

Isomorphisms in a Category of Proofs

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To show how a category of propositions
and *classical* proofs can give rise to
finely grained hyperintensional notions
of sameness of content.

One notion is *very* finely grained
(distinguishing p and $p \wedge p$)
others are is less finely grained.

One of these notions amounts to equivalence in
Richard B. Angell's logic of analytic containment.

To apply distinctively
prooftheoretical methods
to issues in philosophical logic.

Acknowledgements

Thanks to
Rohan French, Lloyd Humberstone,
Dave Ripley, Shawn Standefer
& the Melbourne Logic Seminar
for helpful feedback
on this material.

The Category of Classical Proofs

Isomorphisms

More Proofs from A to A

Matching & Logics of Analytic Containment

Matching as Isomorphism

THE CATEGORY OF CLASSICAL PROOFS

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

Four different derivations,

$$\frac{\frac{p \succ p}{p \wedge q \succ p} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{\frac{q \succ q}{p \wedge q \succ q} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

Four different derivations, two *proofs*

$$\frac{\frac{p \succ p}{p \wedge q \succ p} \wedge L}{p \wedge q \succ p \vee q} \vee R \approx \frac{p \wedge q}{p} \approx \frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{\frac{q \succ q}{p \wedge q \succ q} \wedge L}{p \wedge q \succ p \vee q} \vee R \approx \frac{p \wedge q}{q} \approx \frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

Proof terms are an *invariant*
for derivations under rule permutation.

δ_1 and δ_2 have the same *term* iff
some permutation sends δ_1 to δ_2 .

Four different derivations, two *proof terms*

$$\begin{array}{c}
 \frac{x \curvearrowright y}{x : p \multimap y : p} \wedge L \\
 \frac{\lambda x \curvearrowright y}{x : p \wedge q \multimap y : p} \vee R \\
 \lambda x \curvearrowright \dot{v} y \\
 x : p \wedge q \multimap y : p \vee q
 \end{array}$$

$$\lambda x \curvearrowright \dot{v} y$$

$$\begin{array}{c}
 \frac{x \curvearrowright x}{x : p \multimap y : p} \vee R \\
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 \end{array}$$

Ingredients

λ terms • flow graphs • proof nets

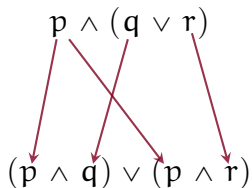
A proof term for $\Sigma \succ \Delta$
encodes the flow of information
in a proof of $\Sigma \succ \Delta$.

Proof Terms

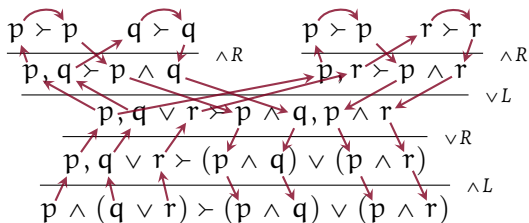
$$\begin{array}{l} \lambda x \rightarrow \lambda \forall y \quad \lambda x \rightarrow \lambda \forall y \quad \forall \lambda x \rightarrow \lambda \forall y \quad \forall \lambda x \rightarrow \lambda \forall y \\ x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r) \end{array}$$

Proof Terms as Graphs on Sequents

$\lambda x \curvearrowright \lambda \vee y \quad \lambda x \curvearrowright \lambda \vee y \quad \vee \lambda x \curvearrowright \lambda \vee y \quad \vee \lambda x \curvearrowright \lambda \vee y$
 $x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$



Finding a Proof Term from a Derivation



Finding a Proof Term from a Derivation

$$\begin{array}{c}
 \frac{p \succ p \quad q \succ q}{p, q \succ p \wedge q} \wedge R \qquad \frac{p \succ p \quad r \succ r}{p, r \succ p \wedge r} \wedge R \\
 \frac{\frac{p, q \succ p \wedge q \quad p, r \succ p \wedge r}{p, q \vee r \succ p \wedge q, p \wedge r} \vee L}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)} \vee R \\
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 \end{array}$$

$$\begin{array}{c}
 p \wedge (q \vee r) \\
 \swarrow \quad \searrow \quad \downarrow \\
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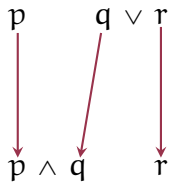
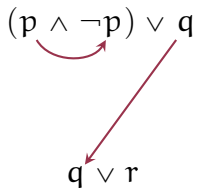
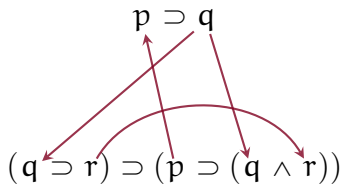
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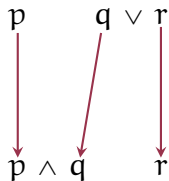
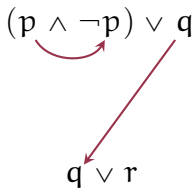
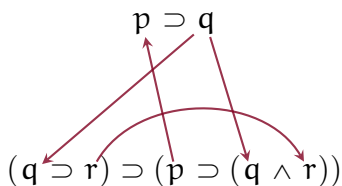
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 \end{array}$$

More Flow Graphs



More Flow Graphs



Links wholly internal to a *premise* or a *conclusion* are called *cups* (↪) and *caps* (↩).

