Proof Theory:
Logical and Philosophical Aspects

Class 4: Hypersequents for Modal Logics

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NASSLLI · JULY 2016 · RUTGERS

To introduce *proof theory*, with a focus in its applications in philosophy, linguistics and computer science.

Explore the behaviour of hypersequent systems for modal logics, including two dimensional modal logic with more than one modal operator.

Flat Hypersequents

Two Dimensional Modal Logic

The modal logic of equivalence relations.

The Modal Logic S5

The modal logic of equivalence relations.

Equivalently, it is the modal logic of *universal* relations.

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Equivalently, it is the modal logic of *universal* relations.

A model is a pair $\langle W, v \rangle$.

$$v_w(\Box A) = 1 \text{ iff for every } u,\ v_u(A) = 1$$

$$v_w(\Diamond A) = 1 \text{ iff for some } u,\ v_u(A) = 1$$

How can we simplify hypersequents for s5?

$$\frac{\mathcal{H}[X \vdash Y \curvearrowright X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \curvearrowright X' \vdash Y']}\;[\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \curvearrowright\ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\;[\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \curvearrowright A \vdash\ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\;[\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \curvearrowright X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \curvearrowright X' \vdash Y']}\;[\Diamond R]$$

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Eliminate the arrows!

flat hypersequents

A *flat hypersequent* is a non-empty multiset of sequents.

$$X_1 \vdash Y_1 \mid X_2 \vdash Y_2 \mid \cdots \mid X_n \vdash Y_n$$

FLAT HYPERSEQUENTS

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \curvearrowright X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \curvearrowright X' \vdash Y']}\ [\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \curvearrowright\ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\ [\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \curvearrowright A \vdash\ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\ [\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \curvearrowright X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \curvearrowright X' \vdash Y']}\ [\Diamond R]$$

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\,[\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \mid \ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\,[\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \mid A \vdash\ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\,[\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']}\,[\Diamond R]$$

There is *subtlety* here—concerning reflexivity.

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\ [\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \mid \ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\ [\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \mid A \vdash \ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\ [\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']}\ [\Diamond R]$$

There is *subtlety* here—concerning reflexivity.

In $\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ the $X \vdash Y$ and $X' \vdash Y'$ can be *the same*.

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\,[\Box L] \qquad \frac{\mathcal{H}[X', A \vdash Y']}{\mathcal{H}[X', \Box A \vdash Y']}\,[\Box L]$$

$$\frac{\mathcal{H}[X \vdash Y \;\mid\; X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \;\mid\; X' \vdash Y']}\,[\Diamond R] \qquad \frac{\mathcal{H}[X' \vdash A, Y']}{\mathcal{H}[X' \vdash \Diamond A, Y']}\,[\Diamond R]$$

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\,[\Box L] \qquad \frac{\mathcal{H}[X', A \vdash Y']}{\mathcal{H}[X', \Box A \vdash Y']}\,[\Box L]$$

$$\frac{\mathcal{H}[X \vdash Y \ \mid\ X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \ \mid\ X' \vdash Y']}\,[\Diamond R] \qquad \frac{\mathcal{H}[X' \vdash A, Y']}{\mathcal{H}[X' \vdash \Diamond A, Y']}\,[\Diamond R]$$

$\mathcal{H}[X \vdash Y \ \mid\ X' \vdash Y']$ is a hypersequent
in which $X \vdash Y$ and $X' \vdash Y'$ are components.

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\ [iKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\ [iKR]$$

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash Y \mid X' \vdash Y']}\ [eK]$$

Forms of Weakening

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\,[iKL] \qquad \frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash A, Y]}\,[iKR]$$

$$\frac{\mathcal{H}[X \vdash Y]}{\mathcal{H}[X \vdash Y \;\mid\; X' \vdash Y']}\,[eK]$$

$$\mathcal{H}[X, A \vdash A, Y] \qquad [axK]$$

Forms of Contraction

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\,[iWL] \qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\,[iWR]$$

