Generality & Existence I

Quantification & Free Logic

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To analyse the *quantifiers*

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(including their interactions with *modals*)

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(including their interactions with *modals*)
using the tools of *proof theory*

To analyse the *quantifiers*
(including their interactions with *modals*)
using the tools of *proof theory*
in order to better understand
quantification, *existence* and *identity*.

Understanding the quantifier rules.

Sequents & Defining Rules

Generality & Classical Quantifiers

Quantifiers & Non-Denoting Terms

Derivations & Systems

Positions & Models

Completeness & Cut

Consequences & Questions

SEQUENTS & DEFINING RULES

$$\Gamma \succ \Delta$$

Don't assert each element of Γ
and deny each element of Δ.

Identity: $A \succ A$

$$Identity: \quad A \succ A$$

$$Weakening: \quad \frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta} \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}$$

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$$\textit{Weakening:} \quad \frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta} \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}$$

$$\textit{Contraction:} \quad \frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta} \qquad \frac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}$$

Structural Rules

Identity: $A \succ A$

Weakening: $\dfrac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta}$ $\dfrac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}$

Contraction: $\dfrac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta}$ $\dfrac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}$

Cut: $\dfrac{\Gamma \succ A, \Delta \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}$

Structural Rules

$$\textit{Identity:} \quad A \succ A$$

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$$\textit{Contraction:} \quad \frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta} \quad \frac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}$$

$$\textit{Cut:} \quad \frac{\Gamma \succ A, \Delta \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}$$

Structural rules govern declarative sentences *as such*.

With Left/Right rules?

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} \ [\wedge L] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta} \ [\wedge R]$$

With Left/Right rules?

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} \; [\wedge L] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta} \; [\wedge R]$$

$$\frac{\Gamma, B \succ \Delta}{\Gamma, A \; tonk \; B \succ \Delta} \; [tonkL] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ A \; tonk \; B, \Delta} \; [tonkR]$$

Use $\succ_{\mathcal{L}}$ to *define* $\succ_{\mathcal{L}'}$.

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Desideratum #1: $\succ_{\mathcal{L}'}$ is *conservative*: $(\succ_{\mathcal{L}'})|_{\mathcal{L}}$ is $\succ_{\mathcal{L}}$.

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Desideratum #1: $\succ_{\mathcal{L}'}$ is *conservative*: $(\succ_{\mathcal{L}'})|_{\mathcal{L}}$ is $\succ_{\mathcal{L}}$.

Desideratum #2: Concepts are defined *uniquely*.

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta} \ [\land Df]$$

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Fully specifies norms governing conjunctions
on the *left* in terms of simpler vocabulary.

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on the *left* in terms of simpler vocabulary.

Identity and *Cut* determine the behaviour
of conjunctions on the *right*.

$$
\cfrac{
  \cfrac{
    \cfrac{\ }{A \wedge B \succ A \wedge B}\ [Id]
    \qquad
    \cfrac{\ }{A, B \succ A \wedge B}\ [\wedge Df]
  }{
    \Gamma \succ A, \Delta
    \qquad
    \cfrac{\Gamma \succ B, \Delta \qquad A, B \succ A \wedge B}{\Gamma, A \succ A \wedge B, \Delta}\ [Cut]
  }{
  }
}{\Gamma \succ A \wedge B, \Delta}\ [Cut]
$$

$$\frac{\overline{A \land B \succ A \land B} \;\; [Id]}{\Gamma \succ A, \Delta \quad \dfrac{\Gamma \succ B, \Delta \quad A, B \succ A \land B}{\Gamma, A \succ A \land B, \Delta} \;\; [\land Df]}{\Gamma \succ A \land B, \Delta} \;\; [Cut]}{[Cut]}$$

$$\cfrac{\cfrac{\quad}{A \wedge B \succ A \wedge B} \ [Id]}{\Gamma \succ B, \Delta \qquad \cfrac{}{A, B \succ A \wedge B} \ [\wedge Df]}$$

$$\cfrac{\Gamma \succ A, \Delta \qquad \cfrac{\Gamma \succ B, \Delta \qquad \overline{A, B \succ A \wedge B}}{\Gamma, A \succ A \wedge B, \Delta} \ [Cut]}{\Gamma \succ A \wedge B, \Delta} \ [Cut]$$

