PROOF TERMS FOR CLASSICAL DERIVATIONS

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Abstract: I give an account of *proof terms* for derivations in a sequent calculus for classical propositional logic. The term for a derivation δ of a sequent $\Sigma \succ \Delta$ encodes *how* the premises Σ and conclusions Δ are related in δ. This encoding is many–to–one in the sense that different derivations can have the same proof term, since different derivations may be different ways of representing the same underlying connection between premises and conclusions. However, not all proof terms for a sequent $\Sigma \succ \Delta$ are the same. There may be *different* ways to connect those premises and conclusions.

Proof terms can be simplified in a process corresponding to the elimination of cut inferences in sequent derivations. However, unlike cut elimination in the sequent calculus, each proof term has a *unique normal form* (from which all cuts have been eliminated) and it is straightforward to show that term reduction is strongly normalising—*every* reduction process terminates in that unique normal form. Furthermore, proof terms are *invariants* for sequent derivations in a strong sense—two derivations δ_1 and δ_2 have the same proof term *if and only if* some permutation of derivation steps sends δ_1 to δ_2 (given a relatively natural class of permutations of derivations in the sequent calculus). Since not every derivation of a sequent can be permuted into every other derivation of that sequent, proof terms provide a non-trivial account of the identity of proofs, independent of the syntactic representation of those proofs.

OUTLINE

This paper has six sections: *Section 1* motivates the problem of proof identity, and reviews the reasons that the question of proof identity for classical propositional logic is difficult.

Section 2 defines the sequent system to be analysed, showing that the standard structural rules of *Contraction* and *Weakening*, and a less familiar structural rule—*Blend*—are height-preserving admissible. It also introduces proof terms annotating sequent derivations, by analogy to the well-understood sense in which λ terms annotate intuitionist natural deduction proofs. This defines for a derivation δ a proof term $\tau(\delta)$, a connection graph on the sequent derived by δ.

Section 3 covers an independent criterion for when a connection graph annotates some derivation. This can be seen as a kind of *soundness* and *completeness* result for proof terms.

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In *Section 4*, we define a family of permutations of inferences in sequent derivations, and show that if a permutation sends δ_1 to δ_2 then $\tau(\delta_1) = \tau(\delta_2)$. This is followed by the converse result—if $\tau(\delta_1) = \tau(\delta_2)$ then some permutation of inference steps sends δ_1 to δ_2. It follows that proof terms a strong kind of *invariant* for derivations under those permutations.

The sequent derivations considered in *Sections 1* to *4* may involve *Cut* inferences. The terms corresponding to these inferences involve what we call 'cut points', marking locations a *Cut formula* is used in a proof. In *Section 5* we define a *reduction process* for proof terms, eliminating these cut points. This reduction procedure is both *confluent* and *strongly normalising*.

Then in the final section, *Section 6*, we will explore the connection between the reduction process for terms and to cut elimination steps for derivations, and revisit the problem of proof identity and cut reduction in the light of reduction for terms.

Together, the results of *Section 2* to *6* provide an answer to the problems of *Section 1*, a natural account of proof identity for classical propositional logic.

1 PROOF IDENTITY

To prove a conclusion from some premises, you show *how* that conclusion follows from those premises. Sometimes, the same conclusion can be proved in a number of different ways, from the same collection of premises. We see this all the time in mathematics. Mathematicians give very different proofs of the one theorem, from the same basic set of mathematical principles. Think, for example, of the different proofs that $\sqrt{2}$ is irrational [16].

The fact of distinctions between proofs for the one argument is not restricted to complex mathematical proofs: It is present at the very foundations of logic, in very simple proofs in basic propositional logic—even in proofs very simple tautologies. For example, we can prove the tautology $p \supset (p \supset (p \land p))$ in a number of different ways. *One* way is to prove the conjunction $p \land p$ from a single assumption p (conjoining that assumption with *itself*), then discharging that assumption of p to derive $p \supset (p \land p)$, and then *vacuously* discharging the assumption of p again. A second proof goes differently. We could make two assumptions of the premise p and then discharge each assumption at different stages. We could use the first assumption to derive the first conjunct of $p \land p$, and discharge that first. The second assumption, used to derive the second conjunct of $p \land p$ could be discharged second.

Using the conventions of Prawitz and his *Natural Deduction* [25], the two proofs look like this:[1]

$$\dfrac{\dfrac{\dfrac{(p)^1 \quad (p)^1}{p \land p}{\scriptstyle \land I}}{p \supset (p \land p)}{\scriptstyle \supset I^1}}{p \supset (p \supset (p \land p))}{\scriptstyle \supset I^2} \qquad \dfrac{\dfrac{\dfrac{(p)^1 \quad (p)^2}{p \land p}{\scriptstyle \land I}}{p \supset (p \land p)}{\scriptstyle \supset I^1}}{p \supset (p \supset (p \land p))}{\scriptstyle \supset I^2}$$

[1]These proof uses the convention that the superscripted numbers indicate the steps in a proof where a formula is discharged. So, in the first proof, the bracketed p assumptions are both discharged at the *first* $\supset I$ step. No assumptions are discharged at the *second* $\supset I$ step—none are marked with a 2—which is why that discharge is *vacuous*.

These two proofs look *very* similar. They differ only in the way that the assumptions are discharged. A proof of the same of shape as the proof on the left could prove the more general formula $q \supset (p \supset (p \wedge p))$, since there is no requirement that the vacuously discharged p be at all connected to the p in the rest of the formula. And a proof of the same shape as the proof on the right could prove $q \supset (p \supset (p \wedge q))$, since there is no requirement in this proof that the first assumption (giving the first conjunct) and the second assumption (giving the second) be the same formula, since they are discharged separately.

One straighforward and perspicuous way to represent the different structures of these proofs is by way of Church's λ-terms [5, 15, 20]. We annotate the proofs with terms as follows:[2]

$$\dfrac{\dfrac{\dfrac{(x:p)^1 \qquad (x:p)^1}{\langle x,x\rangle : p \wedge p}\wedge I}{\lambda x\langle x,x\rangle : p \supset (p \wedge p)}\supset I^1}{\lambda y\lambda x\langle x,x\rangle : p \supset (p \supset (p \wedge p))}\supset I^2 \qquad \dfrac{\dfrac{\dfrac{(x:p)^1 \qquad (y:p)^2}{\langle x,y\rangle : p \wedge p}\wedge I}{\lambda x\langle x,y\rangle : p \supset (p \wedge p)}\supset I^1}{\lambda y\lambda x\langle x,y\rangle : p \supset (p \supset (p \wedge p))}\supset I^2$$

Variables annotate assumptions, and the term constructors of pairing and λ-abstraction correspond to the introduction of conjunctions and conditionals respectively. Now the terms corresponding to the proofs bear the marks of the different proof behaviour. The first proof took an assumption p to $p \wedge p$ (pairing x with itself), and discharged an unneeded extra assumption of p. The second assumed p twice (tagging these assumptions with x and y), conjoined their result (in that order) and disharged them in turn (also in that order). Seeing this, you may realise that there are two other proofs of the same formula. One where the conjuncts are formed in the other order (with term $\lambda y\lambda x\langle y,x\rangle$) and the other, where the *first* discharged assumption of p is vacuous, not the second (with term $\lambda x\lambda y\langle x,x\rangle$). Each of these terms describe a proof of $p \supset (p \supset (p \wedge p))$, and each proves that tautology in a distinctive way.

We have already seen two different ways to represent the same underlying "way of proving"— Prawitz's natural deduction proofs, and its associated λ-term. This should at least hint at the idea that identity of "proof" in this sense is something deeper than any particular syntactic representation. It's one thing to say that something may be proved in different ways because it could be proved in English, or in French or in some other language. (Or on a blackboard or a whiteboard or a paper or screen.) It's another to say that the underlying *proofs* may differ in a deeper sense than their representations. Here is another hint of why this might be the case: These propositional proofs may be represented in yet another way, in Gentzen's sequent calculus. The two proofs can be seen to correspond to the following sequent derivations, which we annotate with λ-terms to display the parallel structure:

$$\dfrac{\dfrac{\dfrac{x:p \succ x:p \qquad x:p \succ x:p}{x:p \succ \langle x,x\rangle : p \wedge p}\wedge R}{\succ \lambda x\langle x,x\rangle : p \supset (p \wedge p)}\supset R}{\succ \lambda y\lambda x\langle x,x\rangle : p \supset (p \supset (p \wedge p))}\supset R \qquad \dfrac{\dfrac{\dfrac{x:p \succ x:p \qquad y:p \succ y:p}{x:p, y:p \succ \langle x,y\rangle : p \wedge p}\wedge R}{y:p \succ \lambda x\langle x,y\rangle : p \supset (p \wedge p)}\supset R}{\succ \lambda y\lambda x\langle x,y\rangle : p \supset (p \supset (p \wedge p))}\supset R$$

[2] Here, and elsewhere in the paper, *terms* are coloured *red* while formulas and derivations are *black*, to depict the two different levels of abstraction.

In both derivations pairing goes with the $\wedge R$ rule, and λ abstraction, with $\supset R$. In the first, the assumed p is tagged with the variable x in both branches, so when the branches join, one single instance of the assumption remains, and it is discharged in the next step, and the remaining discharge is vacuous. The variable y is is not present in $\succ \lambda x\langle x, x\rangle : p \supset (p \wedge p)$, so the discharge at the next $\supset R$ step is vacuous. In the second derivation, the p assumptions are tagged with different variables, so they are discharged in different lines of the derivation, and neither is vacuous.

These two sequent derivations are different representations of the same underlying logical proof structure as the corresponding natural deduction proofs. Proof structure at this level of analysis is not to be identified with the syntactic representation of the proofs. It is some kind of *invariant*, shared by the diffrent representations.

This paper's focus is proof terms as invariants for derivations in the *classical* sequent calculus,[3] and before we attend to the distinctives of the classical system, it is important to attend to some of the distinctives of sequent systems as such. What is possible to do in parallel in natural deduction and in proof terms is sometimes *linearised* in the sequent calculus. Let's weaken our target formula to $(p \wedge q) \supset (p \supset (p \wedge p))$ and consider two different derivations, using $\wedge L$, and the *first projection* fst on terms.

$$\dfrac{\dfrac{\dfrac{x : p \succ x : p \qquad \dfrac{z : p \succ z : p}{y : p \wedge q \succ \mathsf{fst}\, y : p}\wedge L}{x : p, y : p \wedge q \succ \langle x, \mathsf{fst}\, y\rangle : p \wedge p}\wedge R}{y : p \wedge q \succ \lambda x\langle x, \mathsf{fst}\, y\rangle : p \supset (p \wedge p)}\supset R}{\succ \lambda y\lambda x\langle x, \mathsf{fst}\, y\rangle : (p \wedge q) \supset (p \supset (p \wedge p))}\supset R \qquad \dfrac{\dfrac{\dfrac{\dfrac{x : p \succ x : p \qquad z : p \succ z : p}{x : p, z : p \succ \langle x, z\rangle : p \wedge p}\wedge R}{z : p \succ \lambda x\langle x, z\rangle : p \supset (p \wedge p)}\supset R}{y : p \wedge q \succ \lambda x\langle x, \mathsf{fst}\, y\rangle : p \supset (p \wedge p)}\wedge L}{\succ \lambda y\lambda x\langle x, \mathsf{fst}\, y\rangle : (p \wedge q) \supset (p \supset (p \wedge p))}\supset R$$

The $\wedge L$ rule has the following general form

$$\dfrac{\Sigma, z : A, z' : B \succ M(z, z') : C}{\Sigma, y : A \wedge B \succ M(\mathsf{fst}\, y, \mathsf{snd}\, y) : C}\wedge L$$

and it allows one to transform a sequent appealing to (either or both of) $z : A$ and $z' : B$, to derive some consequent C with term $M(z, z')$ into a sequent using $y : A \wedge B$, to derive the same consequent C, but now instead of transforming the information z and z' using M, we transform y, by taking the first projection $\mathsf{fst}\, y$ (to extract A from $A \wedge B$) and the second projection $\mathsf{snd}\, y$ (to extract B from $A \wedge B$) and then proceeding to apply M as before.

Notice that the two different derivations have the same term—$\lambda y\lambda x\langle x, \mathsf{fst}\, y\rangle$—despite applying the rules in different orders. The term $\lambda y\lambda x\langle x, \mathsf{fst}\, y\rangle$ carries no information about the

[3]One reason for the focus on the classical sequent calculus and not classical natural deduction is that there is no well understood and well behaved natural deduction system for classical logic that has all of the symmetries present in the classical sequent calculus. The closest we can get is to use systems of proof with multiple premises and multiple conclusions [30,31,36], either of a traditional natural deduction sort, or more sophisticated *proof nets* [14,29], but introducing one of these systems would be too much trouble for what it would be worth in this paper—and the details of how such systems incorporate the structural rules of weakening and contraction is a delicate matter— so the focus remains on the sequent calculus, which much better understood.

relative order of application of the rules $\wedge L$ and $\wedge R$. This is a general feature of the sequent calculus. Different derivations represent the same underlying proof structure.

Another feature of this sequent calculus with terms is the way that the structural rules of contraction and weakening interact with the discipline of labelling. Consider the general structure of the sequent rule $\wedge R$.

$$\frac{X \succ A \quad Y \succ B}{X, Y \succ A \wedge B} \wedge R$$

If we take X and Y to be *lists* or *multisets* of formulas, then any formulas appearing in both X and Y will appear more than once in the sequent $X, Y \succ A \wedge B$. If we wish to discharge them in one go in a $\supset R$ inference, we apply the rule of contraction to reduce the many instances to one. This explicit appeal to a contraction inference can be avoided if you think of X and Y as *sets* of formulas, with no duplicates. But this choice makes it difficult to discharge different instances of the same formula in different inference steps, as we did in our proof with term $\lambda y \lambda x \langle x, y\rangle$, in which two instances of p are discharged in two successive inferences. The *term* sequent calculus avoids the explicit inference steps of contraction (and weakening) on the one hand, and allows for duplicate copies of the one formula discharged at different stages of the proof by making use of variables labelling formulas. In a term sequent calculus, sequents are composed of *sets* of *labelled* formulas. In the case of $\wedge R$ here, formulas occurring Σ_1 and Σ_2 sharing a variable are immediately contracted into the one instance in the union Σ_1, Σ_2, and those tagged with different variables remain distinct.[4] Terms for proofs contain explicit structure corresponding to the operational rules of the sequent calculus. The structural rules are implicit in the identity and difference of the variables in those terms.

The *Cut* rule in the sequent calculus is also not an operational rule. It corresponds in λ-terms to substitution, in the following way:

$$\frac{\Sigma_1 \succ M : A \qquad \Sigma_2, x : A \succ N(x) : C}{\Sigma_1, \Sigma_2 \succ N(M) : C} \; Cut$$

Substitution of this form gives scope for *simplification* in proofs. Take the case where the cut formula is a complex formula—here, a conjunction—introduced on both premises of the *Cut*:[5]

$$\frac{\dfrac{\Sigma_1 \succ M_1 : A \qquad \Sigma_2 \succ M_2 : B}{\Sigma_{1,2} \succ \langle M_1, M_2\rangle : A \wedge B} \wedge R \qquad \dfrac{\Sigma_3, y : A, z : B \succ N(y, z) : C}{\Sigma_3, x : A \wedge B \succ N(\mathsf{fst}\, x, \mathsf{snd}\, x) : C} \wedge L}{\Sigma_{1\text{–}3} \succ N(\mathsf{fst}\,\langle M_1, M_2\rangle, \mathsf{snd}\,\langle M_1, M_2\rangle) : C} \; Cut$$

Trading in this complex *Cut* for *Cut*s on the subformulas A and B) corresponds to *evaluating* the complexes $\mathsf{fst}\,\langle M_1, M_2\rangle$ and $\mathsf{snd}\,\langle M_1, M_2\rangle$ to M_1 and M_2 respectively.

[4] You can think of the variables in the different sequents $\Sigma_1 \succ A$ and $\Sigma_2 \succ B$ in the $\wedge R$ inference as their 'interface,' dictating how they are to interact when combined. Where is data to be shared between Σ_1 and Σ_2, and where is it kept apart? The same goes for any rule in which sequents are is combined.

[5] In this derivation, and elsewhere in the paper, I use the shorthand notation: $\Sigma_{1,2}$ for $\Sigma_1 \cup \Sigma_2$, and $\Sigma_{1\text{–}3}$ for $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, etc., to save space in sequent derivations.

$$\dfrac{\dfrac{\Sigma_1 \succ M_1 : A \qquad \Sigma_3, y : A, z : B \succ N(y,z) : C}{\Sigma_{1,3}, z : B \succ N(M_1, z) : C}\,\mathit{Cut} \qquad \Sigma_2 \succ M_2 : B}{\Sigma_{1-3} \succ N(M_1, M_2) : C}\,\mathit{Cut}$$

If we think of $\mathsf{fst}\langle M_1, M_2\rangle$ as *identical* to M_1 and $\mathsf{snd}\langle M_1, M_2\rangle$ as identical to M_2, then there is a sense in which $N(\mathsf{fst}\langle M_1, M_2\rangle, \mathsf{snd}\langle M_1, M_2\rangle)$ is just $N(M_1, M_2)$,[6] and these two derivations have the same invariant. Simplifying this *Cut* preserves the underlying connection from premises to the conclusion.

So much is well understood in the case of intuitionistic natural deduction, the intuitionistic sequent calculus and λ-terms.[7] Cut elimination in the classical sequent calculus brings with it a number of complications, primarily due to the availability of the structural rules of contraction and weakening on both sides of the sequent. Consider this simple derivation with a single *Cut*.[8]

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p \vee p \succ p, p}\,\vee L}{p \vee p \succ p}\,W \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \wedge p}\,\wedge R}{p \succ p \wedge p}\,W}{p \vee p \succ p \wedge p}\,\mathit{Cut}$$

We can eliminate the *Cut* by commuting it past the $\vee L$ step on the left, or past the $\wedge R$ on the right. This results in the following two derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \wedge p}\,\wedge R}{p \succ p \wedge p}\,W \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p, p \succ p \wedge p}\,\wedge R}{p \succ p \wedge p}\,W}{p \vee p \succ p \wedge p, p \wedge p}\,\vee L}{p \vee p \succ p \wedge p}\,W \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \succ p \qquad p \succ p}{p \vee p \succ p, p}\,\vee L}{p \vee p \succ p}\,W \qquad \dfrac{\dfrac{p \succ p \qquad p \succ p}{p \vee p \succ p, p}\,\vee L}{p \vee p \succ p}\,W}{p \vee p, p \vee p \succ p \wedge p}\,\wedge R}{p \vee p \succ p \wedge p}\,W$$

and these derivations are *different*. If we think of the process of eliminating cuts as *evaluating* complex terms to a canonical result, it is more than puzzling to have two different answers, depending on the order of evaluation.[9]

Having *weakening* on both sides of the sequent separator has more drastic consequences for cut elimination. Suppose we have two different derivations of the one sequent $\Sigma \succ \Delta$. We can combine them, using *weakening* and *Cut* as follows.[10]

[6] One should attempt to hold these two opposing thoughts at once. The number $2+2$ is identical to the number 4, while the term $2+2$ is not the same as the term 4. Philosophers may be reminded of the Fregean distinction between *sense* and *reference*, while computer scientists may be reminded of the difference between *call by name* and *call by value*. The term $\mathsf{fst}\langle M_1, M_2\rangle$ has the same value (referent) as the term M_1, while they remain different names (have different senses).

[7] This is not to say that it is straightforward or simple. The rules for *disjunction* and the existential quantifier in intuitionistic natural deduction introduce complications of their own [15, Chapter 10].

[8] '*W*' is for *contraction* here, following Curry's notation for the combinators [6, 28].

[9] Unless, of course, those two different derivations are different presentations of the same underlying *answer*. There is no conflict when one calculation returns the decimal number 2, while another returns the binary value 10. Do these derivations present the same underlying *proof*? That remains to be seen.

[10] '*K*' is Curry's name for the *weakening* combinator [6, 28].

$$\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ \Sigma \succ \Delta\end{matrix}}{\Sigma \succ C, \Delta}\,K \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\\ \Sigma \succ \Delta\end{matrix}}{\Sigma, C \succ \Delta}\,K}{\Sigma \succ \Delta}\,Cut$$

The standard procedure for eliminating the *Cut* gives us a choice: We can simplify the derivation by choosing δ_1 as the derivation of $\Sigma \succ \Delta$, or we could choose δ_2. If reduction is *identity* of the underlying logical connection between Σ and Δ, then distinctions between different derivations collapse—if the whole derivation is, to all intents and purposes, identical to δ_1, and it is, to all intents and purposes, identical to δ_2—then to all intents and purposes any derivation for $\Sigma \succ \Delta$ is identical to any other. Distinctions between derivations collapse.

This paper will address these issues. I will introduce a proof invariant for classical sequent derivations that shares many features with λ-terms for intuitionist derivations or natural deduction proofs. Proof terms for derivations with *Cut* involve explicit 'cut points' which may be *evaluated* to generate a proof term for a *Cut*-free derivation. This process of evaluation is strongly normalising (any procedure for evaluating cut points will terminate) and confluent (different reduction processes result in the same reduced proof term).

Proof terms give rise to an analysis of permutations of rules in derivations, and distinct derivations with the same proof term may be permuted into each other by rearranging inference steps which operate on different parts of the sequent. On this analysis, the two different *Cut*-free derivations of $p \vee p \succ p \wedge p$ are distinct representations of the same underlying proof, and one can be permuted into the other by permuting the $\wedge R$ above or below the $\vee L$.[11] Cut reduction on derivations need not be confluent, since many derivations correspond to the one proof term, even in the intuitionist case. Cut reduction on the underlying proof terms remains confluent, and different derivations are different representations of the same underlying logical connection.

For the triviality argument, in which a *Cut* on a weakened-in *Cut*-formula C leads to reductions to two distinct derivations, δ_1 and δ_2, the proof terms introduced here motivate a more sensitive analysis of Cut reduction steps for derivations. Instead of choosing *between* the derivations δ_1 and δ_2 of $\Sigma \succ \Delta$, the reduction procedure chooses *both*. This motivates a different kind of structural rule, here called '*Blend*'[12] which is natural at the level of proof terms, is well suited to classical logic,[13] and which allows reduction on terms to be confluent and non-trivial.

$$\dfrac{\Sigma_1 \succ \Delta_1 \qquad \Sigma_2 \succ \Delta_2}{\Sigma_{1,2} \succ \Delta_{1,2}}\,Blend$$

[11] When we permute a rule above a rule with two premises, it may *duplicate*, as happens here. In this case, the only way to make a derivation with *one* $\wedge R$ and *one* $\vee L$ step is to take those steps intependently and to compose them with a *Cut*.

[12] In the literature on linear logic this is called 'Mix' [14, page 99]. This name is used for a *different* structural rule in the literature on relevant logic, so I have opted for this new name.