Proof Term Facts

Not every directed graph on occurrences of atoms in a sequent is a proof term.

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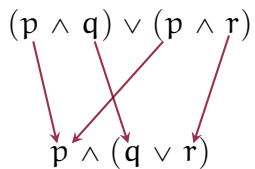
- ▶ They *typecheck*. [An occurrence of p is linked only with an occurrence of p .]
- ▶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]

Proof Term Facts

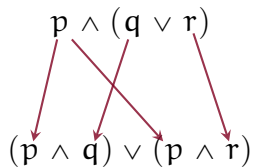
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- ▶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]
- ▶ They must satisfy an “enough connections” condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

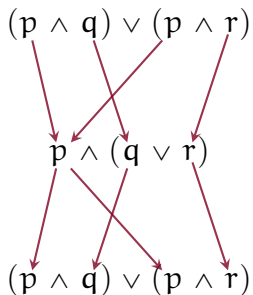
Cut is chaining of proof terms



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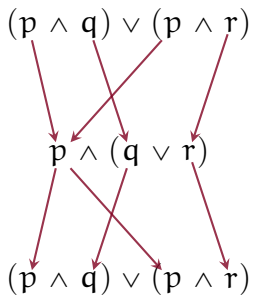


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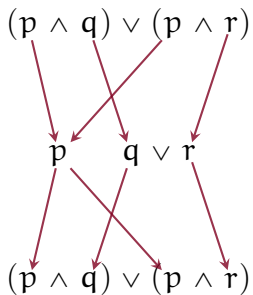


The *cut formula* is no longer a premise or a conclusion in the proof term.

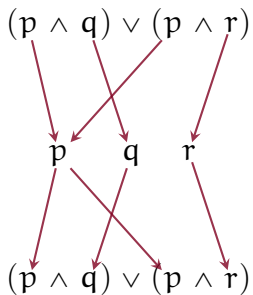
Eliminating Cuts is Local



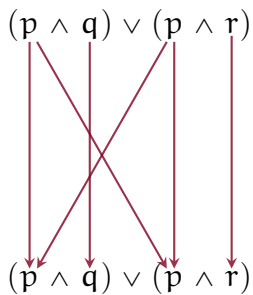
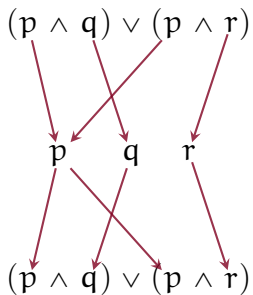
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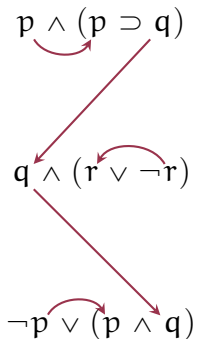
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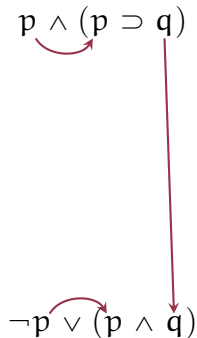
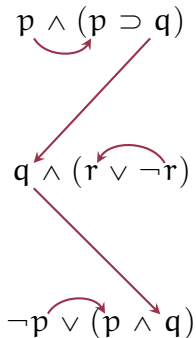
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- ▶ Cut elimination for proof terms is *local*.

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[So it can be understood as a kind of *evaluation*.]
- ▶ Cut elimination for proof terms is *local*.
[So it is easily made parallel.]

Cuts with Caps and Cups



Cuts with Caps and Cups



\mathcal{C} is the Category of Classical Proofs

OBJECTS Formulas — A, B , etc.

ARROWS Cut-Free Proof Terms — $\pi : A \multimap B$.

COMPOSITION Composition of derivations with the elimination of *Cut* — If $\pi : A \multimap B$ and $\tau : B \multimap C$ then $\tau \circ \pi : A \multimap C$.

IDENTITY Canonical identity proofs — $Id(A) : A \multimap A$.

Identity Proofs

$$\begin{array}{c}
 \frac{p \succ p}{p, \neg p \succ} \neg L \qquad \frac{p \succ p \quad p \succ p}{p \supset p, p \succ p} \supset L \\
 \frac{\neg p \succ \neg p}{\neg p \succ \neg p} \neg R \qquad \frac{p \supset p, p \succ p}{p \supset p \succ p \supset p} \supset R \\
 \hline
 \frac{\neg p \succ \neg p}{\neg p \succ \neg p} \neg R \qquad \frac{p \supset p, p \succ p}{p \supset p \succ p \supset p} \supset R \\
 \hline
 \frac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)} \vee L \\
 \hline
 \frac{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)} \vee R
 \end{array}$$

$$\neg p \vee (p \supset p)$$

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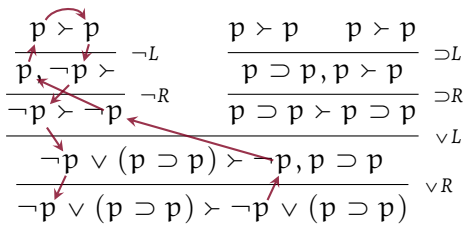
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 \frac{\neg p \succ \neg p}{\neg p \vee (p \supset p) \succ \neg p, p \supset p} \vee L \\
 \frac{\neg p \vee (p \supset p) \succ \neg p, p \supset p}{\neg p \vee (p \supset p) \succ \neg p \vee (p \supset p)} \vee R
 \end{array}$$

