

# Defining Rules, Proofs and Counterexamples

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To present an account of *defining rules*,  
with the aim of explaining these rules  
they play a central role in *analytic proofs*.

Along the way, I'll explain how Kreisel's *squeezing argument* helps us understand the connection between an informal notion of validity and the notions formalised in our accounts of proofs and models, and the relationship between proof-theoretic and model-theoretic analyses of logical consequence.

Positions and Bounds

Definitions

What Proofs Are & What They Do

Counterexamples & Kreisel's Squeeze

# POSITIONS AND BOUNDS

# *Assertions and Denials*

$[X : Y]$

... in a communicative practice

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They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, *etc.*

Assertions and denials take a *stand*  
(*pro* or *con*) on something.

DENIAL clashes with assertion.  
ASSERTION clashes with denial.

## Yes/No Questions

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Ask:  $p$ ?

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**YES**

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Assert  $p$

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YES

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## Yes/No Questions

Ask:  $p$ ?

YES      NO

Assert  $p$     Deny  $p$

Ask:  $p$ ?

YES

NO

Assert  $p$     Deny  $p$

These two answers *clash*.

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on the negative side

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*Lesley:* No.

*Lesley:* She's either in the study or outside.

## **Weak denial**

Retracts  $p$  from the common ground  
on the positive side

Perhaps  $p$ .

Perhaps  $p$ .

Retracts  $p$  from the common ground on the negative side?

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- ▶ CUT: If  $[X, A : Y]$  and  $[X : A, Y]$  are out of bounds, then so is  $[X : Y]$ .
- ▶ A position that is OUT OF BOUNDS is *overcommitted*.

# On *Cut*

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Suppose  $[X : Y]$  is not out of bounds.

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Suppose  $[X, A : Y]$  is out of bounds.

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Ask the question:  $A$ ?

Suppose  $[X : Y]$  is not out of bounds.

Suppose  $[X, A : Y]$  is out of bounds.

Ask the question:  $A$ ?

The answer *no* is forced,  
as a *yes* answer is excluded  
(given our other commitments).



## Structural Rules

$$A \succ A \quad Id \qquad \frac{X \succ A, Y \quad X', A \succ Y'}{X, X' \succ Y, Y'} \quad Cut$$

$$\frac{X \succ Y}{X \succ A, Y} \quad K$$

$$\frac{X \succ Y}{X, A \succ Y} \quad K$$

$$\frac{X \succ A, A, Y}{X \succ A, Y} \quad W$$

$$\frac{X, A, A \succ Y}{X, A \succ Y} \quad W$$

## The Power of Bounds: *Comparatives*

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*strong irreflexivity:*  $s >_{\mathsf{F}} s \succ$

*weak reflexivity:*  $\succ s \geq_{\mathsf{F}} s$

*contraries:*  $s >_{\mathsf{F}} t, t \geq_{\mathsf{F}} s \succ$

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$$Fs \quad s >_F t \quad s \geqslant_F t$$

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*weak transitivity:*  $s \geqslant_F t, t \geqslant_F u \succ s \geqslant_F u$

*strong irreflexivity:*  $s >_F s \succ$

*weak reflexivity:*  $\succ s \geqslant_F s$

*contraries:*  $s >_F t, t \geqslant_F s \succ$

*subcontraries:*  $\succ s >_F t, t \geqslant_F s$



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*contraries:*  $s >_F t, t \geqslant_F s \succ$

*subcontraries:*  $\succ s >_F t, t \geqslant_F s$

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*contraries:*  $s >_F t, t \geqslant_F s \succ$

*subcontraries:*  $\succ s >_F t, t \geqslant_F s$

*strength:*  $s >_F t \succ s \geqslant_F t$

*preservation:*  $\text{Fs}, t \geqslant_F s \succ \text{Ft}$

# DEFINITIONS

How do you *define* a concept?

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By showing people how to *use* it.

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Logical concepts are similarly sharply delimited, but they cannot *all* be given explicit definitions.