From [∧*Df*] to [∧*L/R*]

$$\cfrac{\Gamma \succ A, \Delta \qquad \cfrac{\Gamma \succ B, \Delta \qquad \cfrac{\cfrac{}{A \wedge B \succ A \wedge B} \ [Id]}{A, B \succ A \wedge B} \ [\wedge Df]}{\Gamma, A \succ A \wedge B, \Delta} \ [Cut]}{\Gamma \succ A \wedge B, \Delta} \ [Cut]$$

$$\cfrac{\Gamma \succ A, \Delta \qquad \cfrac{\Gamma \succ B, \Delta \qquad \cfrac{\overline{A \wedge B \succ A \wedge B}}{A, B \succ A \wedge B}^{[Id]}}{\Gamma, A \succ A \wedge B, \Delta}^{[\wedge Df]}}{\Gamma \succ A \wedge B, \Delta}^{[Cut]}}{[Cut]}$$

$$\cfrac{\Gamma \succ A, \Delta \qquad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta} \; [\wedge R]$$

$$\dfrac{\dfrac{A \succ A \quad B \succ B}{A, B \succ A \wedge B} \; [\wedge R] \quad \Gamma, A \wedge B \succ \Delta}{\Gamma, A, B \succ \Delta} \; [Cut]$$

$$\mathcal{L}[\wedge Df, Cut] = \mathcal{L}[\wedge L/R, Cut]$$

Equivalence

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This *generalises*: $\wedge$, $\vee$, $\supset$, $\neg$ work in the same way.

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I want to see how this works for quantifiers.

The Rules

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \; [\forall Df] \qquad \frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \; [\exists Df]$$

(where n is not present in the bottom sequent of both rules)

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \ [\forall Df] \qquad \frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \ [\exists Df]$$

(where n is not present in the bottom sequent of both rules)

For this to work as expected, n must be *deductively general*.

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \ [\forall Df] \qquad \frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \ [\exists Df]$$

(where n is not present in the bottom sequent of both rules)

For this to work as expected, n must be *deductively general*.

Function terms are not deductively general:
$(\forall x)(0 \neq x') \succ 0 \neq 1$, but $(\forall x)(0 \neq x') \nsucc (\forall x)(0 \neq x)$.

A term $\mathfrak{n}$ is *deductively general* for the category $\mathfrak{T}$ iff the rule of *specification* is admissible for each term $\mathfrak{t}$ of category $\mathfrak{T}$.

$$\frac{\Gamma \succ \Delta}{\Gamma[\mathfrak{n} := \mathfrak{t}] \succ \Delta[\mathfrak{n} := \mathfrak{t}]} \; [Spec^{\mathfrak{n}}_{\mathfrak{t}}]$$

A term n is *deductively general* for the category $\mathfrak{T}$ iff the rule of *specification* is admissible for each term t of category $\mathfrak{T}$.

$$\frac{\Gamma \succ \Delta}{\Gamma[n := t] \succ \Delta[n := t]} \; [Spec_t^n]$$

In classical first order predicate logic, names are deductively general.

$$\frac{(\forall x)Fx \succ (\forall x)Fx}{(\forall x)Fx \succ Fn} \ [\forall Df]$$

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How can we derive $(\forall x)Fx \succ Ft$?

$$\frac{(\forall x)Fx \succ (\forall x)Fx}{(\forall x)Fx \succ Fn} \ [\forall Df]$$

How can we derive $(\forall x)Fx \succ Ft$?

We must make explicit use of *specification*.

$$\frac{\dfrac{(\forall x)Fx \succ (\forall x)Fx}{(\forall x)Fx \succ Fn} \ [\forall Df]}{(\forall x)Fx \succ Ft} \ [Spec_t^n]$$

$$\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)}^{\ [Id]}}{(\forall x)A(x) \succ A(n)}^{\ [\forall Df]} \qquad \Gamma, A(n) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}^{\ [Cut]}$$

$$\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)}\ [Id]}{(\forall x)A(x) \succ A(n)}\ [\forall Df] \qquad \Gamma, A(n) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\ [Cut]$$

$$
\dfrac{
  \dfrac{
    \dfrac{}{(\forall x)A(x) \succ (\forall x)A(x)} \; [Id]
  }{(\forall x)A(x) \succ A(\mathfrak{n})} \; [\forall Df]
  \qquad
  \Gamma, A(\mathfrak{n}) \succ \Delta
}{\Gamma, (\forall x)A(x) \succ \Delta} \; [Cut]
$$

$$
\cfrac{
\cfrac{
\overline{(\forall x)A(x) \succ (\forall x)A(x)} \;\; [Id]
}{
(\forall x)A(x) \succ A(n)
} \;\; [\forall Df] \quad \Gamma, A(n) \succ \Delta
}{
\Gamma, (\forall x)A(x) \succ \Delta
} \;\; [Cut]
$$

From [∀*Df*] to [∀*L*]

$$\frac{\dfrac{}{(\forall x)A(x) \succ (\forall x)A(x)} \, [Id]}{(\forall x)A(x) \succ A(n)} \, [\forall Df] \qquad \Gamma, A(n) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} \, [Cut]$$

The resulting rule no longer has the side condition for n, because the premise sequent $\Gamma, A(n) \succ \Delta$ is arbitrary.

$$\frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} \, [\forall L\text{: for names}]$$

However, it only applies to names, not terms.

