MOPSI
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Optimal Trades in Illiquid Markets

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Introduction

Illiquid markets are crucial in trading: there is not always a counterparty to trade with, hence no market-maker. These situations can lead to significant losses. Dark Pools are an example: private trading in market securities. Thus, it is necessary to be prepared for any scenario.

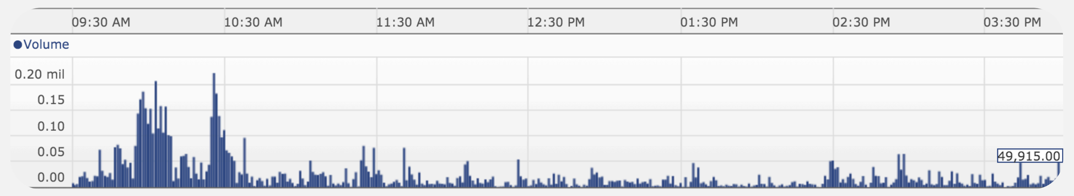
- Suppose we have k shares to sell before a terminal date T.
- Naïve method: issue all shares at the beginning and waiting for counterparties to buy our shares. This strategy is not optimal as leaving your shares on the market has a cost.
- Splitting orders into small batches over the period seems more optimal.

This theory is based on a 2009 article OPTIMAL TRADE EXECUTION IN ILLIQUID MARKETS by Bayraktar and Ludkovski.

Liquidity Model

We use different approaches for modelling the market liquidity:

- Use a Poisson process to model the order book. In this scenario, when someone comes to buy shares, he can buy an unlimited amount.
- Use a Compound Poisson process. By doing so, the order size is constrained by a distribution, here an off-centered Poisson distribution.
- Use Poisson process with different trade regimes. Often, opening and closing hours are denser then mid-day trading. First, we suppose we know the regime. Second, we suppose the regimes are unknown and are deduce using a Bayesian method.
- Use Poisson Process but with continuous sales amounts.



ORPN Volume

Poisson Process

- Arrival times σ_i for the i-th order timestamp.
- The agent has always the possibility of selling everything at terminal date.
- The price impact is modeled through a cost function strictly increasing and strictly convex F. Placing a shares costs $F(a)$.
- There are no constraint on the order size.

The value function follows the below equation:

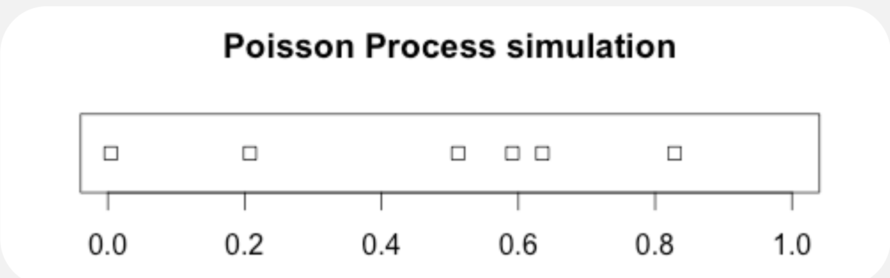
$$v(k,T) = \mathbb{E}[min_{a\epsilon\{1,\dots,k\}}\{v(k-a,T-\sigma)+F(a)\}\cdot 1_{\{\sigma<T\}} + F(k)\cdot 1_{\{\sigma\geq T\}}]$$

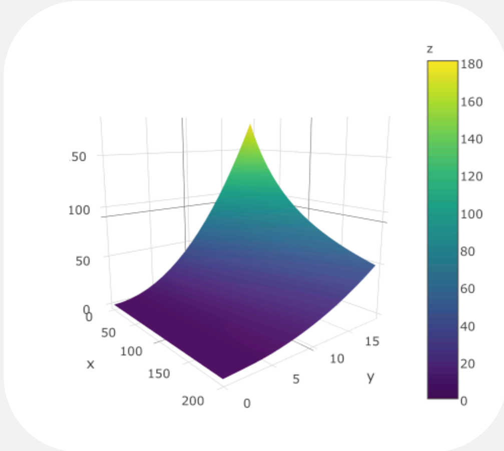
As a first approach, the use of Monte Carlo method was interesting to observe the function's behavior.
Deterministic equation:

$$\partial_T v(k,T) = -\lambda(v(k,T) - min_{a\epsilon\{0,1,\dots,k\}}[v(k-a,T)+F(a)])$$

