

ISOMORPHISMS IN A CATEGORY OF PROPOSITIONS AND PROOFS

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MELBOURNE LOGIC SEMINAR * 2 MARCH 2018

I aim to show how a category of propositions and *classical* proofs can give rise to finely grained hyperintensional notions of sameness of content. One notion is *very* finely grained (distinguishing p and $p \wedge p$) others are less finely grained. Another notion amounts to equivalence in R. B. Angell's logic of analytic containment [1].

1 THE CATEGORY OF CLASSICAL PROOFS

Four different *derivations*, and two *proofs*.

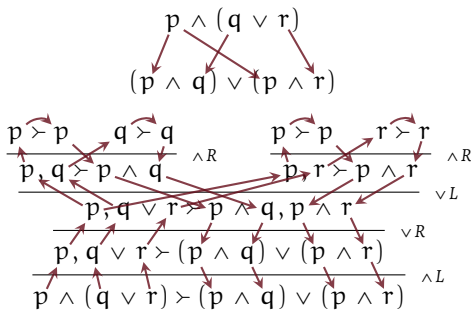
$$\begin{array}{ccc} \frac{p \succ p}{p \wedge q \succ p} \wedge L & \frac{p \wedge q}{p} \wedge R & \frac{p \succ p}{p \succ p \vee q} \vee R \\ \frac{p \wedge q \succ p}{p \wedge q \succ p \vee q} \vee R & \frac{p \wedge q}{p \vee q} \vee L & \frac{p \succ p}{p \wedge q \succ p \vee q} \wedge L \\ \\ \frac{q \succ q}{p \wedge q \succ q} \wedge L & \frac{p \wedge q}{q} \wedge R & \frac{q \succ q}{q \succ p \vee q} \vee R \\ \frac{p \wedge q \succ q}{p \wedge q \succ p \vee q} \vee R & \frac{p \wedge q}{p \vee q} \vee L & \frac{q \succ q}{p \wedge q \succ p \vee q} \wedge L \end{array}$$

MOTIVATING IDEA: *Proof terms* are an invariant for derivations under rule permutation. δ_1 and δ_2 have the same *term* iff some permutation sends δ_1 to δ_2 .

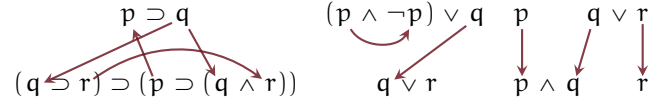
$$\begin{array}{ccc} \frac{x \succ y : p}{x : p \succ y : p} \wedge L & \frac{x \succ x : p}{x : p \succ y : p} \vee R & \frac{x \succ y : p}{x : p \succ y : p \vee q} \wedge L \\ \frac{x \wedge x \succ y : p}{x : p \wedge q \succ y : p} \wedge L & \frac{x \wedge x \succ y : p}{x : p \wedge q \succ y : p \vee q} \wedge L & \frac{x \wedge x \succ y : p}{x : p \wedge q \succ y : p \vee q} \wedge L \\ \frac{x \wedge x \succ y : p}{x : p \wedge q \succ y : p \vee q} \wedge L & \frac{x \wedge x \succ y : p}{x : p \wedge q \succ y : p \vee q} \wedge L & \frac{x \wedge x \succ y : p}{x : p \wedge q \succ y : p \vee q} \wedge L \\ \\ \frac{x \succ y : q}{x : p \wedge q \succ y : q} \wedge L & \frac{x \succ y : q}{x : p \wedge q \succ y : q} \wedge L & \frac{x \succ y : q}{x : p \wedge q \succ y : q} \wedge L \\ \frac{x \wedge x \succ y : q}{x : p \wedge q \succ y : q} \wedge L & \frac{x \wedge x \succ y : q}{x : p \wedge q \succ y : q} \wedge L & \frac{x \wedge x \succ y : q}{x : p \wedge q \succ y : q} \wedge L \\ \frac{x \wedge x \succ y : q}{x : p \wedge q \succ y : p \vee q} \wedge L & \frac{x \wedge x \succ y : q}{x : p \wedge q \succ y : p \vee q} \wedge L & \frac{x \wedge x \succ y : q}{x : p \wedge q \succ y : p \vee q} \wedge L \end{array}$$

