Assignment 7

1) $U(x) = \log(x)$

infinitesimal change in wealth:

$$dW_t = ((r + \pi_t(\mu - r))W_t - c_t)\,dt + \pi_t \sigma W_t\, dz_t$$

Want to maximize:

$$\mathbb{E}\left[\int_t^T e^{-\rho(s-t)} \log(c_s)\,ds + e^{-\rho(T-t)} \underset{\overset{\uparrow}{\varepsilon^\gamma}}{B(T)} \log(W_T)\right]$$

HJB:

$$\hookrightarrow \max_{\pi_t, c_t} \{ \mathbb{E}_t [dV^*(t, W_t) + \log(c_t)\,dt\} = \rho V^*(t, W_t)\,dt$$

After following steps like in book, we get:

$$\pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t}(\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \cdot \sigma^2 W_t} \quad \text{and} \quad c_t^* = \left(\frac{\partial V^*}{\partial W_t}\right)^{-1}$$

$$-1 + \log\left(\frac{\partial V^*}{\partial W_t}\right)^{-1}$$

$$= -1 - \log(\partial$$

$\hookrightarrow$ solve:

$$\frac{\partial V^*}{\partial t} + \cancel{\partial V^*} \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\left(\frac{\partial V^*}{\partial W_t}\right)^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} r W_t - 1 - \log\left(\frac{\partial V^*}{\partial W_t}\right)$$

$$= \rho V^*(t, W_t)$$

boundary condition: $V^*_{(T)} = \varepsilon^\gamma \log(W_t)$

guess solution: $V^*(t, W_t) = f(t)$ ~~[illegible]~~ e^{W_t}

$\hookrightarrow$ $\log(W_t)$?

$$\frac{\partial V^*}{\partial t} = f'(t)\, e^{W_t}$$

$$\frac{\partial V^*}{\partial W_t} = f(t)\, e^{W_t}$$

$$\frac{\partial^2 V^*}{\partial W_t^2} = f(t)\, e^{W_t}$$

$$\hookrightarrow f'(t) e^{W_t} - \frac{(\mu-r)^2}{2\sigma^2} \frac{f(t)^2 e^{2W_t}}{f(t) e^{W_t}} \cdot \cancel{f(t)e} + f(t) e^{W_t} r W_t - 1 - W_t = \rho f(t) e^{W_t}$$

$$f' = \left(\frac{(\mu-r)^2}{2\sigma^2} - rW_t + \rho\right) f(t) + 1 + W_t$$

$$f'(t) = v f(t) + 1 + W_t \quad \text{, where } v = \frac{(\mu-r)^2}{2\sigma^2} - rW_t + \rho$$

boundary condition: $V^*(T) = \varepsilon^\gamma \log(W_t)$

$$\hookrightarrow f(T) = \varepsilon^\gamma \log(W_t)$$

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for ~~$\log(W_t)$~~ $V^*(t, W_t) = f(t) \log(W_t)$

$$\frac{\partial V^*}{\partial t} = f'(t) \log(W_t)$$

$$\frac{\partial V^*}{\partial W_t} = f(t) \cdot \frac{1}{W_t} \qquad \frac{\partial^2 V^*}{\partial W_T^2} = -f(t) \cdot \frac{1}{W_t^2}$$

$$\frac{\partial V^*}{} + \frac{(\mu-r)^2}{} \cdot \left(\frac{\partial V^*}{\partial W_t}\right)^2 + \frac{\partial V^*}{\partial W_t} r W_t - 1 - \log\left(\frac{\partial V^*}{\partial W_t}\right)$$

$$\partial t \qquad \sigma^2 \qquad \frac{\qquad}{\frac{\partial^2 V^*}{\partial W_t^2}} \qquad \partial W_t \qquad = \rho V^*(t, W_t)$$

$$\hookrightarrow \quad f'(t)\log W_t + \frac{(\mu-r)^2}{2\sigma^2} \cancel{t} + f(t) r - 1$$

$$V^*(t, W_t) = \log\left(f(t)^c W_t\right)$$

$$\frac{\partial V^*}{\partial t} = \frac{1}{f(t)^c} c \cdot f(t)^{c-1} \cdot f'(t) = \frac{c}{f(t)} \cdot f'(t)$$

