

Chapitre 1

Dynamic systems and state models

In this first chapter we give first of all the definition of the dynamic systems group which is studied in the book, as well as the terminology and the used notations, and we illustrate it with some examples from engineering sciences. We explain then what recover the notions of modelling and analysis of dynamic systems. The chapter ends by a brief description of the nine other chapters content.

1.1. Definitions and examples

In this book, we will study dynamic systems described by sets of first-order differential equations of the shape

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m), \\ &\vdots \quad \vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_m),\end{aligned}\tag{1.1}$$

where f_i are applications from \mathbb{R}^{n+m} to \mathbb{R} , while the x_i and u_i are scalar functions of time t , which is an independant variable. The quantity \dot{x}_i represents the derivative of the variable x_i relative to time t . The variables x_1, x_2, \dots, x_n are called *state variables* and contain all the necessary information on the *state* of the system in the present to be able to measure its evolution in the future, by means of equations (1.1), knowing the future values of the variables u_1, u_2, \dots, u_m . These are called *entries* of the system, and represent the outside's influence on the studied

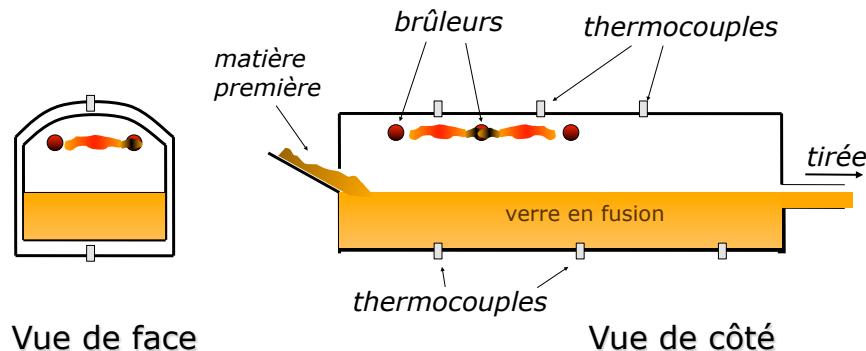


FIGURE 1.1 – A glass furnace

system. We often write, in a condensed way,

$$\dot{x} = f(x, u) \quad (1.2)$$

where f is an application from \mathbb{R}^{n+m} to \mathbb{R}^n while x and u are vectorial functions of time.

Such a system of equations is called *state model*. The goal of this book is to treat the *modelling*, i.e. obtaining such equations for diverse applications of engineering sciences, and the *analysis*, i.e. the determination of main properties of these systems, deducted from the equations. Let us begin with some examples to illustrate our subject.

Example 1.1. A glass furnace

The first example is an industrial process, illustrated in figure 1.1. It is about a furnace whose walls are built in refractory material and in which we melt a mixture of sand, lime and other additives to obtain glass. This fusion is obtained thanks to an energy input inside the furnace, for instance coming from a gas burner placed above the bath of glass. The molten glass is extracted from the furnace in a continuous way to feed machines downstream. By making the assumption that the temperature of the glass is homogeneous in the furnace and that this one is perfectly isolated, we can write the following equations, corresponding to the mass balance and to the energy balance of the process. We thus write that the mass or energy change in the considered system, by unit of time, equals the sum of what goes into the system, in terms of mass and heat, decreased by what gets out of it, always during the same unit of time :

$$\begin{aligned} \frac{dM}{dt} &= P_{in} - P_{out}, \\ \frac{d}{dt}(CTM) &= Q_{in} + C_{in}T_{in}P_{in} - CT P_{out}, \end{aligned} \quad (1.3)$$

with the following meaning of variables and parameters of the model :

- M : mass of the glass in fusion in the furnace (kg),
- T : temperature of the glass in fusion in the furnace (K),
- T_{in} : temperature of the raw material put in the furnace (K),
- C : specific heat of the glass ($J/K \times kg$),
- C_{in} : specific heat of the raw material ($J/K \times kg$),
- Q_{in} : heat quantity supplied by unit of time (J/s),
- P_{in} : mass put in the furnace by unit of time (kg/s),
- P_{out} : mass extracted by unit of time (kg/s).

We indicated units for each of the sizes defined above. The dimensional coherence of the equations is the first check to be made in an exercise of transforming a mathematical model into equations.

In order to put the system of equations (1.3) under the shape of a state model (1.1), we define the state variables :

- $x_1 \triangleq M$: mass of the glass in fusion (kg),
- $x_2 \triangleq CT$: heat quantity by unit of mass of glass in fusion (J/kg),

and the input variables :

- $u_1 \triangleq P_{in}$: mass put in the furnace by unit of time (kg/s),
- $u_2 \triangleq P_{out}$: mass extracted by unit of time (kg/s),
- $u_3 \triangleq Q_{in}$: head supplied by unit of time (J/s).

We obtain the following state model :

$$\begin{aligned}\dot{x}_1 &= u_1 - u_2, \\ \dot{x}_2 &= \frac{u_1(\alpha - x_2) + u_3}{x_1},\end{aligned}\tag{1.4}$$

where the constant parameter $\alpha = C_{in}T_{in}$ is the heat quantity of the material put in the furnace by unit of mass.

Note that other choices for the state variables and input variables are possible (see exercise 1.2). \square

Example 1.2. A chemical reactor

In a chemical reactor (Figure 1.2), a reaction transforming a reactive A into a product B takes place in liquid phase with a certain temperature T . The reactor is fed with reactive A via a valve which introduces the reactive with concentration A_{in} with a variable volume flow q_{in} which is a monotonous increasing function of the

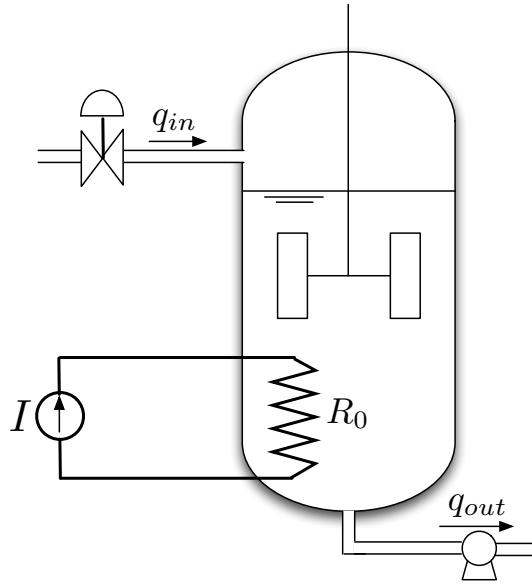


FIGURE 1.2 – Chemical reactor

valve opening $w : q_{in} = \phi(w)$. The content of the reactor is extracted by a pump with a flow q_{out} . We assume that the reaction is endothermic and then requires a heat input W supplied by a heating resistance R_0 fed by a variable current source I as illustrated in the figure. Moreover, we assume that the reactor is perfectly mixed. The reaction (i.e. the transformation of the reactive A in produced B) takes place with a reaction speed which follows a kinetic of the first order, i.e. proportional to the quantity of reactive A in the reactor. The coefficient of proportionality is a function of the temperature and verify the law of Arrhenius, $k(T) = k_0 \exp(-\frac{E}{RT})$.

We describe the evolution of this system by writing the equations of volumetric, mass and heat balances :

$$\begin{aligned}\frac{dV}{dt} &= q_{in} - q_{out}, \\ \frac{d}{dt}(AV) &= q_{in}A_{in} - q_{out}A - k(T)AV, \\ \frac{d}{dt}(BV) &= -q_{out}B + k(T)AV, \\ \frac{d}{dt}(CTV) &= CT_{in}q_{in} - CTq_{out} - hk(T)AV + R_0I^2,\end{aligned}$$

with :

V : liquid volume in the reactor,
 A : concentration of reactive A in the reactor,
 B : concentration of reactive B in the reactor,
 k_0 : reaction speed's constant,
 E : activation energy,
 R : Boltzmann constant,
 C : specific heat,
 h : reaction enthalpy.

The other notations are defined above. We can define the state variables

$x_1 = A$: reactive concentration in the reactor,
 $x_2 = B$: product concentration in the reactor,
 $x_3 = V$: volume of the reactionnal environment,
 $x_4 = T$: temperature of the reactionnal environment,

and the input variables

$u_1 = w$: valve opening,
 $u_2 = q_{out}$: withdrawal rate,
 $u_3 = I$: electrical current supplied in the heating resistance,

we obtain the following state model :

$$\begin{aligned}
 \dot{x}_1 &= \phi(u_1) \frac{A_{in} - x_1}{x_3} - k(x_4)x_1, \\
 \dot{x}_2 &= -\phi(u_1) \frac{x_2}{x_3} + k(x_4)x_1, \\
 \dot{x}_3 &= \phi(u_1) - u_2, \\
 \dot{x}_4 &= \frac{1}{x_3} [\phi(u_1)(T_{in} - x_4) + \frac{R_0}{C}u_3^2] - \frac{h}{C}k(x_4)x_1.
 \end{aligned}$$

Continuous and isotherms reactors constitute an interesting particular case. It is about reactors for which the volume V and the temperature T are maintained constant by adequate regulation devices. The state model is then reduced to the first two equations of the model above :

$$\begin{aligned}
 \dot{x}_1 &= \frac{\phi(u_1)}{V}(A_{in} - x_1) - k(T)x_1, \\
 \dot{x}_2 &= -\frac{\phi(u_1)}{V}x_2 + k(T)x_1.
 \end{aligned} \tag{1.5} \quad \square$$

Example 1.3. Ladybirds and aphids

Aphids are permanent pests for the rosebushes cultures. The biological fight against these devastating species is an alternative to the treatments by pesticides, which are less and less effective in front of the resistances developed by aphids. The *Harmonia axyridis* ladybirds (Fig. 1.3) are used in this biological fight because



FIGURE 1.3 – *Harmonia axyridis*

they feed on aphids with great voracity. They are active by spring, that is by the appearance of the aphids colonies in rose gardens. To increase the predatory efficiency of ladybirds, the French institute of agronomic research (INRA) developed a ladybirds variety « home-bodies » who do not fly (and therefore are not likely to leave the cultures).

We wish to establish a model describing the evolution of the number of aphids $x_1(t)$ and of ladybirds $x_2(t)$ under the following assumptions :

1. if there are no ladybirds, the population of aphids has enough food (rosebushes leaves) to have an exponential growth with a specific constant rate of growth ;
2. ladybirds eat more aphids if their population is bigger ;
3. the predation by ladybirds is the only source of aphids natural mortality ;
4. ladybirds have a constant specific rate of natural mortality ;
5. the gardener, who is not very smart, spreads a pesticide which kills indifferently aphids and ladybirds with a variable manuring rate $u(t)$.

The following state model expresses the balance of the number of aphids and ladybirds :

$$\begin{aligned}\dot{x}_1 &= ax_1 - bx_1x_2 - cux_1, \\ \dot{x}_2 &= dx_1x_2 - ex_2 - fux_2.\end{aligned}\tag{1.6}$$

where a, b, c, d, e, f are positive constants. We can verify that every term of this model formalizes one of the assumptions above. This check is left as exercise.

This type of model was originally introduced by the Italian mathematician V. Volterra who tried to understand the fluctuations in the efficiency of the fishing in Adriatic sea at the beginning of the twentieth century. Obviously, it is a rather big simplification of the reality. The model doesn't take into account numerous factors that can influence the evolution of the populations (weather conditions, other available resources, other predators, migration of the populations etc.). As our example shows it, an important application of this type of model is the fight against pests in the agriculture. It often happens that the harmful population is controlled by predator introductions. The model is then an interesting tool for the design of intervention programs on the ground. \square

Example 1.4. A DC motor

We now examine the electromechanical device shown in figure 1.4. This is a DC

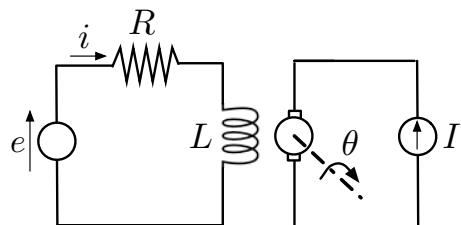


FIGURE 1.4 – DC motor

motor which can be controlled by both the stator voltage e and the rotor current I . The stator circuit equation is given by

$$e = Ri + L \frac{di}{dt}$$

where R and L are the resistance and inductance of the circuit, e is the control voltage and i is the current. The couple exerted on the rotor is given by $C = \Phi I$ where I is the rotor current and Φ is the magnetic flow proportional to the excitation current $\Phi = Ki$. We therefore obtain

$$C = KiI.$$

It remains to model the mechanical part of the system. Noting θ the angular position of the rotor, J its moment of inertia and F the viscous friction coefficient, the application of Newton's law leads to :

$$J \frac{d^2\theta}{dt^2} + F \frac{d\theta}{dt} = C.$$

We can define the state variables $x_1 = \theta, x_2 = \dot{\theta}, x_3 = i$, the inputs $u_1 = e, u_2 = I$, and we obtain the following state model :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{F}{J}x_2 + \frac{K}{J}x_3u_2, \\ \dot{x}_3 &= -\frac{R}{L}x_3 + \frac{1}{L}u_1.\end{aligned}\tag{1.7}$$

□

The four examples we just discussed are intended to show that equation (1.2) actually helps to build models of dynamic systems in various application areas of engineering as we treated successively examples about thermodynamics, chemical engineering, ecology and electronics. They will also help to understand better the terminology that is introduced in the next section.

1.2. Terminology and notations

As the previous examples illustrated it, we study *dynamic systems* whose behavior is described by a *state model* composed by a set of differential equations written under condensed shape :

$$\dot{x} = f(x, u).\tag{1.8}$$

We consider this state model from an initial time t_0 . The state $x \in \mathbb{R}^n$ and the entry $u \in \mathbb{R}^m$ are vectorial functions of time that we can sometimes note $x(t)$ and $u(t)$. However, the argument t is often omitted without confusion.

For a given system, the entry $u(t)$ is a priori any function of the time. However, we always assume that it is a *piecewise continuous* and *bounded* function : $u(t) \in \mathcal{U}$ where \mathcal{U} is a set of piecewise continuous and bounded of \mathbb{R} to \mathbb{R}^m functions.

For a given value of the initial state $x(t_0) = x_0$ and an input $u(t)$ given, the solution $x(t) \quad t \geq t_0$ of the differential system (1.8) is called *path* of the system. Sometimes, when it will be necessary for the presentation clarity, the path will be denoted $x(t, x_0, u)$. We always assume that such a path exists at any time $t \geq t_0$, is unique and is a continuous function of time. Graphically, a path may be displayed by a continuous curve in \mathbb{R}^{n+1} . The path projection in the *state space* \mathbb{R}^n (also called *phase space*) is called an *orbit* of the system.

When the input $u(t)$ can be freely chosen in \mathcal{U} , we say that the system $\dot{x} = f(x, u)$ is a *forced* system, or a *controlled* system. The term *forced* is used to mean that starting from an initial state x_0 , the shape of the path is somehow forced by the choices we made of an input $u(t)$. Similarly, in an automatic context, the term *controlled* means that the system state can be manipulated in the state space by suitable manipulation of the input $u(t)$.

In the next chapters however, we will be interested in the solution of the equation $\dot{x} = f(x, u)$ when the input is actually a constant fixed a priori : $u(t) = \bar{u} \quad \forall t \geq t_0$. In that case, we write the state model as $\dot{x} = f(x, \bar{u})$. We sometimes also write

$$\dot{x} = f_{\bar{u}}(x)$$

to express more clearly that f is only a function of x , parameterized by the constant \bar{u} . In such a case, there is only one possible path moving freely at the start of an initial state x_0 . By fixing in advance the input at a constant value, we have no longer the ability to control the paths of the system and say that the system is a *free* system (we also say an *autonomous* system, or *stationary* system). The path is sometimes called free response of the system.

When we are interested in the solution of the equation $\dot{x} = f(x, u)$ for *one* input $u(t)$ that is not constant over time but specific (for example a sine wave), we may as well forget that we selected an input in \mathcal{U} and simply write :

$$\dot{x} = f(x, t)$$

A dynamic system represented this way is called a *non-autonomous* or *unstationary* system.

We will sometimes consider various special cases of the general state model (1.8). We distinguish in particular ;

Affine systems relative to the input

$$\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x) \triangleq f(x) + G(x)u$$

where f and the g_i are applications from \mathbb{R}^n to \mathbb{R}^n . The state model (1.4) of a glass furnace is of this form with the following definitions :

$$f(x) = 0, \quad G(x) = \begin{pmatrix} 1 & -1 & 0 \\ (\alpha - x_2)/x_1 & 0 & 1/x_1 \end{pmatrix}.$$

Affine systems relative to the state

$$\dot{x} = \sum_{i=1}^m x_i a_i(u) + b(u) \triangleq A(u)x + b(u)$$

where b and the a_i are applications from \mathbb{R}^m to \mathbb{R}^n . The state model of an isothermal continuous chemical reactor (1.5) is of this form with the following definitions :

$$A(u) = \begin{pmatrix} -\left(\frac{\phi(u_1)}{V} + k(T)\right) & 0 \\ k(T) & -\frac{\phi(u_1)}{V} \end{pmatrix},$$

$$b(u) = \begin{pmatrix} \frac{\phi(u_1)A_{in}}{V} \\ 0 \end{pmatrix}.$$

The bilinear systems

These systems are affine relative both to the state and the input

$$\begin{aligned} \dot{x} &= \left(A_0 + \sum_{i=1}^m u_i A_i \right) x + B_0 u, \\ &= A_0 x + (B_0 + \sum_{i=1}^n x_i B_i) u, \end{aligned}$$

where the A_i ($i = 0, \dots, m$) are matrices of dimension $(n \times n)$ and the B_i ($i = 0, \dots, n$) are matrices of dimension $(n \times m)$. The state model of a DC motor (1.7) is of this form with the following definitions :

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & 0 \\ 0 & 0 & -\frac{R}{L} \end{pmatrix} \quad A_1 = 0$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{K}{J} \\ 0 & 0 & 0 \end{pmatrix} \quad B_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L} & 0 \end{pmatrix}.$$

Linear systems

$$\dot{x} = Ax + Bu$$

where A is a matrix of dimension $(n \times n)$ and B is a matrix of dimension $(n \times m)$. If we consider the state model of the DC motor assuming that the rotor current

source is constant ($I = \text{constant}$), we obtain an example of a linear system with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{KI}{J} \\ 0 & 0 & -\frac{R}{L} \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L} & 0 \end{pmatrix}.$$

1.3. Modeling and analysis

A dynamic system, as we conceive it in this book, is thus a part of the *concrete reality* that seems relevant in an engineering problem and that we choose to isolate in thought to describe its behavior in mathematical terms using a model. In particular, we are interested in characterize quantitatively the evolution over time of the *state* of the system. For this, we use [deterministic, lumped parameters state models](#) which consist of ordinary differential equations.

However, other modeling approaches of dynamic systems are possible. For the various examples described in the section 1.1 we could have build [deterministic, distributed parameters state models](#) composed of partial differential equations. This is how, in the example of the glass furnace, we made the hypothesis that the temperature was uniform across the glass bath. This is a simplistic view, but very useful to build simple and effective models for instance in view of the control and optimization of the dynamic behavior of the furnace. However, if this hypothesis is not accepted and we want to study the spatial variations of temperature, we can develop a state model consisting of partial differential equations describing the evolution over time of the temperature and velocity of the fluid fields in the molten glass bath. Neither model is better than the other. They simply are different models obtained for different objectives, usually corresponding to different spatial and temporal scales.

The interaction between the system and the « outside world » is represented by the inputs of the model $u_i(t)$ which are, as we mentioned above, functions of time, real and deterministic. In reality, a given system is often subject to random influences that can be represented by introducing stochastic inputs, i.e. random functions $u_i(t)$ (also called stochastic processes). We then obtain stochastic state models whose state variables are themselves random functions and whose study uses different mathematical techniques than those used in this book.

The state model of a dynamic system is thus a simplified mathematical representation of the behavior of the system. Yet, when it will not affect the clarity of the argumentation, we will often consider these two notions as equivalent in order to simplify the presentation. We will then speak of the dynamic system $\dot{x} = f(x, u)$, implying that we actually speak of a deterministic state model of the system.

The *modeling* of a dynamic system, as we conceive it in this book, is thus the exercise that aims to, given a discursive and qualitative description of the system, establish a mathematical description of it as a state model. Without being unnecessarily complicated, the resulting model should be an effective tool for solving the engineering problem posed for the system considered. The assumptions adopted for modeling should be clearly identified and highlighted.

In the first part of the book, we will show how the modeling approach can be systematized for different relevant classes of systems in engineering. We will successively discuss mechanical systems, electrical and electromechanical systems, **compartment systems** and **reaction systems** in Chapters 2-5. In each case, we will describe the basic physical principles and how they are used to obtain the state models. In Chapter 6, we will see how to define and use *state transformations* to obtain equivalent models of a given system.

However, we do not intend to describe and justify in detail all the physical principles of the different disciplines which constitute the art of engineering. In more complex modeling cases than those covered in this book, the reader should refer to the literature of the disciplines. However, we hope that the unifying character of the state model concept in engineering sciences will be clearly perceived.

After obtaining the model, we can analyze its properties and derive a certain number of lessons, or on the appropriateness of the model itself, or on the properties of the dynamic system which is the subject of modeling. It is to this analysis that the second part of the book is dedicated.

In Chapters 7, 8 and 9, we analyze the behavior of dynamic systems whose inputs are *constant* : $\dot{x} = f(x, \bar{u})$. In Chapter 7, we first examine the conditions of existence of equilibrium states and invariant subsets of the state space. Chapter 8 is devoted to the study of **plan systems**, i.e. systems whose state vector is of dimension 2. We examine in particular the behavior of the system near the equilibrium states, as well as periodic trajectories and bifurcations. The purpose of Chapter 9 is to analyze the stability of equilibrium states by the Lyapunov's method and characterize the basins of attraction.

Finally, in Chapter 10, we look at the issue of controllability of dynamical systems that can be formulated as follows : for a forced dynamic system $\dot{x} = f(x, u)$, under what conditions and how we can determine the input functions $u_i(t)$ to conduct the system of a given initial x_0 state to a given final state x_f in a prescribed time. The answer to this question obviously has important implications in many engineering problems such as **the steering of electro-mechanical devices** or the conduct of industrial processes.

1.4. Exercices

Exercise 1.1. A glass furnace

For the glass melting furnace that has been described in this chapter :

1. Establish a state model whose state variables are the mass M and the stored heat $C_T M$.
2. Establish a state model whose state variables are the temperature T and the stored heat $C_T M$.
3. Indicate how to change the state model to take into account the heat loss to the outside through the walls of the furnace.
4. The state model was built under the implicit assumption of near-instantaneous fusion of the raw material. Imagine how to easily modify the model to explicitly include the fusion (indication : cut the oven into two compartments of variable mass, one containing the not yet molted material and the other one containing the melt).

Exercise 1.2. Beetles and aphids

1. Justify each term of the model (1.6) explaining how it formalizes one of the modeling assumptions.
2. Is the model (1.6) affine in the input, affine in the state, bilinear, linear ?
3. The model (1.6) was established with two populations : the aphids (x_1) and the beetles (x_2). The adult beetle can ingest up to 100 aphids per day, but the larvae is even more voracious, it can ingest up to 150 per day. By formulating additional relevant modeling assumptions, establish a more accurate state model distinguishing adult beetles and larvae (i.e. a model with three state variables : aphids (x_1), larvae (x_2) and adult beetles (x_3)).

Chapitre 2

Articulated mechanical systems

The subject of this chapter is setting in equations state models of mechanical systems formed of a set of rigid bodies connected by joints. The systematic modeling method that we will study is applicable to many practical examples of mechanical systems such as vehicles (cars, trains, planes, ...) or robots. This method results from a systematic application of Newton's law.

In order to simplify the notations and calculations, we will limit ourselves to the establishment of the motion equations in a two-dimensional space (i.e. in a plane). The extension to the case of a movement in a three-dimensional space is conceptually simple but more difficult to visualize.

We first consider the case of a single rigid body without friction. Then, we discuss the modeling of an articulated system consisting of several rigid bodies. The modeling method is described in detail using an example of a [robot manipulator](#) with two degrees of freedom. Finally, we discuss how to extend the model to take into account friction, the elasticity of the joints and the non-holonomic constraints.

2.1. Dynamics of a rigid body in the plane

We consider a rigid body moving in a plane in which an orthonormal inertial basis $0, X_B, Y_b$ is fixed arbitrarily (Fig.2.1). A vector \vec{W} is attached to the body. The position of the body is completely specified by the 3 coordinates x, y, θ :

- x, y are the Cartesian coordinates of the center of mass G in the fixed basis $0, X_b, Y_b$;
- θ is the orientation of the vector \vec{W} with respect to the fixed basis $0, X_b, Y_b$.

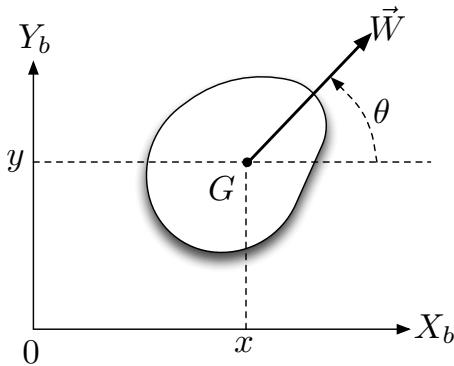


FIGURE 2.1 – Coordinates of a rigid body in plane

We define the three-dimensional vector describing the position of the body :

$$q \triangleq \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}. \quad (2.1)$$

A direct application of Newton's laws, coordinate by coordinate, then leads to the following general equations of motion :

— Equations of translation of the center of mass :

$$\begin{aligned} m\ddot{x} &= F_x, \\ m\ddot{y} &= F_y. \end{aligned}$$

— Equation of rotation around the center of mass :

$$I\ddot{\theta} = T.$$

where m is the mass of the body, I is its moment of inertia with respect to the center mass, F_x and F_y denotes the projections of the resultant of the forces applied to the body on the axes $0X_b$ and $0Y_b$ respectively and T is the resultant of the torques applied for the rotation of the body around the center of mass.

These general equations of motion form the basis of the establishment of the system state model as we will illustrate it in an example.

Example 2.1. Modeling of the dynamics of a rocket.

We consider a rocket moving in a plane perpendicular to the ground. This rocket is propelled by two jet engines arranged symmetrically with respect to the body of the rocket as shown in Figure 2.2. The equations of motion are established under the **modeling assumption** that the rocket is a rigid body of constant mass.

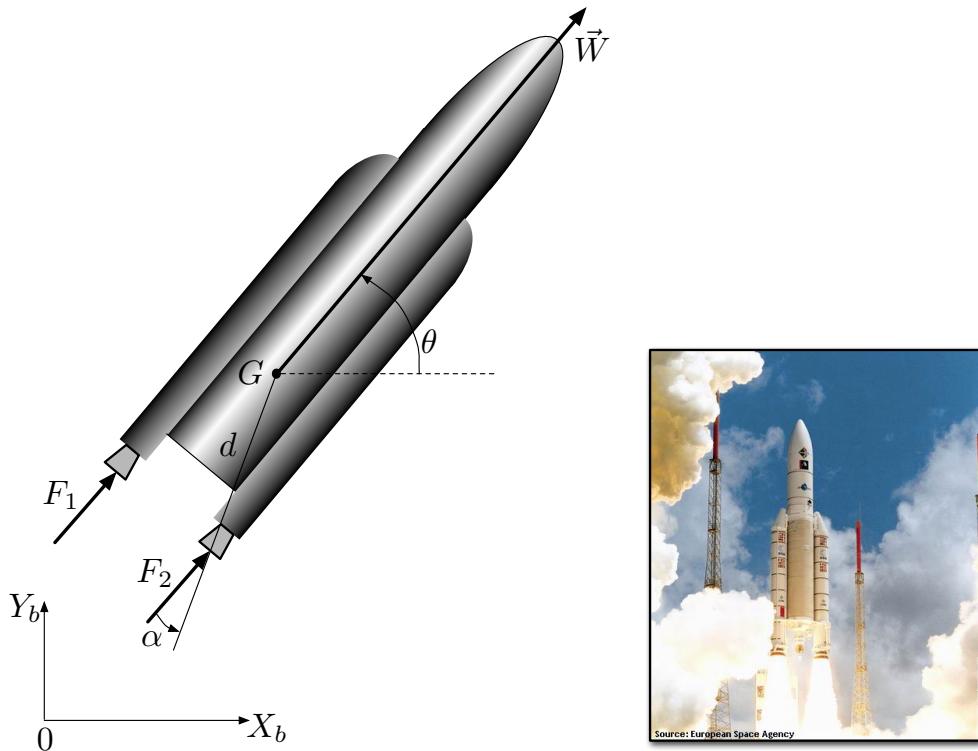


FIGURE 2.2 – Modeling of the dynamics of a rocket -
Photo of the Ariane rocket during takeoff
(© ESA)

— Equations of translation :

$$\begin{aligned} m\ddot{x} &= F_x = (F_1 + F_2) \cos \theta, \\ m\ddot{y} &= F_y = (F_1 + F_2) \sin \theta - mg_0. \end{aligned} \quad (2.2)$$

— Equation of rotation :

$$I\ddot{\theta} = T = (F_2 - F_1)d \sin \alpha. \quad (2.3)$$

In these equations, (x, y) is the position of the center of mass G , θ the angle of the vector \vec{W} with respect to the horizontal, F_1, F_2 **the thrusts of the reactors**, m the mass of the rocket, I its moment of inertia, d, α geometrical parameters (Fig.2.2) and g_0 the gravitational constant.

The equations (2.2)-(2.3) can be put in the standard form of a state model $\dot{x} = f(x, u)$ of dimension 6 with two inputs if we introduce the following notations :

State variables :

$$x_1 = x, \quad x_2 = y, \quad x_3 = \theta, \quad x_4 = \dot{x}, \quad x_5 = \dot{y}, \quad x_6 = \dot{\theta}.$$

Input variables :

$$u_1 = F_1, \quad u_2 = F_2.$$

The state model is written as follows :

$$\begin{aligned}\dot{x}_1 &= x_4, \\ \dot{x}_2 &= x_5, \\ \dot{x}_3 &= x_6, \\ \dot{x}_4 &= \frac{\cos x_3}{m}(u_1 + u_2), \\ \dot{x}_5 &= -g_0 + \frac{\sin x_3}{m}(u_1 + u_2), \\ \dot{x}_6 &= \frac{d \sin \alpha}{I}(u_2 - u_1). \quad \square\end{aligned}$$

A special situation appears when the considered body is subject to a set of forces whose resultant is zero but which are not all applied at the same point. The equations of motion can be written as :

$$\begin{aligned}m\ddot{x} &= 0 \\ m\ddot{y} &= 0 \\ I\ddot{\theta} &= T\end{aligned}$$

In such a case, it is common, in some applications, to not specify the forces that are behind the torque T , but to directly consider this one as the cause of the movement. We say, for simplicity, that the body is subjected to a *torque*. Thus, for example, we will talk about the torque provided by an engine to rotate a [robot manipulator](#) segment.

The state model obtained in the example of the rocket is non-linear with respect to the state variables and affine in the input variables. This will be the case for most applications of interest for which the translational and rotational equations describing the dynamics of a rigid body can be written in the general matrix form :

$$J\ddot{q} + b(q) = B(q)u.$$

In this equation J is the inertia matrix (diagonal and constant), $b(q)$ represents the effect of gravity and $B(q)$ is a matrix (called kinematic) non-linearly depending on the state variables. We deduce that the state model is written in the following general form :

$$\begin{aligned}\dot{q} &= v, \\ \dot{v} &= J^{-1}[-b(q) + B(q)u],\end{aligned}$$

where $v \triangleq \dot{q}$ is called *vector of generalized velocities*.

2.2. Dynamics of articulated mechanical systems

We now consider the case of an articulated mechanical system with N body. The general procedure for setting in equations the state model can be summarized as follows :

1. Fix an inertial reference frame in the system configuration space and N moving frames attached to the centers of mass of the N bodies of the system.
2. Write the **path and bonding constraints** equations faced by the movement of the system. Deduce the number of degrees of freedom.
3. Write the equations of motion (translation and rotation) for each of the coordinates by including the bonding forces related to the constraints (method of Lagrange coefficients).
4. Remove the coefficients of Lagrange and redundant coordinates.

We will now detail the procedure, explaining the new concepts (degrees of freedom, Lagrange coefficients, redundant coordinates) that have been mentioned, and illustrate it with a typical example : the development of the dynamic model of a **robot manipulator** with two degrees of freedom.

First step : Defining coordinate

An inertial reference frame is set in the configuration space Ω of the system. N moving frames are attached to the centers of mass of the N bodies of the system. The position of the system is at any time characterized by the coordinate vector

$$\xi = (x_1 \ y_1 \ \theta_1 \dots x_N \ y_N \ \theta_N)^T$$

of dimension $3N$.

Step Two : Expression of geometric constraints

The movement of an articulated mechanical system may be subject to two types of constraints (called geometric) : **path constraints** on the one hand and bonding constraints between the bodies on the other hand. These constraints are expressed as a set of algebraic relations between the coordinates which we will note

$$\Psi(\xi) = 0,$$

where Ψ is an application $\Omega \rightarrow \mathbb{R}^p$ of class C^1 and p denotes the number of constraints. According to the implicit function theorem, in a neighborhood of any point ξ of the configuration space, there is a partition $\xi = (q, \bar{q})$ of the coordinates vector such that :

- the dimension (noted σ) of \bar{q} is equal to the rank of the Jacobian matrix of the application Ψ :

$$\sigma \triangleq \dim \bar{q} = \text{rank} \frac{\partial \Psi}{\partial \xi};$$

- we can express the coordinates \bar{q} depending on coordinates q :

$$\bar{q} = \phi(q). \quad (2.4)$$

This means that we can use the expression (2.4) to remove the *redundant coordinates* \bar{q} of the system description. The size of the vector q of the coordinates that are preserved is the *number of degrees of freedom* of the system, denoted δ :

$$\delta \triangleq 3N - \sigma.$$

Step Three : Equations of motion

Then we write the equations of motion (translation and rotation) for each of the coordinates by including the bonding forces related to the constraints. The partition (q, \bar{q}) of the coordinates induces a similar partition of the set of equations of motion as follows :

$$J\ddot{q} + b(q, \bar{q}) = B(q, \bar{q})u + w, \quad (2.5)$$

$$\bar{J}\ddot{\bar{q}} + \bar{b}(q, \bar{q}) = \bar{B}(q, \bar{q})u + \bar{w}. \quad (2.6)$$

In these equations, vectors w and \bar{w} represent the bonding forces that ensure that the constraints are satisfied at any time during the movement of the system. It is shown in the mechanical basic works that these binding forces are expressed as follows :

$$\begin{aligned} w &= -A(q)\lambda, \\ \bar{w} &= \lambda. \end{aligned}$$

where λ is the vector of *Lagrange coefficients* (of dimension σ) and $A(q)$ is the matrix of dimension $\delta \times \sigma$ defined as follows :

$$A(q) \triangleq \left(\frac{\partial \phi}{\partial q} \right)^T.$$

Step Four : Eliminate redundant coordinates

In the equation (2.6), λ is expressed as :

$$\lambda = \bar{J}\ddot{\bar{q}} + \bar{b}(q, \bar{q}) - \bar{B}(q, \bar{q})u.$$

By substituting this expression in (2.5) and using (2.4), it is resulted as :

$$\begin{aligned} J\ddot{q} + A(q)\bar{J}\ddot{q} + b(q, \phi(q)) + A(q)\bar{b}(q, \phi(q)) \\ = (B(q, \phi(q)) + A(q)\bar{B}(q, \phi(q)))u. \end{aligned} \quad (2.7)$$

Last step to do is the elimination of \ddot{q} . To obtain this product, it is needed to differentiate two times the expression (2.4) :

$$\dot{\bar{q}} = A^T(q)\dot{q} \quad (2.8)$$

$$\ddot{\bar{q}} = A^T(q)\ddot{q} + \dot{A}^T(q)\dot{q}. \quad (2.9)$$

Substituting this last expression (2.9) in (2.7) and introducing these following notations :

$$\begin{aligned} M(q) &\triangleq J + A(q)\bar{J}A^T(q), \\ f(q, \dot{q}) &\triangleq A(q)\bar{J}A^T(q)\dot{q}, \\ g(q) &\triangleq b(q, \phi(q)) + A(q)\bar{b}(q, \phi(q)), \\ G(q) &\triangleq B(q, \phi(q)) + A(q)\bar{B}(q, \phi(q)), \end{aligned}$$

The general dynamic model of an articulated mechanical system is finally obtained under this form :

$$M(q)\ddot{q} + f(q, \dot{q}) + g(q) = G(q)u. \quad (2.10)$$

In this equation :

- q is the vector (of dimension δ) of coordinates necessary for the description of the system,
- $M(q)$ is the inertia matrix (of dimensions $\delta \times \delta$) symmetrical and defined positive,
- $f(q, \dot{q})$ is the vector (of dimension δ) that represents forces and couples resulting of relative liaisons to constraints ; it could also be written as

$$f(q, \dot{q}) = C(q, \dot{q})\dot{q}$$

where $C(q, \dot{q})$ is the matrix of dimensions $\delta \times \delta$ defined below :

$$C(q, \dot{q}) \triangleq A(q)\bar{J}\dot{A}^T(q),$$

- $g(q)$ is a vector (of dimension δ) representing forces and couples which resulting of gravity,
- u is a vector (of dimension m) of forces and couples applied to the system
- $G(q)$ is a cinematic matrix of dimensions $\delta \times m$.

Once the general dynamic model (2.10) established, the last step remaining is the deduction of the state model of the system :

$$\begin{aligned}\dot{q} &= v, \\ \dot{v} &= M^{-1}(q)[-f(q, v) - g(q) + G(q)u].\end{aligned}$$

In these state equations, q is the coordinates vector of position and $v = \dot{q}$ is the coordinates vector of speed.

Example 2.2. Dynamic model of a manipulator robot .

A manipulator robot is formed by a set of rigid articulated segments. There is two kinds of joints : the revolute joint and the prismatic joint. A revolute allows a relative movement of rotation between two segments. A prismatic joint allows a relative movement of translation between two segments.

Robots are operated by recessed engines producing translation forces for prismatic joints and rotation couples for revolute joints.

Let's consider the manipulator robot represented in the figure 2.3 formed by rigid segment moving horizontally (body 1) which a second rigid segment is articulated. The second segment could perform a rotation movement (body 2). The movement of the system is due to the application of the force F applied horizontally on the first segment and the couple of rotation T applied on the second segment. The inertial basis and the different coordinates are indicated on the figure.

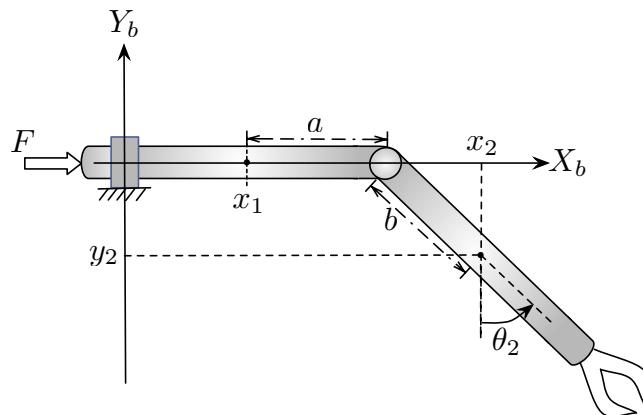


FIGURE 2.3 – Modelisation of a manipulator robot

This system is subjected to these constraints :

Course constraints :

$$\begin{aligned}y_1 &= 0, \\ \theta_1 &= 0.\end{aligned}$$

Liaison constraints :

$$\begin{aligned}x_2 - b \sin \theta_2 - x_1 - a &= 0, \\y_2 + b \cos \theta_2 &= 0.\end{aligned}$$

The course constraints show the fact that the body 1 can only have a horizontal movement. The liaison constraints express the relation existing between the Cartesian coordinates of the centers of mass of these two bodies due to their joint. The Jacobian matrix of constraints is written as followed :

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -b \cos \theta_2 \\ 0 & 0 & 0 & 0 & 1 & -b \sin \theta_2 \end{pmatrix}.$$

It is observed that this matrix is a full row matrix $\sigma = 4$ and so the system has $\delta = 2$ degrees of freedom (as it is expected). It is also observed that the partition (q, \bar{q}) can be described with this coordinates like :

$$q = \begin{pmatrix} x_1 \\ \theta_2 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} y_1 \\ \theta_1 \\ x_2 \\ y_2 \end{pmatrix}.$$

It is easy to verify that in all the configuration space, coordinates \bar{q} can be expressed as an explicit function $\bar{q} = \phi(q)$ of coordinates q :

$$y_1 = 0, \tag{2.11}$$

$$\theta_1 = 0, \tag{2.12}$$

$$x_2 = x_1 + b \sin \theta_2 + a, \tag{2.13}$$

$$y_2 = -b \cos \theta_2. \tag{2.14}$$

It is now possible to eliminate the coordinates $\bar{q} = (y_1, \theta_1, x_2, y_2)^T$ of the description of the system and just keep the coordinates $q = (x_1, \theta_2)^T$. The matrix $A(q)$ is written :

$$A(q) = (\frac{\partial \phi}{\partial q})^T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & b \cos \theta_2 & b \sin \theta_2 \end{pmatrix}$$

The movement equations are written :

$$m_1 \ddot{x}_1 = F - \lambda_3, \tag{2.15}$$

$$I_2 \ddot{\theta}_2 = -\lambda_3 b \cos \theta_2 - \lambda_4 b \sin \theta_2 + T, \tag{2.16}$$

$$m_1 \ddot{y}_1 = -m_1 g_0 + \lambda_1, \tag{2.17}$$

$$I_1 \ddot{\theta}_1 = \lambda_2, \tag{2.18}$$

$$m_2 \ddot{x}_2 = \lambda_3, \tag{2.19}$$

$$m_2 \ddot{y}_2 = -m_2 g_0 + \lambda_4. \tag{2.20}$$

By combining constraints (2.11), (2.12) with the movement equations (2.17), (2.18) the following values λ_1 et λ_2 are deducted :

$$\lambda_1 = m_1 g_0, \quad \lambda_2 = 0.$$

These values express the liaison forces applied on the two bodies to satisfy the course constraints along the movement system.

Moreover, by suppressing λ_3 et λ_4 between the movement equations (2.15), (2.16), (2.19), (2.20), this result is obtained :

$$\begin{pmatrix} m_1 \ddot{x}_1 \\ I_2 \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ b \cos \theta_2 & b \sin \theta_2 \end{pmatrix} \begin{pmatrix} m_2 \ddot{x}_2 \\ m_2 \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} F \\ T - b m_2 g_0 \sin \theta_2 \end{pmatrix}. \quad (2.21)$$

By derivating two times the constraints (2.13), (2.14), the result is obtained :

$$\begin{pmatrix} m_2 \ddot{x}_2 \\ m_2 \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & b \cos \theta_2 \\ 0 & b \sin \theta_2 \end{pmatrix} \begin{pmatrix} m_2 \ddot{x}_1 \\ m_2 \ddot{\theta}_2 \end{pmatrix} + m_2 b \dot{\theta}_2^2 \begin{pmatrix} -\sin \theta_2 \\ \cos \theta_2 \end{pmatrix}. \quad (2.22)$$

By substituting (2.22) in (2.21), the model of the system is obtained under the desired form :

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + g(q) = G(q)u, \quad (2.23)$$

with

$$\begin{aligned} M(q) &= \begin{pmatrix} m_1 + m_2 & m_2 b \cos \theta_2 \\ m_2 b \cos \theta_2 & I_2 + m_2 b^2 \end{pmatrix}, \\ C(q, \dot{q}) &= \begin{pmatrix} 0 & -m_2 b \dot{\theta}_2 \sin \theta_2 \\ 0 & 0 \end{pmatrix}, \\ g(q) &= \begin{pmatrix} 0 \\ b m_2 g_0 \sin \theta_2 \end{pmatrix}, \\ G(q)u &= \begin{pmatrix} F \\ T \end{pmatrix}. \end{aligned}$$

□

2.3. Properties of the inertia matrix

1. The inertia matrix $M(q)$ is symmetrical and defined positive. Indeed, this matrix is the sum of a diagonal matrix J whose the elements are positive and a matrix symmetrical and semi-defined positive $A(q) \bar{J} A^T(q)$.

2. The temporal derivative of the inertia matrix $\dot{M}(q)$ verifies this following relation :

$$\begin{aligned}\dot{M}(q) &= A(q)\bar{J}A^T(q) + \dot{A}(q)\bar{J}A^T(q), \\ &= C(q, \dot{q}) + C^T(q, \dot{q}).\end{aligned}$$

This relation involves that the matrix

$$\dot{M}(q) - 2C(q, \dot{q}) \quad (2.24)$$

is anti-symmetric.

3. The inertia matrix $M(q)$ verifies the following relation :

$$\frac{\partial}{\partial q}(\dot{q}^T M(q) \dot{q}) = \dot{q}^T C(q, \dot{q}). \quad (2.25)$$

The verification of this expression is left as an exercise.

2.4. Elastic joints

Until this moment, it has always been considered that articulated mechanic systems were formed only by rigid bodies without possibilities of flexibility or of suppleness in the liaisons and the joints. But a such hypothesis is not realistic in numerical applications. A simple way to introduce suppleness in the joints of a articulated mechanic system is to place a small spring (fictive) massless in the liaisons between bodies as shown in the figure 2.4. The spring exerts a restoring force on each bodies where the spring is attached. This force is applied on the fixation point of the spring, and is a monotonically increasing function of the spring elongation. This force is added on the other forces applied on the system in the writing of the movement equations. When the elasticity is thus introduced in a joint of two bodies of the system, It seems to be logical that where the constraints of corresponding liaisons disappear, and that the number of degrees of freedom is correlatively increased. The method is illustrated on a simple example of a double bodies system.

Example 2.3. Two bodies system with an elastic joint .

Let's consider the two bodies system represented on the figure 2.4. The movement equations of the two bodies are written as follow :

$$m_1 \ddot{x}_1 = F_1, \quad (2.26)$$

$$m_1 \ddot{y}_1 = F_2, \quad (2.27)$$

$$I_1 \ddot{\theta}_1 = F_2 d_1 \cos \theta_1 - F_1 d_1 \sin \theta_1, \quad (2.28)$$

$$m_2 \ddot{x}_2 = -F_1, \quad (2.29)$$

$$m_2 \ddot{y}_2 = -F_2, \quad (2.30)$$

$$I_2 \ddot{\theta}_2 = F_1 d_2 \sin \theta_2 - F_2 d_2 \cos \theta_2, \quad (2.31)$$

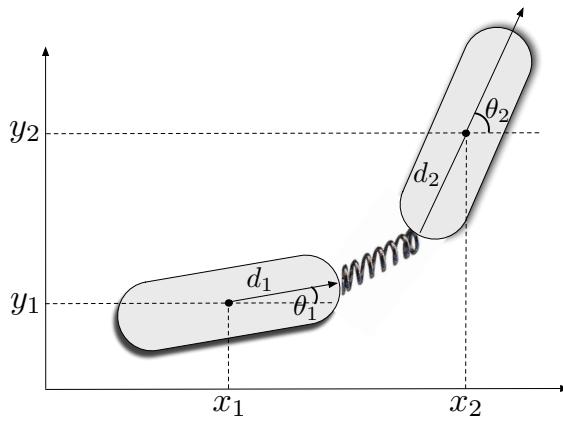


FIGURE 2.4 – Elastic joint modelisation

where F_1 and F_2 indicate the amplitudes of the components of the restoring forces applied on two bodies due to the presence of the spring.

The cartesian coordinates of the fixation points of the spring on the two bodies are expressed like this :

$$\begin{aligned}\tilde{x}_1 &= x_1 + d_1 \cos \theta_1, \\ \tilde{x}_2 &= x_2 - d_2 \cos \theta_2, \\ \tilde{y}_1 &= y_1 + d_1 \sin \theta_1, \\ \tilde{y}_2 &= y_2 - d_2 \sin \theta_2.\end{aligned}$$

The elongation of the spring is defined as the vector of components ϵ_1 and ϵ_2 :

$$\epsilon_1 = \tilde{x}_2 - \tilde{x}_1 \quad \epsilon_2 = \tilde{y}_2 - \tilde{y}_1$$

The restoring forces F_1 et F_2 are modeled as monotone increasing functions of components of the elongation (see Figure 2.5) :

$$F_1 = r(\epsilon_1) \quad F_2 = r(\epsilon_2)$$

Often, for some simplification reasons, the linear model is adopted, ie :

$$\begin{aligned}F_1 &= k_0(\tilde{x}_2 - \tilde{x}_1) = k_0((x_2 - x_1) - (d_1 \cos \theta_1 + d_2 \cos \theta_2)), \\ F_2 &= k_0(\tilde{y}_2 - \tilde{y}_1) = k_0((y_2 - y_1) - (d_1 \sin \theta_1 + d_2 \sin \theta_2)),\end{aligned}$$

where the constant k_0 is called *spring constant*. In this case, the movement equa-

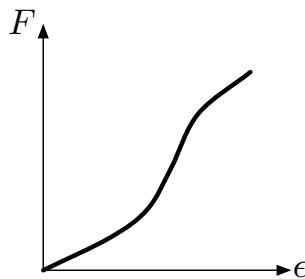


FIGURE 2.5 – elastic joints : restoring force with respect to the elongation

tions (2.26)-(2.31) are rewritten as follows :

$$\begin{aligned}
 m_1\ddot{x}_1 &= k_0((x_2 - x_1) - (d_1 \cos \theta_1 + d_2 \cos \theta_2)), \\
 m_1\ddot{y}_1 &= k_0((y_2 - y_1) - (d_1 \sin \theta_1 + d_2 \sin \theta_2)), \\
 I_1\ddot{\theta}_1 &= k_0 d_1 ((x_1 - x_2 + d_1 \cos \theta_1 + d_2 \cos \theta_2) \sin \theta_1 \\
 &\quad + (y_2 - y_1 - d_1 \sin \theta_1 - d_2 \sin \theta_2) \cos \theta_1), \\
 m_2\ddot{x}_2 &= -k_0((x_2 - x_1) - (d_1 \cos \theta_1 + d_2 \cos \theta_2)), \\
 m_2\ddot{y}_2 &= -k_0((y_2 - y_1) - (d_1 \sin \theta_1 + d_2 \sin \theta_2)), \\
 I_2\ddot{\theta}_2 &= k_0 d_2 ((x_2 - x_1 - d_1 \cos \theta_1 - d_2 \cos \theta_2) \sin \theta_2 \\
 &\quad + (y_1 - y_2 + d_1 \sin \theta_1 + d_2 \sin \theta_2) \cos \theta_2). \quad \square
 \end{aligned}$$

This example shows that in the case of mechanic articulated system, the general dynamic model (2.10) is modified as :

$$M(q)\ddot{q} + f(q, \dot{q}) + g(q) + k(q) = G(q)u \quad (2.32)$$

where appears the additional term $k(q)$ representing the effect of restoring forces due to the presence of elastic joints in the system.

