

# Isomorphisms in a Category of Propositions and Proofs

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To show how a category of propositions  
and *classical* proofs can give rise to  
finely grained hyperintensional notions  
of sameness of content.

One notion is *very* finely grained  
(distinguishing  $p$  and  $p \wedge p$ )  
others are is less finely grained.

Another notion amounts to equivalence in  
R. B. Angell's logic of analytic containment.

To apply distinctively  
*prooftheoretical* methods  
to issues in philosophical logic.

## Acknowledgements

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Thanks to  
*Rohan French,*  
*Dave Ripley,* and  
*Shawn Standefer* for  
helpful conversations  
on this material.

The Category of Classical Proofs

Isomorphisms

More Proofs from  $A$  to  $A$

Matching & Logics of Analytic Containment

Matching as Isomorphism

# THE CATEGORY OF CLASSICAL PROOFS

There can be different ways to prove the same thing

$$p \wedge q \succ p \vee q$$

## Four different derivations,

$$\frac{\frac{p \succ p}{p \wedge q \succ p} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{\frac{q \succ q}{p \wedge q \succ q} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$



## Four different derivations, two *proofs*

$$\frac{\frac{p \succ p}{p \wedge q \succ p} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{p \wedge q}{p} \wedge E$$

$$\frac{\frac{p \succ p}{p \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

$$\frac{\frac{q \succ q}{p \wedge q \succ q} \wedge L}{p \wedge q \succ p \vee q} \vee R$$

$$\frac{p \wedge q}{q} \wedge E$$

$$\frac{\frac{q \succ q}{q \succ p \vee q} \vee R}{p \wedge q \succ p \vee q} \wedge L$$

*Proof terms* are an *invariant*  
for derivations under rule permutation.

$\delta_1$  and  $\delta_2$  have the same *term* iff  
some permutation sends  $\delta_1$  to  $\delta_2$ .

## Four different derivations, two *proof terms*

$$\begin{array}{c}
 \frac{x \curvearrowright y}{x : p \multimap y : p} \wedge L \\
 \frac{x : p \wedge q \multimap y : p}{\wedge x \curvearrowright \vee y} \vee R \\
 x : p \wedge q \multimap y : p \vee q
 \end{array}$$

$$\wedge x \curvearrowright \vee y$$

$$\begin{array}{c}
 \frac{x \curvearrowright x}{x : p \multimap y : p} \vee R \\
 \frac{x \curvearrowright \vee y}{x : p \multimap y : p \vee q} \wedge L \\
 x : p \wedge q \multimap y : p \vee q
 \end{array}$$

$$\begin{array}{c}
 \frac{x \curvearrowright y}{x : q \multimap y : q} \wedge L \\
 \frac{x : p \wedge q \multimap y : q}{\lambda x \curvearrowright \vee y} \vee R \\
 x : p \wedge q \multimap y : p \vee q
 \end{array}$$

$$\lambda x \curvearrowright \vee y$$

$$\begin{array}{c}
 \frac{x \curvearrowright y}{x : q \multimap y : q} \vee R \\
 \frac{x \curvearrowright \vee y}{x : q \multimap y : p \vee q} \wedge L \\
 x : p \wedge q \multimap y : p \vee q
 \end{array}$$

# Ingredients

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$\lambda$  terms

# Ingredients

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$\lambda$  terms        flow graphs

# Ingredients

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$\lambda$  terms    ♦    flow graphs    ♦    proof nets

A *proof term* for  $\Sigma \succ \Delta$   
encodes the flow of information  
in a proof of  $\Sigma \succ \Delta$ .

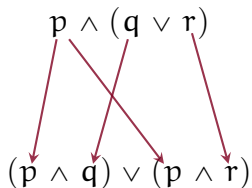
# Proof Terms

$$\begin{array}{l} \lambda x \rightarrow \lambda y \quad \lambda x \rightarrow \lambda y \quad \lambda x \rightarrow \lambda y \quad \lambda x \rightarrow \lambda y \\ x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r) \end{array}$$

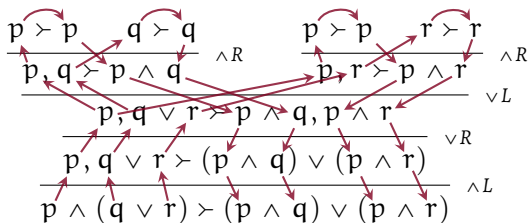


# Proof Terms as Graphs on Sequents

$\lambda x \rightarrow \lambda \vee y \quad \lambda x \rightarrow \lambda \vee y \quad \vee \lambda x \rightarrow \lambda \vee y \quad \vee \lambda x \rightarrow \lambda \vee y$   
 $x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)$



# Finding a Proof Term from a Derivation



# Finding a Proof Term from a Derivation

$$\begin{array}{c}
 \begin{array}{c}
 \frac{p \succ p \quad q \succ q}{p, q \succ p \wedge q} \wedge R \quad \frac{p \succ p \quad r \succ r}{p, r \succ p \wedge r} \wedge R \\
 \hline
 \frac{p, q \succ p \wedge q \quad p, r \succ p \wedge r}{p, q \vee r \succ p \wedge q, p \wedge r} \vee L \\
 \hline
 \frac{p, q \vee r \succ p \wedge q, p \wedge r}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)} \vee R \\
 \hline
 \frac{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)} \wedge L
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 p \wedge (q \vee r) \\
 \swarrow \quad \searrow \quad \downarrow \\
 (p \wedge q) \vee (p \wedge r)
 \end{array}$$

# Finding a Proof Term from a Derivation

$$\begin{array}{c}
 \frac{p \succ p \quad q \succ q}{p, q \succ p \wedge q} \wedge R \qquad \frac{p \succ p \quad r \succ r}{p, r \succ p \wedge r} \wedge R \\
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 \frac{\quad}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)} \vee R \\
 \frac{\quad}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)} \wedge L
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 \frac{\frac{p, q \succ p \wedge q}{p, q \vee r \succ p \wedge q} \vee L \quad \frac{p, r \succ p \wedge r}{q, p \wedge r}}{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)} \vee R \\
 \frac{p, q \vee r \succ (p \wedge q) \vee (p \wedge r)}{p \wedge (q \vee r) \succ (p \wedge q) \vee (p \wedge r)} \wedge L
 \end{array}$$

$$\begin{array}{c}
 p \wedge (q \vee r) \\
 \downarrow \\
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 \end{array}$$