Forms of Contraction

$$\frac{\mathcal{H}[X, A, A \vdash Y]}{\mathcal{H}[X, A \vdash Y]}\,[iWL] \qquad \frac{\mathcal{H}[X \vdash A, A, Y]}{\mathcal{H}[X \vdash A, Y]}\,[iWR]$$

$$\frac{\mathcal{H}[X \vdash Y \;\mid\; X' \vdash Y']}{\mathcal{H}[X, X' \vdash Y, Y']}\,[eWo]$$

Forms of *Cut*

$$\frac{X \vdash A, Y \mid \mathcal{H} \qquad X, A \vdash Y \mid \mathcal{H}}{X \vdash Y \mid \mathcal{H}}\ [aCut]$$

$$\frac{X \vdash A, Y \mid \mathcal{H} \qquad X', A \vdash Y' \mid \mathcal{H}'}{X, X' \vdash Y, Y' \mid \mathcal{H} \mid \mathcal{H}'}\ [mCut]$$

Example Derivation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{A \vdash A}{\Box A \vdash \ \mid\ \vdash A}\,[\Box L]
      }{\Box A, \Box B \vdash \ \mid\ \vdash A}\,[K]
      \qquad
      \dfrac{
        \dfrac{B \vdash B}{\Box B \vdash \ \mid\ \vdash B}\,[\Box L]
      }{\Box A, \Box B \vdash \ \mid\ \vdash B}\,[K]
    }{\Box A, \Box B \vdash \ \mid\ \vdash A \wedge B}\,[\wedge R]
  }{\Box A, \Box B \vdash \Box(A \wedge B)}\,[\Box R]
}{\Box A \wedge \Box B \vdash \Box(A \wedge B)}\,[\wedge R]
$$

More Example Derivations

$$
\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash \;\mid\; \vdash A}\,[\Box L]}{\Box A \vdash \;\mid\; \vdash \Box A}\,[\Box R]}{\Box A \vdash \Box\Box A}\,[\Box R]
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,[\neg L]}{\Box\neg A \vdash \;\mid\; A \vdash}\,[\Box L]}{\vdash \neg\Box\neg A \;\mid\; A \vdash}\,[\neg R]}{\vdash \neg\Box\neg A \;\mid\; A \vdash}\,[sym]}{A \vdash \Box\neg\Box\neg A}\,[\Box R]
$$

Modifying the Hypersequent Rules for S5

$$\frac{\mathcal{H}[X, \Box A \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\,[\Box L] \qquad \frac{\mathcal{H}[X \vdash \Box A, Y \ \mid\ \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]}\,[\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \ \mid\ A \vdash]}{\mathcal{H}[X, \Diamond A \vdash Y]}\,[\Diamond L] \qquad \text{‘}\ \frac{\mathcal{H}[X \vdash \Diamond A, Y \ \mid\ X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \ \mid\ X' \vdash Y']}\,[\Diamond R]$$

Height Preserving Admissibility

With these modified rules,
internal and external *weakening*,
and internal and external *contraction*,
are height-preserving admissible.

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internal and external *weakening*,
and internal and external *contraction*,
are height-preserving admissible.

The von Plato–Negri cut elimination argument
works straightforwardly for this system.
(See Poggiolesi 2008.)

$(m)Cut$ Elimination: the $\Box$ Case

$$
\dfrac{\dfrac{\dfrac{\delta_1}{X \vdash Y \mid\ \vdash A \mid \mathcal{H}}}{X \vdash \Box A, Y \mid \mathcal{H}}[\Box R] \qquad \dfrac{\dfrac{\delta_1}{X' \vdash Y' \mid X'', A \vdash Y'' \mid \mathcal{H}'}}{X', \Box A \vdash Y' \mid X'' \vdash Y'' \mid \mathcal{H}'}[\Box L]}{X, X' \vdash Y, Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}[mCut]
$$

$(m)Cut$ Elimination: the $\Box$ Case

$$\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid\; \vdash A \mid \mathcal{H}}}{X \vdash \Box A, Y \mid \mathcal{H}}\,[\Box R] \qquad \dfrac{\dfrac{\delta_l}{X' \vdash Y' \mid X'', A \vdash Y'' \mid \mathcal{H}'}}{X', \Box A \vdash Y' \mid X'' \vdash Y'' \mid \mathcal{H}'}\,[\Box L]}{X, X' \vdash Y, Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}\,[mCut]$$

simplifies to

$$\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid\; \vdash A \mid \mathcal{H}} \qquad \dfrac{\delta_r}{X' \vdash Y' \mid X'', A \vdash Y'' \mid \mathcal{H}'}}{X \vdash Y \mid X' \vdash Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}\,[mCut]}{X, X' \vdash Y, Y' \mid X'' \vdash Y'' \mid \mathcal{H} \mid \mathcal{H}'}\,[eW]$$