$$\frac{\partial V^*}{\partial W_t} = \frac{1}{W_t}$$

$$\frac{\partial^2 V^*}{\partial W_t^2} = -\frac{1}{W_t^2}$$

$$\hookrightarrow \quad \frac{\partial V^*}{\partial t} + \frac{\frac{(\mu-r)^2}{2\sigma^2}\left(\frac{\partial V^*}{\partial W_t}\right)^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} r W_t - 1 - \log\left(\frac{\partial V^*}{\partial W_t}\right) = \rho V^*(t, W_t)$$

$$\frac{c\, f'(t)}{f(t)} \cancel{t} - \frac{(\mu-r)^2}{2\sigma^2} + r - 1 - \log\left(\frac{1}{W_t}\right) = \rho \log\left(f(t)^c W_t\right)$$

$$c\, f'(t) \quad (\mu-r)^2 \quad + r - 1 + \log(W_t) \quad \rho\left(c \log(f(t)) + \log(W_t)\right)$$

$$\frac{\;\;}{f(t)} - \frac{\;\;}{2\sigma^2} + \; \; = \rho\,(\log(f(t)) + \log W_t)$$

$$V^*(t, W_t) = f(t) + \log W_t$$

$\setminus W \&$ boundary $\quad f(T) = \underset{\approx 1}{B(T)} \log(W_t)$

$$= \log(W_T)$$

$$\hookrightarrow \frac{dV^*}{dt} = f'(t)$$

$$\frac{\partial V^*}{\partial W} = \frac{1}{W_t} \qquad \frac{\partial^2 V^*}{\partial W^2} = -\frac{1}{W_t^2}$$

$$\hookrightarrow f'(t) - \frac{(\mu - r)^2}{2\sigma^2} + r - 1 + \log W_t = \rho f(t) + \rho \log W_t$$

$$f'(t) = \rho f(t) + c \; ; \quad c = (\rho - 1)\log W_t + 1 - r + \frac{(\mu-r)^2}{2\sigma^2}$$

$$\frac{1}{\rho} f'(t) = f(t) + \frac{c}{\rho}$$

$$\hookrightarrow f(t) = c_1 \cdot e^{\rho t} - \frac{c}{\rho}$$

$$f(T) = \log W_T$$

$$\hookrightarrow c_1 = \log W_T + \frac{c}{\rho e^{\rho T}}$$

$\hookrightarrow V^*(t, W_t) = \left(\log W_T + \frac{c}{\rho e^{\rho T}}\right) e^{\rho t} + c$, where $c = (\rho - 1)\log W_t + 1$

$$-\tilde{r} + \frac{(\mu-r)^2}{2\sigma^2}$$

3) exponential decay : $S_0 \cdot e^{-kt}$

$$\frac{S}{2} = S e^{-k\lambda}$$
$$\hookrightarrow e^{k\lambda} = 2$$
$$k = \log 2 / \lambda$$

state: skill, job (y/n), t

- We will want to first learn & then work each day since this will increase our income.

- Our state is always the beginning of each day

- Assume at time t_0, we have skill level s and are employed:

Then with probability p:

~~for each t we want to maximize:~~

We first learn which increase our skill level by:

$$s \to s + \int_0^{(1-\alpha)\cdot 24\cdot 60} g(s)\, dt_{min} \hat{=} s'$$

Then we work which generates the following income:

$$\sum \gamma^c \alpha \cdot 24 \cdot 60 \cdot f(s') \qquad c \text{ is day}$$

$\hookrightarrow$ next state : $(s', t_1, \text{employed})$

BUT with prob $(1-p)$ loose job :

$\longrightarrow$ next state : $(s_{t_0}\, e^{-\log 2/\lambda}, t_1, \text{u.e.})$

If not employed :

with prob $h(s)$ $\rightarrow$ job back and we do above

with prob $1-h(s)$ $\rightarrow$

↳ next state $(s\, e^{-\log 2/\lambda}, t_1, \text{n.e.})$

Let's assume another 1000 days to live.
And initial skill level $S_0 = 1000$ and $\gamma = 0{,}999$

$$W_{t+1} = \cancel{\text{dot}}\left(W_t - x_t\right)(1+r) + x_t \cdot (1+Y_t)$$

$$Y_t \sim N(\mu,\sigma^2)$$

$$V^*(t, W_t) = \max_{x_t}\left\{\mathbb{E}_{Y \sim N(\mu,\sigma^2)} W_T\right\}$$

$$\cancel{V^*_{T-1}}$$

$$V^*$$