$p \wedge (q \vee r)$

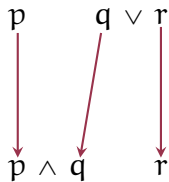
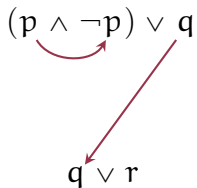
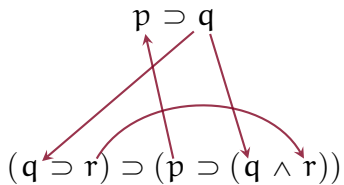
$(p \wedge q) \vee (p \wedge r)$

# Finding a Proof Term from a Derivation

$$\begin{array}{c}
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 \hline p, q \vee r \succ p \wedge q, p \wedge r \vee L \\
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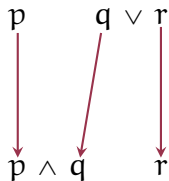
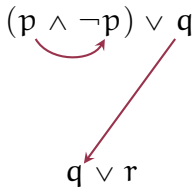
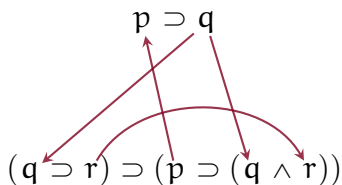
$$\begin{array}{c}
 p \wedge (q \vee r) \\
 \swarrow \quad \searrow \quad \downarrow \\
 (p \wedge q) \vee (p \wedge r)
 \end{array}$$

## More Flow Graphs





## More Flow Graphs



Links wholly internal to a *premise* or a *conclusion* are called *cups* (↪) and *caps* (↩).

## Proof Term Facts

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Not every directed graph on occurrences of atoms in a sequent is a proof term.

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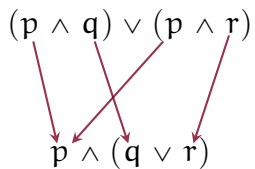
- ▶ They *typecheck*. [An occurrence of  $p$  is linked only with an occurrence of  $p$ .]
- ▶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]

## Proof Term Facts

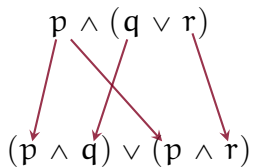
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- ▶ They *typecheck*. [An occurrence of  $p$  is linked only with an occurrence of  $p$ .]
- ▶ They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.]
- ▶ They must satisfy an “enough connections” condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise  $p \vee q$  and conclusion  $p \wedge q$  is not connected enough to be a proof term.]

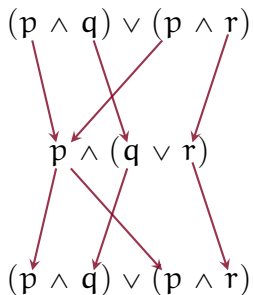
# Cut is chaining of proof terms



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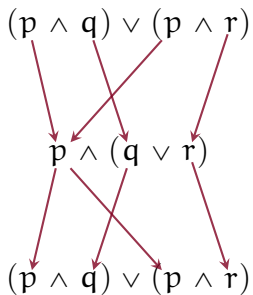
## Cut is chaining of proof terms



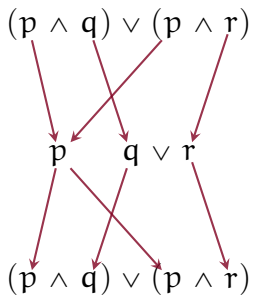
The *cut formula* is no longer a premise or a conclusion in the proof term.



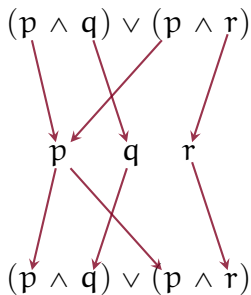
# Eliminating Cuts is Local



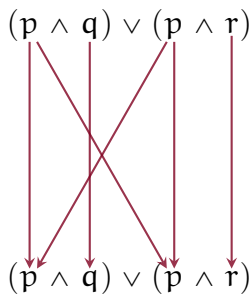
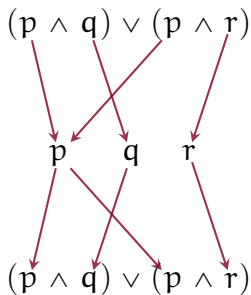
# Eliminating Cuts is Local



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- ▶ Cut elimination is *confluent* and *terminating*.

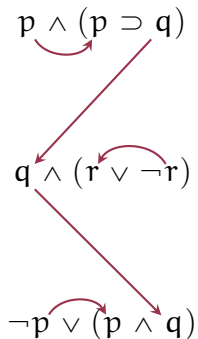
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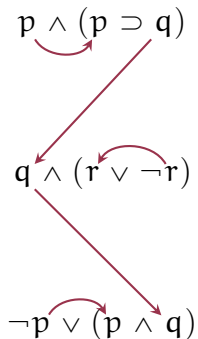
- ▶ Cut elimination is *confluent* and *terminating*.  
[So it can be understood as a kind of *evaluation*.]
- ▶ Cut elimination for proof terms is *local*.  
[So it is easily made parallel.]



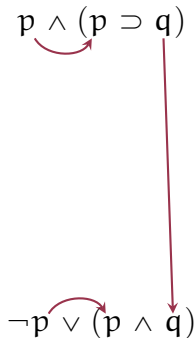
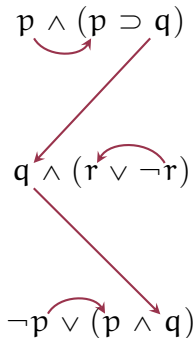
## Cuts with Caps and Cups



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## Cuts with Caps and Cups



# $\mathcal{C}$ is the Category of Classical Proofs

**OBJECTS** Formulas —  $A, B$ , etc.

**ARROWS** Cut-Free Proof Terms —  $\pi : A \succ B$ .

**COMPOSITION** Composition of derivations with the elimination of *Cut* — If  $\pi : A \succ B$  and  $\tau : B \succ C$  then  $\tau \circ \pi : A \succ C$ .