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Identity Proofs



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 \frac{\neg p \supset (p \supset p) \succ \neg p, p \supset p}{\neg p \supset (p \supset p) \succ \neg p \supset (p \supset p)} \vee L \\
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 \end{array}$$

Red arrows indicate the flow of information: from the top-left $p \succ p$ to the $\neg L$ rule, from the top-right $p \succ p$ to the $\supset L$ rule, from the bottom-left $\neg p \vee (p \supset p)$ to the $\vee L$ rule, and from the bottom-right $\neg p \vee (p \supset p)$ to the $\vee R$ rule. A curved red arrow also connects the two $p \succ p$ premises at the top.

$$\begin{array}{c}
 \neg p \vee (p \supset p) \\
 \uparrow \qquad \uparrow \\
 \neg p \vee (p \supset p)
 \end{array}$$

Two vertical red arrows point from the bottom expression $\neg p \vee (p \supset p)$ to the top expression $\neg p \vee (p \supset p)$, indicating a direct identity or equivalence.

Identity Proofs

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Red arrows indicate the flow of the proof: from the top-left $p \succ p$ to the $\neg L$ rule, from the top-right $p \succ p$ to the $\supset L$ rule, from the bottom-left $\neg p \vee (p \supset p)$ to the $\vee L$ rule, and from the bottom-right $\neg p \vee (p \supset p)$ to the $\vee R$ rule. A curved red arrow also connects the two $p \succ p$ premises at the top.

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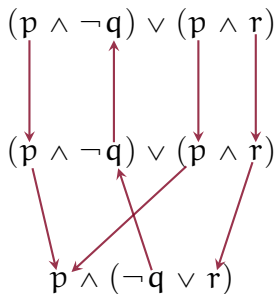
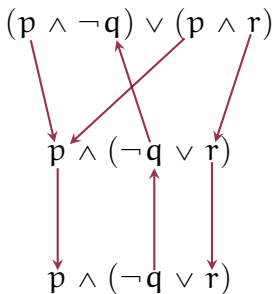
Red arrows indicate the flow of the proof: from the bottom-left $\neg p$ to the top-left $\neg p$, from the bottom-middle $(p \supset p)$ to the top-middle $(p \supset p)$, and from the bottom-right $(p \supset p)$ to the top-right $(p \supset p)$.

Identity Proofs

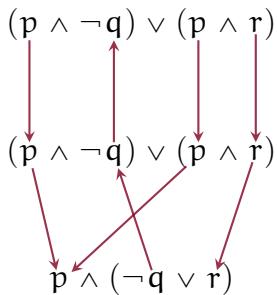
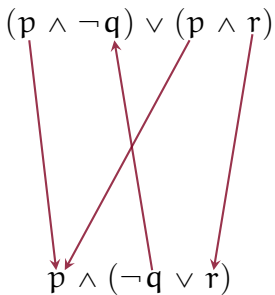
In the identity proof from A to A ,

- ▶ A *positive* occurrence of an atom in the premise linked *to* its mate in the conclusion.
- ▶ A *negative* occurrence of an atom in the premise is linked *from* its mate in the conclusion.
- ▶ There are no other links.

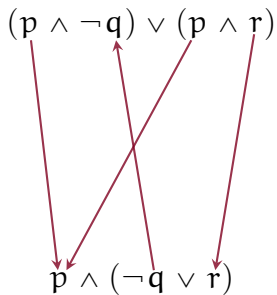
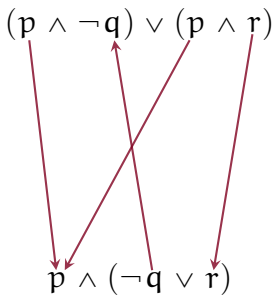
Identity and Composition in \mathcal{C}



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The Category \mathcal{C} ...

- ▶ ... is *symmetric monoidal* and *star autonomous*
- ▶ but not *Cartesian*,
- ▶ with structural *monoids* and *comonoids*,
- ▶ and is enriched in *SLat* (the category of semilattices).

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Being enriched in *SLat* means that proofs terms come ordered by \sqsubseteq , and compose under \cup , and these interact sensibly with composition.

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Being enriched in *SLat* means that proofs terms come ordered by \sqsubseteq , and compose under \cup , and these interact sensibly with composition.

$$\pi \sqsubseteq \pi' \Rightarrow \pi \circ \tau \sqsubseteq \pi' \circ \tau$$

The Category \mathcal{C} ...

