# ALGEBRAIC $K$-THEORY OF GEOMETRIC GROUPS

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Abstract. Let $\Gamma$ be a geometrically finite group of finite asymptotic dimension and let $R$ be a noetherian ring of finite homological dimension. The main result of the paper is that the integral $K$-theoretic assembly map for the group ring $R[\Gamma]$ is an isomorphism. We also include partial results for larger classes of geometric groups.

## Introduction

This paper is a computation of algebraic $K$-groups of a large and useful class of group rings. We are concerned with discrete groups $\Gamma$ which are geometric, in the sense that they arise as fundamental groups of closed manifolds. This is the situation in geometric topology where the $K$-groups of the integral group ring $\mathbb{Z}[\Gamma]$ are the obstruction groups for various constructions. It is also common to have an explicit geometric construction of a finite classifying space for many geometric groups as the quotient of a free action on a contractible manifold.

Our goal is to confirm instances of the Isomorphism Conjecture.

Recall that a ring $R$ is called *regular coherent* if every finitely presented $R$-module has a finite resolution by finitely generated projective $R$-modules. For example, the most important for geometric applications ring of integers $\mathbb{Z}$ is regular coherent.

**The (Integral) Isomorphism Conjecture in Algebraic $K$-theory.** *Given a regular coherent ring $R$ and a torsion-free finitely generated group $\Gamma$, the algebraic $K$-theory of the group ring $R[\Gamma]$ is isomorphic to the homology of the group $\Gamma$ with coefficients in the $K$-theory of $R$,*

$$K_n(R[\Gamma]) \;\cong\; H_n(\Gamma, K(R)),$$

*for all integers $-\infty < n < +\infty$.*

The Fundamental Theorem of algebraic $K$-theory, cf. [41, Theorem 8], was the first result of this type. This is the case of the conjecture when $\Gamma$ is the group of integers $\mathbb{Z}$. Other results include most notably the work of F. Waldhausen [46] on algebraic $K$-theory of amalgamated products and HNN extensions which we generalize in this paper. Waldhausen's work is based on a certain kind of combinatorial description of a group. By contrast we use geometric properties of groups and spaces on which they act, not particular ways of constructing them. On the other hand, our methods are more algebraic and less dependent on specific geometric models compared to most of the recent work such as that of Bartels, Farrell, Jones, Reich [2].

The rational version of the Isomorphism Conjecture has been studied with great success by many authors as surveyed in [25].

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The Isomorphism Conjecture is a consequence of a more particular statement, which we refer to as the integral Borel Conjecture, about the assembly map in algebraic $K$-theory. The use of nonconnective spectra allows us to deal with the Isomorphism Conjecture in all dimensions at once. An overview and our conventions regarding spectra can be found in section 6.2.

Given a ring $R$, one can view an element $\gamma$ of $\Gamma$ as an isomorphism of the trivial $R[\Gamma]$-module with the inverse $\gamma^{-1}$. Following Loday, to each isomorphism $f$ of free finitely generated $R$-modules there corresponds an $R[\Gamma]$-isomorphism $\gamma \otimes f$ of finitely generated $R[\Gamma]$-modules. This induces the assembly map of spectra

$$a(\Gamma, R)\colon h(B\Gamma, K^{-\infty}(R)) \longrightarrow K^{-\infty}(R[\Gamma]).$$

Here $h(B\Gamma, K^{-\infty}(R))$ stands for the group homology spectrum of a discrete group $\Gamma$ with coefficients in the nonconnective algebraic $K$-theory spectrum $K^{-\infty}(R)$ of the ring $R$. The stable homotopy groups of $K^{-\infty}(R)$ coincide with the lower $K$-theory of $R$ in negative dimensions and Quillen $K$-theory in nonnegative dimensions.

**The (Integral) Borel Conjecture in Algebraic $K$-theory.** *The assembly map $a(\Gamma, R)$ is a weak homotopy equivalence for a torsion-free finitely generated group $\Gamma$ and a regular coherent ring $R$.*

The question of computing the integral algebraic $K$-theory of discrete groups is of major importance in geometric topology. For example, it is known from the work of C.T.C. Wall [48] that a topological space $X$ is homotopy equivalent to a finite CW-complex if and only if the finiteness obstruction in the reduced $K$-group $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ is zero. When true, the Isomorphism Conjecture has the following consequence.

