What Proofs are For

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My Aim

To present an account of the nature of proof, with the aim of explaining how proof could actually play the role in reasoning that it does, and answering some long-standing puzzles about the nature of proof.

Along the way, I'll explain how Kreisel's *squeezing argument* helps us understand the connection between an informal notion of validity and the notions formalised in our accounts of proofs and models, and the relationship between proof-theoretic and model-theoretic analyses of logical consequence.

Motivation

Background

What Proofs Are & What They Do

Counterexamples & Kreisel's Squeeze

Consequences for How Proofs Work

MOTIVATION

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Example Proof

Every *drink* (in our fridge) is either a *beer* or a *lemonade* $(\forall x)(Dx \supset (Bx \lor Lx))$. So either every *drink* is a *beer* or some *drink* is a *lemonade* $(\forall x)(Dx \supset Bx) \lor (\exists x)(Dx \land Lx)$. Why? Take an arbitrary *drink*. It's either a *beer* or a *lemonade*.

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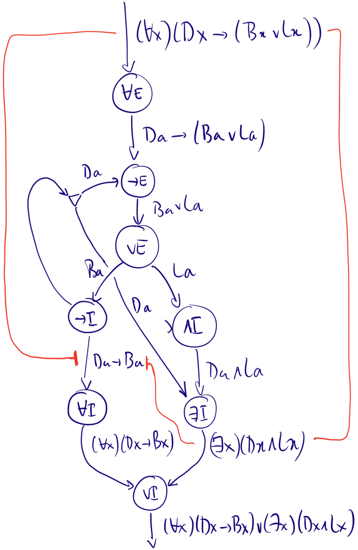
Example Proof (the formal structure)



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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{Da} \succ \mathrm{Da} \quad \dfrac{\mathrm{Da} \succ \mathrm{Da} \quad \dfrac{\mathrm{Ba} \succ \mathrm{Ba} \quad \mathrm{La} \succ \mathrm{La}}{\mathrm{Ba} \lor \mathrm{La} \succ \mathrm{Ba}, \mathrm{La}}\,{\lor}L}{\mathrm{Da} \to (\mathrm{Ba} \lor \mathrm{La}), \mathrm{Da} \succ \mathrm{Ba}, \mathrm{La}}\,{\to}L}{\mathrm{Da} \to (\mathrm{Ba} \lor \mathrm{La}), \mathrm{Da} \succ \mathrm{Ba}, \mathrm{Da} \land \mathrm{La}}\,{\land}R}{\mathrm{Da} \to (\mathrm{Ba} \lor \mathrm{La}) \succ \mathrm{Da} \to \mathrm{Ba}, \mathrm{Da} \land \mathrm{La}}\,{\to}R}{(\forall x)(\mathrm{D}x \to (\mathrm{B}x \lor \mathrm{L}x)) \succ \mathrm{Da} \to \mathrm{Ba}, \mathrm{Da} \land \mathrm{La}}\,\forall L}{(\forall x)(\mathrm{D}x \to (\mathrm{B}x \lor \mathrm{L}x)) \succ \mathrm{Da} \to \mathrm{Ba}, (\exists x)(\mathrm{D}x \land \mathrm{L}x)}\,\exists R}{(\forall x)(\mathrm{D}x \to (\mathrm{B}x \lor \mathrm{L}x)) \succ (\forall x)(\mathrm{D}x \to \mathrm{B}x), (\exists x)(\mathrm{D}x \land \mathrm{L}x)}\,\forall R}{(\forall x)(\mathrm{D}x \to (\mathrm{B}x \lor \mathrm{L}x)) \succ (\forall x)(\mathrm{D}x \to \mathrm{B}x) \lor (\exists x)(\mathrm{D}x \land \mathrm{L}x)}\,\exists R
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But what I say here can be extended to proof relying on other concepts.

Puzzles about proof

- How can proofs expand our knowledge, when the conclusion is (*in some sense*) already present in the premises?

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- How can proofs expand our knowledge, when the conclusion is (*in some sense*) already present in the premises?
- How can we be ignorant of a conclusion which logically follows from what we already know?
- What *grounds* the necessity in the connection between the premises and the conclusion?
- (Notice that these are important questions for proofs in first order predicate logic, as much as for proof more generally.)