- ▶ ... is *symmetric monoidal* and *star autonomous*
- ▶ but not *Cartesian*,
- ▶ with structural *monoids* and *comonoids*,
- ▶ and is enriched in *SLat* (the category of semilattices).

Being enriched in *SLat* means that proofs terms come ordered by \sqsubseteq , and compose under \cup , and these interact sensibly with composition.

$$\begin{aligned}\pi \sqsubseteq \pi' &\Rightarrow \pi \circ \tau \sqsubseteq \pi' \circ \tau \\ \tau \sqsubseteq \tau' &\Rightarrow \pi \circ \tau \sqsubseteq \pi \circ \tau'\end{aligned}$$

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(The sequent calculus plays no essential role here.
You can define proof terms on other proof systems,
e.g. *natural deduction*, *Hilbert proofs*, *tableaux*, *resolution*.)

ISOMORPHISMS

Isomorphisms in Categories

$f : A \rightarrow B$ is an *isomorphism* in a category iff
it has an *inverse* $g : B \rightarrow A$, where
 $g \circ f = id_A : A \rightarrow A$ and $f \circ g = id_B : B \rightarrow B$.

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If g and g' are both inverses, we have
 $g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g'$,
so any inverse is unique. We can call it f^{-1} .

Why Isomorphisms?

If A and B are isomorphic in a category \mathcal{C} ,
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but also, on *proofs*.

The distinctions drawn when you analyse how something is *proved*
(from premises), are not far from what you want to understand
when you ask how something is *made true*.

Isomorphisms in \mathcal{C}

$$p \wedge q \cong q \wedge p$$

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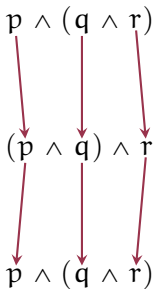


Isomorphisms in \mathcal{C}

$$p \wedge (q \wedge r) \cong (p \wedge q) \wedge r$$

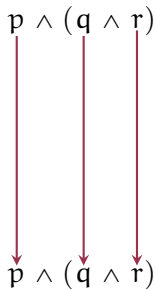
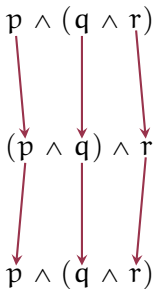
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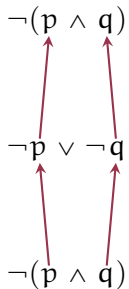


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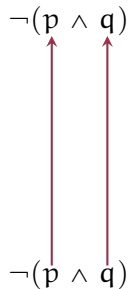
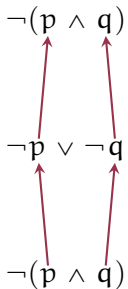
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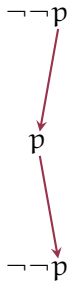


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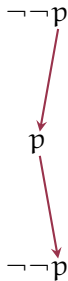
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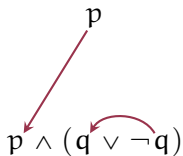
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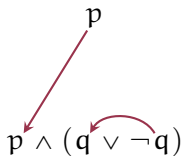


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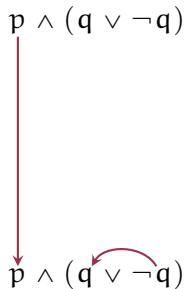
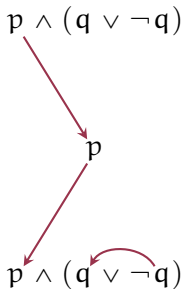
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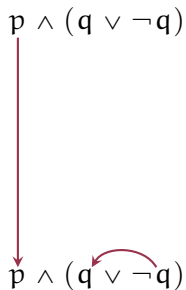
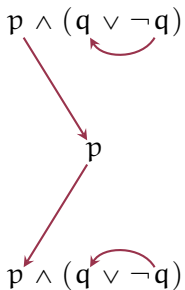
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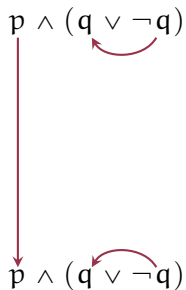
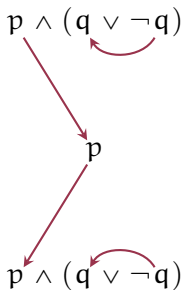
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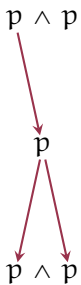
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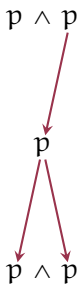
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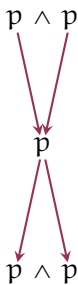
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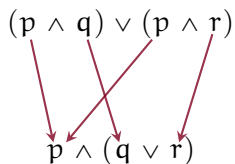
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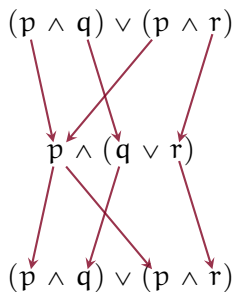
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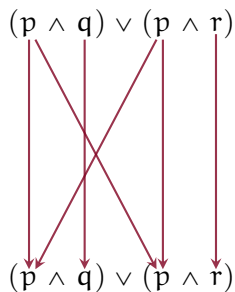
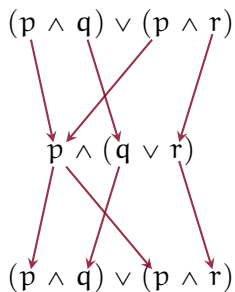
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If A is isomorphic to B in \mathfrak{C}
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If A is isomorphic to B in \mathfrak{C}
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(This condition is *necessary*, not *sufficient*: $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.)