**Corollary 1.** *The Isomorphism Conjecture for a group $\Gamma$ and a ring $R$ implies $K_0(R[\Gamma]) \cong K_0(R)$. In particular, $\widetilde{K}_0(\mathbb{Z}[\Gamma]) = 0$.*

It is also important that we compute the $K$-theory of the group ring using assembly maps from the group homology. These maps fit into a long exact sequence where the intermediate terms are the higher Whitehead groups. In particular, applying our result in lower positive dimensions gives vanishing of the classical Whitehead group.

**Corollary 2.** *If the assembly map $a(\Gamma, R)$ is a weak equivalence then* $\mathrm{Wh}(\Gamma) = 0$.

Our Main Theorem will address the integral Borel Conjecture subject to some constraints on the group $\Gamma$ and the ring $R$.

We will say that a group $\Gamma$ is *geometrically finite* if there is a closed aspherical manifold with fundamental group $\Gamma$.

The *asymptotic dimension* is a global characteristic of a finitely generated group as a metric space with the word-length metric, introduced by M. Gromov [27]. The following classes of groups have been verified to have finite asymptotic dimension:

- Gromov hyperbolic groups [27, 42],
- one-relator groups [4],
- virtually polycyclic groups [5],
- solvable groups with finite rational Hirsch length [21],
- Coxeter groups [19],
- cocompact lattices in connected Lie groups [11],

- arithmetic groups [29],
- $S$-arithmetic groups [30, 31],
- finitely generated linear group over a field of positive characteristic [28],
- relatively hyperbolic groups with the parabolic subgroup of finite asymptotic dimension [37],
- proper isometry groups of finite dimensional CAT(0) cube complexes, for example B(4)–T(4) small cancellation groups [50],
- mapping class groups [6],
- various generalized products of groups from these classes such as fundamental groups of developable complexes of asymptotically finite dimensional groups [3].

We should mention here that there are established examples of asymptotically infinite dimensional groups, including Thompson's group $F$, Grigorchuk's group, and Gromov's group containing an expander. We refer the reader to the discussion of these phenomena in [20].

Recall that a regular noetherian ring $R$ is said to have *finite homological dimension* if there is an integer $n$ such that every left $R$-module has a resolution of length $n$ by finitely generated projective modules. Examples of such rings are the ring of integers and, more generally, principal ideal domains.

Our main result verifies the following case of the Borel Conjecture.

**Main Theorem (the short version).** *Suppose $\Gamma$ is a geometrically finite group of finite asymptotic dimension and suppose $R$ is a regular noetherian ring of finite homological dimension, then the integral $K$-theoretic assembly map $a(\Gamma, R)$ is a weak equivalence.*

The experts might be interested in the following more detailed statement that emphasizes the generality of the result we actually prove.

**Main Theorem (the expanded version).** *Suppose $\Gamma$ is a geometrically finite group. Suppose further that*

(1) *the equivariant assembly map in bounded $K$-theory*

$$h^{lf}(\Gamma, K^{-\infty}(R)) \longrightarrow K^{-\infty}(\Gamma, R)$$

*is a (nonequivariant) equivalence,*

(2) *the group $\Gamma$ and the noetherian ring $R$ have the property that the weak Cartan map*

$$K^{-\infty}(R[\Gamma]) \longrightarrow G^{-\infty}(R[\Gamma]),$$

*defined in* [11] *and reviewed in section* 10*, is an equivalence.*

*Then the integral $K$-theoretic assembly map $a(\Gamma, R)$ is a split surjection.*

*Since assumption* (1) *itself has the consequence that $a(\Gamma, R)$ is a split injection, $a(\Gamma, R)$ is in fact a weak equivalence.*

The short version of the Main Theorem follows from the expanded version in view of two previous results of the authors. We have shown that (1) is satisfied for any torsion-free group $\Gamma$ with finite asymptotic dimension and any ring $R$ in [12]. The same paper verified that the assembly map $a(\Gamma, R)$ is a split injection for these groups. We also showed in [11] that (2) is satisfied by any $\Gamma$ of finite asymptotic dimension and a regular noetherian ring $R$ of finite homological dimension.