$$\frac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)} \; [Id]}{(\forall x)A(x) \succ A(n)} \; [\forall Df]}{\dfrac{(\forall x)A(x) \succ A(t)} \; [Spec_t^n] \qquad \Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} \; [Cut]$$

$$\dfrac{\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)}\ [Id]}{(\forall x)A(x) \succ A(n)}\ [\forall Df]}{(\forall x)A(x) \succ A(t)}\ [Spec_t^n] \qquad \Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\ [Cut]$$

$$\dfrac{\Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\ [\forall L]$$

$$\mathcal{L}[\forall Df, Spec, Cut] = \mathcal{L}[\forall L/R, Spec, Cut]$$

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QUANTIFIERS &
NON-DENOTING
TERMS

$$\frac{1}{0} \qquad \lim_{x \to 0} \frac{\sin x}{x} \qquad \sum_{n=0}^{\infty} f(n)$$

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Non-Denoting Terms

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It is difficult to eliminate non-denoting terms as a matter of *syntax*.

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It is difficult to eliminate non-denoting terms as a matter of *syntax*.

$$(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\succ (\tfrac{1}{0} < 0 \lor \tfrac{1}{0} = 0 \lor \tfrac{1}{0} > 0)$$

$$\frac{1}{0} \qquad \lim_{x \to 0} \frac{\sin x}{x} \qquad \sum_{n=0}^{\infty} f(n) \qquad \textit{Pegasus}$$

It is difficult to eliminate non-denoting terms as a matter of *syntax*.

$$(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\vdash (\tfrac{1}{0} < 0 \lor \tfrac{1}{0} = 0 \lor \tfrac{1}{0} > 0)$$

How can we modify the quantifier rules
to allow for non-denoting terms?

To rule a term *in* is to take it as suitable
to substitute into a quantifier,
i.e., to take the term to *denote*.

To rule a term *out* is to take it as unsuitable
to substitute into a quantifier,
i.e., to take the term to *not denote*.

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to substitute into a quantifier,
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To rule a term *out* is to take it as unsuitable
to substitute into a quantifier,
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We add terms to the LHS and RHS of sequents $\Gamma \succ \Delta$.

Identity: $X \succ X$

Identity: $\quad \mathsf{X} \succ \mathsf{X}$

Weakening: $\quad \dfrac{\Gamma \succ \Delta}{\Gamma, \mathsf{X} \succ \Delta} \qquad \dfrac{\Gamma \succ \Delta}{\Gamma \succ \mathsf{X}, \Delta}$

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$$\textit{Contraction:} \quad \frac{\Gamma, X, X \succ \Delta}{\Gamma, X \succ \Delta} \qquad \frac{\Gamma \succ X, X, \Delta}{\Gamma \succ X, \Delta}$$

$Identity$: $\mathsf{X} \succ \mathsf{X}$

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$Contraction$: $\dfrac{\Gamma, \mathsf{X}, \mathsf{X} \succ \Delta}{\Gamma, \mathsf{X} \succ \Delta}$ $\dfrac{\Gamma \succ \mathsf{X}, \mathsf{X}, \Delta}{\Gamma \succ \mathsf{X}, \Delta}$

Cut: $\dfrac{\Gamma \succ \mathsf{X}, \Delta \quad \Gamma, \mathsf{X} \succ \Delta}{\Gamma \succ \Delta}$

Identity: $X \succ X$

Weakening: $\dfrac{\Gamma \succ \Delta}{\Gamma, X \succ \Delta}$ $\dfrac{\Gamma \succ \Delta}{\Gamma \succ X, \Delta}$

Contraction: $\dfrac{\Gamma, X, X \succ \Delta}{\Gamma, X \succ \Delta}$ $\dfrac{\Gamma \succ X, X, \Delta}{\Gamma \succ X, \Delta}$

Cut: $\dfrac{\Gamma \succ X, \Delta \quad \Gamma, X \succ \Delta}{\Gamma \succ \Delta}$

Here X is either a sentence or a term.

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \,[\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \,[\exists Df]$$

$$\dfrac{\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)} \; [Id]}{(\forall x)A(x), n \succ A(n)} \; [\forall Df]}{(\forall x)A(x), t \succ A(t)} \; [Spec^n_t] \quad \color{red}{\Gamma, A(t) \succ \Delta}}{\dfrac{\Gamma, (\forall x)A(x), t \succ \Delta}{\color{red}{\Gamma, (\forall x)A(x) \succ \Delta}} \; [Cut]} \; [Cut] \quad \color{red}{\Gamma \succ t, \Delta}$$

$$\dfrac{\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)}}{(\forall x)A(x), n \succ A(n)} \; [\forall Df]}{(\forall x)A(x), t \succ A(t)} \; [Spec_t^n] \quad \Gamma, A(t) \succ \Delta}{\dfrac{\Gamma, (\forall x)A(x), t \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} \; [Cut] \quad \Gamma \succ t, \Delta}} \; [Cut]$$

