Terms for Classical Sequents

Proof Invariants & Strong Normalisation

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To introduce a new *invariant*
for classical propositional proofs
and to show how they can be used.

Background

Preterms

Derivations

Terms

Eliminating Cuts

Strong Normalisation

Further Work

BACKGROUND

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L \qquad \dfrac{\dfrac{p \wedge q}{p}\,\wedge E}{p \vee q}\,\vee I \qquad \dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\,\lor R}{p \land q \succ p \lor q}\,\land L \qquad \dfrac{\dfrac{p \land q}{p}\,\land E}{p \lor q}\,\lor I \qquad \dfrac{\dfrac{p \succ p}{p \land q \succ p}\,\land L}{p \land q \succ p \lor q}\,\lor R$$

$$\dfrac{\dfrac{q \succ q}{q \succ p \lor q}\,\lor R}{p \land q \succ p \lor q}\,\land L \qquad \dfrac{\dfrac{p \land q}{q}\,\land E}{p \lor q}\,\lor I \qquad \dfrac{\dfrac{q \succ q}{p \land q \succ q}\,\land L}{p \land q \succ p \lor q}\,\lor R$$

When is π_1 the same proof as π_2?

$$\dfrac{p \lor q \qquad \dfrac{[p]^1}{q \lor p}\,{\lor I} \qquad \dfrac{[q]^1}{q \lor p}\,{\lor I}}{\dfrac{q \lor p}{(q \lor p) \lor r}\,{\lor I}}\,{\lor E^1}
\qquad
\dfrac{p \lor q \qquad \dfrac{\dfrac{[p]^1}{q \lor p}\,{\lor I}}{(q \lor p) \lor r}\,{\lor I} \qquad \dfrac{\dfrac{[q]^1}{q \lor p}\,{\lor I}}{(q \lor p) \lor r}\,{\lor I}}{(q \lor p) \lor r}\,{\lor E^1}$$

Are these *different proofs*, or *different ways of presenting the same proof*?

Girard, Lafont and Taylor: *Proofs and Types*, Chapter 2

Natural deduction is a slightly paradoxical system: it is limited to the intuitionistic case (in the classical case it has no particularly good properties) but it is only satisfactory for the $(\wedge, \Rightarrow, \forall)$ fragment of the language: we shall defer consideration of $\vee$ and $\exists$ until chapter 10. Yet disjunction and existence are the two most *typically* intuitionistic connectors!

The basic idea of natural deduction is an asymmetry: a proof is a vaguely tree-like structure (this view is more a graphical illusion than a mathematical reality, but it is a pleasant illusion) with one or more hypotheses (possibly none) but a single conclusion. The deep symmetry of the calculus is shown by the *introduction* and *elimination* rules which match each other exactly. Observe, incidentally, that with a tree-like structure, one can always decide uniquely what was the *last* rule used, which is something we could not say if there were several conclusions.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{[x : p \supset (q \supset r)] \quad [z : p]}{xz : q \supset r}\supset E
        \qquad
        \dfrac{[y : p \supset q] \quad [z : p]}{yz : q}\supset E
      }{(xz)(yz) : r}\supset E
    }{\lambda z\,(xz)(yz) : p \supset r}\supset I
  }{\lambda y \lambda z\,(xz)(yz) : (p \supset q) \supset (p \supset r)}\supset I
}{\lambda x \lambda y \lambda z\,(xz)(yz) : (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))}\supset I
$$

Contraction and *weakening* are managed by variables

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

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$$\dfrac{\dfrac{\dfrac{x : p \supset (p \supset q) \qquad [y : p]}{xy : p \supset q}\supset E \qquad [y : p]}{(xy)y : q}\supset E}{\lambda y\,(xy)y : p \supset q}\supset I$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,\lor R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,\land L$$

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \lor \neg p}\,\lor R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \land \neg p \succ}\,\land L$$

$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{q \succ q \quad r \succ r}{q \lor r \succ q, r}\,\lor L}{p, q \lor r \succ p \land q, r}\,\land R}{p \land (q \lor r) \succ p \land q, r}\,\land L}{p \land (q \lor r) \succ (p \land q) \lor r}\,\lor R$$

$$X \succ Y \qquad X \succ A, Y \qquad X, A \succ Y$$

Where do you put the *variables*,
and where do you put the *terms*?