**A note about organization.** The paper is written in three parts. In Part 1 we give detailed statements of our results and the proof modulo several facts. The subsequent two parts contain proofs of those facts. Part 2 reviews required results from stable homotopy theory and proves a proper equivariant version of the Spanier–Whitehead or $S$-duality for noncompact manifolds. Part 3 contains details from bounded algebraic $K$-theory and proves finer versions of bounded excision in that theory.

The complete structure of the proof of the Main Theorem is quite complicated. For the benefit of the reader we present the proof in Part 1 in three installments. Each installment is an honest description of what the complete proof accomplishes but contains progressively more details of the nodes and maps in the diagram that we finally assemble in Figures 3–5 in section 5.6.

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## Contents

## Part 1. Proof of the Main Theorem

Here we give the argument modulo a number of facts. The subsequent two parts contain proofs of those facts.

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The first description is given in section 1.2. This is followed by a group of preliminaries in section 2 enough to formulate the second, more detailed proof in section 3. Included in this description is the setting of fibred bounded $G$-theory and the role of bounded excision in the argument. After another stretch of preliminaries in section 4, we give the final required details of the proof in section 5.

### 1. Outline of the Argument

1.1. **Bounded $K$-theory and the Novikov Conjecture.** Our approach to the Borel Conjecture is based on a specific resolution of a related well-known conjecture in $K$-theory with weaker hypotheses and a weaker conclusion.

A map $f\colon \mathcal{S} \to \mathcal{T}$ of spectra is called *split injective* if there is a map $g\colon \mathcal{T} \to \mathcal{S}$ so that the composition $g \circ f$ is a weak equivalence. Note that this means $\pi_i(f)$ is the inclusion onto a direct summand.

**The (Integral) Novikov Conjecture in Algebraic $K$-theory.** *The assembly map*

$$a(\Gamma, R)\colon h(B\Gamma, K^{-\infty}(R)) \longrightarrow K^{-\infty}(R[\Gamma])$$

*is a split injection for a torsion-free finitely generated group* $\Gamma$ *and any ring* $R$.

The authors verified the integral Novikov Conjecture in the case $\Gamma$ has finite classifying space $B\Gamma$ and finite asymptotic dimension [12], cf. Theorem 1.1.7.

The original definition of asymptotic dimension by M. Gromov [27] is a coarse analogue of the covering dimension of topological spaces.

**Definition 1.1.1.** The *asymptotic dimension* of a metric space $X$ is defined as the smallest number $n$ such that for any $d > 0$ there is a uniformly bounded cover $\mathcal{U}$ of $X$ such that any metric ball of radius $d$ in $X$ meets no more than $n+1$ elements of the cover $\mathcal{U}$.

The asymptotic dimension is a quasi-isometry invariant and so is an invariant of a finitely generated group viewed as a metric space with the word-length metric associated to a given presentation. A sample list of classes of groups known to have finite asymptotic dimension is given in the Introduction.

In order to outline the proof of the Main Theorem, we need to recall the ideas behind the proof of the integral Novikov Conjecture [9, 12].

Recall that a *proper metric space* is a metric space where closed bounded subsets are compact. Fix an associative ring $R$ with unity.

**Definition 1.1.2.** Let $\mathcal{C}(M, R)$ be the additive category of *geometric $R$-modules* associated to a proper metric space $M$ and an associative ring $R$ defined in Pedersen [38], Pedersen–Weibel [39]. The objects are functions $F\colon M \to \mathbf{Free}_{fg}(R)$ which are locally finite assignments of free finitely generated $R$-modules $F_m$ to points $m$ of $M$. The local finiteness condition requires precisely that for any bounded subset $S \subset M$ the restriction of $F$ to $S$ has finitely many nonzero values. Let $d$ be the distance function in $M$. The morphisms in $\mathcal{C}(M, R)$ are the $R$-linear homomorphisms

$$\phi\colon \bigoplus_{m \in M} F_m \longrightarrow \bigoplus_{n \in M} G_n$$

with the property that the components $F_m \to G_n$ are zero for $d(m, n) > D$ for some fixed real number $D = D(\phi) \geq 0$.