**IDENTITY** Canonical identity proofs —  $Id(A) : A \succ A$ .

## Identity Proofs

$$\begin{array}{c}
\frac{p \succ p}{p, \neg p \succ} \neg L \qquad \frac{q \succ q \quad p \succ p}{q \supset p, q \succ p} \supset L \\
\frac{}{\neg p \succ \neg p} \neg R \qquad \frac{}{q \supset p \succ q \supset p} \supset R \\
\hline
\frac{}{\neg p \vee (q \supset p) \succ \neg p, q \supset p} \vee L \\
\hline
\frac{}{\neg p \vee (q \supset p) \succ \neg p \vee (q \supset p)} \vee R
\end{array}
\qquad
\begin{array}{l}
\neg p \vee (q \supset p) \\
\\
\\
\\
\\
\neg p \vee (q \supset p)
\end{array}$$

# Identity Proofs

$$\begin{array}{c}
 \frac{p \succ p}{p, \neg p \succ} \neg L \\
 \frac{p, \neg p \succ}{\neg p \succ \neg p} \neg R \\
 \hline
 \frac{\neg p \succ \neg p}{\neg p \vee (q \supset p) \succ \neg p, q \supset p} \vee L \\
 \hline
 \frac{\neg p \vee (q \supset p) \succ \neg p, q \supset p}{\neg p \vee (q \supset p) \succ \neg p \vee (q \supset p)} \vee R
 \end{array}$$

Red arrows indicate the flow of assumptions and discharges in the proof. A curved arrow connects the two occurrences of  $p$  in the top-left derivation. A straight arrow connects the  $\neg p$  in the top-left derivation to the  $\neg p$  in the bottom-left derivation. A straight arrow connects the  $q \supset p$  in the top-right derivation to the  $q \supset p$  in the bottom-right derivation.

$$\neg p \vee (q \supset p)$$

$$\neg p \vee (q \supset p)$$

# Identity Proofs

$$\begin{array}{c}
 \frac{p \succ p}{p, \neg p \succ} \neg L \qquad \frac{q \succ q \quad p \succ p}{q \supset p, q \succ p} \supset L \\
 \frac{\neg p \succ \neg p}{\neg p \succ \neg p} \neg R \qquad \frac{q \supset p, q \succ p}{q \supset p \succ q \supset p} \supset R \\
 \frac{\neg p \succ \neg p \quad q \supset p \succ q \supset p}{\neg p \vee (q \supset p) \succ \neg p, q \supset p} \vee L \\
 \frac{\neg p \vee (q \supset p) \succ \neg p, q \supset p}{\neg p \vee (q \supset p) \succ \neg p \vee (q \supset p)} \vee R
 \end{array}$$

$$\begin{array}{c}
 \neg p \vee (q \supset p) \\
 \uparrow \\
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 \frac{p \succ p}{p, \neg p \succ} \neg L \qquad \frac{q \succ q \quad p \succ p}{q \supset p, q \succ p} \supset L \\
 \frac{\neg p \succ \neg p}{\neg p \succ \neg p} \neg R \qquad \frac{q \supset p, q \succ p}{q \supset p \succ q \supset p} \supset R \\
 \frac{\neg p \succ \neg p \quad q \supset p \succ q \supset p}{\neg p \vee (q \supset p) \succ \neg p, q \supset p} \vee L \\
 \frac{\neg p \vee (q \supset p) \succ \neg p, q \supset p}{\neg p \vee (q \supset p) \succ \neg p \vee (q \supset p)} \vee R
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 \frac{\neg p \succ \neg p}{\neg p \succ \neg p} \neg R \qquad \frac{q \supset p, q \succ p}{q \supset p \succ q \supset p} \supset R \\
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 \frac{\neg p \vee (q \supset p) \succ \neg p, q \supset p}{\neg p \vee (q \supset p) \succ \neg p \vee (q \supset p)} \vee L \\
 \hline
 \neg p \vee (q \supset p) \succ \neg p \vee (q \supset p) \vee R
 \end{array}$$

Red arrows indicate the flow of information: from the top-level assumptions  $p \succ p$  and  $q \succ q$  down to the final conclusion  $\neg p \vee (q \supset p) \succ \neg p \vee (q \supset p)$ . A curved red arrow also connects the two top-level assumptions.

$$\begin{array}{c}
 \neg p \vee (q \supset p) \\
 \uparrow \qquad \uparrow \\
 \neg p \vee (q \supset p)
 \end{array}$$

Red arrows indicate the flow of information from the bottom expression  $\neg p \vee (q \supset p)$  up to the top expression  $\neg p \vee (q \supset p)$ .

# Identity Proofs

$$\begin{array}{c}
 \frac{p \succ p}{p, \neg p \succ} \neg L \qquad \frac{q \succ q \quad p \succ p}{q \supset p, q \succ p} \supset L \\
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 \hline
 \frac{\neg p \vee (q \supset p) \succ \neg p, q \supset p}{\neg p \vee (q \supset p) \succ \neg p \vee (q \supset p)} \vee L \\
 \hline
 \neg p \vee (q \supset p) \succ \neg p \vee (q \supset p) \quad \vee R
 \end{array}$$

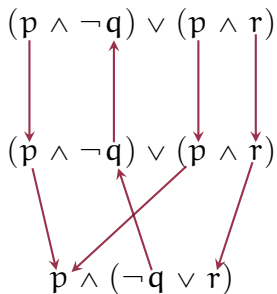
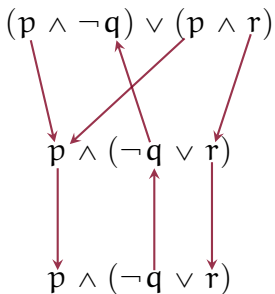
$$\begin{array}{c}
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 \uparrow \qquad \uparrow \qquad \downarrow \\
 \neg p \vee (q \supset p)
 \end{array}$$