Our Choice

$$x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m$$

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Each premise and conclusion is decorated with variables.

Our Choice

$$\frac{\pi(x_1,\ldots,x_n)[y_1,\ldots,y_m]}{x_1 : A_1,\ldots,x_n : A_n \succ y_1 : B_1,\ldots,y_m : B_m}$$

Each premise and conclusion is decorated with variables.

The *sequent* gets the term, connecting inputs & outputs.

Example 1

$$
\dfrac{\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \dfrac{\overset{y\frown y}{y : q \succ y : q} \qquad \overset{z\frown z}{z : r \succ z : r}}{\overset{Lw\frown y \quad Rw\frown z}{w : q \lor r \succ y : q, z : r}}\lor L}{\overset{x\frown Fv \quad Lw\frown Sv \quad Rw\frown z}{x : p, w : q \lor r \succ v : p \land q, z : r}}\land R}{\overset{Fu\frown Fv \quad LSu\frown Sv \quad RSu\frown z}{u : p \land (q \lor r) \succ v : p \land q, z : r}}\land L}{\overset{Fu\frown FLt \quad LSu\frown SLt \quad RSu\frown Rt}{u : p \land (q \lor r) \succ t : (p \land q) \lor r}}\lor R
$$

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fy \quad x\frown Sy}{x : p \succ y : p \land p}}\;{\land}R
\qquad
\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fw\frown z}{w : p \land p \succ z : p}}\;{\land}L
}{
\overset{x\frown F\bullet \quad x\frown S\bullet \quad F\bullet\frown z}{x : p \succ z : p}
}\;Cut
$$

PRETERMS

Variables and Cut Points

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– We use $x, y, z, u, v, w, \ldots; \bullet, \star, *, \sharp, \flat$ as schematic letters for variables and cut points, ommitting type superscripts where possible.

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- A variable x of type A and a cut point $\bullet$ of type A are both A NODES.

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- If n is an $A \lor B$ node, then Fn is an A node and Sn is a B node.
- If n is an $A \supset B$ node, then An is an A node and Cn is a B node.

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- If n is an $A \supset B$ node, then An is an A node and Cn is a B node.
- If n is a $\neg A$ node, then Nn is an A node.
- For each complex node Ln, Rn, Fn, Sn, An, Cn and Nn, n is its IMMEDIATE subnode, and the subnodes of n are also subnodes of the original node.

Linkings, Inputs and Outputs

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  - If $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$ or $\mathsf{C}n$ are in *output position*, n is also in *output position*.
  - L, R, F, S and C each *preserve position*.

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  - If $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$ or $\mathsf{C}n$ are in *input position*, n is also in *input position*.
  - If $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$ or $\mathsf{C}n$ are in *output position*, n is also in *output position*.
  - L, R, F, S and C each *preserve position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *input position*, n is in *output position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *output position*, n is in *input position*.

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  - If Ln, Rn, Fn, Sn or Cn are in *input position*, n is also in *input position*.
  - If Ln, Rn, Fn, Sn or Cn are in *output position*, n is also in *output position*.
  - L, R, F, S and C each *preserve position*.
  - If An or Nn is in *input position*, n is in *output position*.
  - If An or Nn is in *output position*, n is in *input position*.
  - A and N *reverse position*.

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  - – L, R, F, S and C each *preserve position*.
  - If An or Nn is in *input position*, n is in *output position*.
  - If An or Nn is in *output position*, n is in *input position*.
  - – A and N *reverse position*.
- The INPUTS (OUTPUTS) of a linking are the *variables* in INPUT (OUTPUT) position of that linking.

Example Linkings

x of type $((p \supset q) \supset p) \supset p$

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x of type $((p \supset q) \supset p) \supset p$

$AAAx \frown Cx$

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x of type $((p \supset q) \supset p) \supset p$

$\mathsf{AAA}x \frown \mathsf{C}x$

$\mathsf{CA}x \frown \mathsf{C}x$

Preterms

- A PRETERM is a finite set of linkings.