The associated bounded $K$-theory spectrum is denoted by $K(M, R)$, or $K(M)$ when the choice of ring $R$ is implicit, and called the *bounded $K$-theory* of $M$.

**Remark 1.1.3.** There is an inherited ambiguity present in Definition 1.1.2. It is possible that the same finitely generated free module is assigned to different points in $X$. To repair the issue one should think of the objects in $\mathcal{C}(M, R)$ as triples $(\mathcal{F}, F, \iota)$. Here $\mathcal{F}$ is a free $R$-module, not necessarily finitely generated. The locally finite function $F\colon M \to \mathbf{Free}_{fg}(R)$ is to the category of finitely generated free $R$-modules, as before. The $\iota$ is an isomorphism $\iota\colon \bigoplus_{x \in X} F(x) \to \mathcal{F}$.

To simplify the notation, we may refer to $(\mathcal{F}, F, \iota)$ as simply $F$. When we do so, $F$ stands for the module $\mathcal{F}$ together with the *stalks* $F_x$ which are the values of the function $F(x)$ or equivalently the submodules $\iota(f(x))$ of $\mathcal{F}$.

A map between proper metric spaces is *proper* if the preimage of a bounded subset of $Y$ is a bounded subset of $X$. A map $f\colon M_1 \to M_2$ is called *eventually Lipschitz* if, for some number $k \geq 0$ and large enough $s \geq 0$, $d_X(x_1, x_2) \leq s$ implies $d_Y(f(x_1), f(x_2)) \leq ks$.

According to [40], the construction $K(M, R)$ becomes a functor in the first variable on the category of proper metric spaces and proper eventually Lipschitz maps.

**Definition 1.1.4.** On objects of $\mathcal{C}(M_1, R)$,

$$(f_* F)_y = \bigoplus_{z \in f^{-1}(y)} F(z).$$

The proper condition ensures that $f_* F$ is an object of $\mathcal{C}(M_2, R)$. The eventually Lipschitz condition ensures that morphisms are mapped to morphisms of $\mathcal{C}(M_2, R)$.

Examples of proper eventually Lipschitz maps of immediate interest to us are inclusions of metric spaces.

The main result of Pedersen–Weibel [39] is the delooping theorem which can be stated as follows.

**Theorem 1.1.5** (Nonconnective Delooping of Bounded $K$-theory)**.** *Given a proper metric space $M$ and the standard Euclidean metric on the real line $\mathbb{R}$, the natural inclusion $M \to M \times \mathbb{R}$ induces isomorphisms $K_n(M) \simeq K_{n-1}(M \times \mathbb{R})$ for all integers $n > 1$. If one defines the spectrum*

$$K^{-\infty}(M, R) \ = \ \underset{\overrightarrow{k}}{\operatorname{hocolim}} \ \Omega^k K(M \times \mathbb{R}^k),$$

*then the stable homotopy groups of $K^{-\infty}(\text{point}, R)$ coincide with the algebraic $K$-groups of $R$ in positive dimensions and with the Bass negative $K$-theory of $R$ in negative dimensions.*

When we refer to the nonconnective $K$-theory of the ring $R$, we will mean this spectrum $K^{-\infty}(R) = K^{-\infty}(\text{point}, R)$.

Suppose $U$ is a subset of $M$. Let $\mathcal{C}(M, R)_{<U}$ denote the full subcategory of $\mathcal{C}(M, R)$ on the objects $F$ with $F_m = 0$ for all points $m \in M$ with $d(m, U) \leq D$ for some fixed number $D > 0$ specific to $F$. This is an additive subcategory of $\mathcal{C}(M, R)$ with the associated $K$-theory spectrum $K^{-\infty}(M, R)_{<U}$. Similarly, if $U$ and $V$ are a pair of subsets of $M$, then there is the full additive subcategory $\mathcal{C}(M, R)_{<U,V}$ of $F$ with $F_m = 0$ for all $m$ with $d(m, U) \leq D_1$ and $d(m, V) \leq D_2$ for some numbers $D_1, D_2 > 0$. It is easy to see that $\mathcal{C}(M, R)_{<U}$ is in fact equivalent to $\mathcal{C}(U, R)$.

