Collection Frames for Substructural Logics

Greg Restall

THE UNIVERSITY OF MELBOURNE

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Joint work with Shawn Standefer

To *better understand*,
to *simplify* and to *generalise*
the ternary relational semantics
for substructural logics.

Our Plan

Ternary Relational Frames

Multiset Relations

Multiset Frames

Soundness

Completeness

Beyond Multisets

TERNARY RELATIONAL FRAMES

$$\langle P, N, \sqsubseteq, R \rangle$$

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- $N \subseteq P$
- $\sqsubseteq \subseteq P \times P$
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Ternary Relational Frames for Positive Substructural Logics

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4. $y \sqsubseteq y'$ iff $(\exists x)(Nx \wedge Rxyy')$.

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No conditions!

Binary relations are *everywhere*.

Intuitionist Frames

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Where can you find a structure like *that*?

$$\langle P, N, \sqsubseteq, R \rangle$$

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$$N \subseteq P \qquad \sqsubseteq \subseteq P \times P \qquad R \subseteq P \times P \times P$$

$$R^2(xy)zw =_{df} (\exists v)(Rxyv \land Rvzw)$$
$$R'^2x(yz)w =_{df} (\exists v)(Ryzv \land Rxvw)$$

$$
\begin{aligned}
R^2(xy)zw &=_{\mathrm{df}} (\exists v)(Rxyv \wedge Rvzw) \\
R'^2x(yz)w &=_{\mathrm{df}} (\exists v)(Ryzv \wedge Rxvw)
\end{aligned}
$$

$$R^2, R'^2 \subseteq P \times P \times P \times P$$

In RW^+

$$Rxyz \iff Ryxz$$
$$R^2(xy)zw \iff R'^2x(yz)w$$

$$Rxyz \iff Ryxz$$

$$R^2(xy)zw \iff R'^2x(yz)w$$

$$Rxxx$$

The Behaviour of N, $\sqsubseteq$ and R

$$N\,\overline{z} \qquad \underline{x} \sqsubseteq \overline{z} \qquad R\,\underline{x}\underline{y}\overline{z}$$

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- The position of an $\overline{\textit{overlined}}$ variable is closed *upwards* along $\sqsubseteq$.

The Behaviour of N, $\sqsubseteq$ and R

$$\mathsf{R}\,\overline{z} \qquad \underline{x}\,\mathsf{R}\,\overline{z} \qquad \underline{x}\,\underline{y}\,\mathsf{R}\,\overline{z}$$

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- The position of an $\overline{\textit{overlined}}$ variable is closed *upwards* along $\sqsubseteq$.

Collection Relations

$$R\,z \qquad x\,R\,z \qquad xy\,R\,z$$

$$X \; R \; z$$

X is a finite *collection* of elements of P; z is in P.

What kind of finite collection?

Trees *Lists* *Multisets* *Sets* *more …*

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MULTISET RELATIONS

(Finite) Multisets

$$[1, 2] \qquad [1, 1, 2] \qquad [1, 2, 1] \qquad [1] \qquad [\ ]$$

$$R \subseteq \mathcal{M}(P) \times P$$

Finding our Target

$$R \subseteq \mathcal{M}(P) \times P$$

R generalises $\sqsubseteq$.

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So, it should satisfy analogues of *reflexivity* and *transitivity*.

Reflxivity

$$[x] \; R \; x$$

$X \; R \; x$

$$X \; R \; x \qquad [x] \cup Y \; R \; y$$

$$X \; R \; x \qquad [x] \cup Y \; R \; y \qquad\qquad X \cup Y \; R \; y$$

Generalised Transitivity

$$(X \; R \; x \wedge [x] \cup Y \; R \; y) \Rightarrow X \cup Y \; R \; y$$

$$(X\ R\ x \wedge [x] \cup Y\ R\ y) \Rightarrow X \cup Y\ R\ y$$

$X \cup Y\ R\ y$

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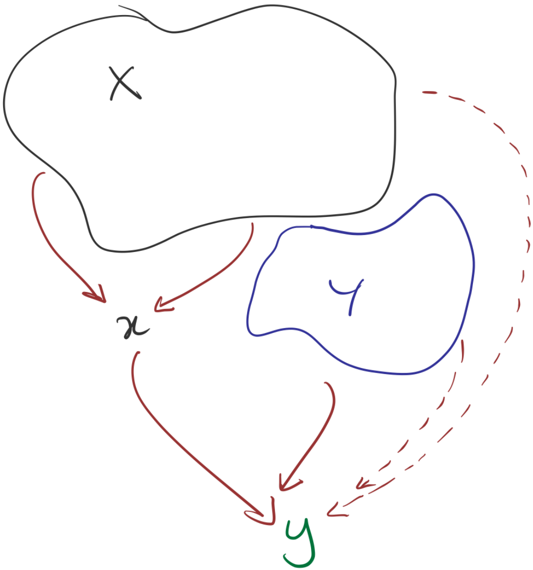
Generalised Transitivity

$$(X \; R \; x \wedge [x] \cup Y \; R \; y) \Rightarrow X \cup Y \; R \; y$$

$$X \cup Y \; R \; y \;\Rightarrow\; (\exists x)(X \; R \; x \;\wedge\; [x] \cup Y \; R \; y)$$

$$(\exists x)(X \mathrel{R} x \wedge [x] \cup Y \mathrel{R} y) \Leftrightarrow X \cup Y \mathrel{R} y$$