This results in a two-premise rule:

$$\dfrac{\Gamma, A(t) \succ \Delta \quad \Gamma \succ t, \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} \; [\forall L]$$

$$\frac{\displaystyle \frac{\overline{(\exists x)A(x) \succ (\exists x)A(x)}}{A(n), n \succ (\exists x)A(x)} \, [\exists Df] }{\displaystyle \frac{\Gamma \succ A(t), \Delta \quad \frac{A(t), t \succ (\exists x)A(x)}{A(t), t \succ (\exists x)A(x)} \, [Spec_t^n]}{\Gamma, t \succ (\exists x)A(x), \Delta}} \, [Id]$$

Rendered proof tree:

$$
\cfrac{
\Gamma, t \succ \Delta \qquad
\cfrac{
\Gamma \succ A(t), \Delta \qquad
\cfrac{
\cfrac{
\cfrac{\quad}{(\exists x)A(x) \succ (\exists x)A(x)} \,[Id]
}{A(n), n \succ (\exists x)A(x)} \,[\exists Df]
}{A(t), t \succ (\exists x)A(x)} \,[Spec_t^n]
}{\Gamma, t \succ (\exists x)A(x), \Delta} \,[Cut]
}{\Gamma \succ (\exists x)A(x), \Delta} \,[Cut]
$$

$$\cfrac{\cfrac{\cfrac{}{(\exists x)A(x) \succ (\exists x)A(x)} \ [Id]}{\cfrac{A(n), n \succ (\exists x)A(x)}{\Gamma \succ A(t), \Delta \quad \cfrac{A(t), t \succ (\exists x)A(x)} \ [Spec^n_t]}}{\cfrac{\Gamma, t \succ \Delta \qquad \Gamma, t \succ (\exists x)A(x), \Delta}{\Gamma \succ (\exists x)A(x), \Delta}} \ [Cut]} \ [Cut]$$

This gives a two-premise [∃R] rule:

$$\cfrac{\Gamma \succ t, \Delta \quad \Gamma \succ A(t), \Delta}{\Gamma \succ (\exists x)A(x), \Delta} \ [\exists R]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t\!\downarrow \, \succ \Delta} \; [\downarrow Df]$$

This results in the obvious $[\downarrow R]$ rule.

$$\frac{\dfrac{\overline{t\!\downarrow \, \succ t\!\downarrow}}{} \; [Id]}{\dfrac{\Gamma \succ t, \Delta \quad t \succ t\!\downarrow}{\Gamma \succ t\!\downarrow, \Delta} \; [Cut]} \qquad \frac{\Gamma \succ t, \Delta}{\Gamma \succ t\!\downarrow, \Delta} \; [\downarrow R]$$

SOLOMON FEFERMAN*

DEFINEDNESS

ABSTRACT. Questions of definedness are ubiquitous in mathematics. Informally, these involve reasoning about expressions which may or may not have a value. This paper surveys work on logics in which such reasoning can be carried out directly, especially in computational contexts. It begins with a general logic of "partial terms", continues with partial combinatory and lambda calculi, and concludes with an expressively rich theory of partial functions and polymorphic types, where termination of functional programs can be established in a natural way.

$$\frac{t_i, \Gamma \succ \Delta}{f(t_1, \ldots, t_n), \Gamma \succ \Delta} \; [fL] \qquad \frac{t_i, \Gamma \succ \Delta}{Ft_1 \cdots t_n, \Gamma \succ \Delta} \; [FL]$$

DERIVATIONS & SYSTEMS

Identity: $\mathsf{X} \succ \mathsf{X}$

Weakening: $\dfrac{\Gamma \succ \Delta}{\Gamma, \mathsf{X} \succ \Delta} \qquad \dfrac{\Gamma \succ \Delta}{\Gamma \succ \mathsf{X}, \Delta}$

Contraction: $\dfrac{\Gamma, \mathsf{X}, \mathsf{X} \succ \Delta}{\Gamma, \mathsf{X} \succ \Delta} \qquad \dfrac{\Gamma \succ \mathsf{X}, \mathsf{X}, \Delta}{\Gamma \succ \mathsf{X}, \Delta}$

Cut: $\dfrac{\Gamma \succ \mathsf{X}, \Delta \quad \Gamma, \mathsf{X} \succ \Delta}{\Gamma \succ \Delta}$

$$\frac{t_i, \Gamma \succ \Delta}{f(t_1, \ldots, t_n), \Gamma \succ \Delta} \ [fL] \qquad \frac{t_i, \Gamma \succ \Delta}{Ft_1 \cdots t_n, \Gamma \succ \Delta} \ [FL]$$

$$\frac{\Gamma \succ \Delta}{\Gamma[n := t] \succ \Delta[n := t]} \ [Spec_t^n]$$

Defining Rules

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} \ [\wedge Df] \qquad \frac{\Gamma \succ A, B, \Delta}{\Gamma \succ A \vee B, \Delta} \ [\vee Df]$$

$$\frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \supset B, \Delta} \ [\supset Df] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma, \neg A \succ \Delta} \ [\neg Df]$$