**Theorem 1.1.6** (Bounded Excision [9])**.** *Given a proper metric space $M$ and a pair of subsets $U$, $V$ of $M$, there is a homotopy pushout diagram*

$$\begin{tikzcd}
K^{-\infty}(M)_{<U,V} \arrow[r] \arrow[d] & K^{-\infty}(U) \arrow[d] \\
K^{-\infty}(V) \arrow[r] & K^{-\infty}(M)
\end{tikzcd}$$

Suppose $X$ is a contractible space with a cocompact free left action of $\Gamma$ by isometries. Then the quotient space $X/\Gamma$ is a finite classifying space $B\Gamma$ for $\Gamma$. The free left action on $X$ induces an action of $\Gamma$ on $K^{-\infty}(X, R)$.

There is an equivariant version of this spectrum which is in fact nonequivariantly weakly homotopy equivalent to $K^{-\infty}(X, R)$ but has better equivariant properties, cf. [9]. We suppress the distinction in this outline.

Let $h^{lf}(X, \mathcal{S})$ stand for the locally finite homology spectrum of the proper metric space $X$ with coefficients in another spectrum $\mathcal{S}$ as defined in [9] or section 7 below. There is an equivariant map

$$A(\Gamma, R)\colon\ h^{lf}(X, K^{-\infty}(R)) \longrightarrow K^{-\infty}(X, R)$$

with the following properties.

(1) The fixed point spectrum $h^{lf}(X, K^{-\infty}(R))^{\Gamma}$ is weakly equivalent to the homology spectrum $h(X/\Gamma, K^{-\infty}(R))$ which can be defined as the smash product $(X/\Gamma)_+ \wedge K^{-\infty}(R)$ of the compact quotient $X/\Gamma$ with a disjoint base point and the coefficient spectrum $K^{-\infty}(R)$.
(2) The spectrum $K^{-\infty}(X, R)^{\Gamma}$ is weakly equivalent to the nonconnective $K$-theory of the group ring $K^{-\infty}(R[\Gamma])$.
(3) The assembly map $a(\Gamma, R)$ is related to the fixed point map $A(\Gamma, R)^{\Gamma}$ via the commutative diagram

$$(\dagger)\qquad \begin{array}{ccc} (X/\Gamma)_+ \wedge K^{-\infty}(R) & \xrightarrow{\ a\ } & K^{-\infty}(R[\Gamma]) \\ \simeq \big\downarrow \sigma & & \simeq \big\downarrow \\ h^{lf}(X, K^{-\infty}(R))^{\Gamma} & \xrightarrow{\ A^{\Gamma}\ } & K^{-\infty}(X, R)^{\Gamma} \end{array}$$

where the vertical arrows are the weak equivalences from parts (1) and (2).

In general, the fixed point spectrum $K^{-\infty}(X, R)^{\Gamma}$ is weakly equivalent to the $K$-theory of the lax limit of the equivariant category $\mathcal{C}(X, R)$, cf. [44].

The fixed points of a spectrum $S$ with an action by $\Gamma$ can be viewed as the fixed points of the equivariant function spectrum as follows. If $S^0 = \text{point}_+$ is the one-point space with the trivial action and a disjoint base point then $S^{\Gamma} = F(S^0, S)^{\Gamma}$. The homotopy fixed point spectrum can be defined similarly as $S^{h\Gamma} = F(E\Gamma_+, S)^{\Gamma}$, where $E\Gamma$ is the universal contractible free left $\Gamma$-space.

The collapse map $E\Gamma_+ \to S^0$ sending the base point to the base point induces the canonical map $\rho\colon S^{\Gamma} \to S^{h\Gamma}$. Using this description, one constructs the commutative square

$$(\dagger\dagger)\qquad \begin{array}{ccc} h^{lf}(X, K^{-\infty}(R))^{\Gamma} & \xrightarrow{\ A^{\Gamma}\ } & K^{-\infty}(X, R)^{\Gamma} \\ \big\downarrow \rho & & \big\downarrow \rho \\ h^{lf}(X, K^{-\infty}(R))^{h\Gamma} & \xrightarrow{\ A^{h\Gamma}\ } & K^{-\infty}(X, R)^{h\Gamma} \end{array}$$

It is further shown in [9] that for $S = h^{lf}(X, K^{-\infty}(R))$ the map

$$\rho\colon\ h^{lf}(X, K^{-\infty}(R))^{\Gamma} \longrightarrow h^{lf}(X, K^{-\infty}(R))^{h\Gamma}$$

is always a weak equivalence.