Hypersequent Validity

$$X_1 \vdash Y_1 \;\mid\; \cdots \;\mid\; X_n \vdash Y_n$$

holds in $\mathfrak{M}$ iff there are no worlds w_i where
each element of X_i is true at w_i
and each element of Y_i is false at w_i.

Hypersequent Validity

$$X_1 \vdash Y_1 \ | \ \cdots \ | \ X_n \vdash Y_n$$

holds in $\mathfrak{M}$ iff there are no worlds w_i where
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and each element of Y_i is false at w_i.

Equivalent *formula*:

$$\neg(\Diamond(\bigwedge X_1 \wedge \neg \bigvee Y_1) \wedge \cdots \wedge \Diamond(\bigwedge X_n \wedge \neg \bigvee Y_n))$$

$$X_1 \vdash Y_1 \ \mid \ \cdots \ \mid \ X_n \vdash Y_n$$

holds in $\mathfrak{M}$ iff there are no worlds w_i where
each element of X_i is true at w_i
and each element of Y_i is false at w_i.

Equivalent *formula*:

$$\neg(\Diamond(\bigwedge X_1 \wedge \neg \bigvee Y_1) \wedge \cdots \wedge \Diamond(\bigwedge X_n \wedge \neg \bigvee Y_n))$$

$$\Box(\bigwedge X_1 \supset \bigvee Y_1) \vee \cdots \vee \Box(\bigwedge X_n \supset \bigvee Y_n)$$

Features of this Proof System

Soundness and Completeness

Separation

Decision Procedure

Easy Extension

TWO DIMENSIONAL MODAL LOGIC

The modal logic of *universal* relations with a distinguished world $w_@$.

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A model is a pair $\langle W, v, w_@ \rangle$.

$$v_w(\Box A) = 1 \text{ iff for every } u, v_u(A) = 1$$

$$v_w(\Diamond A) = 1 \text{ iff for some } u, v_u(A) = 1$$

$$v_w(@A) = 1 \text{ iff } v_{w_@}(A) = 1$$

Hypersequents with @

$$X_1 \vdash Y_1 \;\mid\; \cdots \;\mid\; X_n \vdash Y_n$$

$$X_1 \vdash_{@} Y_1 \;\mid\; \cdots \;\mid\; X_n \vdash Y_n$$

Multisets of sequents where one (at most) is tagged with the label ‘@’.

Hypersequents with @

$$X_1 \vdash Y_1 \mid \cdots \mid X_n \vdash Y_n$$

$$X_1 \vdash_{@} Y_1 \mid \cdots \mid X_n \vdash Y_n$$

Multisets of sequents where one (at most) is tagged with the label '@'.

When you take the union of two hypersequents with @, the @-sequents in the parent hypersequents are *merged*.

$$(X_1 \vdash_{@} Y_1 \mid X_2 \vdash Y_2) \mid (X_1' \vdash_{@} Y_1' \mid X_2' \vdash Y_2') =$$

$$X_1, X_1' \vdash_{@} Y_1, Y_1' \mid X_2 \vdash Y_2 \mid X_2' \vdash Y_2'$$

Rules for the @ operator

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash_{@} Y']}{\mathcal{H}[X, @A \vdash Y \mid X' \vdash_{@} Y']}\ [@L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash_{@} A, Y']}{\mathcal{H}[X \vdash @A, Y \mid X' \vdash_{@} Y']}\ [@R]$$

@-Hypersequent Notation

$\mathcal{H}[X \vdash Y \mid X' \vdash Y']$ — a hypersequent with components $X \vdash Y$ and $X' \vdash Y'$, which may or may not be identical.

