Notes about the relationship between the Stern-Brocot Tree and $PSL(2,\mathbb{Z})$

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The Stern-Brocot tree is isomorphic to the special linear monoid $SL(2,\mathbb{N})$, the 2x2 matrices of determinant 1 with elements in $\mathbb{N} = \{0, 1, 2, \ldots\}$

The Stern-Brocot tree $SL(2,\mathbb{N})$ is the free monoid generated by matrix multiplication on the generators $L = \begin{bmatrix}1&1\\0&1\end{bmatrix}$ and $R = \begin{bmatrix}1&0\\1&1\end{bmatrix}$. In the isomorphism between the Stern-Brocot tree and $SL(2,\mathbb{N})$, matrix multiplication corresponds to the concatination of two finite Stern-Brocot representations (of rationals)

The special linear group $SL(2,\mathbb{Z})$ is the non-free group generated by $\{L, R\}$, and is the 2x2 matrices of determinant 1 with elements in $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2 \ldots\}$.

The identity function is an injective homomorphism of monoids from $SL(2,\mathbb{N})$ to $SL(2,\mathbb{Z})$.

$SL(2,\mathbb{Z})$ doesn't add any elements that "look like" $SL(2,\mathbb{N})$. Define "the positive lobe of $SL(2,\mathbb{Z})$" as those matrices of $SL(2,\mathbb{Z})$ that have nonnegative elements.

$$SL(2,\mathbb{N}) = \text{positive lobe of } SL(2,\mathbb{Z})$$

Every* matrix $A \in SL(2,\mathbb{Z})$ can be written in exactly two ways as $x\tilde{A}y$, with $\tilde{A} \in SL(2,\mathbb{N})$ and $x, y \in \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&-1\\1&0\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix}, \begin{bmatrix}0&1\\-1&0\end{bmatrix}\right\}$, a subgroup of $SL(2,\mathbb{Z})$ isomorphic to $\mathbb{Z}_4$.

* ~~er~~ err, most, other than the subgroup isomorphic to $\mathbb{Z}_4$, which have 4 different ways to write themselves

In general, I don't see a particularly simple, nice way to understand the multiplication $A \cdot B = C$ (in $SL(2,\mathbb{Z})$ *or* $PSL(2,\mathbb{Z})$)
in terms of $x_A \tilde{A} y_A x_B \tilde{B} y_B = x_C \tilde{C} y_C$ (or "$SL(2,\mathbb{N})$ with signs")
without computing C (or $\tilde{A} y_A x_B \tilde{B}$) and then re-normalizing, because which "lobe" the result falls into depends on specific values of $\tilde{A}$ and $\tilde{B}$, and cannot* be computed from the "decorating" x's and y's alone. * except for special cases.

The projective special linear group $PSL(2,\mathbb{Z})$ is $SL(2,\mathbb{Z})$ quotentied by $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix}\right\}$, so that we set $A = -A$ for all matrices A.

Every* matrix $A \in PSL(2,\mathbb{Z})$ can be written uniquely as $x \tilde{A} y$, with $\tilde{A} \in SL(2,\mathbb{N})$ and $x, y \in \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\-1&0\end{bmatrix}\right\}$, a subgroup of $PSL(2,\mathbb{Z})$ that is isomorphic to $\mathbb{Z}_2$

* most other than the $\mathbb{Z}_2$, which have two ways to write themselves.

Comments / Musings

The euclidean algorithm basically uses $PSL(2,\mathbb{Z})$ except that typically the intermediate / final results are constrained to the positive lobe.

Reminds me of one of my favorite constructions for D_4, $D(\mathbb{Z}_4)$, a "dihedral" operator that adds a "hysteresis of chirality" to any group, and returns a non-abelian group assuming there is an element that is not it's own inverse

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"Dihedral" operator on groups
"Adds a hysteresis of chirality"

2022-08-11 Note:
aka. "Semidirect product"
Neither Hungerford nor Fraleigh really cover this in detail

Given a group $\langle G, +_G, \mathrm{inv}_G, 0_G \rangle$

Define $D(G) = \langle G', + \rangle$

where $G' = \{\text{Flip}, \text{Translate}\} \times G$

and $\langle \text{Translate}, x \rangle + \langle \text{Translate}, y \rangle$
$= \langle \text{Translate}, x +_G y \rangle$

$\langle \text{Translate}, x \rangle + \langle \text{Flip}, y \rangle$
$= \langle \text{Flip}, \mathrm{inv}\ x +_G y \rangle$

$\langle \text{Flip}, x \rangle + \langle \text{Translate}, y \rangle$
$= \langle \text{Flip}, x + y \rangle$

$\langle \text{Flip}, x \rangle + \langle \text{Flip}, y \rangle$
$= \langle \text{Translate}, \mathrm{inv}\ x +_G y \rangle$

Then $D(G)$ is a group, and non-abelian iff G is non-abelian, or $\exists x \in G.\ \mathrm{inv}\ x \neq x$

Imagine a train on a track, that can turn 180° at any time. $D(\mathbb{R})$ corresponds to a such a train on an infinitely long track.