A *proof term* for $\Sigma \succ \Delta$ encodes the flow of information in a proof of $\Sigma \succ \Delta$. They can be represented as directed graphs on sequents [2].

$$\frac{\lambda x \wedge \lambda y : y \quad \lambda x \wedge \lambda y : y \quad \lambda x \wedge \lambda y : y \quad \lambda x \wedge \lambda y : y}{x : p \wedge (q \vee r) \succ y : (p \wedge q) \vee (p \wedge r)}$$



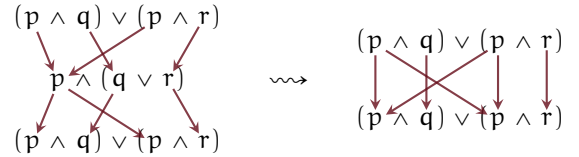
More examples:



Links wholly internal to a *premise* or a *conclusion* are called *cups* (\curvearrowright) and *caps* (\curvearrowleft).

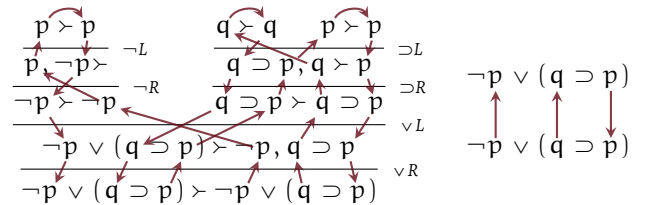
FACTS: Not every directed graph on occurrences of atoms in a sequent is a proof term. \curvearrowright They *typecheck*. [An occurrence of p is linked only with an occurrence of p .] \curvearrowright They *respect polarities*. [Positive occurrences of atoms in premises and negative occurrences of atoms in conclusions are *inputs*. Negative occurrences of atoms in premises and positive occurrences of atoms in conclusions are *outputs*.] \curvearrowright They must satisfy an “enough connections” condition, amounting to a non-emptiness under every *switching*. [e.g. the obvious linking between premise $p \vee q$ and conclusion $p \wedge q$ is not connected enough to be a proof term.]

Cut is chaining of proof terms, composition of graphs.



Cut elimination is *confluent* and *terminating*. [So it can be understood as a kind of *evaluation*.] \curvearrowright Cut elimination for proof terms is *local*. [So it is easily made parallel.]

\mathcal{C} is the *Category of Classical Proofs*. OBJECTS Formulas — A, B , etc. ARROWS Cut-Free Proof Terms — $\pi : A \succ B$. COMPOSITION Composition of derivations with the elimination of *Cut* — If $\pi : A \succ B$ and $\tau : B \succ C$ then $\tau \circ \pi : A \succ C$. IDENTITY Canonical identity proofs — $\text{Id}(A) : A \succ A$.



The category \mathcal{C} is *symmetric monoidal* and *star autonomous*, but not *cartesian*, with structural *monoids* and *comonoids*, and is enriched in *SLat* (the category of semilattices). Being enriched in *SLat* means that proofs terms come ordered by \sqsubseteq , and compose under \cup , and these interact sensibly with composition.

$$\begin{array}{lcl} \pi \sqsubseteq \pi' & \Rightarrow & \pi \circ \tau \sqsubseteq \pi' \circ \tau \\ \tau \sqsubseteq \tau' & \Rightarrow & \pi \circ \tau \sqsubseteq \pi \circ \tau' \\ \pi \circ (\tau \cup \tau') & = & (\pi \circ \tau) \cup (\pi \circ \tau') \\ (\pi \cup \pi') \circ \tau & = & (\pi \circ \tau) \cup (\pi' \circ \tau) \end{array}$$

\mathcal{C} is just *classical* propositional logic, in a categorical setting. (The sequent calculus is playing no essential role here. You can define proof terms on other proof systems, e.g. *natural deduction*, *Hilbert proofs*, *tableaux*, *resolution*.)