2.5. Friction

The presence of friction forces is another physic phenomenon that was neglected until this moment and that is often an important effect on the movement of mechanic systems. In particular, in the case of elastic joints modeled like in the precedent section, the presence of an damping by the friction is essential to avoid the developing of models that would be the oscillations centers which are not really conform to the experiment reality.

There are many ways to introduce the friction in the description of a mechanic articulated system. We will retain here the most simple description supposing that

the movement of each coordinates q_i of the vector of generalized coordinates $q = (q_1, q_2, \dots, q_\delta, \dots)$ is affected by a separated friction force only depending of the speed (\dot{q}_i) of this same coordinate and denoted $h_i(\dot{q}_i)$. The vector of these friction forces is written

$$h(\dot{q}) = \begin{pmatrix} h_1(\dot{q}_1) \\ h_2(\dot{q}_2) \\ \vdots \\ h_\delta(\dot{q}_\delta) \end{pmatrix}$$

in order that the general dynamic model (2.32) is increased as follows :

$$M(q)\ddot{q} + f(q, \dot{q}) + g(q) + k(q) + h(\dot{q}) = G(q)u.$$

The most common form of the functions $h_i(\dot{q}_i)$ est is this one :

$$h_i(\dot{q}_i) = \alpha_i \text{sign}(\dot{q}_i) + \beta_i(\dot{q}_i).$$

In this equation, the first term $\alpha_i \text{sign}(\dot{q}_i)$ represents the dry friction while the second term $\beta_i(\dot{q}_i)$ represents the viscous friction. The α_i coefficient is constant. The function β_i is monotonically increasing with $\beta(0) = 0$. It is noticed that the function h is discontinuous at the origin, that can involve difficulties for the simulation and the analysis of the system. In the applications that will be considered in this book, except inverse indication, it is supposed that the dry friction is neglected ($\alpha_i = 0$).

2.6. Energy and Euler-Lagrange equation

The kinetic energy E_C of a articulated mechanic system is defined as :

$$E_C(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}.$$

The potential energy E_P is the primitive of the sum of forces deriving of a potential, to be more precise, gravity forces and restoring forces of springs :

$$\frac{\partial E_P(q)}{\partial q} = g^T(q) + k^T(q).$$

The total energy E_T is the sum of the kinetic and potential energy :

$$E_T = E_C + E_P.$$

The evolution of the total energy during the movement of the system is examined by calculating his temporal derivative :

$$\begin{aligned} \dot{E}_T &= \frac{\partial E_C}{\partial \dot{q}} \ddot{q} + \frac{\partial E_C}{\partial q} \dot{q} + \frac{\partial E_P}{\partial q} \dot{q} \\ &= \dot{q}^T [M(q)\ddot{q} + \frac{1}{2}\dot{M}(q)\dot{q} + g(q) + k(q)]. \end{aligned} \tag{2.33}$$

By substituting the expression $M(q)\ddot{q}$ extracted of the general equation of the movement (2.23), this result is obtained :

$$\dot{E}_T = \frac{1}{2}\dot{q}^T[\dot{M}(q) - 2C(q, \dot{q})]\dot{q} + \dot{q}^T[G(q)u - h(\dot{q})].$$

The first term of the right member of this equation is equal to zero because the matrix $\dot{M}(q) - 2C(q, \dot{q})$ is anti-symmetrical (see above). It remains :

$$\dot{E}_T = \dot{q}^T[G(q)u - h(\dot{q})].$$

When the system is not subjected by any other forces than forces deriving of a potential, the total energy is constant during the entire movement :

$$G(q)u - h(\dot{q}) = 0 \quad \Rightarrow \quad \dot{E}_T = 0.$$

In this case, It is said that the system is *conservative*.

By using proprieties (2.24) and (2.25), it is verified that the kinetic energy satisfied this following relation :

$$\frac{d}{dt}\left(\frac{\partial E_C}{\partial \dot{q}}\right)^T - \left(\frac{\partial E_C}{\partial q}\right)^T = M(q)\ddot{q} + C(q, \dot{q})\dot{q}.$$

this results in a alternative expression of the general movement equation (2.23) is given by the expression :

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}}\right)^T - \left(\frac{\partial L(q, \dot{q})}{\partial q}\right)^T = G(q)u - h(\dot{q})$$

with :

$$L(q, \dot{q}) \triangleq E_C(q, \dot{q}) - E_P(q, \dot{q}).$$

This equation is generally called the *Euler-Lagrange equation* and the quantity $L(q, \dot{q})$ is called *Lagrangien* of the system.

2.7. Nonholonomic system

The nonholonomic systems are articulated mechanic systems for which course constraints can not only depend of positions q but also speeds \dot{q} . When these constraints can not be integrated to produce course constraints which depend exclusively of configuration coordinates, there are called *nonholonomic*. This situation is produced under various practical applications, more specifically in automotive, robotic and aviation industry. We consider the particular case of one specific having

δ degrees of freedom which is subjected to m nonholonomic independent constraints ($m < \delta$) which are linear with respect to speeds :

$$N^T(q)\dot{q} = 0$$

with the full row matrix $N^T(q)$ of dimensions $(m \times \delta)$. The matrix $S(q)$, of dimensions $\delta \times (\delta - m)$ and of full row is defined as :

$$N^T(q)S(q) = 0.$$

The constraints are equivalents to the fact that the vector of speeds \dot{q} belongs to the space generated by the columns of the matrix $S(q)$ or, in another way, there is a vector η of dimension $(\delta - m)$ as :

$$\dot{q} = S(q)\eta. \quad (2.34)$$

The movement equations are written under the standard form :

$$M(q)\ddot{q} + f(q, \dot{q}) + g(q) + N(q)\lambda = G(q)u,$$

by adding the term $N(q)\lambda$ mouvement that represents the liaison forces which guarantee that the constraints are satisfied along the movement (see section 1.2.3). By suppressing the Lagrange multipliers On λ and by pre-multiplying this equation by $S^T(q)$:

$$S^T(q)M(q)\ddot{q} + S^T(q)[C(q, \dot{q})\dot{q} + g(q)] = S^T(q)G(q)u.$$

Finally, by using the relation (2.34), this expression is obtained :

$$J(q)\dot{\eta} + F(q, \eta) = S^T(q)G(q)u \quad (2.35)$$

avec

$$\begin{aligned} J(q) &= S^T(q)M(q)S(q) \\ F(q, \eta) &= S^T(q)M(q)\{[\partial_q S(q)]S(q)\eta\}\eta + S^T(q)f(q, S(q)\eta). \end{aligned}$$

The general dynamic model of a nonholonomic system consists in equation (2.34) and (2.35) that we can write under a state model :

$$\begin{aligned} \dot{q} &= S(q)\eta \\ \dot{\eta} &= J^{-1}(q)[-F(q, \eta) + S^T(q)G(q)u]. \end{aligned}$$

We can observe that the state vector :

$$\begin{pmatrix} q \\ \eta \end{pmatrix}$$

is of dimension $(2\delta - m)$ with coordinates η homogeneous to speeds.

2.8. Exercices

Exercise 2.1. *Technician robots*

Figure 2.6 represents three configurations of planar robots. For each of these configurations :

1. Establish the dynamic model of the system and the corresponding state model. Make explicit the matrices $M(q)$, $C(q, \dot{q})$ and $G(q)$, and the vector $g(q)$.
2. Check that the model is conservative and that it satisfies Euler-Lagrange equation.
3. Indicate how the equations of the model are modified if the segments are subjected to a viscous friction proportional to the square of the speed. \square

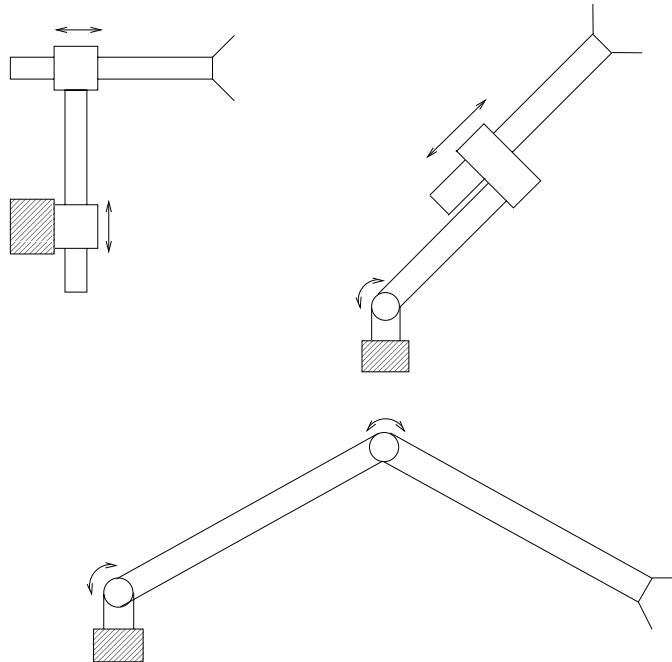
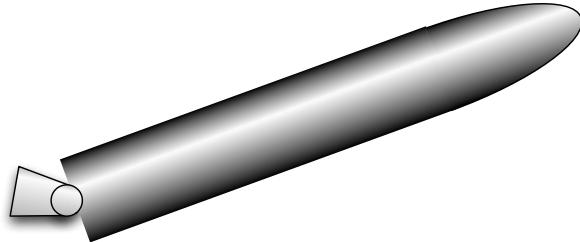


FIGURE 2.6 – Configurations of *technician* planar robots

Exercise 2.2. *Modelling of the dynamics of a rocket*

Let's consider a rocket propelled by an *adjustable* jet engine as indicated on figure 2.7 and moving in a vertical plane. The engine orientation is controlled by

FIGURE 2.7 – Rocket with **adjustable** motor

a hydraulic **actuator** providing a torque T . The engine itself provides a propelling force F .

1. Establish the equations of the state model of the system, assuming that the two parts of the rocket (main body and engine) are rigid bodies of constant mass.

Exercise 2.3. Dynamic modelling of the lunar excursion module

During the Apollo 11 mission, astronauts Armstrong and Aldrin landed on the moon using the LEM (Lunar Excursion Module; Fig. 2.8). We consider the following model assumptions :

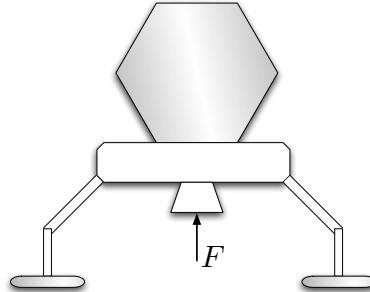


FIGURE 2.8 – Lunar Excursion Module

- a) the LEM is a rigid body
 - b) the movement is vertical
 - c) the forces acting on the system are the thrust F and the lunar attraction
 - d) the mass of fuel onboard is an important (non negligible) part of the total mass of the LEM
 - e) the mass of fuel consumed per unit of time is proportional to F .
1. Establish a state model of the system that satisfies these modelling assumptions.

2. What are the main limits of validity of this model ? □

Exercise 2.4. A pendular train

A pendular train is a train that can move at very high speed in bends without the need of inclining the rails. For this each car is equipped with an active device

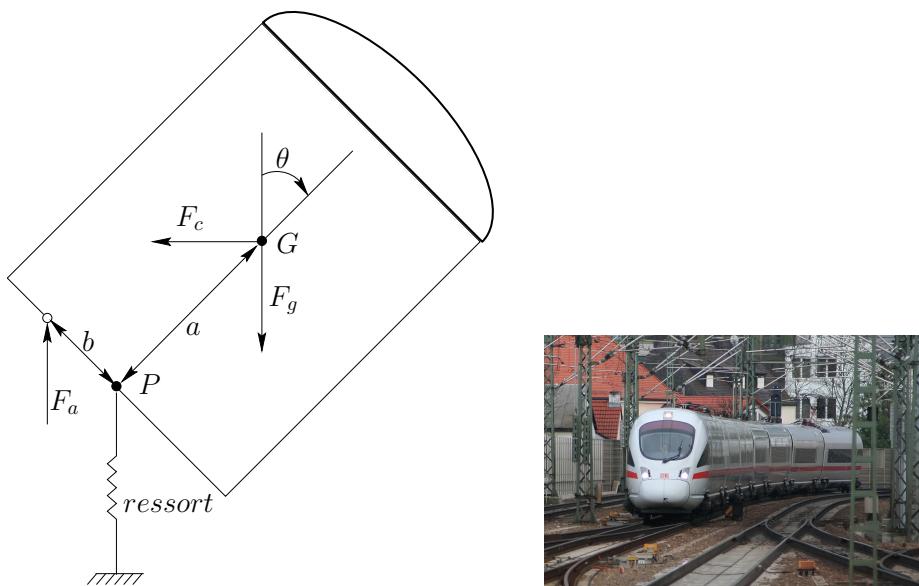


FIGURE 2.9 – A pendular train

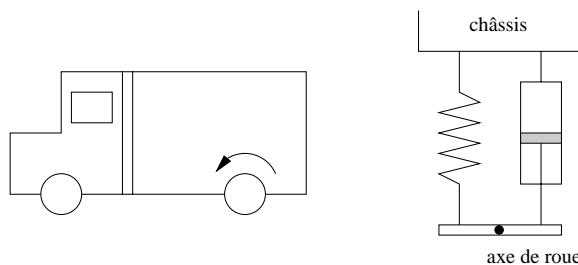


FIGURE 2.10 – Modelling of the dynamics of a truck

that applies a force vertical to the car **body** to compensate the effect of the « centrifugal » force. This is illustrated on figure 2.9 where a section of the **body** of a car is schematically represented with the gravity force F_g (applied to the center of

mass G) the "centrifugal" force F_c and the applied force F_a . It is assumed that the action line of force F_a is vertical irrespective of the angular position θ of the **body**. On the other hand the car's suspension is represented by a vertical spring that exerts a force proportional to its elongation. The application point P of the spring is *forced to move vertically*. *Established the state model of the system.* \square

Exercise 2.5. Modelling of the dynamics of a truck

We consider a truck traveling in a straight line (Fig. 2.10), under the following modelling assumptions :

- a) The truck is an articulated system composed of rigid bodies (**body** and **wheels**)
 - b) The truck is equipped with a rear propulsion (the torque developed by the motor is transmitted to the rear wheels).
 - c) The rear wheels roll without slipping.
 - d) The wheels are connected to the frame by a suspension system composed of a linear spring and a viscous friction shock absorber of negligible mass. This suspension system allows only vertical displacements.
1. Establish a state model of the system that satisfies the modelling assumptions (be limited to two bodies : the frame and a driving wheel)
 2. What are the main limits of validity of this model ? \square

Exercise 2.6. A boat

A boat equipped with an **adjustable** motor of type « speedboat » moves on a river as shown in figure 2.11 (top view). The river is of constant width ($= 2L$). The engine thrust is shown by the vector of length F ($=$ size of the propulsion force) and of orientation β . The boat is also subject to the force of the current of the river which is a parabolic function of axis y : the current is zero at both edges and maximal at the middle of the river. When the engine is stopped, the boat is driven at the speed of the current by the friction force of water on the hull .

1. Establish a state model of the system. For simplicity, we can assume that :
 - a) the boat is a rigid body of constant mass ;
 - b) the stretch of water is almost horizontal and gravity does not influence the movement of the boat ;
 - c) the force exerted by the current is locally applied at the center of mass of the boat (we neglect the fact that the current force can be exerted in variable ways in diverse points of the hull)
2. What should be the thrust capacity of the engine to have the guarantee that the boat will be able to go against the current ?

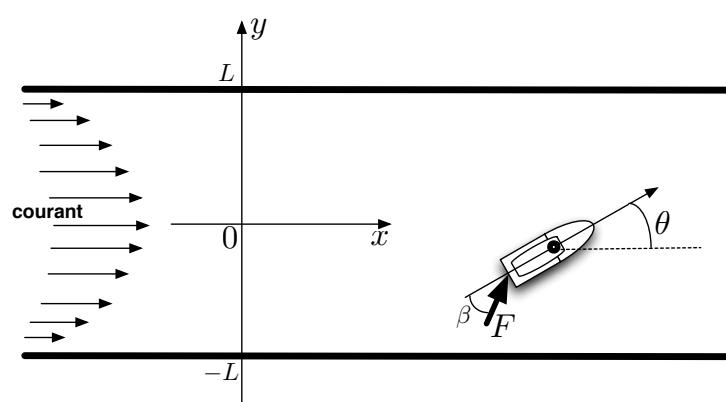


FIGURE 2.11 – A boat

Chapitre 3

Systèmes électriques et électromécaniques

This chapter deals with the modelling of systems whose dynamics are essentially characterised by the presence of electrical currents, i.e. by the movement of electrical charges in conductive materials (metal wires for example). We will first study the [putting in equation](#) of the state model of electrical networks. We will then study electromechanical systems (especially electrical machines) that combine in a unified description the state equations of electrical networks with those of mechanical systems such as we have presented them in the previous chapter.

3.1. Electrical systems

An electrical system is defined as a black box equipped with terminals which are electrical contact points, each of them subject to a voltage V_i and letting a current i in (see fig. 3.1).

The system behavior is given by the set of all possible trajectories $(I_1(t), V_1(t), I_2(t), V_2(t), \dots, I_k(t), V_k(t))_{t \in \mathbb{R}}$ for the system. This set of trajectories has symmetries. Indeed, electricity laws teach us that potentials are only defined up to a constant. In other words, if

$(I_1(t), V_1(t), I_2(t), V_2(t), \dots, I_k(t), V_k(t))_{t \in \mathbb{R}}$ is a possible trajectory, then

$(I_1(t), V_1(t) + V, I_2(t), V_2(t) + V, \dots, I_k(t), V_k(t) + V)_{t \in \mathbb{R}}$ is also a possible trajectory. Furthermore, in almost all the circuits produced today, there is no accumulation of electrical charge : the system remains electrically neutral, which implies that $I_1(t) + I_2(t) + \dots + I_k(t) = 0$ at all time t .

Note that the power transmitted to the circuit by the environment at time t is given by $\sum_i V_i(t)I_i(t)$. This formula has an unequivocal physical sense thanks to

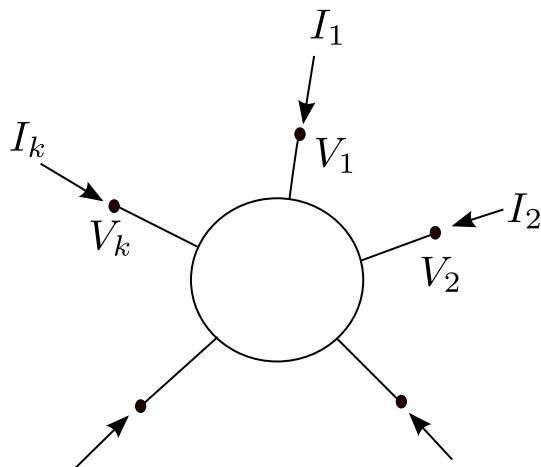


FIGURE 3.1 – Electrical system

the two relations above : translating all voltages by a constant does not change the received power.

The simplest electrical circuits only have two terminals, and that is why they are called dipoles. In that case, two variables are enough to characterise the trajectories : the potential difference, or voltage, $v(t) = V_1 - V_2$ and the current $i(t) = I_1(t) = -I_2(t)$ (see fig. 3.2). The direction of the current and voltage are chosen conventionally. The voltage represents thus the energy required to move a unit of electric charge through the dipole. We can also have tripoles (three terminals systems, such as transistors) or quadrupoles (transformers by example).

In this course, we will only consider circuits whose behaviour (the set of all possible trajectories) can be described by a state model as defined in Chapter 1, where each external variable (voltage or current) is taken either as an input u_i , or as an output y_i . State variables can be voltages, currents, flows or charges, as we will see. This type of state model limited to differential equations is restrictive, in the sense that it forbids us to model contentedly delay elements, whose behavior satisfies a delay equation of the type $y(t) = u(t - 1)$ (thus not differential).

3.2. Elementary dipoles

We consider two types of elementary dipoles : impedances and sources.

Impedances

1. *Resistors.* Resistors are elements that convert electrical energy into heat. They are represented by the symbol of figure 3.3 and are characterised by an algebraic relationship between the voltage $v(t)$ and the current

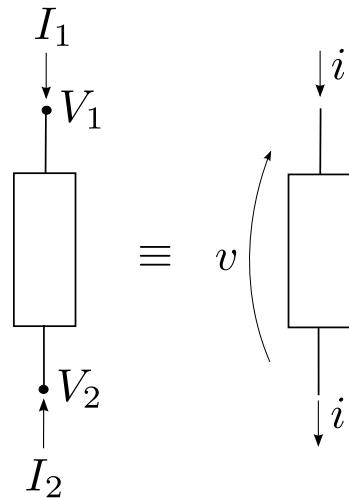


FIGURE 3.2 – Electrical dipole

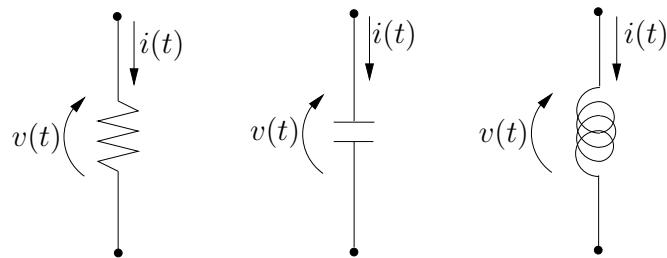


FIGURE 3.3 – Impedances : resistor, capacitor, inductor

$i(t)$:

$$r(v(t), i(t)) = 0.$$

In the case of a linear resistor, this relationship is particularized as follows (Ohm's law) :

$$v(t) = Ri(t).$$

2. *Capacitors.* Capacitors are elements which accumulate electric charges. They are represented by the symbol of figure 3.3 and are characterised by the following relationship between the charge $Q(t)$ and the current $i(t)$:

$$i(t) = \frac{dQ(t)}{dt}.$$

The charge $Q(t)$ is a function of the tension : $Q(v(t))$. This relationship can also be written as follow :

$$i(t) = c(v(t)) \frac{dv(t)}{dt} \quad \text{or} \quad c(v) \triangleq \frac{\partial q}{\partial v}.$$

In the case of a linear capacitor, this relationship can be particularized as follow :

$$Q(t) = Cv(t) \quad \text{or} \quad i(t) = C \frac{dv(t)}{dt}.$$

3. *Inductors.* Inductors are elements which store the energy of a magnetic field. They are represented by the symbol of figure 3.3 and are characterised, according to Faraday's law, by the following relationship between the magnetic field $\phi(t)$ and the tension $v(t)$:

$$v(t) = \frac{d\phi}{dt}. \quad (3.1)$$

We say that the voltage $v(t)$ is *induced* by the flow $\phi(t)$, and that is where the name of inductor comes from. In a general way, this variation in flow can be generated by a magnetic material moving in the vicinity of the inductor, or by a varying electrical current flowing in a conductor located near the inductor. In this section, we will only consider the particular case of *self-inductances*, where the flow is produced only by the current flowing through the dipole itself. In that case, the flow is a function of the current : $\phi(i(t))$ and the relationship (3.1) can be written as :

$$v(t) = l(i(t)) \frac{di(t)}{dt} \quad \text{or} \quad l(i) \triangleq \frac{\partial \phi}{\partial i}.$$

In the case of a linear (self-)inductance, this relationship can be particularized as :

$$\phi(t) = Li(t) \quad \text{or} \quad v(t) = L \frac{di(t)}{dt}.$$

Sources

1. *Tension sources*, represented by the symbol of figure 3.4, are dipoles defined by the tension $v(t)$ independently of the current they produce.
2. *Current sources*, represented by the symbol of figure 3.4, are dipoles defined by the current $i(t)$ that they produce regardless of their terminal voltage.

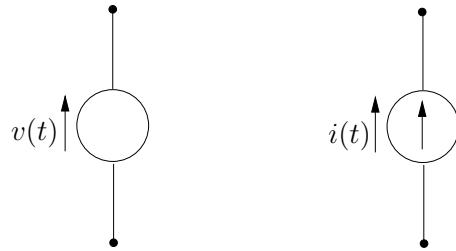


FIGURE 3.4 – Tension and current sources

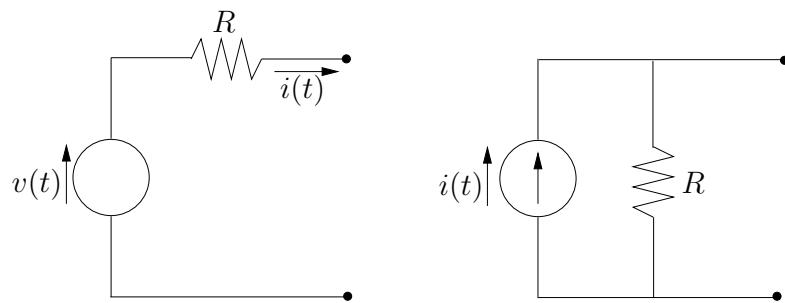


FIGURE 3.5 – Sources with internal resistance

It is important to understand that impedances and sources are ideal conceptual models that have no physical existence. The various elements of which the actual electrical circuits are constituted as for example coils, capacitors or batteries are modeled in practice by suitable assemblies of inductors and sources. For example, a source of tension or of current is always modeled with its unavoidable internal resistance, as indicated on figure 3.5.

3.3. Equivalency between open and closed systems

We have seen an electrical system as “open”, i.e. equipped with inputs and outputs, which are currents or tensions. Adding sources, however, allows us to consider an equivalent closed system. For example, a dipole whose tension $v(t)$ is the input, may be closed by a variable tension source that precisely delivers the tension $v(t)$ (see figure ??). However, for the sake of the analysis, it is comfortable to return to a closed graph.

If the dipole is adapted to supply energy to the outside ($v(t)i(t) > 0$), then it can also close the circuit by the load supposed to absorb the energy, often modeled like a (great) load resistance. For example, an engine is rather of an inductive nature.

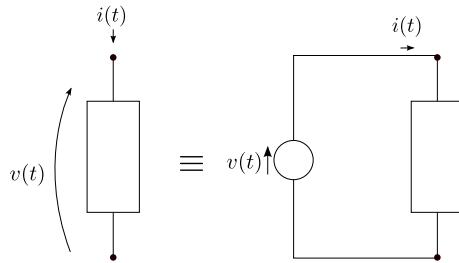


FIGURE 3.6 – Equivalence between an open circuit and closed circuit : the case where a dipole where the input is tension and the output power.

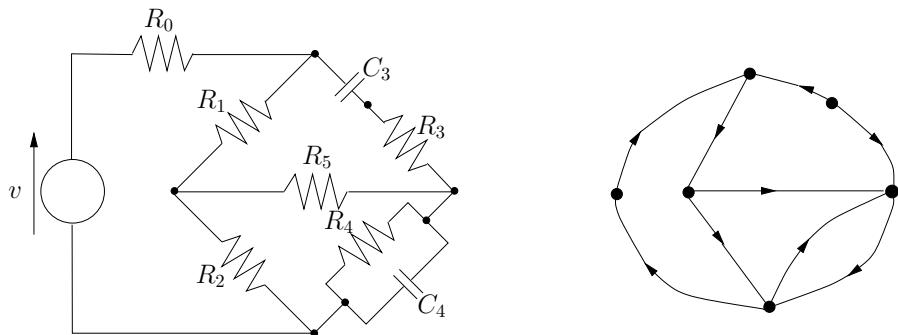


FIGURE 3.7 – Impedance bridge

3.4. Electrical networks and the setting in equation of the state model

Complex electrical systems are generally created by the interconnection of a defined number of elementary systems. In this chapter we will consider networks designed by elementary dipoles (impedances and sources). It is clear that such a network has a graph representation the branches of which are defined by the dipoles. An example of such a network and its graph is given in figure 3.7, representing an impedance bridge. The arrows on the graph indicate the conventional flow chosen by the current in each of the branches.

We are considering closed associated electrical networks (with sources of tension and/or power, even with load resistors) with N nodes and M branches. A graph is considered associated when there is link between any two nodes.

Let's define two concepts that will be used subsequently : meshes and sections of an electrical network.

A *mesh* is a cycle, i.e. a closed path without repeating nodes.

A *section* is an ensemble of branches the extraction of which will split an associated network in at least two associated separate sub-networks.

The organization of a electrical network state model is based on Kirchhoff's laws, as follows :

- Kirchhoff's law on currents : the algebraic sum of the currents in the branches incident to node a is equal to zero.
- Kirchhoff's law on tensions : the algebraic sum of tensions in a mesh is equal to zero.

If the graph is closed, the sum of Kirchhoff's equations on current of all the N nodes gives the trivial equality $0 = 0$, since each current occurs twice with an opposite sign. There are $N - 1$ Kirchhoff's independent equations on the currents. For any section, we also find out that the sum of currents on the brances of the section is equal to zero ; this result is acquired by summing Kirchhoff's current equations on all nodes on one side of the section.

In order to find the number of Kirchhoff's independent equations on meshes in a closed network, let's consider the tree at the origin of the graph, i.e. an associated sub-graph, without any mesh or signal breakdown on any of the nodes. Such a tree has $N - 1$ edges. Every network branch added to this tree creates one and only one independent mesh. Thus, we can create $M - N + 1$ mesh equations which will be independent.

Variables of a network state are the currents in some inductors and tensions at the terminals of certain capacitors. In order to establish a state model for a network, we proceed as follows :

1. Write $N - 1$ Kirchhoff's equations for all currents
2. Write $M - N + 1$ Kirchhoff's linearly independent equations for all tensions.
3. Write definition impedance laws corresponding to the tensions or currents prevailing in Kirchhoff's equations.
4. Eliminate redundant tensions and currents.

It is interesting to note that if a circuit is comprised of several successive branches creating a dipole composed of several elementary dipoles placed in series, Kirchhoff's current laws on intermediary nodes will be negligible. Similarly, Kirchhoff's mesh equations will only take the total dipole tension into consideration, and not isolated elementary tensions. As far as writing Kirchhoff's equations, one can consider this composed dipole as a single branch and two nodes.

If the network is comprised of a mesh of capacitors, it is clear that the tension at the terminals of one of those capacitors can be written as the (signed) sum of the others, thus creating a linear relation in state variables. On the contrary, when a network does not contain a mesh of capacitors, all tensions at the capacitors' terminals are independent state variables. Similarly, any inductors section imposes a linear relationship between currents crossing the section, since their (signed) sum is equal to zero. One of these currents is not considered an independent state variable. On the contrary, when a network is not comprised of inductors section, all currents in inductors are independent state variables.

Example 3.1. Rectifier Circuit with Low Pass Filter

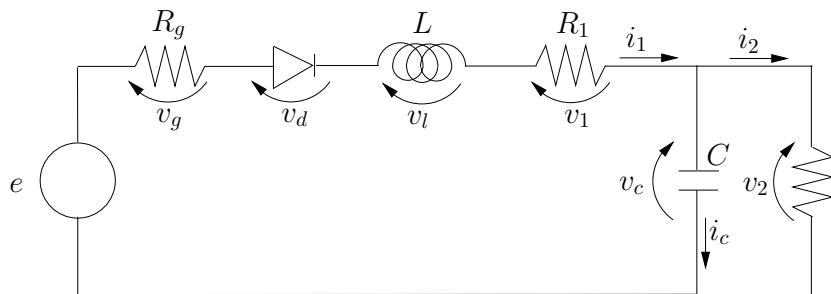


FIGURE 3.8 – Rectifier Circuit

Figure 3.8 represent a diode rectifier circuit with a filter composed of a capacitor and an inductor. The diode is a non-linear resistor where the current-tension characteristic is expressed as follows :

$$i = i_0 [e^{\frac{v}{\alpha}} - 1]$$

where α is a constant proportional to temperature and inversely proportional to the electron charge, whereas i_0 defines the leakage current of the diode. It is clear that the circuit does not contain any capacitors mesh or inductors section. So this state model will have two state variables : tension v_c at the capacitor terminals et current i_1 in the inductor.

The circuit is comprised of $N = 6$ nodes and $M = 7$ branches. However, for convenience, we can group dipoles in series in one branch which leaves us with $N = 2$ nodes and $M = 3$ branches. In order to establish the state model of the system, we would then write $N^{\vee}1 = 1$ Kirchhoff equation for the currents :

$$i_c - i_1 + i_2 = 0,$$

And $M - N + 1$ where 1 is self-inductance. In matrix notation that gives $1 = 2$

Kirchhoff equations for the tensions :

$$\begin{aligned} v_c - v_2 &= 0, \\ v_g + v_d + v_\ell + v_1 + v_c - e &= 0. \end{aligned} \quad (3.2)$$

These equations are completed by equations of the definition of the different elements of the circuit :

$$\begin{aligned} v_g &= R_g i_1, \\ v_d &= \alpha \ln \frac{i_0 + i_1}{i_0}, \\ v_\ell &= L \frac{di_1}{dt}, \\ i_c &= C \frac{dv_c}{dt}, \\ v_1 &= R_1 i_1, \\ v_2 &= R_2 i_2. \end{aligned}$$

By eliminating the 7 variables $i_2, i_c, v_g, v_d, v_\ell, v_1, v_2$ between the 9 equations, we easily get the following equations :

$$\begin{aligned} R_g i_1 + \alpha \ln \frac{i_0 + i_1}{i_0} + L \frac{di_1}{dt} + R_1 i_1 + v_c - e &= 0, \\ C \frac{dv_c}{dt} - i_1 + \frac{v_c}{R_2} &= 0. \end{aligned}$$

By defining the state and input variables as follows :

$$x_1 = i_1, \quad x_2 = v_c, \quad u = e, \quad (3.3)$$

We finally get the following state equations :

$$\begin{aligned} \dot{x}_1 &= -\frac{R_1 + R_g}{L} x_1 - \frac{\alpha}{L} \ln \left(\frac{i_0 + x_1}{i_0} \right) - \frac{1}{L} x_2 + \frac{1}{L} u, \\ \dot{x}_2 &= \frac{1}{C} x_1 - \frac{1}{R_2 C} x_2 \end{aligned}$$

■

When the network contains capacitor meshes or inductor cuts, one searches for a forest (sub-graph without mesh) of maximum size and an inductor non-cut of maximum size. The number of selected capacitors and inductors represent the number of independent state variables.

One can easily demonstrate that is like finding a tree that contains the greatest possible number of capacitors, so that the “co-tree” (i.e. the complementary sub-network of the tree) contains the greatest number of inductors. The number of independent state variables, thus, is the sum of the number of capacitors and number of inductors of the co-tree.

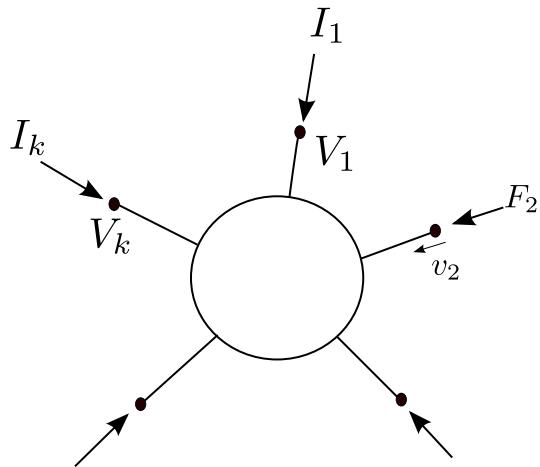


FIGURE 3.9 – Electromechanical system

3.5. Electromechanical systems

Could a mechanical system be defined similarly with an electric system, whose terminals are subjected, not with currents and voltages, but with forces F_i and speeds $v_i = \dot{q}_i$ (not to be confused with the symbol of tension). Translation all speed $v_i(t)$ vers $v_i(t) + v$ do not modify the behavior of the system, by Galilean relativity. The sum of the forces $\sum_i F_i(t)$ in general worthless constantly in the mechanical systems of the engineer. The mechanical energy transmitted to the system under the conditions univocally define as $\sum_i F_i(t)v_i(t)$. Certain forces or speeds are selected like entries, and others like outputs. An example of a mechanical dipole is a combination of two forces $F_1(t) + F_2(t) = 0$ applying to different points which can be summarized at a couple T and an angular velocity ω ! (see Chapter 2)

An electromechanical system is of course defined as a black box whose certain variables are of electric nature and others of mechanical nature (see fig. 3.9).

For example an electromechanical quadrupole whose variables are $v(t)$, $i(t)$, $T(t)$, $w(t)$ can model a generator if $T\omega < 0$, $vi > 0$ electrical energy converted into mechanical energy or an engine si $T\omega > 0$, $vi < 0$ (mechanical energy converted into electrical energy).

Concretely, an electromechanical system often arises as an articulated mechanical system (with p degrees of freedom) which can be built partially in a magnetic material and whose certain bodies carry one or more inductive electrical circuits (coils, winding, ...). The equations constitutive of an electromechanical system consequently comprise a mechanical part and an electric part. The two sections that follow explain, with some electromagnetism refreshers, how the electromagne-

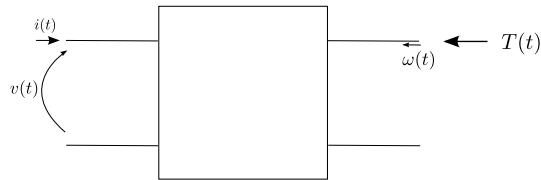


FIGURE 3.10 – Electromechanical quadrupole

tic variables (loads, flow, currents, tensions) and mechanics (generalized positions, speeds, forces) can interact.

3.6. Influence of a mechanical movement on the electric quantities

The Faraday's law $E_i = \frac{d\phi_i}{dt}$ describes the electromotive force (tension) undergone by an inductance crossed by one magnetic flow ϕ_i . Up to now it was supposed that this flow was created only by the current I_i going through it (self-inductance). For a linear relation $\phi_i = L_i I_i$, one obtains the law constitutive $V_i = L_i \dot{I}_i$.

Actually each current I_j generates a magnetic field and thus a flow in the inductance i . This flow depends on I_j , but also on the relative position of the conductors which carry these currents, possibly varying with time because of the mechanical movement of the parts which support the conductors. Generally only the currents which cross an inductance generate a non-negligible magnetic field. If ϕ_i depends linearly of each I_j , one can write

$$\phi_i = \sum_j L_{ij} I_j, \quad (3.4)$$

Where L_{ii} is self-inductance. In matrix notation that gives

$$\Phi = L(q) I, \quad (3.5)$$

where $L(q)$ is a symmetrical matrix of inductance depending on the generalized positions of the mechanical part q . By combining this with the equation of Faraday for each inductance, one arrives at the equations of state

$$E = \sum_k \frac{\partial L}{\partial q_k} \dot{q}_k I + L \dot{I}, \quad (3.6)$$

which mix positions, speeds, currents and tensions.

The electric component of the electromechanical systems discussed in this chapter are often simply reduced to a set of inductances, each one in series with

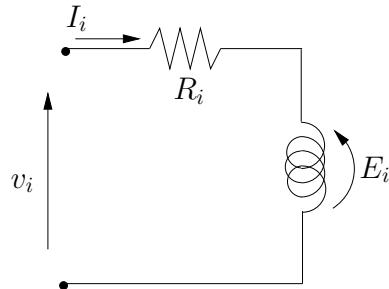


FIGURE 3.11 – Basic circuit

a resistance representing inevitable internal resistance or one resistance of load, which represents the useful consumption of electrical energy possibly generated by the system (see G. 3.11). In this case, voltage at the terminals of the circuits are written in a matrix way

$$V = RI + E \quad (3.7)$$

where R is the diagonal matrix $\text{diag}\{R_i, i = 1, m\}$ and the vectors V, I and E are defined as follows :

$$\begin{aligned} V^T &= (v_1, v_2, \dots, v_m), \\ I^T &= (I_1, I_2, \dots, I_m), \\ E^T &= (E_1, E_2, \dots, E_m). \end{aligned}$$

3.7. Creation of a mechanical movement by electro-magnetic fields

A magnetic field exerts a force known as Force of Lorentz on each carrier of moving load. By integrating this force on all the charge carriers set in movement by an electric current in a conductor one obtains a macroscopic force, known as Laplace force.

In general, one can show that the Laplace force on the coordinate q_k is

$$F_k = \frac{1}{2} \sum_i \frac{\partial \phi_i}{\partial q_k} I_i = \frac{1}{2} I^T \frac{\partial L}{\partial q_k} I, \quad (3.8)$$

the last equality concerning linear inductances. In the latter case case, one can directly assume the expression F_k supposing that the force originates from

a potential energy contained in inductances which generate the magnetic fields, expressed by $\frac{1}{2}I^T L(q_k)I$.

Example 3.2. An electromechanical system of positioning.

The flow diagram of an electromagnet used for a positioning of precision is indicated in figure 3.12. The electromagnet A is equipped with an inductor coil in

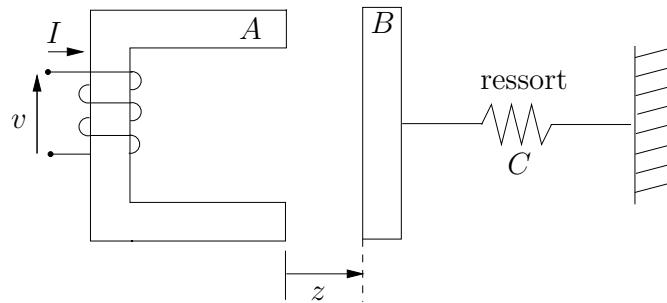


FIGURE 3.12 – An electromechanical system of positioning

which circulates an inductor current I . The metal piece B is mobile and subject to a linear traction force by the spring C

The magnetic flow varies proportionally with the current I and inversely proportionally to the distance z in the gap between A and B .

$$\phi(I, z) = \frac{\alpha I}{1 + \beta z}.$$

This is an instance of the magnetic law of Ohm (or law d'Hopkinson) which expresses the magnetic flow (equivalent of the current flow of electric charges) like the ratio between the magnetomotive flow (proportional of the current) and the reluctance (resistance of material to the magnetic flow) which is proportional to the length of the circuit.

The tension inducted in the circuit is thus written :

$$\begin{aligned} e &= \frac{d\phi}{dt} = \frac{\partial\phi}{\partial I} \frac{dI}{dt} + \frac{\partial\phi}{\partial z} \frac{dz}{dt} \\ &= \frac{\alpha}{1 + \beta z} \frac{dI}{dt} - \frac{\alpha\beta I}{(1 + \beta z)^2} \frac{dz}{dt}. \end{aligned}$$

The electromagnetic force of origin being exerted on the moving part is written :

$$F_{em} = \frac{1}{2} I \frac{\partial\phi}{\partial z} = -\frac{\alpha\beta}{2} \left(\frac{I}{1 + \beta z} \right)^2.$$

In agreement with the physical intuition, one observes that this force tends to bring closer the part B to the electromagnet whatever the direction of the current I . One can then write the dynamic equations of the system.

— *Mechanic equation*

$$m\ddot{z} = k(z_o - z) + F_{em} \quad (3.9)$$

$$= k(z_o - z) - \frac{\alpha\beta}{2} \left(\frac{I}{1 + \beta z} \right)^2 \quad (3.10)$$

where m indicates the mass of the constant part B , k the contant of the rearward movement of the s_0 spring and the position of the spring at rest

— *Electric equation*

$$V = RI + e \quad (3.11)$$

$$= RI + \frac{\alpha}{1 + \beta z} \dot{I} - \frac{\alpha\beta I}{(1 + \beta z)^2} \dot{z} \quad (3.12)$$

where R indicated the resistance of the electric circuit.

By introducing the following definitions of the state variables

$$x_1 = z, \quad x_2 = \dot{z}, \quad x_3 = I$$

and of the variable of entry :

$$u = V,$$

Finally :

$$\dot{x}_1 = x_2, \quad (3.13)$$

$$\dot{x}_2 = \frac{k}{m}(z_o - x_1) - \frac{\alpha\beta}{2m} \left(\frac{x_3}{1 + \beta x_1} \right)^2, \quad (3.14)$$

$$\dot{x}_3 = \frac{\beta x_2 x_3}{1 + \beta x_1} - \frac{R}{\alpha}(1 + \beta x_1)x_3 + \frac{1 + \beta x_1}{\alpha}u. \quad (3.15)$$

■

3.8. Rotating electric machines

The rotating electric machines constitute a particular category of electromechanical systems formed by two bodies. The first one, called *rotor*, is in rotation around an axe whose position is fixed to the second one, called *stator*. These two

pieces are provided with different **field windings** whose utility is to realise electro-mechanical conversions of which the machines are the centre.

When the stator itself is fixed to the inertial landmark, an electrical machine has only one mechanical degree of freedom : the rotor's rotation angle, written θ . The mechanical part of the system shrinks therefore to a scalar equation of the form :

$$J\ddot{\theta} + h(\dot{\theta}) = T_{em} + T_a \quad (3.16)$$

where $h(\dot{\theta})$ represents the friction torque, T_{em} the electromagnetical torque and T_a all the other external torques applied to the rotor.

On the other hand, the electrical part of the dynamic has the general form (3.7) :

$$V = RI + E.$$

In most running machines, when the magnetical saturation effects are negligible (or neglected), it is possible to present the flows ϕ_{ij} by an expression of the form :

$$\phi_{ij} = L_{ij}(\theta)I_i$$

which is linear to the inductor current I_i but that depends on the angular position θ of the rotor, following a law $L_{ij}(\theta)$ usually periodic. The inductance matrix (symmetrical) is defined as :

$$L(\theta) \triangleq [L_{ij}(\theta)]$$

and its derivative to θ is :

$$K(\theta) \triangleq \frac{\partial L(\theta)}{\partial \theta}.$$

Then, the use of the theory that was introduced in the last sections leads to general equations of electromechanical coupling, whose form is the following :

$$E = L(\theta) \frac{dI}{dt} + \frac{d\theta}{dt} K(\theta)I, \quad (3.17)$$

$$T_{em} = \frac{1}{2} I^T K(\theta) I. \quad (3.18)$$

By combining the equations (3.16),(3.7),(3.17) and (3.18) we get the general model of the electrical machines :

$$\begin{aligned} L(\theta)\dot{I} &= -\omega K(\theta)I - RI + V, \\ \dot{\theta} &= \omega, \\ J\dot{\omega} &= \frac{1}{2} I^T K(\theta) I - h(\omega) + T_a. \end{aligned}$$

It is this general model that is on the basis of the particular models establishment in the applications. Often, but it not a standard, the voltage vector V or the torque T_a are parameterized by well selected entrance variables that represent the external influence on the machine's behaviour. Here is an example. Other examples are provided in the exercises.

Example 3.3. Elementary machine with two windings

Let's consider an electrical machine whose rotor and stator are concentric cylinders with a winding on the stator and another on the rotor (Fig. 3.13). The statorical and rotorical auto-inductances L_s and L_r are constants. The mutual inductance L_{sr} is a cosinusoidal periodical function of angle θ

$$L_{sr}(\theta) = L_o \cos \theta.$$

Matrices $L(\theta)$ and $K(\theta)$ are written as follows :

$$L(\theta) = \begin{pmatrix} L_s & L_o \cos \theta \\ L_o \cos \theta & L_r \end{pmatrix} \quad K(\theta) = \begin{pmatrix} 0 & -L_o \sin \theta \\ -L_o \sin \theta & 0 \end{pmatrix}.$$

Vectors of current and induced tensions are noted :

$$I = \begin{pmatrix} I_s \\ I_r \end{pmatrix} \quad E = \begin{pmatrix} e_s \\ e_r \end{pmatrix}.$$

Electromechanical coupling equations (3.17), (3.18) particularize themselves by :

$$\begin{aligned} e_s &= L_s \dot{I}_s + L_o \cos \theta \dot{I}_r - \dot{\theta} L_o \sin \theta I_r, \\ e_r &= L_r \dot{I}_r + L_o \cos \theta \dot{I}_s - \dot{\theta} L_o \sin \theta I_s, \\ T_{em} &= -L_o \sin \theta I_s I_r. \end{aligned}$$

The rotorical circuit is being supplied by a constant current source I_r . Such a machine can be used either as a generator (that transforms the mechanical power provided by the external torque T_a in electrical power delivered by the statoric electromotive power e_s), or as an engine (that transforms the electrical power delivered to the stator by the source v_s in a mechanical power delivered by the electromagnetical torque T_{em}).