$\mathcal{H}[X \vdash Y]$ — a hypersequent with a component $X \vdash Y$, which may or may not be tagged with '@'.

$\mathcal{H}[X \vdash_! Y]$ — a hypersequent with a component $X \vdash Y$, which is *not* tagged with '@.'

$\mathcal{H}[X \vdash_@ Y]$ — a hypersequent with a component $X \vdash_@ Y$, if X or Y are non-empty.

Modal Rules

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']}\ [\Box L] \qquad \frac{\mathcal{H}[X \vdash Y \mid \ \vdash_! A]}{\mathcal{H}[X \vdash \Box A, Y]}\ [\Box R]$$

$$\frac{\mathcal{H}[X \vdash Y \mid A \vdash_! \ ]}{\mathcal{H}[\Diamond A, X \vdash Y]}\ [\Diamond L] \qquad \frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \Diamond A, Y \mid X' \vdash Y']}\ [\Diamond R]$$

Here, can't tag the $A \vdash$ component of $[\Diamond L]$
and the $\vdash A$ component of $[\Box R]$ with @.

(If we tag it, the premise is not general enough.)
We have $\vdash_@ p \supset @p$, but not $\vdash_@ \Box(p \supset @p)$.

… means that the inference step

$$\frac{\vdash_{@} A}{\vdash A}\,[@L]$$

is indeed an instance of $[@L]$ as it is specified.

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash_{@} Y']}{\mathcal{H}[X, @A \vdash Y \mid X' \vdash_{@} Y']}\,[@L]$$

Example Derivations

$$
\dfrac{\dfrac{\dfrac{p \vdash_{@} p \;\mid\; \vdash}{p \vdash_{@} \;\mid\; \vdash @p}\,[@R]}{p \vdash_{@} \Box @p}\,[\Box R]}{\vdash_{@} p \supset \Box @p}\,[\supset R]
$$

Example Derivations

$$\dfrac{\dfrac{\dfrac{p \vdash_{@} p \mid \vdash}{p \vdash_{@} \mid \vdash @p}\,[@R]}{p \vdash_{@} \Box @p}\,[\Box R]}{\vdash_{@} p \supset \Box @p}\,[\supset R]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash_{@} p \mid \vdash}{p \vdash_{@} \mid \vdash @p}\,[@R]}{p \vdash_{@} \Box @p}\,[\Box R]}{\vdash_{@} p \supset \Box @p}\,[\supset R]}{\vdash @(p \supset \Box @p)}\,[@R]}{\vdash \Box @(p \supset \Box @p)}\,[\Box R]$$

(m)*Cut* Elimination is unscathed

$$
\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid X' \vdash_{@} A, Y' \mid \mathcal{H}}}{X \vdash @A, Y \mid X' \vdash_{@} Y' \mid \mathcal{H}}[@R] \quad \dfrac{\dfrac{\delta_r}{X'' \vdash Y'' \mid X''', A \vdash_{@} Y''' \mid \mathcal{H}'}}{X'', @A \vdash Y'' \mid X''' \vdash_{@} Y''' \mid \mathcal{H}'}[@L]}{X, X'' \vdash Y, Y'' \mid X', X''' \vdash_{@} Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'}[mCut]
$$

(m)*Cut* Elimination is unscathed

$$
\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid X' \vdash_@ A, Y' \mid \mathcal{H}}}{X \vdash @A, Y \mid X' \vdash_@ Y' \mid \mathcal{H}}[@R] \quad \dfrac{\dfrac{\delta_r}{X'' \vdash Y'' \mid X''', A \vdash_@ Y''' \mid \mathcal{H}'}}{X'', @A \vdash Y'' \mid X''' \vdash_@ Y''' \mid \mathcal{H}'}[@L]}{X, X'' \vdash Y, Y'' \mid X', X''' \vdash_@ Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'}[mCut]
$$

simplifies to

$$
\dfrac{\dfrac{\dfrac{\delta_l}{X \vdash Y \mid X' \vdash_@ A, Y' \mid \mathcal{H}} \quad \dfrac{\delta_r}{X'' \vdash Y'' \mid X''', A \vdash_@ Y''' \mid \mathcal{H}'}}{X \vdash Y \mid X'' \vdash Y'' \mid X', X''' \vdash_@ Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'}[mCut]}{X, X'' \vdash Y, Y'' \mid X', X''' \vdash_@ Y', Y''' \mid \mathcal{H} \mid \mathcal{H}'}[eW]
$$