*Thanks to Rohan French, Dave Ripley, and Shawn Standefer for helpful conversations on this material. \curvearrowright This research is supported by the Australian Research Council, through Grant DP150103801. \curvearrowright Resources available at <http://consequently.org/presentation/2018/logicmelb-isomorphisms/>.

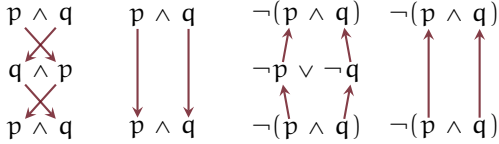
2 ISOMORPHISMS

$f : A \rightarrow B$ is an *isomorphism* in a category iff it has an *inverse* $g : B \rightarrow A$, where $g \circ f = id_A : A \rightarrow A$ and $f \circ g = id_B : B \rightarrow B$. (If g and g' are inverses, $g = id_A \circ g = (g' \circ f) \circ g = g' \circ (f \circ g) = g' \circ id_B = g'$, so any inverse is unique. We can call it f^{-1} .)

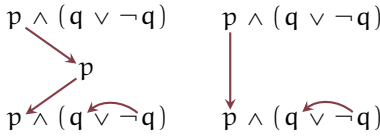
If A and B are isomorphic in a category \mathcal{C} , then what we can do with A (in \mathcal{C}) we can do with B , too.

If A and B are isomorphic in \mathcal{C} , then they agree not only on *provability*, but also, on *proofs*. The distinctions drawn when you analyse how something is *proved* (from premises), are not far from what you want to understand when you ask how something is *made true*.

Isomorphisms in \mathcal{C} : $p \wedge q \cong q \wedge p$; $\neg(p \wedge q) \cong \neg p \vee \neg q$



Non-isomorphisms in \mathcal{C} : $p \wedge (q \vee \neg q) \not\cong p$; $p \wedge p \not\cong p$; $p \wedge (q \vee r) \not\cong (p \wedge q) \vee (p \wedge r)$; $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$



Occurrence Polarity Condition: If A is isomorphic to B in \mathcal{C} then each variable occurs positively [negatively] in A exactly as many times as it occurs positively [negatively] in B . (This condition is *necessary*, not *sufficient*: $p \wedge (p \vee q) \not\cong p \vee (p \wedge q)$.)

A is *isomorphic* to B iff A and B are equivalent in the following calculus:

$$\begin{aligned} A \wedge B &\leftrightarrow B \wedge A, & A \wedge (B \wedge C) &\leftrightarrow (A \wedge B) \wedge C. \\ A \vee B &\leftrightarrow B \vee A, & A \vee (B \vee C) &\leftrightarrow (A \vee B) \vee C. \\ \neg(A \vee B) &\leftrightarrow \neg A \vee \neg B, & \neg(A \wedge B) &\leftrightarrow \neg A \vee \neg B. \\ \neg\neg A &\leftrightarrow A, & A \leftrightarrow B &\Rightarrow C(A) \leftrightarrow C(B). \end{aligned}$$

(This allows for a *negation normal form*, but not DNF or CNF.)

Proof Sketch (Došen and Petrić, 2012 [3]).

