1.13

First show that $Fib(n) = \dfrac{\phi^n - \psi^n}{\sqrt{5}}$, where $\phi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$

For $n = 0$:

$$Fib(0) \stackrel{?}{=} \frac{(\frac{1+\sqrt{5}}{2})^0 - (\frac{1-\sqrt{5}}{2})^0}{\sqrt{5}} \tag{1}$$

$$0 \stackrel{?}{=} \frac{1-1}{\sqrt{5}} \tag{2}$$

$$0 = 0 \quad \checkmark \tag{3}$$

For $n = 1$:

$$Fib(1) \stackrel{?}{=} \frac{(\frac{1+\sqrt{5}}{2})^1 - (\frac{1-\sqrt{5}}{2})^1}{\sqrt{5}} \tag{4}$$

$$1 \stackrel{?}{=} \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \tag{5}$$

$$\sqrt{5} \stackrel{?}{=} \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \tag{6}$$

$$2\sqrt{5} \stackrel{?}{=} 1 + \sqrt{5} - (1 - \sqrt{5}) \tag{7}$$

$$2\sqrt{5} = 2\sqrt{5} \quad \checkmark \tag{8}$$

Assume it holds for $n = k$:

$$Fib(k) = \frac{\phi^k - \psi^k}{\sqrt{5}} \tag{9}$$

Assume it also holds for $n = k + 1$:

$$Fib(k+1) = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} \tag{10}$$

Then for $n = k+2$:

$$Fib(k+2) \stackrel{?}{=} \frac{\phi^{k+2} - \psi^{k+2}}{\sqrt{5}} \tag{11}$$

$$Fib(k+1) + Fib(k) \stackrel{?}{=} \frac{\phi^{k+2} - \psi^{k+2}}{\sqrt{5}} \tag{12}$$

$$\frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} + \frac{\phi^{k} - \psi^{k}}{\sqrt{5}} \stackrel{?}{=} \frac{\phi^{k+2} - \psi^{k+2}}{\sqrt{5}} \tag{13}$$

$$\phi^{k+1} - \psi^{k+1} + \phi^{k} - \psi^{k} \stackrel{?}{=} \phi^{k+2} - \psi^{k+2} \tag{14}$$

$$\phi\phi^{k} - \psi\psi^{k} + \phi^{k} - \psi^{k} \stackrel{?}{=} \phi^2\phi^{k} - \psi^2\psi^{k} \tag{15}$$

$$\psi^2\psi^{k} - \psi\psi^{k} - \psi^{k} \stackrel{?}{=} \phi^2\phi^{k} - \phi\phi^{k} - \phi^{k} \tag{16}$$

$$(\psi^2 - \psi - 1)\psi^{k} \stackrel{?}{=} (\phi^2 - \phi - 1)\phi^{k} \tag{17}$$

$$\left[\left(\frac{1-\sqrt{5}}{2}\right)^2 - \frac{1-\sqrt{5}}{2} - 1\right]\psi^{k} \stackrel{?}{=} \left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1+\sqrt{5}}{2} - 1\right]\phi^{k} \tag{18}$$

$$\left(\frac{1-2\sqrt{5}+5}{4} - \frac{1-\sqrt{5}}{2} - 1\right)\psi^{k} \stackrel{?}{=} \left(\frac{1+2\sqrt{5}+5}{4} - \frac{1+\sqrt{5}}{2} - 1\right)\phi^{k} \tag{19}$$

$$\left(\frac{1-2\sqrt{5}+5-(2-2\sqrt{5})-4}{4}\right)\psi^{k} \stackrel{?}{=} \left(\frac{1+2\sqrt{5}+5-(2+2\sqrt{5})-4}{4}\right)\phi^{k} \tag{20}$$

$$\left(\frac{0}{4}\right)\psi^{k} \stackrel{?}{=} \left(\frac{0}{4}\right)\phi^{k} \tag{21}$$

$$0 = 0 \quad \checkmark \tag{22}$$

$\therefore$ by induction it holds $\forall\ n$.

Next, show that $\forall\ n$, $\left|\frac{\phi^n - \psi^n}{\sqrt{5}} - \frac{\phi^n}{\sqrt{5}}\right| < \frac{1}{2}$

$$\left|\frac{\phi^n - \psi^n}{\sqrt{5}} - \frac{\phi^n}{\sqrt{5}}\right| \stackrel{?}{<} \frac{1}{2} \tag{23}$$

$$\left|\frac{-\psi^n}{\sqrt{5}}\right| \stackrel{?}{<} \frac{1}{2} \tag{24}$$

$$|-\psi|^n \stackrel{?}{<} \frac{\sqrt{5}}{2} \tag{25}$$

$$\left|\frac{\sqrt{5}-1}{2}\right|^n \stackrel{?}{<} \frac{\sqrt{5}}{2} \tag{26}$$

$$\tag{27}$$

Since $\frac{\sqrt{5}-1}{2} < 1 < \frac{\sqrt{5}}{2}$, $\forall\ n \in \mathbb{Z}_{\geq 0}$:

$$\left(\frac{\sqrt{5}-1}{2}\right)^n < \frac{\sqrt{5}}{2} \quad \checkmark \tag{28}$$

$\therefore Fib(n)$ is the closest integer to $\frac{\psi^n}{\sqrt{5}}$, $\forall\ n$ in the domain of $Fib(n)$