Characterising Isomorphisms

A is isomorphic to B
iff *A* and *B* are equivalent
in the following calculus:

$$A \wedge B \leftrightarrow B \wedge A, \quad A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C.$$

$$A \vee B \leftrightarrow B \vee A, \quad A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C.$$

$$\neg(A \vee B) \leftrightarrow \neg A \wedge \neg B, \quad \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B.$$

$$\neg\neg A \leftrightarrow A. \quad A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B).$$

Characterising Isomorphisms

A is *isomorphic* to B
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This allows for a *negation normal form*, but not DNF or CNF.

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- ▶ Replace $l \wedge m$ by a new atom in both A and B , and repeat.
- ▶ This shows how to reconstruct a proof of equivalence for A and B in the syntactic calculus for isomorphic formulas.

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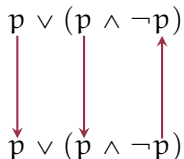
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- ▶ Not even A and $A \wedge A$ are equivalent in *this* sense.
- ▶ Yet, A and $A \wedge A$ seem to have identical *subject matter* (insofar as we understand that notion).
- ▶ Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

MORE PROOFS
FROM A TO A

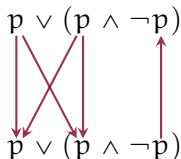
$$Id(p \vee (p \wedge \neg p))$$



In $Id(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion.
Different occurrences of atoms in A are treated differently.

$Id(A), Hz(A), Mx(A)$

$Hz(p \vee (p \wedge \neg p))$

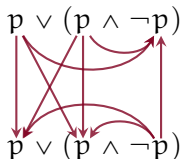


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We treat occurrences of an atom in A —with the same polarity—equally.

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$Mx(p \vee (p \wedge \neg p))$



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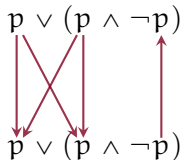
In $Hz(A)$, each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion.
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In $Mx(A)$, each syntactically possible linking is included.
We treat all occurrences of an atom in A equally.

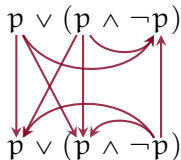
$H\mathbf{z}(\mathbf{A})$, $M\mathbf{x}(\mathbf{A})$, Caps and Cups

Note: $H\mathbf{z}(\mathbf{A})$ is $M\mathbf{x}(\mathbf{A})$ with the caps and cups removed.

$H\mathbf{z}(p \vee (p \wedge \neg p))$



$M\mathbf{x}(p \vee (p \wedge \neg p))$



Let's look at relations like isomorphism,
but which erase distinctions, up to Hx or Mx .

Hz-Matching

Let's say that A and B *Hz-MATCH*, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = \text{Hz}(A)$ and $\pi \circ \pi' = \text{Hz}(B)$.

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We write “ \approx_{Hz} ” for the *Hz*-matching relation, and we write “ $\pi, \pi' : A \approx_{\text{Hz}} B$ ” to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a *Hz*-match between A and B .

Mx-Matching

Let's say that A and B Mx -MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\tau \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$.

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If $\pi : A \succ B$ and $\pi^{-1} : B \succ A$, then
consider $\pi' = \text{Hz}(B) \circ \pi \circ \text{Hz}(A)$
and $\tau' = \text{Hz}(A) \circ \pi^{-1} \circ \text{Hz}(B)$.

These satisfy the Hz-matching criteria,
 $\tau' \circ \pi' = \text{Hz}(A)$ and $\pi' \circ \tau' = \text{Hz}(B)$.

Proof

$$\begin{aligned} Hz(A) &= Id(A) \circ Id(A) \circ Hz(A) \\ &\subseteq Hz(A) \circ Id(A) \circ Hz(A) \\ &= Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \\ &= Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \\ &\subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \\ &= (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \\ &= \tau' \circ \pi' \\ &\subseteq Hz(A) \end{aligned}$$

Proof

$$\begin{aligned} Hz(A) &= Id(A) \circ Id(A) \circ Hz(A) \\ &\subseteq Hz(A) \circ Id(A) \circ Hz(A) \\ &= Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \\ &= Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \\ &\subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \\ &= (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \\ &= \tau' \circ \pi' \\ &\subseteq Hz(A) \end{aligned}$$

...and similarly, $Hz(B) \subseteq \pi' \circ \tau' \subseteq Hz(B)$

$\text{Hz-Matching} \subseteq \text{Mx-Matching}$

If $\pi, \pi' : A \approx_{\text{Hz}} B$, then
consider $\tau = \text{Mx}(B) \circ \pi \circ \text{Mx}(A)$
and $\tau' = \text{Mx}(A) \circ \pi' \circ \text{Mx}(B)$.