[13] The *Blend* rule is not well suited to sequents for intuitionist logic, which have only one formula on the RHS. We cannot straightforwardly blend $\Sigma_1 \succ A$ and $\Sigma_2 \succ B$ because we only have one slot on the RHS in which to put both A and B.

Now, we can eliminate the *Cut* on the weakened-in C in a natural way, *blending* the two derivations δ_1 and δ_2, and not making an arbitrary choice between them.

$$\dfrac{\dfrac{\overset{\delta_1}{\vdots}\quad \Sigma \succ \Delta}{\Sigma \succ C, \Delta}{\scriptstyle K} \qquad \dfrac{\overset{\delta_2}{\vdots}\quad \Sigma \succ \Delta}{\Sigma, C \succ \Delta}{\scriptstyle K}}{\Sigma \succ \Delta}{\scriptstyle Cut} \qquad \textit{becomes} \qquad \dfrac{\overset{\delta_1}{\vdots}\quad \Sigma \succ \Delta \qquad \overset{\delta_2}{\vdots}\quad \Sigma \succ \Delta}{\Sigma \succ \Delta}{\scriptstyle Blend}$$

The remainder of this paper defines proof terms and examines their basic properties. We start with the definition of the sequent calculus and proof terms. In the process of defining the term calculus, we will face a number of choice points, and I will attempt to be explicit about these choices, and indicate where alternative paths could have been taken.

2 DERIVATIONS AND TERMS

Proof terms are to stand to derivations of classical sequents in a similar way to λ-terms for intuitionist sequents. However, there must be differences and distinctive features for the classical case. The classical sequent calculus is highly symmetric, with an underlying duality between premise and conclusion, the left hand side (LHS) and right (RHS). There is no principled difference in logical power between formulas on the LHS and those on the RHS. This differs derivations in the intutionist sequent calculus which have some number of formulas on the LHS (perhaps none) and always single formula on the RHS—-and intuitionist natural deduction proofs, which have some number (perhaps zero) of assumptions and a single conclusion. In an intuitionist derivation (proof) the formulas at the left (assumptions) are tagged with variables, and the formula on the right (conclusion) is tagged with a term, carrying the structure of the *proof.* This must be modified if we are to analyse the classical sequent calculus, and that is our first choice point: we *will* attempt to analyse the full range of classical sequent derivations.

CHOICE 1**:** Terms encode derivations for the full range of classical sequents of the form $X \succ Y$ with zero or more formulas in X (the LHS) and zero or more formulas in Y (the RHS).

This choice is not forced upon us. You could restrict attention to a smaller class of proofs, such as sequents with a single side (for sequents of the form $\succ Y$, or $X \succ$), or even restrict your attention to proofs of single formulas. However, if we can find a perspective from which proofs for sequents of such disparate forms as (1) $\succ A$ (showing that A is a tautology) (2) $B \succ$ (showing that B is a contradiction) (3) $A \succ B$ (showing that B follows from A) (4) $A, B \succ$ (showing that A and B are jointly inconsistent) and (5) $\succ A, B$ (showing that one of A and B must hold) are on equal footings, with none any more fundamental than the other, then this would seem to be consonant with the expressive power of the classical sequent calculus.

Consider, for example, the following two simple sequent derivations, and how we might represent these with something like λ terms:

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}{\scriptstyle \neg L}}{p \land \neg p \succ}{\scriptstyle \land L} \qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}{\scriptstyle \neg R}}{\succ p \lor \neg p}{\scriptstyle \lor R}$$

If at the end of the first derivation we tag the premise $p \land \neg p$ with a variable, then there is no formula in the sequent $p \land \neg p \succ$ to tag with the term of the proof. In the second derivation we have two formulas in the RHS. If one formula in the RHS is to be tagged with the term of the proof, which is it to be? There seems to be no principled option to choose. Instead of tagging a conclusion formula with a term, we tag the sequent separator itself, and tag the formulas on the LHS and RHS with variables. Term sequents, then, will have this structure:[14]

$$x_1 : A_1, \ldots, x_n : A_n \overset{\pi(x_1,\ldots,x_n)[y_1,\ldots,y_m]}{\succ} y_1 : B_1, \ldots, y_m : B_m$$

where the proof term $\pi(x_1, \ldots, x_n)[y_1, \ldots, y_m]$ relates the *inputs* $x_1, \ldots, x_n$ and the *outputs* $y_1, \ldots, y_m$. The proof term π shows how the atoms in the derived sequent $A_1, \ldots, A_n \succ B_1, \ldots B_m$ are related, and how information flows inside the underlying proof. So, we have made the following choice:

CHOICE 2: A proof term for the sequent $X \succ Y$ shows how the formulas in X and Y are related in the proof of that sequent. The premises and conclusions are treated in the same way, encoded as variables, and the proof term encodes their relationship.

We can represent proof terms in a number of different ways. The representation that we will mostly use in this paper is a collection of links between *nodes*, where the nodes in a proof term are representations of instances of *atomic formulas* in a sequent, in a way that we will make precise soon. Here are two examples of derivations annotated with these proof terms:

$$\dfrac{\dfrac{x:p \overset{x \curvearrowright y}{\succ} y:p}{x:p, z:\neg p \overset{x \curvearrowright \grave{\neg}z}{\succ}}\,\neg L}{w:p \land \neg p \overset{\acute{\land}w \curvearrowright \grave{\neg}\grave{\land}w}{\succ}}\,\land L \qquad \dfrac{\dfrac{x:p \overset{x \curvearrowright y}{\succ} y:p}{\overset{\grave{\neg}z \curvearrowright y}{\succ} y:p, z:\neg p}\,\neg R}{\overset{\grave{\neg}\grave{\lor}v \curvearrowright \acute{\lor}v}{\succ} v:p \lor \neg p}\,\lor R$$

Here, $\acute{\land}z \curvearrowright \grave{\neg}\grave{\land}w$ is a proof term with two nodes, $\acute{\land}w$ (picking out the first conjunct of w, in this case the first 'p' in 'p ∧ ¬p'), and $\grave{\neg}\grave{\land}w$ (picking out the negand of second conjunct of w, which is the second 'p' in 'p ∧ ¬p'). In the proof term $\acute{\land}w \curvearrowright \grave{\neg}\grave{\land}w$, the variable w occurs as an input, and it has no outputs. (This corresponds to the fact that the variable w occurs on the left side of the sequent we derived.) For the second derivation, $\grave{\neg}\grave{\lor}v \curvearrowright \acute{\lor}v$ is a proof term with v as an *output* (and no inputs).

In the first derivation, the terms work like this, as the derivation is built up: $x \curvearrowright y$, the axiom, tells us that the input formula (p, tagged with x) matches the output formula (tagged with y). At the second stage, $x \curvearrowright \grave{\neg}z$ tells us that of the two inputs x and z, the x input is the same as the thing *negated* in the z input. So, in $\grave{\neg}z$, the '$\grave{\neg}$' unpacks the negation to return the negand. Then, the final term $\acute{\land}w \curvearrowright \grave{\neg}\grave{\land}w$ says that the *left conjunct* of the input w matches as thing negated in the *right conjunct* of the input w. So here, $\acute{\land}w$ and $\grave{\land}w$ select the *left* and *right*

[14] I first learned of annotated sequents with this structure in Philip Wadler's typescript "Down with the bureaucracy of syntax! Pattern matching for classical linear logic" [38], which introduces terms of this kind for classical linear logic. I since learned that the idea at least dates back to Frank Pfenning's work on structural cut elimination in linear logic [24, page 9].

conjuncts respectively of the conjunction formula tagged with w—and hence the orientation of the accents. The same goes for the term $\grave{\neg}\grave{\vee}v \curvearrowright \acute{\vee}v$ for the derivation of $v : p \vee \neg p$. Here, the thing negated in the *right* disjunct of v is the same as the *left* disjunct of v.

A proof term shows how information flows in the derivation it annotates. Proof terms can also be represented graphically, as a directed graph on the atoms of each sequent in the derivation:

$$\dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L \qquad\qquad \dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R$$

(In this representation, I have elided the variables, though they still play a role in the management of contractions.) Either representation of the underlying flow of information is possible, though in the rest of the paper, I will use the term definition, as the parallel with well-understood λ-terms is most clear. Nonetheless, the heritage in Sam Buss's *flow graphs* for classical derivations is important to acknowledge [2–4], as is Došen's work on generality [7] in proofs, and Lamarche and Straßburger's work on classical proof nets [19].

Enough of examples and motivation—now for the definitions.

DEFINITION 1 [FORMULAS] The formulas in our language are constructed from a countable collection $p_0, p_1, p_2, \ldots$ of atomic formulas, closed under the binary connectives $\wedge$, $\vee$ and $\supset$, the unary operator $\neg$ and the two constants $\top$ and $\bot$, in the usual manner.[15] We use p, q, r, etc., as schematic letters ranging over atomic formulas, and A, B, C, etc., as schematic letters ranging over formulas.

This definition reflects another choice.

CHOICE 3: Each logical constant, $\wedge, \vee, \supset, \neg, \top, \bot$ is treated as primitive, and not defined in terms of other logical constants. Furthermore, the proof term system is designed to be *separable* and *modular*. Rules for a concept will use that concept and will not appeal to other logical constants, so we could take the collection of rules for (as an example) the fragment of the language involving $\wedge$, $\vee$, $\top$ and $\bot$, in which $\neg$ and $\supset$ are not defined, and attend to proofs in this restricted vocabulary.

Other options are psossible: We could take $\neg$ and $\wedge$ (for example), as primitive, and define the other connectives in terms of them. To do this, however, would be to make the relationship between the defined connective and the concepts in terms of which it is defined *invisible* to proof analysis, and it would make it impossible to reason about fragments of the language.

DEFINITION 2 [VARIABLES AND CUT NODES] For each formula A, we have countably many VARIABLES of type A. To be specific $x_1^A, x_2^A, \ldots x_n^A, \ldots$ are variables of type A, and we use u, v, w, x, y, z as schematic letters ranging over variables, omitting the type annotation except where

[15] We could extend things further, to include separate connectives for every definable connective in the propositional language. The only reason for not doing this space. This paper is long enough as it is.

it is useful to mention it. In addition, for each formula A we have a CUT NODE of type A. For definiteness, we use $\bullet^{A}$ for the cut node of type A, and we use $\bullet, \star, *, \sharp, \flat$ as schemas ranging over cut nodes, again, omitting the type annotation except for where it is helpful.

Cut nodes and variables are both kinds of NODES. We use n, m as schemas ranging over nodes.

DEFINITION 3 [NODES AND SUBNODES] A NODE is either a VARIABLE or a CUT NODE, or a COMPLEX NODE defined as follows:

- If n has type $\neg A$, then $\grave{\neg}n$, the *negand* of n, has type A.
- If n has type $A \wedge B$, its *first* and *second conjuncts* $\acute{\wedge}n$ and $\grave{\wedge}n$, have type A and B respectively.
- If n has type $A \vee B$, its *first* and *second disjuncts* $\acute{\vee}n$ and $\grave{\vee}n$ have type A and B respectively.
- If n has type $A \supset B$, its *antecedent* and *consequent* $\acute{\supset}n$ and $\grave{\supset}n$ have type A and B respectively.

For each complex node $\acute{\vee}n$, $\grave{\vee}n$, $\acute{\wedge}n$, $\grave{\wedge}n$, $\acute{\supset}n$, $\grave{\supset}n$ and $\grave{\neg}n$, we say that n is one of its SUBNODES (its IMMEDIATE subnode), and the subnodes of n are the other subnodes of that complex node.

The node/subnode relation is, in a sense, a converse of the formula/subformula relation. For example, if n has type $A \wedge B$, it is a *subnode* of the complex node $\acute{\wedge}n$ of type A, while A is a *subformula* of the formula $A \wedge B$. A complex node is a path into a constituent of a formula.

DEFINITION 4 [LINKS AND INPUT/OUTPUT POSITIONS] A LINK is a pair of nodes $n\curvearrowright m$ of the same type, or a single node $n\curvearrowright$ (where n has type $\bot$), or a single node $\curvearrowright m$ (where m has type $\top$). In the link $n\curvearrowright m$, or in $n\curvearrowright$, or $\curvearrowright m$, we say that left node n is in INPUT position and the right node m is in OUTPUT position. The TYPE of the link is the type of the nodes in the link.

We generalise positions to subnodes in a link as follows:

- If $\acute{\vee}n$, $\grave{\vee}n$, $\acute{\wedge}n$, $\grave{\wedge}n$ or $\grave{\supset}n$ are in input position in a link, the indicated immediate subnode n is also in input position in that link. If $\acute{\vee}n$, $\grave{\vee}n$, $\acute{\wedge}n$, $\grave{\wedge}n$ or $\grave{\supset}n$ are in output position, the indicated immediate subnode n is also in output position in that link. We say that $\acute{\vee}$, $\grave{\vee}$, $\acute{\wedge}$, $\grave{\wedge}$ and $\grave{\supset}$ each PRESERVE position.
- If $\acute{\supset}n$ or $\grave{\neg}n$ is in input position in a link, the indicated immediate subnode n is in output position in that link. If $\acute{\supset}n$ or $\grave{\neg}n$ is in output position in a link, the indicated immediate subnode n is in input position. We say that $\acute{\supset}$ and $\grave{\neg}$ REVERSE position in that link.

Finally we say that the INPUTS in the link (if any) are the *variables* in input position in that link, and the OUTPUTS (if any) are the *variables* in output position in that link. Notice that since each link contains either zero, one or two (occurrences of) variables, each link has at most two inputs and outputs in total.

EXAMPLE 1 [A LINK] Consider, for example, the following link

$$\acute{\supset}\acute{\supset}\acute{\supset}x_1^{((p\supset q)\supset p)\supset p} \curvearrowright \grave{\supset}x_1^{((p\supset q)\supset p)\supset p}$$

This satisfies the conditions to be a link. The node in output position, $\grave{\supset}x_1^{((p\supset q)\supset p)\supset p}$, picks out the consequent of the formula $((p \supset q) \supset p) \supset p$, so it has type p. The inpu t node $\acute{\supset}\acute{\supset}\acute{\supset}x_1^{((p\supset q)\supset p)\supset p}$ also has type p, since p is the antecedent of the antecedent of the antecedent of $((p \supset q) \supset p) \supset p$. Dropping the superscripted type, we have $\acute{\supset}\acute{\supset}\acute{\supset}x_1$ is in *input* position, $\grave{\supset}x_1$ is in *output* position. The x_1 in $\grave{\supset}x_1$ is also in *output* position since $\grave{\supset}$ preserves position, while in $\acute{\supset}\acute{\supset}\acute{\supset}x_1$ (in input position), $\acute{\supset}\acute{\supset}x_1$ (of type $(p \supset q) \supset p$) is in *output* position, $\acute{\supset}x_1$ (of type $p \supset q$) is in *input* position, and x_1 is in *output* position. So, the variable x_1 is the only output of the link $\acute{\supset}\acute{\supset}\acute{\supset}x_1 \curvearrowright \grave{\supset}x_1$, and the link has no inputs.

Were we to replace the variable x_1 in the output node $\grave{\supset}x_1$ by a different variable of the same type (or any variable of type $A \supset p$ for some formula A), the the link would have *two* outputs. If we were to replace the variable in $\grave{\supset}x_1$ by a cut node of a suitable type, to result in $\acute{\supset}\acute{\supset}\acute{\supset}x_1 \curvearrowright \grave{\supset}\bullet$, the result would still be a link with the sole output x_1.

Derivations will be annotated by sets of links satisfying certain conditions—they are the connections between parts of formulas that account for information flow in the derivation. Before explaining those conditions, we will see some examples and develop the notation for representing and working with proof terms.

DEFINITION 5 [PRETERMS, THEIR INPUTS, OUTPUTS AND TYPES] A PRETERM is any finite set of links. The INPUTS of a preterm are the inputs of any of its links, and its OUTPUTS are the outputs of any of its links. We use π, π', etc., as schematic letters ranging over preterms.

If π is a preterm, x is a variable of type A and n is an input node of the same type, then $\pi(x := n)$ is the result of replacing any occurrences of the variable x in input position in π by n. Instead of π and $\pi(x := n)$ we may, as usual, represent these preterms as $\pi(x)$ and $\pi(n)$.[16] We may generalise this to allow for more input variables. Given a preterm $\pi(x_1, \ldots, x_m)$, the preterm $\pi(n_1, \ldots, n_m)$ is the result of replacing the variables $x_1, \ldots, x_m$ in π by $n_1, \ldots, n_m$.

If π is a preterm, y is a variable of type A and m is an output node of the same type, then $\pi[y := m]$ is the result of replacing any occurrences of the variable y in output position in π by m. Instead of π and $\pi[y := m]$ we may, as usual, represent these preterms as $\pi[y]$ and $\pi[m]$.[17] We may generalise this to allow for more input variables. Given a preterm $\pi[y_1, \ldots, y_n]$, the preterm $\pi[m_1, \ldots, m_n]$ is the result of replacing the variables $y_1, \ldots, y_n$ in π by $m_1, \ldots, m_n$.

Finally if π is a preterm and its inputs are among the variables $x_1, \ldots, x_n$ of types $A_1, \ldots, A_n$ respectively, and its outputs are among the variables $y_1, \ldots, y_m$ of types $B_1, \ldots, B_m$ respectively, then we say that π is a preterm OF TYPE $x_1{:}A_1, \ldots, x_n{:}A_n \succ y_1{:}B_1, \ldots, y_m{:}B_m$. A preterm of type $\Sigma \succ \Delta$ is a candidate for describing a derivation of $\Sigma \succ \Delta$.[18]

[16] This is intended to include cases where x does not actually occur in $\pi(x)$ in input position, in which case the preterm will be identical to $\pi(n)$.

[17] This is intended to include cases where y does not occur in $\pi[y]$ in output position, in which case the preterm will be identical to $\pi[m]$.

[18] Not all preterms succeed in describing a derivation, no more than all formulas assert truths. In the next section, we give an independent characterisation of those preterms that describe derivations—the *terms*.

So, consider the preterm

$$\acute{\wedge}x \curvearrowright \grave{\neg}\acute{\vee}y \quad \grave{\wedge}x \curvearrowright \grave{\neg}\grave{\vee}y$$

where x has type $p \wedge q$ and y has type $\neg p \vee \neg q$ (so $\acute{\wedge}x$ and $\grave{\neg}\acute{\vee}y$ have type p, and $\grave{\wedge}x$ and $\grave{\neg}\grave{\vee}y$ have type q). In this term, x and y both occur as inputs. So, the term has type

$$x : p \wedge q, y : \neg p \vee \neg q \succ$$

If we think of this term as $\pi(x)$, singling out x as an input, and if z has type $\neg(p \wedge q)$, then $\grave{\neg}z$ has type $p \wedge q$, the same type as x, so $\pi(\grave{\neg}z)$ is the following preterm

$$\acute{\wedge}\grave{\neg}z \curvearrowright \grave{\neg}\acute{\vee}y \quad \grave{\wedge}\grave{\neg}z \curvearrowright \grave{\neg}\acute{\vee}y$$

which still has y as an input, but now has z as an *output*, since $\grave{\neg}$ *reverses* position.

DEFINITION 6 [COMPLEXITY FOR PRETERMS] The complexity of a *node* is defined as follows. The complexity of a *variable* is 0. The complexity of a *cut node* is 1. If n is an complex node, its complexity is the complexity of its immediate subnode plus one. The complexity of a *link* $n \curvearrowright m$ is the sum of the complexities of n and of m. Finally, the complexity of a preterm is the sum of the complexities of its links.

With preterms defined, we can now define the rules of the sequent calculus, alongside their annotations with terms.

DEFINITION 7 [LABELLED SEQUENTS] A labelled sequent has the form

$$\Sigma \overset{\pi}{\succ} \Delta$$

where π is a preterm of type $\Sigma \succ \Delta$. As is usual in the classical sequent calculus, one of Σ or Δ may be empty.

As is usual in the sequent calculus, the sequent $\Sigma \succ \Delta$ can be seen as claiming that Δ follows from Σ. Here, the labels are added so that the proof term π has some way to express *how* that logical connection is made, connecting the constituents of the formulas in Σ and Δ.

Now we have the resources to provide the rules for derivations in the labelled sequent calculus, along with their proof terms. We start with the axioms.

DEFINITION 8 [BASIC AXIOMS] The BASIC AXIOMS of the sequent calculus are the labelled sequents of the following three forms: $\bot L$, *Identity*, and $\top R$:

$$\Sigma, x : \bot \overset{x \curvearrowright}{\succ} \Delta \qquad \Sigma, x : p \overset{x \curvearrowright y}{\succ} y : p, \Delta \qquad \Sigma \overset{\curvearrowright y}{\succ} y : \top, \Delta$$

Each kind of sequent follows the general scheme for labelled sequents. In $\bot L$ and *Identity* sequents, the variable x appears on the LHS, and it is the input variable in the proof term (either $x^\frown y$ or $x^\frown$). In *Identity* and $\top R$ sequents, the variable y appears on the RHS, and it is the output variable in the proof term (either $x^\frown y$ or $^\frown y$). The other variables present in the sequent (those tagging formulas in Σ or Δ play no role in this justification of the claim to consequence made by the sequent, and do not appear in the proof term—they are irrelevant bystanders).

As mentioned in at the end of the first section of this paper, the proof calculus is closed under the structural rule *Blend*.

$$\dfrac{\Sigma_1 \overset{\pi_1}{\succ} \Delta_1 \qquad \Sigma_2 \overset{\pi_2}{\succ} \Delta_2}{\Sigma_{1,2} \overset{\pi_1\ \pi_2}{\succ} \Delta_{1,2}}\ \textit{Blend}$$

This structural rule is not a primitive rule of the system, but will be shown (after the whole system is introuced) to be an admissible rule. (We show that if there are derivations of the premise sequents, there is also a derivation of the concluding sequent.) At the level of derivability, *Blend* is a consequence of the weakening rule. (If there is some derivation of $\Sigma_1 \succ \Delta_1$ then we can weaken it to the derivation of $\Sigma_{1,2} \succ \Delta_{1,2}$, by the addition of the irrelevent and unused side formulas Σ_2 and Δ_2.) But in the proof term calculus, this is not enough to meet our ends, for the resulting derivation would appeal only to the connections established in the proof π_1, and not to those in π_2. To properly blend the two proofs, we want to use both of them. This is possible with one minor emendation to how the sequent calculus is interpreted. We expand our set of basic axioms just a little, including no new axiomatic sequents, but expanding our family of axiomatic proof terms.

DEFINITION 9 [BLENDED SEQUENTS] Given a family

$$\Sigma_i \overset{\pi_i}{\succ} \Delta_i$$

of labelled sequents for each i in some finite index set I, their BLEND is the labelled sequent

$$\bigcup_{i\in I} \Sigma_i \overset{\bigcup_{i\in I} \pi_i}{\succ} \bigcup_{i\in I} \Delta_i$$

which collects together the LHS and RHS formulas of the basic axioms and combines their proof terms.