If $A \leftrightarrow B$ holds in the calculus, A and B are isomorphic. \P A is isomorphic to B iff there are *diversified* A' and B' where A' and B' are isomorphic, and $A = \sigma A'$ and $B = \sigma B'$ for some substitution σ . \P A is isomorphic to B iff their negation normal forms are isomorphic. (If A is diversified, so is its negation normal form.) \P If A and B are diversified, isomorphic, and in negation normal form, if $l \wedge m$ is a conjunction in A (l and m , literals), a substitution argument (substituting \top and \perp for the *other* atoms) shows that l and m must be conjunctively joined in B , too. The same goes for $l \vee m$. \P Replace $l \wedge m$ by a new atom in both A and B , and repeat. \P This shows how to reconstruct a proof of equivalence for A and B in the syntactic calculus for isomorphic formulas.

Isomorphism is a very tight constraint: If A and B are isomorphic, they can play *essentially* the same role in proof. \P Replacing A by B in a proof (as a premise or conclusion) not only gives you something that is also a proof (mere logical equivalence would do *that*), but it gives you a proof which is *essentially the same*. \P Not even A and $A \wedge A$ are equivalent in *this* sense. \P Yet, A and $A \wedge A$ seem to have identical *subject matter* (insofar as we understand that notion). \P Can we use the fine-grained tools of proof theoretical analysis to generalise this notion of subject matter to arbitrary statements?

3 MORE PROOFS FROM A TO A

$$Id(p \vee (p \wedge \neg p)) \quad \begin{array}{c} p \vee (p \wedge \neg p) \\ \downarrow \quad \downarrow \quad \uparrow \\ p \vee (p \wedge \neg p) \end{array}$$

In $Id(A)$, each occurrence of an atom in the premise is linked with every its corresponding occurrence of in the conclusion. *Different occurrences of atoms in A are treated differently.*

$$Mx(p \vee (p \wedge \neg p)) \quad \begin{array}{c} p \vee (p \wedge \neg p) \\ \downarrow \quad \downarrow \quad \uparrow \\ p \vee (p \wedge \neg p) \end{array}$$

In $Hz(A)$, each positive [negative] occurrence of an atom in the premise is linked with every positive [negative] occurrence in the conclusion. *We treat occurrences of an atom in A —with the same polarity—equally.*

$$Hz(p \vee (p \wedge \neg p)) \quad \begin{array}{c} p \vee (p \wedge \neg p) \\ \downarrow \quad \downarrow \quad \uparrow \\ p \vee (p \wedge \neg p) \end{array}$$

In $Mx(A)$, each syntactically possible linking is included. *We treat all occurrences of an atom in A equally.*

Note: $Hz(A)$ is $Mx(A)$ with the caps and cups removed.

Let's look at relations like isomorphism, but which erase distinctions, up to Hz or Mx .

Let's say that A and B Hz -MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$. We write " \approx_{Hz} " for the Hz -matching relation, and we write " $\pi, \pi' : A \approx_{Hz} B$ " to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a Hz -match between A and B .

Let's say that A and B Mx -MATCH, when there are proofs $\pi : A \succ B$ and $\pi' : B \succ A$ where $\pi' \circ \pi = Mx(A)$ and $\pi \circ \pi' = Mx(B)$. We write " \approx_{Mx} " for the Mx -matching relation, and we write " $\pi, \pi' : A \approx_{Mx} B$ " to say that $\pi : A \succ B$ and $\pi' : B \succ A$ define a Mx -match between A and B .

Isomorphism \subseteq Hz -Matching: If $\pi : A \succ B$ and $\pi^{-1} : B \succ A$, then consider $\pi' = Hz(B) \circ \pi \circ Hz(A)$ and $\tau' = Hz(A) \circ \pi^{-1} \circ Hz(B)$. These satisfy the Hz -matching criteria, $\tau' \circ \pi' = Hz(A)$ and $\pi' \circ \tau' = Hz(B)$.

Hz -Matching \subseteq Mx -Matching: If $\pi, \pi' : A \approx_{Hz} B$, then consider $\tau = Mx(B) \circ \pi \circ Mx(A)$ and $\tau' = Mx(A) \circ \pi' \circ Mx(B)$. These satisfy the Mx -matching criteria, $\tau' \circ \tau = Mx(A)$ and $\tau \circ \tau' = Mx(B)$.