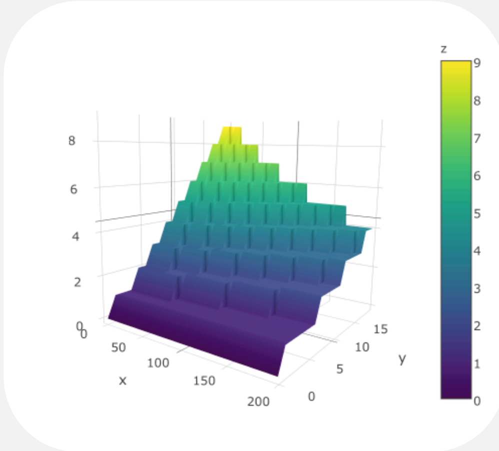
$$a(k,T) = argmin_{a\epsilon\{1,\dots,k\}}\{v(k-a,T)+F(a)\}$$

Using finite difference method and dynamic programming, we compute $v(k,T)$ with Euler explicit method.



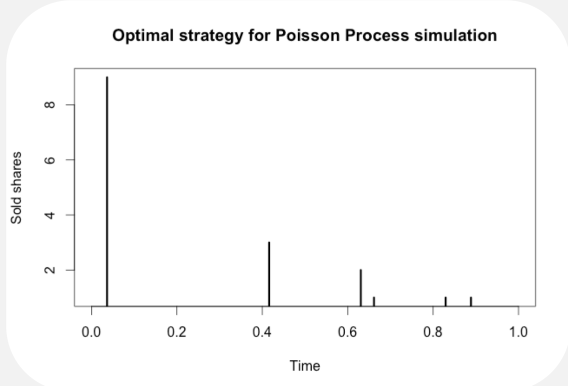


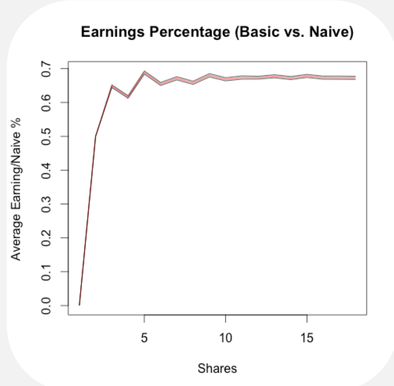
Value function



Optimal Trading Strategy

Now, let's simulate the trajectories. Using R, we simulate the Poisson distribution. The optimal strategy is represented below:





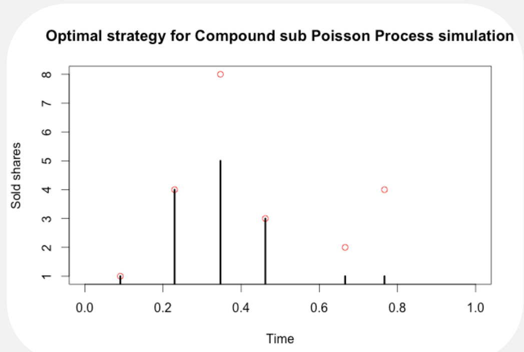
Compound Poisson Process

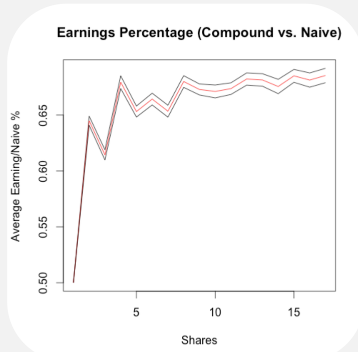
We now constrain order sizes: $0 \leq \xi_{\sigma-} - \xi_\sigma \leq Y_i$ with ξ_σ the number of remaining shares and Y_i the order size random variable. Here Y_i follow a decentered Poisson distribution.
The two following equations are used in parallel to compute the optimal order size amount to place.