With this definition in hand, we fill out the family of axioms.

DEFINITION 10 [AXIOMS] The AXIOMS of the sequent calculus are each of the BLENDs of the basic axioms.

Notice that the blend of a finite set of axiomatic sequents is a sequent which appears in a basic axiom. (All axiomatic sequents, whether basic or not, either has a $\bot$ on the left, or an atomic proposition on both sides, or a $\top$ on the right.) However, the *non*-basic axioms differ from their basic cousins by having proof terms with more than one link. Here is a non-basic axiom.

$$x:p, y:q, z:r \land \neg s \overset{x^\frown x\ \ ^\frown w}{\succ} x:p, y:q, w:\top$$

It is, for example, the blend of these two basic axioms:

$$\overset{x\curvearrowright x}{x:p, z:r\wedge\neg s \succ x:p} \qquad \overset{\curvearrowright w}{y:q \succ y:q, w:\top}$$

in which $z:r\wedge\neg s$ and $y:q$ are irrelevant and unused bystanders. Notice that the sequent $y:q \succ y:q, w:\top$ is *also* the blend of two axioms, and could have been annotated with the proof term $\curvearrowright w\;\; y\curvearrowright y$. In this proof calculus, that is a different axiom, though the underlying sequent is the unchanged. You can think of different proof terms as giving different (axiomatic) justifications of the same underlying sequent. One way to justify $y:q \succ y:q, w:\top$ is to point out that q occurs on the left and the right (that is the link $y\curvearrowright y$). Another is to note the presence of $\top$ on the right (that is the link $\curvearrowright w$). Yet another is to give *both* justifications. These three justifications differ in that they *generalise* in different ways. One ($y\curvearrowright y$) would suffice to ground sequents in which $\top$ does not occur. Another ($\curvearrowright w$) justifies sequents in which q does not occur. The combined ground ($\curvearrowright w\;\; y\curvearrowright y$) does neither. With that understanding of the axioms in place, we can move to the rules of inference.

DEFINITION 11 [INFERENCE RULES] The rules of inference are transitions between labelled sequents, or from a pair of labelled sequents to another labelled sequents. The inference rules are presented in Figure 1. Each rule is given schematically, specifying labelled inference schemata for the premises of the rule, and defining the appropriate substitutions to make to construct the concluding sequent of the rule.

For example, the conjunction right rule:

$$\frac{\overset{\pi_1[x]}{\Sigma_1 \succ x:A, \Delta_1} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y:B, \Delta_2}}{\overset{\pi_1[\acute{\wedge} z]\;\; \pi_2[\grave{\wedge} z]}{\Sigma_{1,2} \succ z:A\wedge B, \Delta_{1,2}}}\;\wedge R$$

is read as follows: where $\Sigma_1 \succ x:A, \Delta_1$ is a labelled sequent with proof term $\pi_1[x]$, and $\Sigma_2 \succ y:B, \Delta_2$ is a labeled sequent with proof term $\pi_2[y]$, then we can conclude $\Sigma_{1,2} \succ z:A\wedge B, \Delta_{1,2}$ with the proof term $\pi_1[\acute{\wedge} z]\;\; \pi_2[\grave{\wedge} z]$, where z is a variable (possibly already occurring in one of the premise sequents, but also possibly fresh), and $\pi_1[\acute{\wedge} z]$ is the proof term found by replacing the output occurrences of x in $\pi_1[x]$ by $\acute{\wedge} z$, and $\pi_2[\grave{\wedge} z]$ is the proof term found by replacing the output occurrences of y in $\pi_2[y]$ by $\grave{\wedge} z$. Recall, Σ_1, Σ_2, Δ_1, Δ_2 are all *sets* of labelled formulas, and $\Sigma_{1,2}$ is the union of the two sets Σ_1 and Σ_2, so some contractions of formulas may occur in this rule, if they share the same label in Σ_1 and Σ_2 (or Δ_1 and Δ_2).

Some caveats in order. In $\pi_1[x]$ and $\pi_2[y]$, the output variables x and y may not actually *appear* in $\pi_1[x]$ and $\pi_2[y]$—consider the case where the sequent $\Sigma_1 \succ x:A, \Delta_1$ is an axiom, grounded in some feature of Σ_1 and Δ_1, where A is an unused bystander. In that case, the variable x will not appear in the proof term. This is why the substitution $\pi_1[\acute{\wedge} z]\;\; \pi_2[\grave{\wedge} z]$ is to be understood as replacing all output instances of x in $\pi_1[x]$ and y in $\pi_2[y]$ by $\acute{\wedge} z$ and $\grave{\wedge} z$ respectively, even when the number of those instances is *zero*. With that understanding in place, we can proceed to some derivations. Here are two different derivations of the same labelled sequent.

CONJUNCTION:

$$\frac{\overset{\pi(x,y)}{\Sigma, x:A, y:B \succ \Delta}}{\overset{\pi(\acute{\wedge}z,\grave{\wedge}z)}{\Sigma, z:A \wedge B \succ \Delta}}\wedge L \qquad \frac{\overset{\pi_1[x]}{\Sigma_1 \succ x:A, \Delta_1} \quad \overset{\pi_2[y]}{\Sigma_2 \succ y:B, \Delta_2}}{\overset{\pi_1[\acute{\wedge}z]\ \pi_2[\grave{\wedge}z]}{\Sigma_{1,2} \succ z:A \wedge B, \Delta_{1,2}}}\wedge R$$

DISJUNCTION:

$$\frac{\overset{\pi_1(x)}{\Sigma_1, x:A \succ \Delta_1} \quad \overset{\pi_2(y)}{\Sigma_2, y:B \succ \Delta_2}}{\overset{\pi_1(\acute{\vee}z)\ \pi_2(\grave{\vee}z)}{\Sigma_{1,2}, z:A \vee B \succ \Delta_{1,2}}}\vee L \qquad \frac{\overset{\pi[x,y]}{\Sigma \succ x:A, y:B, \Delta}}{\overset{\pi[\acute{\vee}z,\grave{\vee}z]}{\Sigma \succ z:A \vee B, \Delta}}\vee R$$

CONDITIONAL:

$$\frac{\overset{\pi_1[x]}{\Sigma_1 \succ x:A, \Delta_1} \quad \overset{\pi_2(y)}{\Sigma_2, y:B \succ \Delta_2}}{\overset{\pi_1[\acute{\supset}z]\ \pi_2(\grave{\supset}z)}{\Sigma_{1,2}, z:A \supset B \succ \Delta_{1,2}}}\supset L \qquad \frac{\overset{\pi(x)[y]}{\Sigma, x:A \succ y:B, \Delta}}{\overset{\pi(\acute{\supset}z)[\grave{\supset}z]}{\Sigma \succ z:A \supset B, \Delta}}\supset R$$

NEGATION:

$$\frac{\overset{\pi[x]}{\Sigma \succ x:A, \Delta}}{\overset{\pi[\grave{\neg}z]}{\Sigma, z:\neg A \succ \Delta}}\neg L \qquad \frac{\overset{\pi(x)}{\Sigma, x:A \succ \Delta}}{\overset{\pi(\acute{\neg}z)}{\Sigma \succ z:\neg A, \Delta}}\neg R$$

CUT:

$$\frac{\overset{\pi_1[x]}{\Sigma_1 \succ x:A, \Delta_1} \quad \overset{\pi_2(y)}{\Sigma_2, y:A \succ \Delta_2}}{\overset{\pi_1[\bullet]\ \pi_2(\bullet)}{\Sigma_{1,2} \succ \Delta_{1,2}}}\mathit{Cut}$$

Figure 1: Inference Rules of the Sequent Calculus

EXAMPLE 2 Two derivations of the sequent $\succ u:(p \wedge q) \supset (p \supset (p \wedge p))$, with the term $\acute{\supset}\grave{\supset}u \curvearrowright \acute{\wedge}\grave{\supset}\grave{\supset}u\ \ \acute{\wedge}\acute{\supset}u \curvearrowright \grave{\wedge}\grave{\supset}\grave{\supset}u$.

$$\frac{\dfrac{\overset{x \curvearrowright x}{x:p \succ x:p} \qquad \dfrac{\overset{z \curvearrowright z}{z:p \succ z:p}}{\overset{\acute{\wedge}y \curvearrowright z}{y:p \wedge q \succ z:p}}\wedge L}{\dfrac{\overset{x \curvearrowright \acute{\wedge}w\ \ \acute{\wedge}y \curvearrowright \grave{\wedge}w}{x:p, y:p \wedge q \succ w:p \wedge p}}{\overset{\acute{\supset}v \curvearrowright \acute{\wedge}\grave{\supset}v\ \ \acute{\wedge}y \curvearrowright \grave{\wedge}\grave{\supset}v}{y:p \wedge q \succ v:p \supset (p \wedge p)}}\supset R}\wedge R$$

$$\frac{\overset{\acute{\supset}v \curvearrowright \acute{\wedge}\grave{\supset}v\ \ \acute{\wedge}y \curvearrowright \grave{\wedge}\grave{\supset}v}{y:p \wedge q \succ v:p \supset (p \wedge p)}}{\overset{\acute{\supset}\grave{\supset}u \curvearrowright \acute{\wedge}\grave{\supset}\grave{\supset}u\ \ \acute{\wedge}\acute{\supset}u \curvearrowright \grave{\wedge}\grave{\supset}\grave{\supset}u}{\succ u:(p \wedge q) \supset (p \supset (p \wedge p))}}\supset R$$

$$\frac{\overset{x \curvearrowright x}{x:p \succ x:p} \qquad \overset{z \curvearrowright z}{z:p \succ z:p}}{\dfrac{\overset{x \curvearrowright \acute{\wedge}w\ \ z \curvearrowright \grave{\wedge}w}{x:p, z:p \succ w:p \wedge p}}{\dfrac{\overset{\acute{\supset}v \curvearrowright \acute{\wedge}\grave{\supset}v\ \ z \curvearrowright \grave{\wedge}\grave{\supset}v}{z:p \succ v:p \supset (p \wedge p)}}{\overset{\acute{\supset}v \curvearrowright \acute{\wedge}\grave{\supset}v\ \ \acute{\wedge}y \curvearrowright \grave{\wedge}\grave{\supset}v}{y:p \wedge q \succ v:p \supset (p \wedge p)}}\wedge L}\supset R}\wedge R$$

$$\frac{\overset{\acute{\supset}v \curvearrowright \acute{\wedge}\grave{\supset}v\ \ \acute{\wedge}y \curvearrowright \grave{\wedge}\grave{\supset}v}{y:p \wedge q \succ v:p \supset (p \wedge p)}}{\overset{\acute{\supset}\grave{\supset}u \curvearrowright \acute{\wedge}\grave{\supset}\grave{\supset}u\ \ \acute{\wedge}\acute{\supset}u \curvearrowright \grave{\wedge}\grave{\supset}\grave{\supset}u}{\succ u:(p \wedge q) \supset (p \supset (p \wedge p))}}\supset R$$

These two derivations use the same inference rules but appeal to them in different orders. They end with the same proof term, in which the antecedent of the consequent of the formula (the second occurrence of p) is linked to the first conjunct of the consequent of the consequent of the formula (the third occurrence of p), and the first conjunct of the antecedent of the formula (the first occurrence of p) is linked to the second conjunct of the consequent of the consequent of the formula (the last occurrence of p). This is an example of how proof terms collapse the merely syntactic difference between two derivations differing in inessential order of application of the rules. On the other hand, we can have *different* proof terms annotating the same sequent, when the sequent is derived in a manner grounded in an appeal to different linkings between the atoms, such as the proof term given in this derivation.

$$\dfrac{\dfrac{\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p} \qquad \dfrac{\overset{z\curvearrowright z}{z:p\succ z:p}}{\overset{\acute{\land}y\curvearrowright z}{y:p\land q\succ z:p}}\,{\land}L}{\overset{x\curvearrowright\grave{\land}w\ \ \acute{\land}y\curvearrowright\acute{\land}w}{x:p,\, y:p\land q\succ w:p\land p}}\,{\land}R}{\overset{\acute{\supset}v\curvearrowright\grave{\land}\grave{\supset}v\ \ \acute{\land}y\curvearrowright\acute{\land}\grave{\supset}v}{y:p\land q\succ v:p\supset(p\land p)}}\,{\supset}R}{\overset{\acute{\supset}\grave{\supset}u\curvearrowright\grave{\land}\grave{\supset}\grave{\supset}u\ \ \acute{\land}\acute{\supset}u\curvearrowright\acute{\land}\grave{\supset}\grave{\supset}u}{\succ u:(p\land q)\supset(p\supset(p\land p))}}\,{\supset}R$$

Here, the linkings are swapped: the second p is linked to the fourth p and the first p is linked to the third.

When reasoning about derivations and inference rules, it will help to follow a distinction introduced by Raymond Smullyan in the 1960s, between α (linear) rules, and β (branching) rules [32, 33]. The α rules for binary connectives (conjunction on the left, and disjunction and the conditional on the right) takes a sequent in which two *formulas* (the subformulas) occur in their specified *position* (the *left* or *right*), tagged with a *variable*. The output of the rule is that same sequent, in which those tagged formulas are deleted, and replaced by the *result* in its specified *position*, tagged with a specified *variable*. In the proof term, the the variables for the specified subformulas are replaced by new nodes, using the specified *constructors* applied to the given *variable*. So, to characterise the rule, we need to specify the variables, formulas and position of the premise formulas, and the variable, position, formula and constructors for the output sequent.

The same holds for the negation rule, which is an α rule for a unary connective. Following Smullyan, I will treat this as a degenerate case of an α rule for a binary connective, in which the two subformulas (and corresponding variables and positions) are the same. This means that we can specify each of the five rules by giving the information in this single table:

RULE	$\{a_1:\alpha_1\}$		$\{a_2:\alpha_2\}$		$\{a:\alpha\}$		$\acute{\alpha}$	$\grave{\alpha}$
$\land L$	left	A	left	B	left	$A\land B$	$\acute{\land}$	$\grave{\land}$
$\lor R$	right	A	right	B	right	$A\lor B$	$\acute{\lor}$	$\grave{\lor}$
$\supset R$	left	A	right	B	right	$A\supset B$	$\acute{\supset}$	$\grave{\supset}$
$\neg L$	right	A	right	A	left	$\neg A$	$\grave{\neg}$	$\grave{\neg}$
$\neg R$	left	A	left	A	right	$\neg A$	$\grave{\neg}$	$\grave{\neg}$

where, for example, in $\supset R$, one subformula (the antecedent) A occurs on the left, while the other (the consequent) B occurs on the right of the premise sequent, while the result has the formula $A \supset B$ occurring on the right. The variables for the premises are prefixed by the constructors $\acute{\supset}$ and $\grave{\supset}$ respectively. Given the information on the row of this table, each rule can be characterised as follows:

$$\frac{\overset{\pi\{a_1\}\{a_2\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a : \alpha\}}}$$

where the premise sequent $\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}$ has variable a_1 tagging formula α_1 in its specified position (given in the table, represented here by the curly braces), and variable a_2 tagging formula α_2 in its specified position. The resulting sequent replaces those occurrences (leaving the rest) and replaces them by variable a tagging formula α in *its* specified position. The proof term $\pi\{a_1\}\{a_2\}$ for the premise sequent is transformed into $\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}$, in which the appropriate occurrences (input or output) of the variables a_1 and a_2 are replaced by the nodes $\acute{\alpha}a$ and $\grave{\alpha}a$ respectively. Inspecting each of the α rules shows them to all be of this form.

The remaining rules are branching, β, rules, with two premises. Each inference has two premise sequents, each with one of the two *formulas* (the *subformulas*, in the case of connective rules, the *Cut* formula in the case of the *Cut* rule) occurring in a specified *position* in each premise sequents, and tagged with a *variable*. The concluding sequent of the rule collects together the side formulas of both sequents, deletes the given tagged formulas (the subformulas or cut formulas) and replaces them with the *resulting* formula in its specified *position*, tagged by the specified *variable*—except in the case of the Cut rule, where there is no resulting formula. The proof term collectes together the proof terms of the premise sequents, where the appropriate variables for the specified subformulas are replaced by new nodes—using the specified *constructors* applied to the given *variable* in the case of a connective rule, or using the specified *cut point* of the type of the cut formula, in the case of the *Cut* rule. As before, we can specify each of the four β rules in a single table:

RULE	$\{b_1 : \beta_1\}$		$\{b_2 : \beta_2\}$		$\{b : \beta\}$		$\acute{\beta}$	$\grave{\beta}$
$\wedge R$	right	A	right	B	right	$A \wedge B$	$\acute{\wedge}$	$\grave{\wedge}$
$\vee L$	left	A	left	B	left	$A \vee B$	$\acute{\vee}$	$\grave{\vee}$
$\supset L$	right	A	left	B	left	$A \supset B$	$\acute{\supset}$	$\grave{\supset}$
Cut	right	A	left	A	—		$\bullet$	$\bullet$

Each β inference then has the following shape, given the choice of *variables*, *positions* and *formulas*:

$$\frac{\overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1 : \beta_1\}} \qquad \overset{\pi_2\{b_2\}}{\mathfrak{S}_2\{b_2 : \beta_2\}}}{\overset{\pi_1\{\acute{\beta}b\} \;\; \pi_2\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{b : \beta\}}}$$

Characterising rules at this level of abstractness simplifies the presentation of our results about the proof calculus. We start with the following theorem for the structural rules.

THEOREM 1 [HEIGHT PRESERVING ADMISSIBILITY] *The structural rules of weakening, contraction and blend are height-preserving admissible in the following senses. For the single premise rules of weakening and contraction*

$$\frac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma, x:A \succ \Delta}}\,KL \qquad \frac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x:A, \Delta}}\,KR \qquad \frac{\overset{\pi(x,y)}{\Sigma, x:A, y:A \succ \Delta}}{\overset{\pi(x,x)}{\Sigma, x:A \succ \Delta}}\,WL \qquad \frac{\overset{\pi[x,y]}{\Sigma \succ x:A, y:A, \Delta}}{\overset{\pi[x,x]}{\Sigma \succ x:A, \Delta}}\,WR$$

if there is a derivation δ of the premise of the rule, of height $h(\delta)$, then there is a derivation δ' of the conclusion of the rule, of the same height, $h(\delta)$. For the two premise rule of blend

$$\frac{\overset{\pi}{\Sigma \succ \Delta} \qquad \overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi\ \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\,Blend$$

if there is a derivation δ of the premise $\Sigma \succ \Delta$ (with term π) and a derivation δ' of the premise $\Sigma' \succ \Delta'$ (with term π') then there is a derivation of the conclusion $\Sigma, \Sigma' \succ \Delta, \Delta'$ (with term $\pi\ \pi'$) with height $\max(h(\delta), h(\delta'))$.

For this theorem, we need to make precise the notion of the height of a derivation.

DEFINITION 12 [DERIVATION HEIGHT] The height of a derivation is the height of its longest branch, counting the number of applications of application of inference rules along that branch. (The height of an axiom is 0.) Given a derivations δ, δ_1 and δ_2 with heights $h(\delta)$, $h(\delta_1)$ and $h(\delta_2)$ respectively the derivation

$$\frac{\begin{array}{c}\delta\\ \vdots\\ \overset{\pi\{a_1\}\{a_2\}}{\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}}\end{array}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a:\alpha\}}} \qquad \frac{\begin{array}{c}\delta_1\\ \vdots\\ \overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1:\beta_1\}}\end{array} \qquad \begin{array}{c}\delta_2\\ \vdots\\ \overset{\pi_2\{b_2\}}{\mathfrak{S}_2\{b_2:\beta_2\}}\end{array}}{\overset{\pi_1\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{b:\beta\}}}$$

have heights $h(\delta) + 1$ and $\max(h(\delta_1), h(\delta_2)) + 1$, respectively.

Proof: We can prove Theorem 1 by induction on the heights of derivations. For *weakening*, the result is trivial. If $\Sigma \succ \Delta$ is an axiom with term π, then the addition of $x:A$ on the left or the right of the sequent gives us another axiom with the same proof term. If the derivation is not an axiom, then it ends in either an α or a β rule, and we may assume that the hypothesis holds for all shorter derivations, including the derivation of the premise(s) of that rule. The derivations end in either an α or a β rule, as follows:

$$\frac{\begin{array}{c}\delta\\ \vdots\\ \overset{\pi\{a_1\}\{a_2\}}{\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}}\end{array}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a:\alpha\}}} \qquad \frac{\begin{array}{c}\delta_1\\ \vdots\\ \overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1:\beta_1\}}\end{array} \qquad \begin{array}{c}\delta_2\\ \vdots\\ \overset{\pi_1\{b_2\}}{\mathfrak{S}_2\{b_2:\beta_2\}}\end{array}}{\overset{\pi_1\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{b:\beta\}}}$$

By hypothesis, we can transform derivations δ, δ_1 and δ_2 with the addition of $x:A$ as a new premise or conclusion without disturbing the proof term. Let those derivations be δ', δ_1' and δ_2' respectively, and let $\mathfrak{S}'$, $\mathfrak{S}_1'$ and $\mathfrak{S}_2'$ be the sequents with $x:A$ added on the left (or the right). The new derivations

$$\dfrac{\begin{array}{c}\delta' \\ \vdots \\ \pi_1\{a_1\}\{a_2\} \\ \mathfrak{S}'\{a_1:\alpha_1\}\{a_2:\alpha_2\}\end{array}}{\begin{array}{c}\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\} \\ \mathfrak{S}'\{a:\alpha\}\end{array}} \qquad \dfrac{\begin{array}{c}\delta_1' \\ \vdots \\ \pi_1\{b_1\} \\ \mathfrak{S}_1'\{b_1:\beta_1\}\end{array} \qquad \begin{array}{c}\delta_2' \\ \vdots \\ \pi_2\{b_2\} \\ \mathfrak{S}_2'\{b_2:\beta_2\}\end{array}}{\begin{array}{c}\pi_1\{\acute{\beta}b\} \quad \pi_2\{\grave{\beta}b\} \\ \mathfrak{S}_{1,2}'\{b:\beta\}\end{array}}$$

are instances of exactly the same α or β rules, and the resulting derivations have the same height as before.

For *contraction*, we reason in a similar way. If $\Sigma, x:A, y:A \succ \Delta$ is an axiomatic sequent with proof term $\pi(x,y)$, then $\Sigma, x:A \succ \Delta$ is an axiomatic sequent with term $\pi(x,x)$: If A is neither an atom nor $\bot$, so it is unlinked in the axiomatic proof term, so x and y don't appear as output variables in $\pi(x,y)$, hence this term *is* $\pi(x,x)$, or A is an atom or $\bot$ and we replace the y-input links in the axiom with x-input links. The same holds for x and y in the RHS instead of the LHS. The result is another axiom, and the derivation has the same height: 0.