The rules above can be understood as instructing the train to

translate x = move (forward or backward) a length "x"
flip x = turn around, then move a length "x"

$D(\mathbb{Z}_2^n) \cong \mathbb{Z}_2^{n+1}$ for all $n \in \mathbb{N}$. Additive portion of Galios Fields.

$D(\mathbb{Z}_n) \cong D_n$. traditional dihedral groups $n \geq 3$, symmetry group of regular n-sided polygons

$D(\mathbb{R}) \cong$ symmetry group of line

$D(\mathbb{R}_{\pi}) \cong$ symmetry group of circle, dynamical systems, chaos theory. numerical analysis.

Notes on multiplication in $PSL(2,\mathbb{Z})$

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Let $Signs = \{1, i\} \subset PSL(2,\mathbb{Z})$, a subgroup
$i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

"$PSL(2,\mathbb{Z})$ is the Stern-Brocot tree with signs"

Every $A \in PSL(2,\mathbb{Z})$ can be written uniquely as $x\tilde{A}y$, with $\tilde{A} \in PSL(2,\mathbb{N})$ and $x, y \in Signs$, except for $A \in Signs$, which each have two non-unique representations.

"Lobes" of $PSL(2,\mathbb{Z})$: every A is in one of four lobes: $\tilde{A}$, $i\tilde{A}$, $\tilde{A}i$, or $i\tilde{A}i$

Given a multiplication $A \cdot B = C$,

consider $x_A \tilde{A} y_A x_B \tilde{B} y_B = x_C \tilde{C} y_C$

- if $y_A x_B = 1$, then $x_C = x_A$ and $y_C = x_B$. We can understand the entire computation as a single multiplication in $PSL(2,\mathbb{N})$ + a side computation of signs
- if $y_A x_B = i$, then we have a more complicated case

Let $a \odot b = aib$ be a binary operation on $PSL(2,\mathbb{Z})$

$\odot$ is associative
$\odot$ has identity i
$\odot$ has left/right inverses

$\langle PSL(2,\mathbb{Z}), \odot \rangle$ is a group

Consider a restriction of domain $\odot : PSL(2,\mathbb{N}) \to PSL(2,\mathbb{N}) \to Im$

what does the image look like?

Consider $A \odot B = C$

Choose A, C: when does B exist?
Choose B, C: when does A exist?

(notice that w, x, y, z, a, b, c, d are all non-negative)

Let A (or B) $= \begin{bmatrix} w & y \\ x & z \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} z & -y \\ -x & w \end{bmatrix}$ let $\tilde{C} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Case I: $C = \tilde{C}$. Corresponding A, B sometimes exists.

Choose A, C. Set $B \in PSL(2,\mathbb{Z}) = iA^{-1}\tilde{C}$ Check $B \in PSL(2,\mathbb{N})$

$$iA^{-1}C \doteq \begin{bmatrix} x & -w \\ z & -y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} xa - wb & xc - wd \\ za - yb & zc - yd \end{bmatrix}$$

These entries are sometimes all non-negative, depending on A and C. ✓

Notes on multiplication in $PSL(2,\mathbb{Z})$

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Case 1 $C = \tilde{C}$.

Choose B, C. Set $A \in PSL(2,\mathbb{Z}) = \tilde{C} B^{-1} i$ Check $A \in PSL(2,\mathbb{N})$

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} y & z \\ -w & -x \end{bmatrix} = \begin{bmatrix} ay-cw & az-cx \\ by-dw & bz-dx \end{bmatrix}$$

These entries are sometimes all non-negative, depending on choice of B, C.

Case 2. $C = i\tilde{C}$

Choose A, C.

$$B = i A^{-1} i \tilde{C}$$
$$= \begin{bmatrix} x & -w \\ z & -y \end{bmatrix}\begin{bmatrix} -b & d \\ -a & -c \end{bmatrix}$$
$$= \begin{bmatrix} xb+wa & xd+wc \\ zb+ya & zb+yc \end{bmatrix}$$

Always non-negative

Choose B, C.

$$A = i\tilde{C} B^{-1} i$$
$$= \begin{bmatrix} b & d \\ -a & -c \end{bmatrix}\begin{bmatrix} y & z \\ -w & -x \end{bmatrix}$$
$$= \begin{bmatrix} by-dw & bz-dx \\ -ay+cw & -az+cx \end{bmatrix}$$

Sometimes non-negative

Case 3. $C = \tilde{C} i$

Choose A, C

$$B = i A^{-1} \tilde{C} i$$
$$= \begin{bmatrix} x & -w \\ z & -y \end{bmatrix}\begin{bmatrix} c & -a \\ d & -b \end{bmatrix}$$
$$= \begin{bmatrix} xc-dw & -xa+wb \\ zc-dy & -za+yb \end{bmatrix}$$