Left to Right



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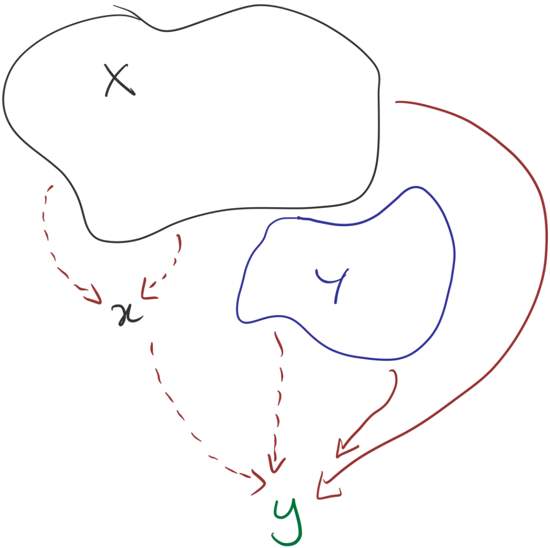

Compositional Multiset Relations

$R \subseteq \mathcal{M}(P) \times P$ is *compositional* iff for each $X, Y \in \mathcal{M}(P)$ and $y \in P$

- $[y]\ R\ y$
- $(\exists x)(X\ R\ x \wedge [x] \cup Y\ R\ y) \iff X \cup Y\ R\ y$

Examples on $\mathcal{M}(\omega) \times \omega$

$X\ R\ y$ iff …

SUM $y = \Sigma X$ (where $\Sigma[\,] = 0$)

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***SOME* SUM** for some $X' \leq X, y = \Sigma X'$

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SUM	$y = \Sigma X$ (where $\Sigma[\,] = 0$)
PRODUCT	$y = \Pi X$ (where $\Pi[\,] = 1$)
***SOME* SUM**	for some $X' \leq X$, $y = \Sigma X'$
***SOME* PROD.**	for some $X' \leq X$, $y = \Pi X'$
MAXIMUM	$y = \max(X)$ (where $\max[\,] = 0$)

Sum

$$X \mathrel{R} y \text{ iff } y = \Sigma X$$

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REFL. $n = \Sigma[n]$

Sum

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REFL. $n = \Sigma[n]$

TRANS. $y = \Sigma(X \cup Y) = \Sigma X + \Sigma Y = \Sigma([\Sigma X] \cup Y)$.

Some Product

$$X \; R \; y \text{ iff for some } X' \leq X, y = \Pi X'$$

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TRANS. $Z \leq X \cup Y$ iff for some $X' \leq X$ and $Y' \leq Y$, $Z = X' \cup Y'$,
so $X \cup Y \mathrel{R} y$ iff for some $X' \leq X$ and $Y' \leq Y$, $y = \Pi(X' \cup Y')$.
But $\Pi(X' \cup Y') = \Pi X' \times \Pi Y' = \Pi([\Pi X'] \cup Y')$, and $X \mathrel{R} \Pi X'$.

Membership?

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Membership is a compositional relation on $\mathcal{M}'(\omega) \times \omega$, on *non-empty* multisets.

Between?

$$\min(X) \leq y \leq \max(X)$$

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This is also compositional on $\mathcal{M}'(\omega) \times \omega$.

MULTISET FRAMES

Consider the binary relation $\sqsubseteq$ on P given by setting $x \sqsubseteq y$ iff $[x]\ R\ y$.

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If $[x]\ R\ y$ and $[y]\ R\ z$,
then since $[x]\ R\ y$ and $[y] \cup [\,]\ R\ z$,
we have $[x]\ R\ z$, as desired.

R respects order

$$\underline{X}\ R\ \overline{y}$$

If $x \Vdash p$ and $[x]\ R\ y$ then $y \Vdash p$

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- $x \Vdash A \lor B$ iff $x \Vdash A$ or $x \Vdash B$.
- $x \Vdash A \to B$ iff for each y, z where $[x, y]Rz$, if $y \Vdash A$ then $z \Vdash B$.