$$a(k,T) = argmin_{a\epsilon\{1,\dots,k\}}\{v(k-a,T)+F(a)\}$$

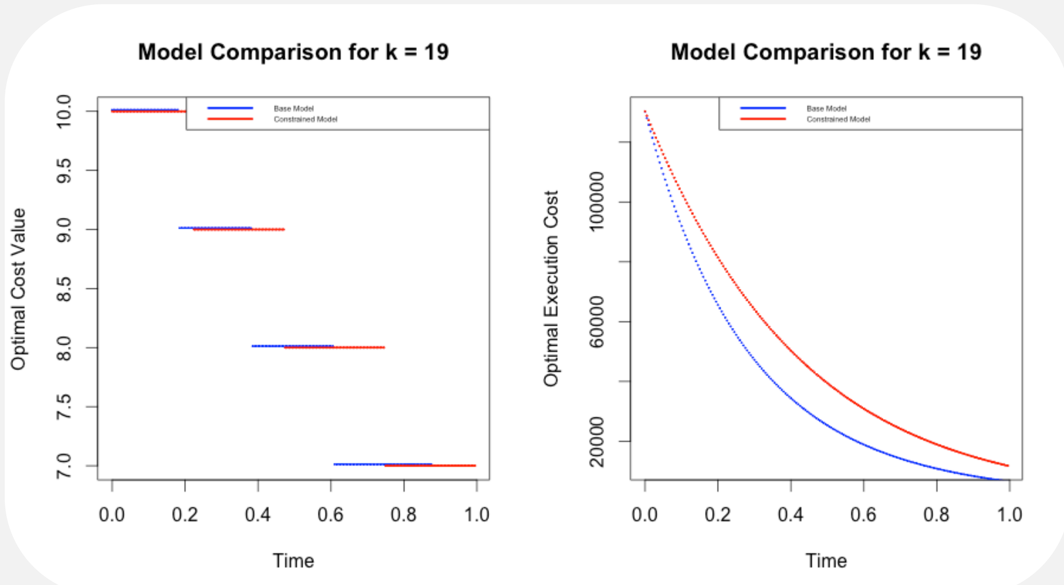
$$\partial_T v(k,T) = -\lambda(v(k,T) - [v(k-a(k,T),T) + F(a(k,T)]\nu[a(k,T),\infty) - \sum_{y=1}^{a(k,T)} \nu(y)[v(k-y,T)+F(y)]])$$

Note that because the model is the constrained model of the simple Poisson process one, $v_{compound} \geq v_{basic}$.