BACKGROUND

Assertions and *Denials*

$[X : Y]$

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They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, *etc*.

Assertions and denials take a *stand* (*pro* or *con*) on something.

DENIAL clashes with assertion.
ASSERTION clashes with denial.

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- CUT: If $[X, A : Y]$ and $[X : A, Y]$ are out of bounds, then so is $[X : Y]$.
- A position that is OUT OF BOUNDS doesn't succeed in taking a stand.

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Logical concepts are similarly sharply delimited, but they cannot all be given explicit definitions.

Definition through a rule for use

$$[X, A \land B : Y] \text{ is out of bounds}$$

if and only if

$$[X, A, B : Y] \text{ is out of bounds}$$

Definition through a rule for use

$[X, A \wedge B : Y]$ is out of bounds

if and only if

$[X, A, B : Y]$ is out of bounds

$$\frac{X, A, B \succ Y}{X, A \wedge B \succ Y}\ \wedge Df$$

What about when to *deny* a conjunction?

$$\dfrac{X \succ A, Y \qquad \dfrac{X \succ B, Y \qquad \dfrac{\overline{A \land B \succ A \land B}\ \textit{Id}}{A, B \succ A \land B}\ {\land}\textit{Df}}{X, A \succ A \land B, Y}\ \textit{Cut}}{X \succ A \land B, Y}\ \textit{Cut}$$

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So, we have

$$\dfrac{X \succ A, Y \qquad X \succ B, Y}{X \succ A \land B, Y}\,{\land}R$$

Definitions for other logical concepts

$$\frac{X \succ A, Y}{X, \neg A \succ Y}\ \neg Df \qquad \frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y}\ {\rightarrow}Df \qquad \frac{X \succ A, B, Y}{X \succ A \vee B, Y}\ {\vee}Df$$

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\begin{array}{c} X \succ A, B, Y \\ \hline\hline X \succ A \vee B, Y \end{array}\ \vee Df
$$

$$
\begin{array}{c} X \succ A|^{x}_{n}, Y \\ \hline\hline X \succ (\forall x)A, Y \end{array}\ \forall Df
\qquad
\begin{array}{c} X, A|^{x}_{n} \succ Y \\ \hline\hline X, (\exists x)A \succ Y \end{array}\ \forall Df
\qquad
\begin{array}{c} X, Fs \succ Ft, Y \\ \hline\hline X \succ s = t, Y \end{array}\ {=} Df
$$

(Where n and F are not present in X and Y.)

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- Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)
- In Brandom's terms, they *make explicit* some of what was implicit in the practice of assertion and denial.

WHAT PROOFS ARE & WHAT THEY DO

A Tiny Proof

If it's Thursday, I'm in Melbourne.

It's Thursday.

Therefore, I'm in Melbourne.

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$$\dfrac{\overline{A \to B \succ A \to B}\ \textit{Id}}{A \to B, A \succ B}\ {\to}\textit{Df}$$

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$$[\mathit{It's\ Thursday} \to \mathit{I'm\ in\ Melbourne}, \mathit{It's\ Thursday} : \mathit{I'm\ in\ Melbourne}]$$

(This is out of bounds.)

The *Undeniable*

Take a context in which I've asserted
it's Thursday $\rightarrow$ *I'm in Melbourne*
and I've asserted *it's Thursday*,
then *I'm in Melbourne* is *undeniable*.

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The *stance* (*pro* or *con*)
on *I'm in Melbourne* was already made.

A *proof* for $X \succ Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.

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In this sense, proofs are *analytic*.

They apply, given the definitions, independently of the positions taken by those giving the proof.

A proof of $A, B \succ C, D$ can be seen as a *proof* of C from $[A, B : D]$, A proof of $A, B \succ C, D$ can be seen

as a *proof* of C from $[A, B : D]$,

and a *refutation* of A from $[B : C, D]$,

and *more*.

COUNTEREXAMPLES & KREISEL'S SQUEEZE

Enlarging Positions

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If $X \succ Y$ is not derivable
then one of $X, A \succ Y$ and $X \succ A \succ Y$
is also not derivable.

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If $X \succ Y$ is not derivable
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If $[X : Y]$ is available, then
so is either $[X, A : Y]$ or $[X : A, Y]$

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$U \succ V$ is not derivable
for any finite $U \subseteq X'$ and $V \subseteq Y'$.