The system has three state variables :

$$\begin{aligned} x_1 &= I_s, \\ x_2 &= \theta, \\ x_3 &= \dot{\theta}, \end{aligned}$$

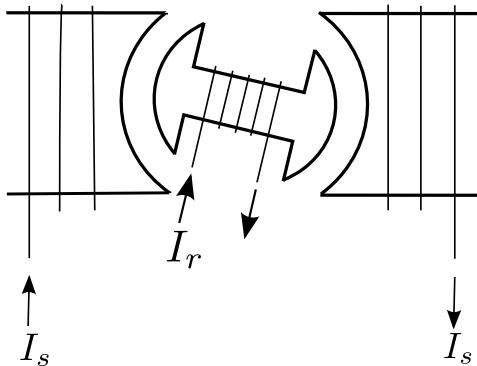


FIGURE 3.13 – Elementary machine with two windings

and two input variables :

$$\begin{aligned} u_1 &= v_s, \\ u_2 &= T_a. \end{aligned}$$

The state-spaced model of the system is noted as follows :

$$\begin{aligned} L_s \dot{x}_1 &= L_o I_r x_3 \sin x_2 - R_s x_1 + u_1, \\ \dot{x}_2 &= x_3, \\ J \dot{x}_3 &= -h(x_3) - L_o I_r x_1 \sin x_2 + u_2. \end{aligned}$$

In the case of generator mode, we can also close the electrical part of the circuit with a resistance of charge R_L , we can then set $u_1 = R_L x$ and we obtain a state-spaced model with only one input $u_2 = T_a$. ■

3.9. Direct current machines

Direct current machines (fig. 3.14) generally include a stator winding and a rotor one.

The stator winding is the inductive circuit whose current is noted by I_s . The rotor winding is the [induced circuit](#) whose current is written by I_r .

So a direct current machine (DC machine) seems similar to the elementary two-winding machine we studied in the previous section. However, there is a fundamental difference : a DC machine is designed with a [commutation](#) system whose effect is to modify the electromechanical coupling. A detailed description of this commutation effect on the coupling equations is beyond the scope of this book. So

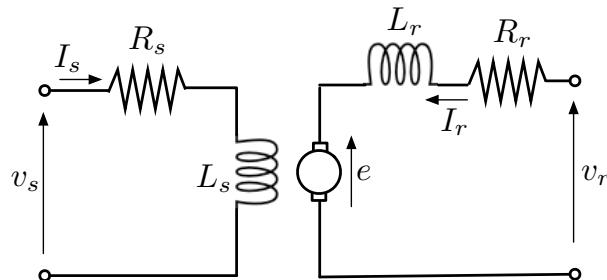


FIGURE 3.14 – Direct current machines

we restrict ourselves to give the results. When the effect of magnetic saturation are negligible and when the commutation doesn't generate significant non-linearities, the equations of electromechanical coupling of a DC machine are written as the following multilinear form :

$$\begin{aligned} e_s &= L_s \frac{dI_s}{dt}, \\ e_r &= L_r \frac{dI_r}{dt} + \frac{d\theta}{dt} K_e I_s, \\ T_{em} &= K_m I_r I_s. \end{aligned}$$

Let's notice the the **resemblance, but not the similarity**, of those equations with the general one (3.17), (3.18) of electrical machines without commutation which have been previously defined. We notice in particular the default of symmetry between the form of e_s and the one of e_r which is precisely due to commutation.

According to the way they are build and implemented, direct current machines can be used as motors or as generators. Here are some usual examples of applications.

General model for DC machine

The system has four state variables :

$$\begin{aligned} x_1 &= \theta, \\ x_2 &= \dot{\theta}, \\ x_3 &= I_s, \\ x_4 &= I_r. \end{aligned}$$

The system inputs are the voltages at the terminals of the inductor circuit v_s and

of the induced circuit v_r as well as the external torque T_a :

$$u_1 = v_s,$$

$$u_2 = v_r,$$

$$u_3 = T_a.$$

The state-spaced model can be written as follows :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ J\dot{x}_2 &= -h(x_2) + K_m x_3 x_4 + u_3, \\ L_s \dot{x}_3 &= -R_s x_3 + u_1, \\ L_r \dot{x}_4 &= -R_r x_4 - K_e x_2 x_3 + u_2.\end{aligned}$$

DC engine controlled by stator.

It's a DC engine whose rotor current is provided by a constant current source (figure 3.15) :

$$I_r = \text{constant}$$

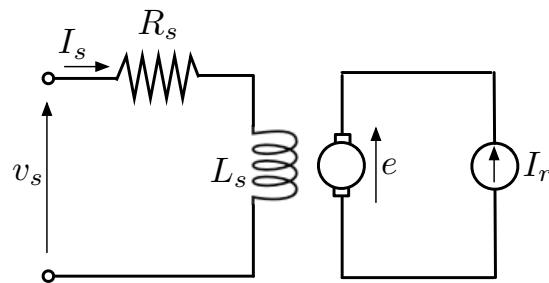


FIGURE 3.15 – DC engine controlled by the stator

The system has three state variables :

$$x_1 = \theta,$$

$$x_2 = \dot{\theta},$$

$$x_3 = I_s.$$

The system inputs are the voltages at the terminals of the stator circuit v_s and the external torque T_a :

$$u_1 = v_s,$$

$$u_2 = T_a.$$

The state-spaced model is written as follows :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ J\dot{x}_2 &= -h(x_2) + K_m I_r x_3 + u_2, \\ L_s \dot{x}_3 &= -R_s x_3 + u_1.\end{aligned}$$

DC engine controlled by the rotor.

It's a DC engine whose stator current is provided by a constant current source (figure 3.16) :

$$I_s = \text{constant}$$

The system has three state variables :

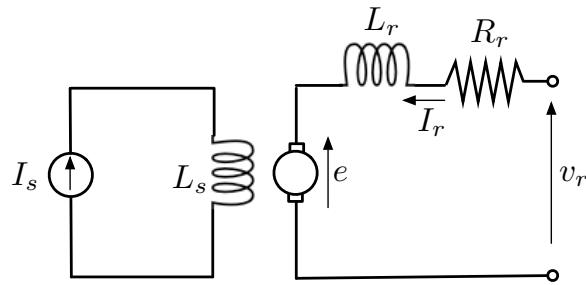


FIGURE 3.16 – DC engine controlled by rotor

$$\begin{aligned}x_1 &= \theta, \\ x_2 &= \dot{\theta}, \\ x_3 &= I_r.\end{aligned}$$

The system inputs are the voltage at the terminals of the rotor circuit v_r and the external torque T_a :

$$\begin{aligned}u_1 &= v_r, \\ u_2 &= T_a.\end{aligned}$$

The state-spaced model is written as follows :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ J\dot{x}_2 &= -h(x_2) + K_m I_s x_3 + u_2, \\ L_r \dot{x}_3 &= -R_r x_3 - K_e I_s x_2 + u_1.\end{aligned}$$

DC generator. The function of a generator is to convert mechanical power in electrical power **delivered** by the rotor circuit on any load impedance of Z_L .

When this impedance is resistive (R_L), the system has three state variables (figure 3.17) :

$$\begin{aligned}x_1 &= I_s, \\x_2 &= I_r, \\x_3 &= \omega.\end{aligned}$$

The system inputs are the voltage at the terminals of the stator circuit v_s and the

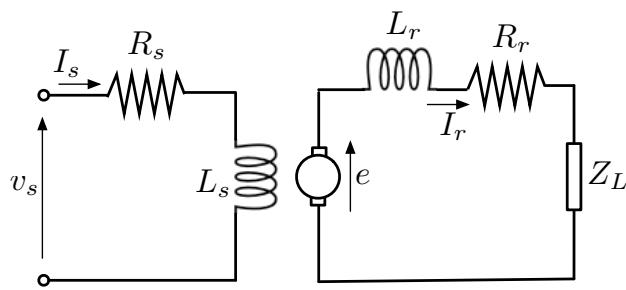


FIGURE 3.17 – DC generator

torque T_a :

$$\begin{aligned}u_1 &= v_s, \\u_2 &= T_a.\end{aligned}$$

The state-spaced model is written as follows :

$$\begin{aligned}L_s \dot{x}_1 &= -R_s x_1 + u_1, \\L_r \dot{x}_2 &= -(R_r + R_L)x_2 - K_e x_3 x_1, \\J \dot{x}_3 &= -h(x_3) + K_m x_1 x_2 + u_2.\end{aligned}$$

3.10. Exercices

Exercise 3.1. Linear circuit

Establish a state-spaced model of the linear circuit shown in figure 3.18 with the applied voltage e and the adjustable resistance r as input variables. \square

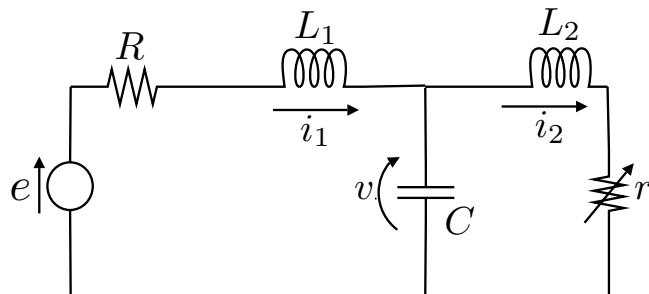


FIGURE 3.18 – Linear circuit

Exercise 3.2. Voltage doubler bridge

The electrical diagram of a voltage doubler bridge is shown in figure 3.19. Establish the state-spaced model of the system assuming that all the dipoles are linear except the diodes. \square

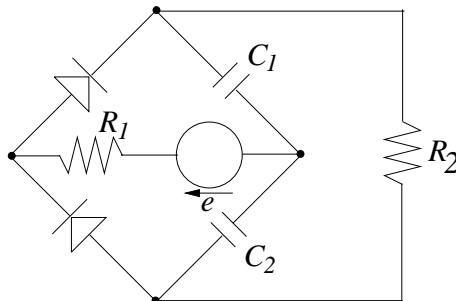


FIGURE 3.19 – Voltage doubler bridge

Exercise 3.3. Transformer The electrical diagram of a transformer is shown in figure 3.20.

1. Has this electrical network some capacity **mesh** and/or inductance **cuts**? Explain your answer.
2. Establish the state-spaced model assuming that the dipoles are linear. \square

Exercise 3.4. A circuit with a tunnel diode

A electrical circuit is described by the following state equations :

$$\begin{aligned} C\dot{x}_1 &= -h(x_1) + x_2, \\ L\dot{x}_2 &= -x_1 - Rx_2 + u. \end{aligned}$$

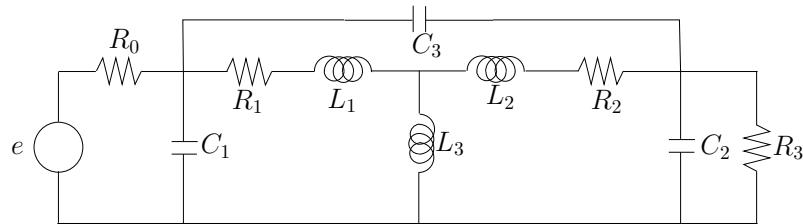


FIGURE 3.20 – Transformer equivalent circuit

x_1 is the voltage to the terminals of a linear capacity, x_2 is the current in a linear inductance, $h(x_1) = x_1^3 - 10x_1^2 + 25x_1$ is the characteristic of a tunnel diode. Establish the circuit diagram. \square

Exercise 3.5. Electromechanical converter

The device illustrated in figure 3.21 transforms an electrical power provided by the voltage source in a mechanical movement of translation. It is made of a cylindrical steel core moving longitudinally inside a solenoid. \square

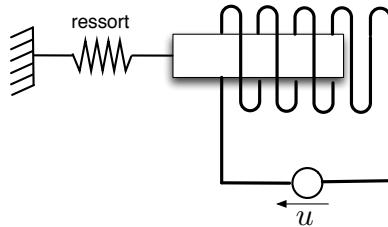


FIGURE 3.21 – electromechanical converter

Suggest a state-spaced model of this system according to the following modeling assumptions :

1. The core movement is forced to be linear by a slide. The friction can be considered as viscous and linear.
2. The core is shorter than the solenoid.
3. The flow in the solenoid is a linear function of the length h of the core part which is inside the solenoid.
4. The flow is a saturated monotonically increasing function of the current.
5. The spring is linear. \square

Exercise 3.6. Clockwork motor A small motor used in watchmaking is presented in figure 3.22. The stator is provided with field winding whose inductance $L(\theta)$ is a sinusoidal function of the rotor angular position θ .

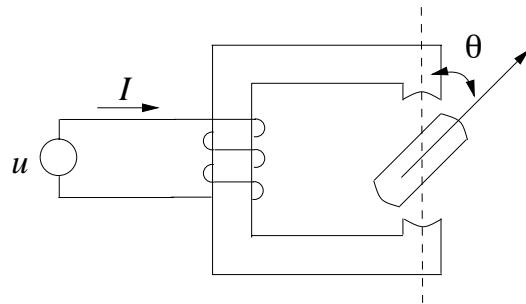


FIGURE 3.22 – Clockwork motor

1. Suggest a model for the function $L(\theta)$ and compute the flow.
2. Establish the state-spaced model of the system assuming a linear viscous friction. The inputs are the voltage u at the terminals of the inductor and the load torque T_a .
3. What has to be the waveform of the input signal u when T_a is constant and so that the rotor rotates at constant speed ? \square

Exercise 3.7. Unipolar off-phase synchronous machine

This is a machine holding two stator windings (labelled a and b) arranged in quadrature and a rotor winding (labelled r). The self and mutual inductances of the stator windings vary depending on the angular position θ of the rotor according to the following laws :

$$\begin{aligned}L_a &= L_o + L_1 \cos 2\theta \\L_b &= L_o - L_1 \cos 2\theta \\L_{ab} &= L_1 \sin 2\theta\end{aligned}$$

The self inductance of the rotor L_r is constant. The mutual inductances between the rotor and the stator windings are expressed in terms of θ :

$$\begin{aligned}L_{ar} &= L_2 \cos \theta \\L_{br} &= L_2 \sin \theta\end{aligned}$$

1. Establish the coupling equations of the electromechanics system (see Section 3.4)
2. The rotor is supplied by a constant current source I_r . Establish the state model of this machine when it works as a motor (you might need to use the example 3.3). \square

Exercise 3.8. Elementary machine with two windings.

Let's consider the elementary machine with two windings working as a generator, such as it is described in the example 3.3. Denote how to revise the state equations under the following hypothesis :

1. The electric charge of the stator circuit is capacitive
2. The rotor winding is closed using a short-circuit. \square

Exercise 3.9. DC motor with self-excitation

A DC motor with self-excitation is designed such as the stator current and the rotor current are supplied by the same power source (see figure 3.23). Establish the state model of this system using the tension source u as the only input variable of this system. \square

Exercise 3.10. DC generator with self-excitation Let's consider a DC generator with self-excitation. The rotor tension induced at constant rate is a *strictly increasing and bounded* function depending on the excitation current $E(I_s)$ such as $E(0) > 0$. The generator provides a capacitive electric charge. The input control variable of the system is the angular velocity of the generator.

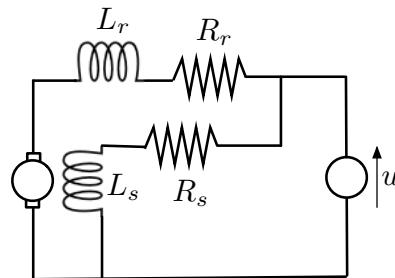


FIGURE 3.23 – DC motor with self-excitation

1. Propose an analytical form for the function $E(I_s)$.
2. Propose a state model for this system.
3. Justify the existence of a non-zero residual voltage $E(0)$. □

Exercise 3.11. DC motor with off-center load

A DC motor with independent excitations and controlled by the rotor current drives an off-center load (the motor shaft do not pass through the center of mass of the load : indeed, there is an unbalanced effect) through a transmission whose flexibility is significant. Propose a state model taking into account this features.

Exercise 3.12. DC-DC converters

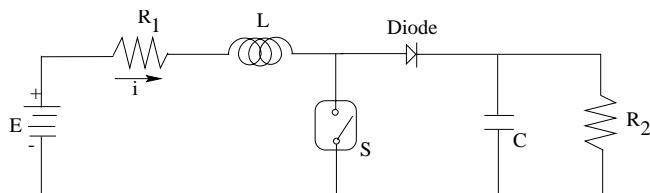


FIGURE 3.24 – DC-DC converter

The circuit depicted at the figure above describes a DC-DC converter. The device labelled "S" pictures an electronic switch of type MOSFET which is open and close periodically.

- Establish a state model of the system under the following modelling hypothesis :
- a) the voltage E of the supply battery is constant
 - b) both resistances R_1, R_2 , the inductance L and the capacitor C are linear
 - c) the input variable is the commutation frequency of the switch. □

Chapitre 4

Compartments systems

The notion of compartments system is used to specify a wide set of systems for which the dynamic can be described by balanced equations. It is used in many engineering fields (such as chemistry engineering, biomedical engineering or ecology), in economy and social sciences as well.

4.1. Definitions and notations

A compartment is a conceptual tank or box for which the content (matter, energy, money, population...) can be quantified. The symbolic notation used is depicted at figure 4.1 where q_{in} and q_{out} are respectively the filling and emptying flows of the compartment expressed in quantity of content by time unit. These flows are always *positive*, by convention.

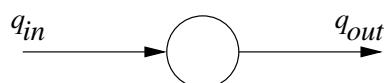


FIGURE 4.1 – Symbolic representation of a compartment.

A compartments system is made of one *network* of compartments interconnected and labelled 1 through n . To be clear, an example of system made of 3 compartments is shown at figure 4.2. The arrows specify the flows of content exchanged by the compartments in the network and with outside of the system.

In general, a compartments system is represented by an *oriented graph* whose nodes correspond to compartments and arcs to flows. The following notations are introduced :

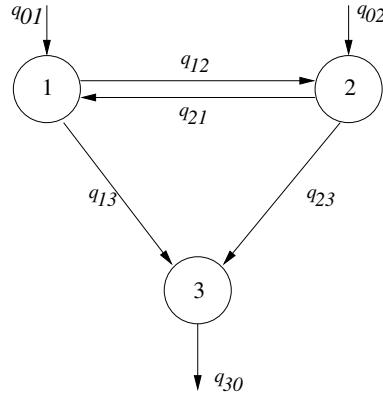


FIGURE 4.2 – An example of a compartments system graph.

x_i is the quantity of content in the compartment of indices i , ($i = 1, \dots, n$).

This quantity is always *positive*. Using a slight abuse of terms, x_i is used to depict the *level* of matter in the compartment i .

q_{ij} specify the flow flowing from compartment i towards the compartment j , ($i = 1, \dots, n; j = 1, \dots, n$). As mentioned above, it is a variable which is always *positive* by convention.

Definition 4.1. Open or close system

A system is **open** when there exists a possibility to exchange matter with outside of the system. In this case :

q_{io} specify the flow from compartment i towards the outside

q_{oi} specify the flow from the outside towards the compartment i

Otherwise, the system is said to be **close** : $q_{io} = q_{oi} = 0$ for all i . □

Definition 4.2. System connected to entrances and exits

A compartment i is *connected to an exit* if there is a path $i \rightarrow j \rightarrow k \rightarrow \dots \rightarrow \ell$ from this compartment ending in a compartment ℓ from which there is an outgoing flow $q_{\ell o}$. The system is *completely connected to the exits* (CCO) if each compartment is connected to an exit.

A compartment ℓ is *connected to an entry* if there is a path $i \rightarrow j \rightarrow k \rightarrow \dots \rightarrow \ell$ ending in this compartment and coming from a compartment i in which there is an entering flow q_{oi} . The system is *completely connected to the entries* (CCI) if each compartment is connected to an entry. □

4.2. State model

The balanced equation of each compartment (also called continuity equation)

$$\dot{x}_i = \sum_{j=0}^n q_{ji}(t) - \sum_{j=0}^n q_{ij}(t) \quad i = 1, \dots, n$$

is the basic statement to establish the state model of a compartments system. This equation tells us that the variation, per unit of time, of the quantity contained in a compartment is the difference between the sum of the entering flows (or debits) and the sum of the outgoing flows (or debits). In practice, of course, the flows which are structurally null are not explicitly in the equation ((4.1)).

Computing the equations of the state model of a compartments system required two fundamental aspects.

First of all, the structure of the graph related to the system determines the number and the structure of the balanced equations ((4.1)); the variables x_i are the state variables whereas the order of the model is the number n of compartments.

To complete the state model, the flows should be specified in terms of the state variables and input variables :

$$q_{ij}(x, u)$$

where x and u are, as usual, the vector of states and entries. This modelling is the point of the next section.

The general form of the state equations of a compartments system is the following :

$$\dot{x}_i = \sum_{j=0}^n q_{ji}(x, u) - \sum_{j=0}^n q_{ij}(x, u) \quad i = 1, \dots, n$$

In this model, the physical meaning of the state variables x_i is obvious : these are the quantities contained in each compartment. But, the input variables u can be of different natures, depending on the applications, as the next examples will show.

If the flows vector $q(x, u)$ is defined as containing, in an arbitrary order, all the flows $q_{ij}(x, u)$ which are not structurally null, then the state model ((4.1)) can also be written in a more compact matrix form :

$$\dot{x} = Lq(x, u) \tag{4.1}$$

where L is the incident matrix of the oriented graph, whose coefficients all belong to $(-1, 0, 1)$.

Example 4.3. For the system depicted at figure 4.2, the state model is written as :

$$\begin{aligned}\dot{x}_1 &= q_{01}(x, u) - q_{12}(x, u) - q_{13}(x, u) + q_{21}(x, u) \\ \dot{x}_2 &= q_{02}(x, u) + q_{12}(x, u) - q_{21}(x, u) - q_{23}(x, u) \\ \dot{x}_3 &= q_{13}(x, u) + q_{23}(x, u) - q_{30}(x, u)\end{aligned}$$

If the flows vector is defined as :

$$q(x, u) \triangleq \begin{pmatrix} q_{01}(x, u) \\ q_{02}(x, u) \\ q_{12}(x, u) \\ q_{13}(x, u) \\ q_{21}(x, u) \\ q_{23}(x, u) \\ q_{30}(x, u) \end{pmatrix}$$

the state model is written in a matrix format ((4.1)) with the matrix L :

$$L \triangleq \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}.$$

□

4.3. Modelling of the flows

Depending on the applications, the functions $q_{ij}(x, u)$ depicting the flows can take different types of forms. However, they must be defined in a way which guarantees the compartment system to be a *positive system*, that is a system for which each state variable remains positive along the trajectories. It is a likelihood guarantee of the model, because the state variables represent measures which do not have a physical meaning if they are negative.

Definition 4.4. Positive vector and positive orthant

A vector $x = (x_1, \dots, x_n)^T$ is positive (notation $x \geq 0$) if each of its component is a positive real number : $x_i \geq 0$ for all i .

The positive orthant of dimension n (written \mathbb{R}_+^n) is the set of all positive vectors of dimension n . □

Definition 4.5. Positive system

A dynamical system $\dot{x} = f(x, u)$ is a positive system if, for every admissible input $u(t)$, its state is confined in the positive orthant when the initial state is positive :

$$x(t_0) \in \mathbb{R}_+^n \text{ et } u(t) \in \mathcal{U} \implies x(t) \in \mathbb{R}_+^n \quad \forall t \geq t_0. \quad \square$$

The following theorem gives a sufficient condition which can be easily used to check that a system is positive.

Theorem 4.6. A dynamical system $\dot{x} = f(x, u)$ is a positive system if $f(x, u)$ is differentiable and if

$$x \in \mathbb{R}_+^n \quad \text{et} \quad x_i = 0 \implies \dot{x}_i \geq 0 \quad \forall i. \quad \square$$

To ensure that a compartments system is a positive system, let's impose the following conditions on the flows functions $q_{ij}(x, u)$:

C1. The functions $q_{ij}(x, u)$ are positive functions of their arguments on their definition domain :

$$q_{ij}(x, u) : \mathbb{R}_+^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$$

C2. The functions $q_{ij}(x, u)$ are continuous and differentiable functions of their arguments on their definition domain.

C3. As there cannot be an outgoing flow from an empty compartment, the functions $q_{ij}(x, u)$ verify the condition :

$$x_i = 0 \implies q_{ij}(x, u) = 0$$

Theorem 4.7. Under conditions C1, C2, C3, a dynamical compartment system $\dot{x} = Lq(x, u)$ is a positive system. ■

Example 4.8. Hydraulic system

Let's consider an hydraulic system made of a set of tanks located at different elevations and whose the liquid content flows "as waterfalls" from the highest tanks to the lowest tanks, thanks to gravity action. An example is illustrated at figure 4.3.

It is clearly a compartments system whose the associated graph is depicted at figure 4.4 and whose the continuity equations are written as :

$$\begin{aligned}\dot{x}_1 &= q_{01} - q_{12} - q_{13} \\ \dot{x}_2 &= q_{12} - q_{23} \\ \dot{x}_3 &= q_{13} + q_{23} - q_{30}\end{aligned}$$

In these equations, the state variables x_1, x_2 et x_3 specify, obviously, the volumes of water contained in the tanks; and the flows q_{ij} depict the debits flowing from the upper tanks toward the lower tanks. In order to complete the model, the flows should be expressed in terms of the state variables and the input signals, correctly chosen. The flow provided by the supply pump of the upper tank can obviously be chosen as an input variable. The outgoing flow q_{ij} of each tank is a positive function of the volume x_i of the tank. The form of this function depends on the

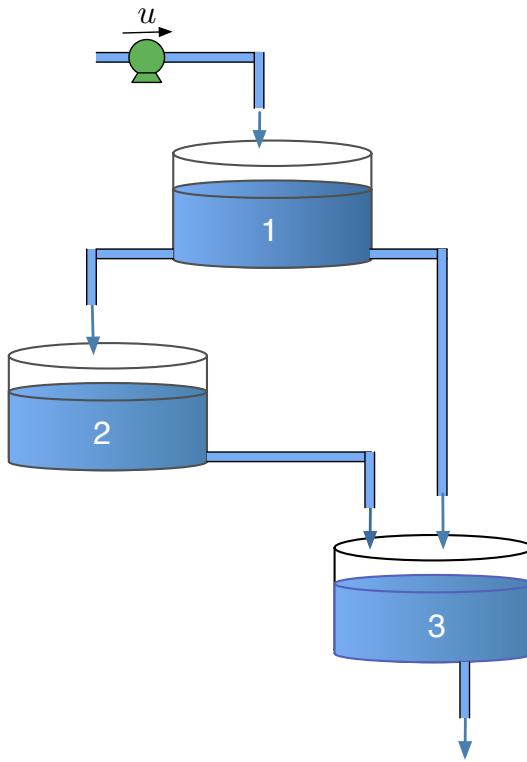


FIGURE 4.3 – Waterfall of tanks.

shape of the tanks and the configuration of the holes through which the water flows. Let's consider the case where the tanks have a constant horizontal section and where the flow goes through a rectangular hole located at the bottom of the tanks. The water elevation in a tank is expressed as :

$$h_i = \frac{x_i}{S_i}$$

where S_i specifies the section of the tank. According to the hydraulic laws, we know that when the elevation of the water h_i is big toward the elevation of the hole, the link between the debit and the elevation of the water is proportional to $\sqrt{h_i}$ (Torricelli's law¹). However, when the elevation of the water is lower than the

1. This law written by Torricelli in 1643 states that the speed v of the outgoing water of a tank of elevation h verifies $v^2 = 2gh$. It can be proven intuitively by analogy with a body in free fall : a elementary volume of water at the surface of the tank has a potential energy ρgh and a kinetic energy $\rho v^2/2$ when it reaches the exit of the tank, where ρ depicts the density. More rigorously, this can be deduced from Bernoulli's theorem without pressure loss or pump $p + \rho g z + \rho v^2/2 = \text{constante}$, where p depicts the pressure and z the elevation.

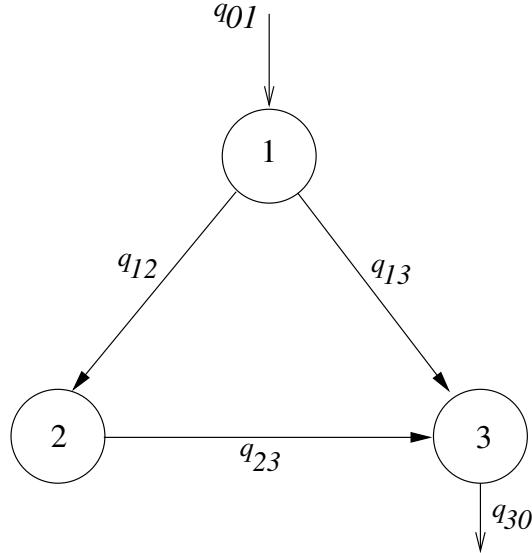


FIGURE 4.4 – Graph of the waterfall of tanks.

elevation of the hole, the flows becomes proportional to $h_i \sqrt{h_i}$ (law of flows for a rectangular tank). A model of the following form can be given :

$$q_{ij} = \frac{\alpha_{ij} h_i \sqrt{h_i}}{\beta_{ij} + h_i}$$

where α_{ij} et β_{ij} are positive constants. Indeed, this model verifies the property telling that, for low water elevations ($h_i \ll \beta_{ij}$), the flow q_{ij} is proportional to $h_i \sqrt{h_i}$ whereas for high water elevations ($h_i \gg \beta_{ij}$), the flow q_{ij} is proportional to $\sqrt{h_i}$. The flows q_{ij} can be expressed in terms of x_i :

$$q_{ij}(x_i) = \frac{k_{ij} x_i \sqrt{x_i}}{S_i \beta_{ij} + x_i} \quad \text{avec } k_{ij} \triangleq \frac{\alpha_{ij}}{\sqrt{S_i}}$$

Finally, the state model can be written as :

$$\begin{aligned} \dot{x}_1 &= -\frac{k_{12} x_1 \sqrt{x_1}}{S_1 \beta_{12} + x_1} - \frac{k_{13} x_1 \sqrt{x_1}}{S_1 \beta_{13} + x_1} + u, \\ \dot{x}_2 &= \frac{k_{12} x_1 \sqrt{x_1}}{S_1 \beta_{12} + x_1} - \frac{k_{23} x_2 \sqrt{x_2}}{S_2 \beta_{23} + x_2}, \\ \dot{x}_3 &= \frac{k_{13} x_1 \sqrt{x_1}}{S_1 \beta_{13} + x_1} + \frac{k_{23} x_2 \sqrt{x_2}}{S_2 \beta_{23} + x_2} - \frac{k_{30} x_3 \sqrt{x_3}}{S_3 \beta_{30} + x_3}. \end{aligned} \tag{4.2}$$

Let's notice that the functions $q_{ij}(x_i)$ verify the positivity conditions C1, C2 and C3. \square

4.4. Linear models driven by controlled external supplies

This is the most represented class of compartmental models within the literature. It is characterized by the following flow definitions :

1. Flows between compartments and system output flows are linear in function of the providing compartment level :

$$q_{ij} = k_{ij}x_i \quad k_{ij} > 0 \quad (i = 1, \dots, n; j = 0, \dots, n)$$

2. The system inputs u_ℓ are proportional to the supply flow :

$$q_{0\ell} = k_{0\ell}u_\ell$$

In that case, the required information used to write the state model is entirely unclosed within the system graph. The state model can be represented as a linear system (see chapter 1) :

$$\dot{x} = Ax + Bu$$

with the following structural features :

1. Matrix A is a *Metzler matrix* i.e. such that $a_{ij} \geq 0$ for all $i \neq j$
2. Matrix A is diagonally dominant i.e.

$$|a_{ii}| \geq \sum_{j \neq i} a_{ji}$$

3. Matrix B is a full rank *elementary matrix*, i.e. a matrix containing at most one non null element per line and per column.

Example 4.9. The compartmental system linear state model corresponds to the graph shown on 4.2 and can be written as :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -(k_{12} + k_{13}) & k_{21} & 0 \\ k_{12} & -(k_{21} + k_{23}) & 0 \\ k_{13} & k_{23} & -k_{30} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} k_{01} & 0 \\ 0 & k_{02} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (4.3)$$

We observe that A is a diagonally dominant Metzler matrix and that B is a full rank elementary matrix ($\text{rank} = 2$). ■

Example 4.10. Physiological modelling Physiologists are often interested in describing and analyzing biological or chemical substance propagation within mammal body. Those substances can stand for medicinal substances (Pharmacokinetic studies) or toxic substances voluntary or accidentally absorbed. They can also be natural substances such as hormones or proteins. Compartmental models are frequently used to process such studies : the mammal body is therefore represented as a more or less diversified group of interconnected vessels.

Let us consider the example on figure 4.5.

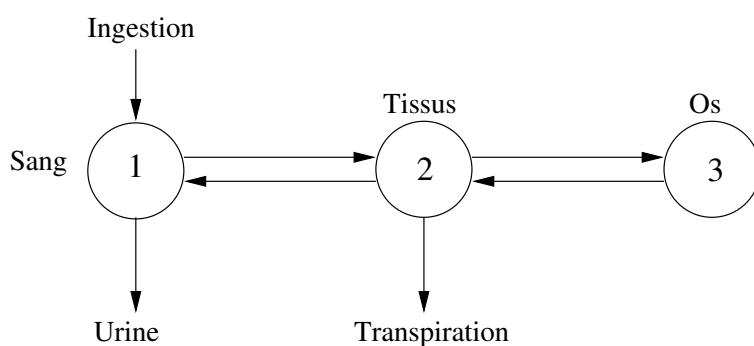


FIGURE 4.5 – Pharmacokinetic compartmental graph model

A toxic substance (lead for example) is absorbed by an animal and permeated in its blood. This substance progressively propagates within the body, from the blood to tissues at first, towards bones afterwards. It is secreted by sweating from one part and by urinating from the other part. The linear compartmental model corresponding to the graph on figure 4.5 is the following model :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -(k_{10} + k_{12}) & k_{21} & 0 \\ k_{12} & -(k_{20} + k_{21} + k_{23}) & k_{32} \\ 0 & k_{23} & -k_{32} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} k_{01} \\ 0 \\ 0 \end{pmatrix} u.$$

In this model, the state variables are : x_1 , x_2 et x_3 , which stands for toxial substance quantities within the three compartments (blood, tissues and bones). The input variable u stands for the body ingestion flow. ■

4.5. Non linear model with controlled flows

We will now consider non linear compartmental systems which flows q_{ij} can be non linear functions whose arguments respect the C1 - C3 conditions. We already approached a non linear model in the vessels cascade example. However, flows between compartments were not depending on input variables u_ℓ in that example. We will here consider a case where flows between compartments are explicit functions of input variables u_ℓ allowing to monitor the debit between compartments. The symbolic representation presented on figure 4.6 shows the presence of such a monitoring variable.

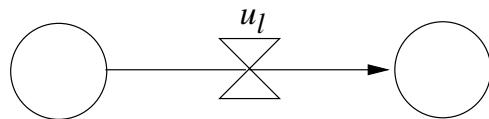


FIGURE 4.6 – Symbolic representation of a monitored flow

Example 4.11. Vessels network Let us consider the hydraulic system represented on figure 4.7. This vessels network corresponds to the vessel cascade example (4.8 with a small modification : the flow between vessel 2 and vessel 3 is now monitored by a pump. As this pump is controllable, we can consider the pumped debit F as an input variable.

The state model (4.2) we obtained for the vessels cascade is therefore simply modified as follows :

$$\begin{aligned}\dot{x}_1 &= -q_{12}(x_1) - q_{13}(x_1) + u_1 \\ \dot{x}_2 &= q_{12}(x_1) - u_2 \\ \dot{x}_3 &= q_{13}(x_1) - q_{30}(x_3) + u_2\end{aligned}\tag{4.4}$$

where the state variables x_i stand for water volumes contained in vessels, input variable u_1 corresponds to the first vessel supply debit, input variable $u_2 = F$ corresponds to the pumped flow from the second vessel towards the third vessel and functions $q_{ij}(x_i)$ are defined as follows :

$$q_{ij}(x_i) = \frac{k_{ij}x_i\sqrt{x_i}}{S_i\beta_{ij} + x_i}$$

We observe that this state model *cannot* represents a compartmental system respecting C1-C3 conditions. The flow $q_{23} = u_2$ does indeed not respect the C3 condition and the system is not positive : simulations of this model can lead to negative vessels levels (even if the pumped debits remain positive) which is obviously conflicting with physical reality.

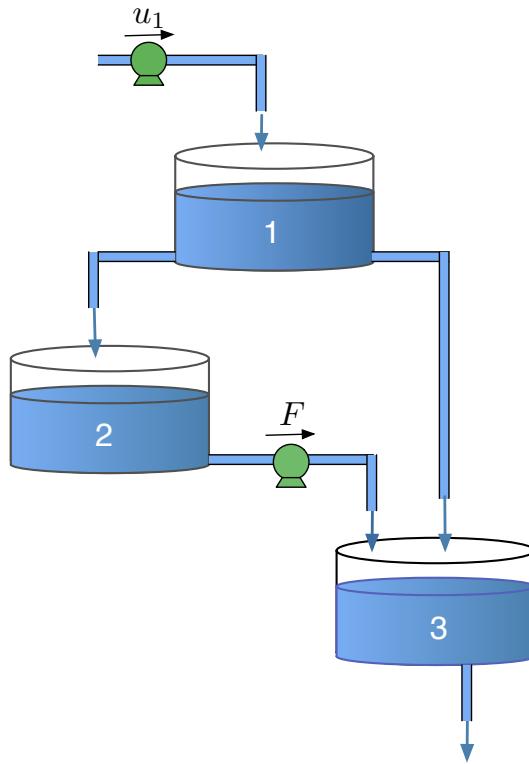


FIGURE 4.7 – Vessels networks

The model as stated indeed allows to pump water in the second vessel even when it is empty !

This problem can be easily avoided if the flow q_{23} (where the pumped debit is F) is modeled such as it respects the physical reality and the C3 condition as :

$$q_{23}(x_2, u_2) = \phi(x_2)u_2$$

where $\phi(x_2)$ is a positive function satisfying $\phi(0) = 0$ and u_2 represents the pump activation.

We therefore obtain a compartment system which graph is presented on figure 4.8 and the state model can be written as :

$$\begin{aligned}\dot{x}_1 &= -q_{12}(x_1) - q_{13}(x_1) + u_1 \\ \dot{x}_2 &= q_{12}(x_1) - \phi(x_2)u_2 \\ \dot{x}_3 &= q_{13}(x_1) - q_{30}(x_3) + \phi(x_2)u_2\end{aligned}$$

■

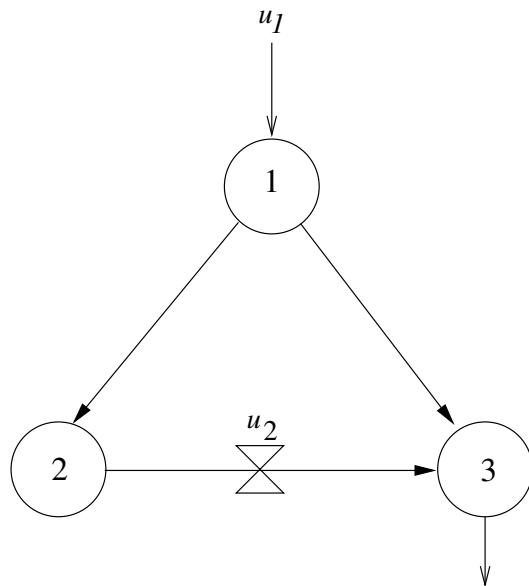


FIGURE 4.8 – Vessels network associated graph

The fundamental structural property of compartments linear systems can be generalized to non linear system with the following theorem.

Theorem 4.12. Given a compartments non linear system which flows q_{ij} satisfy C1-C3 conditions. Therefore, the flows can be written as follows :

$$\begin{aligned} q_{ij}(x, u) &= a_{ij}(x, u)x_i \quad (i = 1, \dots, n; j = 1, \dots, n) \\ q_{i0}(x, u) &= a_{i0}(x, u)x_i \quad (i = 1, \dots, n) \\ q_{0i} &= k_{0i}u_i \end{aligned}$$

where functions $a_{ij}(x, u)$ et $a_{i0}(x, u)$, defined on the positive orthant, are continuous.

Therefore, the system state model can be written as :

$$\dot{x} = A(x, u)x + Bu$$

where the matrix $A(x, u)$ is a diagonally dominant Metzler matrix for all (x, u) in the positive orthant and B an elementary matrix. ■

We will end this chapter with another industrial classical compartmental system example.

Example 4.13. Binary distillation process

A binary distillation process is a process used to split a liquid load composed of two liquid chemical components. A *depropanizer* used to split propane from butane is a typical example of binary distillation process within the petrochemical industry.

The split is made by evaporation in an enclosed vessel called *round-bottom flask* (see figure 4.9). The *distillate* containing mainly the lightest component with

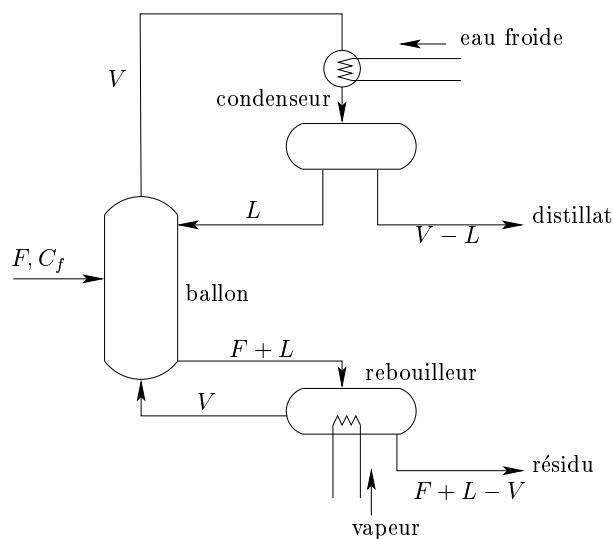


FIGURE 4.9 – Distillation process

a bit of the heaviest one exits from the top of the flask.

The *residue* containing mainly the heaviest component with a bit of the lightest one exits from the bottom of the flask.

The flask is filled by the liquid load with a molar debit F (mol/min). The steam flow spreading out the flask is cooled down and entirely condensed. The outgoing liquid is partially recycled toward the flask with a molar debit L .

The remaining part, called *distillate*, is extracted from the system.

At the bottom of the flask, the outgoing liquid is warmed up a boiler and the produced steam is recycled within the flask. The remaining part, called *residue*, is extracted from the system.

The distillation process dynamic is simplified by the following modeling assumptions and represented below :

1. the load is liquid and has the flask temperature ;
2. the liquid and steam state in the flask and the boiler are homogeneous and at equilibrium ;

3. the flask pressure is constant and there is no steam accumulation ; this assumption allows to omit pressure dependencies in the equations and implies that the steam debit V exiting the flask is equivalent to the input debit ;
4. the liquid extraction debits are adjusted such as the total molar masses of the components in liquid state remain constant : the distillate is therefore extracted with a molar debit $V - L$, the liquid at the bottom of the flask is extracted with a molar debit $F + L$ and the residue is extracted with a molar debit $F + L - V$. Obviously, this implies that the inequality $0 < L < V < F + L$ has to be verified.

Given this definition, the distillation process can be interpreted as a compartments system which dynamic model is based on balance equations of one of the two components in the flask, in the condenser and in the boiler. This compartments system graph is presented on figure 4.10 and the state equations are :

$$\begin{aligned}\dot{x}_1 &= u_2 k(x_2) - u_1 \frac{x_1}{m_1} - (u_2 - u_1) \frac{x_1}{m_1} \\ \dot{x}_2 &= u_1 \frac{x_1}{m_1} - (u_1 + u_3) \frac{x_2}{m_2} + u_2 (k(x_3) - k(x_2)) + u_3 c_f \\ \dot{x}_3 &= (u_1 + u_3) \left(\frac{x_2}{m_2} - \frac{x_3}{m_3} \right) + u_2 \left(\frac{x_3}{m_3} - k(x_3) \right)\end{aligned}$$

State variables x_i stand for the molar mass of the lightest component in the liquid

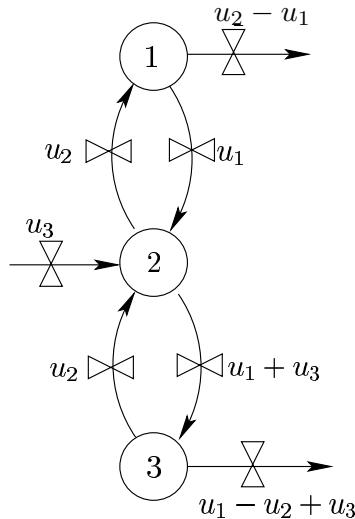


FIGURE 4.10 – Distillation process associated graph

state within the condenser (index 1), the flask (index 2) and the boiler(index 3) ;

Parameters m_i represent those total (and constant) molar masses : the ratio x_i/m_i corresponds to the *molar fraction*; parameter c_f molar fraction of the lightest component within the load ;

Input variables $u_1 = L$, $u_2 = V$ and $u_3 = F$ are, respectively, molar debit of the reflux, the steam production and the supply. Finally, the function $k(x)$ is a liquid-steam equilibrium relationship allowing to link the molar fraction of the lightest component leaving the liquid by vaporization to the molar fraction of the component in liquid state.

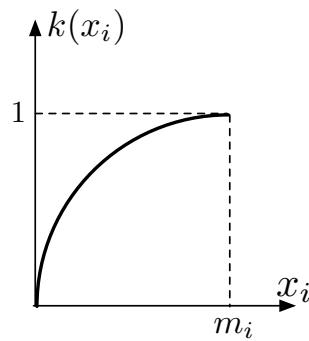


FIGURE 4.11 – Liquid-steam equilibrium relationship

This relationship is classically expressed as follows :

$$k(x_i) \triangleq \frac{\alpha x_i}{m_i + (\alpha - 1)x_i}$$

where the constant parameter $\alpha > 1$ is called separation factor. This function, defined on the interval $[0, m_i]$, checks $k(0) = 0$ and $k(m_i) = 1$ (see figure 4.11). \square

4.6. Exercises

Exercise 4.1. A compartments system

Given the following dynamical system :

$$\begin{aligned}\dot{x}_1 &= x_3 - \log(1 + x_1) \\ \dot{x}_2 &= x_3 - x_2^2 \\ \dot{x}_3 &= x_2^2 - 2x_3 + u\end{aligned}$$

Demonstrate that it is a compartments system. Draw the associated graph. Compute the flows q_{ij} , the matrix L and the matrix $A(x, u)$. \square

Exercise 4.2. A hydraulic system

A hydraulic system containing three vessels and two pumps is presented on figure 4.12.

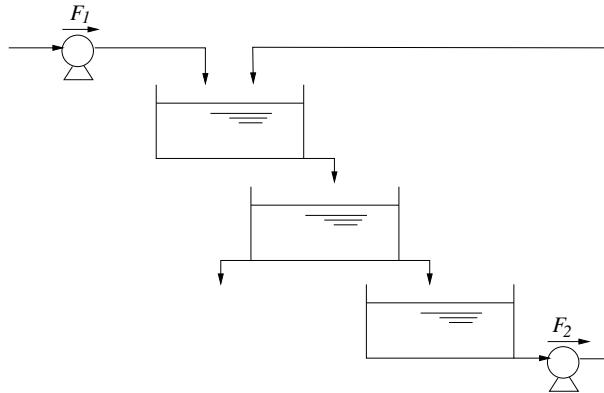


FIGURE 4.12 – Hydraulic system

1. Establish a state model for the system, where the volumetric debits $u_1 = F_1$ and $u_2 = F_2$ are input variables. Show that the obtained system is *not* a positive system.
2. Suggest an alternative definition for the input variable u_2 which ensure a positive system.
3. Draw the obtained compartments graph model. □

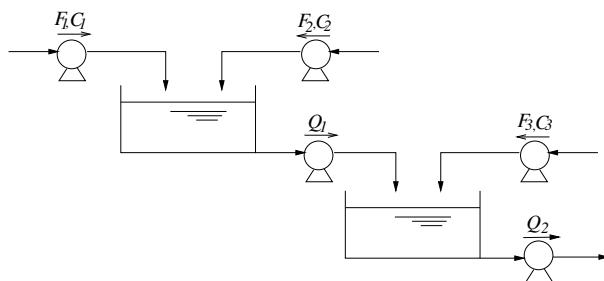


FIGURE 4.13 – Mixing vessels network

Exercise 4.3. A mixing vessels network

The system represented on figure 4.13 is designed for mixing three substances X_1, X_2, X_3 whose supply concentrations are denoted C_1, C_2, C_3 respectively.

The contained volumes in the two vessels are denoted V_1 and V_2 . The pump volumetric debits are denoted as Q_1, Q_2, F_1, F_2, F_3 .

1. Establish a state model of the system with the following input variables : $u_1 = Q_1/V_1, u_2 = Q_2/V_2, u_3 = C_1, u_4 = C_2, u_5 = C_3$. The debits $F_i, i = 1, \dots, 3$, are supposed constants.
2. Justify the input variables u_1 and u_2 form. \square

Exercise 4.4. Compartments linear model

Characterize the graph structure of a compartments linear model whose associated matrix A is :

1. bidiagonal
2. tridiagonal
3. lower triangular \square

Exercise 4.5. Distillation process model

Determine the matrix $A(x, u)$ of the distillation process model. \square

Exercise 4.6. Communicating vessels

A system with two communicating vessels is represented on figure 4.14. The liquid flows freely between the two vessels and towards the outside under the hydrostatic pressure action.

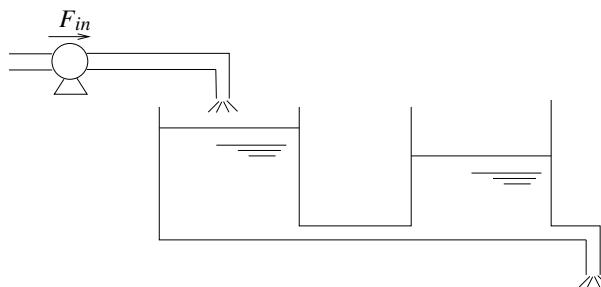


FIGURE 4.14 – Communicating vessels

1. Establish a state model of the system. The provided debit (by the supply pump) is the only input variable.
2. Show that it is a compartments system. Draw the associated graph. Explain the flow between the compartments. \square

Chapitre 5

Reaction systems

The reaction system concept applies for a dynamic systems class used in various engineering domains such as chemistry, biomedical, biotechnologies or ecology. Under a spatial homogeneity assumption, reaction systems dynamic is described by balance differential equations.