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- Its OUTPUTS are the outputs of its linkings.

DERIVATIONS

Annotating Derivations: Identity

$$\Sigma, x : A \overset{x\frown y}{\succ} y : A, \Delta$$

Annotating Derivations: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\wedge L$$

Annotating Derivations: Conjunction

$$\frac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}} \wedge R$$

Excursus on *Weakening* and *Variables*

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\,{\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\,{\supset}I$$

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$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\,{\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\,{\supset}I$$

$$\dfrac{\overset{\pi(x, y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz, Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\,{\wedge}L \qquad \text{can be} \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \wedge B \succ \Delta}}\,{\wedge}L$$

Excursus on *Weakening* and *Variables*

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\,{\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\,{\supset}I$$

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \land B \succ \Delta}}\,{\land}L \qquad \text{can be} \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \land B \succ \Delta}}\,{\land}L$$

In a premise $\pi(x,y)$ the indicated x and y display all of the x and y inputs to the proof term.

There might be *none*.

Annotating Derivations: Negation

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\underset{\Sigma, z : \neg A \succ \Delta}{\pi[\mathsf{N}z]}} \neg L \qquad \frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\underset{\Sigma \succ z : \neg A, \Delta}{\pi(\mathsf{N}z)}} \neg R$$

Annotating Derivations: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}} \vee L \qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Annotating Derivations: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az] \quad \pi'(Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}} \supset R$$

Example Annotation

$$
\dfrac{
  \dfrac{
    \overset{x^\frown x}{x : p \succ x : p}
    \qquad
    \dfrac{
      \overset{y^\frown y}{y : q \succ y : q}
      \qquad
      \overset{z^\frown z}{z : r \succ z : r}
    }{
      \overset{Lw^\frown y \quad Rw^\frown z}{w : q \vee r \succ y : q, z : r}
    }\,\vee L
  }{
    \overset{x^\frown Fv \quad Lw^\frown Sv \quad Rw^\frown z}{x : p, w : q \vee r \succ v : p \wedge q, z : r}
  }\,\wedge R
}{
  \dfrac{
    \overset{Fu^\frown Fv \quad LSu^\frown Sv \quad RSu^\frown z}{u : p \wedge (q \vee r) \succ v : p \wedge q, z : r}
  }{
    \overset{Fu^\frown FLt \quad LSu^\frown SLt \quad RSu^\frown Rt}{u : p \wedge (q \vee r) \succ t : (p \wedge q) \vee r}
  }\,\vee R
}\,\wedge L
$$

Annotating Derivations: Cut

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet] \quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

Example Annotation, with *Cut*

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}{\scriptstyle \lor L} \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}{\scriptstyle \land R}}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}}\,Cut$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \lor q}}\ \lor R}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\ \land L$$

$$\dfrac{\dfrac{p \land q}{p}\ \land E}{p \lor q}\ \lor I$$

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \land q \succ z : p}}\ \lor R}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\ \land L$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \vee q}}\,\vee R}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}\,\wedge L \qquad \dfrac{\dfrac{p \wedge q}{p}\,\wedge E}{p \vee q}\,\vee I \qquad \dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \wedge q \succ z : p}}\,\vee R}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}\,\wedge L$$

$$\dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{w\frown Ry}{w : q \succ y : p \vee q}}\,\vee R}{\overset{Sx\frown Ry}{x : p \wedge q \succ y : p \vee q}}\,\wedge L \qquad \dfrac{\dfrac{p \wedge q}{q}\,\wedge E}{p \vee q}\,\vee I \qquad \dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{Sx\frown w}{x : p \wedge q \succ w : q}}\,\vee R}{\overset{Sx\frown Ry}{x : p \wedge q \succ y : p \vee q}}\,\wedge L$$

$$
\dfrac{p \lor q \qquad \dfrac{[p]^1}{q \lor p}\,{\lor I} \qquad \dfrac{[q]^1}{q \lor p}\,{\lor I}}{\dfrac{q \lor p}{(q \lor p) \lor r}\,{\lor I}}\,{\lor E^1}
$$