Adding Witnesses

If $(\exists x)A$ is added on the left, we also add a *witness* $A|^x_n$, where n is fresh

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If $(\exists x)A$ is added on the left, we also add a *witness* $A|^x_n$, where n is fresh and similarly when $(\forall x)A$ is added on the right.

$$\frac{X, A|^x_n, (\exists x)A \succ Y}{X, (\exists x)A \succ Y}\ \exists Df,W \qquad \frac{X \succ (\forall x)A, A|^x_n, Y}{X \succ (\forall x)A, Y}\ \forall Df,W$$

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$$A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y',$$

$$A \wedge B \in X' \text{ iff } A \in X' \text{ and } B \in X'.$$

$$A \vee B \in X' \text{ iff } A \in X' \text{ or } B \in X'.$$

$$A \rightarrow B \in X' \text{ iff } A \in Y' \text{ or } B \in X'.$$

$$(\forall x)A \in X' \text{ iff } A|^x_n \in X' \text{ for each name } n.$$

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(Things are *little* more delicate when the language contains the identity predicate.)

$X \succ Y$ is derivable
iff there is no *model*
in which each member of X is true
and each member of Y is false.

$X \succ Y$ is *informally* valid

Kreisel's *Squeeze*

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- Axiomatic sequents ($A \succ A$) are informally valid in this sense.
- Structural rules preserve informal validity.
- Defining rules *define* the connectives/quantifiers.

(2) From Informal Validity to Absence of Countermodel

(2) From Countermodel to Informal *In*validity

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- So, there is no clash involved in asserting *any* formulas in X and denying any formulas in Y, by appeal to the defining rules. (This is an induction on the depth of the structure of the formulas. The defining rules reduce clashes involving formulas into clashes involving subformulas.)
- So, a countermodel for a sequent shows *how* there is no clash involved in asserting each member of X and denying each member of Y.

That's the *Completeness Theorem*.

$X \succ Y$ has a derivation

$\Downarrow$

$X \succ Y$ is *informally* valid

$\Downarrow$

$X \succ Y$ has no countermodel

$\Downarrow$

$X \succ Y$ has a derivation.

Informal validity (in virtue of first order logical form), for the language given by the defining rules, is *first order classical logic*, as given by the sequent calculus and Tarski's models.

CONSEQUENCES FOR HOW PROOFS WORK

Observation 0: Proofs are *definitionally analytic*

Validity is grounded in the *rules defining* the concepts used in them.

- The account of consequence does not use the concept of *truth*.

Observation 1: Proofs Preserve Truth

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- However, given plausible (minimal) assumptions concerning T, we can show that (*for example*) if $A, B \succ C$ then $T\langle A\rangle, T\langle B\rangle \succ T\langle C\rangle$.

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- This *follows from* the concepts of consequence and truth.

Observation 2: *Specification* outstrips *Recognition*

Our ability to *specify* concepts and consequence far outstrips our ability to *recognise* that consequence.

Peano Arithmetic and Goldbach's Conjecture

SUCCESSOR AXIOMS:

PA1: $\forall x \forall y (x' = y' \to x = y)$;

PA2: $\forall x (0 \neq x')$.

ADDITION AXIOMS:

PA3: $\forall x (x + 0 = x)$;

PA4: $\forall x \big(x + y' = (x + y)'\big)$.

MULTIPLICATION AXIOMS:

PA5: $\forall x (x \times 0 = 0)$;

PA6: $\forall x \forall y (x \times y' = (x \times y) + x)$.

INDUCTION SCHEME:

PA7: $(\phi(0) \land \forall x(\phi(x) \to \phi(x'))) \to \forall x \phi(x)$.

GOLDBACH'S CONJECTURE:

GC: $\forall x \exists y \exists z (\text{Prime } y \land \text{Prime } z \land 0'' \times x = y + z)$

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Verifying a putative proof is straightforward.
Checking that something *has* a proof is not so easy.

Are we logically omniscient?

Suppose that PA $\succ$ GC is derivable
(but we don't possess that proof)
and that we *know* PA.

Do we know GC?

In a weak sense of ‘know’, *yes*, we *do* know GC

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- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
- There is no epistemic possibility (no circumstance consistent with our knowledge) that leaves GC out.
- The means to come to know GC (the derivation) is “there” to be found.