If the derivation not an axiom, then it ends in either an α or a β rule, and we may assume that the hypothesis holds for all shorter derivations, including the derivations of the premise(s) of that rule. By hypothesis, we have derivations in which all y variables in the conclusion of those derivations are shifted to x. Now consider the conclusion of the α or β inference. If y was not *introduced* as a variable in the inference, we are done: the conclusion has all appropriate ys shifted to x. If y is introduced as a variable in that inference, then we transform the inference to introduce x instead, and we are done. (We can choose the variable to be introduced in an α or β rule at will—it need not be fresh.)

For *blend*, we proceed as follows. The blend of two axiomatic sequents (of height 0) is itself an axiom (also of height 0). Now consider the case where one of the two sequents in the premise of the blend is derived in a derivation of height > 0. In this case it ends in either an α or a β rule. So, the derivation has one of the following two forms (where without loss of generality we consider the case where the second premise of the blend ends in the α or β inference):

$$\dfrac{\begin{array}{c}\delta_1 \\ \vdots \\ \pi_1 \\ \mathfrak{S}_1\end{array} \qquad \dfrac{\begin{array}{c}\delta_2 \\ \vdots \\ \pi_2\{a_1\}\{a_2\} \\ \mathfrak{S}_2\{a_1:\alpha_1\}\{a_2:\alpha_2\}\end{array}}{\begin{array}{c}\pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\} \\ \mathfrak{S}_2\{a:\alpha\}\end{array}}\alpha}{\begin{array}{c}\pi_1 \quad \pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\} \\ \mathfrak{S}_{1,2}\{a:\alpha\}\end{array}}\,\textit{Blend} \qquad \dfrac{\begin{array}{c}\delta_1 \\ \vdots \\ \pi_1 \\ \mathfrak{S}_1\end{array} \qquad \dfrac{\begin{array}{c}\delta_2 \\ \vdots \\ \pi_2\{b_2\} \\ \mathfrak{S}_2\{b_2:\beta_2\}\end{array} \qquad \begin{array}{c}\delta_3 \\ \vdots \\ \pi_3\{b_3\} \\ \mathfrak{S}_3\{b_3:\beta_3\}\end{array}}{\begin{array}{c}\pi_2\{\acute{\beta}b\} \quad \pi_3\{\grave{\beta}b\} \\ \mathfrak{S}_{2,3}'\{b:\beta\}\end{array}}\beta}{\begin{array}{c}\pi_1 \quad \pi_2\{\acute{\beta}b\} \quad \pi_3\{\grave{\beta}b\} \\ \mathfrak{S}_{1-3}'\{b:\beta\}\end{array}}\,\textit{Blend}$$

In either case, we can blend the result of derivation δ_1 with one of the premises of the inference, applying the induction hypothesis on the shorter derivation, and then apply the rule *after* the blend. The only subtlety in this transformation is the possibility that the deferred α or β rule might now capture variables in the proof term π_1 and the sequent $\mathfrak{S}_1$. To avoid this, we transform the derivations δ_2 and δ_3, shifting the variables a_1, a_2, or b_2 and b_3 to new variables (of the same type), not present in $\mathfrak{S}_1$, namely a_1', a_2', or b_2' and b_3' respectively. In this way, no extra variables are captured in the deferred inference step, and the derivations end in the same labelled sequent.

$$
\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots \\ \pi_1 \\ \mathfrak{S}_1\end{matrix} \qquad \begin{matrix}\delta_2\{a_1'/a_1\}\{a_2'/a_2\}\\ \vdots \\ \pi_2\{a_1'\}\{a_2'\} \\ \mathfrak{S}_2\{a_1':\alpha_1\}\{a_2':\alpha_2\}\end{matrix}}{\begin{matrix}\pi_1 \quad \pi_2\{a_1'\}\{a_2'\}\\ \mathfrak{S}_{1,2}\{a_1':\alpha_1\}\{a_2':\alpha_2\}\end{matrix}}\,\mathit{Blend}}{\begin{matrix}\pi_1 \quad \pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\\ \mathfrak{S}_{1,2}\{a:\alpha\}\end{matrix}}\,\alpha
\qquad
\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots \\ \pi_1 \\ \mathfrak{S}_1\end{matrix} \qquad \begin{matrix}\delta_2\{b_2'/b_2\}\\ \vdots \\ \pi_2\{b_2'\} \\ \mathfrak{S}_2\{b_2':\beta_2\}\end{matrix}}{\begin{matrix}\pi_1 \quad \pi_2\{\acute{\beta}b\}\\ \mathfrak{S}_{1,2}\{b_2':\beta_2\}\end{matrix}}\,\mathit{Blend} \qquad \begin{matrix}\delta_3\{b_3'/b_3\}\\ \vdots \\ \pi_3\{b_3'\} \\ \mathfrak{S}_3\{b_3':\beta_3\}\end{matrix}}{\begin{matrix}\pi_1 \quad \pi_2\{\acute{\beta}b\} \quad \pi_3\{\grave{\beta}b\}\\ \mathfrak{S}_{1-3}'\{b:\beta\}\end{matrix}}\,\beta
$$

in which the *Blend* occurs higher in the derivation, and is height preserving admissible, by induction. ■

The proof of the admissibility of *Blend* required the transformation of derivations by replacing free variables. This notion can be formally defined as follows:

DEFINITION 13 [VARIABLE SHIFTING] Given variables x and x' of the same type, and a specified position (input or output) the x'-FOR-x-SHIFT of the derivation δ — $\delta\{x'/x\}$ — is defined inductively on the construction of δ as follows.

- If δ is an axiom then $\delta\{x'/x\}$ is found by replacing any instances of x in δ in the appropriate position by x'. If there are no such instances, then $\delta\{x'/x\}$ is identical to δ.
- If we extend δ by an α step or a β step as follows:

$$
\dfrac{\begin{matrix}\delta\\ \vdots \\ \pi\{a_1\}\{a_2\} \\ \mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\end{matrix}}{\begin{matrix}\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\\ \mathfrak{S}\{a:\alpha\}\end{matrix}}
\qquad
\dfrac{\begin{matrix}\delta_1\\ \vdots \\ \pi_1\{b_1\} \\ \mathfrak{S}_1\{b_1:\beta_1\}\end{matrix} \qquad \begin{matrix}\delta_2\\ \vdots \\ \pi_1\{b_2\} \\ \mathfrak{S}_2\{b_2:\beta_2\}\end{matrix}}{\begin{matrix}\pi_1\{\acute{\beta}b\} \quad \pi_2\{\grave{\beta}b\}\\ \mathfrak{S}_{1,2}\{b:\beta\}\end{matrix}}
$$

and x does not appear in the end sequent of the extended derivation (in the appropriate position), then the x-to-x' shift of that derivation is that derivation itself. (There is no variable in the end sequent that needs shifting. So, x may have been used in the derivation up to that point, but that does not need to shift, as it is not free in the conclusion.) If, on the other hand,

x is in the end sequent of the extended derivation (in its appropriate position), then the x-to-x' shift of the new derivation is defined as follows:

$$
\frac{\begin{array}{c}\delta\{x/x'\}\\ \vdots \\ \scriptstyle \pi'\{a_1\}\{a_2\} \\ \mathfrak{S}'\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}\end{array}}{\begin{array}{c}\scriptstyle \pi'\{\acute{\alpha} a'\}\{\grave{\alpha} a'\} \\ \mathfrak{S}'\{a' : \alpha\}\end{array}}
\qquad
\frac{\begin{array}{c}\delta_1\{x/x'\}\\ \vdots \\ \scriptstyle \pi'_1\{b_1\} \\ \mathfrak{S}'_1\{b_1 : \beta_1\}\end{array} \qquad \begin{array}{c}\delta_2\{x/x'\}\\ \vdots \\ \scriptstyle \pi'_1\{b_2\} \\ \mathfrak{S}'_2\{b_2 : \beta_2\}\end{array}}{\begin{array}{c}\scriptstyle \pi'_1\{\acute{\beta} b'\} \quad \pi'_2\{\grave{\beta} b'\} \\ \mathfrak{S}'_{1,2}\{b' : \beta\}\end{array}}
$$

where in the α inference step, a' is the same as the variable a, unless a is identical to x (in the required position), in which case a' is x'. Similarly, in the β inference step, b' is b, unless b is x (in the required position), in which case b' is x'.

The effect of this definition is that in any derivation δ in which the variable x appears in the nominated position in the endsequent, it is replaced by x' in all of the derivation leading up to that use of x—either from axioms or from α or β rules where it is introduced into a derivation. *Other* uses of that variable, absorbed by α or β rules are left unchanged. The result is a derivation of exactly the same shape, but with an end sequent in which the variable x is replaced by x' in the nominated position.

This is the term sequent calculus. It is a relatively standard sequent calculus for classical propositional logic in which the standard structural rules of weakening and contraction—and the novel structural rule *Blend*—are admissible. The explicit inference rules are the left and right rules for the connectives and *Cut*.

This proof system incorporates one other choice that has not been mentioned so far. The restriction of the identity axioms to atomic formulas is intentional. Of course, one can derive identity sequents of the form $x : A \succ x : A$ for each complex formula A.

DEFINITION 14 [IDENTITY SEQUENTS] For each formula A and variable x of type A, the identity derivation $Id_{(x)[y]}(A)$ with term $x \rightleftarrows y$ is defined inductively in terms of the structure of A, with clauses given in Table 2.

The presence of identity derivations like $Id_{(x)[y]}(A)$ raises a question concerning proof identity. What is the relationship between this proof for $A \succ A$ and the axiomatic derivation consisting of $A \succ A$ itself—if that is included as an axiom? We will not spend time in this paper examining the categorical structure of proof terms (that is reserved for a sequel), but a comment is worth making at this point. The category of proof terms (if it is to be a category) will have for each formula A a canonical identity arrow (proof) $id_A : A \to A$. Given proofs $f : A \to B$ and $g : C \to D$, it is straightforward (in the manner we have seen above) to canonically define a proof $f \wedge g : A \wedge C \to B \wedge D$—and similarly for the other connectives. This raises the question: what is the relationship between $id_{A\wedge B}$ and $id_A \wedge id_B$? In the proof term calculus, we identify $id_{A\wedge B}$ and $id_A \wedge id_B$ by *defining* $id_{A\wedge B}$ to be $id_A \wedge id_B$. The reason for this is twofold. First, it simplifies cut reduction in the term calculus. Second, the problem of classical proof identity is

$$Id(p) \qquad \overset{x \curvearrowright y}{x : p \succ y : p}$$

$$Id(\top) \qquad \overset{\curvearrowright y}{x : \top \succ y : \top} \qquad\qquad Id(\bot) \qquad \overset{x \curvearrowright}{x : \bot \succ y : \bot}$$

$$Id(A \wedge B) \qquad \dfrac{\dfrac{\dfrac{Id_{(x_1)[y_1]}(A)}{\overset{x_1 \rightrightarrows y_1}{x_1 : A \succ y_1 : A}} \quad \dfrac{Id_{(x_2)[y_2]}(B)}{\overset{x_2 \rightrightarrows y_2}{x_2 : B \succ y_2 : B}}}{\overset{x_1 \rightrightarrows \grave{\wedge} y \quad x_2 \rightrightarrows \grave{\wedge} y}{x_1 : A, x_2 : B \succ y : A \wedge B}} \wedge R}{\overset{\acute{\wedge} x \rightrightarrows \acute{\wedge} y \ \grave{\wedge} x \rightrightarrows \grave{\wedge} y}{x : A \wedge B \succ y : A \wedge B}} \wedge L$$

$$Id(A \vee B) \qquad \dfrac{\dfrac{\dfrac{Id_{(x_1)[y_1]}(A)}{\overset{x_1 \rightrightarrows y_1}{x_1 : A \succ y_1 : A}} \quad \dfrac{Id_{(x_2)[y_2]}(B)}{\overset{x_2 \rightrightarrows y_2}{x_2 : B \succ y_2 : B}}}{\overset{\acute{\vee} x \rightrightarrows y_1 \ \grave{\vee} x \rightrightarrows y_2}{x : A \vee B \succ y_1 : A, y_2 : B}} \vee L}{\overset{\acute{\vee} x \rightrightarrows \acute{\vee} y \ \grave{\vee} x \rightrightarrows \grave{\vee} y}{x : A \vee B \succ y : A \vee B}} \vee R$$

$$Id(\neg A) \qquad \dfrac{\dfrac{\dfrac{Id_{(x_1)[y_1]}(A)}{\overset{x_1 \rightrightarrows y_1}{x_1 : A \succ y_1 : A}}}{\overset{\grave{\neg} y \rightrightarrows y_1}{\succ y_1 : A, y : \neg A}} \neg R}{\overset{\grave{\neg} y \rightrightarrows \grave{\neg} x}{x : \neg A \succ y : \neg A}} \neg L$$

$$Id(A \supset B) \qquad \dfrac{\dfrac{\dfrac{Id_{(x_1)[y_1]}(A)}{\overset{x_1 \rightrightarrows y_1}{x_1 : A \succ y_1 : A}} \quad \dfrac{Id_{(x_2)[y_2]}(B)}{\overset{x_2 \rightrightarrows y_2}{x_2 : B \succ y_2 : B}}}{\overset{x_1 \rightrightarrows \acute{\supset} x \quad \grave{\supset} x \rightrightarrows y_2}{x : A \supset B, x_1 : A \succ y_2 : B}} \supset L}{\overset{\acute{\supset} y \rightrightarrows \acute{\supset} x \quad \grave{\supset} x \rightrightarrows \grave{\supset} y}{x : A \supset B \succ y : A \supset B}} \supset R$$

Figure 2: The definition of identity terms and derivations

primarily due to the power of natural identities between proofs. To show as much as possible to be safe without collapse, we admit as many natural identities between proofs as possible, without collapse.

CHOICE 4: There are no primitive identity axioms for complex formulas. We identify the identity proof for the complex formula $A \wedge B$ to itself with the natural composition of identity proofs on A and B (and similarly for all complex formulas).

One consequence of this choice is that in proof terms (those terms generated by derivations), all links have atomic type.[19] If x is a term of type $p \wedge q$, then $x \curvearrowright x$ is a perfectly acceptable link—and an acceptable preterm, according to Definition 5—but such a link will never appear in a derivation. Not every preterm will label a derivation. In the next section we give an independent characterisation of exactly which preterms are describe proofs. We will answer the question: Which preterms are proof terms?

[19] To be more precise, a link $m \curvearrowright n$ has type p for some atomic formula p, a link $m \curvearrowright$ has type $\bot$ and a link $\curvearrowright n$ has type $\top$. In what follows, we will call these ATOMIC LINKS.

3 TERMS AND CORRECTNESS

We have seen one way that a preterm can fail to be a proof term—by containing non-atomic links. Here is another example of a preterm that will not annotate a derivation. If x has type $p \vee q$ and y has type $p \wedge q$, then

$$\acute{\vee}x \frown \acute{\wedge}y \quad \grave{\vee}x \frown \grave{\wedge}y$$

is a set of links—and in this case, a set of links with atomic type. The link $\acute{\vee}x \frown \acute{\wedge}y$ has type p and $\grave{\vee}x \frown \grave{\wedge}y$ has type q. This is never going to be generated by any derivation, for if it were generated by some derivation, that would be a derivation of the labelled sequent $x : p \vee q \succ y : p \wedge q$, and this sequent is has no derivation at all.

It turns out that there is an independent characterisation of those preterms that count as proof terms, and this characterisation can give us some insight into the behaviour of proof terms. One natural way to understand proof terms is to think of them as describing *how* information flows from the inputs of a sequent to its outputs. The input of $\acute{\vee}x \frown \acute{\wedge}y \ \grave{\vee}x \frown \grave{\wedge}y$ has the *disjunctive* type $(p \vee q)$ and the output has the *conjunctive* type $(p \wedge q)$. Successful information flow requires more than just some link between input and output (there are two links from input to output), but for a disjunctive input, we require not only a link from one disjunct, but from both. (We want the link to be maintained if the input information is p, or if it is q.) For a conjunctive output, we require not only a link to one conjunct, but to both. (We want the link to be maintained for the p conjunct, as well as for q.) And we need this to hold in arbitrary combinations. This is what fails in the case of our preterm: there is no route from the p input to the q output.

What is common between conjunction nodes in output position and disjunction nodes in input position in a preterm? They both arise out of branching (β) rules in a derivation. The other nodes in arising out of β rules are conditional nodes input position, and cut points. These are the said to be the *switch* nodes in a preterm.

DEFINITION 15 [SWITCH PAIRS AND SWITCHINGS] The SWITCH PAIRS for a preterm τ are the pairs $[\acute{\wedge}n][\grave{\wedge}n]$ in output position, $(\acute{\vee}n)(\grave{\vee}n)$ in input position, $[\acute{\supset}n](\grave{\supset}n)$ in output and input position respectively, and $[\bullet](\bullet)$ in output and input position respectively, such that *at least one* of the nodes in the pair is a subnode of τ in its nominated position. The *switchings* of preterm τ are given by selecting one node for each switch pair for τ and deleting each link in τ containing that node as a subnode in the required position.

Consider the following preterm:

$$\acute{\wedge}u \frown \acute{\wedge}\acute{\vee}t \ \ \acute{\vee}\grave{\wedge}u \frown \grave{\wedge}\acute{\vee}t \ \ \grave{\vee}\grave{\wedge}u \frown \grave{\vee}t$$

where u has type $p \wedge (q \vee r)$ and t has type $(p \wedge q) \vee r$. It has two switch pairs $[\acute{\wedge}\acute{\vee}t][\grave{\wedge}\acute{\vee}t]$ and $(\acute{\vee}\grave{\wedge}u)(\grave{\vee}\grave{\wedge}u)$. The nodes $\acute{\wedge}u$ and $\grave{\wedge}u$ could occur in a switch pair, but only when they are present in output position. Here they are in input position, so they are not switched. So, the four switchings of this proof term are found by making the following deletions:

1. delete $[\acute{\wedge}\acute{\vee}t]$, delete $(\acute{\vee}\grave{\wedge}u)$ — result: $\cancel{\acute{\wedge}u \frown \acute{\wedge}\acute{\vee}t} \ \ \cancel{\acute{\vee}\grave{\wedge}u \frown \grave{\wedge}\acute{\vee}t} \ \ \grave{\vee}\grave{\wedge}u \frown \grave{\vee}t$

2. delete $[\acute{\land}\acute{\lor}t]$, delete $(\grave{\lor}\grave{\land}u)$ — result: $\cancel{\acute{\land}u \curvearrowright \acute{\land}\acute{\lor}t}\ \ \acute{\lor}\grave{\land}u \curvearrowright \grave{\land}\acute{\lor}t\ \ \cancel{\grave{\lor}\grave{\land}u \curvearrowright \grave{\lor}t}$

3. delete $[\grave{\land}\acute{\lor}t]$, delete $(\acute{\lor}\grave{\land}u)$ — result: $\acute{\land}u \curvearrowright \acute{\land}\acute{\lor}t\ \ \cancel{\acute{\lor}\grave{\land}u \curvearrowright \grave{\land}\acute{\lor}t}\ \ \grave{\lor}\grave{\land}u \curvearrowright \grave{\lor}t$

4. delete $[\grave{\land}\acute{\lor}t]$, delete $(\grave{\lor}\grave{\land}u)$ — result: $\acute{\land}u \curvearrowright \acute{\land}\acute{\lor}t\ \ \cancel{\acute{\lor}\grave{\land}u \curvearrowright \grave{\land}\acute{\lor}t}\ \ \cancel{\grave{\lor}\grave{\land}u \curvearrowright \grave{\lor}t}$

DEFINITION 16 [SPANNED PRETERMS] A preterm is said to be *spanned* if and only if every switching of that term is is non-empty.

DEFINITION 17 [ATOMIC PRETERMS] A preterm is said to be *atomic* if and only if every link in that preterm is either of type p (for some atomic proposition p) or of type $\perp$ or of type $\top$.

THEOREM 2 [TERMS ARE ATOMIC AND SPANNED] *For every derivation δ, the term $\tau(\delta)$ is atomic and spanned.*

Proof: This is shown by an easy induction on the height of the derivation δ. If δ has height 0, then it is an axiom, and the proof term for each axiom is clearly atomic, and it is spanned because it is nonempty and it has no switched pairs.

Now suppose δ has height $n+1$ and the induction hypothesis holds for derivations of height n and smaller. First, the term $\tau(\delta)$ remains atomic because each inference rule preserves the type of links from premise to conclusion.

Second, consider last inference in δ. If it is an α inference, then this inference adds no switched pairs to the proof term, and if any switching of the proof term of the premise is spanned, so is any switching of the proof term of the conclusion, because the terms differ only in non-switched subnodes.

If the last step in δ is a β inference, then its conclusion contains one switched pair, the subnodes $\acute{\beta}b$ and $\grave{\beta}b$ as presented below:

$$\dfrac{\begin{array}{c}\delta_1\\ \vdots\\ \pi_1\{b_1\}\\ \mathfrak{S}_1\{b_1 : \beta_1\}\end{array} \qquad \begin{array}{c}\delta_2\\ \vdots\\ \pi_2\{b_2\}\\ \mathfrak{S}_2\{b_2 : \beta_2\}\end{array}}{\begin{array}{c}\pi_1\{\acute{\beta}b\}\ \ \pi_2\{\grave{\beta}b\}\\ \mathfrak{S}_{1,2}\{b : \beta\}\end{array}}$$

By hypothesis, the terms $\pi_1\{b_1\}$ and $\pi_2\{b_2\}$ are both spanned. The switched pairs in the term $\pi_1\{\acute{\beta}b\}\ \pi_2\{\grave{\beta}b\}$ are the pair $\{\acute{\beta}b\}\{\grave{\beta}b\}$, together with the switched pairs in $\pi_1\{b_1\}$ and those in $\pi_2\{b_2\}$. In any selection of deletions from the pairs other than $\{\acute{\beta}b\}\{\grave{\beta}b\}$, $\pi_1\{b_1\}$ and $\pi_2\{b_2\}$ are nonempty, and therefore, so do $\pi_1\{\acute{\beta}b\}$ and $\pi_2\{\grave{\beta}b\}$. When we delete one of $\{\acute{\beta}b\}\{\grave{\beta}b\}$, at most of $\pi_1\{\acute{\beta}b\}$ and $\pi_2\{\grave{\beta}b\}$ is deleted, and the other remains. So, $\pi_1\{\acute{\beta}b\}\ \pi_2\{\grave{\beta}b\}$, that is, $\tau(\delta)$ is spanned. ■

In the rest of this section, we devote ourselves to the proof of the converse of this theorem.