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \ [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \ [\exists Df]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t\!\downarrow \ \succ \Delta} \ [\downarrow Df]$$

DL[*Df, Cut, Spec*]

Example Derivation

$$\dfrac{\dfrac{(\forall x)(Fx \supset Gx) \succ (\forall x)(Fx \supset Gx)}{(\forall x)(Fx \supset Gx), n \succ Fn \supset Gn} \, [\forall Df] \quad \dfrac{\dfrac{\dfrac{Fn \supset Gn \succ Fn \supset Gn}{Fn \supset Gn, Fn \succ Gn} \, [\supset Df] \quad \dfrac{(\exists x)Gx \succ (\exists x)Gx}{n, Gn \succ (\exists x)Gx} \, [\exists Df]}{Fn \supset Gn, n, Fn \succ (\exists x)Gx} \, [Cut]}{\dfrac{(\forall x)(Fx \supset Gx), n, Fn \succ (\exists x)Gx}{\dfrac{(\forall x)(Fx \supset Gx), (\exists x)Fx \succ (\exists x)Gx}{(\forall x)(Fx \supset Gx) \succ (\exists x)Fx \supset (\exists x)Gx} \, [\supset Df]} \, [\exists Df]} \, [Cut]}$$

Replace the quantifier rules by these *generalised* defining rules:

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \; [\forall Df{\downarrow}] \qquad \frac{\Gamma \succ (\forall x)A(x), \Delta}{\Gamma, t \succ A(t), \Delta} \; [\forall Df{\Uparrow}]$$

$$\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \; [\exists Df{\downarrow}] \qquad \frac{\Gamma, (\exists x)A(x) \succ \Delta}{\Gamma, t, A(t) \succ \Delta} \; [\exists Df{\Uparrow}]$$

Replace the quantifier rules by these *generalised* defining rules:

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \; [\forall Df{\downarrow}] \qquad \frac{\Gamma \succ (\forall x)A(x), \Delta}{\Gamma, t \succ A(t), \Delta} \; [\forall Df{\Uparrow}]$$

$$\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \; [\exists Df{\downarrow}] \qquad \frac{\Gamma, (\exists x)A(x) \succ \Delta}{\Gamma, t, A(t) \succ \Delta} \; [\exists Df{\Uparrow}]$$

$$\text{DL}[\textit{GDf, Cut}]$$

Theorem

A derivation of a sequent $\Gamma \succ \Delta$ in DL[Df, Cut, Spec] can be systematically transformed into a derivation of that sequent in DL[GDf, Cut], and vice versa.

Proof.

All of the rules in DL[*GDf, Cut*], are closed under specification. Take a derivation in DL[*Df, Cut, Spec*], and systematically replace each derivation leading up to the first use of a $Spec_t^n$ rule by transforming that derivation by replacing n by t throughout.

Conversely, the *GDf* rules are a composition of Df rules and *Spec*, so a DL[*GDf, Cut*] derivation can be transformed into a DL[*Df, Cut, Spec*] derivation. $\square$

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta} \ [\wedge L] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta} \ [\wedge R]$$

$$\frac{\Gamma, A \succ \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta} \ [\vee L] \qquad \frac{\Gamma \succ A, B, \Delta}{\Gamma \succ A \vee B, \Delta} \ [\vee R]$$

$$\frac{\Gamma \succ A, \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \supset B \succ \Delta} \ [\supset L] \qquad \frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \supset B, \Delta} \ [\supset R]$$

$$\frac{\Gamma \succ A, \Delta}{\Gamma, \neg A, \succ \Delta} \ [\neg L] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma \succ \neg A, \Delta} \ [\neg R]$$

$$\frac{\Gamma, A(t) \succ \Delta \quad \Gamma \succ t, \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} \; [\forall L] \qquad \frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \; [\forall R]$$

$$\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \; [\exists L] \qquad \frac{\Gamma \succ t, \Delta \quad \Gamma \succ A(t), \Delta}{\Gamma \succ (\exists x)A(x), \Delta} \; [\exists R]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t{\downarrow} \succ \Delta} \; [{\downarrow}L] \qquad \frac{\Gamma \succ t, \Delta}{\Gamma \succ t{\downarrow}, \Delta} \; [{\downarrow}R]$$

$$\frac{\Gamma, A(t) \succ \Delta \quad \Gamma \succ t, \Delta}{\Gamma, (\forall x)A(x) \succ \Delta} \,[\forall L] \qquad \frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta} \,[\forall R]$$

$$\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta} \,[\exists L] \qquad \frac{\Gamma \succ t, \Delta \quad \Gamma \succ A(t), \Delta}{\Gamma \succ (\exists x)A(x), \Delta} \,[\exists R]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t\!\downarrow \,\succ \Delta} \,[\downarrow L] \qquad \frac{\Gamma \succ t, \Delta}{\Gamma \succ t\!\downarrow, \Delta} \,[\downarrow R]$$

DL[*L/R, Cut*]

Theorem

A derivation of a sequent $\Gamma \succ \Delta$ in DL[GDf, Cut] can be systematically transformed into a derivation of that sequent in DL[L/R, Cut], and vice versa.