Mx -Matching \subseteq Logical Equivalence: If $A \approx_{Mx} B$ then there are proofs $\pi : A \succ B$ and $\tau : B \succ A$.

Matching Relations are Equivalences: REFLEXIVE $Hz(A), Hz(A) : A \approx_{Hz} A$. $Mx(A), Mx(A) : A \approx_{Mx} A$. \P SYMMETRIC If $\pi, \pi' : A \approx_{Hz} B$, then $\pi', \pi : B \approx_{Hz} A$. If $\pi, \pi' : A \approx_{Mx} B$, then $\pi', \pi : B \approx_{Mx} A$. \P TRANSITIVE If $\pi, \pi' : A \approx_{Hz} B$ and $\tau, \tau' : B \approx_{Hz} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Hz} C$. If $\pi, \pi' : A \approx_{Mx} B$ and $\tau, \tau' : B \approx_{Mx} C$, then $(\tau \circ \pi), (\pi' \circ \tau') : A \approx_{Mx} C$.

Matchings: $p \vee p \approx_{Hz} p \approx_{Hz} p \wedge p$; $p \wedge (q \vee r) \approx_{Hz} (p \wedge q) \vee (p \wedge r)$.

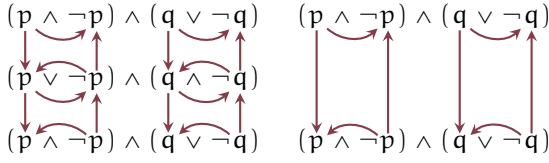
Mx -Matching \subset Logical Equivalence: If an atom p occurs positively [negatively] in A but not in B , then A and B do not Mx -match.

PROOF: $Mx(A) : A \succ A$ contains the link from [to] that occurrence of p in the premise A to [from] its corresponding occurrence in the conclusion A . \P No proof from A to B contains a link from [to] that occurrence to anything in B (since there is no positive [negative] occurrence in B at all). \P So, in the composition proof

from A to A, there is no link from [to] the premise occurrence to [from] the conclusion occurrence. No proof from A to B and back can recreate $Mx(A)$.

COROLLARY: $p \not\approx_{Mx} p \wedge (q \vee \neg q)$; $p \wedge \neg p \not\approx_{Mx} q \wedge \neg q$.

$Hz\text{-matching} \subset Mx\text{-matching}$: $(p \wedge \neg p) \wedge (q \vee \neg q) \approx_{Mx} (p \vee \neg p) \wedge (q \wedge \neg q)$.



However, $(p \wedge \neg p) \wedge (q \vee \neg q) \not\approx_{Hz} (p \vee \neg p) \wedge (q \wedge \neg q)$. So:

$Isomorphism \subset Hz\text{-Matching} \subset Mx\text{-Matching} \subset Logical\ Equivalence$

4 MATCHING & LOGICS OF ANALYTIC CONTAINMENT

Angell's Logic of Analytic Containment: [AC1] $A \leftrightarrow \neg\neg A$ [AC2] $A \leftrightarrow (A \wedge A)$ [AC3] $(A \wedge B) \leftrightarrow (B \wedge A)$ [AC4] $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$ [AC5] $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$ [RI] $A \leftrightarrow B, C(A) \Rightarrow C(B)$

Here, $A \vee B$ is shorthand for $\neg(\neg A \wedge \neg B)$. You can define $A \rightarrow B$ as $A \leftrightarrow (A \wedge B)$.

The first degree fragment of *Parry's Logic of Analytic Containment* is found by adding $(A \vee (B \wedge \neg B)) \rightarrow A$ to Angell's Logic. (Parry's logic still satisfies this relevance constraint: $A \rightarrow B$ is provable only when the atoms in B are present in A.)

