac{\dfrac{p \land q}{q}\,\land E}{p \lor q}\,\lor I \qquad \dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{Sx\frown w}{x : p \land q \succ w : q}}\,\lor R}{\overset{Sx\frown Ry}{x : p \land q \succ y : p \lor q}}\,\land L$$

When is π_1 the same proof as π_2 (revisited)?

$$\cfrac{\cfrac{p \lor q \quad \cfrac{[p]^1}{q \lor p}\lor I \quad \cfrac{[q]^1}{q \lor p}\lor I}{q \lor p}\lor E^1}{(q \lor p) \lor r}\lor I$$

$$\cfrac{\cfrac{\cfrac{\overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Rz}{x : p \succ z : q \lor p}}\lor R \quad \cfrac{\overset{y\frown y}{y : q \succ y : q}}{\overset{y\frown Lz}{y : q \succ z : q \lor p}}\lor R}{\overset{Lw\frown Rz \quad Rw\frown Lz}{w : p \lor q \succ z : q \lor p}}\lor L}{\overset{Lw\frown RLu \quad Rw\frown LLu}{w : p \lor q \succ u : (q \lor p) \lor r}}\lor R$$

When is π_1 the same proof as π_2 (revisited)?

$$\cfrac{p \lor q \quad \cfrac{\cfrac{[p]^1}{q \lor p}\lor I}{(q \lor p) \lor r}\lor I \quad \cfrac{\cfrac{[q]^1}{q \lor p}\lor I}{(q \lor p) \lor r}\lor I}{(q \lor p) \lor r}\lor E^1$$

$$\cfrac{\cfrac{\cfrac{\overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Rz}{x : p \succ z : q \lor p}}\lor R}{\overset{x\frown RLu}{x : p \succ u : (q \lor p) \lor r}}\lor R \quad \cfrac{\cfrac{\overset{y\frown y}{y : q \succ y : q}}{\overset{x\frown LLu}{y : q \succ z : q \lor p}}\lor R}{\overset{y\frown LLu}{y : q \succ u : (q \lor p) \lor r}}\lor R}{\overset{Lw\frown RLu \quad Rw\frown LLu}{w : p \lor q \succ u : (q \lor p) \lor r}}\lor L$$

Sequentialisable Preterms

DEFINITION: A preterm is SEQUENTIALISABLE iff it is the conclusion of some derivation.

TERMS

Nonsequentialisable Preterms

$$\overset{Lx\frown Fy \quad Rx\frown Sy}{x : p \lor q \succ y : p \land q}$$

This is connected, but it is not connected *enough*.

Switching Example

$$\overset{Lx\frown Fy \quad Rx\frown Sy}{x : p \lor q \succ y : p \land q}$$

$$\overset{Lx\frown Fy \quad \cancel{Rx}\frown \cancel{Sy}}{x : p \lor - \succ y : p \land -}$$

$$\overset{Lx\frown \cancel{Fy} \quad \cancel{Rx}\frown Sy}{x : p \lor - \succ y : - \land q}$$

$$\overset{\cancel{Lx}\frown Fy \quad Rx\frown \cancel{Sy}}{x : - \lor q \succ y : p \land -}$$

$$\overset{\cancel{Lx}\frown \cancel{Fy} \quad Rx\frown Sy}{x : - \lor q \succ y : - \land q}$$

Switchings

- The SWITCHINGS of a preterm π are found by selecting for each pair of subterms Ln and Rn in *input* position; Fn and Sn in *output position*, An in *output* position and Cn in *input position*; or the cut point $\bullet$ (in both *input* and *output position*), one item of the pair to keep, and the other to DELETE.
- A LINKING in a switching of a preterm π SURVIVES if and only if neither side of the link involves a deletion.
- A preterm is SPANNED if every switching has at least one surviving linking.

Example

$$Fu\frown FLt \quad LSu\frown SLt \quad RSu\frown Rt$$

This has two pairs for switching:

LSu/RSu in *input position*. FLt/SLt in *output position*.

$$Fu\frown \cancel{FLt} \quad \cancel{LSu}\frown SLt \quad RSu\frown Rt$$

$$Fu\frown FLt \quad \cancel{LSu}\frown \cancel{SLt} \quad RSu\frown Rt$$

$$Fu\frown \cancel{FLt} \quad LSu\frown SLt \quad \cancel{RSu}\frown Rt$$

$$Fu\frown FLt \quad LSu\frown \cancel{SLt} \quad \cancel{RSu}\frown Rt$$

Terms

DEFINITION: A preterm π is a TERM when it is SPANNED.

Theorem: Sequentialisable Preterms are Terms

By induction on the derivation sequentialising π.