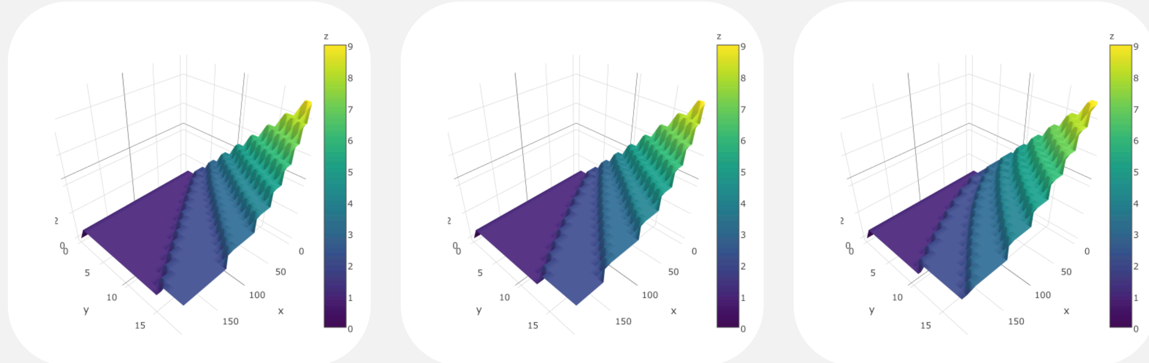
Comparing both models



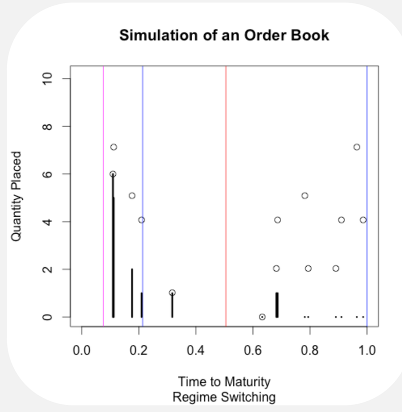
Regime Switching Setting

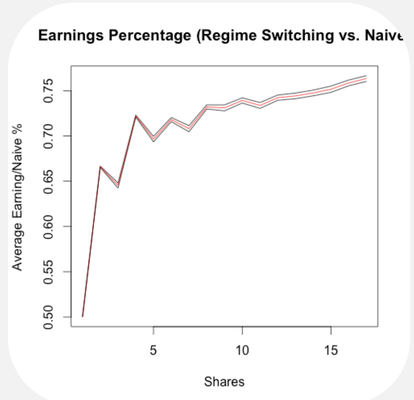
Then, as one knows, intraday trading has different regimes. Often, opening and closing hours are denser then mid-day trading. Let M be a Markov chain with an infinitesimal generator $Q = (q_{ij})$. The 3 regimes {High, Med, Low}, have different intensities (λ_i, ν_i). Thus, the order flow (arrival times, order sizes) is given by M.

$$\partial_T v(k,T;i) = -\lambda_i(v(k,T;i) - min_a[v(k-a,T;i)+F(a)]) + \sum_{j\epsilon E\{i\}} q_{ij}\,(v(k,T;i) - v(k,T;i))$$