First Degree Entailment (FDE) is found by adding $A \rightarrow (A \vee B)$ to Angell's Logic. \mathfrak{J} FDE allows for a disjunctive (or conjunctive) normal form. The only difference from classical logic is that $p \vee \neg p$, and $q \wedge \neg q$ are both non-trivial, and ineliminable. \mathfrak{J} A simple translation encodes FDE inside classical logic. Choose, for each atom p , a fresh atom p' , its *shadow*. For each FDE formula A , its translation is the formula A' found by replacing the negative occurrences of atoms p in A by their shadows. An argument is FDE valid iff its translation is classically valid.

DEFINITION: $Mx(A, B)$ is the set of all possible linkings which could occur in any proof from A to B. \mathfrak{J} That is, it contains a link from {positive atoms in A, negative atoms in B} to matching {positive atoms in B, negative atoms in A}.

FACT: $Mx(A, B)$ is a proof iff there is some proof from A to B. (And if so, it is the maximal such proof.)

$Mx(p \vee \neg p, p \wedge \neg q)$ is not a proof:



LEMMA: If $A \approx_{Mx} B$, then $Mx(A, B)$ and $Mx(B, A)$ are proofs, and $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

PROOF: If $\pi, \pi' : A \approx_{Mx} B$, then $\pi \subseteq Mx(A, B)$ and $\pi' \subseteq Mx(B, A)$, so $Mx(A, B)$ and $Mx(B, A)$ are both proofs. \mathfrak{J} Since $\pi' \circ \pi = Mx(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Mx(B, A) \circ Mx(A, B) \subseteq Mx(A)$, and similarly, $Mx(B) = Mx(A, B) \circ Mx(B, A)$, so $Mx(A, B), Mx(B, A) : A \approx_{Mx} B$.

FACT: If A is classically equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Mx-match*—and conversely.

PROOF: If A is logically equivalent to B, then $Mx(A, B)$ and $Mx(B, A)$ are both proofs. \mathfrak{J} It suffices to show that $Mx(B, A) \circ Mx(A, B) = Mx(A)$ (and similarly for B). To show this, we need

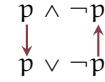
to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Mx(A, B)$ composed with a link in $Mx(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Mx(A, B)$ and $Mx(B, A)$. \mathfrak{J} Conversely, if $A \approx_{Mx} B$, we have already seen that A and B must be equivalent, and no atom occurs positively [negatively] in A but not B.

This is *not* Equivalence in Parry's Logic. A is equivalent to B in Parry's logic of analytic containment iff A is classically equivalent to B and the atoms present in A are the atoms present in B. \mathfrak{J} $(p \wedge \neg p) \wedge q \not\approx_{Mx} (p \wedge \neg p) \wedge \neg q$, but this pair satisfies Parry's variable sharing criteria.

QUESTION: Does the equivalence relation of *Mx-matching* occur elsewhere in the literature?

DEFINITION: $Hz(A, B)$ is the set of all possible linkings which could occur in any proof from A to B, excluding caps and cups. \mathfrak{J} That is, it contains a link from positive atoms in A to corresponding positive atoms in B and negative atoms in A to corresponding negative atoms in B.

$Hz(p \wedge \neg p, q \vee \neg q)$ contains no links. $Hz(p \wedge \neg p, p \vee \neg p)$ is a proof, but not the maximal one:



FACT: $Hz(A, B)$ is a proof iff A entails B in FDE.

PROOF: From FDE-validity to *Hz*-proof: straightforward induction on an FDE-axiomatisation. \mathfrak{J} From the *Hz*-proof $Hz(A, B)$ to FDE-validity: Notice that no negative occurrences of atoms in A or B are linked to any positive occurrences of atoms in A or B. So, there is another *Hz*-proof $Hz(A', B')$ for the FDE translations for A and B.