# Identity Proofs

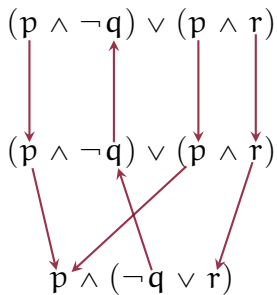
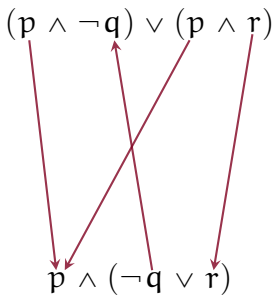
In the identity proof from  $A$  to  $A$ ,

- ▶ A *positive* occurrence of an atom in the premise linked *to* its mate in the conclusion.
- ▶ A *negative* occurrence of an atom in the premise is linked *from* its mate in the conclusion.
- ▶ There are no other links.

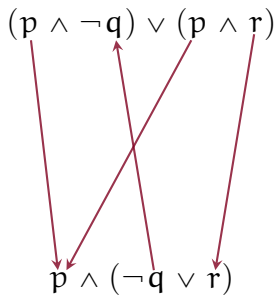
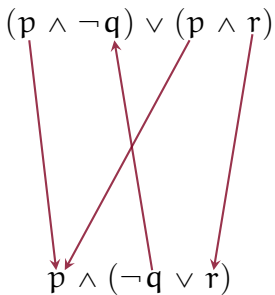
# Identity and Composition in $\mathcal{C}$



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## The Category $\mathcal{C}$ ...

- ▶ ... is *symmetric monoidal* and *star autonomous*
- ▶ but not *cartesian*,
- ▶ with structural *monoids* and *comonoids*,
- ▶ and is enriched in *SLat* (the category of semilattices).



## The Category $\mathcal{C}$ ...

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Being enriched in *SLat* means that proofs terms come ordered by  $\sqsubseteq$ , and compose under  $\cup$ , and these interact sensibly with composition.

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Being enriched in *SLat* means that proofs/terms come ordered by  $\sqsubseteq$ , and compose under  $\cup$ , and these interact sensibly with composition.

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(The sequent calculus is playing no essential role here.  
You can define proof terms on other proof systems,  
e.g. *natural deduction*, *Hilbert proofs*, *tableaux*, *resolution*.)

# ISOMORPHISMS



# Isomorphisms in Categories

$f : A \rightarrow B$  is an *isomorphism* in a category iff  
it has an *inverse*  $g : B \rightarrow A$ , where  
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If  $g$  and  $g'$  are both inverses, we have  
 $g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g'$ ,  
so any inverse is unique. We can call it  $f^{-1}$ .

## Why Isomorphisms?

If  $A$  and  $B$  are isomorphic in a category  $\mathcal{C}$ ,  
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they agree not only on *provability*,  
but also, on *proofs*.

The distinctions drawn when you analyse how something is *proved*  
(from premises), are not far from what you want to understand  
when you ask how something is *made true*.

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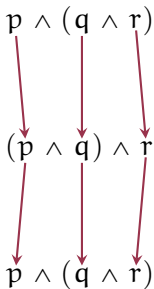


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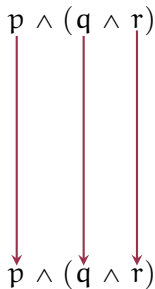
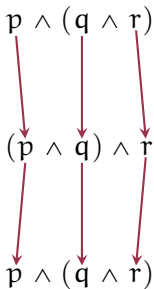
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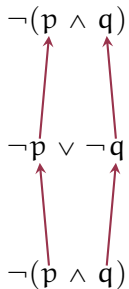


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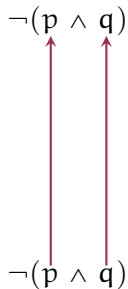
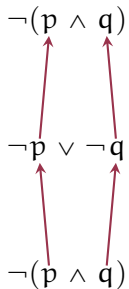
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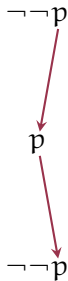


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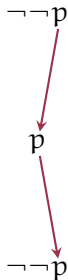
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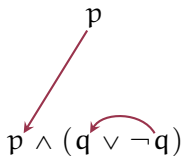
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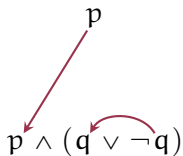
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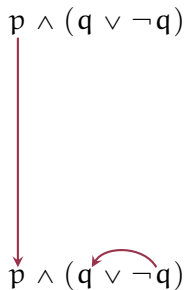
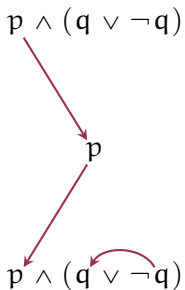
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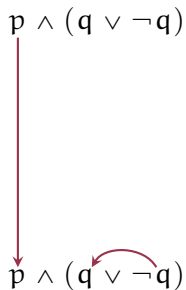
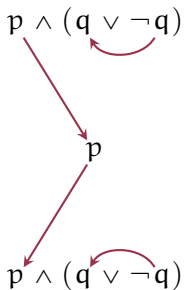
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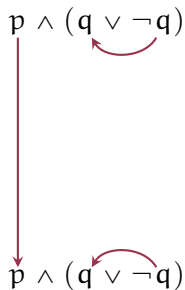
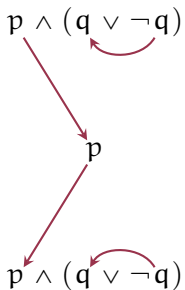
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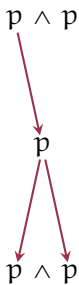
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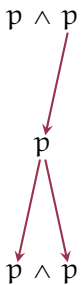
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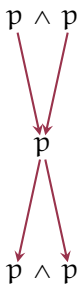
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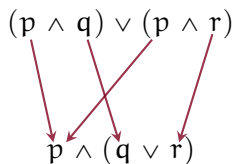
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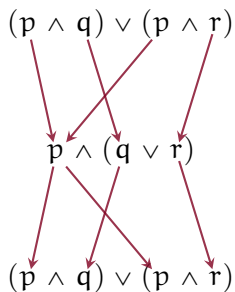
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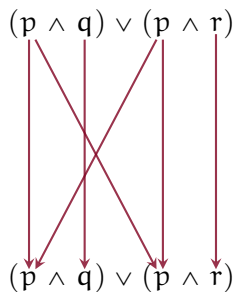
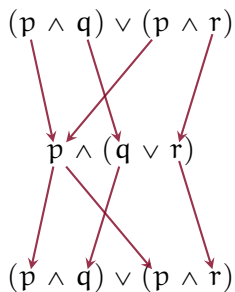
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## Occurrence Polarity Condition