In a not-so-weak sense, we don't know GC

- Do we *believe* GC?

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In a not-so-weak sense, we don't know GC

- Do we *believe* GC?
- If we do believe it, do we believe it *in the right way*?
- There "is" evidence for GC (its proof from PA, for example), but if that evidence plays no role in our belief…

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- Here, p transforms warrants for the premises into warrant for the conclusion.
- This works only for *categorical*, *conclusive* warrants (*grounds*), not for *defeasible* warrants.

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$$[\ (\exists x)(Tx \wedge Wx),\ (\forall x)(Tx \equiv (x = t_1 \vee x = t_2 \vee \cdots \vee x = t_{1\,000\,000}))\ :\ Wt_1, Wt_2, \ldots, Wt_{1\,000\,000}\ ]$$

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$$
\begin{aligned}
[\ &(\exists x)(Tx \land Wx), \\
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&\qquad\qquad\qquad\qquad : \ Wt_1, Wt_2, \ldots, Wt_{1\,000\,000}\ ]
\end{aligned}
$$

We have a very high degree of confidence in each part.

Each component is highly probable.

But the whole position is out of bounds.

Observation 4: Achilles and the Tortoise

"Well, now, let's take a little bit of the argument in that First Proposition—just *two* steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let's call them *A*, *B*, and *Z*:—

(*A*) Things that are equal to the same are equal to each other.

(*B*) The two sides of this Triangle are things that are equal to the same.

(*Z*) The two sides of this Triangle are equal to each other.

Readers of Euclid will grant, I suppose, that *Z* follows logically from *A* and *B*, so that any one who accepts *A* and *B* as true, *must* accept *Z* as true?"

"Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be till some two thousand years later—will grant *that*."

"And if some reader had *not* yet accepted *A* and *B* as true, he might still accept the *sequence* as a *valid* one, I suppose?"

Observation 4: Achilles and the Tortoise

"No doubt such a reader might exist. He might say 'I accept as true the Hypothetical Proposition that, *if* A and B be true, Z must be true; but, I *don't* accept A and B as true.' Such a reader would do wisely in abandoning Euclid, and taking to football."

"And might there not *also* be some reader who would say 'I accept A and B as true, but I *don't* accept the Hypothetical'?"

"Certainly there might. *He*, also, had better take to football."

"And *neither* of these readers," the Tortoise continued, "is *as yet* under any logical necessity to accept Z as true?"

"Quite so," Achilles assented.

"Well, now, I want you to consider *me* as a reader of the *second* kind, and to force me, logically, to accept Z as true."

"A tortoise playing football would be—" Achilles was beginning

"—an anomaly, of course," the Tortoise hastily interrupted. "Don't wander from the point. Let's have Z first, and football afterwards!"

"I'm to force you to accept Z, am I?" Achilles said musingly. "And your present position is that you accept A and B, but you *don't* accept the Hypothetical—"

"Let's call it C," said the Tortoise.

"—but you *don't* accept

(C) If A and B are true, Z must be true."

"That is my present position," said the Tortoise.

"Then I must ask you to accept C."

$$A, A \to Z \succ Z$$

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This *doesn't* mean when I accept A
and I accept $A \to Z$,
I ought to also accept Z.

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I ought to also accept Z.

However, if I assert A and $A \to Z$ then Z is *undeniable*.

If I assert A and *if* A *then* Z and *deny* Z, then I am using '*if* …*then*' in a way that deviates from the defining rule for $\rightarrow$, or I am explicitly contradicting myself.

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$$\frac{A \rightarrow B \succ A \rightarrow B}{A \rightarrow B, A \succ B} \rightarrow Df$$

Questions and Answers

If I assert A and *if* A *then* Z and ask whether Z holds?

- We need to understand connections between defining rules and norms for questions and answers, as well as assertions and denials.

Upshot

An account of the logical concepts given in terms of defining rules governing assertions and denials

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An account of the logical concepts given in terms
of defining rules governing assertions and denials

helps explain how (*first order predicate logic*) proof works,

how possessing a proof can expand our knowledge,

while proofs *make explicit* what is *implicit* in what we know.

THANK YOU!

`https://consequently.org/presentation/2018/what-proofs-are-for-melbourne-glasgow/`

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