These satisfy the Mx-matching criteria,
 $\tau' \circ \pi' = \text{Mx}(A)$ and $\pi' \circ \tau' = \text{Mx}(B)$.

Proof

$$\begin{aligned} Mx(A) &= Id(A) \circ Id(A) \circ Mx(A) \\ &\subseteq Mx(A) \circ Hz(A) \circ Mx(A) \\ &= Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \\ &= Mx(A) \circ (\pi' \circ Id(B) \circ Id(B) \circ \pi) \circ Mx(A) \\ &\subseteq Mx(A) \circ (\pi' \circ Mx(B) \circ Mx(B) \circ \pi) \circ Mx(A) \\ &= (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \\ &= \tau' \circ \tau \\ &\subseteq Mx(A) \end{aligned}$$

Proof

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...and similarly, $Mx(B) \subseteq \pi' \circ \tau' \subseteq Mx(B)$

If $A \approx_{Mx} B$ then there are proofs
 $\pi : A \multimap B$ and $\tau : B \multimap A$.

Matching Relations are Equivalence Relations

REFLEXIVE $Hx(A), Hx(A) : A \approx_{Hx} A.$

$Mx(A), Mx(A) : A \approx_{Mx} A.$

Matching Relations are Equivalence Relations

REFLEXIVE $\text{Hz}(A), \text{Hz}(A) : A \approx_{\text{Hz}} A.$

$\text{Mx}(A), \text{Mx}(A) : A \approx_{\text{Mx}} A.$

SYMMETRIC If $\pi, \pi' : A \approx_{\text{Hz}} B$, then $\pi', \pi : B \approx_{\text{Hz}} A.$

If $\pi, \pi' : A \approx_{\text{Mx}} B$, then $\pi', \pi : B \approx_{\text{Mx}} A.$

Matching Relations are Equivalence Relations

REFLEXIVE $H_z(A), H_z(A) : A \approx_{H_z} A$.

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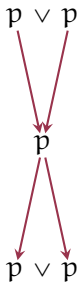
If $\pi, \pi' : A \approx_{M_x} B$, then $\pi', \pi : B \approx_{M_x} A$.

TRANSITIVE If $\pi, \pi' : A \approx_{H_z} B$ and $\tau, \tau' : B \approx_{H_z} C$, then
 $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{H_z} C$.

If $\pi, \pi' : A \approx_{M_x} B$ and $\tau, \tau' : B \approx_{M_x} C$, then
 $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{M_x} C$.

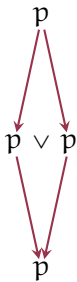
More Matchings

$$p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$$



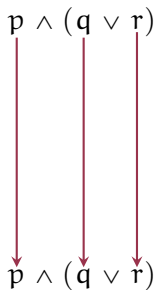
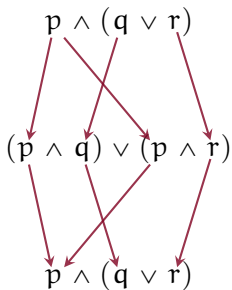
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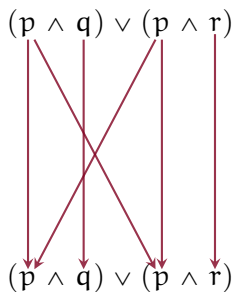
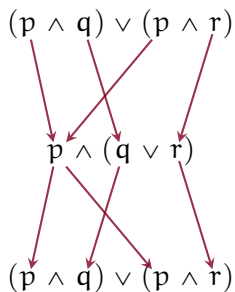
More Matchings

$$p \wedge (q \vee r) \approx_{\text{Hz}} (p \wedge q) \vee (p \wedge r)$$



More Matchings

$$p \wedge (q \vee r) \approx_{\text{Hz}} (p \wedge q) \vee (p \wedge r)$$



Mx -Matching \subset Logical Equivalence

FACT: If an atom p occurs positively in A but not in B , then A and B do not Mx -match.

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So, in the composition proof from A to A , there is no link from the premise occurrence to the conclusion occurrence. No proof from A to B and back can recreate $Mx(A)$.