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}x \frown x \\ x : p \succ x : p\end{array}}{\begin{array}{c}x \frown Rz \\ x : p \succ z : q \lor p\end{array}}\,{\lor R} \qquad \dfrac{\begin{array}{c}y \frown y \\ y : q \succ y : q\end{array}}{\begin{array}{c}y \frown Lz \\ y : q \succ z : q \lor p\end{array}}\,{\lor R}}{\begin{array}{c}Lw \frown Rz \quad Rw \frown Lz \\ w : p \lor q \succ z : q \lor p\end{array}}\,{\lor L}}{\begin{array}{c}Lw \frown RLu \quad Rw \frown LLu \\ w : p \lor q \succ u : (q \lor p) \lor r\end{array}}\,{\lor R}
$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{p \vee q \qquad \dfrac{\dfrac{[p]^1}{q \vee p}\,{\vee I}}{(q \vee p) \vee r}\,{\vee I} \qquad \dfrac{\dfrac{[q]^1}{q \vee p}\,{\vee I}}{(q \vee p) \vee r}\,{\vee I}}{(q \vee p) \vee r}\,{\vee E^1}$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} x^\frown x \\ x : p \succ x : p \end{array}}{\begin{array}{c} x^\frown Rz \\ x : p \succ z : q \vee p \end{array}}\,{\vee R}}{\begin{array}{c} x^\frown RLu \\ x : p \succ u : (q \vee p) \vee r \end{array}}\,{\vee R} \qquad \dfrac{\dfrac{\begin{array}{c} y^\frown y \\ y : q \succ y : q \end{array}}{\begin{array}{c} x^\frown LLu \\ y : q \succ z : q \vee p \end{array}}\,{\vee R}}{\begin{array}{c} y^\frown LLu \\ y : q \succ u : (q \vee p) \vee r \end{array}}\,{\vee R}}{\begin{array}{c} Lw^\frown RLu \quad Rw^\frown LLu \\ w : p \vee q \succ u : (q \vee p) \vee r \end{array}}\,{\vee L}$$

Sequentialisable Preterms

A preterm is SEQUENTIALISABLE
iff it is the conclusion of some derivation.

TERMS

Nonsequentialisable Preterms

$$\begin{array}{c} \color{darkred}{Lx\frown Fy \quad Rx\frown Sy} \\ \color{darkred}{x} : p \vee q \succ \color{darkred}{y} : p \wedge q \end{array}$$

This is connected, but it is not connected *enough*.

Switching Example

$$\overset{\color{darkred}{Lx^\frown Fy}}{} \quad \overset{\color{darkred}{Rx^\frown Sy}}{}$$

$$x : p \vee q \succ y : p \wedge q$$

Switching Example

$$\overset{Lx\frown Fy}{x : p \vee q} \succ \overset{Rx\frown Sy}{y : p \wedge q}$$

$$\overset{Lx\frown Fy}{x : p \vee -} \succ \overset{\cancel{Rx}\frown\cancel{Sy}}{y : p \wedge -}$$

$$\overset{Lx\frown\cancel{Fy}}{x : p \vee -} \succ \overset{\cancel{Rx}\frown Sy}{y : - \wedge q}$$

$$\overset{\cancel{Lx}\frown Fy}{x : - \vee q} \succ \overset{Rx\frown\cancel{Sy}}{y : p \wedge -}$$

$$\overset{\cancel{Lx}\frown\cancel{Fy}}{x : - \vee q} \succ \overset{Rx\frown Sy}{y : - \wedge q}$$

Switchings

- The SWITCHINGS of a preterm π are found by selecting for each pair of subterms $\mathsf{L}n$ and $\mathsf{R}n$ in *input* position; $\mathsf{F}n$ and $\mathsf{S}n$ in *output position*, $\mathsf{A}n$ in *output* position and $\mathsf{C}n$ in *input position*; or the cut point $\bullet$ (in both *input* and *output position*), one item of the pair to keep, and the other to DELETE.

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- A LINKING in a switching of a preterm π SURVIVES if and only if neither side of the link involves a deletion.
- A preterm is SPANNED if every switching has at least one surviving linking.