If  $A$  is isomorphic to  $B$  in  $\mathfrak{C}$   
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(This condition is *necessary*, not *sufficient*:  $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$ .)



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$A$  is *isomorphic* to  $B$   
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This allows for a *negation normal form*, but not DNF or CNF.

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- ▶ Replace  $l \wedge m$  by a new atom in both  $A$  and  $B$ , and repeat.
- ▶ This shows how to reconstruct a proof of equivalence for  $A$  and  $B$  in the syntactic calculus for isomorphic formulas.



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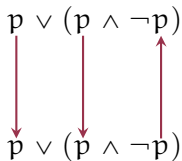
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- ▶ Yet,  $A$  and  $A \wedge A$  seem to have identical *subject matter* (insofar as we understand that notion).
- ▶ Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

MORE PROOFS  
FROM  $A$  TO  $A$

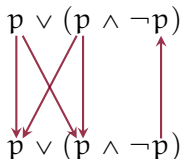
$$Id(p \vee (p \wedge \neg p))$$



In  $Id(A)$ , each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion.  
*Different occurrences of atoms in  $A$  are treated differently.*

# $Id(A), Hz(A), Mx(A)$

$Hz(p \vee (p \wedge \neg p))$



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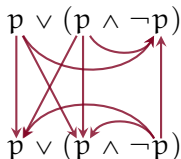
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$Mx(p \vee (p \wedge \neg p))$



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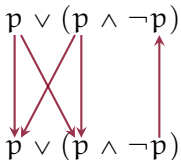
In  $Hz(A)$ , each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion.  
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In  $Mx(A)$ , each syntactically possible linking is included.  
*We treat all occurrences of an atom in  $A$  equally.*

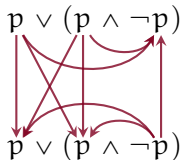
# $H\mathbf{z}(\mathbf{A})$ , $M\mathbf{x}(\mathbf{A})$ , Caps and Cups

*Note:*  $H\mathbf{z}(\mathbf{A})$  is  $M\mathbf{x}(\mathbf{A})$  with the caps and cups removed.

$$H\mathbf{z}(p \vee (p \wedge \neg p))$$



$$M\mathbf{x}(p \vee (p \wedge \neg p))$$



Let's look at relations like isomorphism,  
but which erase distinctions, up to  $Hx$  or  $Mx$ .

## Hz-Matching

Let's say that  $A$  and  $B$  *Hz-MATCH*, when there are proofs  $\pi : A \succ B$  and  $\pi' : B \succ A$  where  $\pi' \circ \pi = \text{Hz}(A)$  and  $\pi \circ \pi' = \text{Hz}(B)$ .

## Mx-Matching

Let's say that  $A$  and  $B$   $Mx$ -MATCH, when there are proofs  $\pi : A \succ B$  and  $\pi' : B \succ A$  where  $\tau \circ \pi = Mx(A)$  and  $\pi \circ \pi' = Mx(B)$ .

We write “ $\approx_{Mx}$ ” for the  $Mx$ -matching relation, and we write “ $\pi, \pi' : A \approx_{Mx} B$ ” to say that  $\pi : A \succ B$  and  $\pi' : B \succ A$  define a  $Mx$ -match between  $A$  and  $B$ .

## Isomorphism $\subseteq$ Hz-Matching

If  $\pi : A \succ B$  and  $\pi^{-1} : B \succ A$ , then  
consider  $\pi' = \text{Hz}(B) \circ \pi \circ \text{Hz}(A)$   
and  $\tau' = \text{Hz}(A) \circ \pi^{-1} \circ \text{Hz}(B)$ .

These satisfy the Hz-matching criteria,  
 $\tau' \circ \pi' = \text{Hz}(A)$  and  $\pi' \circ \tau' = \text{Hz}(B)$ .

## Proof

$$\begin{aligned} Hz(A) &= Id(A) \circ Id(A) \circ Hz(A) \\ &\subseteq Hz(A) \circ Id(A) \circ Hz(A) \\ &= Hz(A) \circ (\pi^{-1} \circ \pi) \circ Hz(A) \\ &= Hz(A) \circ (\pi^{-1} \circ Id(B) \circ Id(B) \circ \pi) \circ Hz(A) \\ &\subseteq Hz(A) \circ (\pi^{-1} \circ Hz(B) \circ Hz(B) \circ \pi) \circ Hz(A) \\ &= (Hz(A) \circ \pi^{-1} \circ Hz(B)) \circ (Hz(B) \circ \pi \circ Hz(A)) \\ &= \tau' \circ \pi' \\ &\subseteq Hz(A) \end{aligned}$$

## Proof

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...and similarly,  $Hz(B) \subseteq \pi' \circ \tau' \subseteq Hz(B)$



## $\text{Hz-Matching} \subseteq \text{Mx-Matching}$

If  $\pi, \pi' : A \approx_{\text{Hz}} B$ , then  
consider  $\tau = \text{Mx}(B) \circ \pi \circ \text{Mx}(A)$   
and  $\tau' = \text{Mx}(A) \circ \pi' \circ \text{Mx}(B)$ .

These satisfy the Mx-matching criteria,  
 $\tau' \circ \pi' = \text{Mx}(A)$  and  $\pi' \circ \tau' = \text{Mx}(B)$ .