These equations are obtained from a combination of a *reaction network* which encodes the reactions that are supposed to occur in the system with two basic physical phenomena : the *reactions kinetic* from one part and the *exchange dynamics* from the other part. Those various elements describing reaction systems will be presented in the following sections, starting by reaction networks.

5.1. Reaction networks

A reaction system is characterized by a given number of *reactions* between chemical or biological components. The number of components is finite and we denote those components using the following symbols :

$$X_1, X_2, X_3, \dots, X_n.$$

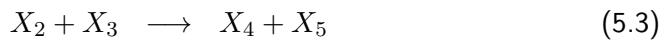
The number of reactions is also a finite number m and those reactions occur inside a geometrically well defined domain. For example, a chemical reactor if it occurs between chemical components or an ecological niche between animal species. The domain boundary is also well defined and split the system from the external world.

The easiest way to introduce the reaction network concept is to start with an example.

Example 5.1. Chemical reaction

The reaction mechanism between nitric oxide and hydrogen is described using the following reaction network which has $m = 4$ reactions employing $n = 6$

chemical components :



The six components are : $X_1 = NO$, $X_2 = N_2O_2$, $X_3 = H_2$, $X_4 = N_2$, $X_5 = H_2O_2$, $X_6 = H_2O$. \square

A reaction network is therefore a set of m reactions in the following form :

$$\sum_{i=1}^n \gamma_{ij} X_i \longrightarrow \sum_{i=1}^n \delta_{ij} X_i \quad j = 1, \dots, m \quad \gamma_{ij} \geq 0 \quad \delta_{ij} \geq 0.$$

The coefficients γ_{ij} and δ_{ij} are positive real numbers called *stoichiometric coefficients*. They represent the nominal quantity for the component X_i which is consumed or produced by the j^{th} reaction. For example, the fourth network reaction above means : one mole of X_3 combined to one mole of X_5 produces two moles of X_6 .

We introduce the following matrix notations :

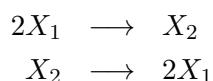
$$\begin{aligned} \Gamma &= [\gamma_{ij}] \quad \text{matrix } n \times m \text{ with elements } \gamma_{ij} \\ \Delta &= [\delta_{ij}] \quad \text{matrix } n \times m \text{ with elements } \delta_{ij} \end{aligned}$$

The *stoichiometric matrix* is defined as :

$$C = \Delta - \Gamma.$$

The matrix rank p is called the *reaction network rank*. It corresponds to the numbers of independent reactions.

As a convention, all reactions are denoted with an arrow from the left to the right. In the example above, the *reversible* reaction $2X_1 \rightleftharpoons X_2$ is encoded as two simple distinct reactions :



Reactants and products

The *reactants* are the components X_i which are written on the left hand side of the arrow with a coefficient $\gamma_{ij} > 0$.

The *products* are the components X_i which are written on the right hand side of the arrow with a coefficient $\delta_{ij} > 0$.

A component X_i can either be a reactant in a reaction and a product in another or the same reaction. This is the case of the component X_5 in the example 5.1.

A *terminal product* is a component produced by at least one reaction but which is a reactant of none reaction.

An *initial reactant* is a component consumed by at least one reaction but which is produced by no reaction.

As an example, in the reaction network (5.1) - (5.4), we identify the following subsets :

Reactants	:	X_1, X_2, X_3, X_5
Products	:	X_1, X_2, X_4, X_5, X_6
Initial reactant	:	X_3
Terminal products	:	X_4, X_6

Catalysts et autocatalysts

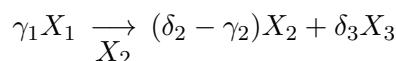
As we just explained, a given component can be on both sides of a reaction. The component X_2 is an example in the following reaction :



If $\gamma_2 = \delta_2$ the component X_2 is a *catalyst*, in other words a component which is neither consumed nor produced but whose presence is required to perform the reaction.

If $\gamma_2 < \delta_2$, the component X_2 is an *autocatalyst*, a component which is a catalyst in its own production.

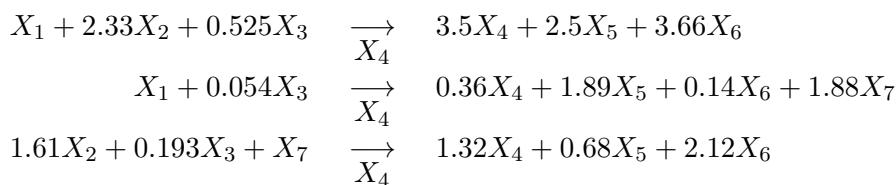
We also use an alternative representation for catalytic and autocatalytic reactions. It consists in not writing the catalyst on the left side of the reaction but indicating it under the arrow without coefficient and balancing it on the right side of the reaction with the coefficient $\delta_2 - \gamma_2$:



Among the most typical autocatalytic reaction examples we can quote polymerization reactions or microbial growth reactions as described in the following example.

Example 5.2. Alcoholic fermentation

The alcoholic fermentation underlying mechanism can be described by the following reaction network :



The seven components are : glucose X_1 , oxygen X_2 , ammoniac X_3 , yeasts X_4 , carbon dioxide X_5 , water X_6 , ethanol X_7 . \square

5.2. Reaction systems state model

The presence of each component inside the system can be quantified. The component X_i concentration denoted $x_i(t)$ corresponds to its amount in the system divided by the mixture volume. The concentrations vector, which is also the model state vector, is denoted :

$$x(t) \triangleq (x_1(t), x_2(t), \dots, x_n(t))^T.$$

The reaction rates, also called *reaction kinetics*, describes the speed at which reactants are consumed and products are produced per volume unit in the system, according to the reaction network. A reaction rate r_j is associated to each reaction of the network ($j = 1, \dots, m$).

The reaction rates depend on each component concentrations x_i , but they can eventually also be influenced by other system physico-chemical factors, such as temperature, catalysts or pressure.

We will here consider reactions depending only on state x . The reaction kinetics vector is denoted :

$$r(x) \triangleq (r_1(x), r_2(x), \dots, r_m(x))^T.$$

Each function $r_j : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ has positive values and is defined on the positive orthant. A reaction cannot occurs unless all the reactants are presents in the system. In other words, the reaction rate is therefore null if one of the reactants is missing. Mathematically, this condition can be stated as :

Hypothesis 5.3.

$$1) r_j(x) \geq 0 \quad \forall j \quad \forall x \in \mathbb{R}_+, \tag{5.5}$$

$$2) r_j(x) = 0 \text{ si } x_i = 0 \text{ for a value of } i, \in I^{r_j} \tag{5.6}$$

where I^{r_j} stands for the index of the reactants used in reaction j set (including catalysts). \square

Based on the reaction network description and the reaction rates, we can easily check that the quantitative balance of each component inside the system bounds can be written as :

$$\dot{x}_i = \sum_{j=1}^m (\delta_{ij} - \gamma_{ij}) r_j(x(t)) + \frac{1}{V} (Q_{0i}(t) - Q_{i0}(t)).$$

In this equation, the notations δ_{ij} , γ_{ij} (stoichiometric coefficients) and $r_j(x(t))$ (reaction rate) were defined earlier. The notation V stands for the volume (assumed constant) of the domain. The notations $Q_{0i}(t)$ and $Q_{io}(t)$ stand for the flows of the component X_i through the domain boundary :

$Q_{io}(t)$ is the flow going from the domain towards the outside,

$Q_{oi}(t)$ is the flow going from the outside towards the domain.

This continuity equation states that the variation, per time unit, of the component concentration X_i comes from two distinct mechanisms :

- $\sum_{j=1}^m (\delta_{ij} - \gamma_{ij}) r_j(x(t))$ express the difference, per volume unit, between the product quantities sum and consumed quantities sum in the reaction where this component X_i is a product or a reactant respectively ;
- $Q_{0i}(t) - Q_{io}(t)$ express the difference between the incoming flow and the outgoing flow from the same component X_i towards the domain boundary.

The system is said to be *closed* when $Q_{io}(t) = Q_{oi}(t) = 0$ for all i and for all t , in other words, when there is no exchange with the outside. In the opposite situation, the system is said *open*.

A reaction system state modeling has three fundamental aspects.

First, the reaction network determines the number of state variables, the structure and the numerical values of the stoichiometric matrix coefficients.

Second, we might ask ourselves how to model reaction rates $r_j(x)$ according to the state variables x_i . The modeling will be presented in the following section.

We will then model the incoming and outgoing flows according to the state and the input variables :

$$Q_{0i}(x, u) \quad Q_{io}(x, u)$$

This modeling will be illustrated using various examples.

Reaction system dynamic is therefore represented by the following state model :

$$\dot{x} = Cr(x) + q_{in}(x, u) - q_{out}(x, u) \quad (5.7)$$

where the definition of vectors $q_{in}(x, u)$ and $q_{out}(x, u)$ is obvious. This state model can then only be defined on the positive orthant. We can easily show that, under the 5.3 assumption, the system (5.7) is a positive system with a compartments system structure.

For a closed system, vectors q_{in} and q_{out} are identically zeros and the state model can be reduced to the equation :

$$\dot{x} = Cr(x).$$

Hypothesis 5.4. The preservation principle The stoichiometric matrix kernel contains a positive vector :

$$\exists \omega = (\omega_1, \dots, \omega_n)^T \quad \omega_i > 0 \quad i = 1, \dots, n \quad \text{tel que } \omega \in \ker C^T.$$

□

Under this assumption, we easily check that the quantity

$$z = \sum_{i=1}^m \omega_i x_i = \omega^T x$$

is a system invariant (5.2) (in other words, $z(t)$ is constant along system solutions).

$$\dot{z} = \sum_{i=1}^n \omega_i \dot{x}_i = [\omega^T C] r(x) = 0$$

and the quantity between brackets is indeed zero thanks to the assumption 5.4.

This assumption is essential because it express the fact that, according to the reality, a closed reaction system is a preservative system ; the total quantity inside the system is a constant : the produced quantities are equal to the consumed quantities (using appropriate normalization coefficients). As Lavoisier said, « rien ne se perd, rien ne se crée ».

5.3. Reaction kinetics modeling

If a reaction respects the *Law mass action*, a classical general expression, respecting the conditions (5.5) - (5.6), can be written :

$$r_j(x) = k_j \prod_{i \in I^{rj}} x_i^{\nu_{ij}}$$

where k_j is the *rate constant* of the j^{th} reaction. The mass action principle consists in expressing each reaction rate as being proportional to the product of the reactants concentrations in the reaction (including catalysts), each concentration being set to the ν_{ij} positive power called *order* of the $j^{reaction}$ s.

The mass action Law corresponds to the particular case where $\nu_{ij} = \gamma_{ij} \quad \forall(i, j)$, where the reaction orders match the reactants stoichiometric coefficients.

However, the mass action principle is often not enough to match the experimentally observed reaction rates. We therefore have to generalize the model :

$$r_j(x) = k_j \prod_{i \in I^{rj}} \rho_{ij}(x_i)$$

where functions $\rho_{ij} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ respect the following conditions :

$$\begin{aligned} \rho_{ij}(x_i) &\geq 0 \quad \forall x_i \geq 0 \\ \rho_{ij}(0) &= 0 \end{aligned}$$

Functions $\rho_{ij}(x_i)$ are often monotonically increasing function, as shown in figure 5.1. One of the most famous example is called the Michaelis-Menten kinetic

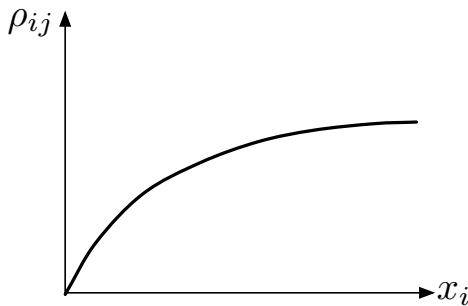


FIGURE 5.1 – Monotonically increasing kinetic function

and is represented by the function :

$$\rho_{ij}(x_i) = \frac{x_i}{K_{ij} + x_i}.$$

Inhibitors and activators

A reaction can be slowed down by a reaction product of any other component being in the reaction network. Such an *inhibitor* effect is modeled by adding an additional multiplicative term in the kinetic model. This term is a decreasing function of the inhibitor component concentration. The two most frequent models are the following :

$$\text{hyperbolic inhibition : } \rho_{ij}(x_i) = \frac{K_{ij}}{K_{ij} + x_i}, \quad (5.8)$$

$$\text{exponential inhibition : } \rho_{ij}(x_i) = e^{-(K_{ij}x_i)}. \quad (5.9)$$

Example 5.5. Let consider the following reaction network :



Suppose that kinetics respect the mass action law and that the first reaction is the most inhibited by the X_4 product of the second reaction, following an exponential law (5.9). Both reaction kinetics will have the following form :

$$\begin{aligned} r_1(x) &= k_1 x_1 x_2 e^{-(Kx_4)}, \\ r_2(x) &= k_2 x_3^2. \end{aligned} \quad (5.11) \quad \square$$

A component in the reaction can also have a speed effect without being required to the reaction (this component is nor a reactant nor a catalyst of the reaction). Such *activator* effect is modeled by adding an additional multiplicative term in the kinetic model. This term is an increasing function of the activator component concentration (which is non zero at start).

5.4. Perfectly Stirred Tank Reactors

Perfectly stirred chemical or biological reactors are one of the most typical examples of reaction systems. These reactors are made of a tank containing a liquid reaction medium mixed permanently by a suitable agitation system. The composition of the tank is considered homogeneous. The various reagents can be provided to reactor in liquid or gaseous form. Reaction products are created in solution in the reaction medium. Some of these products can be easily gasifiable and can freely escape from the reactor in gaseous form. The reaction medium is [withdrawn](#) to collect products.

5.4.1. Continuous Reactors

A perfectly stirred tank reactor work in *continuous-flow mode* when feed and withdrawal rate are adapted so that the volume V of the reaction medium is constant. This is called *continuous stirred tank reactor* (acronym CSTR). An example of this kind of reactor is illustrated in the figure 5.2. The tank is fitted with a supply pipe

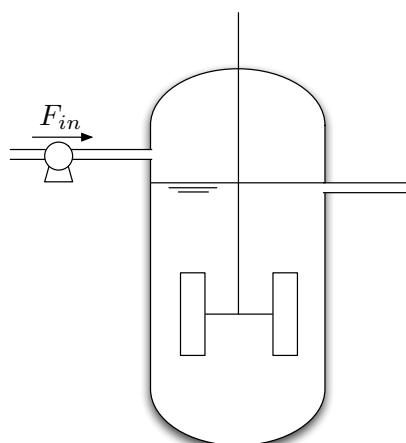


FIGURE 5.2 – Perfectly stirred tank reactor

and a suitable overflow to keep constant volume.

It is assumed this reactor is the place of a set of m reactions involving n chemical species X_1, X_2, \dots, X_n . The species concentrations in the reaction medium are noted x_i . The various species are supplied to reactor in solution or in suspension in the feed flow with concentrations noted x_i^{in} . The feed volumetric flow is noted F_{in} . With these notations and definitions, balance equation of the species in the reactor can be noted in the following matrix form :

$$\dot{x}V = Cr(x)V - F_{in}x + F_{in}x^{in}$$

where C is the stoichiometric matrix of the reaction network, $r(x)$ the vector of reactions rates, x the vector of concentrations x_i and x^{in} the vector of feed concentrations x_i^{in} .

By defining the input variable

$$u \triangleq \frac{F_{in}}{V}$$

which is the feed rate per unit of volume, also called **dilution ratio** (the inverse of dilution ratio is the **retention time**), we get the state model of a perfectly stirred tank reactor :

$$\dot{x} = Cr(x) - ux + ux^{in}.$$

It is observed this model has the structure (5.7) with the following definitions :

$$q_{out}(x, u) = ux \quad q_{in}(x, u) = ux^{in}.$$

Example 5.6.

We consider a perfectly stirred tank reactor in which the two reactions (6.1) run simultaneously in the liquid phase with kinetics (5.11). The reactor is fed by the two initial reagents X_1 and X_2 in solution with feed concentrations x_1^{in} and x_2^{in} .

The state model is written

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 2 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 x_1 x_2 e^{-(Kx_4)} \\ k_2 x_3^2 \end{pmatrix} + u \begin{pmatrix} x_1^{in} - x_1 \\ x_2^{in} - x_2 \\ -x_3 \\ -x_4 \end{pmatrix}$$

where state variables x_1, x_2, x_3 and x_4 represent the concentrations of the species in the reaction medium. \square

5.4.2. Reactor with Variable Volume

Now, let's consider the reactor illustrated in the figure 5.3. It's the same as the previous one but the overflow is replaced by a withdrawal pipe, the volumetric flow F_{out} of which is regulated with a pump. In the particular case where the flow rate F_{out} may be intermittently zero (no withdraw), we say the reactor works in **batch mode (discontinuous mode)**.

The matrix equation of species mass balance is written as follows :

$$\frac{d}{dt}(xV) = Cr(x)V - F_{out}x + F_{in}x^{in}.$$

The reaction medium volume can vary if feed and withdrawal rate are different. The volume variations are described by the volumetric balance equation :

$$\dot{V} = F_{in} - F_{out}.$$

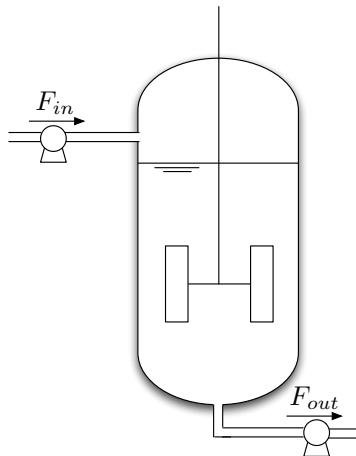


FIGURE 5.3 – Variable volume reactor

If both volumetric rates F_{in} and F_{out} are chosen as input variables u_1 and u_2 , we get the following state model :

$$\begin{aligned}\dot{x} &= Cr(x) + \frac{u_1}{x_{n+1}}(x^{in} - x), \\ \dot{x}_{n+1} &= u_1 - u_2,\end{aligned}$$

with the additional state variable x_{n+1} referring the volume V .

It is interesting to choose flow rates by unit of volume $u_1 = F_{in}/V$ and $u_2 = F_{out}/V$ as input variables. In this case, the state model is written

$$\begin{aligned}\dot{x} &= Cr(x) + u_1(x^{in} - x), \\ \dot{x}_{n+1} &= (u_1 - u_2)x_{n+1}.\end{aligned}$$

The first of this two equations describe the changes of the reactor composition. This is independent of u_2 and x_{n+1} and is identical to that we get for a reactor with a constant volume (5.4.1).

5.4.3. Non Isothermal Reactors

The rate of a chemical reaction also depends on the temperature of the reaction medium. Up to now we have not taken this dependence into account in modeling : it was implicitly assumed the temperature is regulated so we maintain a perfectly constant temperature. Absent such regulation, it is the rate constant which depends on the temperature. The general form of the speed of the j -th reaction is written

$$r_j(x, T) = k_j(T)\rho_j(x)$$

where T stands for the temperature (in Kelvin) of the reaction medium and the function $\rho_j(x)$ satisfies the hypothesis conditions (5.5)-(5.6). The function $k_j(T)$ is positive, bounded and $k_j(0) = 0$. A typical example is given by the *Arrhenius law* illustrated in the figure 5.4 :

$$k_j(T) = k_{0j} \exp\left(-\frac{E_j}{RT}\right)$$

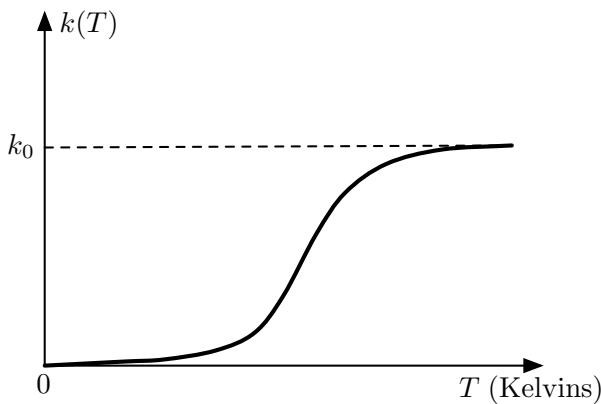


FIGURE 5.4 – Arrhenius law

where k_{0j} is a constant, E_j the activation energy of the reaction and R the Boltzmann constant. This function of the temperature is a monotonously increasing and bounded. In some applications (particularly in biotechnology) the function k_j can also be monotonic.

The state model of a non isothermal reactor is achieved by adding an energy balance equation to mass and volume balance equations. For example, let's consider a continuous reactor equipped with a heat exchanger. The energy balance equation is written as follows :

$$\delta c_p V \dot{T} = \left(\sum_{j=1}^m \Delta H_j r_j(x, T) \right) V + \delta c_p F_{in} (T_{in} - T) + Q$$

where δ is the reaction medium density, c_p the heat capacity, ΔH_j the reaction heat, T_{in} the feed flow temperature and Q the heat flow exchanged. If it is assumed the parameters δ , c_p and ΔH_j are constant, we get the heat balance equation :

$$\dot{T} = \sum_{j=1}^m h_j r_j(x, T) + d(T_{in} - T) + q$$

where $h_j = \Delta H_j / c_p \delta V$ is the specific heat of reaction, $d = F_{in}/V$ is the dilution ratio and $q = Q/c_p \delta V$.

The parameters h_j can be positive or negative. If h_j is negative, the reaction is endothermic : it consumes heat provided by heat exchanger. If h_j is positive, the reaction is exothermic : heat is generated and reactor has to be cooled by the heat exchanger.

The specific heat flow exchanged q is a function of the temperature T . A simple model is the statement q is proportional to the difference between reactor and input temperature, respectively T and T_w :

$$q = e(T_w - T)$$

In this case, the global state model of the reactor is written :

$$\begin{aligned}\dot{x} &= Cr(x) + d(x^{in} - x), \\ \dot{x}_{n+1} &= h^T r(x) + d(T_{in} - x_{n+1}) + e(T_w - x_{n+1}).\end{aligned}$$

with the additive state variable x_{n+1} referring temperature T . As input variables, we can choose, for example, the dilution ratio d and the heat transfer coefficient e which is proportional to the flow of the heat exchanger.

5.5. Ecological Systems

The reaction formalism and the state model (5.7) are also suitable to describe an important class of ecological systems (or ecosystem) in which populations of living organisms (vegetable or animal) share a same habitat.

The mathematical model of an ecosystem is presented as a particular case of reaction system in which :

- the reaction network describes interactions between species : inert resource consumption, grazing on vegetable resource, predation, etc. Reactions are necessarily auto-catalytic.
- inflows represent the supply of resources to the system by external agent and immigration of some species.
- outflows represent emigration of species outwards, the capture by external agents (hunting, fishing, harvesting, gathering, ...) or simply natural mortality in species.

We begin with one simple example.

Example 5.7. Algae in the lagoon

An organic nutrient coming for example from domestic waste water or agricultural fertilizer is released to the lagoon. A population of floating single-celled algae (phytoplankton) grows on the surface water feeding on this nutrient. This situation can be described by the reaction :



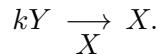
expressing that in the growing mechanism of algae, the nutrient Y is transformed in living matter (or biomass) X with a yield k^{-1} . Algae may also die as all living beings.

As consequence, the lagoon can be considered as a wide reactor transforming reagent Y (the nutrient) in product X (the biomass). The reactor is fed by a reagent inflow (the nutrient released to the lagoon) while mortality causes a product outflow. Under the spatial homogeneity assumption, dynamic of this reactor is described by the state model

$$\begin{aligned}\dot{y} &= -kr(x, y) + v, \\ \dot{x} &= r(x, y) - dx.\end{aligned}\tag{5.13}$$

where y represent the nutrient concentration, x the density of the algae population, v the nutrient feed rate (per unit of volume) of the lagoon, dx the assumed mortality proportional to the population density (coefficient d is the specific mortality rate) and $r(x, y)$ the reaction speed, namely the growing speed of algae. \square

From a broader standpoint, reaction (5.12) can represent the growing of any population of living organisms (vegetable or animal) X that in a determined habitat, consumes a food resource Y . This food resource can be inert matter (organic or inorganic) as in the above example. It could also be another living species (vegetable or animal) : this is called *prey - predator* model in which resource species Y is the prey and the consumer one is X the predator. Obviously, this growing reaction is auto-catalytic because X represent necessarily a population of self-reproductive living beings :



It is therefore natural for there to consider that the growth speed is proportional to the predator population density and to represent the function $r(x, y)$ with this kind of model :

$$r(x, y) \triangleq \mu(x, y)x$$

where the function $\mu(x, y)$ is called the *specific growth rate*. This function has to be defined so that the reaction rate verifies conditions (5.5) - (5.6), namely :

— $\mu(x, y)$ is a positive definite function on the positive orthant :

$$\mu(x, y) \geq 0 \quad \forall (x, y) \in \mathbb{R}_+^2.$$

— $\mu(x, 0) = 0$: no food supply means no growth.

The specific growth rate can depend on many environmental factors. Two typical non linear effects are *satiety* and *overpopulation* effects.

Satiety effect : When food supply is scarce, it is usually observed that the specific growth rate is an increasing function of available amount of supply. However it exists a physiological limit on the resource consumption rate and so on the growth rate. This can be modeled with $\mu(x, y)$ as an increasing saturated function of y , so that the growth speed is independent of y beyond a critical concentration y_c :

$$\frac{\partial \mu(x, y)}{\partial y} \geq 0, \quad \mu(x, y) = \mu(x, y_c) \quad \forall y \geq y_c.$$

Overpopulation effect : Even when there is a food over-supply, the population density is usually limited by the available space. This can be modeled with $\mu(x, y)$ as a decreasing function of density x which becomes zero when the population reaches its maximal value x_m :

$$\frac{\partial \mu(x, y)}{\partial x} \leq 0, \quad \mu(x, y) = 0 \quad \forall x \geq x_m.$$

Example 5.8. The Contois model. This is classic model of specific rate used to describe the growth of micro-organisms populations :

$$\mu(x, y) = \frac{\mu_0 y}{y + Kx}.$$

It is observed this model is an increasing bounded function of y (identical to the Michaelis-Menten model for x fixed) and a decreasing (hyperbolic) function of x . **However satiety and overpopulation concentration limits (y_c and x_m respectively) are rejected to infinity.** \square

Example 5.9. The logistic model

It's common to adopt the following multiplicative structure for the specific growth rate :

$$\mu(x, y) = \sigma(y)\phi(x).$$

This structure allows to model separately satiety and overpopulation effects, for example in the following way :

$$\sigma(y) = \begin{cases} \alpha y & \forall y \leq y_c \\ \alpha y_c & \forall y \geq y_c \end{cases}$$

$$\phi(x) = \begin{cases} (1 - \frac{x}{x_m}) & \forall x \leq x_m \\ 0 & \forall x \geq x_m \end{cases}$$

It is noticeable that the functions σ and ϕ are linear and saturated (see figure 5.5). With these definitions, when $y \leq y_c$ and $x \leq x_m$, the prey - predator model (5.13)

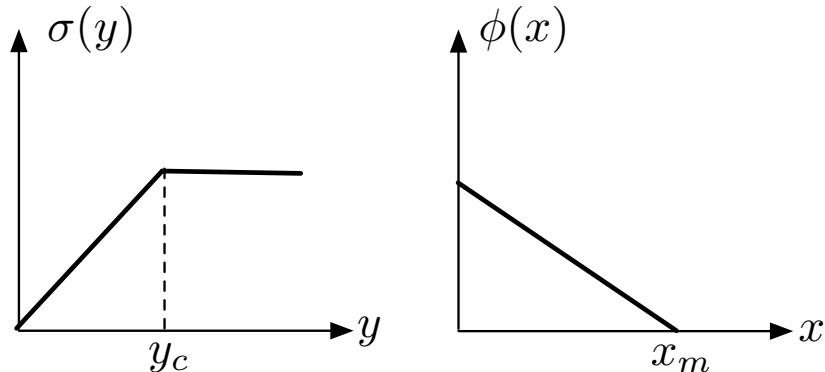


FIGURE 5.5 – Specific growth rate for the logistic model

can be written as

$$\begin{aligned}\dot{y} &= -k\alpha xy\left(1 - \frac{x}{x_m}\right) + v, \\ \dot{x} &= \alpha xy\left(1 - \frac{x}{x_m}\right) - dx.\end{aligned}$$

On the other hand, when the food supply is provided to the system in sufficiently large quantities to maintain concentration above its critical value ($y(t) \geq y_c \ \forall t$), then the dynamic of the predator population becomes *independent of the available amount of food supply* and can be written as :

$$\dot{x} = \sigma_c x \left(1 - \frac{x}{x_m}\right) - dx \quad (5.14)$$

where $\sigma_c = \alpha y_c$. The function $\phi(x) = (1 - x/x_m)$ is usually called *logistic model* in the literature. By extension, the model (5.14) is called logistic model of a population growth on a non-limiting food supply. \square

Up to now we have considered a simple model which only involves two species X and Y . This description can be extended without difficulty to more complex ecosystems in which several biological species (vegetable or animal) can coexist and interact within the same habitat. Here is an example.

Example 5.10. An aquatic ecosystem

An aquatic ecosystem, as all other natural ecological system, is usually characterized by the cohabitation of three kinds of biological species : vegetable, herbivorous animal and carnivorous animal species. For example, let's us consider a pond (see figure 5.6) in which is released an organic nutrient X_1 . A population of algae (phytoplankton) X_2 grows by consumption of this nutrient. A population of tiny herbivorous crustaceans X_3 grazes on the phytoplankton which is its principal food

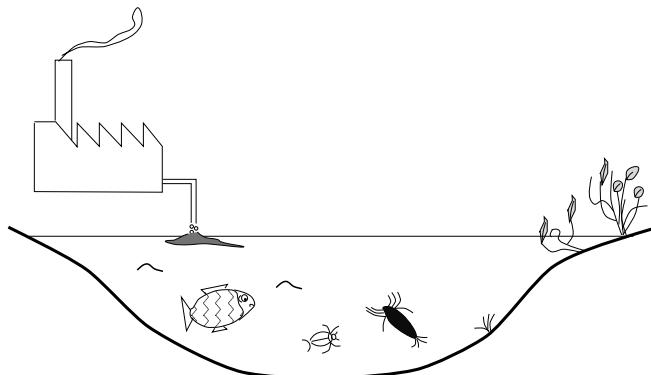
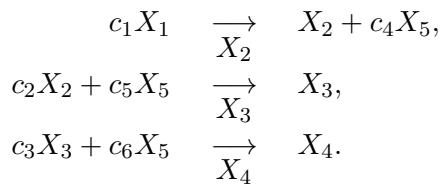


FIGURE 5.6 – Aquatic ecosystem

supply. A population of carnivorous fish X_4 ensure its development and its growth by consumption of the crustaceans. The animal breathing consumes oxygen X_5 in the water produced by the photosynthesis . This description is represented by the following reaction network :



A state model of this system is realized under the following hypothesis and with these notations :

- The organic nutrient is released with a flow rate per unit of volume v .
- The three biological species are subject to natural mortality. The mortality coefficients are labeled $d_i, i = 2, 3, 4$. The coefficient of proportionality is noted d_1 .
- The growth kinetic of algae is described by the logistic model, with a Michaelis Menten dependency to nutrient concentration
- Both kinetics of animal populations are described by the Contois model with a Michaelis Menten dependency to the oxygen concentration in the water.

The state model of this aquatic ecosystem is written as :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} -c_1 & 0 & 0 \\ 1 & -c_2 & 0 \\ 0 & 1 & -c_3 \\ 0 & 0 & 1 \\ c_4 & -c_5 & -c_6 \end{pmatrix} \begin{pmatrix} \frac{\mu_1 x_1 x_2}{x_1 + K_1} \left(1 - \frac{x_2}{x_{2c}}\right) \\ \frac{\mu_2 x_2 x_3}{x_2 + K_2 x_3} \frac{x_5}{x_5 + K_4} \\ \frac{\mu_3 x_3 x_4}{x_3 + K_3 x_4} \frac{x_5}{x_5 + K_5} \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & 0 & d_1 + d_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The state variables x_2, x_3, x_4 denote the density of the three biological populations while x_1 et x_5 denote respectively the nutrient and oxygen concentrations. \square

5.6. Exercices

Exercice 5.1. Un procédé chimique

Une installation de génie chimique est représentée à la figure 5.7. Une réaction réversible $A + B \leftrightarrow C$, obéissant à la loi d'action des masses, se déroule dans le réacteur. Le séparateur est supposé opérer une séparation parfaite et instantanée

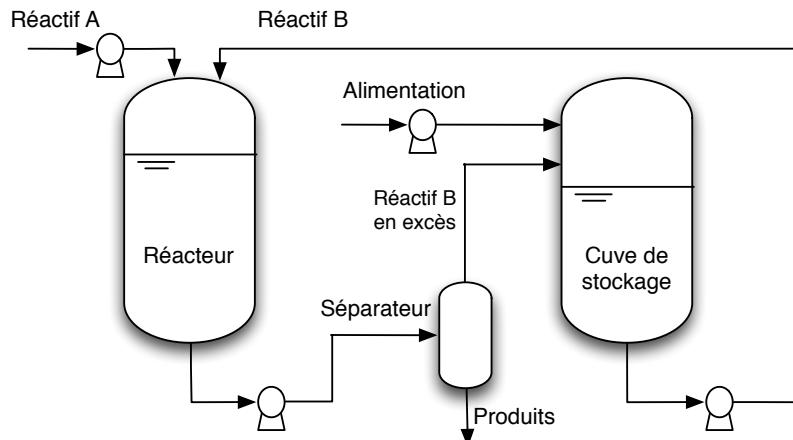


FIGURE 5.7 – Un procédé chimique

des trois espèces chimiques. Le réactif B est recyclé via une cuve de stockage. Le

réactif A et le produit C sont soutirés du système. Proposer un modèle d'état du système. \square

Exercice 5.2. Réacteur avec alimentations séparées

Nous avons considéré dans ce chapitre que les différentes espèces qui alimentent un réacteur sont fournies ensemble par une canalisation unique (voir par exemple la figure 5.2). Un tel dispositif peut avoir l'inconvénient de voir les réactions débuter dans la canalisation d'amenée avant d'atteindre le réacteur. Cet inconvénient est évité si les réactifs sont introduits dans le réacteur par des canalisations séparées. Reconsidérons l'exemple 5.6 avec des alimentations séparées pour les deux réactifs X_1 et X_2 (voir figure 5.8) :

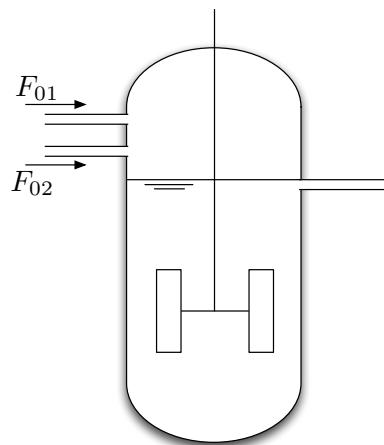


FIGURE 5.8 – Réacteur continu avec alimentations séparées

1. Etablir un modèle d'état du système si les variables d'entrée sont les deux débits volumique d'alimentation F_{01} et F_{02} .
2. Un cas particulier intéressant est celui où le réacteur est alimenté à débit volumique total constant ($F_{01} + F_{02} = \text{constante}$). Seule la composition de l'alimentation est variable. En pratique cela peut être réalisé en ajustant complémentairement les deux débits F_{01} et F_{02} avec une vanne à quatre voies (voir figure 5.9) de manière que leur somme soit constante. On choisit le débit F_{01} comme unique variable d'entrée et on définit le taux de dilution constant $d = (F_{01} + F_{02})/V$. Etablir le modèle d'état du système et montrer qu'il s'écrit sous la forme (5.7). \square

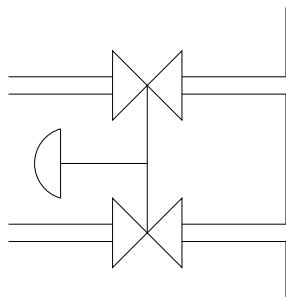


FIGURE 5.9 – Alimentations séparées avec vanne à quatre voies

Exercise 5.3. Gazeous reagents and products

The state model (5.4.1) of a continuous reactor perfectly mixed can be extended for gaseous reagents or products. Assume, first, that the reactor is supplied by a reagent X in gaseous form (for example by oxygen) with a mass flow Q_{in} (see figure 5.10). [The reagent splashes in the liquid environment where it is partially dissolved](#). The undissolved reagent excess escapes freely from the reactor in gaseous form with a mass flow Q_{out} . The amount of reagent put in solution by unit time is $Q_{in} - Q_{out}$. Basing on the Henry's law and neglecting the gas-liquid dynamic transfer, we can model this amount as being proportional to the gaseous feed flow on the one hand and to the saturation deficit on the other hand :

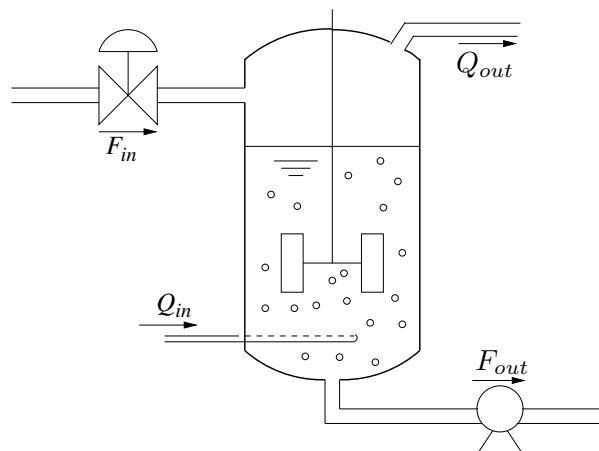


FIGURE 5.10 – Reactor with gaseous reagents and products

[The reagent splashes in the liquid environment where it is partially dissolved](#). The undissolved reagent excess escapes freely from the reactor in gaseous form with a mass flow Q_{out} . The amount of reagent put in solution by unit time is $Q_{in} - Q_{out}$. Basing on the Henry's law and neglecting the gas-liquid dynamic transfer, we can model this amount as being proportional to the gaseous feed flow on the one hand and to the saturation deficit on the other hand :

$$Q_{in} - Q_{out} = aQ_{in}(x^{sat} - x)$$

Where x is the concentration of the species X in solution and x^{sat} is the saturation concentration of this species in the liquid phase.

Consider now that a reaction product X (for example CO_2) formed in solution is **gasifiable**. It escapes the reaction medium with a mass flow Q_{out} . With a balance hypothesis between the liquid and gaseous phase, we can consider that this flow is proportional to the concentration of the product X in solution in a reactional environment :

$$Q_{out} = dx$$

1. As in the Example 5.6, we consider a continuous reactor in which the two reactions (6.1)-(??) take place simultaneously in the liquid phase with the kinetic (5.11). This time, we assume that the reagent X_2 and the product X_4 are in gaseous form. It is asked to establish the state model of the system under the following modeling hypotheses :
 - The reactor is supplied by the initial reagent X_1 in solution with the volumetric flow F_{in} and a feed concentration X_1^{in} .
 - The reagent X_2 is injected in the reactor in gaseous form. The amount of reagent X_2 put in solution by unit of time is noted $aQ_{in}(x_2^{sat} - x_2)$.
 - The products X_3 and X_4 are formed in solution in the reaction medium. The product X_4 is **gasifiable** and escapes the reactor with a gaseous flow dx_4 .
2. If the input variables are the liquid feed volumetric flow per unit of reaction medium's volume $u_1 = F_{in}/V$ and the gaseous feed mass flow per unit of reaction medium's volume $u_2 = Q_{in}/V$, show that the state model has the structure (5.7). \square

Exercise 5.4. A biochemical reactor

A biochemical reactor working in CSTR mode involves 3 species : a bacterial population X_1 , glucose X_2 and lactose X_3 .

The reactor's dynamic is described by the following state model (x_i designates the concentration of species x_i) :

$$\begin{aligned}\dot{x}_1 &= x_1 x_2 - ux_1, \\ \dot{x}_2 &= -x_1 x_2 + x_1 x_3 - ux_2, \\ \dot{x}_3 &= -x_1 x_3 + u(c - x_3) \quad c > 0.\end{aligned}$$

1. What is the reaction scheme ?
2. The input u is positive : $u > 0$. **What does it represent physically ?**
3. Show that the system is positive \square

Exercise 5.5. Ladybugs and aphids

Show that the system (1.6) of the chapter 1 modeling the interaction between the ladybug and aphid populations is a reaction system. \square

Exercise 5.6. An aerobic biological treatment plant

An aerobic biological treatment plant is represented at the figure 5.11. The

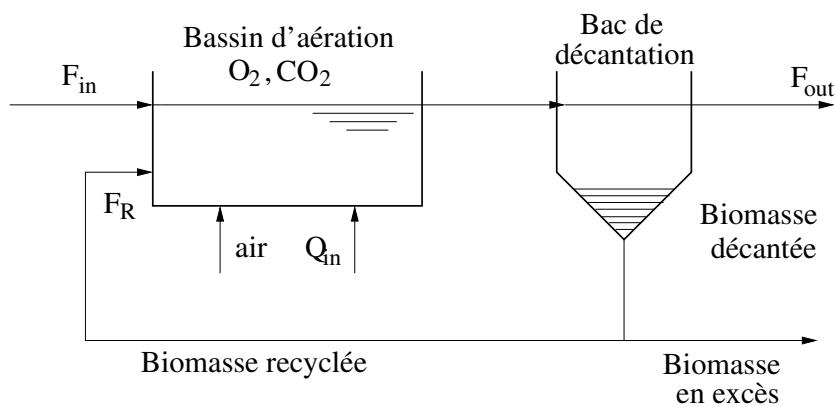


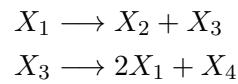
FIGURE 5.11 – Anaerobic biological treatment plant

aeration basin is supplied by waste water (flow F_{in}) containing a pollutant organic substrate (concentration S). This organic substrate is degraded by aerobic micro-organisms (concentration X). This degradation requires oxygen dissolved in the water (concentration O) and products carbon dioxide (concentration C) in dissolved form but which easily gasifies and goes out of the system in gaseous form. The dissolved oxygen is provided by an aeration system (air flow Q_{in}). We assume that the [transfer processes](#) between the gaseous and liquid phase are negligible (instant reaction).

The output of the aeration basin is connected to a sedimentation tank ([decantation](#)) where the biomass (i.e. the mass of micro-organisms) is isolated. The clarified water is evacuated of the system (flow F_{out}). The biomass is recycled towards the aeration basin (flow F_R). However, we consider eliminating excess biomass (flow F_S). Levels in the aeration basin and in the decanter are supposed constant. The aeration basin is supposed perfectly mixed. The decantation basin (which cannot be perfectly mixed !) is modeled by two tanks (compartments) perfectly mixed (one for the clarified water, one for the decanted biomass). We so assume that there is no biologic reaction in the decanter. It is asked to establish a state model of the system. \square

Exercise 5.7. A non-conservative system

Consider the following [reaction chain](#) :



1. Establish a state model of a closed reactional system under the following modeling hypotheses : principle of masses action for the first reaction with a 2 order reaction speed for all the reagents, Michaelis-Menten's kinetic for the second reaction with hyperbolic inhibition by X_2 .
2. Show that the system is non-conservative. Give a physical justification.
3. Show that it's sufficient to add an initial reagent in the first or second reaction to make the system conservative. □

Chapitre 6

State transformation

In the previous chapters, we have shown how the modeling process can be systematized for different classes of systems relating to engineering. For each type of system, a general state model has been established. The state variables in these models have precise physical meaning : positions and speeds for the mechanical systems, currents and tensions for the electrical systems, total amount for the systems with compartments, concentrations, volume and temperature for the reaction systems. However, it's often helpful to analyze the behavior of a dynamic system to proceed of a state transformation leading to an equivalent model of the system but expressed with new state variables.

In addition to state transformations, it's also interesting to use graphical depictions to see easily some structural features of the system. Among the most common representations, the *function diagram* and the *graph of the system* defined below will be mentioned.

6.1. Function diagram

The function diagram of a dynamic system is an oriented graph where each node is one of the two function blocks represented at the figure 6.1.

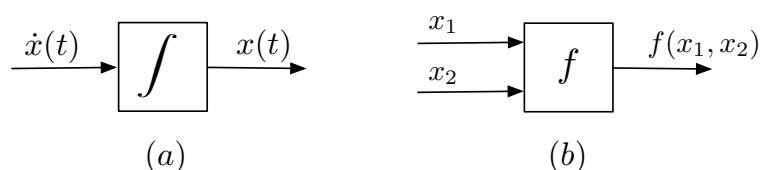


FIGURE 6.1 – function blocks : (a) integrator, (b) function

- The function block Fig. 6.1 (a) represents an integrator where the input variable is the derivative of the output variable.
- The function block Fig. 6.1 (b) represents a function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ where the output variable $z(t)$ is a function of the input variables :

$$z(t) = f(x_1(t), x_2(t), \dots, x_p(t)).$$

In some cases, the drawing of this block is particularized in order to make explicit the function which it represents. Three examples are shown in the figure 6.2. The function diagram of a dynamic system contains necessarily n integrators for which the outputs are the n state variables of the system. These integrators are interconnected via function blocks representing the different functions appearing in the state equations. **Function diagram's arcs are interpreted as instant transmission of variables attached to them.**

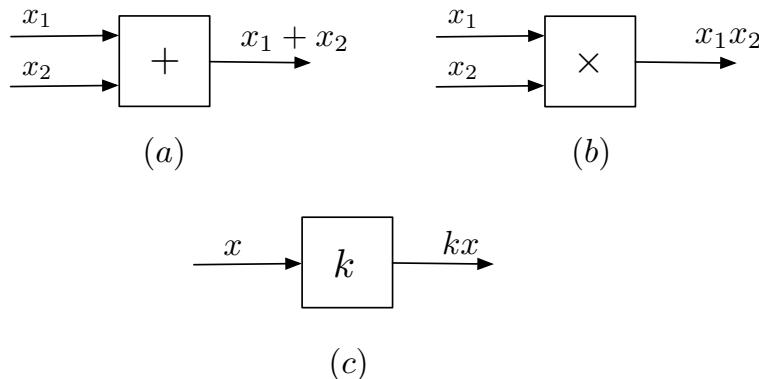


FIGURE 6.2 – examples of function blocks : (a) adder, (b) multiplier, (c) product by a constant.

In addition to their importance for the analyze of the dynamic systems, function diagrams are also a programming fundamental tool in standard languages of dynamic simulation MATLAB/Simulink or VisSim.

Example 6.1. Algae in the lagoon (continuation)

In the previous chapter, we have established a simple model describing the growing process of an algae population in a lagoon. Assuming that the growing kinetic follows a bilinear law $r(x_1, x_2) = x_1x_2$, this model is written :

$$\begin{aligned}\dot{x}_1 &= -kx_1x_2 + u, \\ \dot{x}_2 &= x_1x_2 - dx_2.\end{aligned}$$

The corresponding function diagram is represented at the figure 6.3. □

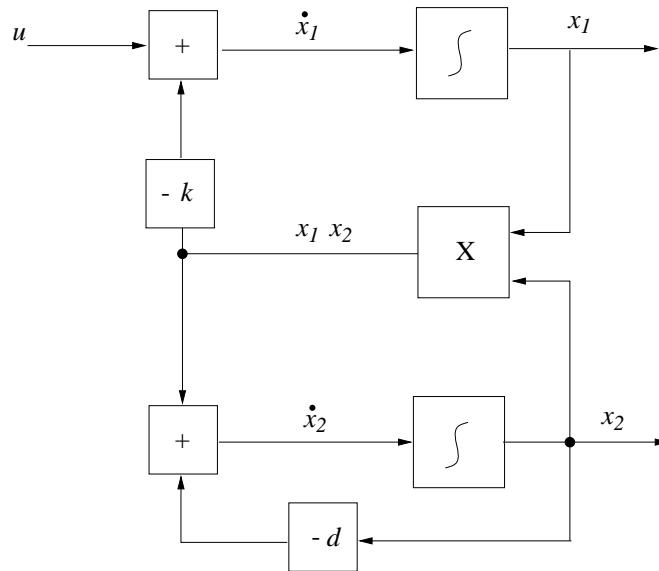


FIGURE 6.3 – Function diagram of the algae growing model

6.2. Graph of a dynamic systeme

The graph of a dynamic system is, in some way, the complementary graph of the function diagram. Indeed, nodes of the graph contain the state variables x_i and the input variables u_j while (oriented) arcs represent the functional relationship between these variables.

The construction's rules of a graph of a dynamic system are the following :

1. The graph contains $n+m$ nodes respectively labeled by the n state variables x_1, x_2, \dots, x_n and the m input variables u_1, u_2, \dots, u_m .
2. There is an oriented arc from x_i to x_j (or from u_k to x_j) if the variable x_i (or u_k) appears explicitly in the equation of the derivative of \dot{x}_j .