Example

$\mathsf{Fu}^\frown\mathsf{FLt}\quad \mathsf{LSu}^\frown\mathsf{SLt}\quad \mathsf{RSu}^\frown\mathsf{Rt}$

This has two pairs for switching:

$\mathsf{LSu}/\mathsf{RSu}$ in *input position*. $\mathsf{FLt}/\mathsf{SLt}$ in *output position*.

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$$\mathsf{Fu}\frown\cancel{\mathsf{FLt}}\quad \mathsf{LSu}\frown\mathsf{SLt}\quad \cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\mathsf{FLt}\quad \mathsf{LSu}\frown\cancel{\mathsf{SLt}}\quad \cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

A preterm π is a TERM when it is SPANNED.

Sequentialisable Preterms are Terms

By induction on the derivation sequentialising π.

Sequentialisable Preterms are Terms: Identity

$$\Sigma, x : A \overset{x \frown y}{\succ} y : A, \Delta$$

Sequentialisable Preterms are Terms: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \land B \succ \Delta}}\land L \qquad \dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \land B, \Delta, \Delta'}}\land R$$

Sequentialisable Preterms are Terms: Negation

$$\frac{\Sigma \succ \overset{\pi[x]}{x : A}, \Delta}{\Sigma, \underset{}{\overset{\pi[\mathsf{N}z]}{z : \neg A}} \succ \Delta}\,\neg L \qquad \frac{\Sigma, \overset{\pi(x)}{x : A} \succ \Delta}{\Sigma \succ \overset{\pi(\mathsf{N}z)}{z : \neg A}, \Delta}\,\neg R$$

Sequentialisable Preterms are Terms: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}} \vee L \qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Sequentialisable Preterms are Terms: Conditional

$$\dfrac{\Sigma \succ \overset{\pi[x]}{x} : A, \Delta \qquad \Sigma', \overset{\pi'(y)}{y} : B \succ \Delta'}{\Sigma, \Sigma', \overset{\pi[Az] \quad \pi'(Cz)}{z} : A \supset B \succ \Delta, \Delta'} \supset L \qquad \dfrac{\Sigma, x : A \succ \overset{\pi(x)[y]}{y} : B, \Delta}{\Sigma \succ \overset{\pi(Az)[Cz]}{z} : A \supset B, \Delta} \supset R$$

Sequentialisable Preterms are Terms: Cut

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet] \quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ Cut$$

By induction on the number of pairs for switching in π.

Except ...

$$x \frown y \quad u \frown v$$

ELIMINATING CUTS

Conjunction Cut Reduction

$$
\dfrac{
\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}}\wedge R
\qquad
\dfrac{\overset{\pi''(u, v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \wedge B \succ \Delta''}}\wedge L
}{
\overset{\pi[F\bullet] \quad \pi'[S\bullet] \quad \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}
}\mathit{Cut}
$$

Conjunction Cut Reduction

$$\dfrac{\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}}{\scriptstyle \wedge R} \qquad \dfrac{\overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \wedge B \succ \Delta''}}{\scriptstyle \wedge L}}{\overset{\pi[F\bullet] \quad \pi'[S\bullet] \quad \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}\mathit{Cut}$$

reduces to

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \dfrac{\overset{\pi'[y]}{\Sigma' \succ y : B, \Delta} \qquad \overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi'[\star] \quad \pi''(u, \star)}{\Sigma', \Sigma'', u : A \succ \Delta', \Delta''}}\mathit{Cut}}{\overset{\pi[*] \quad \pi'[\star] \quad \pi''(*, \star)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}\mathit{Cut}$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y \frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \quad \bullet \frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\ Cut$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y\frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \quad \bullet\frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\ Cut$$

reduces to

$$\overset{\pi[z]}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}$$

Difficult Cases: Contraction

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,{\vee L}
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \wedge p}}\,{\wedge R}
}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \vee p \succ z : p \wedge p}}\,\mathit{Cut}
$$

Difficult Cases: Contraction

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L}{\overset{Ly\frown Fz \quad Ry\frown Fz \quad Ly\frown Sz \quad Ry\frown Sz}{y : p \lor p \succ z : p \land p}}\,{\land}R$$