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Mx -Matching \subset Logical Equivalence: Examples

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Mx -Matching \subseteq Logical Equivalence: Examples

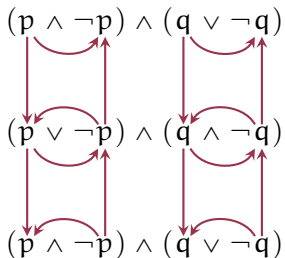
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$$p \wedge \neg p \not\approx_{Mx} q \wedge \neg q.$$

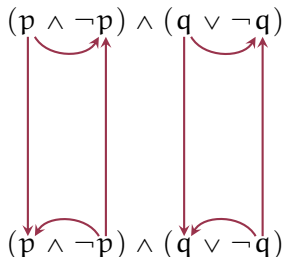
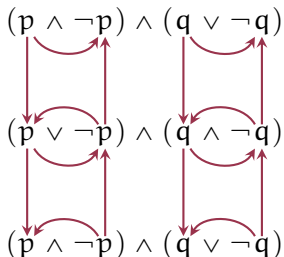
$Hz\text{-matching} \subset Mx\text{-matching}$

$$(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$$



$H\mathbf{z}$ -matching $\subset M\mathbf{x}$ -matching

$$(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{M\mathbf{x}} (p \vee \neg p) \wedge (q \wedge \neg q)$$



Isomorphism \subset Hz-Matching \subset Mx-Matching \subset Logical Equivalence

So what *are* the *matching* relations?

MATCHING & LOGICS OF ANALYTIC CONTAINMENT

Angell's Logic of Analytic Containment

$$\text{AC1 } A \leftrightarrow \neg\neg A$$

$$\text{AC2 } A \leftrightarrow (A \wedge A)$$

$$\text{AC3 } (A \wedge B) \leftrightarrow (B \wedge A)$$

$$\text{AC4 } A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$$

$$\text{AC5 } A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$$

$$\text{R1 } A \leftrightarrow B, C(A) \Rightarrow C(B)$$

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Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$.

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Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$.

You can define $A \rightarrow B$ as $A \leftrightarrow (A \wedge B)$.

Famously, $A \rightarrow (A \vee B)$ is not derivable in Angell's logic.
We cannot prove $A \leftrightarrow (A \wedge (A \vee B))$.

Extensions of Angell's Logic

- ▶ The first degree fragment of *Parry's* Logic of Analytic Containment is found by adding $(A \vee (B \wedge \neg B)) \rightarrow A$ to Angell's Logic.
 - ▶ Parry's logic still satisfies this relevance constraint: $A \rightarrow B$ is provable only when the atoms in B are present in A .
- ▶ *First Degree Entailment* (FDE) is found by adding $A \rightarrow (A \vee B)$ to Angell's Logic.
 - ▶ FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that $p \vee \neg p$, and $q \wedge \neg q$ are both non-trivial, and ineliminable.
 - ▶ A simple translation encodes FDE inside classical logic. Choose, for each atom p , a fresh atom p' , its *shadow*. For each FDE formula A , its translation is the formula A' found by replacing the negative occurrences of atoms p in A by their shadows. An argument is FDE valid iff its translation is classically valid.

$$Mx(A, B)$$

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FACT: $Mx(A, B)$ is a proof iff
there is some proof from A to B .

$Mx(A, B)$ examples

$Mx(p, q)$

p

q

$Mx(A, B)$ examples

$Mx(p, q)$

p

q

No links.

$Mx(A, B)$ examples

$Mx(p, q)$

p

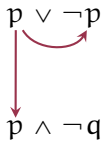
q

No links.

$Mx(p \vee \neg p, p \wedge \neg q)$

$p \vee \neg p$

$p \wedge \neg q$



$Mx(A, B)$ examples

$Mx(p, q)$

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q

No links.

$Mx(p \vee \neg p, p \wedge \neg q)$

$p \vee \neg p$

$p \wedge \neg q$

Not a proof.

$Mx(A, B)$ and matching

LEMMA: If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$

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PROOF: If $\pi, \pi' : A \approx_{Mx} B$, then $\pi \subseteq Mx(A, B)$ and $\pi' \subseteq Mx(B, A)$, so $Mx(A, B)$ and $Mx(B, A)$ are both proofs.

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Since $\pi' \circ \pi = Mx(A)$, we have
 $Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$,
and similarly, $Mx(B) = Mx(A, B) \circ Mx(B, A)$,
so $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

FACT: If A is classically logically equivalent to B ,
and all atoms occurring positively [negatively] in A
also occur positively [negatively] in B ,
and *vice versa*, then A and B Mx -match
— and conversely.

Proof

If A is logically equivalent to B , then $Mx(A, B)$ and $Mx(B, A)$ are both proofs.

It suffices to show that $Mx(B, A) \circ Mx(A, B) = Mx(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Mx(A, B)$ composed with a link in $Mx(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Mx(A, B)$ and $Mx(B, A)$.

Conversely, if $A \approx_{Mx} B$, we have already seen that A and B must be equivalent, and no atom occurs positively [negatively] in A but not B .

This is *not* Equivalence in Parry's Logic

FACT: A is equivalent to B
in Parry's logic of analytic containment iff
 A is classically equivalent to B
and the atoms present in A are present in B
and *vice versa*.

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FACT: A is equivalent to B
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$$(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$$

But this pair satisfies Parry's variable sharing criterion.

Open Question

Does the equivalence relation of Mx -matching occur elsewhere in the literature?

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That is, it contains a link from positive atoms in A to corresponding positive atoms in B and negative atoms in A to corresponding negative atoms in B .