Sequentialisable Preterms are Terms: Identity

$$\Sigma, \overset{x \frown y}{x : A \succ y : A}, \Delta$$

Sequentialisable Preterms are Terms: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,\,Sz)}{\Sigma, z : A \land B \succ \Delta}}\land L \qquad \dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz]\quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \land B, \Delta, \Delta'}}\land R$$

Sequentialisable Preterms are Terms: Negation

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\overset{\pi[Nz]}{\Sigma, z : \neg A \succ \Delta}}\neg L \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Nz)}{\Sigma \succ z : \neg A, \Delta}}\neg R$$

Sequentialisable Preterms are Terms: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz)\quad \pi'(Rz)}{\Sigma, \Sigma', z : A \lor B \succ \Delta, \Delta'}}\lor L \qquad \dfrac{\overset{\pi[x,\,y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz,\,Rz]}{\Sigma \succ z : A \lor B, \Delta}}\lor R$$

Sequentialisable Preterms are Terms: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az]\quad \pi'(Cz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}}\supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}}\supset R$$

Sequentialisable Preterms are Terms: Cut

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet]\quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\mathit{Cut}$$

Theorem: Terms are Sequentialisable

By induction on the number of pairs for switching in π.

Except …

$$x \frown y \quad u \frown v$$

ELIMINATING CUTS

Conjunction Cut Reduction

$$\dfrac{\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[Fz]\ \ \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}}{\scriptstyle \wedge R} \qquad \dfrac{\overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \wedge B \succ \Delta''}}{\scriptstyle \wedge L}}{\overset{\pi[F\bullet]\ \ \pi'[S\bullet]\ \ \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}{\scriptstyle Cut}$$

reduces to

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \dfrac{\overset{\pi'[y]}{\Sigma' \succ y : B, \Delta} \qquad \overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi'[\star]\ \ \pi''(u,\star)}{\Sigma', \Sigma'', u : A \succ \Delta', \Delta''}}{\scriptstyle Cut}}{\overset{\pi[\ast]\ \ \pi'[\star]\ \ \pi''(\ast,\star)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}{\scriptstyle Cut}$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y \frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet]\ \ \bullet \frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}{\scriptstyle Cut}$$

reduces to

$$\overset{\pi[z]}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}$$

Difficult Cases: Contraction

$$\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{Ly \frown x\ \ Ry \frown x}{y : p \vee p \succ x : p}}{\scriptstyle \vee L} \qquad \dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Fz\ \ x \frown Sz}{x : p \succ z : p \wedge p}}{\scriptstyle \wedge R}}{\overset{Ly \frown \bullet\ \ Ry \frown \bullet\ \ \bullet \frown Fz\ \ \bullet \frown Sz}{y : p \vee p \succ z : p \wedge p}}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{Ly \frown x\ \ Ry \frown x}{y : p \vee p \succ x : p}}{\scriptstyle \vee L} \qquad \dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{Ly \frown x\ \ Ry \frown x}{y : p \vee p \succ x : p}}{\scriptstyle \vee L}}{\overset{Ly \frown Fz\ \ Ry \frown Fz\ \ Ly \frown Sz\ \ Ry \frown Sz}{y : p \vee p \succ z : p \wedge p}}{\scriptstyle \wedge R}$$

Difficult Cases: Contraction

$$\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{Ly \frown x\ \ Ry \frown x}{y : p \vee p \succ x : p}}{\scriptstyle \vee L} \qquad \dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Fz\ \ x \frown Sz}{x : p \succ z : p \wedge p}}{\scriptstyle \wedge R}}{\overset{Ly \frown \bullet\ \ Ry \frown \bullet\ \ \bullet \frown Fz\ \ \bullet \frown Sz}{y : p \vee p \succ z : p \wedge p}}{\scriptstyle Cut}$$

$$\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Fz\ \ x \frown Sz}{x : p \succ z : p \wedge p}}{\scriptstyle \wedge R} \qquad \dfrac{\overset{x \frown x}{x : p \succ x : p} \qquad \overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Fz\ \ x \frown Sz}{x : p \succ z : p \wedge p}}{\scriptstyle \wedge R}}{\overset{Ly \frown Fz\ \ Ry \frown Fz\ \ Ly \frown Sz\ \ Ry \frown Sz}{y : p \vee p \succ z : p \wedge p}}{\scriptstyle \vee L}$$

Difficult Cases: Weakening

$$\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma', y : A \succ \Delta'}}}{\overset{\pi\ \ \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}{\scriptstyle Cut}$$

$$\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma, \Sigma' \succ \Delta, \Delta'}}{\scriptstyle Weak} \qquad \dfrac{\overset{\pi}{\Sigma \succ \Delta} \qquad \overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi\ \ \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}{\scriptstyle Mix} \qquad \dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}{\scriptstyle Weak}$$

Back to Sequentialisation

$$x \frown y \quad u \frown v$$

$$\dfrac{\overset{x \frown y}{x : A \succ y : A} \qquad \overset{u \frown v}{u : B \succ v : B}}{\overset{x \frown y\ \ u \frown v}{x : A, u : B \succ y : B, v : A}}{\scriptstyle Mix}$$

Sequentialisation: Terms with *No* Switchings

The term contains no Ln, Rn, Cn and $\bullet$ in input position or Fn, Sn, An and $\bullet$ in output position.