Difficult Cases: Contraction

$$\dfrac{\dfrac{\overset{x\frown x}{x:p \succ x:p} \qquad \overset{x\frown x}{x:p \succ x:p}}{\overset{Ly\frown x\quad Ry\frown x}{y:p\lor p \succ x:p}}{\lor}L \qquad \dfrac{\overset{x\frown x}{x:p \succ x:p} \qquad \overset{x\frown x}{x:p \succ x:p}}{\overset{x\frown Fz\quad x\frown Sz}{x:p \succ z:p\land p}}{\land}R}{\overset{Ly\frown\bullet\quad Ry\frown\bullet\quad \bullet\frown Fz\quad \bullet\frown Sz}{y:p\lor p \succ z:p\land p}}\mathit{Cut}$$

$$\dfrac{\dfrac{\overset{x\frown x}{x:p \succ x:p} \qquad \overset{x\frown x}{x:p \succ x:p}}{\overset{x\frown Fz\quad x\frown Sz}{x:p \succ z:p\land p}}{\land}R \qquad \dfrac{\overset{x\frown x}{x:p \succ x:p} \qquad \overset{x\frown x}{x:p \succ x:p}}{\overset{x\frown Fz\quad x\frown Sz}{x:p \succ z:p\land p}}{\land}R}{\overset{Ly\frown Fz\quad Ry\frown Fz\quad Ly\frown Sz\quad Ry\frown Sz}{y:p\lor p \succ z:p\land p}}{\lor}L$$

Difficult Cases: Weakening

$$\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma \succ \Delta}}{\overset{\pi'}{\Sigma, y : A \succ \Delta}}}{\overset{\pi\ \ \pi'}{\Sigma \succ \Delta}}\ \mathit{Cut}$$

Difficult Cases: Weakening

$$\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma \succ \Delta}}{\overset{\pi'}{\Sigma, y : A \succ \Delta}}}{\overset{\pi \;\; \pi'}{\Sigma \succ \Delta}}\;Cut$$

$$\dfrac{\overset{\pi}{\Sigma \succ \Delta} \qquad \overset{\pi'}{\Sigma \succ \Delta}}{\overset{\pi \;\; \pi'}{\Sigma \succ \Delta}}\;Mix$$

Back to Sequentialisation

$$x \frown y \quad u \frown v$$

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$$\dfrac{\overset{x \frown y}{x : A \succ y : A} \qquad \overset{u \frown v}{u : B \succ v : B}}{\overset{x \frown y \quad u \frown v}{x : A, u : B \succ y : B, v : A}}\ \mathit{Mix}$$

Sequentialisation: Terms with *No* Switchings

The term contains no Ln, Rn, Cn and • in input position
or Fn, Sn, An and • in output position.

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$\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

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It has a derivation using the linear rules
$\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

$$\mathsf{Fy}\frown\mathsf{Lz} \quad \mathsf{NRz}\frown\mathsf{Lz} \quad \mathsf{Sy}\frown\mathsf{Rz}$$

$$y : p \wedge \neg p \succ z : p \vee \neg p$$

Terms with No Switchings: Example

$$
\begin{prooftree}
\AxiomC{$\begin{array}{c} x \frown x \\ x : p \succ x : p \end{array}$}
\RightLabel{$\lor R$}
\UnaryInfC{$\begin{array}{c} x \frown Lz \\ x : p \succ z : p \lor \neg p \end{array}$}
\RightLabel{$\land L$}
\UnaryInfC{$\begin{array}{c} Fy \frown Lz \\ y : p \land \neg p \succ z : p \lor \neg p \end{array}$}
\AxiomC{$\begin{array}{c} u \frown u \\ u : p \succ u : p \end{array}$}
\RightLabel{$\neg R$}
\UnaryInfC{$\begin{array}{c} Nv \frown u \\ \succ u : p, v : \neg p \end{array}$}
\RightLabel{$\lor R$}
\UnaryInfC{$\begin{array}{c} NRz \frown Lz \\ \succ z : p \lor \neg p \end{array}$}
\AxiomC{$\begin{array}{c} v \frown v \\ v : \neg p \succ v : \neg p \end{array}$}
\RightLabel{$\lor R$}
\UnaryInfC{$\begin{array}{c} v \frown Rz \\ v : \neg p \succ z : p \lor \neg p \end{array}$}
\RightLabel{$\land L$}
\UnaryInfC{$\begin{array}{c} Sy \frown Rz \\ y : p \land \neg p \succ z : p \lor \neg p \end{array}$}
\RightLabel{Mix}
\BinaryInfC{$\begin{array}{c} NRz \frown Lz \quad Sy \frown Rz \\ y : p \land \neg p \succ z : p \lor \neg p \end{array}$}
\RightLabel{Mix}
\BinaryInfC{$\begin{array}{c} Fy \frown Lz \quad NRz \frown Lz \quad Sy \frown Rz \\ y : p \land \neg p \succ z : p \lor \neg p \end{array}$}
\end{prooftree}
$$