Different regimes

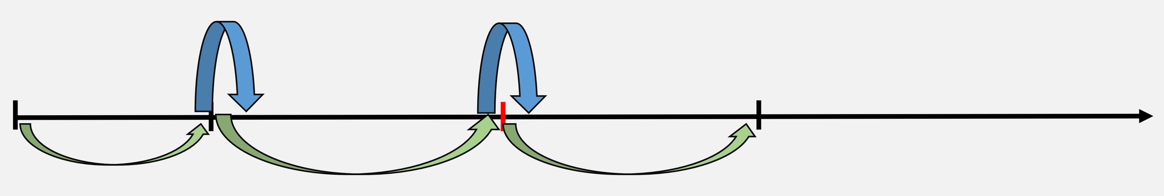


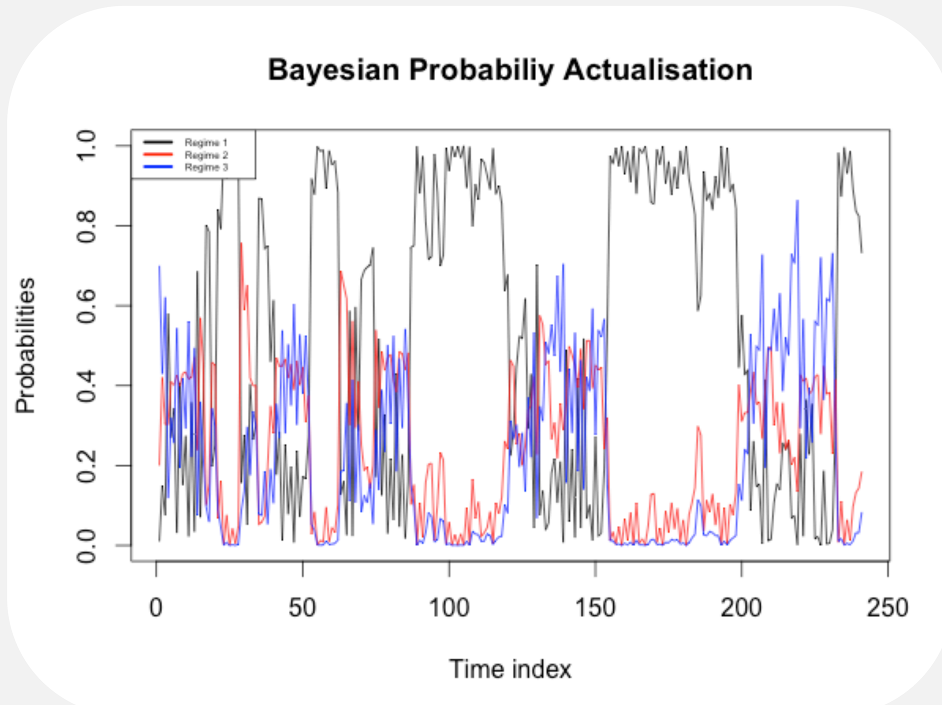


Partially Observed Setting

For a more realistic solution, let M be a hidden Markov chain. Therefore, decreasing frequency of trades may point to an impending liquidity crisis and thus force agents to place larger trades to avoid being stuck with an unfortunate position. Let $\vec{\pi}$ be the beliefs about the market liquidity variable M. Using a Bayesian method, the beliefs are updated using the two formulas below.

$$\Pi_i(t+u) = \left.\frac{\mathbb{P}^{\vec{\pi}}(\sigma_1>u, M_u=i)}{\mathbb{P}^{\vec{\pi}}(\sigma_1>u)}\right|_{\vec{\pi}=\overrightarrow{\Pi(t)}} ; \Pi_i(\sigma_{l+}) = \frac{\lambda_i\nu_i(Y_l)\Pi_i(\sigma_{l-})}{\sum_{j\in E}\lambda_j\nu_j(Y_l)\Pi_j(\sigma_{l-})}$$



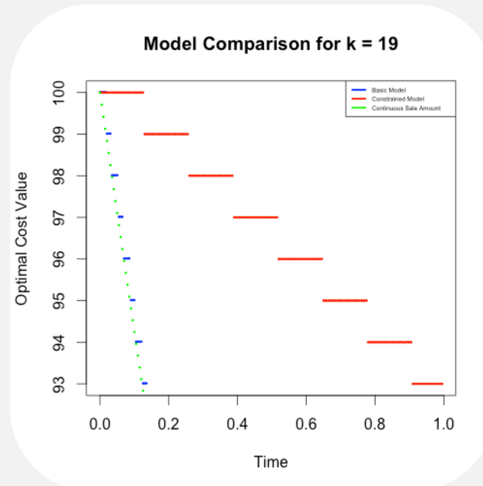


Continuous Sale Amounts

The amount of shares being sold can now be a real number. For a continuous sale amount, we consider the cost function as : $F(x) = x^\gamma$. The optimal order size is given by:

$$\partial_T a(T) = \frac{\lambda}{\gamma-1}\,a(T)(1-a(T))((1-a(T))^{\gamma-1}-1) \text{ and } a(0) = 1/2$$

$$a(k,T) \approx k\cdot a(T)$$



Conclusion

- Employed methods : Probabilistic approach (Monte Carlo) and determinist method.
- The most naïve method is clearly not optimal.
- Another naïve method is to split the number of shares by the average number of agents $\lambda\cdot T$ arriving during T epochs. This method proved to be as efficient as the one developed in this poster. Thus, these computation may not be necessary.
- Issues encountered:
  - In C++, there were some issues of number types. Also, computing a factorial proved to be quite delicate as division was not very accurate.
  - In R, simulating the random variable and C++ conversion.
- Prospects:
  - Determine the value function in Regime Switching setting with hidden Markov chain.
  - Time actualization.
  - Modulate $\lambda(t)$ within a day.
- Acknowledgement

We would like to thank Prof. Aurélien Alfonsi for his useful help in understanding the paper and for his precious guidance

- Bibliography:

- Optimal Trade Execution in Illiquid Markets, Erhan Bayraktar & Mike Ludkovski, 2009
- Introduction au calcul stochastique appliqué à la finance, Damien Lamberton & Bernard Lapeyre
- Outils stochastiques des marchés financiers, El Karoui & Gobet.

Additional information

Please visit: www.mopsi.ml

Link towards a website dedicated to the project.