# Proof

$$\begin{aligned} Mx(A) &= Id(A) \circ Id(A) \circ Mx(A) \\ &\subseteq Mx(A) \circ Hz(A) \circ Mx(A) \\ &= Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \\ &= Mx(A) \circ (\pi' \circ Id(B) \circ Id(B) \circ \pi) \circ Mx(A) \\ &\subseteq Mx(A) \circ (\pi' \circ Mx(B) \circ Mx(B) \circ \pi) \circ Mx(A) \\ &= (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \\ &= \tau' \circ \tau \\ &\subseteq Mx(A) \end{aligned}$$

## Proof

$$\begin{aligned} Mx(A) &= Id(A) \circ Id(A) \circ Mx(A) \\ &\subseteq Mx(A) \circ Hz(A) \circ Mx(A) \\ &= Mx(A) \circ (\pi' \circ \pi) \circ Mx(A) \\ &= Mx(A) \circ (\pi' \circ Id(B) \circ Id(B) \circ \pi) \circ Mx(A) \\ &\subseteq Mx(A) \circ (\pi' \circ Mx(B) \circ Mx(B) \circ \pi) \circ Mx(A) \\ &= (Mx(A) \circ \pi' \circ Mx(B)) \circ (Mx(B) \circ \pi \circ Mx(A)) \\ &= \tau' \circ \tau \\ &\subseteq Mx(A) \end{aligned}$$

...and similarly,  $Mx(B) \subseteq \pi' \circ \tau' \subseteq Mx(B)$

If  $A \approx_{Mx} B$  then there are proofs  
 $\pi : A \multimap B$  and  $\tau : B \multimap A$ .

# Matching Relations are Equivalence Relations

**REFLEXIVE**  $Hx(A), Hx(A) : A \approx_{Hx} A.$

$Mx(A), Mx(A) : A \approx_{Mx} A.$

# Matching Relations are Equivalence Relations

**REFLEXIVE**  $\text{Hz}(A), \text{Hz}(A) : A \approx_{\text{Hz}} A.$

$\text{Mx}(A), \text{Mx}(A) : A \approx_{\text{Mx}} A.$

**SYMMETRIC** If  $\pi, \pi' : A \approx_{\text{Hz}} B$ , then  $\pi', \pi : B \approx_{\text{Hz}} A.$

If  $\pi, \pi' : A \approx_{\text{Mx}} B$ , then  $\pi', \pi : B \approx_{\text{Mx}} A.$

# Matching Relations are Equivalence Relations

**REFLEXIVE**  $H_z(A), H_z(A) : A \approx_{H_z} A.$

$M_x(A), M_x(A) : A \approx_{M_x} A.$

**SYMMETRIC** If  $\pi, \pi' : A \approx_{H_z} B$ , then  $\pi', \pi : B \approx_{H_z} A.$

If  $\pi, \pi' : A \approx_{M_x} B$ , then  $\pi', \pi : B \approx_{M_x} A.$

**TRANSITIVE** If  $\pi, \pi' : A \approx_{H_z} B$  and  $\tau, \tau' : B \approx_{H_z} C$ , then  
 $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{H_z} C.$

If  $\pi, \pi' : A \approx_{M_x} B$  and  $\tau, \tau' : B \approx_{M_x} C$ , then  
 $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{M_x} C.$

## More Matchings

$$p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$$

$$p \wedge (q \vee r) \approx_{Hz} (p \wedge q) \vee (p \wedge r)$$



## $Mx$ -Matching $\subset$ Logical Equivalence

**FACT:** If an atom  $p$  occurs positively in  $A$  but not in  $B$ , then  $A$  and  $B$  do not  $Mx$ -match.

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So, in the composition proof from  $A$  to  $A$ , there is no link from the premise occurrence to the conclusion occurrence. No proof from  $A$  to  $B$  and back can recreate  $Mx(A)$ .

## $Mx$ -Matching $\subset$ Logical Equivalence

FACT: If an atom  $p$  occurs positively [negatively] in  $A$  but not in  $B$ , then  $A$  and  $B$  do not  $Mx$ -match.

PROOF:  $Mx(A) : A \succ A$  contains the link from [to] that occurrence of  $p$  in the premise  $A$  to [from] its corresponding occurrence in the conclusion  $A$ .

No proof from  $A$  to  $B$  contains a link from [to] that occurrence to anything in  $B$  (since there is no positive [negative] occurrence in  $B$  at all).

So, in the composition proof from  $A$  to  $A$ , there is no link from [to] the premise occurrence to [from] the conclusion occurrence. No proof from  $A$  to  $B$  and back can recreate  $Mx(A)$ .

**FACT:** If an atom occurs positively  
[negatively] in A but not in B  
then A and B do not  $Mx$ -match.

## $Mx$ -Matching $\subseteq$ Logical Equivalence: Examples

**FACT:** If an atom occurs positively  
[negatively] in  $A$  but not in  $B$   
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**COROLLARY:**  $p \not\approx_{Mx} p \wedge (q \vee \neg q)$ .

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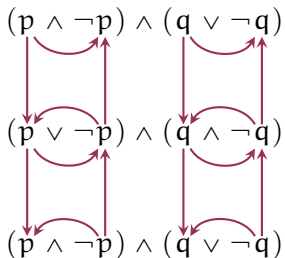
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$$p \wedge \neg p \not\approx_{Mx} q \wedge \neg q.$$



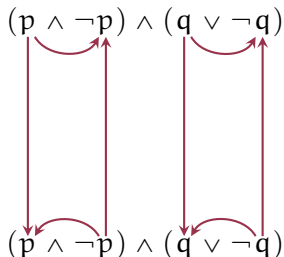
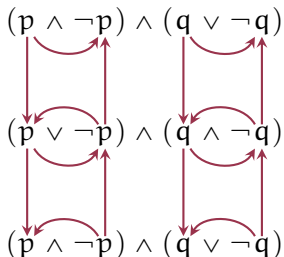
## $Hz\text{-matching} \subset Mx\text{-matching}$

$$(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$$



## $H\mathbf{z}$ -matching $\subset M\mathbf{x}$ -matching

$$(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{M\mathbf{x}} (p \vee \neg p) \wedge (q \wedge \neg q)$$



*Isomorphism  $\subset$  Hz-Matching  $\subset$  Mx-Matching  $\subset$  Logical Equivalence*

So what *are* the *matching* relations?