A 2D model is a pair $\langle W, v\rangle$.

$$v_{w,w'}(\Box A) = 1 \text{ iff for every } u;\ v_{u,w'}(A) = 1$$

$$v_{w,w'}(\Diamond A) = 1 \text{ iff for some } u;\ v_{u,w'}(A) = 1$$

$$v_{w,w'}(@A) = 1 \text{ iff } v_{w',w'}(A) = 1$$

Two Dimensional Modal Logic: Relativising the Actual

A 2D model is a pair $\langle W, v \rangle$.

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$$v_{w,w'}(\Diamond A) = 1 \text{ iff for some } u;\ v_{u,w'}(A) = 1$$

$$v_{w,w'}(@A) = 1 \text{ iff } v_{w',w'}(A) = 1$$

$$v_{w,w'}(\mathsf{F}A) = 1 \text{ iff for every } u,\ v_{w,u}(A) = 1$$

Martin Davies & Lloyd Humberstone

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1						
w_2						
w_3						
$\vdots$						
w_n						
$\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$					
w_2						
w_3						
$\vdots$						
w_n						
$\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$	A	A	$\cdots$	A	$\cdots$
w_2						
w_3						
$\vdots$						
w_n						
$\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A	A	A	$\cdots$	A	$\cdots$
w_2 w_3 $\vdots$ w_n $\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $\mathrm{F@}B$	A	A	$\cdots$	A	$\cdots$
w_2						
w_3						
$\vdots$						
w_n						
$\vdots$						

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $F@B$	A	A	$\cdots$	A	$\cdots$
w_2	$@B$					
w_3	$@B$					
$\vdots$	$\vdots$					
w_n	$@B$					
$\vdots$	$\vdots$					

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $F@B$ $@B$	 A	 A	 $\cdots$	 A	 $\cdots$
w_2	$@B$					
w_3	$@B$					
$\vdots$	$\vdots$					
w_n	$@B$					
$\vdots$	$\vdots$					

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$					
	A	A	A	$\cdots$	A	$\cdots$
	$F@B$					
	$@B,\ B$					
w_2	$@B$					
w_3	$@B$					
$\vdots$	$\vdots$					
w_n	$@B$					
$\vdots$	$\vdots$					

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $F@B$ $@B,\ B$	A	A	$\cdots$	A	$\cdots$
w_2	$@B$	B				
w_3	$@B$		B			
$\vdots$	$\vdots$			$\vdots$		
w_n	$@B$				B	
$\vdots$	$\vdots$					$\vdots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	A	A	$\cdots$	A	$\cdots$
w_2		B				
w_3			B			
$\vdots$				$\vdots$		
w_n					B	
$\vdots$						$\vdots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	 A	 A	 $\cdots$	 A	 $\cdots$
w_2		B				
w_3			B			
$\vdots$				$\vdots$		
w_n					B	
$\vdots$						$\vdots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	A	A	$\cdots$	A	$\cdots$
w_2		B				
w_3			B			
$\vdots$				$\vdots$		
w_n					B	
$\vdots$						$\vdots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$ A $[K]B$ B	A	A	$\cdots$	A	$\cdots$
w_2		B				
w_3			B			
$\vdots$				$\vdots$		
w_n					B	
$\vdots$						$\vdots$

$\Box$ and F@ — the *necessary* and the *fixedly actual*

	w_1	w_2	w_3	$\cdots$	w_n	$\cdots$
w_1	$\Box A$					
	A	A	A	$\cdots$	A	$\cdots$
	$[K]B$					
	B					
w_2		B				
w_3			B			
$\vdots$				$\vdots$		
w_n					B	
$\vdots$						$\vdots$