## Definition through a *rule for use*

$[X, A \otimes B : Y]$  is out of bounds

if and only if

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$$\frac{\frac{X, A, B \succ Y}{X, A \otimes B \succ Y}}{\otimes Df}$$

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$$\frac{\frac{X \succ A, Y \quad \frac{\frac{X' \succ B, Y' \quad \frac{\frac{A \otimes B \succ A \otimes B}{Id}}{A, B \succ A \otimes B}{\otimes Df}}{X', A \succ A \otimes B, Y'}{Cut}}{X, X' \succ A \otimes B, Y, Y'}{Cut}}$$

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So, we have

$$\frac{X \succ A, Y \quad X' \succ B, Y'}{X, X' \succ A \otimes B, Y, Y'} \otimes R$$



We have given norms governing  $\otimes$  judgements in terms of norms governing simpler judgements.

## Definitions for other logical concepts

$$\frac{\frac{X \succ A, Y}{\phantom{X, \neg A \succ Y}}}{X, \neg A \succ Y} \neg Df$$

$$\frac{\frac{X, A \succ B, Y}{\phantom{X \succ A \rightarrow B, Y}}}{X \succ A \rightarrow B, Y} \rightarrow Df$$

$$\frac{\frac{X \succ A, B, Y}{\phantom{X \succ A \oplus B, Y}}}{X \succ A \oplus B, Y} \oplus Df$$

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$$\frac{\frac{X \succ A, Y \quad X \succ B, Y}{\phantom{X \succ A \wedge B, Y}}}{X \succ A \wedge B, Y} \wedge Df \qquad \frac{\frac{X, A \succ Y \quad X, B \succ Y}{\phantom{X, A \vee B \succ Y}}}{X, A \vee B \succ Y} \vee Df$$

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$$\frac{\frac{X, A \succ B, Y}{\phantom{X \succ A \rightarrow B, Y}}}{X \succ A \rightarrow B, Y} \rightarrow Df$$

$$\frac{\frac{X \succ A, B, Y}{\phantom{X \succ A \oplus B, Y}}}{X \succ A \oplus B, Y} \oplus Df$$

$$\frac{\frac{X \succ A, Y \quad X \succ B, Y}{\phantom{X \succ A \wedge B, Y}}}{X \succ A \wedge B, Y} \wedge Df$$

$$\frac{\frac{X, A \succ Y \quad X, B \succ Y}{\phantom{X, A \vee B \succ Y}}}{X, A \vee B \succ Y} \vee Df$$

$$\frac{\frac{X \succ A|_n^x, Y}{\phantom{X \succ (\forall x)A, Y}}}{X \succ (\forall x)A, Y} \forall Df$$

$$\frac{\frac{X, A|_n^x \succ Y}{\phantom{X, (\exists x)A \succ Y}}}{X, (\exists x)A \succ Y} \forall Df$$

$$\frac{\frac{X, Fs \succ Ft, Y}{\phantom{X \succ s = t, Y}}}{X \succ s = t, Y} = Df$$

(Where  $n$  and  $F$  are not present in  $X$  and  $Y$ .)

## How does this work?

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How do concepts defined in this way *work*?

# Transforming Systems of Rules

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- ▶ Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)
- ▶ In Brandom's terms, they *make explicit* some of what was implicit in the practice of assertion and denial.



# WHAT PROOFS ARE & WHAT THEY DO

## A Tiny Proof

*If it's Thursday, I'm in Melbourne.*

*It's Thursday.*

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*[It's Thursday  $\rightarrow$  I'm in Melbourne, It's Thursday : I'm in Melbourne]*

(This is out of bounds.)

## The Undeniable

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The *stance* (*pro* or *con*)  
on *I'm in Melbourne* was already made.

**A *proof* for  $X \succ Y$  shows that the position  $[X : Y]$  is out of bounds, by way of the defining rules for the concepts involved in the proof.**



**A proof for  $X \succ Y$  shows that the position  $[X : Y]$  is out of bounds, by way of the defining rules for the concepts involved in the proof.**

In this sense, proofs are *analytic*.

They apply, given the definitions, independently of the positions taken by those giving the proof.