Example 6.2. DC electrical machine

We consider the general model of a DC machine presented in chapter 3, section 3.5. It's a system with four state variables and three input variables whose model state is written :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= J^{-1}(-h(x_2) + K_m x_3 x_4 + u_3), \\ \dot{x}_3 &= L_f^{-1}(-R_f x_3 + u_1), \\ \dot{x}_4 &= L_a^{-1}(-R_a x_4 - K_e x_2 x_3 + u_2).\end{aligned}$$

The graph of this system is represented at the figure 6.4. \square

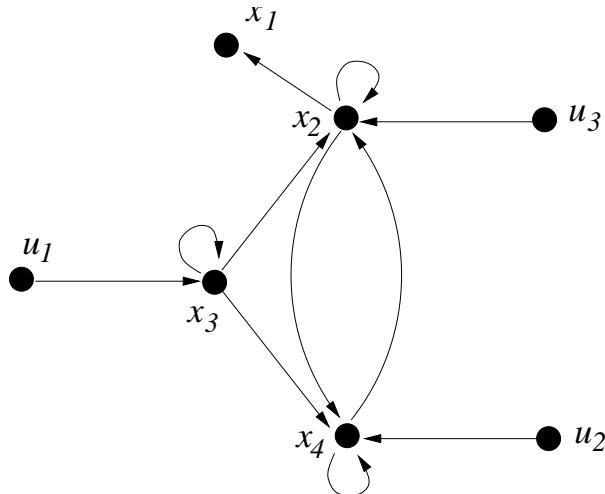


FIGURE 6.4 – graph of the state model of a DC engine

The graph of a dynamic system is a tool to verify easily if the considered system has some interesting structural features. We will see an illustration in section 6.7 when we will study triangular systems.

6.3. State linear transformation

For a dynamic system $\dot{x} = f(x, u)$, a state linear transformation is a linear application $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is bijective and transforms the state of the system $x \in \mathbb{R}^n$ in a new state $z \in \mathbb{R}^n$ following the rule :

$$z = Tx$$

where T is a regular ($n \times n$) matrix.

In the new coordinates z , the system's state model is transformed as :

$$\dot{z} = T\dot{x} = Tf(x, u)$$

Expressing that $x = T^{-1}z$ we obtain :

$$\dot{z} = g(z, u) \quad \text{with} \quad g(z, u) \triangleq Tf(T^{-1}z, u).$$

In particular, a linear state model $\dot{x} = Ax + Bu$ is transformed in another linear model :

$$\dot{z} = Fz + Gu \quad \text{with} \quad F \triangleq TAT^{-1}, \quad G \triangleq TB.$$

Example 6.3. DC generator

At chapter 3 (Section 3.6.) we have established the state model of a DC generator. When the generator rotates at constant speed ω , the state model is linear and is written :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{R_s}{L_s} & 0 \\ \frac{K_e \omega}{L_r} & -\frac{R_r + R_L}{L_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L_s} \\ 0 \end{pmatrix} u$$

where the state variables x_1 and x_2 represent respectively the stator and rotor current, while the input u is the applied tension to the stator system.

We define new state variables z_1 and z_2 which can be interpreted as magnetic flow ϕ_s and ϕ_r through the stator and rotor systems respectively :

$$\begin{aligned} z_1 &= \phi_s = L_s x_1, \\ z_2 &= \phi_r = L_r x_2 + K_e x_1. \end{aligned}$$

We see that it's well a linear state transformation :

$$T = \begin{pmatrix} L_s & 0 \\ K_e & L_r \end{pmatrix}.$$

The matrix T is invertible ($\det T = L_s L_r > 0$) the reverse transformation is written :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{L_s} & 0 \\ -\frac{K_e}{L_s L_r} & \frac{1}{L_r} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

In the new coordinates (z_1, z_2) , the state model is written :

$$\begin{aligned} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} &= \begin{pmatrix} -\frac{R_s}{L_s} & 0 \\ \frac{K_e \omega}{L_r} + \frac{K_e(R_r + R_L)}{L_r L_s} - \frac{K_e R_s}{L_s^2} & -\frac{R_r + R_L}{L_r} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &+ \begin{pmatrix} 1 \\ \frac{K_e}{L_s} \end{pmatrix} u \end{aligned}$$
□

Example 6.4. Compartments linear model

We are interested in linear compartments models as described in the section 4.4. Remind that the general form of equations state is :

$$\dot{x}_i = \sum_{j=1}^n k_{ji}x_j - \sum_{\ell=0}^n k_{i\ell}x_i + b_i u_i, \quad i = 1, n$$

or in matrix form :

$$\dot{x} = Ax + Bu$$

with A a Metzler's matrix diagonally dominant and x_i the total amount contained in the compartment i . We want express the model in terms of concentrations. We introduce following notations :

$$\begin{aligned} V_i &: \text{volume of compartment } i, \\ a_{ij} &\triangleq k_{ij}V_i, \\ z_i &= \frac{x_i}{V_i} : \text{concentration in the compartment } i. \end{aligned}$$

With these notations, we can rewrite the model as :

$$\begin{aligned} \dot{x}_i &= \sum_{j=1}^n \frac{a_{ji}}{V_j}x_j - \sum_{l=0}^n \frac{a_{il}}{V_i}x_i + b_i u_i, \\ \dot{x}_i &= \sum_{j=1}^n a_{ji}z_j - \sum_{\ell=0}^n a_{i\ell}z_i + b_i u_i, \end{aligned}$$

and therefore :

$$\dot{z}_i = \sum_{j=1}^n \frac{a_{ji}}{V_i}z_j - \sum_{\ell=0}^n \frac{a_{i\ell}}{V_i}z_i + \frac{b_i}{V_i}u_i$$

We have done a state transformation converting the total amount x_i to concentrations z_i as state variables. In matrix form the state transformation is written :

$$z = V^{-1}x \text{ avec } V \triangleq \text{diag}\{V_i, i = 1, \dots, n\}$$

In the concentration coordinates, the model becomes :

$$\dot{z} = Fz + Gu$$

with $F \triangleq V^{-1}AV$ et $G \triangleq V^{-1}B$. We can verify that the matrix F^T is also a Metzler matrix diagonally dominant. \square

Example 6.5. Diagonalisation et constantes temps

On considère un modèle linéaire $\dot{x} = Ax + Bu$ dont la matrice A a toutes ses valeurs propres λ_i réelles, distinctes et non-nulles. Elle est alors diagonalisable, c'est à dire qu'il existe une matrice T telle que

$$D \triangleq TAT^{-1} = \text{diag}(\lambda_i, i = 1, n)$$

Si on définit une transformation d'état :

$$z = Tx$$

le système est transformé en :

$$\dot{z} = Dz + TBu$$

ou, encore composante par composante :

$$\dot{z}_i = \lambda_i z_i + \beta_i u \quad i = 1, n$$

où β_i est la i -ème ligne de la matrice TB . Les grandeurs $\tau_i = |\lambda_i|^{-1}, i = 1, \dots, n$, sont les *constantes de temps du système*.

We have now operate a state transformation going from total quantities x_i to concentrations z_i as state variables. In a matrix format the state transformation can be written as :

$$z = V^{-1}x \text{ avec } V \triangleq \text{diag}\{V_i, i = 1, \dots, n\}$$

In the concentration coordinates, the model becomes :

$$\dot{z} = Fz + Gu$$

with $F \triangleq V^{-1}AV$ et $G \triangleq V^{-1}B$. We can check that the matrix F^T is also a diagonally dominant Metzler matrix. \square

Example 6.6. Diagonalization and time constants

Let us consider a linear model $\dot{x} = Ax + Bu$ in which the eigen values λ_i of the matrix A are real, distinct and non-zero. Then the matrix A is diagonalizable, which means there is a matrix T such that

$$D \triangleq TAT^{-1} = \text{diag}(\lambda_i, i = 1, n)$$

If we define a state transformation :

$$z = Tx$$

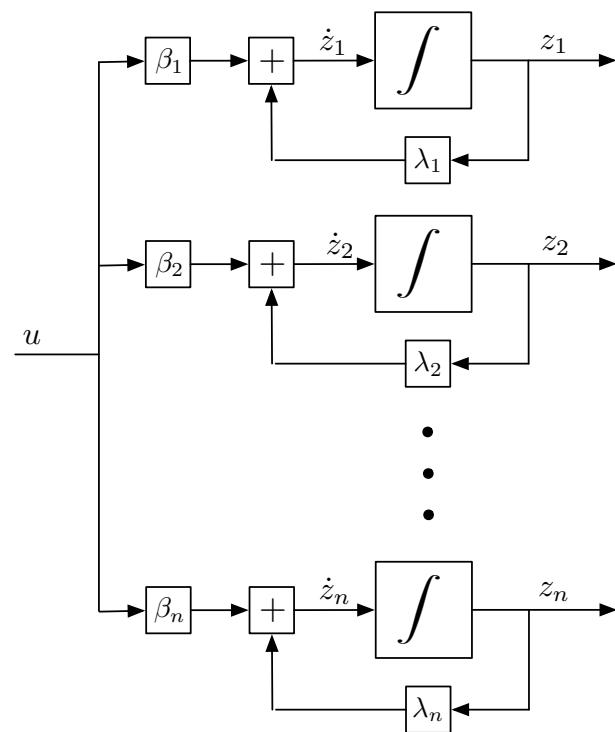


FIGURE 6.5 – Schéma fonctionnel d'un système diagonalisé à une entrée

The system is then transformed into :

$$\dot{z} = Dz + TBu$$

or, components by components :

$$\dot{z}_i = \lambda_i z_i + \beta_i u \quad i = 1, n$$

where β_i is the i -th row of the matrix TB . The parameters $\tau_i = |\lambda_i|^{-1}$, $i = 1, \dots, n$, are the *time constants of this system*.

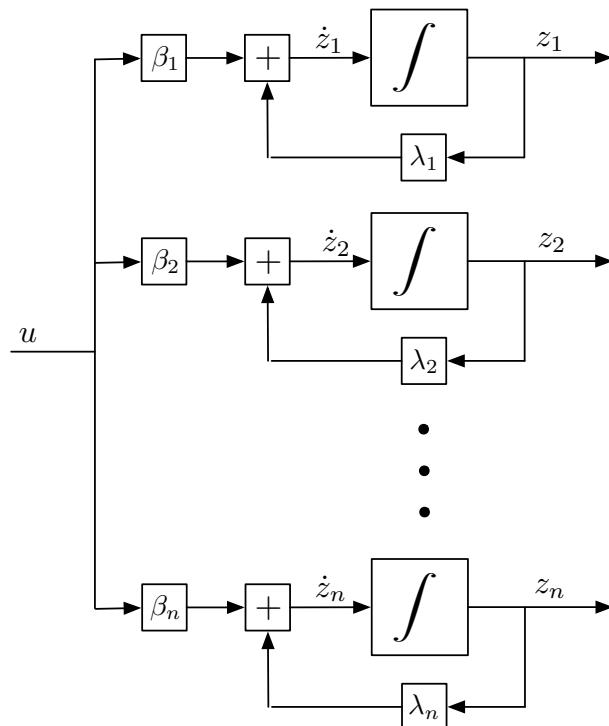


FIGURE 6.6 – Block diagram of a diagonalized system with one entry

We have now replaced the initial model in which the state variables can be strongly related by a collection of first-order systems completely separated from each other as we can see on the block diagram illustrated in figure 6.6.

For instance let us consider a DC motor controlled by the stator (see chapter 3, section 3.6) with $h(\omega) = B\omega$:

$$\frac{d}{dt} \begin{pmatrix} I_s \\ \omega \end{pmatrix} = \begin{pmatrix} -\frac{R_s}{L_s} & 0 \\ \frac{K_m I_r}{J} & -\frac{B}{J} \end{pmatrix} \begin{pmatrix} I_s \\ \omega \end{pmatrix} + \begin{pmatrix} \frac{1}{L_s} u_1 \\ \frac{1}{J} u_2 \end{pmatrix},$$

we check that time constants are

$$\tau_e = \frac{L_s}{R_s} \text{ electric time constant,}$$

$$\tau_m = \frac{J}{B} \text{ mechanical time constant.}$$

□

Example 6.7. Reaction systems as multi-compartment systems

In chapter 5, we have seen that the state model for reaction systems can be written as

$$\dot{x} = Cr(x) + q_{in}(x, u) - q_{out}(x, u).$$

Let us introduce the following notations for entry and output vectors :

$$q_{in}(x, u) \triangleq \left(q_{o1}(x, u), q_{o2}(x, u), \dots, q_{on}(x, u) \right)^T,$$

$$q_{out}(x, u) \triangleq \left(q_{1o}(x, u), q_{2o}(x, u), \dots, q_{no}(x, u) \right)^T.$$

Let us now suppose that the system is conservative and that the flows q_{oi} and q_{io} meet the conditions C1, C2 and C3 from chapter 4. Then the reaction system is equivalent to a multi-compartments system with the following linear state transformation :

$$z = Tx, \quad T \triangleq \text{diag}\{\omega_2, \omega_2, \dots, \omega_n\}.$$

To illustrate that property, let us consider again the example of the perfectly mixed chemical reactor (see example 5.6). In that reactor, the two reactions



occur simultaneously in the liquid phase with kinetics

$$\begin{aligned} r_1(x) &= k_1 x_1 x_2 e^{-(Kx_4)}, \\ r_2(x) &= k_2 x_3^2. \end{aligned} \tag{6.2}$$

The reactor is fed by the two initial reactives X_1 and X_2 in solution with feeding concentrations x_1^{in} and x_2^{in} .

The state model can be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \\ 2 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 x_1 x_2 e^{-(Kx_4)} \\ k_2 x_3^2 \end{pmatrix} + u \begin{pmatrix} x_1^{in} - x_1 \\ x_2^{in} - x_2 \\ -x_3 \\ -x_4 \end{pmatrix}$$

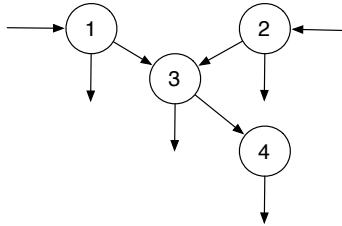


FIGURE 6.7 – Compartment representation of a reaction system

where the state variables x_1, x_2, x_3 and x_4 represents the species concentrations in the reaction medium.

We can easily check that the system is conservative with the normalization vector $\omega = (1, 1, 1, 2)$. Thus, we define the linear state transformation

$$z_1 = x_1, \quad z_2 = x_2, \quad z_3 = x_3, \quad z_4 = 2x_4.$$

In those new coordinates, we obtain a multi-compartment system of which the graph is illustrated in figure 6.7 and of which the state model is :

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} -k_1 z_2 \varphi - u & 0 & 0 & 0 \\ 0 & -k_1 z_1 \varphi - u & 0 & 0 \\ k_1 z_2 \varphi & k_1 z_1 \varphi & -2k_2 z_3 - u & 0 \\ 0 & 0 & 2k_2 z_3 & -u \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} ux_1^{in} \\ ux_2^{in} \\ 0 \\ 0 \end{pmatrix}$$

with

$$\varphi \triangleq \exp\left(-\frac{K}{2}z_4\right). \quad \square$$

6.4. Non-linear state transformations

For a non-linear state model $\dot{x} = f(x, u)$, it is often more interesting to consider non-linear state transformations. However it is generally not possible to define *global* transformations that are valid for every $x \in \mathbb{R}^n$. We are then interested in *local* transformations that are defined in a susbset of \mathbb{R}^n .

Definition 6.8. Non-linear state transformation

Let U and V be two open subset of \mathbb{R}^n . A non-linear state transformation is an application $T : U \rightarrow V$ that transforms the state of the system $x \in U$ in a new state $z \in V$:

$$z = T(x)$$

and that possess the following properties :

- a) the application T is bijective, which means that there is an inverse function $T^{-1} : V \rightarrow U$ such that $x = T^{-1}(z)$,
- b) $T(x)$ and $T^{-1}(z)$ are functions of class C^1 , that is to say continuous and differentiable.

The state transformation is said to be *global* if $U = V = \mathbb{R}^n$. \square

A transformation T possessing those properties is called a diffeomorphism. Its bijectivity is necessary to reverse the state variables change and to go back to the initial state variables. The property b) (T and T^{-1} are of classes C^1) is necessary to express the state model in the new coordinates as follows :

$$\dot{z} = \frac{\partial T}{\partial x} \dot{x} = \frac{\partial T}{\partial x} f(x, u)$$

where, by using $x = T^{-1}(z)$, we obtain

$$\dot{z} = g(z, u)$$

with :

$$g(z, u) \triangleq \left[\frac{\partial T}{\partial x} f(x, u) \right]_{x=T^{-1}(z)}.$$

In a similar way, we can express :

$$f(x, u) \triangleq \left[\frac{\partial T^{-1}}{\partial z} g(z, u) \right]_{z=T(x)}.$$

The properties given in the following lemma can be useful to demonstrate the existence of a non-linear state transformation.

Lemma 6.9.

1. If the jacobian matrix $[\partial T / \partial x]$ is non-singular at x_0 , then, by the inverse function theorem, there is a neighborhood U around x_0 such that the application T restricted to U is a diffeomorphism on U .
2. T is a global diffeomorphism if and only if :
 - a) $[\partial T / \partial x]$ is non-singular for every x in \mathbb{R}^n ;
 - b) $\lim_{\|x\| \rightarrow \infty} \|T(x)\| = \infty$. \square

6.5. Mechanical systems

As we have seen in chapter 2, the state vector of a mechanical system is made of two parts : the position coordinates q and the speed coordinates $v = \dot{q}$

$$x = \begin{pmatrix} q \\ v \end{pmatrix}.$$

In numerous applications, it is interesting to consider different sets of position coordinates. The state transformation is then made through two steps. First we transform the position coordinates :

$$p = \phi(q)$$

where $\phi : U_1 \rightarrow V_1$ is a diffeomorphism and $\partial\phi/\partial q$ is of full rank $\forall q \in U_1$.

The new state vector is then formed by the new position coordinates p and their derivatives $w = \dot{p}$:

$$z = \begin{pmatrix} p \\ w \end{pmatrix}.$$

The state transformation is then defined as follows :

$$z = T(x), \quad \begin{pmatrix} p \\ w \end{pmatrix} = \begin{pmatrix} \phi(q) \\ \frac{\partial\phi}{\partial q} v \end{pmatrix}.$$

The inverse state transformation is :

$$x = T^{-1}(z), \quad \begin{pmatrix} q \\ v \end{pmatrix} = \begin{pmatrix} \phi^{-1}(p) \\ \left(\frac{\partial\phi}{\partial q}\right)^{-1}_{q=\phi^{-1}p} w \end{pmatrix}.$$

Example 6.10. Polar and cartesian coordinates

In the method described in chapter 2 to establish a state model for articulate mechanical systems, the position of each body's center of mass is given by its cartesian coordinates $q = (x, y)$, as showed on the figure 6.8. Another set of

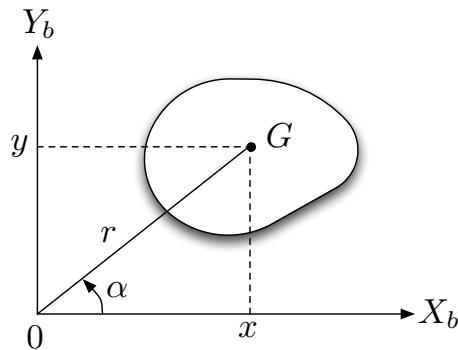


FIGURE 6.8 – Cartesian coordinates and polar coordinates

frequently used coordinates are the polar coordinates r and α : r is the distance between the origin and the center of mass and α is the angle between the axis OX_b and the vector \vec{OG} .

The transformation allowing to go from cartesian coordinates to polar coordinates can be written as :

$$q = \begin{pmatrix} x \\ y \end{pmatrix} \quad p = \begin{pmatrix} r \\ \alpha \end{pmatrix},$$

$$p = \phi(q) : \begin{cases} r = \sqrt{x^2 + y^2}, \\ \alpha = \arcsin \frac{y}{\sqrt{x^2 + y^2}} \end{cases} .$$

The inverse transformation $q = \phi^{-1}(p)$ is written :

$$\begin{aligned} x &= r \cos \alpha, \\ y &= r \sin \alpha. \end{aligned}$$

We notice that the change of coordinates $p = \phi(q)$ is not defined at the origin, that is when $x = 0$ and $y = 0$. We also check that

$$\det \left[\frac{\partial \phi^{-1}}{\partial p} \right] = r$$

is zero when $r = 0$ (which is the origin also). It follows that the transformation of coordinates is not global but only valid on the following sets :

$$\begin{aligned} U_1 &= \mathbb{R}^2 \setminus \{(0, 0)\}, \\ V_1 &= \mathbb{R}^2 \setminus \{(r, \alpha) : r = 0\}. \end{aligned}$$

Finally, the complete state transformation between the state (q, v) and the state (p, w) is written as follows :

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \alpha &= \arcsin \frac{y}{\sqrt{x^2 + y^2}}, \\ \dot{r} &= \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}}, \\ \dot{\alpha} &= \frac{x\dot{y} - y\dot{x}}{x^2 + y^2}, \end{aligned}$$

and the inverse transformation :

$$\begin{aligned} x &= r \cos \alpha, \\ y &= r \sin \alpha, \\ \dot{x} &= \dot{r} \cos \alpha - r \dot{\alpha} \sin \alpha, \\ \dot{y} &= \dot{r} \sin \alpha + r \dot{\alpha} \cos \alpha. \end{aligned}$$

□

Example 6.11. Articular coordinates and robotic task coordinates

For manipulator robots consisting of as many actuators as degrees of freedom, with rotoid joints, articular coordinates from chapter 2 are » natural » coordinates for the description of the system : every coordinate gives the position of an arm compared to the previous one. Generally, with those coordinates, the model becomes quite simple. The articular models are adequate for the conception of systems designed to control robots.

On the point of view of the user, interested for instance by planning trajectories, the task coordinates, that is the coordinates of the effector, are more interesting. Let us consider for instance a planar robot with two degrees of freedom moving in a horizontal plane (see figure 6.9). The articular coordinates are the angles α_1

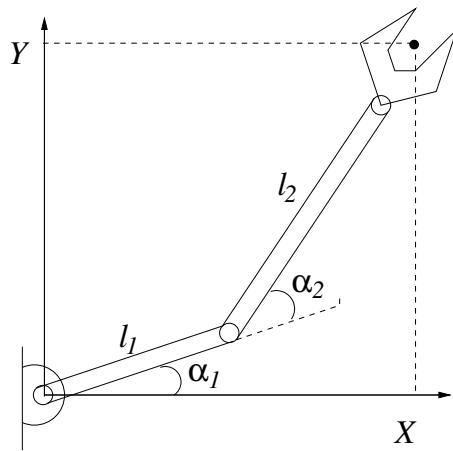


FIGURE 6.9 – Articular coordinates and task coordinate for a robot with two degrees of freedom.

and α_2 , the tasks coordinates are the cartesian coordinates X and Y . Then we have : $q = (\alpha_1, \alpha_2)$ et $p = \phi(q) = (X, Y)$. The transformation allowing to go from articular coordinates to task coordinates are written

$$X = l_1 \cos \alpha_1 + l_2 \cos(\alpha_1 + \alpha_2), \quad (6.3)$$

$$Y = l_1 \sin \alpha_1 + l_2 \sin(\alpha_1 + \alpha_2). \quad (6.4)$$

We easily check that this transformation can not be injective : a position (X, Y) of the effector corresponds to two distinct and symmetrical positions of the robot. To correctly define a transformation of coordinates, we must define the domains U and V corresponding to the application ϕ and its inverse.

We first observe that the image of the application ϕ is necessarily restricted to the circle of accessible positions for the robot, that is (if $l_2 > l_1$) a circle of radius

$l_1 + l_2 :$

$$V_1 \triangleq \{(X, Y) : (l_2 - l_1)^2 < X^2 + Y^2 < (l_1 + l_2)^2\}.$$

On the other hand, the domain of ϕ must be chosen so the application is injective. A possible choice is the following :

$$U_1 \triangleq \{(\alpha_1, \alpha_2) : -\pi < \alpha_1 < \pi \quad 0 < \alpha_2 < \pi\}.$$

With those definitions, we can check that the application

$$\phi : U \longrightarrow V$$

defined by the equations (6.3)-(6.4) is a diffeomorphism.

Then we need to complete the transformation to extend it to the speed coordinates. The state vectors written in articular coordinates and in task coordinates are defined as follows :

$$x^T = (\alpha_1, \alpha_2, \dot{\alpha}_1, \dot{\alpha}_2), \quad z^T = (X, Y, \dot{X}, \dot{Y}).$$

The state transformation $z = T(x)$ can finally be written as :

$$\begin{aligned} X &= l_1 \cos \alpha_1 + l_2 \cos(\alpha_1 + \alpha_2), \\ Y &= l_1 \sin \alpha_1 + l_2 \sin(\alpha_1 + \alpha_2), \\ \dot{X} &= -l_1 \dot{\alpha}_1 \sin \alpha_1 - l_2 \dot{\alpha}_1 \sin(\alpha_1 + \alpha_2) - l_2 \dot{\alpha}_2 \sin(\alpha_1 + \alpha_2), \\ \dot{Y} &= l_1 \dot{\alpha}_1 \cos \alpha_1 + l_2 \dot{\alpha}_1 \cos(\alpha_1 + \alpha_2) + l_2 \dot{\alpha}_2 \cos(\alpha_1 + \alpha_2). \end{aligned}$$
 \square

6.6. Electrical machines

In chapter 3, we have obtained a general model for rotating electrical machines of the form :

$$\begin{aligned} L(\theta) \dot{I} &= -\omega K(\theta) I - RI + V, \\ \dot{\theta} &= \omega, \\ J \dot{\omega} &= \frac{1}{2} I^T K(\theta) I - h(\omega) + T_a, \end{aligned}$$

with

$$K(\theta) \triangleq \frac{\partial L(\theta)}{\partial \theta}.$$

These equations naturally lead to establish state models in which the state vector

$$x^T = (I^T, \theta, \omega)$$

is made of currents I , angular position θ and angular speed ω . Other choices of state variables can be used to ease the mathematical study of electrical machines. A current transformation consists of replacing currents by flows :

$$\phi = L(\theta)I,$$

that is transforming the state vector $x^T = (I^T, \theta, \omega)$ into the state vector $z^T = (\phi^T, \theta, \omega)$. This transformation is actually a diffeomorphism because the inductance matrix $L(\theta)$ is invertible for all θ .

In the new state variables z , the equations (6.6) can be written :

$$\begin{aligned}\dot{\phi} &= -RL^{-1}(\theta)\phi + V, \\ \dot{\theta} &= \omega, \\ J\dot{\omega} &= \frac{1}{2}\phi^T G(\theta)\phi - h(\omega) + T_a,\end{aligned}$$

avec $G(\theta) \triangleq L^{-1}(\theta)K(\theta)L^{-1}(\theta)$.

6.7. Triangular systems

A system with *only one* entry (mono-entry system)

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n \quad u \in \mathbb{R} \tag{6.5}$$

is said to be *triangular* if it meets the following definition.

Definition 6.12. Triangular system

A dynamic mono-entry system is triangular if there is a state variable x_i such that the shortest path from u to x_i in the system's graph is of length n . \square

For a triangular system, it is therefore always possible to renumber the state variable such that the state model can be written as :

$$\begin{aligned}\dot{x}_1 &= g_1(x_1, x_2), \\ \dot{x}_2 &= g_2(x_1, x_2, x_3), \\ &\vdots \\ \dot{x}_i &= g_i(x_1, x_2, \dots, x_{i+1}), \\ &\vdots \\ \dot{x}_{n-1} &= g_{n-1}(x_1, x_2, \dots, x_n), \\ \dot{x}_n &= g_n(x_1, x_2, \dots, x_n, u).\end{aligned} \tag{6.6}$$

We observe that the number of state variables on the right increases progressively from 2 to n (which is why it's called a triangular form). Besides, the entry u appears only in the last equation.

Example 6.13. Manipulative robot with one degree of freedom and an elastic joint

The state model of such a robot with an elastic rotoid joint and negligible friction torques can be written as :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ J_1 \dot{x}_2 &= -mgd \sin x_1 - k(x_1 - x_3), \\ \dot{x}_3 &= x_4, \\ J_2 \dot{x}_4 &= k(x_1 - x_3) + u.\end{aligned}\tag{6.7}$$

where

- x_1 is the angular position coordinate of the arm,
- x_2 is the angular speed of the arm,
- x_3 is the angular position coordinate of the motor,
- x_4 is the angular speed of the motor,
- J_1 and J_2 are the inertia momentum of the arm and the motor,
- d is the distance between the joint and the center of mass,
- k is the elastic spring constant ,
- u is the commanded torque developed by the motor.

The graph of the system is represented on figure 6.10 and we can check that the state equations have the wanted triangular structure. \square

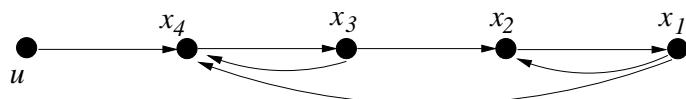


FIGURE 6.10 – Graph of the model of a single arm robot with an elastic joint.

6.8. Brunovski canonical form

Definition 6.14.

A dynamic mono-entry system (6.5) can be written under the Brunovski canonical form if there is a state transformation $T : U \rightarrow V$ and an open interval $W \subset \mathbb{R}$ such that, in the new state variables $z = T(x)$, the system takes on the

following particular triangular form :

$$\begin{aligned}\dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ &\vdots \\ \dot{z}_n &= \alpha(z_1, z_2, \dots, z_n, u),\end{aligned}$$

where the function α is continuous and invertible according to u over W for all $z \in V$. \square

We observe that the system is therefore made of a chain of integrators such that

$$\dot{z}_i = z_{i+1} \quad i = 1, \dots, n-1$$

and that all system non-linearities are focused on the only non-linear scalar function $\alpha(z_1, z_2, \dots, z_n, u)$. The Brunovski canonical form can also be schematized as indicated on figure 6.11. The Brunovski form is interesting because it allows to easily plan trajectories as we will see in chapter 10.

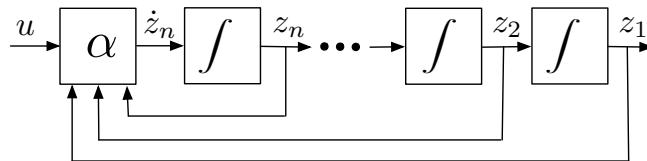
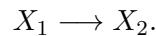


FIGURE 6.11 – Functionnal scheme of Brunovski canonical form

Example 6.15. A chemical reactor

Let us consider a perfectly mixed continuous reactor with constant volume in which occurs an irreversible chemical reaction using two species X_1 and X_2 :



The reactor is only fuelled with X_1 , with constant concentration c . The input variable is the specific volumetric input flow rate of the reactor. The kinetics obeys the law of mass action. According to the modelisation principles established in chapter 5, we get a bilinear state model :

$$\begin{aligned}\dot{x}_1 &= -kx_1 + u(c - x_1), \\ \dot{x}_2 &= kx_1 - ux_2.\end{aligned}$$

We define the following state transformation $z = T(x)$:

$$\begin{aligned} z_1 &= \frac{x_2}{c - x_1}, \\ z_2 &= \frac{kx_1(c - x_1 - x_2)}{(c - x_1)^2}. \end{aligned}$$

The domain U and the image V of the application $T : U \rightarrow V$ are defined according to :

$$\begin{aligned} U &= \{(x_1, x_2) : x_1 > 0, x_2 > 0, x_1 + x_2 < c\}, \\ V &= \{(z_1, z_2) : 0 < z_1 < 1, z_2 > 0\}. \end{aligned}$$

We can then show that the state transformation $z = T(x)$ hereby defined is a diffeomorphism and its inverse is :

$$\begin{aligned} x_1 &= \frac{cz_2}{k(1 - z_1) + z_2}, \\ x_2 &= \frac{ckz_1(1 - z_1)}{k(1 - z_1) + z_2}. \end{aligned}$$

In the new coordinates, the state model is under Brunovski canonical form :

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -\left(z_2 + \frac{(k+1)z_2^2}{k(1-z_1)}\right) + (k(1-z_1) + z_2)u. \end{aligned}$$

The function α is invertible according to u over W . □

This example shows that it is difficult to determine *a priori* if a given dynamic system can be put under Brunovski form and to find the appropriate state transform. However, if the system is already given in a triangular form, here is a sufficient condition to put it in the Brunovski form :

Lemma 6.16.

A triangular dynamic system described by the state model (6.6) can be put under Brunovski canonical form around (x_0, u_0) if the following inequalities :

$$\begin{aligned} \frac{\partial g_i}{\partial x_{i+1}} &\neq 0 \quad i = 1, \dots, n-1, \\ \frac{\partial g_n}{\partial u} &\neq 0, \end{aligned}$$

are satisfied in (x_0, u_0) . □

Example 6.17. Manipulative robot with one degree of freedom and an elastic joint (continued)

Let us consider again the model (6.7) of the example 6.13. We easily check that the conditions of the Lemma 6.16 are satisfied for all $x \in \mathbb{R}^4$ and naturally lead to the state transformation :

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= -J_1^{-1}[mgd \sin x_1 + k(x_1 - x_3)] \\ z_4 &= -J_2^{-1}[mgdx_2 \cos x_1 + k(x_2 - x_4)]. \end{aligned}$$

The inverse state transformation is written as :

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= (mgdk^{-1} \sin z_1 + z_1 + J_1k^{-1}z_3) \\ x_4 &= (mgdk^{-1}z_2 \cos z_1 + z_2 + J_2k^{-1}z_4) \end{aligned}$$

We observe that there is a global diffeomorphism \mathbb{R}^4 in \mathbb{R}^4 . With the new state variables, the model is written following the Brunovski form :

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= J_2^{-1}[mgd(z_2^2 \sin z_1 - z_3 \cos z_1) - kz_3] \\ &\quad + kJ_2^{-2}[mgd \sin z_1 + J_1z_3 - u] \end{aligned}$$

We also observe that the function α is always invertible on \mathbb{R} with respect to u . The Brunovski form is thus overall valid. \square

For control-affine system that are not triangular, the following lemma gives usefull conditions to find the state transformation.

Lemma 6.18. A control-affine system

$$\dot{x} = f(x) + g(x)u \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}$$

can be put in Brunovski form in a domain $U \subset \mathbb{R}^n$ if there exists a state transformation $z = T(x)$ that verify the following conditions :

$$\begin{aligned} T_{i+1}(x) &= \frac{\partial T_i}{\partial x} f(x) \quad i = 1, 2, \dots, n-1, \\ \frac{\partial T_i}{\partial x} g(x) &= 0 \quad i = 1, 2, \dots, n-1, \\ \frac{\partial T_n}{\partial x} g(x) &\neq 0, \end{aligned}$$

for every $x \in U$. □

Example 6.19. A chemical reactor (continuation) We show here how to use the previous lemma to find the state transformation that has been given in the example 6.15. The model can be written :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} c - x_1 \\ -x_2 \end{pmatrix} u \triangleq f(x) + g(x)u$$

First of all we consider the following partial differential equation.

$$\frac{\partial T_1}{\partial x} g(x) = 0 \Rightarrow \frac{\partial T_1}{\partial x_1}(c - x_1) = \frac{\partial T_1}{\partial x_2} x_2$$

This equation admits the following solution :

$$T_1(x) = \frac{x_2}{c - x_1}$$

We calculate then :

$$T_2(x) = \frac{\partial T_1}{\partial x} f(x) \Rightarrow T_2(x) = \frac{kx_1(c - x_1 - x_2)}{(c - x_1)^2}$$

We determine the domain U and the image V of the so defined application T : $U \rightarrow V$. At last, we check that the condition $(\partial T_2 / \partial x)g(x) \neq 0$ is satisfied everywhere on U :

$$\frac{\partial T_2}{\partial x} g(x) = \frac{c(c - x_1 - x_2)}{(c - x_1)^2} \neq 0 \quad \square$$

6.9. Exercises

Exercise 6.1. A glass melting oven

In chapter 1, Exemple 1.1 and Exercice 1.1, we have introduced three different sets of state variables for a model of glass melting furnace. Determine the three corresponding state transformations and give their definition domain. □

Exercise 6.2. An electromagnetic relay

We consider the electromagnetic relay which state model has been given in Chapter 3, Exemple 3.2.

1. We choose the new following state variables : $y_1 = z$, $y_2 = \dot{z}$, $y_3 = \phi(I, z)$. Show that it is a valid state transformation. Give the state model expressed with these new variables.

2. Show that the system can be put in Brunovski canonical form. Determine the state transformations and give a physical interpretation of the state variables. \square

Exercise 6.3. An elevator shaft

On the beside figure, we represented an elevator shaft that hangs from an elastic cable of negligible mass.

Notations :

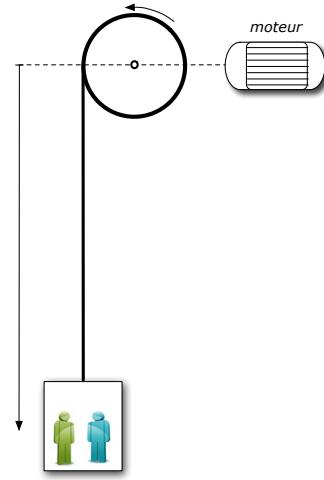
y = length of the cable
 ω = angular velocity of the pulley
 R = radius of the pulley
 m = mass of the elevator shaft

The force in the cable is modeled by the Hooke's law :

$$T = \frac{k(y - z)}{z}$$

where z is an auxiliary state variable. The derivative of z is the peripheral speed of the pulley : $\dot{z} = R\omega$.

- Find a state model with 4 state variables : y, \dot{y}, z, ω . We assume the friction is negligible. The input variable is the torque u applied at the pulley.
- Show that the system can be put in Brunovski form. Give the state transformation. \square



Exercise 6.4. Ladybugs and aphids

Show that there exists a state transformation such that the system (1.6) in chapter 1 describing the interaction between the ladybug and aphid populations can be put in the form of a compartment system. Draw the associated graph. Determine the flows q_{ij} , the matrix L and the matrix $A(x, u)$. \square

Exercise 6.5. A biochemical reactor

Let's consider a continuous stirred tank reactor at constant volume. In this reactor, there is an irreversible autocatalytic reaction with two species A and B :

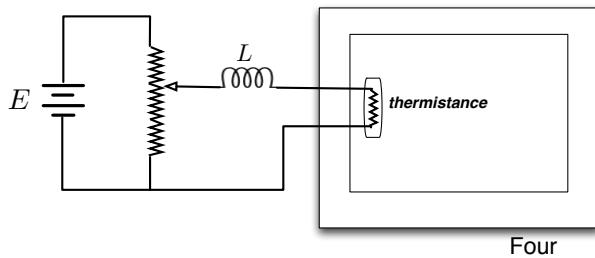


The reactor is only supplied with the species A , at constant concentration. The input variable is the volumetric feed flow of the reactor. Kinetics follows the law of mass action.

1. Give the equations of the state system.
2. Show that the system is conservative.
3. Determine a state transformation which put the system in Brunovski canonical form.
4. Determine the state transformation which put the system in compartment system form.
5. Same questions if the reaction is reversible . □

Exercise 6.6. An electric oven

An electric oven is heated by a thermistor as mentionned on the figure below.



1. Develop a state model of the system with the following modelisation assumptions :
 - a) The thermistor is an electrical resistance. Its value depend on the remperature according to the Reinhart-Hart's law :

$$\frac{1}{T} = a + b \ln R + c(\ln R)^3$$

where a, b, c are constants given by the manufacturer.

- b) As shown on the figure, the thermistor is powered by a battery of constant voltage E through a constant inductance (linear) and an adjustable linear resistance which is the input of the system.
 - c) The oven is heated by the thermistor. The heat which is lost through the wall of the oven is proportionnal to the difference between the inside temperature and the outside temperature. The outside temperature is supposed to be constant.
2. Show that the system can be put in Brunovski form. Give the state transformation. □

Exercise 6.7. A two-compartment system

We consider a linear system with two compartments. The graph is given on (figure 6.12). Determine the state transformation diagonalizing the system. Give the time constant. □

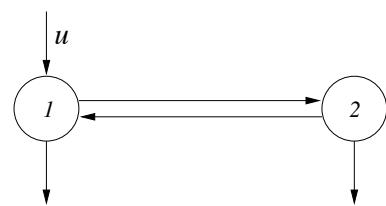


FIGURE 6.12 – Graph of a system with two compartments.

Chapitre 7

Equilibria and invariants

In chapters 7, 8 and 9, we are going to study the dynamic systems behaviour $\dot{x} = f(x, u)$ when input variables are constant. In this particular chapter, we first take a look at the existence condition of equilibrium states and invariant subsets in the state space.

7.1. Equilibria : definition and examples

Definition 7.1. *Equilibrium*

The pair (\bar{x}, \bar{u}) is an *équilibrium* of the system $\dot{x} = f(x, u)$ if

$$f(\bar{x}, \bar{u}) = 0.$$

□

This definition implies that if input signals are constant since t_0 :

$$u(t) = \bar{u} \quad \forall t \geq t_0$$

and if the system state equals \bar{x} at t_0 :

$$x(t_0) = \bar{x}$$

then the system state stays constant and is equal to \bar{x} for all later moments :

$$x(t) = \bar{x} \quad \forall t \geq t_0.$$

In some works, especially those related to the process engineering, an equilibrium is also called a *steady state*. Similarly, the state \bar{x} of equilibrium (\bar{x}, \bar{u}) is sometimes called equilibrium point or *steady state* solution.

Definition 7.2. *Isolated equilibrium*

The pair (\bar{x}, \bar{u}) is an *isolated* equilibrium if, for \bar{u} set, there exists a neighbourhood of \bar{x} in \mathbb{R}^n which contains only one vector \tilde{x} such as $f(\tilde{x}, \bar{u}) = 0$. □

Following examples show the large diversity of possible equilibrium configurations from simple model systems characterized by balance equations.

Example 7.3. Free-flowing tank

We consider a tank with constant section supplied by a pump whose volumetric flow u is the input variable whereas the flow is free. (Fig. 7.1).

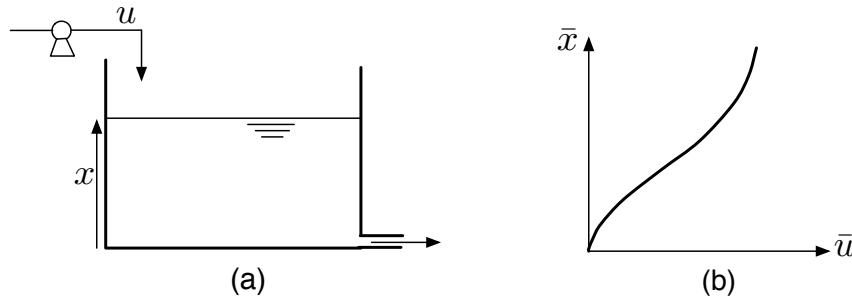


FIGURE 7.1 – (a) Free-flowing tank (b) Equilibrium diagram

The system state model was first shown in chapter 4 :

$$\dot{x} = -\frac{kx\sqrt{x}}{S\beta + x} + u,$$

where x represents the volume of liquid in the tank. The system equilibria satisfy the relation $k\bar{x}\sqrt{\bar{x}} = \bar{u}(S\beta + \bar{x})$ whose graph in \mathbb{R}^2 is called *equilibrium diagram* (Fig. 7.1). We can see on this graph a distinct equilibrium state \bar{x} for each distinct value $\bar{u} \geq 0$ and all equilibria are isolated. \square

Example 7.4. Forced flow tank

Let's now suppose we still have the same tank as earlier, but this time the flow is forced by a pump whose flow rate F_0 is constant (Fig. 7.2). The state model becomes :

$$\dot{x} = -F_0 + u.$$

As in previous example, the system reaches equilibrium when the input flowrate matches exactly the output flowrate :

$$\bar{u} = F_0.$$

This time, there is only one possible input value u for which we have an equilibrium. However, the equilibrium state \bar{x} can take any positive value. The equilibrium diagram is shown in figure 7.2. We observe that the equilibria are not isolated since \bar{x} is undetermined. \square

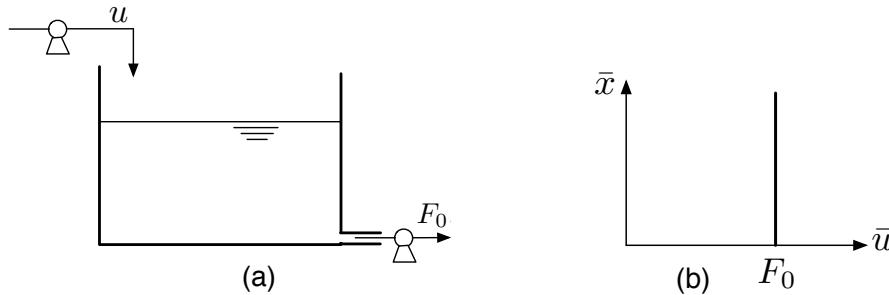


FIGURE 7.2 – (a) Forced flow tank. (b) Equilibria diagram.

Example 7.5. Continuous stirred-tank reactor

Let's consider a continuous stirred-tank reactor with constant volume V (Fig.7.3(a)). The feed rate carries a substance in solution (e.g., a dye) with concentration x_{in} .

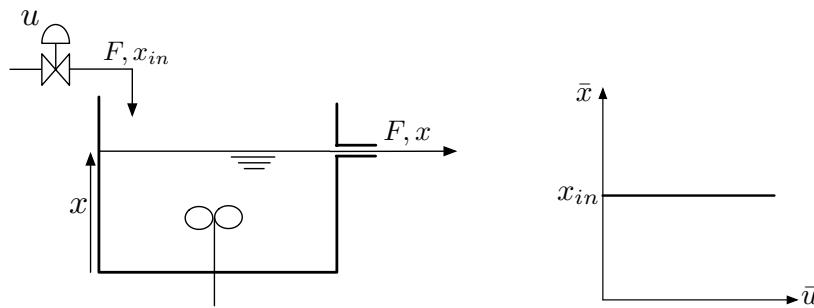


FIGURE 7.3 – (a) Continuous stirred-tank reactor. (b) Equilibria diagram.

The feed rate F is controlled by a valve :

$$F = ku + b \quad k > 0, \quad b > 0,$$

where u denotes the opening of the valve.

The system state is the dye concentration x in the reactor and the state model is :

$$\dot{x} = (x_{in} - x) \frac{ku + b}{V}.$$

The system is at equilibrium when the mass flow rate of the dye is exactly the same as the output mass flow rate :

$$\frac{ku + b}{V} x_{in} = \frac{ku + b}{V} \bar{x},$$

this implies $\bar{x} = x_{in}$. The equilibrium diagram shown in figure 7.3(b) shows that the equilibrium state is fixed and isolated but the corresponding constant input is undetermined. \square

Example 7.6. Forced flow mixing tank

Up until now, we have only considered examples where the state vector has dimension 1. In systems of higher dimension, the different configurations described earlier can coexist as we now illustrate with the example of a forced flow mixing tank (Fig. 7.4). This system state model, noting x_1 the volume of the tank and

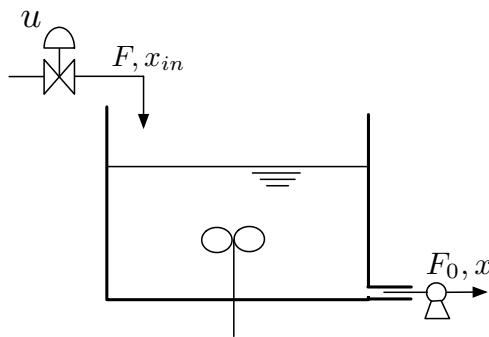


FIGURE 7.4 – Forced flow mixing tank

x_2 the dye concentration in the tank, is :

$$\begin{aligned}\dot{x}_1 &= -F_0 + ku + b, \\ \dot{x}_2 &= (x_{in} - x_2) \frac{ku + b}{x_1}.\end{aligned}$$

In this case, the equilibrium diagram has 3 dimensions (Fig. 7.5) and we can see there is only one input value giving an equilibrium, $\bar{u} = (F_0 - b)/k$, and for this value \bar{u} , the volume of the equilibrium \bar{x}_1 is undetermined while the concentration at equilibrium is $\bar{x}_2 = x_{in}$. \square

- Examples with 1 or 2 dimensions considered so far have shown situations where
- either the system has an isolated equilibrium for each input value \bar{u} ,
 - or the system has an infinity of non-isolated equilibria corresponding to a precise value of \bar{u} .

For non linear systems, other configurations are possible. In particular, we can observe a few isolated equilibria corresponding to a same value of \bar{u} as in the following example.

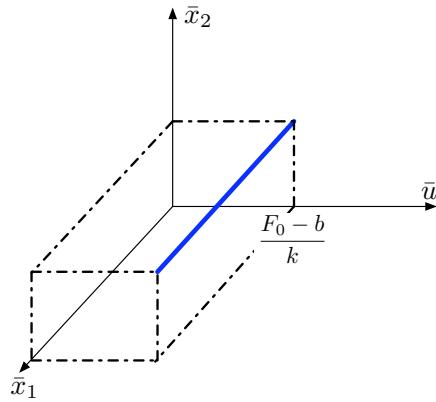


FIGURE 7.5 – Equilibrium diagram for the forced flow mixing tank

Example 7.7. Chemical reactor

Let's consider a continuous stirred-tank reactor where an irreversible exothermic reaction occurs $A \rightarrow B$. The state model is (read Chapters 1 and 5) :

$$\begin{aligned}\dot{x}_A &= -kx_A e^{-\frac{\alpha}{T}} + D(x_A^{in} - x_A), \\ \dot{x}_B &= kx_A e^{-\frac{\alpha}{T}} - Dx_B, \\ \dot{T} &= h k x_A e^{-\frac{\alpha}{T}} - qT + u,\end{aligned}$$

where x_A and x_A^{in} are the concentrations of reactant A in the reactor and in the supply, x_B is the concentration of product B , D is the constant volumetric flow rate, T is the temperature and u the heat input per unit of time.