Proof.

Using *Cut* and *Id*, each (generalised) defining rule can mimic a Left/Right pair, and *vice versa.* □

$$\frac{\dfrac{(\forall x)A(x) \succ (\forall x)A(x)}{(\forall x)A(x), \mathfrak{n} \succ A(\mathfrak{n})} \, [\forall Df]}{(\forall x)A(x) \succ (\forall' x)A(x)} \, [\forall' Df]$$

$$\frac{\dfrac{(\forall' x)A(x) \succ (\forall' x)A(x)}{(\forall' x)A(x), \mathfrak{n} \succ A(\mathfrak{n})} \, [\forall' Df]}{(\forall' x)A(x) \succ (\forall x)A(x)} \, [\forall Df]$$

To show that L/R rules are conservative additions,
we eliminate *Cut*, since the other rules do not
introduce new connectives, quantifiers or predicates.

Then, any derivation of a sequent $\Gamma \succ \Delta$ in a system
will use only the rules involving the connectives,
quantifiers and predicates in that sequent.

POSITIONS & MODELS

$$[\Gamma : \Delta]$$

A pair of *sets*, Γ and Δ where for no
$\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ do we have $\Gamma' \succ \Delta'$.

$$[\Gamma : \Delta]$$

A pair of *sets*, Γ and Δ where for no
$\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ do we have $\Gamma' \succ \Delta'$.

Here Γ and Δ can be infinite,
unlike sequents.

$[\Gamma_2 : \Delta_2]$ is a REFINEMENT of $[\Gamma_1 : \Delta_1]$ iff $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$.

$$\frac{\dfrac{\Gamma, A \wedge B, A, B \succ \Delta}{\Gamma, A \wedge B, A \wedge B \succ \Delta} \, [\wedge Df]}{\Gamma, A \wedge B \succ \Delta} \, [W]$$

$$\frac{\dfrac{\Gamma, A \wedge B, A, B \succ \Delta}{\Gamma, A \wedge B, A \wedge B \succ \Delta} \, [\wedge Df]}{\Gamma, A \wedge B \succ \Delta} \, [W]$$

If $[\Gamma, A \wedge B : \Delta]$ is a position, so is $[\Gamma, A \wedge B, A, B : \Delta]$.

$$\frac{\dfrac{\Gamma, A \wedge B, A, B \succ \Delta}{\Gamma, A \wedge B, A \wedge B \succ \Delta} \ [\wedge Df]}{\Gamma, A \wedge B \succ \Delta} \ [W]$$

$$\frac{\dfrac{\Gamma \succ A, A \wedge B, \Delta \quad \Gamma \succ B, A \wedge B, \Delta}{\Gamma \succ A \wedge B, A \wedge B, \Delta} \ [\wedge R]}{\Gamma \succ A \wedge B, \Delta} \ [W]$$

If $[\Gamma, A \wedge B : \Delta]$ is a position, so is $[\Gamma, A \wedge B, A, B : \Delta]$.

$$\frac{\dfrac{\Gamma, A \wedge B, A, B \succ \Delta}{\Gamma, A \wedge B, A \wedge B \succ \Delta} \; [\wedge Df]}{\Gamma, A \wedge B \succ \Delta} \; [W]$$

$$\frac{\dfrac{\Gamma \succ A, A \wedge B, \Delta \quad \Gamma \succ B, A \wedge B, \Delta}{\Gamma \succ A \wedge B, A \wedge B, \Delta} \; [\wedge R]}{\Gamma \succ A \wedge B, \Delta} \; [W]$$

If $[\Gamma, A \wedge B : \Delta]$ is a position, so is $[\Gamma, A \wedge B, A, B : \Delta]$.

If $[\Gamma : A \wedge B, \Delta]$ is a position, so is (at least)
one of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$.

POSITION	REFINEMENTS
$[\Gamma, A \wedge B : \Delta]$	$[\Gamma, A \wedge B, A, B : \Delta]$
$[\Gamma : A \wedge B, \Delta]$	One of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$

Refinement for Connectives

POSITION	REFINEMENTS
$[\Gamma, A \wedge B : \Delta]$	$[\Gamma, A \wedge B, A, B : \Delta]$
$[\Gamma : A \wedge B, \Delta]$	One of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$
$[\Gamma, A \vee B : \Delta]$	One of $[\Gamma, A, A \vee B : \Delta]$ and $[\Gamma, B, A \vee B : \Delta]$
$[\Gamma : A \vee B, \Delta]$	$[\Gamma : A, B, A \vee B, \Delta]$