# MATCHING & LOGICS OF ANALYTIC CONTAINMENT

# Angell's Logic of Analytic Containment

$$\text{AC1 } A \leftrightarrow \neg\neg A$$

$$\text{AC2 } A \leftrightarrow (A \wedge A)$$

$$\text{AC3 } (A \wedge B) \leftrightarrow (B \wedge A)$$

$$\text{AC4 } A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$$

$$\text{AC5 } A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$$

$$\text{R1 } A \leftrightarrow B, C(A) \Rightarrow C(B)$$

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Here,  $A \vee B$  is shorthand for  $\neg(\neg A \wedge \neg B)$ .

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You can define  $A \rightarrow B$  as  $A \leftrightarrow (A \wedge B)$ .



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Here,  $A \vee B$  is shorthand for  $\neg(\neg A \wedge \neg B)$ .

You can define  $A \rightarrow B$  as  $A \leftrightarrow (A \wedge B)$ .

Famously,  $A \rightarrow (A \vee B)$  is not derivable in Angell's logic.  
We cannot prove  $A \leftrightarrow (A \wedge (A \vee B))$ .

## Extensions of Angell's Logic

- ▶ The first degree fragment of *Parry's* Logic of Analytic Containment is found by adding  $(A \vee (B \wedge \neg B)) \rightarrow A$  to Angell's Logic.
  - ▶ Parry's logic still satisfies this relevance constraint:  $A \rightarrow B$  is provable only when the atoms in  $B$  are present in  $A$ .
- ▶ *First Degree Entailment* (FDE) is found by adding  $A \rightarrow (A \vee B)$  to Angell's Logic.
  - ▶ FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that  $p \vee \neg p$ , and  $q \wedge \neg q$  are both non-trivial, and ineliminable.
  - ▶ A simple translation encodes FDE inside classical logic. Choose, for each atom  $p$ , a fresh atom  $p'$ , its *shadow*. For each FDE formula  $A$ , its translation is the formula  $A'$  found by replacing the negative occurrences of atoms  $p$  in  $A$  by their shadows. An argument is FDE valid iff its translation is classically valid.

$$Mx(A, B)$$

DEFINITION:  $Mx(A, B)$  is the set of all possible linkings which could occur in any proof from  $A$  to  $B$ .

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That is, it contains a link from  
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to matching {positive atoms in  $B$ , negative atoms in  $A$ }.

## $Mx(A, B)$

DEFINITION:  $Mx(A, B)$  is the set of all possible linkings which could occur in any proof from  $A$  to  $B$ .

That is, it contains a link from  
    {positive atoms in  $A$ , negative atoms in  $B$ }  
to matching {positive atoms in  $B$ , negative atoms in  $A$ }.

FACT:  $Mx(A, B)$  is a proof iff  
there is some proof from  $A$  to  $B$ .

## $Mx(A, B)$ examples

$Mx(p, q)$

$p$

$q$

## $Mx(A, B)$ examples

$Mx(p, q)$

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*No links.*

# $Mx(A, B)$ examples

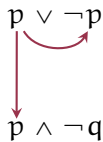
$Mx(p, q)$

$p$

$q$

*No links.*

$Mx(p \vee \neg p, p \wedge \neg q)$





# $Mx(A, B)$ examples

$Mx(p, q)$

$p$

$q$

*No links.*

$Mx(p \vee \neg p, p \wedge \neg q)$

$p \vee \neg p$

$p \wedge \neg q$

*Not a proof.*

## $Mx(A, B)$ and matching

LEMMA: If  $A \approx_{Mx} B$ , then  $Mx(A, B)$  and  $Mx(B, A)$  are proofs, and  $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$

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PROOF: If  $\pi, \pi' : A \approx_{Mx} B$ , then  $\pi \subseteq Mx(A, B)$  and  $\pi' \subseteq Mx(B, A)$ , so  $Mx(A, B)$  and  $Mx(B, A)$  are both proofs.

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Since  $\pi' \circ \pi = Mx(A)$ , we have  
 $Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$ ,  
and similarly,  $Mx(B) = Mx(A, B) \circ Mx(B, A)$ ,  
so  $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$ .

FACT: If  $A$  is classically logically equivalent to  $B$ ,  
and all atoms occurring positively [negatively] in  $A$   
also occur positively [negatively] in  $B$ ,  
and *vice versa*, then  $A$  and  $B$   $Mx$ -match  
— and conversely.

# Proof

If  $A$  is logically equivalent to  $B$ , then  $Mx(A, B)$  and  $Mx(B, A)$  are both proofs.

It suffices to show that  $Mx(B, A) \circ Mx(A, B) = Mx(A)$  (and similarly for  $B$ ). To show this, we need to show that each positive [negative] occurrence of an atom in  $A$  is linked to any positive [negative] occurrence of that atom in  $A$  by way of some link in  $Mx(A, B)$  composed with a link in  $Mx(B, A)$ . But since that atom occurs positively [negatively] also in  $B$  at least once, the links to accomplish this occur in  $Mx(A, B)$  and  $Mx(B, A)$ .

Conversely, if  $A \approx_{Mx} B$ , we have already seen that  $A$  and  $B$  must be equivalent, and no atom occurs positively [negatively] in  $A$  but not  $B$ .

## This is *not* Equivalence in Parry's Logic

**FACT:**  $A$  is equivalent to  $B$  in Parry's logic of analytic containment iff  $A$  is classically equivalent to  $B$  and the atoms present in  $A$  are the atoms present in  $B$ .

## This is *not* Equivalence in Parry's Logic

FACT: A is equivalent to B in Parry's logic of analytic containment iff A is classically equivalent to B and the atoms present in A are the atoms present in B.

$$(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$$

But these satisfy Parry's variable sharing criteria.



# Open Question

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Does the equivalence relation of  $Mx$ -matching  
occur elsewhere in the literature?