Sometimes all non-negative

Choose B, C.

$$A = \tilde{C} i B^{-1} i$$
$$= \begin{bmatrix} c & -a \\ d & -b \end{bmatrix}\begin{bmatrix} y & z \\ -w & -x \end{bmatrix}$$
$$= \begin{bmatrix} cy+aw & cz+ax \\ dy+bw & dz+bx \end{bmatrix}$$

always non-negative

Case 4. $C = i\tilde{C} i$

Choose A, C

$$B = i A^{-1} i \tilde{C} i$$
$$= \begin{bmatrix} x & -w \\ z & -y \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \begin{bmatrix} xd+wc & -xb-wa \\ zd+yc & -zb-ya \end{bmatrix}$$

never has all non-negative entries

Choose B, C

$$A = i\tilde{C} i B^{-1} i$$
$$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} y & z \\ -w & -x \end{bmatrix}$$
$$= \begin{bmatrix} dy+bw & dz+bx \\ -cy-aw & -cz-ax \end{bmatrix}$$

Thus, the image of $\odot$ on $PSL(2,\mathbb{N})$ consists of the entirety of the lobes $i\hat{C}$ and $\hat{C} i$, at least some of $\tilde{C}$ and none of the lobe $i\tilde{C}i$ (other than $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, maybe, which is also part of the positive lobe)

Notes on multiplication in $PSL(2,\mathbb{Z})$

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Thus, from the lobes of A and B, we can correctly predict that the lobe of $A \cdot B$ is either definitely determined by the lobes alone, *or* that the result cannot be in a given lobe, in which of the 3 lobes it ends up in depends on the particular choice of A and B.
"re-normalization" always results in 0, or 1 sign changes, not 2.

$A, B, C \in SL(2,\mathbb{N})$

$A \cdot B$	A	iA	Ai	iAi
B	C	iC	$\overline{iCi}$	$\overline{Ci}$
Bi	Ci	iCi	$\overline{iC}$	$\overline{C}$
iB	$\overline{iCi}$	$\overline{Ci}$	C	iC
iBi	$\overline{iC}$	$\overline{C}$	Ci	iCi

Questions I currently have unanswered:

is $\mathrm{Im}[\odot, SL(2,\mathbb{N})]$ "onto" the positive lobe of $SL(2,\mathbb{Z})$?
(special case of above)

No* → Does there exist $A, B \in SL(2,\mathbb{N})$ such that $A \odot B = 1$

2022-03-14 Yes → Does there exist $A, B \in SL(2,\mathbb{N})^*$ such that $A \odot B = i$?

It seems like for "most" choices of A, B, $A \odot B$ ends up in the positive lobe. There probably is a better understanding left to be found, about where/how this change of lobe occurs.

* non-trivial members of $SL(2,\mathbb{N})$, with $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ removed

* A^{-1} is unique, and $\widetilde{A^{-1}}$ is in opposite lobe, i.e.

$$A^{-1} = x_{A^{-1}} \widetilde{A^{-1}} y_{A^{-1}} = i\, x_A \widetilde{A^{-1}} y_A\, i$$

Therefore, $\forall A, B$, $A \odot B \neq 1$. Also, may follow from a stronger case 4? That $A \odot B \neq 1$ because $i\tilde{1}i$ is part of the opposite lobe?

A free presentation of $PSL_2\mathbb{Z}$

Leon P Smith 2022-03-14 p1/1

I realized recently that I hadn't been fully owning the Stern-Brocot Representation, and that all my questions have simple answers.

Define $T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ("i" in previous notes, but Indra's Pearls dissuaded me of that)

Then every $x \in SL_2\mathbb{N}$ can be written as a string in $\{L, R\}^*$

By direct calculation, we find ($L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$)

$LTL = R$

$RTR = L$

$LTR = T$

$RTL = T$

This can be interpreted as a simple recursive program, answering all the unanswered questions on last page.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is only element in positive lobe of $PSL_2\mathbb{Z}$ that cannot be written as an element of $(SL_2\mathbb{N})T(SL_2\mathbb{N})$
2. if you pick two reasonably large elements $x, y \in SL_2\mathbb{N}$ at random, uncorrelated, then xTy is very likely in $SL_2\mathbb{N}$. For xTy to end up in a non-positive lobe, x would have to be a partial inverse of y, and vice-versa.

This provides us with a free presentation of $PSL_2\mathbb{Z}$.

$PSL_2\mathbb{Z} \cong \langle L, R, T \mid T^2 = 1,\ LTL = R,\ RTR = L,\ LTR = RTL = T \rangle$

Free presentations correspond to short exact sequences, which I am curious to learn about.

Moreover, $PSL_2\mathbb{Z} \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

This free product (a.k.a coproduct) is syntactically equivalent to the pointwise product $\{1, T\} \cdot SL_2\mathbb{N} \cdot \{1, T\}$, computable by small and simple mealy-type machine. Should synthesize w/ Aluffi Chap 0