It has a derivation using the linear rules $\land L$, $\neg L$, $\neg R$, $\lor R$ and $\supset R$ and *mixes*.

$$Fy\frown Lz \quad NRz\frown Lz \quad Sy\frown Rz$$

$$y : p\land\neg p \succ z : p\lor\neg p$$

Terms with No Switchings: Example

$$\dfrac{\dfrac{\dfrac{\overset{x\frown x}{x:p\succ x:p}}{\overset{x\frown Lz}{x:p\succ z:p\lor\neg p}}\,\lor R}{\overset{Fy\frown Lz}{y:p\land\neg p\succ z:p\lor\neg p}}\,\land L \qquad \dfrac{\dfrac{\dfrac{\overset{u\frown u}{u:p\succ u:p}}{\overset{Nv\frown u}{\succ u:p,v:\neg p}}\,\neg R}{\overset{NRz\frown Lz}{\succ z:p\lor\neg p}}\,\lor R \quad \dfrac{\dfrac{\overset{v\frown v}{v:\neg p\succ v:\neg p}}{\overset{v\frown Rz}{v:\neg p\succ z:p\lor\neg p}}\,\lor R}{\overset{Sy\frown Rz}{y:p\land\neg p\succ z:p\lor\neg p}}\,\land L}{\overset{NRz\frown Lz\ \ Sy\frown Rz}{y:p\land\neg p\succ z:p\lor\neg p}}\,\textit{Mix}}{\overset{Fy\frown Lz\ \ NRz\frown Lz\ \ Sy\frown Rz}{y:p\land\neg p\succ z:p\lor\neg p}}\,\textit{Mix}$$

Terms *with* Switchings

By induction on the number of switched pairs.

Take a switched pair at the *adjacent to variables* or *cut points* (peel away unswitched steps if there aren't any).

$$\dfrac{\overset{\pi[x](-)}{\Sigma\succ x:A,\Delta} \qquad \overset{\pi[-](y)}{\Sigma',y:B\succ\Delta'}}{\overset{\pi[Az](Lz)}{\Sigma,\Sigma',z:A\supset B\succ\Delta,\Delta'}}\,\supset L$$

Back to Eliminating Cuts: Cuts can be Complicated

$$\dfrac{\dfrac{\overset{\pi[x,u]}{\succ x:A\land B,u:A} \quad \overset{\pi'[x,v]}{\succ x:A\land B,v:B}}{\overset{\pi[x,Fx]\ \ \pi'[x,Sx]}{\succ x:A\land B}}\,\land R \qquad \dfrac{\overset{\pi''(y,z,x)}{y:A,z:B,x:A\land B\succ}}{\overset{\pi''(Fx,Sx,x)}{x:A\land B\succ}}\,\land L}{\overset{\pi[\bullet,F\bullet]\ \ \pi'[\bullet,S\bullet]\ \ \pi''(F\bullet,S\bullet,\bullet)}{\Sigma\succ\Delta}}\,\textit{Cut}$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $n\frown\bullet$ and $\bullet\frown m$ by $n\frown m$.
- *conjunction*: for each $F\bullet/S\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.

$$Sz\frown F\bullet \quad Fz\frown S\bullet \quad F\bullet\frown Sx \quad S\bullet\frown Fx \quad Ny\frown\bullet \quad \bullet\frown v$$

$$Sz\frown\star \quad Fz\frown * \quad \star\frown Sx \quad *\frown Fx \quad FNy\frown Sx \quad SNy\frown Fx \quad Ny\frown v \quad Sz\frown Fv \quad Fz\frown Sv$$

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- *negation*: for each $N\bullet$, add a new cut point $\star$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.
- *disjunction*: for each $L\bullet/R\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.
- *conditional*: for each $A\bullet/C\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.

STRONG NORMALISATION

Any reduction for π terminates in a unique* term π^*

- There is *some* terminating reduction process.
- Proof reduction is confluent.
  - If $\pi \rightsquigarrow_\bullet \pi'$ and $\pi \rightsquigarrow_\star \pi''$ then there is a π''' where $\pi' \rightsquigarrow_\star \pi'''$ and $\pi'' \rightsquigarrow_\bullet \pi'''$. (This is where α equivalence is required.)

FURTHER WORK

To Do List

- Are these genuine *invariants*? (Can we show that if two derivations have the same term, some set of permutations permute one to the other?)
- Apply these terms to other kinds of proofs (Fitch, Lemmon, tableaux, Hilbert, resolution…)
- *Categories* (The class of *single input*, *single output* terms with composition by defined by *Cut + reduction* is a category. What are its properties?)
- Apply terms to theories of warrants.
- Extend beyond propositional logic.

THANK YOU!

`https://consequently.org/presentation/2016/terms-for-classical-sequents-gothenburg/`

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