THEOREM 3 [SPANNED PRETERMS ARE TERMS] *If a preterm is spanned and atomic, then it is the term of some derivation.*

The proof for this theorem will be more complex than the straightforward induction on the derivation, but not condsiderably so. Since we have the preterm in our hands, the induction will be on the complexity of the preterm. We show that preterms with no complex nodes correspond to derivations (these correspond to axioms), and that *complex* terms can be decomposed from the outside in. To do this, we need to pay a little more attention to the structure of preterms. Consider the following (spanned and atomic) preterm

$$\acute{\wedge}x \curvearrowright \bullet \quad \bullet \curvearrowright \grave{\neg}\acute{\vee}y \quad \grave{\wedge}x \curvearrowright \grave{\neg}\grave{\vee}y$$

where x has type $p \wedge q$, $\bullet$ has type p and y has type $\neg p \vee \neg q$. This is a complex preterm, with complex nodes $\acute{\wedge}x$, $\grave{\wedge}x$ (in input position) $\grave{\neg}\acute{\vee}y$, $\grave{\neg}\grave{\vee}y$ (in output position) $\acute{\vee}y$, $\grave{\vee}y$ (in input position) as well as the *Cut* node, $\bullet$ (occurring in both input and output position). It follows that any derivation with this term involves at least one $\wedge L$ inference, at least one $\vee L$ inference, two $\neg L$ inferences, and at least one *Cut*. However, there is some flexibility in the order in which such inferences can occur. There is *some* flexibility, but not complete freedom. The nesting inside the nodes—in particular, in $\grave{\neg}\acute{\vee}y$ and $\grave{\neg}\grave{\vee}y$ demands that the negation rules occur *above* the disjunction rule, which stands to reason, as the result will be the construction of the formula $\neg p \vee \neg q$, where the negations occur inside the scope of the disjunction. So, no derivation having this term can end in application of a negation rule. These considerations motivate the following definition:

DEFINITION 18 [SURFACE SUBNODES IN A PRETERM] The SURFACE SUBNODES in a preterm τ are the cut points occurring in τ, as well as all complex subnodes in τ whose immediate subnodes are variables.

So, the surface subnodes in our preterm are indicated by underlining.

$$\underline{\acute{\wedge}x} \curvearrowright \underline{\bullet} \quad \underline{\bullet} \curvearrowright \grave{\neg}\underline{\acute{\vee}y} \quad \underline{\grave{\wedge}x} \curvearrowright \grave{\neg}\underline{\grave{\vee}y}$$

The surface subnodes indicate points in the preterm that may have been introduced in the last step of a derivation. The last step of this derivation could be $\wedge L$, $\vee L$ or *Cut*. Figure 3 contains three derivations with this term—ending with $\wedge L$, $\vee L$ and with *Cut*, respectively.

Our proof for Theorem 3 will go by way of an induction on the complexity of the preterm. If it has no surface subnodes, it will correspond to an axiom. If it has a surface subnode, then this corresponds either to an α rule or a β rule. Consider the $\acute{\wedge}x$ and $\grave{\wedge}x$ in the term we are discussing. These correspond to the α rule $\wedge L$. If we replace $\acute{\wedge}x$ and $\grave{\wedge}x$ by new variables (say t and u) of the same types, p and q respectively, then the resulting preterm:

$$t \curvearrowright \bullet \quad \bullet \curvearrowright \grave{\neg}\acute{\vee}y \quad u \curvearrowright \grave{\neg}\grave{\vee}y$$

has a lower complexity, and it is still spanned and atomic, so by hypothesis, it is the term of some derivation. Our target term is then the term of *that* derivation, extended with one application of $\wedge L$, replacing t and u by $\acute{\wedge}x$ and $\grave{\wedge}x$.

The case for β rules is a little more complex. Consider the preterm $x \curvearrowright \bullet \quad \bullet \curvearrowright \bullet \quad \bullet \curvearrowright x$ (where all nodes have type p). The preterm is atomic (all links have type p). The only surface nodes are the $\bullet$ nodes. Its only switch pair is $[\bullet](\bullet)$, so there are two switchings. Making the deletions, we have

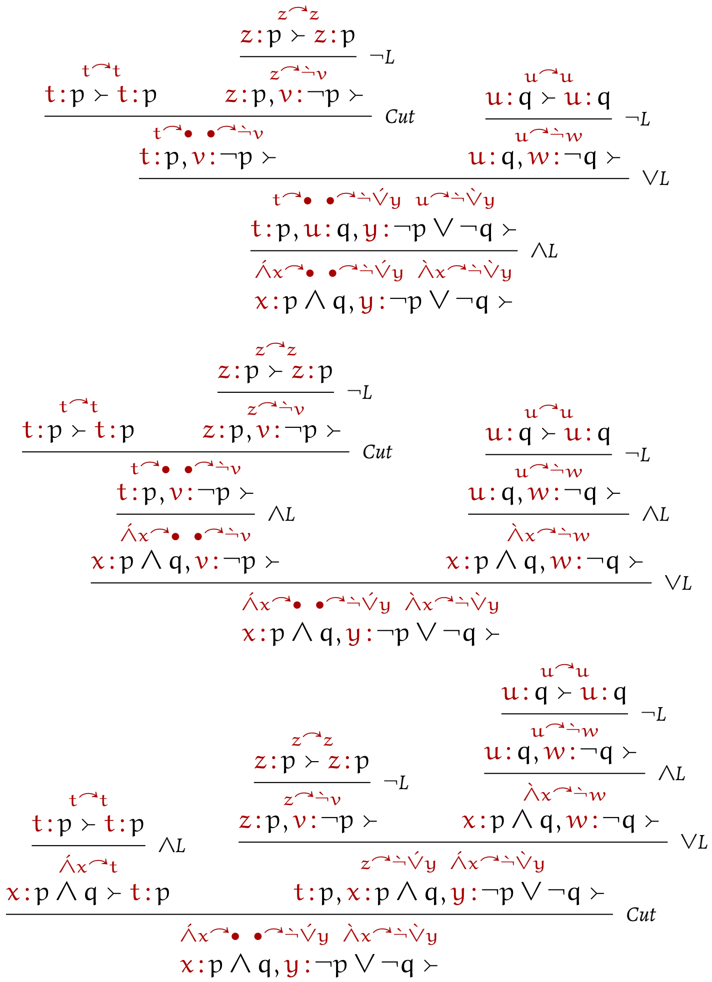

Figure 3: Three derivations with the term $\acute{\wedge}x \frown \bullet \;\; \bullet \frown \dot{\neg}\acute{\vee}y \;\; \grave{\wedge}x \frown \dot{\neg}\grave{\vee}y$

1. delete $[\bullet]$ — result: $\cancel{x \frown \bullet} \;\; \cancel{\bullet \frown \bullet} \;\; \bullet \frown x$
2. delete $(\bullet)$ — result: $x \frown \bullet \;\; \cancel{\bullet \frown \bullet} \;\; \cancel{\bullet \frown x}$

So, the preterm is spanned: all switchings are nonempty. So, if our Theorem 3 is to hold, this must be the result of some derivation. Since it contains a $\bullet$, the derivation must contain at least one *Cut*, on p, since $\bullet$ has type p in input and output positions. However, it is not hard to see that any such derivation must contain more than one *Cut*. If π and π' are terms from *Cut* free derivations, then applying a cut gives us a new proof term, in which the cut point is insterted in output position to links from π and input position to links from π'. There is no way to insert cut points into a link with one *Cut* with result $\bullet \frown \bullet$, for here, one *Cut* point is input position, and the other is in output position. This link must have been involved in two *Cuts*. This is one simple derivation with the required effect:

$$\dfrac{\dfrac{\overset{x\frown x}{x{:}p \succ x{:}p} \qquad \overset{y\frown y}{y{:}p \succ y{:}p}}{\overset{x\frown\bullet\ \ \bullet\frown y}{x{:}p \succ y{:}p}}\,Cut \qquad \overset{x\frown x}{x{:}p \succ x{:}p}}{\overset{x\frown\bullet\ \ \bullet\frown\bullet\ \ \bullet\frown x}{x{:}p \succ x{:}p}}\,Cut$$

Tracing the link $\bullet\frown\bullet$ in the conclusion of the derivation, it appears as $\bullet\frown y$ in the left premise of the second *Cut*, where it came from the $y\frown y$ in the second axiom.

This feature, where two applications of a rule are required to account for the nodes in a term is perhaps starkest in the case of *Cut*, but it holds for other β rules too. Here is an example replacing the *Cut* with a $\neg R/\wedge R$ combination.

$$\dfrac{\dfrac{\overset{x\frown x}{x{:}p \succ x{:}p} \qquad \dfrac{\overset{y\frown y}{y{:}p \succ y{:}p}}{\overset{\grave{\neg}z\frown y}{\succ z{:}\neg p, y{:}p}}\,\neg R}{\overset{x\frown\acute{\wedge}w\ \ \grave{\neg}\grave{\wedge}w\frown y}{x{:}p \succ w{:}p\wedge\neg p, y{:}p}}\,\wedge R \qquad \dfrac{\overset{x\frown x}{x{:}p \succ x{:}p}}{\overset{\grave{\neg}z\frown x}{\succ z{:}\neg p, x{:}p}}\,\neg R}{\overset{x\frown\acute{\wedge}w\ \ \grave{\neg}\grave{\wedge}w\frown\acute{\wedge}w\ \ \grave{\neg}\grave{\wedge}w\frown x}{x{:}p \succ w{:}p\wedge\neg p, x{:}p}}\,\wedge R$$

As with other β rules, $\wedge R$ introduces $\acute{\wedge}$ nodes to links from the *left* premise of the rule, and $\grave{\wedge}$ nodes to links from the *right* premise of the rule. Here, the link $\grave{\neg}\grave{\wedge}w\frown\acute{\wedge}w$ contains *both* kinds of nodes, and therefore must have passed through a $\wedge R$ rule *twice* in any derivation.

Let's consider, then, how one might decompose a pair of β surface subnodes in a preterm, such as a cut point, $\bullet$ in antecedent and consequent position. Take a spanned, atomic preterm $\tau(\bullet)[\bullet]$ in which $\bullet$ occurs as a cut point, with the indicated positions in which $\bullet$ occurs in input and output position marked. Consider the links in $\tau(\bullet)[\bullet]$, and let's call $\tau(\bullet)[-]$ the preterm containing all links in $\tau(\bullet)[\bullet]$ *except* for those in which $\bullet$ occurs in output position. Similarly, $\tau(-)[\bullet]$ the preterm containing all links in $\tau(\bullet)[\bullet]$ *except* for those in which $\bullet$ occurs in input position. If the preterm contains *no* links in which $\bullet$ occurs in both input and output position, then $\tau(\bullet)[\bullet]$ is identical to the union $\tau(-)[\bullet] \cup \tau(\bullet)[-]$. Notice, too, that if we replace $\bullet$ by a fresh variable x in each position, then $\tau(-)[x]$ and $\tau(x)[-]$ are both atomic and spanned, and so, by hypothesis, they correspond to derivations. We can compose these derivations in a *Cut* to construct a derivation for $\tau(\bullet)[\bullet]$.

$$\dfrac{\overset{\tau(-)[x]}{\Sigma_1 \succ x{:}A, \Delta_1} \qquad \overset{\tau(x)[-]}{\Sigma_2, x{:}A \succ \Delta_2}}{\overset{\tau(-)[\bullet]\ \ \tau(\bullet)[-]}{\Sigma_{1,2} \succ \Delta_{1,2}}}\,Cut$$

If $\tau(-)[\bullet] \cup \tau(\bullet)[-]$ is $\tau(\bullet)[\bullet]$ we are done, we have decomposed the *Cut*. If not, we must do a little more work, and use two *Cuts*. For this, we use the fact that $\tau(\bullet)[\bullet]$ is still identical to the union $\tau(-)[\bullet] \cup \tau(\bullet)[\bullet] \cup \tau(\bullet)[-]$.

$$\dfrac{\overset{\tau(-)[x]}{\Sigma_1 \succ x : A, \Delta_1} \qquad \dfrac{\overset{\tau(x)[y]}{\Sigma_2, x : A \succ y : A, \Delta_2} \qquad \overset{\tau(-)[y]}{\Sigma_3, y : A \succ \Delta_3}}{\overset{\tau(x)[\bullet] \quad \tau(\bullet)[-]}{\Sigma_{2,3}, x : A \succ \Delta_{2,3}}}\,\textit{Cut}}{\overset{\tau(-)[\bullet] \quad \tau(\bullet)[\bullet] \quad \tau(\bullet)[-]}{\Sigma_{1,2,3} \succ \Delta_{1,2,3}}}\,\textit{Cut}$$

The terms $\tau(-)[x]$, $\tau(x)[y]$, $\tau(-)[y]$ are each atomic, spanned and of lower complexity than $\tau(\bullet)[\bullet]$,[20] and by hypothesis, they are constructed by derivations. So, successive *Cuts* suffice to construct a derivation for $\tau(-)[\bullet] \cup \tau(\bullet)[\bullet] \cup \tau(\bullet)[-]$, which is $\tau(\bullet)[\bullet]$.

The cases we have seen are enough to motivate the general structure of the proof of Theorem 3. Here is the proof in its full generality.

Proof: Suppose τ is a spanned atomic preterm with no surface subnodes. Then each of its links contain variables alone, and it is the term annotating some axiom. For each variable x occurring as an input of a link of type A (where A is either an atomic formula or $\perp$), add $x : A$ to the LHS of the sequent. For each variable y occurring as the output of a link of type B (where B is either an atomic formula or $\top$), add $y : B$ to the RHS of the sequent. The result is an axiomatic sequent, annotated by this term.

Now suppose τ is a spanned atomic preterm with a surface subnode, and suppose, for an induction that all simpler spanned atomic preterms are terms constructed by some derivation.

If τ has a surface subnode of some α type, it has the form $\tau\{\acute{\alpha}a\}\{\grave{\alpha}a\}$. Choose fresh variables a_1 and a_2 of the same types as $\acute{\alpha}a$ and $\grave{\alpha}a$. The preterm $\tau\{a_1\}\{a_2\}$ is simpler than $\tau\{\acute{\alpha}a\}\{\grave{\alpha}a\}$, and it is atomic and spanned ($\acute{\alpha}a$ and $\grave{\alpha}a$ do not introduce any switch pairs), so it corresponds to some derivation. We can then extend this by one application of the relevant α rule as follows:

$$\dfrac{\overset{\tau\{a_1\}\{a_2\}}{\mathfrak{S}\{a_1 : \alpha_1\}\{a_2 : \alpha_2\}}}{\overset{\tau\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a : \alpha\}}}$$

On the other hand, if τ has a surface subnode of some β type, it has the form $\tau\{\acute{\beta}b\}\{\grave{\beta}b\}$. Choose fresh variables b_1 and b_2 of the same types as $\acute{\beta}b$ and $\grave{\beta}b$ respectively. The preterms $\tau\{b_1\}\{-\}$, $\tau\{b_1\}\{b_2\}$ and $\tau\{-\}\{b_2\}$ are each simpler than $\tau\{\acute{\beta}b\}\{\grave{\beta}b\}$, they are each atomic and spanned—since $\{\acute{\beta}b\}\{\grave{\beta}b\}$ is a switch pair and in each of the term's switchings, one of $\{\acute{\beta}b\}$ and $\{\grave{\beta}b\}$ is deleted—so each of these preterms corresponds to some derivation. We can then extend these

[20] Remember, cut points have complexity 1 and variables have complexity 0.

derivations by two applications of the relevant β rule as follows:

$$
\dfrac{\overset{\tau\{b_1\}\{-\}}{\mathfrak{S}_1\{b_1:\beta_1\}} \qquad \dfrac{\overset{\tau\{b_1\}\{b_2\}}{\mathfrak{S}_2\{b_1:\beta_1\}\{b_2:\beta_2\}} \qquad \overset{\tau\{-\}\{b_2\}}{\mathfrak{S}_3\{b_2:\beta_2\}}}{\overset{\tau\{\acute{\beta}b\}\{b_2\}\;\; \tau\{-\}\{\grave{\beta}b\}}{\mathfrak{S}_{2,3}\{b:\beta\}\{b_2:\beta_2\}}}\,\beta}{\overset{\tau\{\acute{\beta}b\}\{-\}\;\; \tau\{\acute{\beta}b\}\{\grave{\beta}b\}\;\; \tau\{-\}\{\grave{\beta}b\}}{\mathfrak{S}_{1,2,3}\{b:\beta\}}}\,\beta
$$

In either case (α rule or β rule), the result is a derivation ending in our preterm τ, so we have proved that τ is generated by a derivation, and this gives our result in full generality. ∎

We have shown, therefore, that the spanned atomic preterms are exactly the terms that describe derivations. We have two independent characterisations of our proof invariants.

4 PERMUTATIONS AND INVARIANTS

We have seen that there are many different derivations constructing the same proof term. In many cases, the difference between these derivations can be seen as a simple matter of the order of application of the rules. A simple and straightforward permutation will send one derivation into the other. For example, the different derivations in Figure 3 on page 27 can be seen to be simple permutations of each other, differing only in the order of application of the rules contained in them. Our aim in this section is to generalise that observation and to prove the following theorem.

THEOREM 4 [PROOF TERMS ARE INVARIANTS] $\delta_1 \approx \delta_2$ *if and only if* $\tau(\delta_1) = \tau(\delta_2)$*. That is, some permutation sends* δ_1 *to* δ_2 *if and only if the* δ_1 *and* δ_2 *have the same proof term.*

To prove this, we need to isolate the appropriate class of permutations for derivations. Some permutations are straightforward. They are the permutations that shift the order between successive rule applications. We have seen examples of these transformations in Figure 3. Another straightforward permutation for derivations is relabelling of interior variables.[21] Consider these two derivations:

$$
\dfrac{\dfrac{\overset{x \curvearrowright y}{x:p \succ y:p}}{\overset{x \curvearrowright \neg z}{x:p,\, z:\neg p \succ}}\,\neg L}{\overset{\acute{\wedge}w \curvearrowright \neg \grave{\wedge}w}{w:p \wedge \neg p \succ}}\,\wedge L
\qquad\qquad
\dfrac{\dfrac{\overset{u \curvearrowright v}{u:p \succ v:p}}{\overset{u \curvearrowright \neg x}{u:p,\, x:\neg p \succ}}\,\neg L}{\overset{\acute{\wedge}w \curvearrowright \neg \grave{\wedge}w}{w:p \wedge \neg p \succ}}\,\wedge L
$$

[21] This is the notion for proof terms analogous to α equivalence, or relabelling of *bound* variables in the λ calculus. However, it would be a mistake to think of variables as being *bound* in α and β inference steps. In a β rule where we step from $\mathfrak{S}_1\{b_1:\beta_1\}$ and $\mathfrak{S}_2\{b_2:\beta_2\}$ to $\mathfrak{S}_{1,2}\{b:\beta\}$, the variable b_1 is not necessarily absent from $\mathfrak{S}_2\{b_2:\beta_2\}$ or from the conclusion of the derivation, nor need b_1 be absent from $\mathfrak{S}_1$. Don't think of the inference step as binding b_1 or b_2 in the derivation. They are merely substituted for in the step to the conclusion.

These two derivations prove the same sequent, with the same term. The only difference is the variables used in the *interior* of each derivation. (We used x, y and z for the first; u, v, x for the second.) Shuffling around *interior* variables in a derivation (those no longer present in the concluding sequent of that derivation) will count as another kind of permutation of derivations.

DEFINITION 19 [INTERIOR RELABELLING] The equivalence relation of relabelling equivalence for derivations is defined inductively on the structure of the derivation.

- An axiom is relabelling equivalent only to itself. (It has no 'interior' in which to reshuffle variables.)
- If δ (a derivation of $\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}$ with term $\pi\{a_1\}\{a_2\}$) is relabelling equivalent to δ' (a derivation of the same sequent with the same term), and a_1' and a_2' are also variables of type α_1 and α_2 respectively, distinct if and only if a_1 and a_2 are distinct, then the following two derivations are also relabelling equivalent:[22]

$$\dfrac{\begin{array}{c}\delta\\ \vdots\\ {\scriptstyle \pi\{a_1\}\{a_2\}}\\ \mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\end{array}}{\begin{array}{c}{\scriptstyle \pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}}\\ \mathfrak{S}\{a:\alpha\}\end{array}} \qquad \dfrac{\begin{array}{c}\delta'\{a_1'/a_1\}\{a_2'/a_2\}\\ \vdots\\ {\scriptstyle \pi\{a_1'\}\{a_2'\}}\\ \mathfrak{S}\{a_1':\alpha_1'\}\{a_2':\alpha_2'\}\end{array}}{\begin{array}{c}{\scriptstyle \pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}}\\ \mathfrak{S}\{a:\alpha\}\end{array}}$$

- If δ_1 (a derivation of $\mathfrak{S}_1\{b_1:\beta_1\}$ with term $\pi_1\{b_1\}$) is relabelling equivalent to δ_1' (a derivation of the same sequent with the same term), and δ_2 (a derivation of $\mathfrak{S}_2\{b_2:\beta_2\}$ with term $\pi_2\{b_2\}$) is relabelling equivalent to δ_2' (a derivation of the same sequent with the same term), and b_1' and b_2' are also variables of type β_1 and β_2 respectively, then the following two derivations are also relabelling equivalent:

$$\dfrac{\begin{array}{c}\delta_1\\ \vdots\\ {\scriptstyle \pi_1\{b_1\}}\\ \mathfrak{S}_1\{b_1:\beta_1\}\end{array} \qquad \begin{array}{c}\delta_2\\ \vdots\\ {\scriptstyle \pi_2\{b_2\}}\\ \mathfrak{S}_2\{b_2:\beta_1\}\end{array}}{\begin{array}{c}{\scriptstyle \pi\{\acute{\beta}b\}\{\grave{\beta}b\}}\\ \mathfrak{S}_{1,2}\{b:\beta\}\end{array}} \qquad \dfrac{\begin{array}{c}\delta'\{b_1'/b_1\}\\ \vdots\\ {\scriptstyle \pi\{b_1'\}\{b_2'\}}\\ \mathfrak{S}_1\{b_1':\beta_1'\}\end{array} \qquad \begin{array}{c}\delta'\{b_2'/b_2\}\\ \vdots\\ {\scriptstyle \pi\{b_1'\}\{b_2'\}}\\ \mathfrak{S}_2\{b_2':\beta_2'\}\end{array}}{\begin{array}{c}{\scriptstyle \pi\{\acute{\beta}b\}\{\grave{\beta}b\}}\\ \mathfrak{S}_{1,2}\{b:\beta\}\end{array}}$$

The effect of this definition of relabelling equivalence is to allow for arbitrary relabelling of *interior* variables in a derivation, without changing the free variables in the endsequent. As you can see by inspection of the definition, if δ and δ' are relabelling equivalent, then they are derivations of the same endsequent, with the same proof term.

So, relabelling equivalence is one component of the equivalence relation $\approx$ between derivations. Two derivations that are relabelling equivalent make the same logical connections. They

[22] Recall Definition 13 on page 21, where the substitution $\{x'/x\}$ is defined for labelled derivations.

represent them in only marginally different ways. The next component of the relation of permutation equivalence will be permutation of the order of different rules in the derivation. Given that every inference rule in a derivation is either an α or a β rule, we will have three kinds of permutations. Exchanging two α inferences, exchanging an α and a β inference, and exchanging two β inferences. We take these three kinds of permutations in turn. The first kind of permutation is most straightforward.