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That is, it contains a link from positive atoms in  $A$  to corresponding positive atoms in  $B$  and negative atoms in  $A$  to corresponding negative atoms in  $B$ .

## $H_z(A, B)$ examples

$$H_z(p \wedge \neg p, q \vee \neg q)$$

## $Hz(A, B)$ examples

$$p \wedge \neg p$$

$$Hz(p \wedge \neg p, q \vee \neg q)$$

$$q \vee \neg q$$

## $H_z(A, B)$ examples

$$p \wedge \neg p$$

$$H_z(p \wedge \neg p, q \vee \neg q)$$

*No links!*

$$q \vee \neg q$$

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$$p \wedge \neg p$$

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## $Hz(A, B)$ examples

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$$\begin{array}{c} p \wedge \neg p \\ \downarrow \\ p \vee \neg p \\ \uparrow \\ p \wedge \neg p \end{array}$$



## $Hz(A, B)$ examples

$$Hz(p \wedge \neg p, q \vee \neg q)$$

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A proof, but not  
the maximal one.

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the argument from  $A$  to  $B$  is FDE valid.

- ▶ From FDE-validity to  $Hz$ -proof: straightforward induction on an FDE-axiomatisation.
- ▶ From the  $Hz$ -proof  $Hz(A, B)$  to FDE-validity. Notice that no negative occurrences of atoms in  $A$  or  $B$  are linked to any positive occurrences of atoms in  $A$  or  $B$ . So, there is another  $Hz$ -proof  $Hz(A', B')$  for the FDE translations for  $A$  and  $B$ .

## $Hz(A, B)$ and $Hz$ -matching

LEMMA: If  $A \approx_{Hz} B$ , then  $Hz(A, B)$  and  $Hz(B, A)$  are proofs, and  $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$

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PROOF: If  $\pi, \pi' : A \approx_{Hz} B$ , then then since  $\pi' \circ \pi = Hz(A)$  and  $\pi \circ \pi' = Hz(B)$ ,  $\pi$  and  $\pi'$  are cap- and cup-free, so  $\pi \subseteq Hz(A, B)$  and  $\pi' \subseteq Hz(B, A)$ , so  $Hz(A, B)$  and  $Hz(B, A)$  are both proofs.

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Since  $\pi' \circ \pi = Hz(A)$ , we have  
 $Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$ ,  
and similarly,  $Hz(B) = Hz(A, B) \circ Hz(B, A)$ ,  
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FACT: If  $A$  is FDE-equivalent to  $A$ ,  
and all atoms occurring positively [negatively] in  $A$   
also occur positively [negatively] in  $B$ ,  
and *vice versa*, then  $A$  and  $B$  *Hz*-match  
— and conversely.



# Proof

If  $A$  is FDE-equivalent to  $B$ , then  $\text{Hz}(A, B)$  and  $\text{Hz}(B, A)$  are both proofs.

It suffices to show that  $\text{Hz}(B, A) \circ \text{Hz}(A, B) = \text{Hz}(A)$  (and similarly for  $B$ ). To show this, we need to show that each positive [negative] occurrence of an atom in  $A$  is linked to any positive [negative] occurrence of that atom in  $A$  by way of some link in  $\text{Hz}(A, B)$  composed with a link in  $\text{Hz}(B, A)$ . But since that atom occurs positively [negatively] also in  $B$  at least once, the links to accomplish this occur in  $\text{Hz}(A, B)$  and  $\text{Hz}(B, A)$ .

Conversely, if  $A \approx_{\text{Hz}} B$ , we have already seen that  $A$  and  $B$  must be FDE-equivalent, and no atom occurs positively [negatively] in  $A$  but not  $B$ .

## *Hz-matching* $\equiv$ Angellic Equivalence

FACT: (Fine, Ferguson)  $A$  is equivalent to  $B$  in Angell's logic of analytic containment iff  $A$  is FDE equivalent to  $B$ , and any atom occurs positively [negatively] in  $A$  iff it occurs positively [negatively] in  $B$ .

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So,  $Hz\text{-matching}$  is equivalence  
in Angell's Logic of Analytic Containment.

# MATCHING AS ISOMORPHISM

## $H_z(A)$ and $M_x(A)$ are Idempotents

- ▶  $H_z(A) \circ H_z(A) = H_z(A)$ ,  $M_x(A) \circ M_x(A) = M_x(A)$ .

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- ▶  $Hx(A) \circ Hx(A) = Hx(A)$ ,  $Mx(A) \circ Mx(A) = Mx(A)$ .
- ▶ For any category  $\mathcal{C}$ , if  $i_A$  is an idempotent for each object  $A$ , we can form a new category  $\mathcal{C}_i$  with the same objects as  $\mathcal{C}$ , and with arrows  $i_B \circ f \circ i_A : A \rightarrow B$ .

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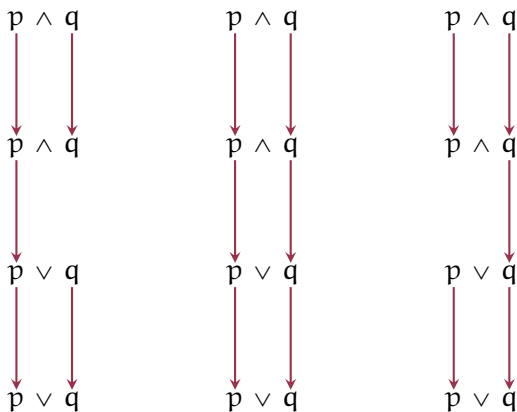
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- ▶  $Hz$ -matching is *isomorphism* in  $\mathfrak{C}_{Hz}$ .

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## $\mathfrak{C}_{Mx}$ and $\mathfrak{C}_{Hz}$ are nontrivial, nonetheless



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- ▶ Proof theoretical resources *for classical logic* provide tools for fine-grained hyperintensional distinctions.
- ▶ One question is how to relate these results to *models* of logics of content.
- ▶ Another next step is these results to first order logic is an open question.



THANK YOU!

# Thank you!

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or *email* at `restall@unimelb.edu.au`