## What Proofs Prove

A proof of  $A, B \succ C, D$  can be seen  
as a *proof* of  $C$  from  $[A, B : D]$ ,

A proof of  $A, B \succ C, D$  can be seen  
as a *proof* of  $C$  from  $[A, B : D]$ ,  
and a *refutation* of  $A$  from  $[B : C, D]$ ,  
and *more*.

COUNTEREXAMPLES  
& KREISEL'S  
SQUEEZE

# Enlarging Positions

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If  $[X : Y]$  is available, then  
so is either  $[X, A : Y]$  or  $[X : A, Y]$

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we can extend it into  
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$U \succ V$  is not derivable  
for any finite  $U \subseteq X'$  and  $V \subseteq Y'$ .

## Adding Witnesses

If  $(\exists x)A$  is added on the left, we also  
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## Adding Witnesses

If  $(\exists x)A$  is added on the left, we also add a *witness*  $A|_n^x$ , where  $n$  is fresh and similarly when  $(\forall x)A$  is added on the right.

$$\frac{X, A|_n^x, (\exists x)A \succ Y}{X, (\exists x)A \succ Y} \exists Df, W$$

$$\frac{X \succ (\forall x)A, A|_n^x, Y}{X \succ (\forall x)A, Y} \forall Df, W$$

# Witnessed Limit Positions give rise to Models

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$$A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y',$$

$$A \wedge B \in X' \text{ iff } A \in X' \text{ and } B \in X'.$$

$$A \vee B \in X' \text{ iff } A \in X' \text{ or } B \in X'.$$

$$A \rightarrow B \in X' \text{ iff } A \in Y' \text{ or } B \in X'.$$

$$(\forall x)A \in X' \text{ iff } A|_n^x \in X' \text{ for each name } n.$$

$$(\exists x)A \in X' \text{ iff } A|_n^x \in X' \text{ for some name } n.$$

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$$A \vee B \in X' \text{ iff } A \in X' \text{ or } B \in X'.$$

$$A \rightarrow B \in X' \text{ iff } A \in Y' \text{ or } B \in X'.$$

$$(\forall x)A \in X' \text{ iff } A|_n^x \in X' \text{ for each name } n.$$

$$(\exists x)A \in X' \text{ iff } A|_n^x \in X' \text{ for some name } n.$$

This is a *model*, where the *true formulas* are in  $X'$  and the *false formulas* are in  $Y'$ , and whose *domain* is the set of names.

## Witnessed Limit Positions give rise to Models

$$A \in X' \text{ iff } \neg A \notin X' \text{ iff } \neg A \in Y',$$

$$A \wedge B \in X' \text{ iff } A \in X' \text{ and } B \in X'.$$

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(Things are *little* more delicate when the language contains the identity predicate.)



$X \succ Y$  is derivable  
iff there is no *model*  
in which each member of  $X$  is true  
and each member of  $Y$  is false.

$X \succ Y$  is *informally* valid

$X \succ Y$  has a derivation



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- To say that  $X \succ Y$  is *informally valid* means that is a clash involved in asserting each member of  $X$  and denying each member of  $Y$ .

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- ▶ Defining rules *define* the connectives/quantifiers.

## (2) From Informal Validity to Absence of Countermodel

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- ▶ So, there is no clash involved in asserting *any* formulas in  $X$  and denying any formulas in  $Y$ , by appeal to the defining rules. (This is an induction on the depth of the structure of the formulas. The defining rules reduce clashes involving formulas into clashes involving subformulas.)
- ▶ So, a countermodel for a sequent shows *how* there is no clash involved in asserting each member of  $X$  and denying each member of  $Y$ .



### (3) From Absence of Countermodel to Derivability

That's the *Completeness Theorem*.

$X \succ Y$  has a derivation



$X \succ Y$  is *informally* valid



$X \succ Y$  has no countermodel



$X \succ Y$  has a derivation.

Informal validity (in virtue of first order logical form), for the language given by the defining rules, is *first order classical logic*, as given by the sequent calculus and Tarski's models.

# THANK YOU!

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