The equilibria of this system are characterized by those equations

$$\begin{aligned}\bar{x}_A &= \frac{Dx_A^{in}}{ke^{-\alpha/\bar{T}} + D}, \\ \bar{x}_B &= \frac{k\bar{x}_A e^{-\alpha/\bar{T}}}{D}, \\ \bar{T} &= \frac{1}{q} \left(\frac{Dx_A^{in} h k e^{-\alpha/\bar{T}}}{ke^{-\alpha/\bar{T}} + D} + \bar{u} \right).\end{aligned}$$

The third equation allows us to determine \bar{T} as a function of \bar{u} . The first two then allow us to deduce from \bar{T} values of equilibrium for \bar{x}_A and \bar{x}_B . The equilibrium diagram representing \bar{T} as a function of \bar{u} is illustrated in figure 7.6. According to the values of \bar{u} , we see that there is one, two or three isolated equilibria. \square

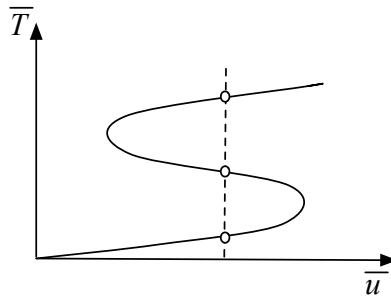


FIGURE 7.6 – Equilibrium diagram for a simple chemical reactor

7.2. Equilibria of linear systems

Let the linear system

$$\dot{x} = Ax + Bu$$

for which the equation defining the equilibria becomes

$$A\bar{x} + B\bar{u} = 0.$$

The equilibria of a linear system are totally characterized by the following theorem.

Theorem 7.8.

- If matrix A is regular, then for each \bar{u} , the pair $(-A^{-1}B\bar{u}, \bar{u})$ is an isolated equilibrium.
- If matrix A is singular, the system (7.1) has an infinity of equilibria (non-isolated) provided that $B\bar{u} \in \text{Im } A$. Those equilibria are an affine manifold solution of system $A\bar{x} = -B\bar{u}$. On the other side, for each \bar{u} such as $B\bar{u} \notin \text{Im } A$, the system (7.1) does not have an equilibrium. \square

For dynamic linear systems, we can't have several isolated equilibria corresponding to the same input value \bar{u} . Finally, note that the pair $(\bar{x}, \bar{u}) = (0, 0)$ is always an equilibrium for dynamic linear systems of the form (7.1).

Example 7.9. Linear models of DC machines

Several models of direct current machines (motors and generators) were introduced in section 3.6. Under general assumptions of linear viscous friction and nonsturation of the streams, some of those models are linear. We examine below their configuration of equilibrium.

DC generator driven by the stator

On considère le modèle d'état d'une génératrice à courant continu tournant à vitesse ω constante. En notant $x_1 = I_s$, le courant statorique, $x_2 = I_r$, le courant rotorique et $u = v_s$ la tension aux bornes du circuit statorique, le modèle d'état est linéaire et s'écrit comme suit :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{R_s}{L_s} & 0 \\ K_e\omega & -\frac{R_r + R_L}{L_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{L_s} \\ 0 \end{pmatrix} u$$

The matrix A of this linear system is invertible and the generator therefore has an isolated equilibrium state for each value of the input voltage \bar{u} :

$$\begin{aligned} \bar{x}_1 &= \frac{L_s}{R_s} \bar{u} \\ \bar{x}_2 &= \frac{L_r}{R_r + R_L} \frac{L_s}{R_s} K_e \omega \bar{u} \end{aligned}$$

DC motor controlled by the rotor

With state variables for this system $x_1 = \theta$, angular position of the rotor, $x_2 = \dot{\theta} = \omega$, angular velocity and $x_3 = I_r$, the rotor current, the following model is obtained :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{K_m I_s}{J} \\ 0 & -\frac{K_e I_s}{L_r} & -\frac{R_r}{L_r} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{J} \\ \frac{1}{L_r} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where u_1 is the resisting torque and u_2 the control voltage. We observe that :

- the state matrix A of the system is singular,
- $B\bar{u} = (0 \quad \bar{u}_2/J \quad \bar{u}_1/L_r)^T \notin \text{Im } A$ unless $\bar{u}_1/\bar{u}_2 = -R_r/K_m I_s$ or if $\bar{u}_1 = \bar{u}_2 = 0$.

The first case corresponds to a rotor control voltage which creates a torque exactly compensating for the load torque. The rotational speed is then zero and the rotor angular position is undetermined. The equilibrium value of the rotor current is given by $\bar{x}_3 = \bar{I}_r = \bar{u}_2/K_m I_s$. In the second case, the equilibria are of the form $\bar{x}_1 = \bar{\theta}$, $\bar{x}_2 = 0$, $\bar{x}_3 = 0$, i.e. that the engine is stopped with the rotor in any angular position.

We can also consider equilibria of the subsystem whose states are the speed ω and the current I_r :

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\frac{B}{J} & \frac{K_m I_s}{J} \\ -\frac{K_e I_s}{L_r} & -\frac{R_r}{L_r} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{J} \\ \frac{1}{L_r} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The state matrix of this system is invertible (all constants are positive and the determinant is not zero) and each value of the input vector \bar{u} will correspond to an equilibrium value of the state vector $(\bar{x}_1 \ \bar{x}_2)^T$. This equilibrium, which is not contradictory with the previous one, corresponds to the case of a DC motor which drives a load by rotating at constant speed. \square

7.3. Invariants

The notion of invariant we will define in this section is a generalization of the concept of equilibrium.

Definition 7.10. Invariant

The subset $\mathcal{X} \times U \subset \mathbb{R}^n \times \mathbb{R}^m$ is an invariant of the dynamic system $\dot{x} = f(x, u)$ si :

$$\left\{ \begin{array}{l} x(t_0) \in \mathcal{X} \\ u(t) \in U \quad \forall t \geq t_0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x(t) \text{ existe} \\ x(t) \in \mathcal{X} \end{array} \right\} \forall t \geq t_0 \quad \square$$

This definition means that if the state of the system is in \mathcal{X} at an initial time, it will remain for all subsequent moments as long as the input signal $u(t)$ will be maintained in U .

We have already seen several examples of invariants in the previous chapters. The simplest example is the set of equilibria of a system corresponding to a constant input \bar{u} .

In this case, the subset $U = \{\bar{u}\}$ is reduced to a singleton and \mathcal{X} contains the corresponding equilibrium states \bar{x} .

Another typical example is the positive orthant ($\mathcal{X} = \mathbb{R}_+^n \times (U \subset \mathbb{R}^m)$) which is, by definition, an invariant for positive systems (see Definition 4.3 and Theorem 4.4.).

There are various ways to characterize the invariants of a dynamical system depending on the particular form of the subset \mathcal{X} . We will show two remarkable characterizations : in the first one \mathcal{X} is an open subspace of \mathbb{R}^n , in the second one \mathcal{X} is an hypersurface in \mathbb{R}^n .

- **\mathcal{X} is an open subspace in \mathbb{R}^n**

Let \mathcal{X} an open subset in \mathbb{R}^n whose border $\partial\mathcal{X}$ is regular enough. If at any point y on the border $\partial\mathcal{X}$, the vector $f(y, v)$ points inwards \mathcal{X} for all $v \in U$, then the subset $\mathcal{X} \times U$ is an invariant of the system $\dot{x} = f(x, u)$.

This characterization will be discussed further in chapter 8 (section 8.4).

- **\mathcal{X} is an hypersurface in \mathbb{R}^n**

We call *first integral* a function $z = h(x)$ of class C^2 such that :

$$\frac{\partial h}{\partial x} f(x, u) = 0 \quad \forall u \in U. \quad (7.1)$$

We define the subset \mathcal{X} as follows :

$$\mathcal{X} \triangleq \{x \in \mathbb{R}^n : h(x) = c\}$$

with c any real constant. This set \mathcal{X} is an hypersurface in \mathbb{R}^n . As the condition (7.1) implies that the function $z = h(x)$ is constant along the trajectories, it is clear that the subset $\mathcal{X} \times U$ is an invariant of the system $\dot{x} = f(x, u)$. The *reaction invariants* are a typical example.

Example 7.11. Reaction invariants

As we saw in chapter 5, the model of a continuous stirred-tank reactor is written as follows :

$$\dot{x} = Cr(x) + u(x^{in} - x) \quad (7.2)$$

where x is the composition of the reaction medium, u the feed flow rate, x^{in} the composition (supposed constant) of the feed flow, C is the stoichiometric matrix and $r(x)$ the vector of the reaction kinetics.

The flow u is positive and bounded by the maximum capacity of the feed pump u_{\max} , so we define U as the closed interval :

$$U = [0, u_{\max}]$$

On the other hand, the subset \mathcal{X} is defined as follows :

$$\mathcal{X} = \{x : x \in \mathbb{R}_+^n, Lx = Lx^{in}\}$$

where L is a matrix $(n-p) \times n$ such as $LC = 0$. In other words, the rows of the matrix are a base of the kernel of the stoichiometric matrix transposed C .

The subset $\mathcal{X} \times U$ thus defined is an invariant of the system (7.2). To check our assumptions, we consider the partial linear transformation of state :

$$z = Lx$$

and we compute its evolution along the system trajectories :

$$\dot{z} = LCr(x) + u(Lx^{in} - Lx) = u(Lx^{in} - Lx) \text{ car } LC = 0$$

According to the definition of \mathcal{X} , we immediately observe that, if $Lx(t_0) = Lx^{in}$, then $\dot{z} = 0$ along the system trajectories and therefore $Lx(t) = Lx^{in} \quad \forall t \geq t_0$, independently of the input signal $u(t)$.

Moreover, the system (7.2) is a positive system, therefore $x(t) \in \mathbb{R}_+^n \quad \forall t \geq t_0$ if $x(t_0) \in \mathbb{R}_+^n$ and if $u(t) \in U \quad \forall t \geq t_0$.

Invariants defined this way are called reaction invariants or chemical invariants in the literature. \square

7.4. Exercises

Exercise 7.1. An electromagnetic relay

Determine the equilibria of the state model of an electromagnetic relayd in chapter 3, example 3.2 (see also exercise 6.2). \square

Exercise 7.2. Direct current generator

We consider the model of a DC generator (see chapter 3, section 3.6) feeding into a resistive load with a linear viscous friction.

1. Find the equilibria based on inputs \bar{u}_1 and \bar{u}_2
2. Determine the optimal operating points that maximize the current supplied by the generator.. \square

Exercise 7.3. A phase-locked loop

A phase-locked loop (Phase-Locked Loop) used in communication networks is described by the equation

$$\ddot{y} + (a + b \cos y)\dot{y} + u \sin y = 0$$

avec $a > b > 0$ et $u(t) > 0 \forall t$.

1. Put the system as a state model.
2. Determine the equilibria.

Exercise 7.4. A boat

Determine the equilibria of the state model of the boat for the exercise 2.7. What is the physical meaning of these equilibria ? \square

Exercise 7.5. An industrial shredder

The operation of an industrial grinding circuit (fig. 7.7) is expressed by the state model :

$$\begin{aligned}\dot{x}_1 &= -\gamma_1 x_1 + (1 - \alpha)\phi(x_3), \\ \dot{x}_2 &= -\gamma_2 x_2 + \alpha\phi(x_3), \\ \dot{x}_3 &= \gamma_2 x_2 - \phi(x_3) + u.\end{aligned}$$

with the following notations :

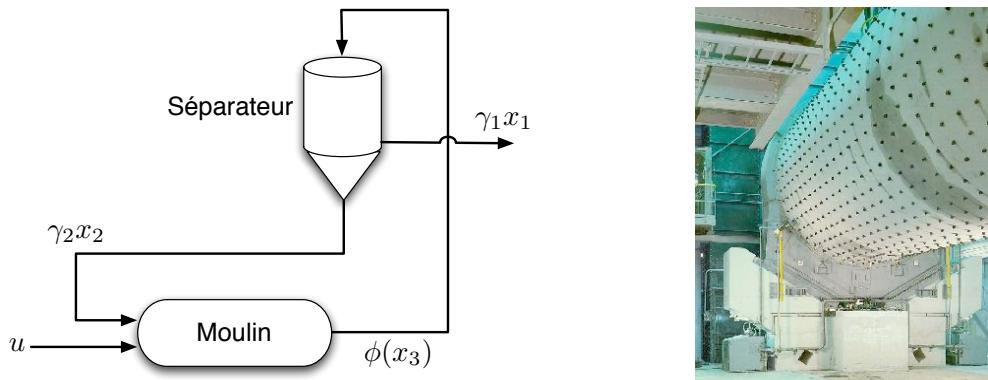


FIGURE 7.7 – Grinding circuit - Photo of an industrial shredder

- x_1 = quantity of end product in the separator ;
- x_2 = amount of recycled material in the separator ;
- x_3 = amount of material in the mill :
- u = mill feed rate.

The parameter α is the characteristic constant of separator. ($0 < \alpha < 1$). The function of grinding $\phi(x_3)$ has the following form :

$$\phi(x_3) = k_1 x_3 e^{-k_2 x_3}$$

where k_1 et k_2 are positive constants.

1. Show that this is a compartment system and give the graphs of system.
2. Determine the equilibria of system.
3. The set that is described by the following inequalities represents a jam situation. Show it is an invariant system.

$$\begin{aligned} x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \\ (1-\alpha)\phi(x_3) &\leq \gamma_1 x_1 < \bar{u}, \\ \alpha\phi(x_3) &\leq \gamma_2 x_2, \\ \frac{\partial\phi(x_3)}{\partial x_3} &< 0. \end{aligned}$$

□

Exercise 7.6. A biochemistry reactor

We consider the state model of a biochemistry reactor from exercise 6.2.

1. Determine the system equilibria and sketch equilibria diagrams.
2. Determine the invariants of the reactor.

3. Same questions if the reactor is reversible. □

Exercise 7.7. Dynamics of a viral infection

The dynamics of a viral infection with lytic and non-lytic actions of immunization is described by the following model state :

$$\begin{aligned}\dot{x}_1 &= \lambda - dx_1 - \frac{\beta x_1 x_2}{1 + qx_3}, \\ \dot{x}_2 &= \frac{\beta x_1 x_2}{1 + qx_3} - ax_2 - px_2 x_3, \\ \dot{x}_3 &= cx_2 - bx_3.\end{aligned}$$

In these equations, x_1 , x_2 et x_3 are respectively the quantities of healthy cells, infected and immune. The infected cells produce viral particles. λ is the healthy cell production rate and d their mortality rate. Lytic components of the anti-viral activity kill infected cells while non-lytic components inhibit replication of the viral particles. The infected cells are killed at the speed px_3 with p represents the intensity of anti-viral lytic activity. The production of infected cells is represented by the term

$$\frac{\beta x_1 x_2}{1 + qx_3}$$

where qx_3 represents the intensity of inhibition replication by the anti-viral non-lytic activity. The death rate of infected cells is a and the death rate of immune cells is b . cx_2 is the production rate of immune cells.

1. Show that the model state is a reaction system.
2. Show that the model state is equivalent to a compartment systems.
3. Find the equilibria of the system in the positive orthant.

Exercise 7.8. Mechanical system

We consider the modeling of a mechanical system with one degree of freedom :

$$\ddot{\theta} + c\dot{\theta} + r \sin \theta = 0$$

1. Wite the model state of the system ($x_1 = \theta$).
2. Determine the equilibria.
3. Show with the condition $c^2 \geq 4r$, there exists an invariant bound (which interior is non-empty) in the orthant $\{x_1 \geq 0, x_2 \leq 0\}$. □

Chapitre 8

Planar systems

Within this chapter, we study in detail the behavior of 2D dynamical system trajectories (planar systems) when entry $u(t)$ is constant : $u(t) = \bar{u}$. These systems are described by the following equations :

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \bar{u}), \\ \dot{x}_2 &= f_2(x_1, x_2, \bar{u}).\end{aligned}$$

A main motivation of this restriction to planar systems is to easily show the results representing the orbits in the phase plane, i.e. the plane of state variables x_1 and x_2 . Besides, planar systems describe most of characteristic behaviors showing the difference between linear and nonlinear systems. We will successively study the trajectories of linear systems, the behavior of nonlinear system trajectories near equilibrium points. We will focus on periodic trajectories and limit cycles. Finally, we will have an overview about bifurcation theory.

8.1. Linear planar systems

Let us consider planar systems when the entry $u(t)$ is constant : $u(t) = \bar{u}$. These systems are represented by the equation

$$\dot{x} = Ax + B\bar{u},$$

where A is a matrix 2×2 . We assume there is at least an equilibrium state \bar{x} corresponding to \bar{u} .

Thanks to a relevant state transformation, $z = M^{-1}(x - \bar{x})$, we get

$$\dot{z} = A'z$$

with

$$A' = M^{-1}AM.$$

The eigenvalues of A' are the same ones than A and A' has one of the following forms :

a.

$$A' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

This form correspond to the case where A has two distinct real eigenvalues λ_1 and λ_2 or a double real eigenvalue with a geometric multiplicity of 2.

b.

$$A' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

This form corresponds to the case where A has a double real eigenvalue of λ with a geometric multiplicity of 1. This is the "Jordan form" related to A .

c.

$$A' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \beta > 0$$

This form corresponds to the case where A has two complex conjugate eigenvalues $\alpha \pm \beta i$.

Thanks to these new coordinates, we are able to easily compute trajectories. They are described by the following equations :

a.

$$\begin{aligned} z_1(t) &= z_1(0)e^{\lambda_1 t}, \\ z_2(t) &= z_2(0)e^{\lambda_2 t}. \end{aligned}$$

b.

$$\begin{aligned} z_1(t) &= z_1(0)e^{\lambda t} + tz_2(0)e^{\lambda t}, \\ z_2(t) &= z_2(0)e^{\lambda t}. \end{aligned}$$

c.

$$\begin{aligned} z_1(t) &= e^{\alpha t}(z_1(0)\cos \beta t + z_2(0)\sin \beta t), \\ z_2(t) &= e^{\alpha t}(z_2(0)\cos \beta t - z_1(0)\sin \beta t). \end{aligned}$$

The tables 8.1 à 8.3 show orbits as functions of one of these three forms and of the sign of eigenvalues. These orbits are represented in the plane (z_1, z_2) and in the plane (x_1, x_2) , centered around the equilibrium point (\bar{x}_1, \bar{x}_2) . In this second case, the directions in the figures correspond to the eigenvectors of the matrix A .

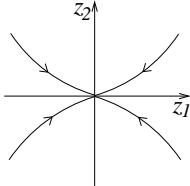
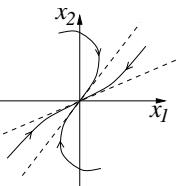
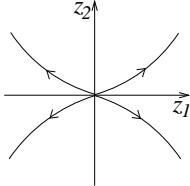
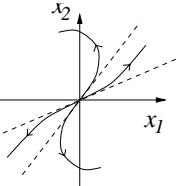
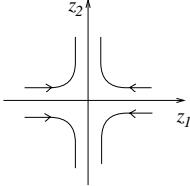
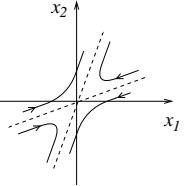
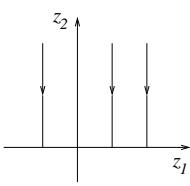
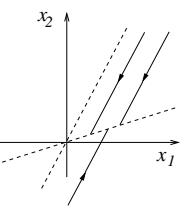
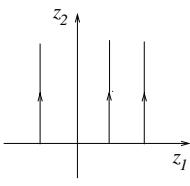
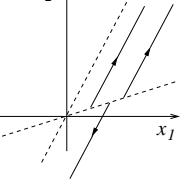
Type of equilibrium	Appearance of trajectories (z_1, z_2)	Appearance of trajectories (x_1, x_2)	Conditions on the eigenvalues
Attractive node			$\lambda_2 \leq \lambda_1 < 0$
Repulsive node			$0 < \lambda_1 \leq \lambda_2$
Saddle point			$\lambda_1 < 0 < \lambda_2$
Non isolated attractive equilibrium			$\lambda_1 = 0, \lambda_2 < 0$
Non isolated repulsive equilibrium			$\lambda_1 = 0, \lambda_2 > 0$

TABLE 8.1 – Orbits of linear planar systems : case a.

Remarks 8.1.

1. In the first two cases (see table 8.1), when $\lambda_1 = \lambda_2$, the trajectories are straight and can be thus represented by a beam of lines coming from the origin.
2. If one of the eigenvalue is zero, the equilibrium is not isolated. The eigenvector corresponding to the eigenvalue zero defines a line of equilibrium points and all the trajectories are straight and converge to or come from a point of this line of equilibriums.

Definition 8.2. When the equilibrium is such that the trajectories converge to this equilibrium, we say it is an attractive equilibrium. ■

The eigenvalues λ_1 and λ_2 of matrix A are the roots of the characteristic polynomial

$$\begin{aligned} p(x) &= x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2 \\ &= x^2 - \text{tr}(A)x + \det A. \end{aligned}$$

We observe that to determine the appearance of trajectories, it is not needed to compute explicitly these eigenvalues. Figure 8.1 show the nature of equilibrium (and thus the appearance of trajectories) as function of the two coefficients of characteristic polynomial that are respectively equal to the sum and product of eigenvalue.

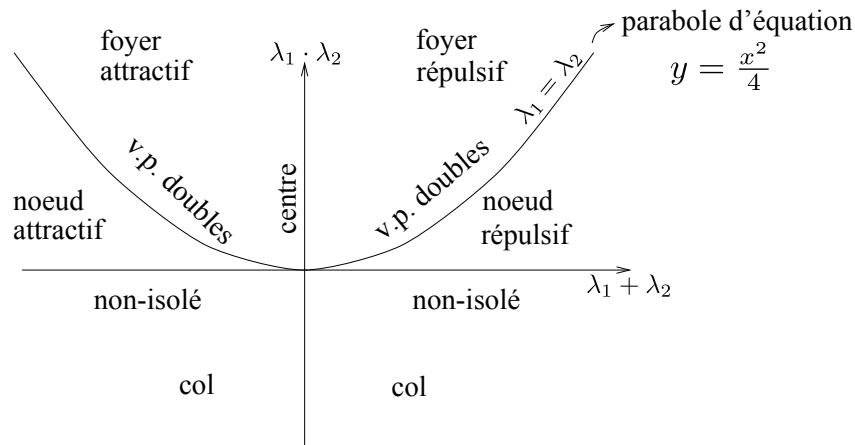


FIGURE 8.1 – Characteristics of equilibriums as function of the sum and product of eigenvalues.

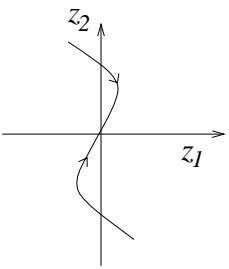
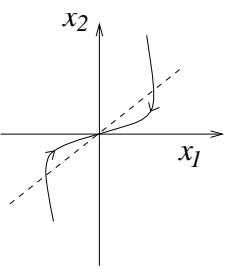
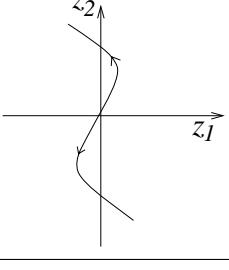
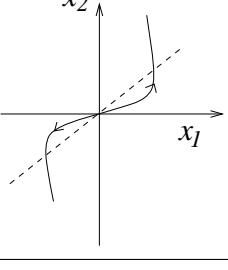
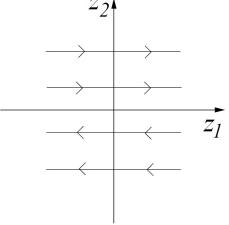
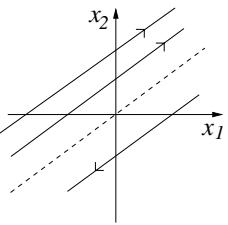
Type of equilibrium	Appearance of trajectories (z_1, z_2)	Appearance of trajectories (x_1, x_2)	Conditions on the eigenvalues
Degenerated attractive node			$\lambda = 0$ (Jordan)
Degenerated repulsive node			$\lambda > 0$ (Jordan)
Non isolated equilibrium			$\lambda = 0$ (Jordan)

TABLE 8.2 – Orbits of planar linear systems : case b.

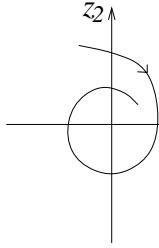
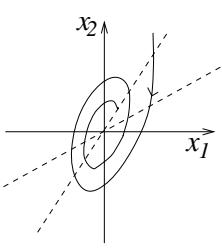
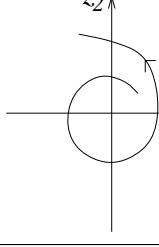
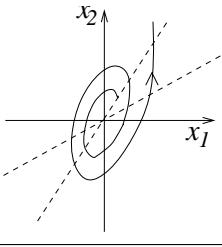
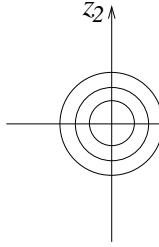
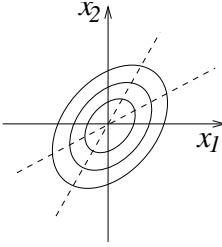
Type of equilibrium	Appearance of trajectories (z_1, z_2)	Appearance of trajectories (x_1, x_2)	Conditions on the eigenvalues
Attractive spiral			$\lambda_{1,2} = \alpha \pm \beta i$ $\alpha < 0, \beta \neq 0$
Repulsive spiral			$\lambda_{1,2} = \alpha \pm \beta i$ $\alpha > 0, \beta \neq 0$
Center			$\lambda_{1,2} = \pm \beta i$ $\beta \neq 0$

TABLE 8.3 – Orbits of planar linear systems : case c.

We may wonder how the nature of these trajectories is sensitive to perturbations of the system. In order to answer to this question, we consider a nominal linear system $\dot{x} = Ax + Bu$ and a perturbation of the nominal system with the form $\dot{x} = (A + \Delta A)x + Bu$. If the matrix A has distinct eigenvalues, we can show that they are continuously dependent of coefficients of A , which means that for every positive number ϵ , there exists a positive number δ such that if every coefficient of the perturbation ΔA is smaller than δ , the eigenvalues of the perturbed matrix $A + \Delta A$ will be in the interior of balls of radius ϵ centered in the eigenvalues of A . Therefore, every eigenvalue is initially in the interior of the left ($Re(\lambda) < 0$) or right ($Re(\lambda) > 0$) half plane will stay in the same half plane for perturbations ΔA that are sufficiently small and, qualitatively, the trajectories of the perturbed system will look like to those of the nominal system : an attractive spiral stays an attractive spiral, a repulsive node stays a repulsive node, a saddle point stays a saddle point,... In this case, we say that such systems (or equilibriums) are *structurally stable*.

Unfortunately, this is not the same if the equilibrium corresponds to a *center*. In this case, the systems has periodic elliptic trajectories and pure imaginary eigenvalues. For this kind of systems, even small perturbation of the matrix A can move out eigenvalues from imaginary axis. The corresponding trajectories become then attractive or repulsive spiral. A linear system having an equilibrium of center type is thus not structurally stable.

The case of linear system having one or two zero eigenvalues generally drives to a changement of property of trajectories under the effect of perturbations arbitrary small. When the system has a double eigenvalue that is different of zero, small perturbations can drive to real or complex conjugate eigenvalues, but the localisation within one of the half plane will not be changed. A degenerated attractive (repulsive) node can become an attractive (repulsive) node or an attractive (repulsive) spiral.

The previous analysis shows well that the imaginary axis can have some problems. We then introduce the following definition.

Definition 8.3. If all the eigenvalues of A have a non zero real part, the system $\dot{x} = Ax$ (or the equilibrium point) is hyperbolic. \square

Actually, an hyperbolic system is structurally stable and the trajectories will have the same properties with small perturbations. In the case of a double (non zero) eigenvalue, small perturbations can produce either a spiral, either a node ; but the attractive or repulsive character of the equilibrium will be preserved. Those observations have high value for the analysis of nonlinear systems.

8.2. Linéarisation des systèmes non linéaires

Orbits illustrated in the table of the previous section are not only valid in the neighborhood of the point of equilibrium (returned to the origine). We characte-

rized well thanks to these table all the possible orbits of the linear systems plans, whatever is the initial condition. This observation establishes a fundamental difference between linear and not linear systems. Indeed, we saw in the previous chapter that the non-linear systems can present several different isolated equilibrium for the same value of the entry \bar{u} . This implies that, contrary to the case of the linear systems, the behavior of orbits in the neighborhood of an equilibrium will keep most of the time a *local character* and could never be expand to the whole plan of phase. For this limitation, an important result allows however to extend to the non-linear systems a part of the analysis which we have just developed for the linear systems.

Set the dynamic system described by

$$\dot{x}_1 = f_1(x_1, x_2, \bar{u}), \quad (8.1)$$

$$\dot{x}_2 = f_2(x_1, x_2, \bar{u}). \quad (8.2)$$

or, under condensed form,

$$\dot{x} = f(x, \bar{u}), \quad (8.3)$$

for which we assumed the existence of an equilibrium (\bar{x}, \bar{u}) such that $f(\bar{x}, \bar{u}) = 0$. Assume further that the function $f(x, \bar{u})$ is sufficiently regular in the neighborhood of this equilibrium for there allow a Taylor expansion convergent :

$$\dot{x} = f(\bar{x}, \bar{u}) + \left(\frac{\partial f(x, \bar{u})}{\partial x} \right)_{\bar{x}} (x - \bar{x}) + \dots$$

The *linear approximation* of this system at the neighborhood of the equilibrium (\bar{x}, \bar{u}) , obtained by neglecting the terms of order greater or equal to 2 in the Taylor expansion of $f(x, \bar{u})$ around (\bar{x}, \bar{u}) , is given by

$$\dot{\tilde{x}} = \left(\frac{\partial f(x, \bar{u})}{\partial x} \right)_{\bar{x}} \tilde{x} \quad (8.4)$$

where $\tilde{x} = x - \bar{x}$. Let $A = \left(\frac{\partial f(x, \bar{u})}{\partial x} \right)_{\bar{x}}$, the Jacobian matrix of f at equilibrium. We can then generalize the definition 8.3 as follow :

Definition 8.4. Hyperbolic equilibrium

The equilibrium (\bar{x}, \bar{u}) of the non-linear systeme (8.3) is said hyperbolic if all the eigen values of A have a non-zero real part ($Re(\lambda_i(A)) \neq 0, \forall i$). \square

It must be clear that it is the *equilibrium* (\bar{x}, \bar{u}) which is (or which is not) hyperbolic, and not the non-linear system. Indeed, this system can have several isolated equilibrium for the same value \bar{u} , some being hyperbolic and others not. In which mesure the study of linear approximation of a non-linear system at the neighborhood of an equilibrium allows to deduct the behavior of the non-linear system ? To specify what we understand by behavior, we want to be able to compare trajectories and then introduce the following definition.

Definition 8.5. The trajectories (or orbits) of two dynamical systems are *topologically equivalent* if it exists an *homeomorphism* (une bijection bicontinue) which allows to move from a trajectory of the first system to a trajectory of the second one. \square

Theorem 8.6. Hartman-Grobman, 1959

If the equilibrium (\bar{x}, \bar{u}) is hyperbolic, then the trajectories of the non-linear system (8.3) *in a neighborhood of the equilibrium* (\bar{x}, \bar{u}) are topologically equivalent to the ones of the linear approximation (8.4). Specifically, it exists a neighborhood X of \bar{x} , a neighborhood \tilde{X} of 0 and an homeomorphism $h : X \rightarrow \tilde{X}$ with $h(\bar{x}) = 0$ such that if $t \mapsto x(t)$ is a trajectory of the non-linear-system (8.3) contain in X (for a certain time interval), then $t \mapsto h(x(t))$ is a trajectory of the linear system (8.4). \square

Topologically equivalent trajectories have the same allure. So we can talk of node or attractive or repulsive center, or saddle, for the equilibrium of nonlinear systems by studying the eigenvalues of the linear approximation matrix, but **no** center.

Remarks 8.7.

1. The interest of this theorem is obvious. Its main limitation, its local character, is not less. In particular, this theorem doesn't give any indication about the size of the basin of attraction of an attrive equilibria.
2. In the case of a non-hyperbolic equilibrium, Dans le cas d'un équilibre non hyperbolique, it is the higher order terms, those that have been neglected, that will determine the local behavior of trajectories.
3. The tools developed so far in this chapter are not specific to plan systems. Classification of linear systems, linearization, Hartman-Grobman theorem generalize without problems in any dimension
4. In the linearization (8.4), we keep $u = \bar{u}$ constant. We could also linearize f around $u = \bar{u}$ to obtain a linearized type $\tilde{x} = A\tilde{x} + B\tilde{u}$. While $\tilde{u} = u - \bar{u}$ stay sufficiently small, and pour a sufficiently small interval of time, the trajectories of a non-linear and linear systems remain close, but there is no simple variation of the Hartman-Grobman theorem in this case. ■

8.3. Beyond plans systems

The above considerations are not specific to plans systems.

The Hartman-Grobman theorem, for example, is true in any dimension $n \geq 2$, and classification of linear systems is similar.

Consider a real matrix A of arbitrary dimension. If all its eigenvalues are distinct then it can be diagonalized by real blocks 1 *times* 1 which contain a real eigenvalue, or 2×2 , of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

which encode a pair of conjugate eigenvalues $\alpha \pm \beta i$.

In this case the linear system (or linearised) can be described as direct product¹ of one-dimensional system or two-dimensional as seen in this chapter.

Briefly, the case of Jordan blocks is a little different and behaves as follows. A real Jordan block, for example

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & \\ & & & \lambda \end{pmatrix}$$

generate a dynamic close to the two-dimentional case, linear combination of $e^{\lambda t}$, $te^{\lambda t}$, $t^2e^{\lambda t}$:

$$\begin{aligned} z_1(t) &= z_1(0)e^{\lambda t} + tz_2(0)e^{\lambda t} + t^2z_3(0)e^{\lambda t}, \\ z_2(t) &= z_2(0)e^{\lambda t} + tz_3(0)e^{\lambda t}, \\ z_3(t) &= z_3(0)e^{\lambda t}. \end{aligned}$$

A Jordan block complex combines with its conjugate to form a block that looks eg like this :

$$\begin{pmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ & & \alpha & \beta \\ & & -\beta & \alpha \end{pmatrix} \quad (8.5)$$

The solutions then look like this :

$$\begin{aligned} z_1(t) &= e^{\alpha t}(z_1(0)\cos \beta t + z_2(0)\sin \beta t + t(z_3(0)\cos \beta t + z_4(0)\sin \beta t)), \\ z_2(t) &= e^{\alpha t}(z_2(0)\cos \beta t - z_1(0)\sin \beta t + t(z_4(0)\cos \beta t - z_3(0)\sin \beta t)), \\ z_3(t) &= e^{\alpha t}(z_3(0)\cos \beta t + z_4(0)\sin \beta t), \\ z_4(t) &= e^{\alpha t}(z_4(0)\cos \beta t - z_3(0)\sin \beta t). \end{aligned}$$

Now illustrate the previous sections with some examples of non-linear 2nd order systems.

1. A union of decoupled systems

8.3.1. Mechanical systems of one degree of freedom

The state equations of a mechanical system with one degree of freedom can be written (see chapter 2) :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ m\dot{x}_2 &= -g(x_1) - k(x_1) - h(x_2) + u,\end{aligned}$$

where x_1 is the coordinate position of the moving body, x_2 is the velocity, m means the mass or inertia and u represents a force or couple applied to an external system. Scalar functions $g(x_1)$ and $k(x_1)$ correspond to the gravity and elasticity while $h(x_2)$ (such that $h(0) = 0$) represents the viscous friction. Dry friction is neglected. note also (see chapter 2, section 2.7) that

$$g(x_1) + k(x_1) = \frac{\partial E_p}{\partial x_1}$$

where E_p refers to the potential energy of the system.

The equilibria of the system are characterized by

$$\begin{aligned}\bar{x}_2 &= 0, \\ g(\bar{x}_1) + k(\bar{x}_1) &= \bar{u}.\end{aligned}$$

Without lost of generality, we consider the particular case where $m = 1$. The Jacobian matrix of the system to equilibrium $(\bar{x}_1, 0, \bar{u})$ is written :

$$A = \begin{pmatrix} 0 & 1 \\ -\left(\frac{\partial^2 E_p}{\partial x_1^2}\right)_{\bar{x}_1} & -h'(0) \end{pmatrix}.$$

The characteristic polynomial of the matrix is

$$p(x) = x^2 + h'(0)x + \left(\frac{\partial^2 E_p}{\partial x_1^2}\right)_{\bar{x}_1}.$$

The product and the sum of the eigenvalues are given by

$$\lambda_1 \lambda_2 = \left(\frac{\partial^2 E_p}{\partial x_1^2}\right)_{\bar{x}_1}, \quad \lambda_1 + \lambda_2 = -h'(0).$$

The derivative $h'(0)$ the viscous friction is by nature non-negative : $\lambda_1 + \lambda_2 = -h'(0) \leq 0$.

The equilibria of the system are hyperbolic if

$$\begin{aligned}h'(0) &> 0 \quad \text{et} \quad \left(\frac{\partial^2 E_p}{\partial x_1^2}\right)_{\bar{x}_1} \neq 0, \\ \text{or if } h'(0) &= 0 \quad \text{et} \quad \left(\frac{\partial^2 E_p}{\partial x_1^2}\right)_{\bar{x}_1} < 0.\end{aligned}$$

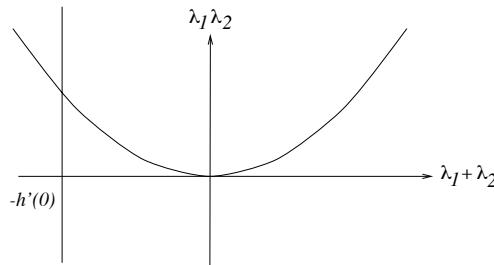


FIGURE 8.2 – Eigenvalues location of equilibria of a mechanical system with one degree of freedom

We observe that the equilibria are *not* hyperbolics if potential energy $E_p(x_1)$ is a linear function of the position x_1 , or more generally if the equilibrium corresponds to an inflection point $E_p(x_1)$. It's also the case when $h'(0) = 0$ and $\left(\frac{\partial^2 E_p}{\partial x_1^2}\right)_{\bar{x}_1} \geq 0$.

Hyperbolic equilibria of a mechanical system with one degree of freedom can then be fully characterized as shown in table 8.4 (see also figure 8.2). In particular, we observe that an hyperbolic equilibrium can never be a node or a repulsive center.

The simple electrical circuits containing only inductance and capacitance are generally referred to *RLC circuits* in the literature. In reference books Electrical Engineering or circuit theory, they are subject to a thorough study because they constitute the basic configuration of many practical devices (filters, oscillators, ...).

The *RLC series circuit* shown in Figure 8.3 is a typical example. Applying the

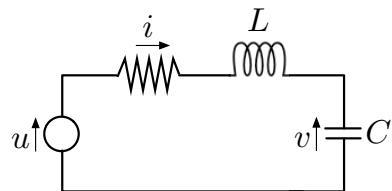


FIGURE 8.3 – RLC series circuits

principles discussed in chapter 3, the dynamic behavior of this circuit is described by a model of 2-dimensional state :

$$\begin{aligned} L\dot{x}_1 &= -r(x_1) - x_2 + u \\ C\dot{x}_2 &= x_1 \end{aligned}$$

where $x_1 = i$ is the current in the linear inductance L , $x_2 = v$ is the voltage across the linear capacity C and $r(x_1)$ is the voltage-current characteristic (eventually non-linear) of the resistance.

Characterization	Nature of hyperbolic equilibria
$0 < [h'(0)]^2 < 4 \left(\frac{\partial^2 E_p}{\partial x_1^2} \right)_{\bar{x}_1}$	stable center
$0 < 4 \left(\frac{\partial^2 E_p}{\partial x_1^2} \right)_{\bar{x}_1} \leq [h'(0)]^2$	stable node
$\left(\frac{\partial^2 E_p}{\partial x_1^2} \right)_{\bar{x}_1} < 0$	csaddle

TABLE 8.4 – Hyperbolic equilibria of mechanical systems with one degree freedom

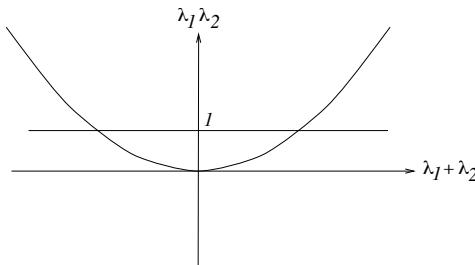


FIGURE 8.4 – Eigenvalues location of equilibria of a RLC circuit

The equilibria of the system are characterized by the equations :

$$\begin{aligned}\bar{x}_2 + r(0) &= \bar{u}, \\ \bar{x}_1 &= 0.\end{aligned}$$

Without loss of generality, consider the special case $L = 1$ and $C = 1$. The Jacobian matrix of the system to equilibrium $(0, \bar{x}_2, \bar{u})$ is written :

$$A = \begin{pmatrix} -r'(0) & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of the matrix is :

$$\begin{aligned}p(x) &= \lambda^2 + r'(0)\lambda + 1 \\ \text{ou } r'(0) &\triangleq (\partial r / \partial x_1)_{x_1=0}.\end{aligned}$$

	Nature of hyperbolics equilibria
$r'(0) \geq 2$	attractive node
$0 < r'(0) < 2$	attractive center
$-2 < r'(0) < 0$	repulsive center
$r'(0) \leq -2$	repulsive node

TABLE 8.5 – Hyperbolic equilibria of an RLC circuit

The product and the sum of the eigenvalues are given by

$$\lambda_1 \lambda_2 = 1, \quad \lambda_1 + \lambda_2 = -r'(0).$$

The equilibria of the system are hyperbolic if $r'(0) \neq 0$, i.e. if the derivative of the characteristic of the resistance is not zero at the origin. We observe that this is especially the case for linear resistance.

Hyperbolic equilibria of a series RLC circuit are then completely characterized as indicated in table 8.5 (see also figure 8.4). In particular we note that a hyperbolic equilibrium of a series RLC circuit can never be a saddle.

Consider two compartments systems whose graph is shown in figure 8.5. The

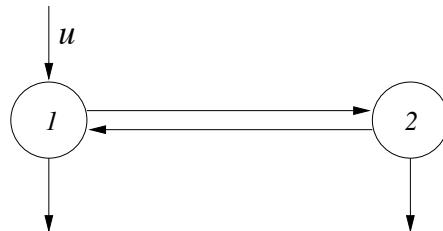


FIGURE 8.5 – Système à deux compartiments

input signal u is the first compartment volumetric flow rate. We assume that the

flows exchanged between the compartments meet the modeling conditions $C1 - C4$ of chapter 4 (Section 4.3). The dynamics of the system is then described by a two dimensional state model of the general form :

$$\begin{aligned}\dot{x}_1 &= -q_{12}(x_1, x_2) + q_{21}(x_2, x_1) - q_{10}(x_1) + u \\ \dot{x}_2 &= q_{12}(x_1, x_2) - q_{21}(x_2, x_1) - q_{20}(x_2)\end{aligned}$$

The functions q_{ij} satisfy the following conditions on the positive orthant :

$$q_{ij}(0, x_j) = 0 \quad \frac{\partial q_{ij}}{\partial x_i} \geq 0 \quad \frac{\partial q_{ij}}{\partial x_j} \leq 0 \quad (8.6)$$

Under these conditions, the system has an infinite number of isolated positive equilibria $(\bar{x}_1, \bar{x}_2, \bar{u})$. The Jacobian matrix around one of these equilibria is written :

$$A = \begin{pmatrix} -(a+c) & b \\ a & -(b+d) \end{pmatrix}$$

with the following simplified notation (all the partial derivatives are measured at equilibrium state) :

$$\begin{aligned}a &\triangleq \frac{\partial q_{12}}{\partial x_1} - \frac{\partial q_{21}}{\partial x_1} & c &\triangleq \frac{\partial q_{10}}{\partial x_1} \\ b &\triangleq \frac{\partial q_{21}}{\partial x_2} - \frac{\partial q_{12}}{\partial x_2} & d &\triangleq \frac{\partial q_{20}}{\partial x_2}\end{aligned}$$

Under the conditions (8.6), we immediately observe that $a, b, c, d \geq 0$. The characteristic polynomial of the Jacobian matrix is :

$$p(x) = x^2 + (a+b+c+d)x + (ad+bc+cd)$$

The product and the sum of the eigenvalues are given by :

$$\lambda_1 \lambda_2 = ad + bc + cd \quad \lambda_1 + \lambda_2 = -(a+b+c+d)$$

The equilibria of the system are hyperbolic if the following inequalities are satisfied :

$$a + b + c + d > 0 \quad \text{et} \quad ad + bc + cd > 0$$

It is easy to show that, under these conditions, the following inequality is satisfied :

$$0 < 4(ad + bc + cd) \leq (a + b + c + d)^2$$

We deduce that hyperbolic equilibrium of a two-compartments system may be only attractive nodes (see figure 8.6).

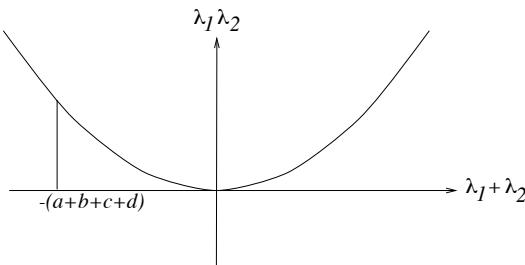


FIGURE 8.6 – Eigenvalues location of equilibria of a two-compartment system

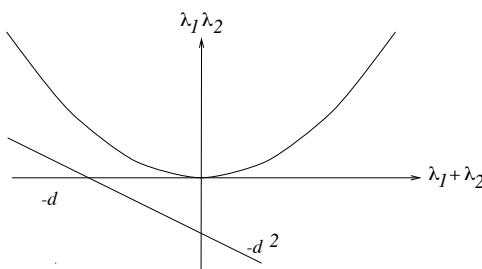


FIGURE 8.7 – Eigenvalues location of the equilibria of a two species reactional system

The simplest reaction systems involve two species. For example this is the case of an irreversible reaction converting a reagent X_1 in a product X_2 :



Assume that this reaction takes place in a continuous reactor perfectly mixed with constant volume. The reactor is fed with the species X_1 , at strictly positive constant volumetric flow rate. As we saw in Chapter 5, the reactor state model can be written as follows :

$$\begin{aligned}\dot{x}_1 &= -r(x_1, x_2) + d(u - x_1) \\ \dot{x}_2 &= r(x_1, x_2) - dx_2\end{aligned}$$

where x_1 et x_2 represent the concentrations of the species X_1 and X_2 in the reactional environment, d is the rate of dilution in the reactional environment and u is the concentration of the reagent X_1 in the diet. The reaction kinetics $r(x_1, x_2)$ is assumed to be a function of the concentrations of the two species.

The equilibria of the system are characterized by the equations :

$$d\bar{x}_2 = d(\bar{u} - \bar{x}_1) = r(\bar{x}_1, \bar{x}_2)$$

These equations involve at the equilibrium $\bar{x}_1 + \bar{x}_2 = \bar{u}$, it signifies that the sum of the concentrations of the species X_1 and X_2 in the reactor is equal to the concentration of the reagent X_1 in the diet. This observation is obviously in agreement with the mass conservation principle.

The Jacobian matrix around equilibrium can be written :

$$A = \begin{pmatrix} -a - d & -b \\ a & b - d \end{pmatrix}$$

with the following simplified notation :

$$a \triangleq \left(\frac{\partial r(x_1, x_2)}{\partial x_1} \right)_{\bar{x}_1, \bar{x}_2} \quad b \triangleq \left(\frac{\partial r(x_1, x_2)}{\partial x_2} \right)_{\bar{x}_1, \bar{x}_2}$$

The characteristic polynomial of the Jacobian matrix is :

$$p(x) = x^2 + (a - b + 2d)x + (a - b)d + d^2$$

The product and the sum of the eigenvalues are given by :

$$\lambda_1 \lambda_2 = (a - b)d + d^2 \quad \lambda_1 + \lambda_2 = -(a - b + 2d)$$

Because of the dilution rate d is a strictly positive quantity, we can check after some calculations that the equilibria of the system is hyperbolic if $(a - b) \neq -d$. We observe that

- si $\lambda_1 + \lambda_2 = -[(a - b) + 2d] > 0$, then necessarily $\lambda_1 \lambda_2 = d[(a - b) + d] < 0$ and therefore the equilibrium is a saddle.
- Si $\lambda_1 + \lambda_2 = -[(a - b) + 2d] \leq 0$, the the equilibrium is a saddle if $-2d \leq (a - b) < -d$, and an attractive node if $(a - b) > -d$. However, the equilibrium can not be a center, because it is impossible to have $\lambda_1 \lambda_2 \geq \frac{1}{4}(\lambda_1 + \lambda_2)^2$.

This analysis is summarized in the table 8.6 and the figure 8.7.

From the tables of the sections 8.2, we can make the following observations.

1. For a linear system of dimension 2, the *attractive* equilibriums either are a node or a spiral or a line of uninsulated equilibriums. in each of these cases, the basin of attraction is the entire phase map.
2. When the equilibrium is repulsive, the system trajectories diverge when the time t tend to infinity.
3. When the equilibrium of a linear system is a center, all the system trajectories are periodic and the radius pf the trajectories depend on the initial conditions. A linear system with periodi trajectories is structurally unstable, and thus any perturbation of the system can remove these periodic trajectories.

	Nature of hyperbolic equilibria
$(a - b) < -d$	saddle
$(a - b) > -d$	attractive node

TABLE 8.6 – Hyperbolic equilibria of a reaction system
with two species

None of this observations is generically verified in the case of nonlinear systems. Indeed, the first two concern a *global* behaviour of the trajectories, and we have seen that this is only locally, in the neighborhood of a hyperbolic equilibrium, that trajectories of a nonlinear system behave like those of the linear approximation of the system.

The purpose of this section is to show that, for non linear systems, there are other attractive sets, and in particular periodic trajectories. Furthermore, we will show that these attractive sets are structurally stable. This is a very interesting property of the linear systems which is used for the conception of oscillator circuits.