By induction on the number of switched pairs.

Take a switched pair at the *adjacent to variables* or *cut points* (peel away unswitched steps if there aren't any).

$$\dfrac{\overset{\pi[x](-)}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi[-](y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az](Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L$$

Back to Eliminating Cuts: Cuts can be Complicated

$$
\dfrac{
\dfrac{
\begin{array}{c}\pi[x,u]\\ \succ x : A \land B, u : A\end{array}
\quad
\begin{array}{c}\pi'[x,v]\\ \succ x : A \land B, v : B\end{array}
}{
\begin{array}{c}\pi[x,Fx] \quad \pi'[x,Sx]\\ \succ x : A \land B\end{array}
}{\land R}
\qquad
\dfrac{
\begin{array}{c}\pi''(y,z,x)\\ y : A, z : B, x : A \land B \succ\end{array}
}{
\begin{array}{c}\pi''(Fx,Sx,x)\\ x : A \land B \succ\end{array}
}{\land L}
}{
\begin{array}{c}\pi[\bullet,F\bullet] \quad \pi'[\bullet,S\bullet] \quad \pi''(F\bullet,S\bullet,\bullet)\\ \Sigma \succ \Delta\end{array}
}{Cut}
$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $\mathfrak{n}\frown\bullet$ and $\bullet\frown\mathfrak{m}$ by $\mathfrak{n}\frown\mathfrak{m}$.

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown * \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x$$

Cut Reductions

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet{}^\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}{}^\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

$$\mathsf{S}z{}^\frown\mathsf{F}\bullet \quad \mathsf{F}z{}^\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet{}^\frown\mathsf{S}x \quad \mathsf{S}\bullet{}^\frown\mathsf{F}x \quad \mathsf{N}y{}^\frown\bullet \quad \bullet{}^\frown\nu$$

$$\mathsf{S}z{}^\frown\star \quad \mathsf{F}z{}^\frown* \quad \star{}^\frown\mathsf{S}x \quad *{}^\frown\mathsf{F}x \quad \mathsf{FN}y{}^\frown\mathsf{S}x \quad \mathsf{SN}y{}^\frown\mathsf{F}x \quad \mathsf{N}y{}^\frown\nu$$

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown* \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x \quad \mathsf{FN}y\frown\mathsf{S}x \quad \mathsf{SN}y\frown\mathsf{F}x \quad \mathsf{N}y\frown\nu \quad \mathsf{S}z\frown\mathsf{F}\nu \quad \mathsf{F}z\frown\mathsf{S}\nu$$

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- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

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- *disjunction*: for each $\mathsf{L}\bullet/\mathsf{R}\bullet$, add new cut points $\star$ and $*$. For any $\bullet^\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}^\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

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- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.
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- *conditional*: for each $\mathsf{A}\bullet/\mathsf{C}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

STRONG NORMALISATION

Any reduction for π terminates in a unique* term π^*

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- There is *some* terminating reduction process.

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- Proof reduction is confluent.
  - If $\pi \rightsquigarrow_\bullet \pi'$ and $\pi \rightsquigarrow_\star \pi''$ then there is a π''' where $\pi' \rightsquigarrow_\star \pi'''$ and $\pi'' \rightsquigarrow_\bullet \pi'''$.

FURTHER WORK

To Do List

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- Apply terms to theories of warrants.
- Extend beyond propositional logic.

THANK YOU!

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