DEFINITION 20 [α/α' PERMUTATIONS] If, in a derivation, an α rule is followed by another α rule (here labelled with α'), the derivation is said to be (α/α') PERMUTATION EQUIVALENT with the derivation in which the order of the two inferences is reversed, as follows:

$$\dfrac{\dfrac{\overset{\pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\}}{\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{a'_1:\alpha'_1\}\{a'_2:\alpha'_2\}}}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{a'_1\}\{a'_2\}}{\mathfrak{S}\{a:\alpha\}\{a'_1:\alpha'_1\}\{a'_2:\alpha'_2\}}}\,\alpha}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}'a'\}\{\grave{\alpha}'a'\}}{\mathfrak{S}\{a:\alpha\}\{a':\alpha'\}}}\,\alpha' \quad\approx\quad \dfrac{\dfrac{\overset{\pi\{a_1\}\{a_2\}\{a'_1\}\{a'_2\}}{\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{a'_1:\alpha'_1\}\{a'_2:\alpha'_2\}}}{\overset{\pi\{a_1\}\{a_2\}\{\acute{\alpha}'a'\}\{\grave{\alpha}'a'\}}{\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{a':\alpha'\}}}\,\alpha'}{\overset{\pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}'a'\}\{\grave{\alpha}'a'\}}{\mathfrak{S}\{a:\alpha\}\{a':\alpha'\}}}\,\alpha$$

The only side condition on the application of this permutation is that the subformulas of both α and α' rules are present in the first sequent—that is, the *second* α rule does not operate on the result of the *first* α rule.

You can see, by construction, swapping the order between two α rules leaves no mark on the proof term of the conclusion. That proof term is unchanged. We simply process the two independent parts of the initial sequent in different orders, and end up in the same place.[23]

Similarly we can swap the order of an α and a β rule.

DEFINITION 21 [α/β PERMUTATIONS] If, in a derivation, a β inference is followed by an α inference, the derivation is said to be (α/β) PERMUTATION EQUIVALENT with the derivation in which the order of the two inferences is reversed, as follows:

$$\dfrac{\dfrac{\overset{\pi_1\{a_1\}\{a_2\}\{b_1\}}{\mathfrak{S}_1\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b_1:\beta_1\}} \qquad \overset{\pi_2\{a_1\}\{a_2\}\{b_2\}}{\mathfrak{S}_2\{a_1:\alpha_1\}\{a_2:\alpha_1\}\{b_2:\beta_2\}}}{\overset{\pi_1\{a_1\}\{a_2\}\{\acute{\beta}b\}\quad \pi_2\{a_1\}\{a_2\}\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b:\beta\}}}\,\beta}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\beta}b\}\quad \pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{a:\alpha\}\{b:\beta\}}}\,\alpha$$

[23] Since there are 5 different α rules in the calculus, there are $5^2 = 25$ different permutations of rules of this form. Thanks to Smullyan's categorisation of the rules in the calculus, we can see the essential form of these 25 rule permutations in one go.

$$\approx \dfrac{\dfrac{\overset{\pi_1\{a_1\}\{a_2\}\{b_1\}}{\mathfrak{S}_1\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b_1:\beta_1\}}}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{b_1\}}{\mathfrak{S}_1\{a:\alpha\}\{b_1:\beta_1\}}}\alpha \qquad \dfrac{\overset{\pi_2\{a_1\}\{a_2\}\{b_2\}}{\mathfrak{S}_2\{a_1:\alpha_1\}\{a_2:\alpha_1\}\{b_2:\beta_2\}}}{\overset{\pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{b_2\}}{\mathfrak{S}_2\{a:\alpha\}\{b_2:\beta_2\}}}\alpha}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\beta}b\}\ \ \pi_2\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{a:\alpha\}\{b:\beta\}}}\beta$$

Here, again, the proviso is only that the rules operate on independent parts of the intial sequents.

The equivalence is symmetric, and if α inferences of the same shape occur above the β rule, they may be permuted back, to reverse the order. However, it may be that we have an α inference above one premise of a β inference and not the other. We may still reverse the order in that case, with another (α/β) PERMUTATION EQUIVALENCE.

$$\dfrac{\dfrac{\overset{\pi_1\{a_1\}\{a_2\}\{b_1\}}{\mathfrak{S}_1\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b_1:\beta_1\}}}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{b_1\}}{\mathfrak{S}_1\{a:\alpha\}\{b_1:\beta_1\}}}\alpha \qquad \overset{\pi_2\{b_2\}}{\mathfrak{S}_2\{b_2:\beta_2\}}}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\beta}b\}\ \ \pi_2\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{a:\alpha\}\{b:\beta\}}}\beta$$

$$\approx \dfrac{\dfrac{\overset{\pi_1\{a_1\}\{a_2\}\{b_1\}}{\mathfrak{S}_1\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b_1:\beta_1\}} \qquad \overset{\pi_2\{b_2\}}{\mathfrak{S}_2\{b_2:\beta_2\}}}{\overset{\pi_1\{a_1\}\{a_2\}\{\acute{\beta}b\}\ \ \pi_2\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{b:\beta\}}}\beta}{\overset{\pi_1\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\beta}b\}\ \ \pi_2\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{a:\alpha\}\{b:\beta\}}}\alpha$$

Notice that here the α rule may be repeated in both premises of the β rule, while it occurs only once if it is *after* the β step. (However, the permutation is still possible of the α rule occurs only over one premise of the β rule (say the left) and the α components do not occur in the right sequent.). The same holds yet again with two β rules. Notice, too, that in each case the proof term is identical for the conclusion of this part of the derivation, regardless of the order in which the rules are applied.

DEFINITION 22 [β/β' PERMUTATIONS] For β/β' PERMUTATION EQUIVALENCE, consider first the case where the same β rule occurs in both premises above a β' rule (with the usual proviso that the rules operate on different components of the premise sequents).

$$\dfrac{\dfrac{\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1:\beta_1\}\{b_1':\beta_1'\}} \quad \overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2:\beta_2\}\{b_1':\beta_1'\}}}{\overset{\pi_1\{\acute{\beta}b\}\{b_1'\}\ \ \pi_2\{\grave{\beta}b\}\{b_1'\}}{\mathfrak{S}_{1,2}\{b:\beta\}\{b_1':\beta_1'\}}}\beta \quad \dfrac{\overset{\pi_3\{b_1\}\{b_2'\}}{\mathfrak{S}_3\{b_1:\beta_1\}\{b_2':\beta_2'\}} \quad \overset{\pi_4\{b_2\}\{b_2'\}}{\mathfrak{S}_4\{b_2:\beta_2\}\{b_2':\beta_2'\}}}{\overset{\pi_3\{\acute{\beta}b\}\{b_2'\}\ \ \pi_2\{\grave{\beta}b\}\{b_2'\}}{\mathfrak{S}_{3,4}\{b:\beta\}\{b_2':\beta_2'\}}}\beta}{\overset{\pi_1\{\acute{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_2\{\grave{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_3\{\acute{\beta}b\}\{\grave{\beta}'b'\}\ \ \pi_4\{\grave{\beta}b\}\{\grave{\beta}'b'\}}{\mathfrak{S}_{1-4}\{b:\beta\}\{b':\beta'\}}}\beta'$$

$$\approx \dfrac{\dfrac{\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1:\beta_1\}\{b_1':\beta_1'\}} \qquad \overset{\pi_3\{b_1\}\{b_2'\}}{\mathfrak{S}_3\{b_1:\beta_1\}\{b_2':\beta_2'\}}}{\overset{\pi_1\{b_1\}\{\acute{\beta}'b'\}\ \ \pi_3\{b_1\}\{\grave{\beta}'b'\}}{\mathfrak{S}_{1,3}\{b_1:\beta_1\}\{b':\beta'\}}}\,\beta' \quad \dfrac{\overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2:\beta_2\}\{b_1':\beta_1'\}} \qquad \overset{\pi_4\{b_2\}\{b_2'\}}{\mathfrak{S}_4\{b_2:\beta_2\}\{b_2':\beta_2'\}}}{\overset{\pi_2\{b_2\}\{\acute{\beta}'b'\}\ \ \pi_4\{b_2\}\{\grave{\beta}'b'\}}{\mathfrak{S}_{2,4}\{b_2:\beta_2\}\{b':\beta'\}}}\,\beta'}{\overset{\pi_1\{\acute{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_2\{\grave{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_3\{\acute{\beta}b\}\{\grave{\beta}'b'\}\ \ \pi_4\{\grave{\beta}b\}\{\grave{\beta}'b'\}}{\mathfrak{S}_{1-4}\{b:\beta\}\{b':\beta'\}}}\,\beta$$

Consider now the case where the β rule occurs in only one of the premises of the β' inference. We have the following permutation:

$$\dfrac{\dfrac{\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1:\beta_1\}\{b_1':\beta_1'\}} \qquad \overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2:\beta_2\}\{b_1':\beta_1'\}}}{\overset{\pi_1\{\acute{\beta}b\}\{b_1'\}\ \ \pi_2\{\grave{\beta}b\}\{b_1'\}}{\mathfrak{S}_{1,2}\{b:\beta\}\{b_1':\beta_1'\}}}\,\beta \qquad \overset{\pi_3\{b_2'\}}{\mathfrak{S}_3\{b_2':\beta_2'\}}}{\overset{\pi_1\{\acute{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_2\{\grave{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_3\{\grave{\beta}'b'\}}{\mathfrak{S}_{1-3}\{b:\beta\}\{b':\beta'\}}}\,\beta'$$

$$\approx \dfrac{\dfrac{\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1:\beta_1\}\{b_1':\beta_1'\}} \qquad \overset{\pi_3\{b_2'\}}{\mathfrak{S}_3\{b_2':\beta_2'\}}}{\overset{\pi_1\{b_1\}\{\acute{\beta}'b'\}\ \ \pi_3\{b_1\}\{\grave{\beta}'b'\}}{\mathfrak{S}_{1,3}\{b_1:\beta_1\}\{b':\beta'\}}}\,\beta' \quad \dfrac{\overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2:\beta_2\}\{b_1':\beta_1'\}} \qquad \overset{\pi_3\{b_2'\}}{\mathfrak{S}_3\{b_2':\beta_2'\}}}{\overset{\pi_2\{b_2\}\{\grave{\beta}'b'\}\ \ \pi_3\{\acute{\beta}'b'\}}{\mathfrak{S}_{2,3}\{b_2:\beta_2\}\{b':\beta'\}}}\,\beta'}{\overset{\pi_1\{\acute{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_2\{\grave{\beta}b\}\{\acute{\beta}'b'\}\ \ \pi_3\{\grave{\beta}'b'\}}{\mathfrak{S}_{1-3}\{b:\beta\}\{b':\beta'\}}}\,\beta$$

and as before, the concluding sequent has the same proof term, though in this case.

The next example of a proof manipulation is when an inference includes two instances of the *same* inference rule, operating on the same labelled formula. Consider a case where an inference uses two α rules, in this case introducing the labelled formula $\{a:\alpha\}$ in both steps, once from the labelled subformulas $\{a_1:\alpha_1\}$ and $\{a_2:\alpha_2\}$, and then from the same formulas with different labels, $\{a_1':\alpha_1\}$ and $\{a_2':\alpha_2\}$. In that case, we could have instead done the two inferences at once, had we relabelled the second instances of the variables:

DEFINITION 23 [α/α EXPANSION AND RETRACTION] The following two derivations are α/α EXPANSION/RETRACTION equivalent.

$$\dfrac{\dfrac{\begin{matrix}\delta\\ \vdots\end{matrix}\atop\overset{\pi\{a_1\}\{a_2\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{a_1':\alpha_1\}\{a_2':\alpha_2\}}}{\overset{\tau\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{a_1'\}\{a_2'\}}{\mathfrak{S}\{a:\alpha\}\{a_1':\alpha_1\}\{a_2':\alpha_2\}}}\,\alpha}{\overset{\tau\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a:\alpha\}}}\,\alpha \quad \approx \quad \dfrac{\begin{matrix}\delta\{a_1/a_1'\}\{a_2/a_2'\}\\ \vdots\end{matrix}\atop\overset{\pi\{a_1\}\{a_2\}\{a_1\}\{a_2\}}{\mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\{a_1:\alpha_1\}\{a_2:\alpha_2\}}}{\overset{\tau\{\acute{\alpha}a\}\{\grave{\alpha}a\}\{\acute{\alpha}a\}\{\grave{\alpha}a\}}{\mathfrak{S}\{a:\alpha\}}}\,\alpha$$

Expansion and retraction for stacked β rules is more complex.

DEFINITION 24 [β/β EXPANSION AND RETRACTION] Stacked instances of β rules which introduce the same labelled formula but with different variables can similarly be simplified into the one pair of variables:

$$\dfrac{\dfrac{\overset{\pi_1\{b_1\}\{b_1'\}}{\mathfrak{S}_1\{b_1:\beta_1\}\{b_1':\beta_1\}} \qquad \overset{\pi_2\{b_2\}\{b_1'\}}{\mathfrak{S}_2\{b_2:\beta_2\}\{b_1':\beta_1\}}}{\overset{\pi_1\{\acute{\beta}b\}\{b_1'\}\quad \pi_2\{\grave{\beta}b\}\{b_1'\}}{\mathfrak{S}_{1,2}\{b:\beta\}\{b_1':\beta_1\}}}\,\beta \qquad \overset{\pi_3\{b_2'\}}{\mathfrak{S}_3\{b_2':\beta_2\}}}{\overset{\pi_1\{\acute{\beta}b\}\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}\{\acute{\beta}b\}\quad \pi_3\{\grave{\beta}b\}}{\mathfrak{S}_{1-3}\{b:\beta\}}}\,\beta$$

$$\approx\quad \dfrac{\overset{\pi_1\{b_1\}\{b_1\}}{\mathfrak{S}_1\{b_1:\beta_1\}} \qquad \dfrac{\overset{\pi_2\{b_2\}\{b_1\}}{\mathfrak{S}_2\{b_2:\beta_2\}\{b_1:\beta_1\}} \qquad \overset{\pi_3\{b_2\}}{\mathfrak{S}_3\{b_2:\beta_2\}}}{\overset{\pi_2\{b_1\}\{\acute{\beta}b\}\quad \pi_3\{\grave{\beta}b\}}{\mathfrak{S}_{2,3}\{b:\beta\}\{b_2:\beta_2\}}}\,\beta}{\overset{\pi_1\{\acute{\beta}b\}\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}\{\acute{\beta}b\}\quad \pi_3\{\grave{\beta}b\}}{\mathfrak{S}_{1-3}\{b:\beta\}}}\,\beta$$

however, as we saw in the previous section, sometimes it is essential for there to be two stacked β rules in order to produce the required proof terms. *More* stacked β rules (in this case, *three*) can be collapsed into the stack of 2, as follows:

$$\dfrac{\dfrac{\overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1:\beta_1\}} \qquad \overset{\pi_2\{b_1\}\{b_2\}}{\mathfrak{S}_2\{b_1:\beta_1\}\{b_2:\beta_2\}}}{\overset{\pi_1\{\acute{\beta}b\}\quad \pi_2\{b_1\}\{\grave{\beta}b\}}{\mathfrak{S}_{1,2}\{b:\beta\}\{b_1:\beta_1\}}}\,\beta \qquad \dfrac{\overset{\pi_3\{b_1\}\{b_2\}}{\mathfrak{S}_3\{b_1:\beta_1\}\{b_2:\beta_2\}} \qquad \overset{\pi_4\{b_2\}}{\mathfrak{S}_4\{b_2:\beta_2\}}}{\overset{\pi_3\{\acute{\beta}b\}\{b_2\}\quad \pi_2\{\grave{\beta}b\}\{b_2\}}{\mathfrak{S}_{3,4}\{b:\beta\}\{b_2:\beta_2\}}}\,\beta}{\overset{\pi_1\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}\{\acute{\beta}b\}\quad \pi_3\{\acute{\beta}b\}\{\grave{\beta}b\}\quad \pi_4\{\grave{\beta}b\}}{\mathfrak{S}_{1-4}\{b:\beta\}}}\,\beta$$

$$\approx\quad \dfrac{\overset{\pi_1\{b_1\}}{\mathfrak{S}_1\{b_1:\beta_1\}} \qquad \dfrac{\dfrac{\overset{\pi_2\{b_1\}\{b_2\}}{\mathfrak{S}_2\{b_1:\beta_1\}\{b_2:\beta_2\}} \qquad \overset{\pi_3\{b_1\}\{b_2\}}{\mathfrak{S}_3\{b_1:\beta_1\}\{b_2:\beta_2\}}}{\overset{\pi_2\{b_1\}\{b_2\}\quad \pi_3\{b_1\}\{b_2\}}{\mathfrak{S}_{2,3}\{b_1:\beta_1\}\{b_2:\beta_2\}}}\,\textit{Blend} \qquad \overset{\pi_4\{b_2\}}{\mathfrak{S}_4\{b_2:\beta_2\}}}{\overset{\pi_2\{\acute{\beta}b\}\{b_2\}\quad \pi_3\{\grave{\beta}b\}\{b_2\}\quad \pi_4\{\grave{\beta}b\}}{\mathfrak{S}_{2-4}\{b_2:\beta_2\}\{b:\beta\}}}\,\beta}{\overset{\pi_1\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}\{\acute{\beta}b\}\quad \pi_3\{\acute{\beta}b\}\{\grave{\beta}b\}\quad \pi_4\{\acute{\beta}b\}}{\mathfrak{S}_{1-4}\{b:\beta\}}}\,\beta$$

and this process can be extended in a natural way—any stack of β inferences can be permuted down into the triangle of two β steps, where the left premise contains the blend of those premise sequents in which only the *left* component of the β rule is processed (here, $\{b_1:\beta_1\}$), the right premise contains the blend of those premise sequents in which only the *right* component is processed (here, $\{b_2 : \beta_2\}$) and the *middle* premise contains the blend of all those sequents where both components are processed. Then, the triangle of two applications of the β rule produce the concluding proof term, and we are done.

The final β/β permutation is the EXPANSION from one to two inferences, which will ensure that *all* applications of a β rule can have the stacked form we have seen:

$$
\dfrac{\begin{matrix}\delta_1\\ \vdots \\ \scriptstyle\pi_1\{b_1\} \\ \mathfrak{S}_1\{b_1 : \beta_1\}\end{matrix} \quad \begin{matrix}\delta_2\\ \vdots \\ \scriptstyle\pi_2\{b_2\} \\ \mathfrak{S}_2\{b_2 : \beta_2\}\end{matrix}}{\begin{matrix}\scriptstyle\pi_1\{\acute{\beta}b\}\ \ \pi_2\{\grave{\beta}b\} \\ \mathfrak{S}_{1,2}\{b : \beta\}\end{matrix}}\,\beta
\quad\approx\quad
\dfrac{\begin{matrix}\delta_1\\ \vdots \\ \scriptstyle\pi_1\{b_1\} \\ \mathfrak{S}_1\{b_1 : \beta_1\}\end{matrix} \quad \dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots \\ \scriptstyle\pi_1\{b_1\} \\ \mathfrak{S}_1\{b_1 : \beta_1\}\end{matrix} \quad \begin{matrix}\delta_2\\ \vdots \\ \scriptstyle\pi_2\{b_2\} \\ \mathfrak{S}_2\{b_2 : \beta_2\}\end{matrix}}{\begin{matrix}\scriptstyle\pi_1\{b_1\}\ \ \pi_2\{b_2\} \\ \mathfrak{S}_{1,2}\{b_1 : \beta_1\}\{b_2 : \beta_2\}\end{matrix}}\,\mathit{Blend} \quad \begin{matrix}\delta_2\\ \vdots \\ \scriptstyle\pi_2\{b_2\} \\ \mathfrak{S}_2\{b_2 : \beta_2\}\end{matrix}}{\begin{matrix}\scriptstyle\pi_1\{\acute{\beta}b\}\ \ \pi_2\{b_2\}\ \ \pi_2\{\grave{\beta}b\} \\ \mathfrak{S}_{1,2}\{b : \beta\}\{b_2 : \beta_2\}\end{matrix}}\,\beta}{\begin{matrix}\scriptstyle\pi_1\{\acute{\beta}b\}\ \ \pi_2\{\grave{\beta}b\} \\ \mathfrak{S}_{1,2}\{b : \beta\}\end{matrix}}\,\beta
$$

so here, in this derivation as in others, the leftmost premise of the β rules contains β_1, the rightmost premise of the β rules contains β_2, and the middle premise contains both β_1 and β_2.

All of the derivation transformations all have the property that the premises of a derivation are unchanged (up to relabelling of variables). Not all natural derivation transformations have this property. In particular, the following two derivations have the same proof term, but they have different premises:

$$
\dfrac{\begin{matrix}\scriptstyle x{\frown}x\ \ v{\frown}v \\ x : p, v : r \succ x : p, v : r\end{matrix} \qquad \begin{matrix}\scriptstyle y{\frown}y \\ y : q \succ y : q\end{matrix}}{\begin{matrix}\scriptstyle x{\frown}\acute{\wedge}z\ \ y{\frown}\grave{\wedge}z\ \ v{\frown}v \\ x : p, y : q, v : r \succ z : p \wedge q, v : r\end{matrix}}\,{\wedge}R
\qquad
\dfrac{\begin{matrix}\scriptstyle x{\frown}x \\ x : p \succ x : p\end{matrix} \qquad \begin{matrix}\scriptstyle y{\frown}y\ \ v{\frown}v \\ y : q, v : r \succ y : q, v : r\end{matrix}}{\begin{matrix}\scriptstyle x{\frown}\acute{\wedge}z\ \ y{\frown}\grave{\wedge}z\ \ v{\frown}v \\ x : p, y : q, v : r \succ z : p \wedge q, v : r\end{matrix}}\,{\wedge}R
$$

Here we have a single application of a conjunction right rule (a β rule) which involve the $x{\frown}x$ and $y{\frown}y$ link, to form $x{\frown}\acute{\wedge}z$ and $y{\frown}\grave{\wedge}z$. However, there is an 'innocent bystander' $v{\frown}v$ link, which comes down from the left premise in the case of the first derivation, and the right premise, in the case of the second. These derivations generate the same proof term, and they both generate the same proof term as the following derivation:

$$
\dfrac{\begin{matrix}\scriptstyle x{\frown}x\ \ v{\frown}v \\ x : p, v : r \succ x : p, v : r\end{matrix} \qquad \begin{matrix}\scriptstyle y{\frown}y\ \ v{\frown}v \\ y : q, v : r \succ y : q, v : r\end{matrix}}{\begin{matrix}\scriptstyle x{\frown}\acute{\wedge}z\ \ y{\frown}\grave{\wedge}z\ \ v{\frown}v \\ x : p, y : q, v : r \succ z : p \wedge q, v : r\end{matrix}}\,{\wedge}R
$$

in which the uninvolved $v{\frown}v$ (and the corresponding $v : r$ on both sides of the sequent separator) is included on both sides of the sequent.