Refinement for Connectives

POSITION	REFINEMENTS
$[\Gamma, A \wedge B : \Delta]$	$[\Gamma, A \wedge B, A, B : \Delta]$
$[\Gamma : A \wedge B, \Delta]$	One of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$
$[\Gamma, A \vee B : \Delta]$	One of $[\Gamma, A, A \vee B : \Delta]$ and $[\Gamma, B, A \vee B : \Delta]$
$[\Gamma : A \vee B, \Delta]$	$[\Gamma : A, B, A \vee B, \Delta]$
$[\Gamma, A \supset B : \Delta]$	One of $[\Gamma, A \supset B : A, \Delta]$ and $[\Gamma, B, A \supset B : \Delta]$
$[\Gamma : A \supset B, \Delta]$	$[\Gamma, A : B, A \vee B, \Delta]$

Refinement for Connectives

POSITION	REFINEMENTS
$[\Gamma, A \wedge B : \Delta]$	$[\Gamma, A \wedge B, A, B : \Delta]$
$[\Gamma : A \wedge B, \Delta]$	One of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$
$[\Gamma, A \vee B : \Delta]$	One of $[\Gamma, A, A \vee B : \Delta]$ and $[\Gamma, B, A \vee B : \Delta]$
$[\Gamma : A \vee B, \Delta]$	$[\Gamma : A, B, A \vee B, \Delta]$
$[\Gamma, A \supset B : \Delta]$	One of $[\Gamma, A \supset B : A, \Delta]$ and $[\Gamma, B, A \supset B : \Delta]$
$[\Gamma : A \supset B, \Delta]$	$[\Gamma, A : B, A \vee B, \Delta]$
$[\Gamma, \neg A : \Delta]$	$[\Gamma, \neg A : A, \Delta]$
$[\Gamma : \neg A, \Delta]$	$[\Gamma, A : \neg A, \Delta]$

POSITION	REFINEMENTS
$[\Gamma, Ft_1 \cdots t_n : \Delta]$	$[\Gamma, Ft_1 \cdots t_n, t_1, \ldots, t_n : \Delta]$
$[\Gamma, f(t_1, \ldots, t_n) : \Delta]$	$[\Gamma, f(t_1, \ldots, t_n), t_1, \ldots, t_n : \Delta]$
$[\Gamma, (\forall x)A(x) : \Delta]$	One of $[\Gamma, (\forall x)A(x), A(t) : \Delta]$, $[\Gamma, (\forall x)A(x) : t, \Delta]$ for each term t in $[\Gamma, (\forall x)A(x) : \Delta]$.
$[\Gamma : (\forall x)A(x), \Delta]$	$[\Gamma, n : A(n), (\forall x)A(x), \Delta]$, for some n.
$[\Gamma, (\exists x)A(x) : \Delta]$	$[\Gamma, (\exists x)A(x), A(n), n : \Delta]$, for some n.
$[\Gamma : (\exists x)A(x), \Delta]$	One of $[\Gamma : A(t), (\exists x)A(x), \Delta]$, $[\Gamma : t, (\exists x)A(x), \Delta]$ for each term t in $[\Gamma : (\exists x)A(x), \Delta]$.
$[\Gamma, t\downarrow : \Delta]$	$[\Gamma, t\downarrow, t : A, \Delta]$
$[\Gamma : t\downarrow, \Delta]$	$[\Gamma, A : t, t\downarrow, \Delta]$

A position $[\Gamma : \Delta]$ is FULLY REFINED
when it is closed under each of these conditions.

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Theorem

Any DL[*LR*] *position* $[\Gamma : \Delta]$ *is extended by some fully refined position.*

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Theorem

Any DL[*LR*] *position* $[\Gamma : \Delta]$ *is extended by some fully refined position.*

Proof.

Use the usual tableaux method. □

Models

A MODEL for the logic DL is a structure $\mathfrak{M}$ consisting of

1. A *domain* D.
2. An n-ary predicate F is interpreted as a subset $F^{\mathfrak{M}}$ of D^n (as usual).
3. An n-ary function symbol f is interpreted as a *partial function* $f^{\mathfrak{M}} : D^n \rightharpoonup D$.

- α is a (partial) assignment of values to variables.
- $[\![x]\!]_{\mathfrak{M},\alpha} = \alpha(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_{\mathfrak{M},\alpha} = f^{\mathfrak{M}}([\![t_1]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha})$ if each $[\![t_i]\!]_{\mathfrak{M},\alpha}$ is defined, and $f^{\mathfrak{M}}$ is defined on the inputs $[\![t_1]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha}$.