$H_z(A, B)$ examples

$$H_z(p \wedge \neg p, q \vee \neg q)$$

$Hz(A, B)$ examples

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$$p \wedge \neg p$$

$$H_z(p \wedge \neg p, q \vee \neg q)$$

No links!

$$q \vee \neg q$$

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$$H_z(p \wedge \neg p, p \vee \neg p)$$

$$\begin{array}{ccc} p \wedge \neg p & & \neg p \\ \downarrow & & \uparrow \\ p & & p \vee \neg p \end{array}$$

$Hz(A, B)$ examples

$$Hz(p \wedge \neg p, q \vee \neg q)$$

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No links!

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$$Hz(p \wedge \neg p, p \vee \neg p)$$

$$\begin{array}{c} p \wedge \neg p \\ \downarrow \\ p \vee \neg p \end{array}$$

A proof, but not
the maximal one.

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- ▶ From FDE-validity to Hz -proof: straightforward induction on an FDE-axiomatisation.
- ▶ From the Hz -proof $Hz(A, B)$ to FDE-validity. Notice that no negative occurrences of atoms in A or B are linked to any positive occurrences of atoms in A or B . So, there is another Hz -proof $Hz(A', B')$ for the FDE translations for A and B .

$Hz(A, B)$ and Hz -matching

LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$

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PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, π and π' are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$, so $Hz(A, B)$ and $Hz(B, A)$ are both proofs.

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Since $\pi' \circ \pi = Hz(A)$, we have
 $Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$,
and similarly, $Hz(B) = Hz(A, B) \circ Hz(B, A)$,
so $Hz(A, B), Hz(B, A) : A \approx_{Mx} B$.

FACT: If A is FDE-equivalent to B ,
and all atoms occurring positively [negatively] in A
also occur positively [negatively] in B ,
and *vice versa*, then A and B *Hz*-match
— and conversely.

Proof

If A is FDE-equivalent to B , then $\text{Hz}(A, B)$ and $\text{Hz}(B, A)$ are both proofs.

It suffices to show that $\text{Hz}(B, A) \circ \text{Hz}(A, B) = \text{Hz}(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $\text{Hz}(A, B)$ composed with a link in $\text{Hz}(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $\text{Hz}(A, B)$ and $\text{Hz}(B, A)$.

Conversely, if $A \approx_{\text{Hz}} B$, we have already seen that A and B must be FDE-equivalent, and no atom occurs positively [negatively] in A but not B .

$Hz\text{-matching} \equiv \text{Angelic Equivalence}$

FACT: (Fine, Ferguson) A is equivalent to B in Angell's logic of analytic containment iff A is FDE equivalent to B , and any atom occurs positively [negatively] in A iff it occurs positively [negatively] in B .

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So, $Hz\text{-matching}$ is equivalence
in Angell's Logic of Analytic Containment.

MATCHING AS ISOMORPHISM

$H_z(A)$ and $Mx(A)$ are Idempotents

- ▶ $H_z(A) \circ H_z(A) = H_z(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.

$Hx(A)$ and $Mx(A)$ are Idempotents

- ▶ $Hx(A) \circ Hx(A) = Hx(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.
- ▶ For any category \mathcal{C} , if i_A is an idempotent for each object A , we can form a new category \mathcal{C}_i with the same objects as \mathcal{C} , and with arrows $i_B \circ f \circ i_A : A \rightarrow B$.

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- ▶ In this new category, the idempotents i_A are the new identity arrows.

$Hz(A)$ and $Mx(A)$ are Idempotents

- ▶ $Hz(A) \circ Hz(A) = Hz(A)$, $Mx(A) \circ Mx(A) = Mx(A)$.
- ▶ For any category \mathcal{C} , if i_A is an idempotent for each object A , we can form a new category \mathcal{C}_i with the same objects as \mathcal{C} , and with arrows $i_B \circ f \circ i_A : A \rightarrow B$.
- ▶ In this new category, the idempotents i_A are the new identity arrows.
- ▶ So, \mathcal{C}_{Hz} and \mathcal{C}_{Mx} are both categories — like \mathcal{C} , but less discriminating, with fewer arrows.

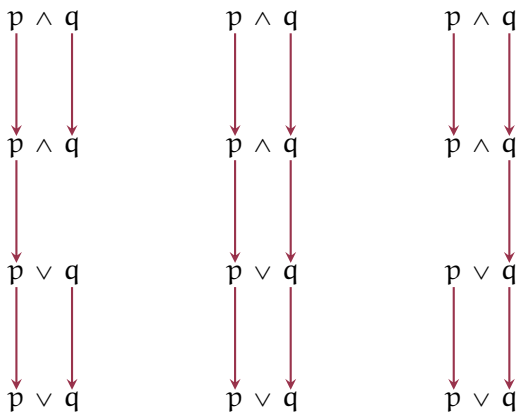
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- ▶ Relate these results to *models* of logics of content.
- ▶ Extend these results to first order logic, and beyond!

THANK YOU!

Thank you!

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or *email* at `restall@unimelb.edu.au`