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$$[x, y]Rz \Leftrightarrow [y, x]Rz$$

$$(\exists v)([x, y]Rv \wedge [v, z]Rw) \Leftrightarrow (\exists u)([y, z]Ru \wedge [x, u]Rw)$$

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5. $Rxyz \Leftrightarrow Rxyz$
6. $(\exists v)(Rxyv \wedge Rvzw) \Leftrightarrow (\exists u)(Ryzu \wedge Rxuw)$

Multiset Frames for RW^+

$$\langle P, R \rangle$$

- P: a non-empty set
- $R \subseteq \mathcal{M}(P) \times P$

1. R is compositional. That is, $[x]\ R\ x$ and $(\exists x)(X\ R\ x \wedge [x] \cup Y\ R\ y) \Leftrightarrow X \cup Y\ R\ y$

SOUNDNESS

Soundness Proof

Standard argument, by induction on the length of a proof.

It is straightforward in a natural deduction sequent system for RW^+.

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Show that if $\Gamma \succ A$ is derivable, then for any model, if $x \Vdash \Gamma$ then $x \Vdash A$.

Soundness Proof

Standard argument, by induction on the length of a proof.

It is straightforward in a natural deduction sequent system for RW^+.

Show that if $\Gamma \succ A$ is derivable, then for any model, if $x \Vdash \Gamma$ then $x \Vdash A$.

Extend $\Vdash$ to structures by setting

$$x \Vdash \epsilon \text{ iff } [\,]\ R\ x$$

$$x \Vdash \Gamma, \Gamma' \text{ iff } x \Vdash \Gamma \text{ and } x \Vdash \Gamma'$$

$$x \Vdash \Gamma; \Gamma' \text{ iff for some } y, z \text{ where } [y, z]\ R\ x,\ y \Vdash \Gamma \text{ and } y \Vdash \Gamma'$$

COMPLETENESS

The canonical RW^+ frame is a multiset frame.

BEYOND MULTISETS

Membership, Betweenness, . . .

Non-Empty Multisets

Membership, *Betweenness*, ...

$$(\exists x)(X \mathrel{R} x \land [x] \cup Y \mathrel{R} y) \Leftrightarrow X \cup Y \mathrel{R} y$$

Membership, Betweenness, . . .

$$(\exists x)(X\ R\ x \wedge [x] \cup [\,]\ R\ y) \Leftrightarrow X \cup [\,]\ R\ y$$

Membership, *Betweenness*, . . .

$$(\exists x)(X \mathrel{R} x \land Y(x) \mathrel{R} y) \Leftrightarrow Y(X) \mathrel{R} y$$

Non-Empty Multisets

Membership, Betweenness, . . .

$$(\exists x)(X \; R \; x \land Y(x) \; R \; y) \Leftrightarrow Y(X) \; R \; y$$

If $Y(x)$ is a multiset containing x and X is a multiset, $Y(X)$ is the multiset found by *replacing* x in $Y(x)$ by X, in the natural way.

e.g., if $Y(x)$ is $[1, 2, 3, x]$ then $Y([3, 4])$ is $[1, 2, 3, 3, 4]$.

Frames on non-empty multisets model RW^+ without t.

There are *no* normal points.

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There are *no* normal points.

They model *entailment* but not *logical truth*.

(Sequents $\Gamma \succ A$ with a non-empty right hand side.)

Sets

$$R \subseteq \mathcal{P}^{\text{fin}}(P) \times P$$

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$$\{x\}\ R\ x$$

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$$\{x\}\ R\ x$$

$$(\exists x)(X\ R\ x \land Y(x)\ R\ y) \Leftrightarrow Y(X)\ R\ y$$

Contraction

Since $\{x\}\ R\ x$, we have $\{x, x\}\ R\ x$.

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Set frames are models of R^+.

OPEN QUESTION: Is the logic of set frames *exactly* R^+?

We can take collections to be *lists* (order matters) or *leaf-labelled binary trees* (associativity matters), and the generalisation works well.

We can model the Lambek Calculus (lists),

or the basic substructural logic B^+ (trees).

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We can model the Lambek Calculus (lists), or the basic substructural logic B^+ (trees).

The *empty list* is straightforward and natural.

The *empty tree* is less straightforward.

(To get the logic B^+ take the empty tree to be a *left* but not a *right* identity.)

There is a general mathematical theory of finite structures.
(The theory of *species*.)

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What *other* finite structures give rise
to natural logics like these?

The Upshot

- The collection of conditions on N, $\sqsubseteq$, R in ternary frames are not *ad hoc*, but arise out of a single underlying phenomenon, the *compositional relation*.

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- Identifying compositional relations on structures is a way to look for *natural* models of substructural logics.

The Upshot

- The collection of conditions on N, $\sqsubseteq$, R in ternary frames are not *ad hoc*, but arise out of a single underlying phenomenon, the *compositional relation*.
- Identifying compositional relations on structures is a way to look for *natural* models of substructural logics.
- Different logics are found by varying the *collections* being related, whether sets, multisets, lists, trees or something else.

THANK YOU!