Different Alternatives

$$\Box p \vdash \quad | \quad \vdash p$$

$$[K]p \vdash \quad \| \quad \vdash_{@} p$$

In fact, we will have the following sort of derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash_{@} p}{[\mathrm{K}]p \vdash \ \| \ \vdash_{@} p}}{[\mathrm{K}]p \vdash \ | \ \vdash [\mathrm{K}]p}\,[[\mathrm{K}]R]}{[\mathrm{K}]p \vdash \Box[\mathrm{K}]p}\,[\Box R]}{\vdash [\mathrm{K}]p \supset \Box[\mathrm{K}]p}\,[\supset R]$$

2D Hypersequents

$$
\begin{array}{ccccccccc}
X_1^1 \vdash_{@} Y_1^1 & \mid & X_2^1 \vdash Y_2^1 & \mid & \cdots & \mid & X_{m_1}^1 \vdash Y_{m_1}^1 & \| \\
X_1^2 \vdash_{@} Y_1^2 & \mid & X_2^2 \vdash Y_2^2 & \mid & \cdots & \mid & X_{m_2}^2 \vdash Y_{m_2}^2 & \| \\
\vdots & & \vdots & & & & \vdots & \\
X_1^n \vdash_{@} Y_1^n & \mid & X_2^n \vdash Y_2^n & \mid & \cdots & \mid & X_{m_n}^n \vdash Y_{m_n}^n &
\end{array}
$$

2D Hypersequent Notation

$$\mathcal{H}[X \vdash Y \mid X' \vdash Y']$$

$$\mathcal{H}[X \vdash Y \parallel X' \vdash Y']$$

2D Hypersequent Rules

$$\frac{\mathcal{H}[X \vdash Y \;\parallel\; X', A \vdash_{@} Y']}{\mathcal{H}[X, [K]A \vdash Y \;\parallel\; X' \vdash_{@} Y']}\,[\text{APK}\,L]$$

$$\frac{\mathcal{H}[\,\vdash_{@} A \;\parallel\; X \vdash Y]}{\mathcal{H}[X \vdash [K]A, Y]}\,[\text{APK}\,R]$$

Example Derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{p \vdash_{@} p}{p \vdash_{@} @p}\,[@R]}{\vdash_{@} p \supset @p}\,[\supset R]}{\vdash [\mathrm{K}](p \supset @p)}\,[[\mathrm{K}]R]}{\vdash \Box[\mathrm{K}](p \supset @p)}\,[\Box R]
$$

Cut Elimination is standard

$$\dfrac{\dfrac{\dfrac{\delta_1}{\mathcal{H}[\ \vdash_@ A \ \parallel\ X \vdash Y \ \parallel\ X' \vdash_@ Y']}}{\mathcal{H}[X \vdash [K]A, Y \ \parallel\ X' \vdash_@ Y']}\,[\text{APK } R] \qquad \dfrac{\dfrac{\delta_2}{\mathcal{H}[X \vdash Y \ \parallel\ X', A \vdash_@ Y']}}{\mathcal{H}[X, [K]A \vdash Y \ \parallel\ X' \vdash_@ Y']}\,[\text{APK } L]}{\mathcal{H}[X \vdash Y \ \parallel\ X' \vdash_@ Y']}\,[aCut]$$

$$\dfrac{\dfrac{\dfrac{\delta_1}{\mathcal{H}[\ \vdash_@ A \ \parallel\ X \vdash Y \ \parallel\ X' \vdash_@ Y']} \qquad \dfrac{\delta_2}{\mathcal{H}[X \vdash Y \ \parallel\ X', A \vdash_@ Y']}}{\mathcal{H}[X \vdash Y \ \parallel\ X' \vdash_@ Y' \ \parallel\ X' \vdash_@ Y']}\,[aCut]}{\mathcal{H}[X \vdash Y \ \parallel\ X' \vdash_@ Y']}\,[eW]$$

Proof Search for invalid sequents generates models

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(For more details on this construction, see tomorrow.)

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- Is this a *virtue* or a *vice*?

What we've done

We've seen how the hypersequent calculus is not only a general technique for giving a sequent style proof theory for a range of propositional modal logics, but it can also be *tailored* to give simple proof systems for specific modal logics, with separable rules, and structural features neatly matched to the frame conditions for those logics.

Semantics and beyond

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Speech Acts and Norms

Proofs and Models

Where to go from here

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THANK YOU!

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