Example 8.8. Diode tunnel RLC circuits

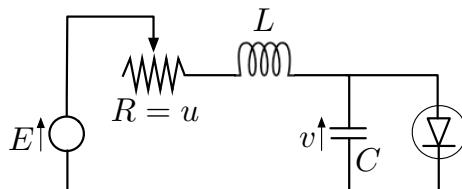


FIGURE 8.8 – Tunnel diode oscillator

The figure 8.8 shows a tunnel diode oscillator. This is an electric circuit with linear dipoles (a constant voltage source E , a variable linear resistance R , a linear inductance $L = 1H$, a linear capacitor $C = 1F$) and a non linear resistance (tunnel diode) whose current-voltage characteristic $i = h(v) = 2v^3 - 6v^2 + 5v$ has the shape of the curve shown in figure 8.11. The input of this system is the variable resistance R .

As we have seen in chapter 3, the state variables of the system are the current $x_1 = i$ in the inductance and the voltage $x_2 = v$ at the terminals of the capacitor. One obtains the following state equations :

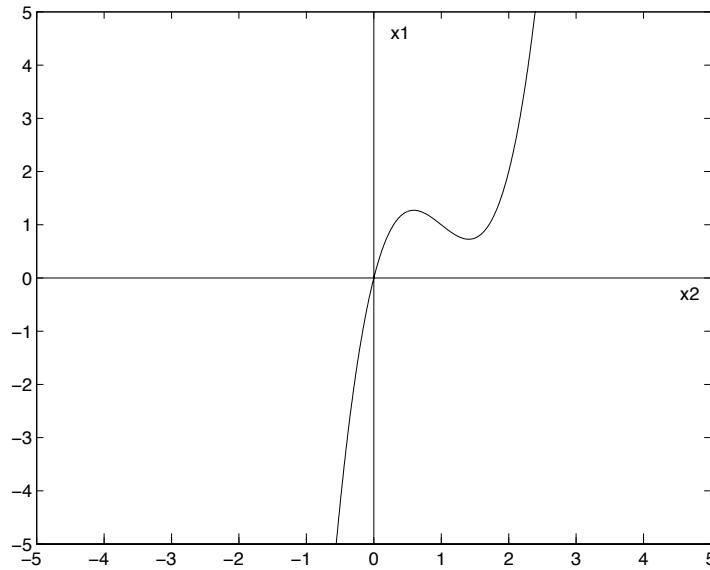


FIGURE 8.9 – Tunnel diode current-voltage characteristic.

$$\begin{aligned}\dot{x}_1 &= -ux_1 - x_2 + E \\ \dot{x}_2 &= x_1 - h(x_2),\end{aligned}$$

and the possible equilibriums are characterized by

$$\begin{aligned}\bar{x}_1 &= \frac{E - \bar{x}_2}{\bar{u}} \\ \bar{x}_1 &= h(\bar{x}_2).\end{aligned}$$

By showing in the phase plane the graphs of the curves \$\bar{x}_1 = (E - \bar{x}_2)/\bar{u}\$ et \$\bar{x}_1 = h(\bar{x}_2)\$, one notes that, for a given characteristic of diode, two configurations are possible according to the respective values of

\bar{u} and E . If the line $(-1/\bar{u})$ is steep enough, there will be only one equilibrium point (figure 8.10.a). On the other side, if this slope is below the one of the tangent at the inflection point of the curve, there will be one, two or three possible equilibriums according to the value of E . (figure 8.10.b).

We can study again the shape of the trajectories at the neighborhood of the equilibrium by calculating the eigenvalues of the Jacobian matrix of the system :

$$A = \begin{pmatrix} -\bar{u} & -1 \\ 1 & -h'(\bar{x}_2) \end{pmatrix}.$$

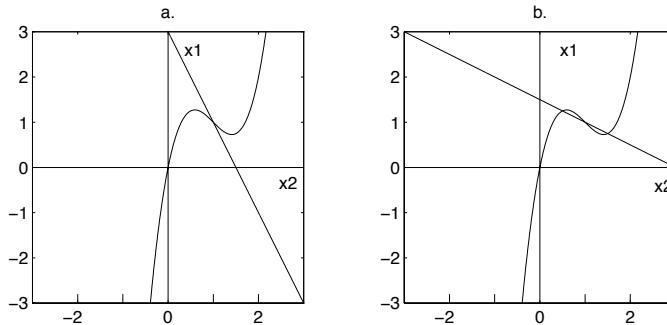


FIGURE 8.10 – Equilibrium configurations for the tunnel diode circuit

The product and the sum of the eigenvalues are given by

$$\lambda_1 \lambda_2 = \bar{u} h'(\bar{x}_2) + 1, \quad \lambda_1 + \lambda_2 = -(\bar{u} + h'(\bar{x}_2)),$$

and one observes that the sign of the eigenvalues doesn't depend on E but only depends on the respective slopes of the two graphs of either of figures 8.10.

Let's examine in details the equilibriums :

- a. For the figure 8.10.a, there is only one equilibrium. If this one is to the left of the local maximum of the curve $h(x_2)$ or to the right of the local minimum of this one, the product of the eigenvalues is positive, and thus the corresponding equilibrium is a node or a stable spiral.
- b. Always for the first figure, if the equilibrium stand between the local maximum and the local minimum, we have $-1/\bar{u} < h'(\bar{x}_2) < 0$ and the product of the eigenvalues is thus positive. Regarding the sum, it will be negative and the corresponding equilibrium will therefore be attractive if $|h'(\bar{x}_2)| < \bar{u}$ (wich corresponds to an important value of \bar{u} ie a a strongly dissipative resistance which ensures the stability of the circuit). On the other side, if $|h'(\bar{x}_2)| > \bar{u}$, the sum of the eigenvalues is positive and the corresponding equilibrium is repulsive.
- c. Regarding the figure 8.10.b., the equilibriums to the left of the local maximum of $h(x_2)$ and to the left of the local minimum are such that the product of the eigenvalues is positive and the sum of the eigenvalues is negative. The corresponding equilibrium is either a node or a stable spiral.
- d. Regarding the eventual equilibrium between the maximum and the minium, it is it verifies $h'(\bar{x}_2) < -1/\bar{u} < 0$. The product of the eigenvalues is negative and the corresponding equilibrium is a saddle.

As we can note, the equilibrium is repulsive in different cases. We can then question about what the trajectories moving away from this equilibrium point be-

come.

Let's consider the following particular numerical values :

$$\begin{aligned}\bar{u} &= 0.5, \\ E &= 1.5, \\ h(v) &= 2v^3 - 6v^2 + 5v.\end{aligned}$$

We can check that for these particular values, $(\bar{x}_1, \bar{x}_2, \bar{u}) = (1, 1, 0.5)$ is the only equilibrium of the system. and that it is a repulsive equilibrium (case b. above).

By simulating the system of two differential equations for different initial conditions, we obtain the orbits illustrated in figure 8.11. It clearly appears that all the calculated orbits (we could think that others would behave the same way) wrap around a periodical orbit. This system has no attractive equilibrium, but there exists a *closed orbit* which is attractive. This is what is called a limit cycle.

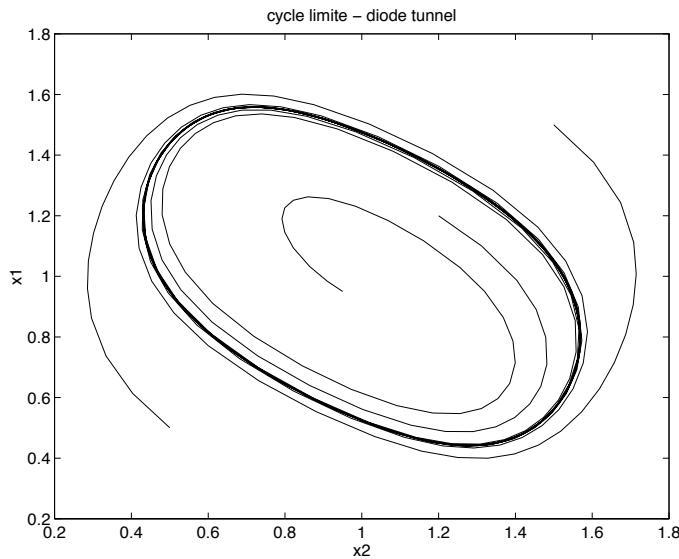


FIGURE 8.11 – Limit cycle for a tunnel diode circuit

Figure 8.12 shows the trajectories (state as function of the time) and shows that they (quickly) converge toward periodic trajectories whose period and amplitude don't depend on the initial conditions.

Asymptotically, therefore, the system will undergo amplitude oscillations constant, whatever the value of the initial conditions, contrary to what happens for a linear system with a center type of equilibrium.

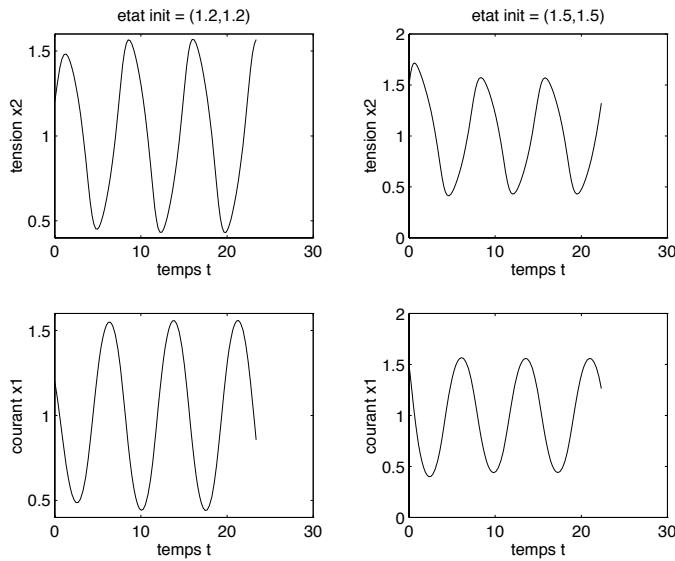


FIGURE 8.12 – Trajectories of the tunnel diode circuit

In fact, this is exactly what is sought when building an oscillator : Oscillations of constant amplitude regardless of initial conditions, we can not get that with a non-linear system. Finally, it is also show that the limit cycle is structurally stable, which is also an interesting property for the design of an oscillator. \square

We formalize below some of the concepts that have just been described in the preceding example. Consider a planar system

$$\dot{x} = f(x, \bar{u})$$

with constant input \bar{u} and let's note $x(t, x_0, \bar{u})$ the solution at time t with $x(0) = x_0$.

Definition 8.9. Limit point

The point z is a *limit point* of y for the dynamical system submitted to a constant input \bar{u} if there exists a sequence $\{t_n\}$ in \mathbb{R} such that $t_n \rightarrow \infty$ when $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} x(t_n, y, \bar{u}) = z$. \square

According to this definition, a attractif equilibrium is therefore an endpoint from any point in its basin of attraction. But the notion of limit point is more general as we will see below.

Definition 8.10. Limit cycle

A limit cycle is a closed orbit γ such that a point of γ is a limit point of another point of the phase plane not in γ . \blacksquare

This definition shows that when closed orbit is a limit cycle, while point of the orbit is a boundary point, and thus the trajectory of the system will approach more and more each point of the closed orbit, to instants determined. We can now state some results to establish the existence periodic trajectories and limit cycles. These results are only valid for planar systems (while there are also limitations cycles systems of higher order). The reason is that these demonstrations results are based on the fact that in two dimension, a closed orbit in the plane of stage divides the plan into an inner region in the orbit and an outer region, This is of course true in a phase space dimension greater than 2. The first result is a sufficient condition for non-existence path periodic (and therefore limit cycle).

Theorem 8.11. Bendixson Dulac, 1901 et 1933

Let D be a simply connex domain² If Si

$$\operatorname{div} f \triangleq \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

is non identically zero in a subdomain of D and does not change sign in this subdomain, then D does not contain any closed orbit. ■

Let's recall that that the divergence $\operatorname{div} f$ describes the remoteness ($\operatorname{div} f > 0$) or the rapprochement ($\operatorname{div} f < 0$) from $\dot{x} = f(x)$. This theorem is proved simply by contradiction : suppose that there exists a closed path γ in the domain, whose interior is $D_\gamma \subseteq D$. Thus the integral of the divergence inside of γ , $\int_{D_\gamma} \operatorname{div} f$ is equal by the Green-Stokes theorem to the integral of the flux through the frontier $\gamma \int_\gamma \langle f, n \rangle$, where $\langle ., . \rangle$ is the scalar product and n is the unitary normal vector exiting from γ . This integral is zero, since f is tangent everywhere in every point of the trajectory γ .

It follows that the divergence $\operatorname{div} f$ can not be negative everywhere or positive everywhere in D_γ . The second result allows to bring out the existence of a limit cycle.

Theorem 8.12. Poincaré-Bendixson, 1901

If E is a closed and bounded subset of \mathbb{R}^2 , invariant for the system $\dot{x} = f(x, \bar{u})$, and if γ is an orbit starting in E , then :

- i) If E doesn't contain any equilibrium point, then γ either is a periodic orbit or converge toward a limit cycle.
- ii) If E doesn't contain any periodic orbit but contains a unique equilibrium point, this equilibrium is globally attractive in E . □

Ensuite, si on a pu exclure la présence d'équilibres dans cet ensemble, celui-ci doit nécessairement contenir un cycle limite, ou ne contenir que des trajectoires périodiques.

2. a connex domain in \mathbb{R}^2 whose frontier can be obtained by deformation of a circle.

This theorem can be used to prove the existence of a limit cycle. To do this, we first seek a closed, bounded and invariant set. To check that the set is invariant, we show that on the border of this set, the vector field points inwards. Then, if we have been able to exclude an equilibrium in this set, this one must necessarily contain a limit cycle, or contain only periodic trajectories.

It is important to notice that these two theorems, unlike other results of this chapter, are specific to the planar systems, and strongly restrict the possible dynamics in two dimensions : the trajectories converge to a point, a cycle, or are unbounded. The higher dimensions have a richer behaviour which exceed the purpose of this course : chaos, strange attractors, etc.

Example 8.13. Tunnel diode circuit

We consider again the circuit already described with the same numerical values than previously, which lead to a unique repulsive equilibrium $(\bar{x}_1, \bar{x}_2, \bar{u}) = (1, 1, E-1)$, with $E > 1$. Let's now consider in the planar phase a circle centred in $(0, 0)$ and with a large enough radius, and let's show that, on the circle, the vector field points inwards.

It is therefore to show that the scalar product of the vector field and the normal vector of the circle is negative : $PS = x_1 f_1(x_1, x_2, \bar{u}) + x_2 f_2(x_1, x_2, \bar{u}) < 0$. Let's choose as a radius $r = \sqrt{2} \frac{E}{E-1}$ (see figure 8.13).

The scalar product is equal to $PS = -(E-1)x_1^2 + Ex_1 - x_2 h(x_2)$. Note that the quantity $-x_2 h(x_2)$ is always nonpositive except for $x_2 = 0$. ≤ 0 , $PS < 0$. Likewise for $x_1 \geq \frac{E}{E-1}$, $Ex_1 \leq (E-1)x_1^2$ and $PS < 0$.

It remains to study the portion of the circle where $x_1 < \frac{E}{E-1}$, $x_2 > \frac{E}{E-1}$. A small calculation allows to check that $|h(x_2)| > |x_2|$ and that the following inequalities are thus checked :

$$\begin{aligned} x_2 h(x_2) &> x_2^2 &> \frac{E^2}{(E-1)^2} \\ Ex_1 &< \frac{E^2}{E-1} &< \frac{E^2}{(E-1)^2} \end{aligned}$$

and then $PS < 0$. On the circle of radius r , the vector field points thus inwards. Otherwise, given that the equilibrium $(1, 1, E-1)$ is repulsive, we can take a small enough circle around this equilibrium such that the vector field evaluated on the circle points outwards. If we now consider that the domain formed by the ring (not centered) between the small circle and the big one, it is an invariant set because on the border of this set, the vector field points inward the domain. Given that the domain doesn't include any equilibrium, it must contain a limit cycle (or contain periodic trajectories only). \square

In this chapter we chose to study the type of systems planning trajectories for a constant value of the input \bar{u} . This value is not necessarily *before* determined,

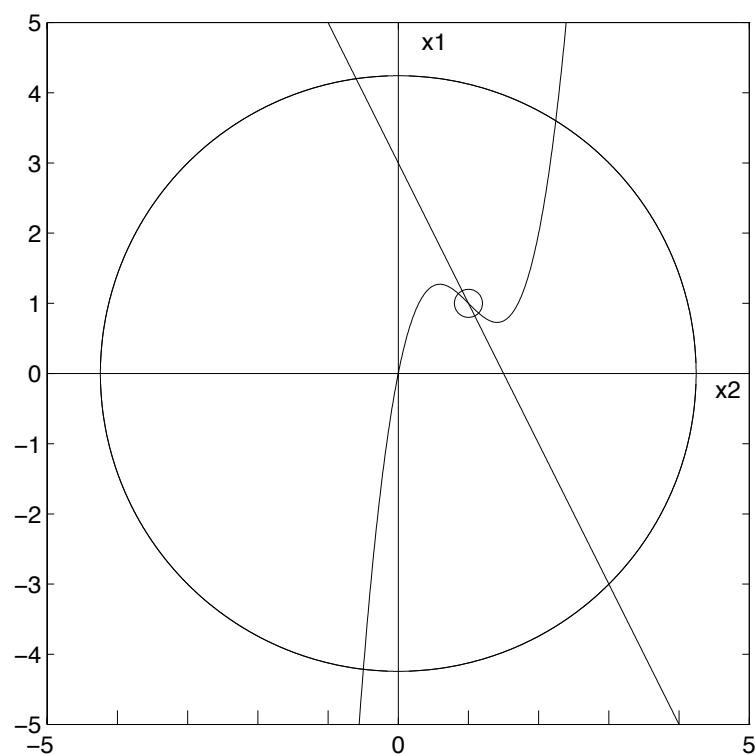


FIGURE 8.13 – Invariant set for the tunnel diode circuit

it is interesting to analyze how the trajectories will be influenced by changing \bar{u} . The theorem 8.6 already gives us an indication. As long as the equilibrium around which the hyperbolic trajectories are analyzed, small variations in \bar{u} won't move much the eigenvalues of the state matrix of the linear approximation of the system, and the curves of the trajectories remain similar.

But by varying the constant input \bar{u} , it may happen that eigenvalues of the state matrix reach the imaginary axis of the complex plane, and in this case, a fundamental change is expected in the shape of trajectories.

More generally, equilibrium diagrams studied in the previous chapter also show that by varying \bar{u} we can change the number of system equilibrium points as well as their kind.

The study of changes in the nature and/or the number of equilibria depending on the evolution of the system input falls under what is called the bifurcation theory and the constant input \bar{u} is then called *parameter bifurcation*. We illustrate below this concept having four types of bifurcations that occur in planes systems.

8.3.2. Hopf bifurcation

Example 8.14. Diode tunnel circuit (ctd)

Resuming again the example of the diode tunnel circuit by varying the input \bar{u} (i.e. the variable resistor R), with a constant voltage source $E = 1.5$.

The figure 8.14 illustrates how the unique equilibrium moves when \bar{u} changes. The following table characterizes the equilibrium type in function of \bar{u} .

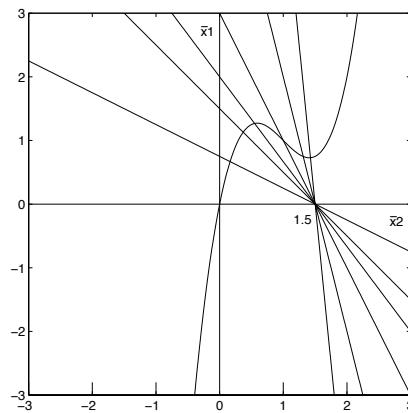


FIGURE 8.14 – Equilibrium of the tunnel diode circuit
when the resistance R varies.

\bar{u}	\bar{x}_2	$h'(\bar{x}_2)$	eigen values	equilibrium type
$\bar{u} > 0.7139$	$\bar{x}_2 < 0.5918$	$h'(\bar{x}_2) > 0$	$\lambda_{1,2} \in C^-$	focus attractive
$0.1261 < \bar{u} < .7139$	$0.5918 < \bar{x}_2 < 1.4082$	$h'(\bar{x}_2) < 0$	$\lambda_{1,2} \in C^+$	focus répulsif
$\bar{u} < 0.1261$	$\bar{x}_2 > 1.4082$	$2.5 > h'(\bar{x}_2) > 0$	$\lambda_{1,2} \in C^-$	focus attractive

Therefore, if one starts with a value of the variable resistor \bar{u} large enough such that the equilibrium point located to the left of the first peak of the characteristic curve of the diode, and that it gradually decreases this value, it cycles through the following configurations : an attractive focus, repellent focus (associated with a limit cycle), an attractive focus. At the time of the two transitions between attractive repellent focus, the system passes through a value such that the equilibrium point is not hyperbolic. \square

The bifurcation we just highlighted (passage of an attractive focus for a repellent focus accompanied by a limit cycle, or vice versa) is called *Hopf bifurcation*. The following theorem guarantees elsewhere the existence of a limit cycle. In order to state it precisely, we formalize the above statement. Suppose a plan system with a unique equilibria family (\bar{x}, \bar{u}) parameterized by \bar{u} . Suppose that there exists a value of \bar{u}^* of *baru* such that the eigenvalues of the Jacobian matrix evaluated in this balance have a zero real part and a non-zero imaginary part. These values depend continuously on \bar{u} , at least in the neighborhood of \bar{u}^* , and are so noted

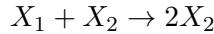
$$\lambda_i(\bar{u}) = \alpha(\bar{u}) \pm i\beta(\bar{u}).$$

suppose further that $\frac{d\alpha}{d\bar{u}}(\bar{u}^*)d\bar{u} > 0$.

Theorem 8.15. With the assumptions above, if the values \bar{u} are close to \bar{u}^* , the equilibrium is attractive for $\bar{u} < \bar{u}^*$ and repellent for $\bar{u} > \bar{u}^*$ then it is a closed orbit for $\bar{u} > \bar{u}^*$ or $\bar{u} < \bar{u}^*$. In particular, if (\bar{x}^*, \bar{u}^*) is locally attractive, there exists an attractive limit cycle around (\bar{x}, \bar{u}) for all $\mu = \bar{u} - u^* Bar > 0$ sufficiently small. In addition, the amplitude of the limit cycle increases when μ increases. \square

Remark 8.16. The statement of the theorem remains ambiguous about the nature (attractive or repulsive) of the closed orbit that appears. We can remove this ambiguity with a more technical statement showing explicitly the terms of the third order non-linear system (see for example Guckenheimer et Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, 1983).

Consider the reaction



that is taking place in a constant volume reactor, fed by a reactive X_1 at a concentration x_1^{in} , and with a dilution rate of \bar{u} .

The system state model (assuming reaction kinetics described by the law of mass action) is given by

$$\begin{aligned}\dot{x}_1 &= -kx_1x_2 + u(x_1^{in} - x_1) \\ \dot{x}_2 &= kx_1x_2 - ux_2.\end{aligned}$$

The system has two separate balance for each constant value of entry $\bar{u} \neq kx_1^{in}$: $(x_1^{in}, 0, \bar{u})$ and $(\bar{u}/k, x_1^{in} - \bar{u}/k, \bar{u})$, as shown in Figure 8.15.

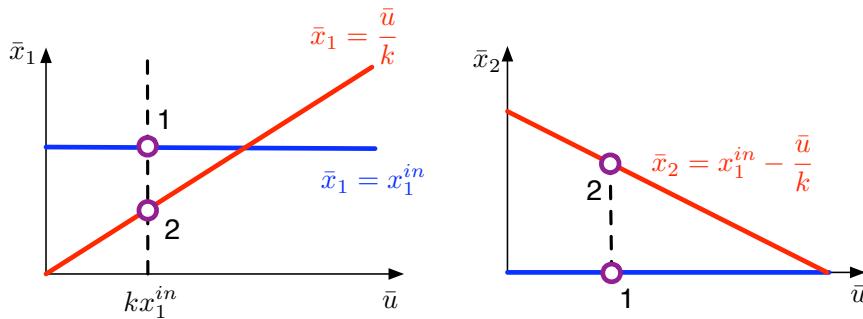


FIGURE 8.15 – Equilibrium diagram - Transcritical bifurcation

We easily verify that the first equilibrium is attractive if $\bar{u} > kx_1^{in}$ and is otherwise a pass. Conversely, the second balance is attractive for small values of \bar{u} and becomes a pass if $\bar{u} > kx_1^{in}$. There is therefore here also has a bifurcation, simpler, however, the characteristics of the two equilibria are exchanged when the bifurcation parameter \bar{u} crosses the critical value kx_1^{in} . This bifurcation is called *transcritical bifurcation*. We also verify that at this critical value, the (unique) equilibrium is not hyperbolic.

8.3.3. Pass-node bifurcation

The third type of bifurcation is illustrated by the example of the exothermic chemical reactor described in section 7.1. Recall that the equilibrium diagram connecting the temperature of balance of the reactor, \bar{T} , to the heat external heat input, \bar{u} , has the appearance shown in figure 8.16.

We notice that for small values of \bar{u} , the system has a single equilibrium point corresponding to an equilibrium of lower temperature and at big concentration of

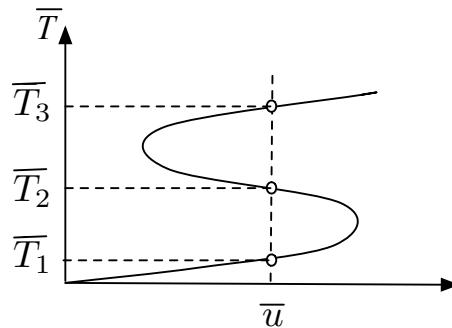


FIGURE 8.16 – Equilibrium diagram - Pass-node bifurcation

reactive in the reactor (and therefore a low concentration of the reaction product). We can verify that this equilibrium is attractive. Then, for a critical value of \bar{u} that is easily identified on the equilibrium diagram, the system proceeds to three equilibrium values for the temperature, the middle corresponding to an attractive balance and the other two repellent equilibria.

Finally, by further increasing \bar{u} , we cross a new critical value beyond which the system also no longer has a single balance. This is the *saddle-node bifurcation*. From a critical value of the input (i.e. the bifurcation parameter) appear two new equilibria, one of them being an attractive node, the other being a pass. At the critical value, the balance is not hyperbolic.

8.3.4. Fork bifurcation

The mechanism shown in figure 8.17 is a « Watt's regulator ». This device can be used to measure a rotation speed from a fixed pointer on the vertical axis, or, and that's why it was invented, to regulate this speed if the pointer is connected to a motor supply valve rotating the device .

We can verify that the equations describing the move of the system are written :

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u^2 \cos x_1 \sin x_1 - k \sin x_1 - Kx_2\end{aligned}$$

where $x_1 = \theta$ is the angular position of the symmetric pendulums and u its rotational speed.

This device has an equilibrium in $(x_1, x_2, u) = (0, 0, \bar{u})$ and, if $\bar{u}^2 > k$, another equilibrium in $(\bar{x}_1 = \arccos \frac{k}{\bar{u}^2}, 0, \bar{u})$ with $\bar{x}_1 \in [0, \frac{\pi}{2}]$. Actually $(-\bar{x}_1, 0, \bar{u})$ is an equilibrium that would be the permutation of two pendulums, which is (conceptually) but not physically possible, according to the above equations .

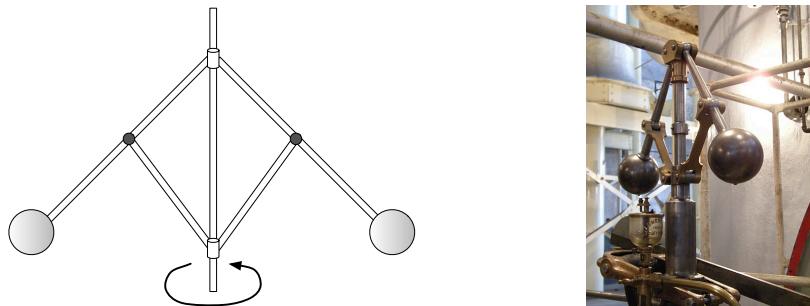


FIGURE 8.17 – Watt's regulator

The Jacobian matrix of the system around the equilibrium $(0, 0, \bar{u})$ is written :

$$A = \begin{pmatrix} 0 & 1 \\ \bar{u}^2 - k & -K \end{pmatrix}$$

This equilibrium is attractive for $\bar{u}^2 < k$ and repellent for $\bar{u}^2 > k$. For $\bar{u}^2 = k$, the equilibrium is not hyperbolic .

Around the other two equilibria, the Jacobian matrix becomes :

$$A = \begin{pmatrix} 0 & 1 \\ \frac{k^2}{\bar{u}^2} - \bar{u}^2 & -K \end{pmatrix} \text{ avec } \bar{u}^2 > k \Rightarrow \bar{u}^4 > k^2$$

These equilibria are therefore attractive. The bifurcation diagram can then be illustrated as shown in figure 8.18. This is a fork type bifurcation.

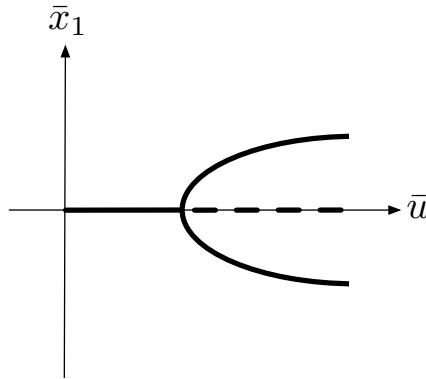


FIGURE 8.18 – Equilibria diagrams - fork bifurcation

8.3.5. Generalisations

In this section we have described bifurcations for the systems of order two depending on a parameter (the value of \bar{u}). These bifurcations are characterized by

crossing the imaginary axis complex plane by a real eigenvalue of the linear approximation or a pair of complex conjugate eigenvalues (Hopfs bifurcation). When systems of larger order than two depends on a variable parameter, it is rare that more than one real eigenvalue (or more than one pair of combined complex eigenvalues) crosses the imaginary axis for the same value of the bifurcation parameter. What we have just described occurs therefore also, in phase spaces more complicated to visualize, for systems of higher order.

8.4. Exercices

Exercise 8.1. A mechanical system

We consider a manipulator robot made from a segment connected to a fixed frame by a rotating joint. The robot moves in a vertical plane. It is powered by a motor producing a torque applied to the joint and is subjected to a torque of viscous friction. Flexibility is neglected.

1. Establish the state system model
2. Determine the equilibria configurations
3. Analyze the behavior of the trajectories in the neighborhood of equilibria with linear viscous friction when the couple applied is constant.
4. What can you say about the equilibrium when the viscous friction is quadratic?

Exercise 8.2. A chemical reactor

Suppose a perfectly mixed continuous reactor with constant volume in which an irreversible chemical reaction takes place using two species A and B :



The reactor is fed only with species A , at strictly positive constant volume flow. The input variable is the reactor feed concentration. The reaction kinetics is a function of the concentrations of the two species : $r(x_A, X_B)$.

1. Establish the state system model.
2. Show that, at constant input, the equilibrium is unique and stable if the kinetics obeys the law of mass action with hyperbolic inhibition by the product. Is that a node or a focus?
3. Show that the system can have unstable equilibria if the kinetics is a monotonically increasing function of its arguments.

Exercise 8.3. A compartment system

What are the conditions on the structure of a two-compartment linear graph system so that the system has one or two null eigenvalues? What is then the behavior of the system (detail the possible various cases)?

Exercise 8.4. DC generator with self-excitation

Consider a DC generator with self-excitation. The induced voltage is at constant speed, a *monotone increasing bounded* function of the excitation current $E(I_S)$ such that $E(0) > 0$. The generator delivers a resistive load. The system entrance control is the speed of rotation of the generator.

1. Determine the state system model
2. Show that we can choose the direction of the reference current for the system to be positive.
3. Of what type must the function $E(I_S)$ be for there to be three isolated hyperbolic equilibria at constant rotational speed. Discuss the stability of these equilibria.
4. Study the bifurcation configuration in function of the rotational speed equilibrium.

Exercise 8.5. RLC electric circuit

We consider the following linear electric circuit :

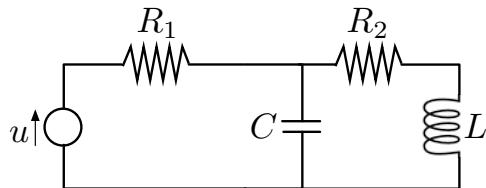


FIGURE 8.19 – RLC electric circuit

where $R_2 = 1\Omega$, $C = 1F$ and $L = 1H$.

1. Write a state model.
2. Determine the equilibria.
3. What are the conditions on R_1 for each equilibrium to be a node, a focus or a pass?

On considère le même circuit électrique mais avec $R_1 = 1\Omega$ et R_2 une résistance non linéaire décrite par la relation tension-courant $v_r = i_r^3 - 3i_r^2 + i_r$. We consider the same circuit but with $R_1 = 1\Omega$ and R_2 a non linear resistance described by the current-voltage relationship $v_r = i_r^3 - 3i_r^2 + i_r$.

1. Calculate the equilibria of the system.
2. Characterize the behavior of the system in the vicinity of these equilibria.

Exercise 8.6. Modeling of a fishing activity.

In a lake lives a fish species whose growth follows a logistic law. The fish are caught by fishermen following a principle of action of the masses. Fishermen are drawn to the lake with a rate directly proportional to the amount of fish in the lake. As against the fishermen are discouraged to fish with a rate directly proportional to the number of fishermen already at the lake.

1. Establish a state system model.
2. Study the existence and the stability of the equilibria states.

Chapitre 9

Equilibrium's stability

This chapter is about the stability of equilibriums. Precisely, we are interested in the behaviour of the trajectories of the system in the neighbourhood of the equilibriums. Let a dynamic system be described by its state model :

$$\dot{x} = f(x, u). \quad (9.1)$$

We assume that the system owns an equilibrium in (\bar{x}, \bar{u}) . We ask ourselves the two following questions :

- a : If the entry is kept at its equilibrium's value \bar{u} and if the initial state $x(t_0)$ is in the neighbourhood of the equilibrium's value \bar{x} , how will the trajectories of the system behave? Under which conditions will the trajectories converge towards \bar{x} ?
- b : If the entry $u(t)$ is close to \bar{u} (but not necessarily constant), what can we say about the system's trajectories? Under which conditions will the trajectories $x(t)$ stay close to \bar{x} ?

9.1. Definitions

Definition 9.1. Stable equilibrium

The equilibrium (\bar{x}, \bar{u}) is a *stable equilibrium* of the system (9.1) if

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ s.t. } \|x(t_0) - \bar{x}\| < \delta \Rightarrow \|x(t, x(t_0), \bar{u}) - \bar{x}\| < \epsilon \ \forall t \geq t_0.$$

If this condition is not satisfied, the equilibrium is *unstable*. □

This definition is to be interpreted as follows. We want to characterize how the trajectory $x(t)$ stays close to the equilibrium point \bar{x} for every $t \geq t_0$ when the entry is constant ($u(t) = \bar{u} \forall t \geq t_0$). For that, we measure the proximity with the norm $\| \cdot \|$ and we impose that the solutions $x(t)$ stay within the region bound by $\|x(t) - \bar{x}\| < \epsilon$, meaning in a "tube" of radius ϵ around the trajectory $x(t) = \bar{x}$. If this goal is achievable for an initial condition $x(t_0)$ close to the equilibrium (meaning that $\|x(t_0) - \bar{x}\| < \delta$), then we say that the equilibrium is stable. The equilibrium is said unstable otherwise.

The previous definition is the weakest form of stability considered in this chapter. In particular, it does not imply that the trajectories $x(t)$ converge to the equilibrium point.

Definition 9.2. Attractive equilibrium

The equilibrium (\bar{x}, \bar{u}) is an *attractive equilibrium* of (9.1) if

$$\exists \delta > 0 \text{ s.t. } \|x(t_0) - \bar{x}\| < \delta \Rightarrow \lim_{t \rightarrow \infty} \|x(t, x(t_0), \bar{u}) - \bar{x}\| = 0. \quad \square$$

An attractive equilibrium \bar{x} is therefore a point to which the solutions $x(t)$ converge if they start close enough of \bar{x} . It has to be noted that stability and attractiveness are two different notions and that they don't imply each other.

Definition 9.3. Asymptotically stable equilibrium

The equilibrium (\bar{x}, \bar{u}) is an *asymptotically stable equilibrium* if it is stable and attractive. \square

A set of initial states $x(t_0)$ from which the trajectories converge to an asymptotically stable equilibrium is called *basin of attraction*. The asymptotic stability is the property that is usually sought for in practice. However, we have to notice that this definition above does not indicate the speed at which the trajectory $x(t)$ converge towards the equilibrium. That's why the notion of exponential stability was introduced, which allows to characterize that speed.

Definition 9.4. Exponential stability

The equilibrium (\bar{x}, \bar{u}) is an *exponentially stable equilibrium* if

$$\forall \epsilon > 0 \ \exists a > 0, b > 0 \text{ and } \delta > 0 \text{ s.t.}$$

$$\|x(t_0) - \bar{x}\| < \delta \Rightarrow \|x(t, x(t_0), \bar{u}) - \bar{x}\| \leq a \|x(t_0) - \bar{x}\| e^{-bt} \ \forall t \geq t_0. \quad \square$$

It is obvious that the exponential stability implies the asymptotic stability but the opposite is not necessarily true.

9.2. Lyapunov's first method (indirect method)

Lyapunov's first method is based on the examination of the system's linearization $\dot{x} = f(x, \bar{u})$ around the equilibrium (\bar{x}, \bar{u}) . Precisely, we study the eigenvalues $\lambda_i(A)$ of the Jacobian matrix estimated at the equilibrium :

$$A = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}).$$

Under this method, the stability properties of (\bar{x}, \bar{u}) are expressed as follows.

Theorem 9.5. Lyapunov's first method.

1. If all the eigenvalues of the Jacobian matrix have a real part strictly negative ($\forall i, \operatorname{Re}(\lambda_i(A)) < 0$), the equilibrium (\bar{x}, \bar{u}) is exponentially stable.
2. If the Jacobian matrix owns at least one eigenvalue with a real part strictly positive ($\exists i, \operatorname{Re}(\lambda_i(A)) > 0$), the equilibrium (\bar{x}, \bar{u}) is unstable. \square

This theorem does not allow to say if the equilibrium is stable or unstable if the Jacobian matrix owns at least one eigenvalue with a real part equal to zero and no eigenvalue with a eigenvalue with a strictly positive real part. In this case, the system's trajectories converge to a sub-set (a variety) whose dimension is the number of eigenvalues of the Jacobian equal to zero and the equilibrium's stability can be examined in this sub-set with the second method.

9.3. Lyapunov's second method (direct method)

As we juste saw, Lyapunov's first method is simple to apply but it only allows to study the equilibrium's stability partially. Furthermore, it does not give any indication on the size of the basins of attraction. The second method is more difficult to implement but, in return, is more general in scope. It is based on the definition of a particular function noted $V(x)$ and called *Lyapunov's function*, which is decreasing along the system's trajectories within the basin of attraction. Before giving the different stability theorems, we start with an example.

Example 9.6. A robot's arm with one degree of freedom.

We consider a robot's arm with one degree of freedom, with linear viscous friction and under a constant moment (see following figure). The state model of the the system is the following (see chapter 2 and 8, section 8.3.1) :

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= J^{-1}(-mgbs \sin x_1 - kx_2 + \bar{u}).\end{aligned}$$

In these equations, x_1 is the angular position, x_2 the angular speed, J the inertia, m the mass, b the distance between the anchor point and the center of mass, k the friction coefficient and \bar{u} the constant moment applied to the robot's arm.

Let's consider the case where $0 < \bar{u} < mgb$. The system owns two equilibriums verifying the two following relations :

$$\bar{x}_1 = \arcsin\left(\frac{\bar{u}}{mgb}\right), \quad \bar{x}_2 = 0.$$

In accordance with the physical intuition and as we can verify by examining the Jacobian matrix's eigenvalues, there is an asymptotically stable equilibrium in the "lower" position and an unstable equilibrium in the "higher" position (see figure 9.2).

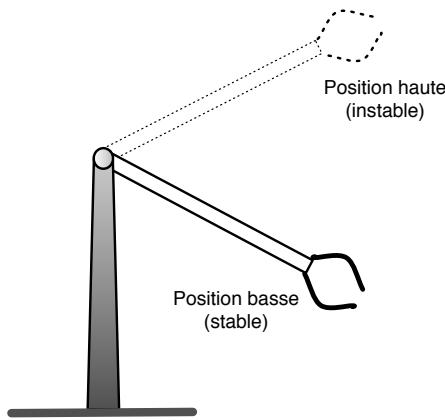


FIGURE 9.1 – The robot's arm in a vertical plane

Let's consider the following function :

$$V(x_1, x_2) = \frac{J}{2}x_2^2 + mgb(1 - \cos x_1) - \bar{u}x_1.$$

This function has the dimension of an energy. Indeed, the first term ($Jx_2^2/2$) is the kinetic energy, the second one ($mgb(1 - \cos x_1)$) is the potential energy and the third one ($\bar{u}x_1$) is the energy spent by the moment \bar{u} pour elevate the arm to the angular position x_1 . The equilibrium in the "lower" position belongs to the domain

$$D = \{(x_1, x_2) : -\pi/2 < x_1 < \pi/2, -a < x_2 < a\}$$

(a is any real positive number). In this domain, the function $V(x_1, x_2)$ is a function that satisfies the following conditions :

- (i) $V(x_1, x_2) : D \rightarrow \mathbb{R}$ is continually differentiable.
- (ii) $V(x_1, x_2) > V(\bar{x}_1, \bar{x}_2)$ for all $(x_1, x_2) \neq (\bar{x}_1, \bar{x}_2)$ in D (meaning V is minimum at equilibrium).

(iii) $\dot{V}(x_1, x_2) \leq 0$ outside the equilibrium in D because

$$\begin{aligned}\dot{V}(x_1, x_2) &= \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 \\ &= [mgb \sin x_1 - \bar{u}] [x_2] + [Jx_2] [J^{-1}(-mgb \sin x_1 - kx_2 + \bar{u})] \\ &= -kx_2^2.\end{aligned}$$

Under these conditions, as we will see with the next theorem, there is a bound neighbourhood of the equilibrium in which $V(x_1, x_2)$ decreases along the system's trajectories as long as the speed $x_2 \neq 0$ and moves towards the minimum of V that corresponds to the equilibrium. We can see that the equilibrium is stable according to the definition 9.1. However, we notice that $V(x_1, x_2)$ ceases to decrease if $x_2 = 0$ (speed equal to zero). Could we have a speed identically equal to zero elsewhere than the equilibrium? In fact we can't, because a speed identically equal to zero implies an acceleration identically equal to zero, which implies that the system is at an equilibrium. \square

This example illustrates the key part of Lyapunov's second method which is as follows.

Theorem 9.7. « Lyapunov's »stability.

The equilibrium (\bar{x}, \bar{u}) of the system $\dot{x} = f(x, \bar{u})$ is stable if there is a function $V(x) : D \rightarrow \mathbb{R}$ continually differentiable with the following properties :

- (i) D is an open set of \mathbb{R}^n and $\bar{x} \in D$;
- (ii) $V(x) > V(\bar{x}) \forall x \neq \bar{x}$ in D ($V(x)$ is minimum at \bar{x});
- (iii) $\dot{V}(x) \leq 0 \forall x \neq \bar{x}$ in D . \square

In other words, this theorem means that a sufficient condition to have the stability of the equilibrium (\bar{x}, \bar{u}) is to have a positive-definite function $V(x) - V(\bar{x})$ whose temporal derivative $\dot{V}(x)$ is negative semi-definite in a neighbourhood of \bar{x} . The temporal derivative $\dot{V}(x)$ is calculated as follows :

$$\dot{V}(x) = \frac{dV}{dt} = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x, \bar{u}) = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x, \bar{u}).$$

The conditions (ii) and (iii) of the theorem 9.7 imply that, for a constant c close enough to $V(\bar{x})$, the set :

$$\Omega_c = \{x \in D : V(\bar{x}) \leq V(x) \leq c\}$$

is a compact (meaning a closed and bound set) invariant. To demonstrate it, let's choose a positive constant r such that the closed ball

$$B_r = \{x \in R^n : \|x - \bar{x}\| \leq r\}$$

is continuous in D . Let's define :

$$\alpha = \min_{\|x\|=r} V(x).$$

We only have to choose any constant c in the open interval $(0, \alpha)$ to define a set Ω_c which is included in B_r and therefore compact. Let's now assume that $x(t_0) \in \Omega_c$. Then, by the condition (iii), we have :

$$\dot{V}(x) \leq 0 \Rightarrow V(\bar{x}) \leq V(x(t)) \leq V(x(t_0)) \leq c \quad \forall t,$$

which shows that Ω_c is invariant.

The example 9.6 is an application of this theorem that shows that the robot's arm's equilibrium est stable. En reality, we know intuitively that the equilibrium is *asymptotically* stable (meaning stable and attractive). One way to demonstrate that an equilibrium is asymptotically stable is to have a Lyapunov function whose temporal derivative $\dot{V}(x)$ is strictly negative-defined (and not only negative semi-defined as in the example 9.6). Indeed, in this case, the Lyapunov function strictly decreases along the system's trajectories, until it reaches (asymptotically) the minimum corresponding exactly to the equilibrium.

Theorem 9.8. Asymptotical stability.

The equilibrium (\bar{x}, \bar{u}) of the system $\dot{x} = f(x, \bar{u})$ is asymptotically stable if there is a function $V(x) : D \rightarrow R$ continuously differentiable with the following properties :

- (i) D is an open set R^n and $\bar{x} \in D$;
- (ii) $V(x) > V(\bar{x}) \quad \forall x \neq \bar{x}$ in D ($V(x)$ is minimum at \bar{x});
- (iii) $\dot{V}(x) < 0 \quad \forall x \neq \bar{x}$ in D .

□

As we have seen in the robot's arm example, we usually find a Lyapunov function whose derivative is only negative semi-definite, which does not allow to conclude that the system is stable, using the previous theorem. The difficulty comes notably from the fact that, by analyzing the function $\dot{V}(x)$, we don't use the fact that the different state variables x_i are not independent but are linked by the equations of the system's dynamic. LaSalle studied this matter in detail and has come with a invariance principle which allows to analyze the asymptotic stability of the equilibriums in the case of a negative semi-definite function $\dot{V}(x)$.

Theorem 9.9. Lasalle invariance principle.

The equilibrium (\bar{x}, \bar{u}) of the system $\dot{x} = f(x, \bar{u})$ is asymptotically stable if there is a function $V(x) : D \rightarrow R$ continuously differentiable with the following properties :

- (i) D is an open set of R^n and $\bar{x} \in D$;
- (ii) $V(x) > V(\bar{x}) \forall x \neq \bar{x}$ in D ($V(x)$ is minimum at \bar{x});
- (iii) $\dot{V}(x) \leq 0 \forall x$ in D ;
- (iv) the set $S \subset D$ such that $\dot{V}(x) = 0$ does not contain any trajectory of the system other than $x(t) = \bar{x}$. □

Example 9.10. Un bras de robot à un degré de liberté (suite).

Let's consider again the robot's arm model (see Example 9.6)

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= J^{-1}(-mgb \sin x_1 - kx_2 + \bar{u}),\end{aligned}\tag{9.2}$$

for which the energy function

$$V(x_1, x_2) = \frac{J}{2}x_2^2 + mgb(1 - \cos x_1) - \bar{u}x_1$$

is a Lyapunov function. Among the system's trajectories, the Lyapunov function's derivative is negative semi-definite :

$$\dot{V}(x_1, x_2) = -kx_2^2 \leq 0.$$

The set $S \subset D$ such that $\dot{V}(x) = 0$ is therefore the following :

$$S = \{(x_1, x_2) \in D : x_2 = 0\}.$$

We can observe that every trajectory within S is such that the speed $x_2(t)$ is identically equal to zero (which we note $x_2(t) \equiv 0$). This immediately implies that $\dot{x}_2(t) \equiv 0$. By equation (9.2), this implies that $mgb \sin x_1(t) - \bar{u} \equiv 0$. And therefore that the only trajectory within S is indeed the equilibrium's trajectory. Thus the equilibrium is asymptotically stable. The reasoning that we just did is formalized as follows :

$$\begin{aligned}\text{trajectory } (x_1(t), x_2(t)) \in S &\Rightarrow x_2(t) \equiv 0 \\ &\Rightarrow \dot{x}_2(t) \equiv 0 \\ &\Rightarrow mgb \sin x_1(t) - \bar{u} \equiv 0 \\ &\Rightarrow x_1(t) = \bar{x}_1, x_2(t) = \bar{x}_2.\end{aligned}$$

□

9.4. Basin of attraction and global convergence

In the demonstration of the theorem 9.7, we have seen that the domain Ω_c is an invariant of the system. If the equilibrium is asymptotically stable, it means that every trajectory that starts at any point of Ω_c converges to the equilibrium. That is the reason why this set is called "basin of attraction". Lyapunov second method allows us to characterize the size of the basin of attraction, information that we could not get with the first method. That's why it can be interesting to look for a Lyapunov function, even if the equilibrium's stability can be easily demonstrated with the linearization.

One particularly interesting case is when the equilibrium point is unique and the basin of attraction contains the whole state space. That case is referred as *global asymptotic stability* whose existence conditions are developed in the following theorem.

Theorem 9.11. Global asymptotic stability.

The equilibrium (\bar{x}, \bar{u}) of the system $\dot{x} = f(x, \bar{u})$ is globally asymptotically stable if it is asymptotically stable and if :

- (i) $D = \mathbb{R}^n$;
- (ii) $|x| \rightarrow \infty \Rightarrow |V(x)| \rightarrow \infty$.

□

9.5. The energy as a Lyapunov function

The choice of a Lyapunov function that is appropriate for the analysis of the equilibriums stability of a dynamic system is usually quite difficult. As we have shown with the robot with one degree of liberty example, the energy can be a good start for several physical systems. Let's examine that on a few examples.