Interpreting a Language

- $\mathfrak{M} \vDash_\alpha t{\downarrow}$ iff $\llbracket t \rrbracket_{\mathfrak{M},\alpha}$ is defined.
- $\mathfrak{M} \vDash_\alpha Ft_1 \cdots t_n$ iff for each i, the value $\llbracket t_i \rrbracket_{\mathfrak{M},\alpha}$ is defined, and the n-tuple $\langle \llbracket t_n \rrbracket_{\mathfrak{M},\alpha}, \ldots, \llbracket t_n \rrbracket_{\mathfrak{M},\alpha} \rangle \in F^{\mathfrak{M}}$
- $\mathfrak{M} \vDash_\alpha A \wedge B$ iff $\mathfrak{M} \vDash_\alpha A$ and $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha A \vee B$ iff $\mathfrak{M} \vDash_\alpha A$ or $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha A \supset B$ iff $\mathfrak{M} \nvDash_\alpha A$ or $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha \neg A$ iff $\mathfrak{M} \nvDash_\alpha A$.
- $\mathfrak{M} \vDash_\alpha (\forall x)A(x)$ iff $\mathfrak{M} \vDash_{\alpha[x:=d]} A(x)$ for every d in D.
- $\mathfrak{M} \vDash_\alpha (\exists x)A(x)$ iff $\mathfrak{M} \vDash_{\alpha[x:=d]} A(x)$ for some d in D.

$\mathfrak{M}$ is a MODEL OF THE POSITION $[\Gamma : \Delta]$ iff
every sentence in Γ is true in $\mathfrak{M}$,
every term in Γ is defined in $\mathfrak{M}$,
every sentence in Δ is false in $\mathfrak{M}$
and every term in Δ is undefined in $\mathfrak{M}$.

For any fully refined position [Γ : Δ] the model where

(1) the domain D is the set of terms in Γ

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by setting $f(t_1, \ldots, t_n)$ to be defined iff it is in Γ,
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(3) the n-ary function symbol f is interpreted
by setting $f(t_1, \ldots, t_n)$ to be defined iff it is in Γ,
and then it takes *itself* as its value

is said to be the model *from* $[\Gamma : \Delta]$.

COMPLETENESS & CUT

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The model from *a fully refined position is a model* for *that position.*

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Proof.

Inspect the conditions for satisfaction in a model. □

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Corollary

Each position has some model.

Completeness

Theorem

The model from *a fully refined position is a model* for *that position.*

Proof.

Inspect the conditions for satisfaction in a model. □

Corollary

Each position has some model.

Proof.

Extend $[\Gamma : \Delta]$ into a fully refined position. Take the model from that position. It is a model for $[\Gamma : \Delta]$. □

Theorem

If $\Gamma \succ \Delta$ is derivable in DL[*LR, Cut*] *then* $[\Gamma : \Delta]$ *has no model.*

Admissibility of Cut

Theorem

If Γ ⊢ Δ *is derivable in* DL[*LR, Cut*] *then* [Γ : Δ] *has no model.*

Proof.

Induction on the length of the derivation. The special case is *Cut*: If [Γ : X, Δ] and [Γ, X : Δ] have no model, then neither does [Γ : Δ]. □

Admissibility of Cut

Theorem

If Γ ≻ Δ *is derivable in* DL[*LR, Cut*] *then* [Γ : Δ] *has no model.*

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Induction on the length of the derivation. The special case is *Cut*: If [Γ : X, Δ] and [Γ, X : Δ] have no model, then neither does [Γ : Δ]. □

Corollary

If Γ ≻ Δ *derivable in* DL[*LR, Cut*]*, it derivable in* DL[*LR*] *too.*

Admissibility of Cut

Theorem

If $\Gamma \succ \Delta$ *is derivable in* DL[*LR, Cut*] *then* [$\Gamma : \Delta$] *has no model.*

Proof.

Induction on the length of the derivation. The special case is *Cut*: If [$\Gamma : X, \Delta$] and [$\Gamma, X : \Delta$] have no model, then neither does [$\Gamma : \Delta$]. □

Corollary

If $\Gamma \succ \Delta$ *derivable in* DL[*LR, Cut*], *it derivable in* DL[*LR*] *too.*

Proof.

If $\Gamma \succ \Delta$ is not derivable in DL[*LR*], then [$\Gamma : \Delta$] has a model. So it is not derivable in DL[*LR, Cut*] either. □

CONSEQUENCES & QUESTIONS

Defining Rules, with *Generality,*
give insight into the quantifiers.

If we allow for 'non-denoting' terms,
defining rules for free logic are straightforward.

These are *also* defining rules:

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\Pi x)A(x), \Delta} \; [\Pi Df] \qquad \frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\Sigma x)A(x) \succ \Delta} \; [\Sigma Df]$$

Wider Quantifiers?

These are *also* defining rules:

$$\frac{\Gamma \succ A(\mathfrak{n}), \Delta}{\Gamma \succ (\Pi x)A(x), \Delta} \ [\Pi Df] \qquad \frac{\Gamma, A(\mathfrak{n}) \succ \Delta}{\Gamma, (\Sigma x)A(x) \succ \Delta} \ [\Sigma Df]$$

Are they meaningful?

Modality

Modality

Identity

THANK YOU!

http://consequently.org/presentation/2015/
generality-and-existence-1

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