DEFINITION 25 [COPYING UNINVOLVED LINKS OVER β STEPS]

$$\dfrac{\begin{matrix}\delta_1\\ \vdots\\ \pi_1\{b_1\}\quad c\frown d\\ \mathfrak{S}_1\{b_1:\beta_1\}\end{matrix}\qquad\begin{matrix}\delta_2\\ \vdots\\ \pi_2\{b_2\}\\ \mathfrak{S}_2\{b_2:\beta_1\}\end{matrix}}{\begin{matrix}\pi\{\acute{\beta}b\}\{\grave{\beta}b\}\quad c\frown d\\ \mathfrak{S}_{1,2}\{b:\beta\}\end{matrix}} \approx \dfrac{\begin{matrix}\delta_1\\ \vdots\\ \pi_1\{b_1\}\quad c\frown d\\ \mathfrak{S}_1\{b_1:\beta_1\}\end{matrix}\qquad\begin{matrix}\delta_2'\\ \vdots\\ \pi_2\{b_2\}\quad c\frown d\\ \mathfrak{S}_2'\{b_2:\beta_1\}\end{matrix}}{\begin{matrix}\pi\{\acute{\beta}b\}\{\grave{\beta}b\}\quad c\frown d\\ \mathfrak{S}_{1,2}\{b:\beta\}\end{matrix}}$$

We could impose the same kind of permutation for larger pieces of proof terms, with complex nodes uninvolved with b_1 and b_2. However, we do not need to do so, as the following lemma will show.

LEMMA 5 [β FILLING] *In for any pair of derivations δ_1 and δ_2 extended with a β inference as follows*

$$\dfrac{\begin{matrix}\delta_1\\ \vdots\\ \pi_1\{b_1\}\\ \mathfrak{S}_1\{b_1:\beta_1\}\end{matrix}\qquad\begin{matrix}\delta_2\\ \vdots\\ \pi_2\{b_2\}\\ \mathfrak{S}_2\{b_2:\beta_1\}\end{matrix}}{\begin{matrix}\pi_1\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}\\ \mathfrak{S}_{1,2}\{b:\beta\}\end{matrix}}$$

using the class of permutations defined up to now, there are derivations δ_1' and δ_2' such that the following derivation is permutation equivalent to the original derivation.

$$\dfrac{\begin{matrix}\delta_1'\\ \vdots\\ \pi_1\{b_1\}\quad \pi_2\{-\}\\ \mathfrak{S}_1'\{b_1:\beta_1\}\end{matrix}\qquad\begin{matrix}\delta_2'\\ \vdots\\ \pi_1\{-\}\quad \pi_2\{b_2\}\\ \mathfrak{S}_2'\{b_2:\beta_1\}\end{matrix}}{\begin{matrix}\pi_1\{\acute{\beta}b\}\quad \pi_2\{\grave{\beta}b\}\\ \mathfrak{S}_{1,2}\{b:\beta\}\end{matrix}}$$

Proof: The process of generating δ_1' and δ_2' is straightforward. Consider the each link $n\frown m$ in $\pi_2\{-\}$ (each link in $\pi_2\{b_2\}$ not inovlving b_2, in its designated position) and each link in $\pi_1\{-\}$ (each link in $\pi_1\{b_1\}$ not inovlving b_1 in its designated position). Each such link is generated from an axiom link and built up by way of connective rules—those connective rules generating parts of the nodes in n and m. Since b_2 (or b_1) is not involved in the link in any way, the rule for the application of that connective may be permuted *below* the β rule, using and α/β or β/β' permutation. Do so, until only the axiom link remains above the β rule, and then use the permutation of copying uninvolved links over β steps to copy the atomic link over the inference. Now permute the connective rules to generate the nodes in $n\frown m$ back over the β rule and now the link is created on *both sides* of the β inference. This process suffices to duplicate all of $\pi_2\{-\}$ on the left premise of the β rule, and $\pi_1\{-\}$ on the right premise of the β rule, generating the required derivation. ■

This is enough to generate or our class of transformations to define permutation of derivations.

DEFINITION 26 [PERMUTATION EQUIVALENCE] $\delta_1 \approx \delta_2$ if some series of the following permutations sends δ_1 to δ_2. (1) *interior relabelling*, (2) α/α' *permutation*, (3) α/β *permutation*, (4) β/β' *permutation*, (5) α/α *expansion and retraction*, (6) β/β *expansion and retraction*, (7) *Copying uninvolved links over* β *steps*.

Now we have the capacity to prove Theorem 4 (see page 30), to the effect that If $\delta_1 \approx \delta_2$ if and only if $\tau(\delta_1) = \tau(\delta_2)$.

Proof: One direction, *soundness*, is straightforward. If $\delta_1 \approx \delta_2$ then $\tau(\delta_1) = \tau(\delta_2)$. By construction, each basic permutation preserves proof term, so any series of them keeps proof terms fixed.

The other direction, *completeness*, is a little more involved. We show, by induction on the complexity of the term $\tau(\delta_1)$ that if $\tau(\delta_1) = \tau(\delta_2)$ then there is some derivation δ where $\delta_1 \approx \delta \approx \delta_2$. The general strategy is simple to describe. If δ_1 is an axiom, it is equivalent ot δ_2 immediately. Otherwise, suppose that the induction hypotheiss holds for derivations with terms of a lower complexity. Consider the last inference in δ_1. It corresponds to surface nodes in $\tau(\delta_1)$ Perhaps there are other inferences in δ_1 which also introduce that same node. Permute those inferences down to the bottom of the derivation (using the relevant α/α', β/β' or α/β permutations), then use relabelling and retraction moves to collapse them as far as possible, to find δ, another proof with the same term which ends in *one* α rule introducing this node, or *two* β rules introducing that node. Do the same with δ_2, permuting the inferences that introduce this node to the bottom. The result is two derivations ending in the same rule, with the same proof term. If the ending rule is an α rule, we can reason as follows: both derivations ending in the α rule has the following shape:

$$\dfrac{\begin{array}{c}\delta\\ \vdots\\ {\scriptstyle \pi\{a_1\}\{a_2\}}\\ \mathfrak{S}\{a_1:\alpha_1\}\{a_2:\alpha_2\}\end{array}}{\begin{array}{c}{\scriptstyle \pi\{\acute{\alpha}a\}\{\grave{\alpha}a\}}\\ \mathfrak{S}\{a:\alpha\}\end{array}}\,\alpha$$

where the variable a is introduced in the α rule, and is absent from the sequent in the premise of that rule. It follows that the *premise* of the rule in both cases are derived with derivations with the proof term $\pi\{a_1\}\{a_2\}$, which, by induction, are permutation equivalent.

If the ending rules are β rules, the reasoning is only slightly more complex. Using β/β expansions and contractions, we can transform both derivations into derivations with the following shape

$$
\dfrac{
\begin{array}{c}\delta_l\\ \vdots\\ \scriptstyle \pi_l\{b_1\}\\ \mathfrak{S}_l\{b_1:\beta_1\}\end{array}
\qquad
\dfrac{
\begin{array}{c}\delta_m\\ \vdots\\ \scriptstyle \pi_m\{b_1\}\{b_2\}\\ \mathfrak{S}_m\{b_1:\beta_1\}\{b_2:\beta_2\}\end{array}
\qquad
\begin{array}{c}\delta_r\\ \vdots\\ \scriptstyle \pi_r\{b_2\}\\ \mathfrak{S}_r\{b_2:\beta_2\}\end{array}
}{
\begin{array}{c}\scriptstyle \pi_m\{\acute{\beta}b\}\{b_2\}\quad \pi_r\{\grave{\beta}b\}\\ \mathfrak{S}_{m,r}\{b_2:\beta_2\}\{b:\beta\}\end{array}
}\,\beta
}{
\begin{array}{c}\scriptstyle \pi\{\acute{\beta}b\}\{\grave{\beta}b\}\\ \mathfrak{S}\{b:\beta\}\end{array}
}\,\beta
$$

where the final proof term $\pi\{\acute{\beta}b\}\{\grave{\beta}b\}$ is identical to

$$\pi_l\{\acute{\beta}b\}\quad \pi_m\{\acute{\beta}b\}\{\grave{\beta}b\}\quad \pi_r\{\grave{\beta}b\}$$

given the particular split 'left', 'middle', 'right' in each particular derivation. Using β FILLING, we can transform both derivations into a permutation equivalent derivations with the following shape:

$$
\dfrac{
\begin{array}{c}\delta_l'\\ \vdots\\ \scriptstyle \pi\{b_1\}\{-\}\\ \mathfrak{S}_l'\{b_1:\beta_1\}\end{array}
\qquad
\dfrac{
\begin{array}{c}\delta_m'\\ \vdots\\ \scriptstyle \pi\{b_1\}\{b_2\}\\ \mathfrak{S}_m'\{b_1:\beta_1\}\{b_2:\beta_2\}\end{array}
\qquad
\begin{array}{c}\delta_r'\\ \vdots\\ \scriptstyle \pi\{-\}\{b_2\}\\ \mathfrak{S}_r'\{b_2:\beta_2\}\end{array}
}{
\begin{array}{c}\scriptstyle \pi\{\acute{\beta}b\}\{b_2\}\quad \pi\{-\}\{\grave{\beta}b\}\\ \mathfrak{S}_{m,r}'\{b_2:\beta_2\}\{b:\beta\}\end{array}
}\,\beta
}{
\begin{array}{c}\scriptstyle \pi\{\acute{\beta}b\}\{\grave{\beta}b\}\\ \mathfrak{S}\{b:\beta\}\end{array}
}\,\beta
$$

but now, the sequents and the proof terms at the conclusions of the three derivations, δ_l', δ_m' and δ_r' are fixed by the proof term $\pi\{\acute{\beta}b\}\{\grave{\beta}b\}$, and are the same in both derivations. So, these sub-derivations are simpler derivations, to which the induction hypothesis applies, and hence, they are permutation equivalent. This completes the induction, and the theorem is proved. If two derivations have the same proof term, some combination of interior relabellings, α/α', α/β and β/β' permutations, α/α and β/β expansions and retractions, and copying uninvolved links over β steps transforms one derivation into the other. ∎

We have, therefore, an independent characterisation of proof terms in terms of permutations of derivations. There are two independent characterisations of what it is for two derivations to have the same essential underlying structure. (1) They have the same proof term. (2) Some series of permutations sends one into the other.

5 TERMS AND REDUCTIONS

Now we turn to cut elimination. For terms, this corresponds to the elimination of cut points. In this section I define a procedure for eliminating cut points and prove it to be strongly normalising (every reduction path terminates) and confluent.

DEFINITION 27 [•-REDUCTION FOR TERMS] Given an atomic preterm τ containing the cut point $\bullet$, we say that τ $\bullet$-reduces to τ' (written $\tau \rightsquigarrow_\bullet \tau'$) where τ' is defined in the following way, depending on the type of the cut point $\bullet$.

- *Type* $A \wedge B$: replace $\acute{\wedge}\bullet$ and $\grave{\wedge}\bullet$ in τ by the cut points of type A and B respectively.
- *Type* $A \vee B$: replace $\acute{\vee}\bullet$ and $\grave{\vee}\bullet$ in τ by the cut points of type A and B respectively.
- *Type* $A \supset B$: replace $\acute{\supset}\bullet$ and $\grave{\supset}\bullet$ in τ by the cut points of type A and B respectively.
- *Type* $\neg A$: replace $\grave{\neg}\bullet$ in τ by the cut point of type A.
- *Atomic type*: For each pair of links $n\curvearrowright\bullet$ and $\bullet\curvearrowright m$ in τ (where n and m are not $\bullet$), add $n\curvearrowright m$ to τ and then delete all links involving $\bullet$.
- *Type* $\bot$: Delete all links of type $\bullet\curvearrowright$. These are the only links in τ involving this cut point.
- *Type* $\top$: Delete all links of type $\curvearrowright\bullet$.

The ONE-STEP REDUCTION RELATION from τ is defined as the union of all reduction relations applying to τ. That is, τ one-step reduces to τ' (written $\tau \rightsquigarrow \tau'$) if and only if there is some cut point $\bullet$ in τ where $\tau \rightsquigarrow_\bullet \tau'$. The REDUCTION RELATION $\rightsquigarrow^*$ is the transitive closure of the one-step reduction relation. We say that τ' is a NORMAL FORM for τ if $\tau \rightsquigarrow^* \tau'$, and τ' has no cut points.

It is straightforward to show that reduction corresponds intuitively to the process of cut reduction in derivations when the cut formula is either atomic, or principal in both premises. Here is the case for conjunction. The cut on the conjunction:

$$\dfrac{\dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x:A, \Delta_1} \qquad \overset{\pi_2[y]}{\Sigma_2 \succ y:B, \Delta_2}}{\overset{\pi_1[\acute{\wedge}z]\ \ \pi_2[\grave{\wedge}z]}{\Sigma_{1,2} \succ z:A \wedge B, \Delta_{1,2}}}\wedge R \qquad \dfrac{\overset{\pi_3(x,y)}{\Sigma_3, x:A, y:B \succ \Delta_3}}{\overset{\pi_3(\acute{\wedge}z, \grave{\wedge}z)}{\Sigma_3, z:A \wedge B \succ \Delta_3}}\wedge L}{\overset{\pi_1[\acute{\wedge}\bullet]\ \ \pi_2[\grave{\wedge}\bullet]\ \ \pi_3(\acute{\wedge}\bullet, \grave{\wedge}\bullet)}{\Sigma_{1-3} \succ \Delta_{1-3}}}\mathit{Cut}$$

is here replaced by cuts on the conjuncts.

$$\dfrac{\overset{\pi_1[x]}{\Sigma_1 \succ x:A, \Delta_1} \qquad \dfrac{\overset{\pi_2[y]}{\Sigma_2 \succ y:B, \Delta_2} \qquad \overset{\pi_3(x,y)}{\Sigma_3, x:A, y:B \succ \Delta_3}}{\overset{\pi_2[\flat]\ \ \pi_3(x,\flat)}{\Sigma_{2,3}, x:A \succ \Delta_{2,3}}}\mathit{Cut}}{\overset{\pi_1[\sharp]\ \ \pi_2[\flat]\ \ \pi_3(\sharp,\flat)}{\Sigma_{1-3} \succ \Delta_{1-3}}}\mathit{Cut}$$

The resulting change in the proof terms is replacing the cut point for $A \wedge B$—which can only occur in the scope of $\acute{\wedge}$ and $\grave{\wedge}$, since each link in the term has an atomic type, and $A \wedge B$ is not itself an atom—by the cut point for A (when in the scope of $\acute{\wedge}$) and B (in the scope of $\grave{\wedge}$) respectively. The same holds for *Cuts* on other kinds of complex formulas.

For cuts on *identities*, the behaviour is similar. Consider this concrete example of a derivation ending in two *Cuts*:

$$\dfrac{\dfrac{\dfrac{\overset{v_1\curvearrowright x}{v_1:p\succ x:p} \qquad \overset{v_2\curvearrowright x}{v_2:p\succ x:p}}{\overset{\acute{\vee}v\curvearrowright x\ \ \grave{\vee}v\curvearrowright x}{v:p\vee p\succ x:p}}\vee L \qquad \overset{y\curvearrowright y}{y:p\succ y:p}}{\overset{\acute{\vee}v\curvearrowright\bullet\ \ \grave{\vee}v\curvearrowright\bullet\ \ \bullet\curvearrowright y}{v:p\vee p\succ y:p}}\mathit{Cut} \qquad \dfrac{\overset{z\curvearrowright w_1}{z:p\succ w_1:p} \qquad \overset{z\curvearrowright w_2}{z:p\succ w_2:p}}{\overset{z\curvearrowright\acute{\wedge}w\ \ z\curvearrowright\grave{\wedge}w}{z:p\succ w:p\wedge p}}\wedge R}{\overset{\acute{\vee}v\curvearrowright\bullet\ \ \grave{\vee}v\curvearrowright\bullet\ \ \bullet\curvearrowright\bullet\ \ \bullet\curvearrowright\acute{\wedge}w\ \ \bullet\curvearrowright\grave{\wedge}w}{v:p\vee p\succ w:p\wedge p}}\mathit{Cut}$$

The reduction for the concluding term $\acute{\vee}v\curvearrowright\bullet\ \ \grave{\vee}v\curvearrowright\bullet\ \ \bullet\curvearrowright\bullet\ \ \bullet\curvearrowright\acute{\wedge}w\ \ \bullet\curvearrowright\grave{\wedge}w$ is

$$\acute{\vee}v\curvearrowright\acute{\wedge}w\quad \acute{\vee}v\curvearrowright\grave{\wedge}w\quad \grave{\vee}v\curvearrowright\acute{\wedge}w\quad \grave{\vee}v\curvearrowright\grave{\wedge}w$$

—we delete the $\bullet\curvearrowright\bullet$ term, and compose all other links through $\bullet$. This corresponds neatly to the result of eliminating the *Cuts* in the derivation:

$$\dfrac{\dfrac{\overset{v_1\curvearrowright w_1}{v_1:p\succ w_1:p} \qquad \overset{v_2\curvearrowright w_1}{v_2:p\succ w_1:p}}{\overset{\acute{\vee}v\curvearrowright w_1\ \ \grave{\vee}v\curvearrowright w_1}{v:p\vee p\succ w_1:p}}\vee L \qquad \dfrac{\overset{v_1\curvearrowright w_2}{v_1:p\succ w_2:p} \qquad \overset{v_2\curvearrowright w_2}{v_2:p\succ w_2:p}}{\overset{\acute{\vee}v\curvearrowright w_1\ \ \grave{\vee}v\curvearrowright w_1}{v:p\vee p\succ w_2:p}}\vee L}{\overset{\acute{\vee}v\curvearrowright\acute{\wedge}w\ \ \acute{\vee}v\curvearrowright\grave{\wedge}w\ \ \grave{\vee}v\curvearrowright\acute{\wedge}w\ \ \grave{\vee}v\curvearrowright\grave{\wedge}w}{v:p\vee p\succ w:p\wedge p}}\wedge R$$

Now we prove that the reduction relation is strongly normalising and confluent. These results are extremely simple, *unlike* the case for cut reduction in the classical sequent calculus [11, 21, 37].

THEOREM 6 [CUT REDUCTION IS STRONGLY NORMALISING] *For a preterm τ, there is no infinite chain*

$$\tau \rightsquigarrow \tau_1 \rightsquigarrow \tau_2 \rightsquigarrow \cdots$$

of reductions. On the contrary, every reduction path from τ terminates.

Proof: If $\tau \rightsquigarrow \tau'$ then τ' is less complex than τ.[24] If the cut reduced in τ is complex, a cut node together with a prefix is replaced by a new cut node, resulting in a less complex preterm. If the cut reduced is atomic, we replace links with the cut node (with complexity 1) with links with atomic variables (with complexity 0). In every case, complexity reduces. The complexity of a preterm is finite, and bounded below by 0, so the process can take no more than n steps where n is the complexity of τ. ■

[24] Recall the definition of complexity for preterms, Definition 6 on page 13.

THEOREM 7 [CUT REDUCTION IS CONFLUENT] *In fact, it is confluent in a very strong sense: If* $\tau \rightsquigarrow_{\bullet} \tau'$ *and* $\tau \rightsquigarrow_{\star} \tau''$ *where* $\bullet \neq \star$ *then there is some* τ''' *where* $\tau' \rightsquigarrow_{\star} \tau'''$ *and* $\tau'' \rightsquigarrow_{\bullet} \tau'''$*. It follows that every reduction path for* τ *terminates in the same normal preterm* τ^**.*

Proof: Suppose $\tau \rightsquigarrow_{\bullet} \tau'$ and $\tau \rightsquigarrow_{\star} \tau''$ for distinct cut points $\bullet$ and $\star$ in τ. If both are of complex type, then τ''' is found by replacing both cut points by the cut points for the subformulas of their type, and the result is in the same regardless of order. If one is atomic and the other is complex, then τ''' is found by replacing the complex one by cut points for the subformulas of its type, and links involving the other cut point are deleted (in the case of the cut point appearing in both sides) or composed (threading a link leading *to* a cut point into a link leading *from* that cut point). Again, these can be done in either order with the same result. Finally, if *both* are atomic, then again, the result can be done in any order. The only possible complication is if a link involves *both* cut points, but this is impossible unless the two cut points are identical—for there is only one cut point of each type.

The reduction process $\rightsquigarrow$ has the diamond property, and hence, so does $\rightsquigarrow^*$. Since $\rightsquigarrow^*$ is also strongly normalising, it follows that it terminates in a unique normal form. ∎

Let us use the convention that τ^* is the unique normal form of the preterm τ, the result of eliminating all cut points from τ. The function sending τ to τ^* gives us a new cut elimination algorithm for derivations.

1. Take a derivation δ, and find its term $\tau(\delta)$.
2. Reduce τ to its normal form τ^*.
3. Use Theorem 3 to construct a derivation with term τ^*.

For this to serve as a cut elimination procedure, we need to verify that τ^* is actually a *term* (and not merely a preterm) and that the derivation for its normal form τ^* is a derivation for the same sequent.

THEOREM 8 [REDUCTION PRESERVES MEANING FOR TERMS] *If* τ *is a term of type* $\Sigma \succ \Delta$ *and* $\tau \rightsquigarrow \tau'$*, then* τ' *is also a term of type* $\Sigma \succ \Delta$*.*

Proof: First we show that if $\tau \rightsquigarrow \tau'$ and τ is atomic, then so is τ'. This is immediate. By inspection you can verify that the types of the links in τ' are unchanged from those in τ.

Next, we show that if τ is spanned, then so is τ'. If the cut point reduced was of complex type, it's straightforward to see that the switch pairs of τ correspond neatly to the switch pairs of τ'. Consider the case where $\bullet$ is of conjunctive type, and the introduced cut points in τ' are $\sharp$ and $\flat$. So τ has the form $\tau(\acute{\wedge}\bullet, \grave{\wedge}\bullet)[\acute{\wedge}\bullet, \grave{\wedge}\bullet]$, and τ' is then $\tau(\sharp, \flat)[\sharp, \flat]$. Then one switch pair in τ $[\bullet](\bullet)$, and another is $[\acute{\wedge}\bullet][\grave{\wedge}\bullet]$. It follows that for any choice τ^s of switches the remainder of τ. the following preterms themselves are non-empty:

$$\tau^s(-,-)[\acute{\wedge}\bullet,-] \quad \tau^s(-,-)[-,\grave{\wedge}\bullet] \quad \tau^s(\acute{\wedge}\bullet,\grave{\wedge}\bullet)[-,-]$$

The *four* switch settings for τ' are

$$\tau^s(-,-)[\sharp,\flat] \quad \tau^s(-,\flat)[\sharp,-] \quad \tau^s(\sharp,-)[-,\flat] \quad \tau^s(\sharp,\flat)[-,-]$$

and these must all be nonempty, given that the switchings for τ are. The same holds for the other β rules with switched nodes, which have the same shape.