Mechanical systems

The general equation of the dynamics of a mechanical system is (cfr Chapter 2)

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + k(q) + h(\dot{q}) = G\bar{u}$$

Here we consider the particular case where the cinematic matrix G is constant. Let's take as function $V(q, \dot{q})$ the following function :

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + E_p(q) - q^T G\bar{u}$$

The first therm is the kinetic energy, the second one is the potential energy and the third one is the work realized by the forces and moments applied. The derivative

of this function along the trajectories is calculated as follows :

$$\begin{aligned}\dot{V} &= \frac{1}{2}\dot{q}^T[\dot{M}(q) - 2C(q, \dot{q})]\dot{q} - \dot{q}^T h(\dot{q}) \\ &= -\dot{q}^T h(\dot{q}),\end{aligned}$$

(because the matrix $\dot{M}(q) - 2C(q, \dot{q})$ is antisymmetric, see chapter 2). This value is well negative semi-definite for reasonable choice of viscous friction models.

Electrical circuits

Let's take for example the non-linear RLC circuit from Chapter 8 (sec. 8.3.1). The state equations are :

$$\begin{aligned}L\dot{x}_1 &= -r(x_1) - x_2 + \bar{u}, \\ C\dot{x}_2 &= x_1.\end{aligned}$$

In these equations, x_1 denotes the current and x_2 the tension. L and C are the inductance et capacitor of the circuit while $r(x_1)$ is a non-linear resistor. We assume that the function $r(x_1)$ is monotonously increasing and passes through the origin $r(0) = 0$.

Let's take for Lyapunov function, the following function which has the dimension of an energy :

$$V(x_1, x_2) = \frac{1}{2}Lx_1^2 + \frac{1}{2}C(x_2 - \bar{u})^2 \geq 0$$

This function is positive and minimum at the equilibrium $(\bar{x}_1, \bar{x}_2) = (0, \bar{u})$: $V(\bar{x}_1, \bar{x}_2) = 0$. On the other hand, we have :

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= -x_1 r(x_1) \leq 0.\end{aligned}$$

The equilibrium is therefore stable and, using the Lasalle invariant principle, we can even conclude the asymptotic stability.

9.6. Linear systems

Let a system $\dot{x} = Ax + B\bar{u}$ and an equilibrium (\bar{x}, \bar{u}) . We define as Lyapuno function $V(x) = (x - \bar{x})^T P(x - \bar{x})$ where P is a symmetric positive-definite matrix.

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P(x - \bar{x}) + (x - \bar{x})^T P \dot{x} \\ &= (\bar{u}^T B^T + x^T A^T)P(x - \bar{x}) + (x - \bar{x})^T P(Ax + B\bar{u}) \\ &= (x - \bar{x})^T A^T P(x - \bar{x}) + (x - \bar{x})^T P A(x - \bar{x}) \\ &= -(x - \bar{x})^T Q(x - \bar{x}),\end{aligned}$$

$$\text{with } -Q = A^T P + PA. \quad (9.3)$$

This last equation is called « Lyapunov matrix equation ». If it has a solution Q that is positive-definite, then the function V will be a Lyapunov function for the system. The reasoning can also be reversed : we take a matrix Q positive-definite and we use the following theorem to conclude that the function V is indeed a Lyapunov function for the system et that the equilibrium is asymptotically stable.

Theorem 9.12. Let a n -order real matrix A . For every matrix Q positive-definite, (9.3) owns a unique solution P positive-definite if and only if A is a Hurwitz matrix (all its eigenvalues have a strictly negative real part). \square

9.7. « Bound entry - Bound state »stability

It would often be illusory to be able to apply a perfectly constant entry to a real dynamic system. In practice, because of various sources of perturbation and uncertainty, the entry will usually be a signal $u(t)$ slightly varying in the neighbourhood of the wanted equilibrium value. It is therefore relevant to take interest in the evolution of the system's state when $u(t)$ is a bound signal close to \bar{u} . We begin with the case of a linear system

$$\dot{x} = Ax + Bu. \quad (9.4)$$

For an given initial condition $x(t_0) = x_0$ and a given entry $u(t)$, the system's trajectory is explicitly given as

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Let's consider the equilibrium $(\bar{x} = 0, \bar{u} = 0)$. This equilibrium is asymptotically stable if and only if the matrix A is a Hurwitz matrix. In that case, $\|e^{At}\|$ is bound for all t and there is positive constants k and λ such that

$$\|e^{A(t-t_0)}\| \leq ke^{-\lambda(t-t_0)}.$$

It can be inferred that

$$\begin{aligned} \|x(t)\| &\leq ke^{-\lambda(t-t_0)}\|x_0\| + \int_{t_0}^t e^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|d\tau \\ &\leq ke^{-\lambda(t-t_0)}\|x_0\| + \frac{k\|B\|}{\lambda} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|. \end{aligned} \quad (9.5)$$

We can immediately see that a bound entry $u(t)$, whatever its amplitude, does generate a bound state $x(t)$. We can also see that the effect of the initial condition

x_0 fades away with time and that the « ultimate boundary » of $x(t)$ is simply proportional to the boundary of $u(t)$:

$$\limsup_{t \rightarrow +\infty} \|x(t)\| \leq \frac{k\|B\|}{\lambda} \|u\|_{\mathcal{L}_\infty}.$$

Let's now see how these results extend to non-linear systems. We consider the system

$$\dot{x} = f(x, u) \text{ with the equilibrium } (\bar{x}, \bar{u}) \quad (9.6)$$

and we assume that the function $f(x, u)$ is continuously differentiable in a neighbourhood of the equilibrium.

Theorem 9.13. (local) EBEB stability

If the equilibrium (\bar{x}, \bar{u}) of the system (9.6) is asymptotically stable,

- (i) there are three positive constants c_1, c_2 et c_3 such that, for all initial state x_0 with $\|x_0 - \bar{x}\| < c_1$ and all entry signal $u(t)$ with $\|u(t) - \bar{u}\| < c_2$, the solution $x(t)$ is bound : $\|x(t) - x_0\| < c_3 \forall t \geq t_0$;
- (ii) there is a positive constant c_0 and a continuous function $\alpha : [0, a) \rightarrow [0, +\infty)$ passing through the origin (meaning $\alpha(0) = 0$) and increasing¹ such that, for all entry signal $u(t)$ with $\|u(t) - \bar{u}\| < c_0 \forall t \geq t_0$, the « ultimate boundary » of $x(t)$ is an increasing function of the boundary of $u(t)$:

$$\limsup_{t \rightarrow +\infty} \|x(t)\| \leq \alpha(\|u\|_{\mathcal{L}_\infty}). \quad \square$$

In the case where the system (9.6) is globally defined and owns an unique equilibrium, we also have the following global property.

Theorem 9.14. (global) EBEB stability

If the function $f(x, u)$ is globally continuously differentiable and is globally a Lipschitz function in (x, u) , if the equilibrium (\bar{x}, \bar{u}) is globally exponentially stable, then

- (i) for all initial condition x_0 and all entry signal $u(t)$, the solution $x(t)$ is bound ;
- (ii) the « ultimate boundary » of $x(t)$ is an increasing function of the boundary of $u(t)$. \square

We have to notice that this last theorem is quite restrictive. There are in fact numerous dynamic systems for which the function $f(x, u)$ is not globally a Lipschitz function and yet own a global EBEB property. However, the exponential stability condition for the equilibrium is crucial. Indeed, if the equilibrium is globally asymptotically stable but not globally exponentially stable, then the system (9.6) is not necessarily EBEB stable, even if $f(x, u)$ is globally a Lipschitz function.

1. such a function is called a \mathcal{K} class function

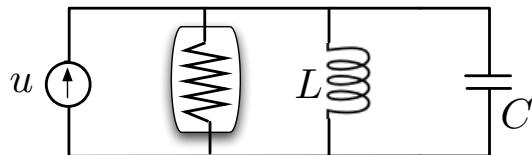
9.8. Exercises

Exercise 9.1. A chemical reactor Two chemical reactions in a liquid phase at constant volume that concern three chemical species X_1 , X_2 , X_3 are happening in a continuous reactor. The dynamic of the reactor is described by the following state model (x_i is the concentration of X_i) :

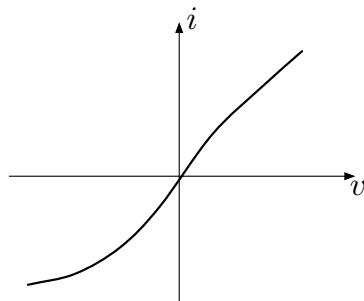
$$\begin{aligned}\dot{x}_1 &= -x_1^2 x_2 - dx_1 + du, \\ \dot{x}_2 &= x_1^2 x_2 - (d + k)x_2, \\ \dot{x}_3 &= 2kx_2 - dx_3.\end{aligned}$$

1. What kind of reactions are they ? (law of action of the masses)
2. In the positive orthant ($x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$) find the equilibrium(s) for $\bar{u} > 0$. How many equilibrium values are there for each value of \bar{u} ? And what are the existence conditions ?
3. Analyse the stability of the equilibriums using the first method of Lyapunov.

Exercise 9.2. RLC Circuit Here is a parallel RLC circuit :



The current-tension $i = g(v)$ characteristic of the non-linear resistance is an increasing function such as represented on the following figure :

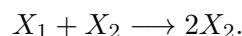


1. Establish a state model of the system.

2. Calculate the equilibria
3. Using the energy as a Lyapunov function, analyse the global stability of the equilibria using the second method of Lyapunov.

Exercise 9.3. Chemical reactor

Granted is a chemical reactor of type CSTR where the following reaction is happening :



1. Establish a state model under the following modelisation hypothesis :
 - kinetic is described by the law of action of the masses
 - the reactor is alimented with X_1 only ($x_{1,in} = \text{constant}$)
 - the volumetric feed rate is the input.
2. Show that this system has an equilibrium in the positive orthant.
3. Show that $V = x_1 - \bar{x}_1 \ln x_1 + x_2 - \bar{x}_2 \ln x_2$ is a Lyapunov function in the positive orthant.
4. Prove that the equilibrium is globally asymptotically stable in the positive orthant.

Exercise 9.4. A mechanical system

We consider a mechanical system with one degree of liberty. The position variable is x_1 . This system is subject to a force deriving from a potential and to a linear viscous friction.

The potential energy is given by

$$E_p(x_1) = \int_0^{x_1} \frac{\sigma}{K + |\sigma|} d\sigma.$$

1. Establish a state model of the system.
2. Calculate the equilibria.
3. Analyse the stability of the equilibria by the direct method of Lyapunov.

Exercise 9.5. Modelisation of a neuron

The model of Naka-Rushton describing the dynamic of a neuron in the short-term memory is given by the following state equations :

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{ux_2}{1+x_2}, \\ \dot{x}_2 &= -x_2 + \frac{ux_1}{1+x_1}.\end{aligned}$$

1. Show that the system is positive
2. Analyse the existence and the stability of the equilibriums in the positive orthant (first method of Lyapunov)
3. For u constant, $0 < \bar{u} < 1$, show that all the paths in the positive orthant converge to the origin using the Lyapunov function $V = (1/2)(x_1^2 + x_2^2)$.

Exercise 9.6. Here is a dynamic system :

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2, \\ \dot{x}_2 &= -x_1^3(1+u^2).\end{aligned}$$

With a Lyapunov function of the form

$$V = ax_1^\alpha + bx_2^\beta$$

where a, b, α, β are positive constants to determine, show that for each constant input $u(t) = \bar{u}$, this system has a unique equilibrium and is globally asymptotically stable.

Exercise 9.7. Here is a dynamic system :

$$\begin{aligned}\dot{x}_1 &= -\phi(x_1) + \phi(x_2), \\ \dot{x}_2 &= \phi(x_1) - 2\phi(x_2) + u.\end{aligned}$$

The function $\phi(x) : \mathbb{R} \rightarrow \mathbb{R}$ has the following properties :

- a) $\phi(x)$ is a bijection,
- b) $\phi(x)$ is C^∞ ,
- c) $\phi(x)$ is strictly increasing ($d\phi/dx > 0, \forall x \in \mathbb{R}$),
- d) $\phi(x)$ passes through the origin ($\phi(0) = 0$).

1. Prove that it's a system with compartments
2. For a constant strictly increasing input ($\bar{u} > 0$), explain the conditions under which the system has a unique equilibrium in the positive orthant.
3. Show that this equilibrium, if it exists, is globally asymptotically stable using the function of Lyapunov

$$V(x_1, x_2) = \int_{\bar{x}_1}^{x_1} (\phi(s) - \bar{u}) ds + \int_{\bar{x}_2}^{x_2} (\phi(s) - \bar{u}) ds.$$

Chapitre 10

Controllability and path planning

In the three precedents chapter, we have studied in detail the behavior of the *free* dynamic systems whose inputs are *constants* : $\dot{x} = f(x, \bar{u})$. In this last chapter, we will consider *controlled* dynamic systems $\dot{x} = f(x, u)$ and we will be particularly interested in the existence and determination of input functions $u(t)$ that can vary with time and that allow to pilot the state system and to plan paths.

10.1. Definitions

In practice, it often happens that we desire to pilot a dynamic system from an initial state x_0 to a final state x_f . This is what we call a *path planning problem*. In order to solve such a problem, there must exist at least one input function that products a path of the system that goes through the states x_0 and x_f .

Definition 10.1. Reachable states

For the dynamic system $\dot{x} = f(x, u)$, the final state x_f is reachable from the initial state x_0 if there exists a finite time T and an input function $u(t) : [t_0, t_0 + T] \rightarrow \mathbb{R}^n$ such as $x(t_0 + T, x_0, u) = x_f$. \square

This notion of reachability leads to the concept of controllability of a dynamic system as explained in the following definition.

Definition 10.2. Controllability

The system $\dot{x} = f(x, u)$ is *locally controllable* in x_f if there exists a neighbourhood of x_f such that x_f is reachable from each element in the neighbourhood.

The system is *globally controllable* if each state $x_f \in \mathbb{R}^n$ is reachable from each initial state $x_0 \in \mathbb{R}^n$. \square

The subject of this chapter is to study the controllability and the path planning of the dynamic systems $\dot{x} = f(x, u)$. As we will see, the analysis of the reachability and of the controllability is completely elucidated for linear systems though a lot of open questions remain for non-linear systems. On the other hand, the problem of path planning is completely solved for linear systems, while we only know the solution for a restricted class of non-linear systems that we call (*differentially*) *flat system* and that are, in a sense, equivalents to linear systems.

10.2. Controllability : linear systems

In order to check if a linear system $\dot{x} = Ax + Bu$ is completely controllable, we can use one of the two criterion given by the following theorem.

Theorem 10.3. Controllability of linear systems

The linear system $\dot{x} = Ax + Bu$ is completely controllable if and only if one of the two following criterion is satisfied :

1. (Kalman's criterion) The matrix $C = (B \ AB \ A^2B \ \dots \ A^{(n-1)}B)$ is regular (this matrix is called *controllability matrix*);
2. (Popov-Belevitch-Hautus's criterion) The rank of the matrix $(sI - A \ B)$ is equal to n for each $s \in \mathbb{C}$. \square

If a linear system is not completely controllable, we can define a state transformation in order to bring the non-controllable part of the state vector out.

Let's suppose that the controllability matrix is of rank $d < n$. We can define a matrix $T = (T_a \ T_b)$ such that T_a contains d linearly independent columns of C and T_b completes the matrix by $n - d$ vectors independent of the columns of T_a . The inverse matrix T^{-1} can be written :

$$T^{-1} = \begin{pmatrix} U_a \\ U_b \end{pmatrix}$$

where the matrix U_a and U_b are chosen such so that :

$$T^{-1}T = \begin{pmatrix} U_a T_a & U_a T_b \\ U_b T_a & U_b T_b \end{pmatrix} = \begin{pmatrix} I_d & 0 \\ 0 & I_{n-d} \end{pmatrix}$$

We define the state transformation :

$$z = \begin{pmatrix} z_a \\ z_b \end{pmatrix} = \begin{pmatrix} U_a x \\ U_b x \end{pmatrix}$$

In these new state variable, we have the following state model :

$$\begin{aligned}\dot{z}_a &= U_a A T_a z_a + U_a A T_b z_b + U_a B u \\ \dot{z}_b &= U_b A T_b z_b\end{aligned}$$

Indeed $U_b T_a = 0$ implies that $U_b B = 0$ and $U_b A T_a = 0$ because the columns of B and $A T_a$ are linear combinations of the columns of T_a . We notice that the part z_b of the state vector isn't influenced by the input u : it represents the non-controllable part of the system's state.

10.3. Controllability : non-linear systems

The study of the controllability of non-linear systems is far more complicated than the one of linear systems. We begin this study by examining the conclusion that we can get from the controllability of a linearised of a non-linear system in the neighbourhood of an equilibrium.

Theorem 10.4. Local controllability (1)

Let's consider the linearised of the system $\dot{x} = f(x, u)$ around an equilibrium (\bar{x}, \bar{u}) :

$$\dot{x} = Ax + Bu \quad \text{avec} \quad A = \left(\frac{\partial f}{\partial x} \right)_{(\bar{x}, \bar{u})} \quad \text{et} \quad B = \left(\frac{\partial f}{\partial u} \right)_{(\bar{x}, \bar{u})}. \quad (10.1)$$

If the linear system (10.1) is controllable, then, for each $\epsilon > 0$, all of the reachable states x_f from \bar{x} with inputs $u(t) : u(t) - \bar{u} < \epsilon$, contains one neighbourhood of \bar{x} . \square

This local property of controllability of non-linear systems has a limited range. As we will see in the following example, there are indeed completely controllable non-linear systems whose linearised is not completely controllable in the neighbourhood of an equilibrium !

Example 10.5. A car

Let's consider an « FWD » car whose front wheels are director and driving wheels. The cinematic model is written :

$$\begin{aligned}\dot{\xi}_1 &= \sin \theta_1 \cos \theta_2 u_1 \\ \dot{\xi}_2 &= -\cos \theta_1 \cos \theta_2 u_1 \\ \dot{\theta}_1 &= \sin \theta_2 u_1 \\ \dot{\theta}_2 &= u_2\end{aligned}$$

where (ξ_1, ξ_2) are the cartesian coordinates of the middle of the back axle, θ_1 is the orientation of the chassis θ_2 is the front wheel orientation, u_1 is the propulsion speed and u_2 is the speed of the front wheels. This system has an infinity of non-isolated equilibriums of the form $(\bar{\xi}_1, \bar{\xi}_2, \bar{\theta}_1, \bar{\theta}_2, 0, 0)$. The matrices (A, B) of the linearised of the system around any of these equilibrium are written :

$$A = 0 \quad B = \begin{pmatrix} \sin \bar{\theta}_1 \cos \bar{\theta}_2 & 0 \\ -\cos \bar{\theta}_1 \cos \bar{\theta}_2 & 0 \\ \sin \bar{\theta}_2 & 0 \\ 0 & 1 \end{pmatrix}$$

We notice immediately that the linearised system is not controllable (rank $C = 2$) while the physical intuition obviously indicates that a car is a controllable dynamic system which can be moved from any initial position to any final one in an obstacle-free environment. \square

As this example indicates it, a non-linear system can have controllability properties that are not apparent in the linearised. The analysis of the properties is eased by the use of concepts and notations of differential geometry which are briefly explained in the appendices. We begin their study with the presentation of a procedure that allows to bring the state variables out when a system is *not* controllable.

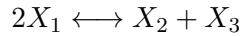
Let's suppose that for a given system $\dot{x} = f(x, u)$, there exists a state transformation $z = \phi(x)$ such that, in the newstate variables, the system can be written as following :

$$z = \begin{pmatrix} z_a \\ z_b \end{pmatrix} \quad \begin{cases} \dot{z}_a = \tilde{f}_a(z_a, z_b, u) \\ \dot{z}_b = \tilde{f}_b(z_b) \end{cases}$$

It is clear, in this case that the part z_b of the state vector isn't influenced by the input u and that the system isn't completely controllable. Here is a simple example.

Example 10.6. A chemical reactor

Let's consider an isotherm chemical reactor that is perfectly mixed in which the following reversible reaction is happening :



The reactor is alimented by the specie X_1 with a constant volumetric flow rate and a variable concentration. The state model is written :

$$\begin{aligned} \dot{x}_1 &= -2k_1x_1^2 + 2k_2x_2x_3 - dx_1 + du \\ \dot{x}_2 &= k_1x_1^2 - k_2x_2x_3 - dx_2 \\ \dot{x}_3 &= k_1x_1^2 - k_2x_2x_3 - dx_3 \end{aligned}$$

Granted is the following linear state transformation :

$$\begin{array}{ll} z_1 = x_1 & x_1 = z_1 \\ z_2 = x_2 & \longleftrightarrow x_2 = z_2 \\ z_3 = x_2 - x_3 & x_3 = z_2 - z_3 \end{array}$$

With the new state variables, the model is rewritten ;

$$\begin{aligned} \dot{z}_1 &= -2k_1 z_1^2 + 2k_2 z_2(z_2 - z_3) - dz_1 + du \\ \dot{z}_2 &= k_1 z_1^2 - k_2 z_2(z_2 - z_3) - dz_2 \\ \dot{z}_3 &= -dz_3 \end{aligned}$$

The system is not completely controllable because the paths of z_3 (which is the difference between the concentrations of the species X_2 and X_3) can not be influenced by the input u (which is the alimentation concentration of the specie X_1). \square

A sufficient existence condition of the non-controllable part of the state is given in the following theorem for affine systems in the input :

$$\dot{x} = f(x) + \sum_{j=1}^m g_j(x)u_j \quad (10.2)$$

Theorem 10.7. If, in the neighbourhood U of a point x_0 , there exists one regular distribution $\Delta(x)$ of dimension d such that :

1. $\Delta(x)$ is involutive
2. $\Delta(x)$ contains $\text{span}\{g_1(x), g_2(x), \dots, g_m(x)\}$
3. $\Delta(x)$ is invariant with respect to $f(x)$ and $g_1(x), g_2(x), \dots, g_m(x)$,

then there exists a state transformation $\phi : U \rightarrow V = \phi(U)$ such that, with the new state variables $z = \phi(x)$, the system (10.2) is rewritten :

$$\begin{aligned} \dot{z}_a &= \tilde{f}_a(z_a, z_b) + \sum_{j=1}^m \tilde{g}_j(z_a, z_b)u_j \\ \dot{z}_b &= \tilde{f}_b(z_b) \end{aligned}$$

with $\dim z_b = (n - d)$. \square

The following theorem allows then to determine the smaller distribution $\Delta^*(x)$ that verifies the upper conditions and that allows to determine the maximum dimension of the non-controllable part.

Theorem 10.8. In a neighbourhood U of x_0 , the distributions sequence is defined :

$$\begin{aligned}\Delta_0(x) &= \text{span}\{g_1(x), g_2(x), \dots, g_m(x)\} \\ \Delta_k(x) &= \Delta_{k-1}(x) + [f(x), \Delta_{k-1}(x)] + \sum_{j=1}^m [g_j(x), \Delta_{k-1}(x)].\end{aligned}$$

Then $\Delta^*(x) = \Delta_{k^*}(x)$ with k^* the smallest integer such that $\Delta_{k^*}(x)$ is regular on U and invariant with respect to $f(x)$ and $g_1(x), g_2(x), \dots, g_m(x)$. If all distributions $\Delta_k(x), 0 \leq k \leq k^*$ are regular on U , then $k^* \leq n$. \square

The dimension of Δ^* is called the *reachability rank* of the system in the neighbourhood of x_0 . The statement of the theorem 10.8 implicitly contains a procedure for the determination of the reachability rank which consists of generating successively the distributions $\Delta_k(x)$. The procedure stops as soon as we find one that is regular and invariant with respect to f and g_i . There is no need to check that this distribution is involutive. It is also interesting to observe that in the case of a linear system $\dot{x} = Ax + Bu$, we have :

$$\Delta_k = \text{span} \{B \ AB \ \dots \ A^{k-1}B\}$$

and therefore that the reachability rank coincides with the rank of the controllability matrix \mathcal{C} .

A system whose reachability rank is maximum (i.e. equal to n) in the neighbourhood of x_0 then owns a property of local controllability similar to the one of theorem 10.4, even if x_0 is not an equilibrium state and even if the linearized of the system is not controllable.

Theorem 10.9. Local controllability (2)

For the system (10.2), there exists a neighbourhood of x_0 in which all states are reachable from x_0 if and only if the reachability rank of the system in the neighbourhood of x_0 is equal to n . \square

Finally we have the complete controllability property for a subclass of systems.

Theorem 10.10. Complete controllability

If $f(x) \in \text{span}\{g_1(x), g_2(x), \dots, g_m(x)\}$ for all $x \in \mathbb{R}^n$ (this is especially true if $f(x) = 0$) and if the reachability rank is n in the neighbourhood of all $x \in \mathbb{R}^n$, then the system (10.2) is completely controllable. \square

These two theorems are illustrated in the following example.

Example 10.11. A motor vehicle

Let us consider again the model of the motor vehicle of the example 10.5 which is written :

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

with :

$$g_1(x) = \begin{pmatrix} \sin \theta_1 \cos \theta_2 \\ -\cos \theta_1 \cos \theta_2 \\ \sin \theta_2 \\ 0 \end{pmatrix} \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We calculate the Lie bracket :

$$\begin{aligned} g_3(x) = [g_1(x), g_2(x)] &= \begin{pmatrix} -\sin \theta_1 \sin \theta_2 \\ \cos \theta_1 \sin \theta_2 \\ \cos \theta_2 \\ 0 \end{pmatrix} \\ g_4(x) = [g_3(x), g_1(x)] &= \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

We can check that $f(x) = 0$ and that the matrix $[g_1(x), g_2(x), g_3(x), g_4(x)]$ is regular for all x in \mathbb{R}^4 . These two conditions are enough so that the assumptions of the two previous theorems are verified. The motor vehicle is therefore completely controllable even if its linear model is not. \square

The results presented in this section can seem restrictive because they only can be applied to systems which are affine in the input. Their reach is however more general because any given system $\dot{x} = f(x, u)$ can always be increased by a *dynamic extension* to return it affine in the input. It is indeed enough to consider u as an additional set of state variables and to define a new vector v of input variables such as :

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{u} &= v \end{aligned}$$

With the augmented state vector $\xi^T = (x^T, u^T)$, the system is written :

$$\begin{aligned} \dot{\xi} &= \varphi(\xi) + \sum_{j=1}^m g_j v_j = \varphi(\xi) + Gv & (10.3) \\ \text{où : } \varphi(\xi) &= \begin{pmatrix} f(x, u) \\ 0 \end{pmatrix} \quad G = \begin{pmatrix} 0 \\ I_m \end{pmatrix} \end{aligned}$$

The controllability of the augmented system (10.3), which can be verified with the previous theorems, is obviously sufficient to guarantee the controllability of the original system.

10.4. Trajectory planning

In the previous sections, we studied the conditions and criteria that allow to know whether a system is controllable or not. It is obviously even more interesting to be able to determine the input function $u(t)$ that effectively allows to lead the system from an initial state x_0 to a final state x_f in a reasonable time. This is the so-called *trajectory planning* problem that we will explain now.

10.4.1. Brunovski single-input systems

We consider here the systems affine in the input $\dot{x} = f(x) + g(x)u, u \in \mathbb{R}$ that can be put in Brunovski form and that are characterized by the following theorem :

Theorem 10.12. A system $\dot{x} = f(x) + g(x)u$ can be put in Brunovski form in a domain $U \subset \mathbb{R}^n$ if and only if :

- 1) The matrix $\mathcal{D} = [g(x) \ ad_f g(x) \ ad_f^2 g(x) \dots ad_f^{n-1} g(x)]$ is regular $\forall x \in U$;
- 2) The distribution $\Delta(x) = \text{span } \{g(x) \dots ad_f^{n-2} g(x)\}$ is involutive in U . \square

If these conditions are satisfied, a state transformation exists $z = \varphi(x)$, $z : U \rightarrow V$ such that the system can be rewritten in the triangular form (called Brunovski) :

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= \alpha(z) + \beta(z)u \quad \beta(z) \neq 0 \quad \forall z \in V\end{aligned}\tag{10.4}$$

We deduce immediately from this theorem that a single-input *linear system* $\dot{x} = Ax + bu$ can be put in Brunovski form if and only if it is completely controllable. Indeed, in this case, the matrix \mathcal{D} is the controllability matrix of the system :

$$\mathcal{D} = \mathcal{C} = [b \ Ab \ A^2b \dots A^{(n-1)}b]$$

and the distribution Δ is necessarily involutive since it only contains constant vectors. Then there is a *linear* state transformation :

$$z = Tx$$

such that the system is rewritten in Brunovski form :

$$\begin{aligned}\dot{z}_i &= z_{i+1} \quad i = 1, \dots, n \\ \dot{z}_n &= -\sum_{i=1}^n \alpha_i z_i + \beta u\end{aligned}$$

The matrix T is defined as follow :

$$T = \begin{pmatrix} h^T \\ h^T A \\ \vdots \\ h^T A^{n-1} \end{pmatrix}$$

where the vector h is the last column of the transpose of the inverse of the controllability matrix \mathcal{C}^{-T} .

Once the system, whether it is linear or non-linear, is in Brunovski form, the problem of trajectory planning becomes very easy to solve. We start by showing the solution in the particular case of any system of dimension two.

Example 10.13. A system of dimension 2

Let the system be :

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) + g_1(x_1, x_2)u \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u\end{aligned}$$

The problem is to find an input function $u(t)$ that leads this system from an initial state $(x_1(0), x_2(0))$ to a final state $(x_1(T), x_2(T))$.

It is assumed that there is a state transformation :

$$\begin{aligned}z_1 &= \phi_1(x_1, x_2) \\ z_2 &= \phi_2(x_1, x_2)\end{aligned}$$

that puts the system in Brunovski form :

$$\dot{z}_1 = z_2 \tag{10.5}$$

$$\dot{z}_2 = \alpha(z_1, z_2) + \beta(z_1, z_2)u \tag{10.6}$$

Now the problem is to find a function $u(t)$ which leads the system (10.5)-(10.6) from the initial state $z_1(0) = \phi_1(x_1(0), x_2(0))$, $z_2(0) = \phi_2(x_1(0), x_2(0))$ to the final state $z_1(T) = \phi_1(x_1(T), x_2(T))$, $z_2(T) = \phi_2(x_1(T), x_2(T))$. For the state variable $z_1(t)$, we define a polynomial trajectory as :

$$z_1(t) = \lambda_3\left(\frac{t}{T}\right)^3 + \lambda_2\left(\frac{t}{T}\right)^2 + \lambda_1\left(\frac{t}{T}\right) + \lambda_0$$

where the coefficients λ_i are currently unknown. We deduce from Brunovski form that the trajectory of $z_2(t)$ must have the form :

$$z_2(t) = \dot{z}_1(t) = \frac{3}{T}\lambda_3\left(\frac{t}{T}\right)^2 + \frac{2}{T}\lambda_2\left(\frac{t}{T}\right) + \frac{1}{T}\lambda_1$$

By clarifying the expressions of $z_1(t)$ et $z_2(t)$ at times $t = 0$ and $t = T$, we then observe that the coefficients λ_i are the solutions of the a linear equation system :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{T} & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & \frac{1}{T} & \frac{2}{T} & \frac{3}{T} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} z_1(0) \\ z_2(0) \\ z_1(T) \\ z_2(T) \end{pmatrix} \quad (10.7)$$

The λ_i being also determined, we know now a trajectory $z_1(t), z_2(t)$ which links the desired initial and final states and we can calculate the corresponding input $u(t)$:

$$u(t) = \frac{\dot{z}_2(t) - \alpha(z_1(t), z_2(t))}{\beta(z_1(t), z_2(t))}$$

with

$$\dot{z}_2(t) = \frac{6}{T^2} \lambda_3(\frac{t}{T}) + \frac{2}{T^2} \lambda_2$$

The problem of the trajectory planning is thus solved. \square

The particular case of a system of dimension 2 that we just studied is easily generalized in dimension n . Let's recall that we assume that the system is in Brunovski form :

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_n &= \alpha(z) + \beta(z)u \quad \beta(z) \neq 0 \end{aligned}$$

For $z_1(t)$ it is sufficient to define a polynomial trajectory of the form :

$$z_1(t) = \sum_{i=0}^{2n-1} \lambda_i (\frac{t}{T})^i$$

By calculating the successive derivatives of $z_1(t)$, we obtain the expressions of $z_j(t)$, $j = 2, \dots, n$:

$$z_j(t) = \sum_{i=j-1}^{2n-1} \frac{i!}{(i-j+1)!} \frac{\lambda_i}{T^{j-1}} (\frac{t}{T})^{-j+1}$$

Then by clarifying these expressions at the instants $t = 0$ et $t = T$, we obtain a linear equation system that generalizes the system (10.7) and makes it possible to calculate the λ_i . Then only the input $u(t)$ remains to be calculated :

$$u(t) = \frac{\dot{z}_n(t) - \alpha(z(t))}{\beta(z(t))}$$

Remark 10.14. We presented above a solution for the trajectory planning problem based on the use of polynomial functions of order $2n - 1$ to generate the trajectories of the system. The choice of such polynomial functions is however nothing imperative. More generally, as it can be easily deduced from the previous developments, we can use linear combinations of $2n - 1$ linearly independent arbitrary functions.

10.4.2. Multi-input linear systems

We now consider multi-input linear systems of the following form :

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$$

We assume that the $\text{rank}(B) = m$ and that the system is controllable. We define the *indices of controllability* $\delta_1, \delta_2, \dots, \delta_m$:

$$\delta_i = \text{card}[m_j \geq i : j \geq 0]$$

with

$$\begin{aligned} m_0 &= \text{rank } B \\ m_1 &= \text{rank}[B, AB] - \text{rank } B \\ &\vdots \\ m_{n-1} &= \text{rank}[B, \dots, A^{n-1}B] - \text{rank}[B, \dots, A^{n-2}B] \end{aligned}$$

By definition, we have :

$$\delta_1 \geq \delta_2 \geq \dots \geq \delta_m \quad \text{and} \quad \sum_{j=1}^m \delta_j = n$$

Then there is a transformation state $z = Tx$ that allows to put the system in a generalized Brunovski form composed of m blocks each having the following triangular form :

$$\begin{aligned} \dot{z}_{j1} &= z_{j2} \\ \dot{z}_{j2} &= z_{j3} \\ &\vdots \\ \dot{z}_{j\delta_{j-1}} &= z_{j\delta_j} \\ \dot{z}_{j\delta_j} &= \sum_{\substack{i=1, m \\ i=1, \delta_j}} \alpha_{ji} z_{ji} + \sum_{k=1, m} \beta_{jk} u_k \end{aligned} \tag{10.8}$$

The state vector z is composed of n variables z_{ji} , $j = 1 \dots m$, $i = 1 \dots \delta_j$. The matrix $G = [\beta_{jk}]$ is squared and invertible. This multi-input form of Brunovski can be used, as in the mono-input case, to solve trajectory planning problems.

10.4.3. Brunovski's outputs

By introducing the notation $y_j = z_{j1}$, the state model (10.8) can also be written under the more compact form

$$\dot{y}_j^{(\delta_j+1)} = \sum_{\substack{i=1, m \\ i=1, \delta_j}} \alpha_{ji} y_j^{(i)} + \sum_{k=1, m} \beta_{jk} u_k \quad j = 1, \dots, m$$

i.e. under the form of m differential linear equations of order $(\delta_j + 1)$. The variables y_j are linear combinations of the state x and are called *Brunovski's outputs*. We notice that the number of Brunovski's outputs is equal to the number of inputs of the system.

10.4.4. Multi-input non-linear systems

Let's now consider a multi-input non-linear system, affine in the input :

$$\dot{x} = f(x) + \sum_{j=1}^m g_j(x) u_j.$$

For this system, we can extend the concept of the Brunovski multi-input form if there exists a non-linear state transformation $z = T(x)$ that allows to put the system under the block triangular form.

$$\begin{aligned} \dot{z}_{j1} &= z_{j2} \\ \dot{z}_{j2} &= z_{j3} \\ &\vdots && j = 1, \dots, m \\ \dot{z}_{j\delta_{j-1}} &= z_{j\delta_j} \\ \dot{z}_{j\delta_j} &= \alpha_j(z) + \sum_{k=1, m} \beta_{jk}(z) u_k \end{aligned}$$

where the state vector z is composed of n variables $z_{ji}, j = 1 \dots m, i = 1 \dots \delta_j$ and the square matrix $G(z) = [\beta_{jk}(z)]$ is invertible. In this case, Brunovski's outputs are non-linear function of the state ($y_j = z_{1j} = h_j(x)$) and the model can be written as the following system of non-linear differential equations

$$\dot{y}_j^{(\delta_j+1)} = \alpha_j(z) + \sum_{k=1, m} \beta_{jk}(z) u_k \quad j = 1, \dots, m \quad (10.9)$$

where the vector z is now defined as follows :

$$z = (y_1, \dot{y}_1, \dots, y_1^{(\delta_1)}, \dots, y_m, \dot{y}_m, \dots, y_m^{(\delta_m)}).$$

Contrary to the linear case, the controllable non-linear systems cannot always be written as the Brunovski multi-input form. It is beyond the scope of this paper to discuss the conditions under which the transformation is possible. This is as a matter of fact an issue that is not fully clarified and which is still the subject of active research at this time. We will only present the two examples below. The first is a simple example where the system is naturally in Brunovski multi-input form (10.9). The second example is more complex. It will show a controllable system for which it is necessary, by a *dynamic extension*, to use an expanded Brunovski form whose size is greater than the dimension of the system itself.

Example 10.15. A manipulator robot

We consider again the model of the manipulator robot with two degrees of freedom that we studied in chapter 2 (Example 2.2). In examining the model, we easily observe that it is straight away given in a Brunovski multi-input form with the two position coordinates $y_1 = x_1$ et $y_2 = \theta_2$ as Brunovski's outputs. To prevent explicitly inverting the inertia matrix, we can write the model in a matrix format as follows :

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & m_2 b \cos y_2 \\ m_2 b \cos y_2 & I_2 + m_2 b^2 \end{pmatrix}^{-1} \begin{pmatrix} m_2 b \dot{y}_2^2 \sin y_2 + u_1 \\ -m_2 b g_o \sin y_2 + u_2 \end{pmatrix}.$$

The controllability indices are here $\delta_1 = \delta_2 = 2$. . The matrix $G(z)$ is the inverse of the inertia matrix. \square

Example 10.16. Rocket dynamics

In chapter 2 (Exemple 2.1), we established the dynamic model of a rocket as follows :

— Translational equations

$$\begin{aligned} m\ddot{x} &= (F_1 + F_2) \cos \theta \\ m\ddot{y} &= (F_1 + F_2) \sin \theta - mg_0 \end{aligned}$$

— Rotational equations

$$I\ddot{\theta} = (F_2 - F_1)d \sin \alpha$$

In these equations, (x, y) is the position of the centre of mass of the rocket, θ the angle of the rocket relative to the horizontal, F_1 and F_2 the thrusts of the engines, m the weight of the rocket, I its moment of inertia, d and α some geometric parameters and g_0 the gravitational acceleration. To simplify the writing without loss of generality, we define the inputs

$$u_1 = \frac{F_1 + F_2}{m}, \quad u_2 = \frac{(F_2 - F_1)d \sin \alpha}{I}.$$

With these notations, the state model is simply written :

$$\begin{aligned}\ddot{x} &= u_1 \cos \theta \\ \ddot{y} &= u_1 \sin \theta - g_0 \\ \ddot{\theta} &= u_2\end{aligned}$$

This system is fully controllable according to the Theorem 10.10. Intuitively, we might think that the coordinates x and y are Brunovski's outputs. We will see that this intuition is justified, but it involves an extended definition of the concept of Brunovski form.

Let's calculate the third derivatives of the coordinates x and y :

$$\begin{aligned}\dddot{x} &= \dot{u}_1 \cos \theta - u_1 \dot{\theta} \sin \theta \\ \dddot{y} &= \dot{u}_1 \sin \theta + u_1 \dot{\theta} \cos \theta\end{aligned}$$

It is clear that these expressions can not be used to build a Brunovski form such as (10.9) because they do not contain the input u_2 . However, if we consider the input u_1 as an additional state variable and add two integrators in the system input, then we can show that the extended system has a multi-input Brunovski form with the coordinates $y_1 = x$ and $y_2 = y$ as Brunovski's outputs. The extended system can be written in the following format :

$$\begin{aligned}\ddot{x} &= u_1 \cos \theta \\ \ddot{y} &= u_1 \sin \theta \\ \ddot{\theta} &= u_2 \\ \ddot{u}_1 &= w_1\end{aligned}\tag{10.10}$$

This is a system of 8 dimensions (while the original system was a 6 dimension system) with two entries w_1 and w_2 . Calculate the fourth derivative of the Brunovski outputs x and y :

$$\begin{aligned}\left(\begin{array}{c} \ddot{x} \\ \ddot{y} \end{array} \right) &= \left(\begin{array}{c} -2\dot{u}_1 \dot{\theta} \sin \theta - u_1 \dot{\theta}^2 \cos \theta \\ 2\dot{u}_1 \dot{\theta} \cos \theta + u_1 \dot{\theta}^2 \sin \theta \end{array} \right) \\ &\quad + \left(\begin{array}{cc} \cos \theta & -u_1 \sin \theta \\ \sin \theta & u_1 \cos \theta \end{array} \right) \left(\begin{array}{c} w_1 \\ u_2 \end{array} \right).\end{aligned}\tag{10.11}$$

It's now clear that the system can be written in a form of multi-input Brunovski :

$$\left(\begin{array}{c} \dddot{y}_1 \\ \ddot{y}_2 \end{array} \right) = \alpha(z) + G(z) \left(\begin{array}{c} w_1 \\ u_2 \end{array} \right).$$

Indeed, from the state model (10.10), the different terms appearing in the equation (10.11) can be expressed (after some calculations!) as a function of Brunovski outputs $y_1 = x$ et $y_2 = y$ and to their derivatives, as follows :

$$\begin{aligned} u_1 \cos \theta &= \ddot{y}_1, & u_1 \sin \theta &= \ddot{y}_2 + g_o, \\ \sin \theta &= \frac{\ddot{y}_2 + g_o}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2 + g_o)^2}}, & \cos \theta &= \frac{\ddot{y}_1}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2 + g_o)^2}}, \\ \dot{\theta} &= \frac{\ddot{y}_1 \ddot{y}_2 - (\ddot{y}_2 + g_o) \ddot{y}_1}{\ddot{y}_1^2 + (\ddot{y}_2 + g_o)^2}, \\ \dot{u}_1 \cos \theta &= \ddot{y}_1 + \dot{\theta}(\ddot{y}_2 + g_o), & \dot{u}_1 \sin \theta &= \ddot{y}_2 - \dot{\theta}\ddot{y}_1. \end{aligned}$$

On the other hand, the matrix D'autre part, la matrice

$$G(z) = \begin{pmatrix} \cos \theta & -u_1 \sin \theta \\ \sin \theta & u_1 \cos \theta \end{pmatrix}$$

is non-singular for all θ and for all $u_1 \neq 0$ (i.e. as long as the total thrust $F_1 + F_2$ provided by the rocket engines is different of zero) \square

The non-linear systems which may be written in the form of mutli-inputs Brunovski form, thanks to a possible dynamic extension, are called in the literature, (*differentially*) *flat system*, because they are in a sense, equivalent to linear systems as showed by the computation of the planning trajectory. For this reason, the Brunovski outputs are sometimes called *flat ouputs*.

1. Vector field

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

2. Lie algebra

$$[f(x), g(x)] = \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)$$

$$[f(x), g(x)] = -[g(x), f(x)]$$

3. Iterative Notation

$$ad_f g = [f, g]$$

$$\begin{aligned} ad_f^2 g &= [f, ad_f g] = [f, [f, g]] \\ &\vdots \\ ad_f^k g &= [f, ad_f^{k-1} g] \end{aligned}$$

4. Distribution = set of vector fields

$$\Delta(x) = \{f_1(x), f_2(x), \dots, f_d(x)\}$$

5. Distribution Δ is *involutive* if $[f_1, f_2] \in \Delta \quad \forall f_1 \in \Delta, f_2 \in \Delta$

6. Distribution Δ is *invariant* in relation to g if

$$\forall f \in \Delta \Rightarrow [g, f] \in \Delta$$

10.5. Exercises

Exercise 10.1. A hot-air balloon¹

We consider the following model for a hot-air balloon :

$$\begin{aligned} \dot{\theta} &= -\frac{1}{\tau_1}\theta + u \\ \dot{v} &= -\frac{1}{\tau_2}v + \sigma\theta \\ \dot{h} &= v \end{aligned}$$

où θ est l'écart de température de l'air par rapport à la température d'équilibre,
 u est la commande (proportionnelle à la quantité d'énergie utilisée pour chauffer
l'air du ballon),

v est la vitesse verticale (vitesse ascensionnelle),

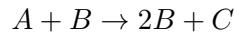
h est la hauteur.

1. Problem from "Analyse et commande de systèmes dynamiques" by F. Bonnans et P. Rouchon, Manual of 'Ecole Polytechnique' (France), 2003 edition.

1. Commenter les équations.
2. Le système est-il commandable ?
3. Planifier une trajectoire de $\begin{pmatrix} 0 \\ 0 \\ h_0 \end{pmatrix}$ à $\begin{pmatrix} 0 \\ 0 \\ h_1 \end{pmatrix}$ en $T = 1$.

Exercice 10.2. Un réacteur biochimique

Soit un réacteur à volume constant (unitaire) alimenté en réactif A (débit d , concentration x_A^{in}) dans lequel se déroule la réaction



La cinétique de la réaction est donnée par la loi d'action des masses.

1. Donner une représentation d'état du système réactionnel.
2. Trouver, si c'est possible, un difféomorphisme mettant en évidence les modes non commandables du système. Examiner les 2 cas suivants :

La commande est d

La commande est x_A^{in}

Exercice 10.3. Contrôle d'attitude d'un satellite

Le contrôle de l'orientation d'un satellite (appelé contrôle d'attitude) peut avoir divers objectifs : pointer une antenne, un appareil de mesure ou un panneau solaire dans la bonne direction, éviter la déterioration par les rayons solaires d'éléments sensibles, orienter le satellite en vue de manœuvres orbitales etc...

On considère un satellite dans l'espace dont les équations du mouvement s'écrivent :

$$\begin{aligned}\dot{x}_1 &= a_1 x_2 x_3 + b_1 u_1 \\ \dot{x}_2 &= a_2 x_1 x_3 + b_2 u_2 \\ \dot{x}_3 &= a_3 x_1 x_2\end{aligned}$$

Etudier la commandabilité locale (Théorème 10.9) de ce système ($a_i \neq 0$ $b_i \neq 0$).

Exercice 10.4. Un plongeur²

On considère le modèle ci-dessous décrivant la dynamique verticale d'un plongeur équipé d'un gilet stabilisateur contenant une quantité réglable d'air, notée q (exprimée en moles) :

$$\begin{aligned} M \frac{d^2h}{dt^2} &= Mg - \rho g \left(V_0 + \frac{qRT}{P_0 + \rho h} \right) \\ \frac{dq}{dt} &= u \end{aligned}$$

avec les notations suivantes :

h : profondeur du plongeur mesurée positivement depuis la surface

M	masse du plongeur	constantes
P_0	pression atmosphérique	
T	température	
R	constante de Boltzmann	
ρ	masse spécifique de l'eau	
V_0	volume du plongeur	

La première équation est un bilan de force selon l'axe vertical. Ce bilan comprend la poussée d'Archimède $\rho g(V_0 + V_g)$ où V_g est le volume du gilet obtenu en fonction de la pression $p = P_0 + \rho h$ par la loi des gaz parfaits $PV = qRT$.

La deuxième équation représente le remplissage du gilet par l'air des bouteilles ($u > 0$) ou la purge du gilet ($u < 0$).

1. Montrer que le système peut être mis sous forme de Brunovski.
2. Montrer que le système est commandable.
3. On désire remonter de façon contrôlée entre deux paliers stabilisés. Le palier de départ ($t = 0$) est à la profondeur \bar{h}_1 . Le palier d'arrivée ($t = t_f$) est à la profondeur \bar{h}_2 . Indiquer comment calculer l'entrée $u(t)$ qui assure la transition entre ces deux équilibres.

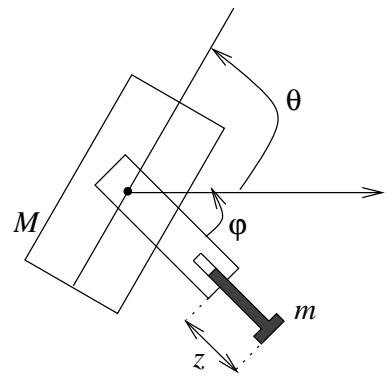
Exercice 10.5. Un robot sauteur

On considère un « robot sauteur » schématisé comme ci-dessous. Ce robot est formé d'un corps de masse M muni d'une jambe de masse m .

La jambe est articulée et on peut en contrôler l'orientation φ et l'extension z . La conservation du moment angulaire autour du centre de masse instantané s'écrit ($d = \text{constante}$) :

$$M\dot{\theta} + m(z + d)^2(\dot{\theta} + \dot{\varphi}) = 0$$

2. Problème extrait de "Analyse et commande de systèmes dynamiques" par F. Bonnans et P. Rouchon, Manuel de l' Ecole Polytechnique (France), édition de 2003.



Les deux entrées de commande du système sont les vitesses d'orientation et d'élongation de la jambe.

$$u_1 = \dot{\varphi} \quad (2) \qquad u_2 = \dot{z} \quad (3)$$

1. Ecrire les 3 équations (1) à (3) sous la forme d'un modèle d'état dont les entrées sont u_1 et u_2 .
2. Examiner si le système décrit par ce modèle est complètement commandable.