LEMMA: If $A \approx_{Hz} B$, then $Hz(A, B)$ and $Hz(B, A)$ are proofs, and $Hz(A, B), Hz(B, A) : A \approx_{Hz} B$.

PROOF: If $\pi, \pi' : A \approx_{Hz} B$, then then since $\pi' \circ \pi = Hz(A)$ and $\pi \circ \pi' = Hz(B)$, π and π' are cap- and cup-free, so $\pi \subseteq Hz(A, B)$ and $\pi' \subseteq Hz(B, A)$, so $Hz(A, B)$ and $Hz(B, A)$ are both proofs. \mathfrak{J} Since $\pi' \circ \pi = Hz(A)$, we have $Mx(A) = \pi' \circ \pi \subseteq Hz(B, A) \circ Hz(A, B) \subseteq Hz(A)$, and similarly, $Hz(B) = Hz(A, B) \circ Hz(B, A)$, so $Hz(A, B), Hz(B, A) : A \approx_{Mx} B$.

FACT: If A is FDE-equivalent to B, and all atoms occurring positively [negatively] in A also occur positively [negatively] in B, and *vice versa*, then A and B *Hz-match*—and conversely.

PROOF: If A is FDE-equivalent to B, then $Hz(A, B)$ and $Hz(B, A)$ are both proofs. \mathfrak{J} It suffices to show that $Hz(B, A) \circ Hz(A, B) = Hz(A)$ (and similarly for B). To show this, we need to show that each positive [negative] occurrence of an atom in A is linked to any positive [negative] occurrence of that atom in A by way of some link in $Hz(A, B)$ composed with a link in $Hz(B, A)$. But since that atom occurs positively [negatively] also in B at least once, the links to accomplish this occur in $Hz(A, B)$ and $Hz(B, A)$. \mathfrak{J} Conversely, if $A \approx_{Hz} B$, we have already seen that A and B must be FDE-equivalent, and no atom occurs positively [negatively] in A but not B.

FACT: (Ferguson 2016 [4]; Fine [5]) A is equivalent to B in Angell's logic of analytic containment iff A is FDE equivalent to B, and any atom occurs positively [negatively] in A iff it occurs positively [negatively] in B.

So, $Hz\text{-matching} \equiv Angellic\ Equivalence$.

5 MATCHING AS ISOMORPHISM

$\text{Hz}(A)$ and $\text{Mx}(A)$ are Idempotents: $\text{Hz}(A) \circ \text{Hz}(A) = \text{Hz}(A)$,
 $\text{Mx}(A) \circ \text{Mx}(A) = \text{Mx}(A)$.

For any category \mathcal{C} , if i_A is an idempotent for each object A , we can form a new category \mathcal{C}_i with the same objects as \mathcal{C} , and with arrows $i_B \circ f \circ i_A : A \rightarrow B$. ¶ In this new category, the idempotents i_A are the new identity arrows. ¶ So, \mathcal{C}_{Hz} and \mathcal{C}_{Mx} are both categories — like \mathcal{C} , but less discriminating, with fewer arrows.

Hz -matching is isomorphism in \mathcal{C}_{Hz} .

Mx -matching is isomorphism in \mathcal{C}_{Mx} .

\mathcal{C}_{Mx} and \mathcal{C}_{Hz} are nontrivial, nonetheless.

$$\begin{array}{ccc} p \wedge q & p \wedge q & p \wedge q \\ \downarrow & \downarrow \downarrow & \downarrow \\ p \vee q & p \vee q & p \vee q \end{array}$$

These are each different proofs in \mathcal{C}_{Mx} and \mathcal{C}_{Hz} .

6 CONCLUSION

- Extending these results to include the units \top and \perp are not difficult. (They were left out only to ease the presentation).
- Proof theoretical resources for *classical logic* provide tools for fine-grained hyperintensional distinctions.
- One question is how to relate these results to *models* of logics of content.
- Another next step is these results to first order logic is an open question.

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