If the cut point reduced was of atomic type, then if τ is spanned, so is τ'. Suppose τ' isn't spanned. Then it disappears under some switching. Consider all of the links of the form $n_i \curvearrowright m_j$ in τ' where $n_i \curvearrowright \bullet$ and $\bullet \curvearrowright m_j$ are in τ. If there is some switching in which all of the links of the form $n_i \curvearrowright m_j$ disappear (along with the rest of τ'), then either on that switching all of the nodes n_i dissapear, or on that switching, all of the nodes in m_j disappear, (If some n_i and some m_j survives under that switching, then the link $n_i \curvearrowright m_j$ survives too, but no link survives.) If all of the n_i terms disappear, then consider *that* switching, and also switch off $\bullet$ in *input* position. Under this switching each $n_i \curvearrowright m_j$ disappears since n_i disappears. Eachg $\bullet \curvearrowright m_i$ disappears because $\bullet$ is deleted. The link $\bullet \curvearrowright \bullet$ disappears (if present), and the remaining links (if any) are unchanged from τ', so they disappear too. So in this case, τ would have to be empty. The same goes for the case where each m_j disappears from τ'. In that case, consider the switching for τ where $\bullet$ is deleted from *output* position. That would make τ empty, but our assumption is that it isn't. So, if τ is spanned and it reduces in one step to τ', then τ' is spanned too.

Finally, the reduction process does not introduce any input or output variables. It may delete them, but it does not introduce them. So, if τ has type $\Sigma \succ \Delta$, and $\tau \rightsquigarrow \tau'$, then τ' also has type $\Sigma \succ \Delta$. ∎

As hinted at in this proof process of cut reduction may change the input or output variables present in a term. For example, the term $x \curvearrowright x \;\; \bullet \curvearrowright y$, where x has type p and $\bullet$ and y have type q, is a term of type $x:p \succ x:p, y:q$. It reduces, on the other hand, to the term $x \curvearrowright x$, which has lost its output variable y. The cut reduction step is straightforward:

$$\dfrac{\overset{x \curvearrowright x}{x:p \succ x:p, y:q} \qquad \overset{y \curvearrowright y}{y:q \succ y:q}}{\overset{x \curvearrowright x \quad \bullet \curvearrowright y}{x:p \succ x:p, y:q}}\ \mathit{Cut} \qquad \text{reduces to} \qquad \overset{x \curvearrowright x}{x:p \succ x:p, y:q}$$

but in this case, the term $x \curvearrowright x$ *also* has type $x:p \succ x:p$, without y making an appearance.[25]

[25]This kind of 'weakened in' output variable y in $x \curvearrowright x \;\; \bullet \curvearrowright y$ is a curious feature of the proof term. The variable y is certainly *there*, but is in not an output variable under every switching of the term. (If we switch the input $\bullet$ off, then y disappears.) This trick can be turned with the use of switched nodes other than *Cut* nodes, too. Consideer this derivation:

$$\dfrac{\overset{x \curvearrowright x}{x:p, u:r \succ x:p} \qquad \overset{v \curvearrowright y}{v:q \succ y:q}}{\overset{x \curvearrowright x \quad \acute{\vee} z \curvearrowright y}{x:p, z:q \vee r \succ x:p, y:q}}\ \vee L$$

Here, z and y (and any connection between them) play only a weak role in the concluding sequent. They are present, but not in every switching. Perhaps it makes sense to say that an input (or output) variable STRONGLY PRESENT in a preterm if it is present as input (or output) in every switching of that preterm. In the switching example given on page 24, notice that the input and output variables are present in every switching.

6 DERIVATIONS AND CUTS

In the remaining section we will revisit the considerations on classical cut reduction broached in the first section of the paper, and explore the relationship between cut reduction for terms and classical cut elimination arguments in the sequent calculus.

Consider again the so-called triviality argument for classical cut reduction (from page 6), which considers a cut on a weakened-in formula C, to allow for the merging of two arbitrary sequents. Annotating the results with terms, we see something important:

$$\dfrac{\dfrac{\begin{matrix}\delta_1\\ \vdots\\ \overset{\pi_1}{\Sigma \succ \Delta}\end{matrix}}{\overset{\pi_1}{\Sigma \succ x:C,\Delta}}\,K \qquad \dfrac{\begin{matrix}\delta_2\\ \vdots\\ \overset{\pi_2}{\Sigma \succ \Delta}\end{matrix}}{\overset{\pi_2}{\Sigma, y:C \succ \Delta}}\,K}{\overset{\pi_1\ \ \pi_2}{\Sigma \succ \Delta}}\,Cut$$

In this derivation, there is no cut point for the cut formula C. We have displayed the derivation with a weakening step in both branches, but this is redundant. The derivation δ_1 with term π_1 could be systematically modified to be a derivation *with the same term* π_1 but with the extra conclusion $x:C$, in the manner of Theorem 1. The same is true for δ_2 and π_2. The weakened-in Cs which are cut out play no role in the proof term, so acccording to the term, there is no *Cut* there to eliminate. Instead, we have a *Blend* of the two derivations. Let's consider what this means in a concrete case, so we can see how the *Blend* of two derivations looks in practice. Here is a cut on two different derivations of the one sequent $x:p \wedge p \succ y:p$.

$$\dfrac{\dfrac{\overset{x_1 \curvearrowright y}{x_2:p \succ y:p, u:C}}{\overset{\acute{\wedge}x \curvearrowright y}{x:p\wedge p \succ y:p, u:C}}\,\wedge L \qquad \dfrac{\overset{x_2 \curvearrowright y}{x_2:p, v:C \succ y:p}}{\overset{\grave{\wedge}x \curvearrowright y}{x:p\wedge p, v:C \succ y:p}}\,\wedge L}{\overset{\acute{\wedge}x \curvearrowright y\ \ \grave{\wedge}x \curvearrowright y}{x:p\wedge p \succ y:p}}\,Cut$$

In this case, the term $\acute{\wedge}x \curvearrowright y\ \grave{\wedge}x \curvearrowright y$ is already in normal form, and it corresponds to this *Cut*-free derivation:

$$\dfrac{\overset{x_1 \curvearrowright y\ \ x_2 \curvearrowright y}{x_1:p, x_2:p \succ y:p}}{\overset{\acute{\wedge}x \curvearrowright y\ \ \grave{\wedge}x \curvearrowright y}{x:p\wedge p \succ y:p}}\,\wedge L$$

among others. We can eliminate *Cuts* in the presence of weakening on both sides of a sequent without collapsing distinctions between proofs (or proof terms). The terms $\acute{\wedge}x \curvearrowright y$, $\grave{\wedge}x \curvearrowright y$ and

I will not make this paper any longer by stopping to analyse the difference between those variables that are strongly present in a term and those which are not, but it seems to me that this notion is worth further consideration.

their union, $\acute{\wedge}x\curvearrowright y\ \grave{\wedge}x\curvearrowright y$ are all distinct proof terms which describe different ways one could prove the sequent $x:p\wedge p \succ y:p$.

Now consider the case of supposed divergence in Cut elimination, due to contraction on both sides of a sequent, introduced on page 6. Here the derivation is as follows, when annotated with proof terms.

$$\dfrac{\dfrac{\overset{x_1\curvearrowright y}{x_1:p \succ y:p} \qquad \overset{x_2\curvearrowright y}{x_2:p \succ y:p}}{\overset{\acute{\wedge}x\curvearrowright y\quad \grave{\wedge}x\curvearrowright y}{x:p\vee p \succ y:p}}\,\vee L \qquad \dfrac{\overset{y\curvearrowright z_1}{y:p \succ z_1:p} \qquad \overset{y\curvearrowright z_2}{y:p \succ z_2:p}}{\overset{y\curvearrowright\acute{\vee}z\quad y\curvearrowright\grave{\vee}z}{y:p \succ z:p\wedge p}}\,\wedge R}{\overset{\acute{\wedge}x\curvearrowright\bullet\quad \grave{\wedge}x\curvearrowright\bullet\quad \bullet\curvearrowright\acute{\vee}z\quad \bullet\curvearrowright\grave{\vee}z}{x:p\vee p \succ z:p\wedge p}}\,\mathit{Cut}$$

The concluding term can be reduced. Its $\bullet$-reduction is

$$\acute{\wedge}x\curvearrowright\acute{\vee}z\quad \grave{\wedge}x\curvearrowright\acute{\vee}z\quad \acute{\wedge}x\curvearrowright\grave{\vee}z\quad \grave{\wedge}x\curvearrowright\grave{\vee}z$$

And this, indeed, is the term of *both* of the derivations proffered as the result of eliminating the *Cut* from our derivation, using standard procedures. The first is:

$$\dfrac{\dfrac{\overset{x_1\curvearrowright z_1}{x_1:p \succ z_1:p} \qquad \overset{x_1\curvearrowright z_2}{x_1:p \succ z_2:p}}{\overset{x_1\curvearrowright\acute{\vee}z\quad x_1\curvearrowright\grave{\vee}z}{x:p \succ z:p\wedge p}}\,\wedge R \qquad \dfrac{\overset{x_2\curvearrowright z_1}{x_2:p \succ z_1:p} \qquad \overset{x_2\curvearrowright z_2}{x_2:p \succ z_2:p}}{\overset{x_2\curvearrowright\acute{\vee}z\quad x_2\curvearrowright\grave{\vee}z}{x:p \succ z:p\wedge p}}\,\wedge R}{\overset{\acute{\wedge}x\curvearrowright\acute{\vee}z\quad \grave{\wedge}x\curvearrowright\acute{\vee}z\quad \acute{\wedge}x\curvearrowright\grave{\vee}z\quad \grave{\wedge}x\curvearrowright\grave{\vee}z}{x:p\vee p \succ z:p\wedge p}}\,\vee L$$

where we choose to process the conjunctions before the disjunctions. The second is:

$$\dfrac{\dfrac{\overset{x_1\curvearrowright z_1}{x_1:p \succ z_1:p} \qquad \overset{x_2\curvearrowright z_1}{x_2:p \succ z_1:p}}{\overset{\acute{\wedge}x\curvearrowright z_1\quad \grave{\wedge}x\curvearrowright z_1}{x:p\vee p \succ z_1:p}}\,\vee L \qquad \dfrac{\overset{x_1\curvearrowright z_2}{x_1:p \succ z_2:p} \qquad \overset{x_2\curvearrowright z_2}{x_2:p \succ z_2:p}}{\overset{\acute{\wedge}x\curvearrowright z_1\quad \grave{\wedge}x\curvearrowright z_2}{x:p\vee p \succ z_2:p}}\,\vee L}{\overset{\acute{\wedge}x\curvearrowright\acute{\vee}z\quad \grave{\wedge}x\curvearrowright\acute{\vee}z\quad \acute{\wedge}x\curvearrowright\grave{\vee}z\quad \grave{\wedge}x\curvearrowright\grave{\vee}z}{x:p\vee p \succ z:p\wedge p}}\,\wedge R$$

where we process the rules in the opposite order. These are, to be sure, different derivations, but they differ only up to permutation. A straightforward permutation (in this case, a single β/β permutation) sends one derivation to the other. They are, in this sense, two different presentations of the one underlying connection between the premise $p\vee p$ and the conclusion $p\wedge p$.

Now, consider the relationship between elimination of cut points in terms and procedures for the elimination of cuts in derivations. If a cut elimination process for derivations follows the cut reduction process for terms (that is, if δ reduces to δ', then $\tau(\delta) \rightsquigarrow \tau(\delta')$), then it follows that the cut reduction process is also strongly normalising. All reduction paths will terminate in a normal form. And since, all reduction paths for terms reduce to a single normal form, then the resulting derivations will be identical—up to permutation. However, most cut elimination procedures do not necessarily correspond *directly* to cut reduction for terms. Cut reduction for terms eliminates *all* the cut points of a given type in one step. The elimination of a *Cut* in a derivation may leave behind other instances of cuts on the same formula in different parts of the proof. This reduction in cuts, on a local level in a derivation may indeed produce a larger term on the way to eliminating all of the cuts and ending up in a smaller term. Here is a concrete example:

$$\dfrac{\dfrac{\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p}\quad \overset{y\curvearrowright y}{y:q\succ y:q}}{\overset{x\curvearrowright\acute{\wedge}w\;\; y\curvearrowright\grave{\wedge}w}{x:p,y:q\succ w:p\wedge q}}\wedge R \quad \dfrac{\overset{u\curvearrowright u}{u:p\succ u:p}}{\overset{\acute{\wedge}w\curvearrowright u}{w:p\wedge q\succ u:p}}\wedge L}{\overset{x\curvearrowright\acute{\wedge}\bullet\;\; y\curvearrowright\grave{\wedge}\bullet\;\; \acute{\wedge}\bullet\curvearrowright u}{x:p,y:q\succ u:p}}\mathit{Cut} \qquad \dfrac{\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p}\quad \overset{y\curvearrowright y}{y:q\succ y:q}}{\overset{x\curvearrowright\acute{\wedge}w\;\; y\curvearrowright\grave{\wedge}w}{x:p,y:q\succ w:p\wedge q}}\wedge R \quad \dfrac{\overset{v\curvearrowright v}{v:q\succ v:q}}{\overset{\grave{\wedge}w\curvearrowright v}{w:p\wedge q\succ v:q}}\wedge L}{\overset{x\curvearrowright\acute{\wedge}\bullet\;\; y\curvearrowright\grave{\wedge}\bullet\;\; \grave{\wedge}\bullet\curvearrowright v}{x:p,y:q\succ v:q}}\mathit{Cut}}{\overset{x\curvearrowright\acute{\wedge}\bullet\;\; y\curvearrowright\grave{\wedge}\bullet\;\; \acute{\wedge}\bullet\curvearrowright\acute{\wedge}z\;\; \grave{\wedge}\bullet\curvearrowright\grave{\wedge}z}{x:p,y:q\succ z:p\wedge q}}\wedge R$$

Cut reduction for the term results in $x\curvearrowright\sharp\;\; y\curvearrowright\flat\;\; \sharp\curvearrowright\acute{\wedge}z\;\; \flat\curvearrowright\grave{\wedge}z$, and then $x\curvearrowright\acute{\wedge}z\;\; y\curvearrowright\grave{\wedge}z$, as expected, each step resulting in a term less complex than the one before. This is not the case if we simplify one of the *Cuts* in the derivation before the other.

$$\dfrac{\dfrac{\dfrac{\overset{y\curvearrowright y}{y:q\succ y:q}\quad \overset{x\curvearrowright x}{x:p,y:q\succ x:p}}{\overset{x\curvearrowright x\;\; y\curvearrowright\flat}{x:p,y:q\succ x:p}}\mathit{Cut} \quad \overset{u\curvearrowright u}{u:p\succ u:p}}{\overset{x\curvearrowright\sharp\;\; y\curvearrowright\flat\;\; \sharp\curvearrowright u}{x:p,y:q\succ u:p}}\mathit{Cut} \qquad \dfrac{\dfrac{\overset{x\curvearrowright x}{x:p\succ x:p}\quad \overset{y\curvearrowright y}{y:q\succ y:q}}{\overset{x\curvearrowright\acute{\wedge}w\;\; y\curvearrowright\grave{\wedge}w}{x:p,y:q\succ w:p\wedge q}}\wedge R \quad \dfrac{\overset{v\curvearrowright v}{v:q\succ v:q}}{\overset{\grave{\wedge}w\curvearrowright v}{w:p\wedge q\succ v:q}}\wedge L}{\overset{x\curvearrowright\acute{\wedge}\bullet\;\; y\curvearrowright\grave{\wedge}\bullet\;\; \grave{\wedge}\bullet\curvearrowright v}{x:p,y:q\succ v:q}}\mathit{Cut}}{\overset{x\curvearrowright\sharp\;\; y\curvearrowright\flat\;\; \sharp\curvearrowright\acute{\wedge}z\;\; x\curvearrowright\acute{\wedge}\bullet\;\; y\curvearrowright\grave{\wedge}\bullet\;\; \grave{\wedge}\bullet\curvearrowright\grave{\wedge}z}{x:p,y:q\succ z:p\wedge q}}\wedge R$$

The resulting term is longer, not shorter. So proof terms do not provide an *immediate* way for proving strong normalisation for all classical cut reduction strategies.

A second observation is worth underlining. Not all classical cut reduction strategies correspond to term reduction. As we have seen in the discussion of the reduction of cuts with weakened-in formulas, classical cut reduction on the blended cut of two derivations δ_1 and δ_2 typically chooses either δ_1 or δ_2 as the result of the reduction. In either case, the term may be considerably changed—connections present in the initial derivation (using the *Cut*) may be sundered apart in the new derivation. This is not just a feature of the interaction between weakening and *Cut* in the sequent calculus. It is there, too, in the connective rules in Gentzen's original calculus. In the reduction case for conjunction in Gentzen's original paper [13, Paragraph 3.113.31], the reduction process for a conjunction is effectively[26] as follows: we transform

[26] Translated into the current context—I annotate with proof terms, elide structural rules, and translate into modern notation. The rest is unchanged.

$$
\dfrac{
\dfrac{
\overset{\pi_1^l[x]}{\Sigma_1 \succ x : A, \Delta_1}
\qquad
\overset{\pi_1^r[y]}{\Sigma_1 \succ y : B, \Delta_1}
}{
\overset{\pi_1^l[\acute{\wedge} z]\quad \pi_1^r[\grave{\wedge} z]}{\Sigma_1 \succ w : A \wedge B, \Delta_1}
}\wedge R
\qquad
\dfrac{
\overset{\pi_2(z)}{\Sigma_2, z : A \succ \Delta_2}
}{
\overset{\pi_2(\acute{\wedge} w)}{\Sigma_2, w : A \wedge B \succ \Delta_2}
}\wedge L
}{
\overset{\pi_1^l[\acute{\wedge}\bullet]\quad \pi_1^r[\grave{\wedge}\bullet]\quad \pi_2(\acute{\wedge}\bullet)}{\Sigma_{1,2} \succ \Delta_{1,2}}
}\mathit{Cut}
$$

this *Cut* in a derivation (or, in general, this *Mix*, but the difference is not important here) into the following simpler *Cut*.

$$
\dfrac{
\overset{\pi_1^l[x]}{\Sigma_1 \succ x : A, \Delta_1}
\qquad
\overset{\pi_2(z)}{\Sigma_2, z : A \succ \Delta_2}
}{
\overset{\pi_1^l[\sharp]\quad \pi_2(\sharp)}{\Sigma_{1,2} \succ \Delta_{1,2}}
}\mathit{Cut}
$$

In this transformation, the entire derivation of $\Sigma_1 \succ y : B, \Delta_1$ in the left branch of the original *Cut* is deleted, and the connections present in $\pi_1^r[\grave{\wedge}\bullet]$ are deleted from the proof term. This kind of reduction step violates local preservation of proof terms. It could, of course, be repaired. The reduction procedure for conjunctions shown on page 40 does not needlessly delete material, and corresponds more closely to cut reduction for terms. There seems to be considerable scope for understanding the dynamics of different cut elimination procedures in terms of way they transform proof terms.

EPILOGUE

We have answered some questions concering classical proof identity, and shown that there is a robust and independently characterisable sense in which there are different ways to prove a sequent. Proof terms and permutations of derivations give rise to a natural non-trivial proof identity relation. However, many questions remain.

- As mentioned at the end of the previous section, different classical cut elimination strategies would seem to merit closer examination in terms of the dynamics of proof terms. Some cut elimination strategies (like Gentzen's) discard components of proof terms that are kept in others. Can new—and simpler—strong normalisation proofs be given in terms of proof term complexity?

- Proof terms are not essentially tied to the sequent calculus. It would be straightforward to define proof terms for tableaux proofs for classical logic, and it would be not too difficult to define proof terms for natural deduction proofs, resolution proofs and even Hilbert proofs, in a natural way. A question which arises is the expressive power of each different proof system. Proof systems may be complete (in the sense that they allow for the derivation of every valid argument) without being complete for proofs—not providing a proof for every proof term. Perhaps the conceptual vocabulary of proof terms can provide new insight into the expressive power of different systems.

- It is easy to construct a category of proofs on the set of formulas. The objects are the formulas, and an arrow between A and B is a reduced (*cut*-free) proof term $\pi(x)[y]$ of type $x:A \succ y:B$. The identity arrow is the canonical identity proof $Id_A : x:A \succ y:A$, and the composition of $\pi_1(x)[y]$ of type $x:A \succ y:B$ and $\pi_2(x)[y]$ of type $x:B \succ y:C$ is the term $(\pi_1(x)[\bullet]\ \pi_2(\bullet)[y])^*$ (we cut on the intermediate formula B and reduce). This satisfies the conditions to be a category, since composition is associative and Id is indeed an identity for composition. (If we did not restrict our attention to the reduced terms, then Id would not be an identity, because the composition of Id_p with itself, for example—$x^{\frown}\bullet\ \ \bullet^{\frown}y$—is not *identical* to Id_p.) It is not a Cartesian closed category (since boolean cartesian closed categories are trivial [20]). What kind of category is it?[27] An immediate answer distinguishing this category from traditional categories is to note that $\bot$ and $\top$ are neither initial nor terminal objects respectively. An initial object in a category is an object such that there is a *unique* arrow from that object to any object. While $x^{\frown}$ is indeed an arrow from $\bot$ to any object at all (it is a term for any sequent $x:\bot \succ y:A$), there is *another* arrow from $\bot$ to $\top$. On this analysis of proof identity, ${}^{\frown}y$ and $x^{\frown}$ are different proof terms for the sequent $x:\bot \succ y:\top$, one appealing to the fact that $\top$ is a tautology, and the other appealing to the fact that $\bot$ is a contradiction. This is analagous to the different proofs one could have for the sequent $x:p \wedge \neg p \succ y:q \vee \neg q$. One has proof term $x^{\frown}\grave{\neg}\grave{\wedge}x$, and the other has proof term $\grave{\neg}\grave{\vee}y^{\frown}y$ (and a third is the blend of the two).

- A natural question is how to generalise proof terms to analyse derivations in classical predicate logic, and beyond, to higher order logics and also to other, non-classical logics. Do proof terms for *intuitionist* sequent calculus behave any differently to λ-terms?

- Proof terms seem like a natural way to represent information flow between premises and conclusions of a sequent. There is scope for further work in understanding and interpreting them, in a simliar way to the interpretation of λ terms and intuitionist natural deduction proofs as describing *constructions* or *verifications* [22, 23, 26, 27, 34, 35]. Can an analagous interpretation be found for proof terms? In the light of any such interpretation, perhaps a deeper understanding can be found of the behaviour of input and output variables, and what it means for inputs and outputs to be strongly present.

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