It is known from [12] that $A^{h\Gamma}$ is a weak equivalence when $\Gamma$ is a group of finite asymptotic dimension. This fact follows from the general observation that when an equivariant map is a nonequivariant weak equivalence, the induced map on homotopy fixed points is also a weak equivalence. The proof that $A$ is a nonequivariant weak equivalence is the main geometric feature of this approach.

Putting the two diagrams together we see that $a(\Gamma, R)$ is the first map in the composition which is homotopic to the composite weak equivalence $A^{h\Gamma} \circ \rho \circ \sigma$. This suffices to show that $a(\Gamma, R)$ a split injection.

**Theorem 1.1.7** (Main Theorem of [12])**.** *If $\Gamma$ is a geometrically finite group of finite asymptotic dimension and $R$ is an arbitrary associative ring then the assembly map*

$$a(\Gamma, R)\colon B\Gamma_+ \wedge K^{-\infty}(R) \longrightarrow K^{-\infty}(R[\Gamma])$$

*is a split injection.*

From the proof of Theorem 1.1.7, the canonical map

$$\rho\colon\; K^{-\infty}(X, R)^{\Gamma} \longrightarrow K^{-\infty}(X, R)^{h\Gamma}$$

induces surjections on homotopy groups.

It follows from the combination of diagrams (†) and (††) that whenever $\rho$ is also a split injection, $\rho$ is a weak equivalence. Therefore, the assembly map $a(\Gamma, R)$ is also a weak equivalence.

The goal of this paper is to prove that $\rho$ is a split injection under the assumptions described in the Main Theorem.

1.2. **The Proof in Broad Strokes.** Section 5 contains the complete proof of the Main Theorem. It requires a number of prerequisite constructions and results in order to describe the details of the argument. All of these prerequisites are provided in section 2. At this point, in order to motivate this rather lengthy development, we give a short outline of the main idea.

Let $\widetilde{M}$ be the universal cover of a closed aspherical manifold $M$ of dimension $n$ with fundamental group $\Gamma$. We will study the splitting

$$\rho\colon\; K^{-\infty}(\widetilde{M}, R)^{\Gamma} \longrightarrow K^{-\infty}(\widetilde{M}, R)^{h\Gamma}$$

of the assembly map $a(\Gamma, R)$. As in section 1.1, we identify $\rho$ up to homotopy with the map

$$K^{-\infty}(R[\Gamma]) \longrightarrow M_+ \wedge K^{-\infty}(R).$$

Since the manifold $M$ is aspherical, its universal cover $\widetilde{M}$ is a model for the universal free $\Gamma$-space $E\Gamma$. Recall that $\rho$ is induced as the fixed point map from the fixed point spectrum $K^{-\infty}(\widetilde{M}, R)^{\Gamma} = F(\text{point}, K^{-\infty}(\widetilde{M}, R))^{\Gamma}$ to the homotopy fixed point spectrum $F(\widetilde{M}, K^{-\infty}(\widetilde{M}, R))^{\Gamma}$ from the canonical equivariant map

$$K^{-\infty}(\widetilde{M}, R) \longrightarrow F(\widetilde{M}, K^{-\infty}(\widetilde{M}, R))$$

given by the collapse of $\widetilde{M}$ to the point.

Suppose the closed manifold $M$ is embedded in a sphere $S^{n+k}$ for sufficiently large $k$. Let $\widetilde{N}$ be the universal cover of the total space $T(\nu)$ of the normal bundle $\nu$ to the embedding. The main result of section 7 is the following proper equivariant version of Spanier–Whitehead duality.

**Theorem 1.2.1** (a.k.a. Theorem 7.4.12). *Let $\mathcal{S}$ be a spectrum with a left action by $\Gamma$. There is a weak homotopy equivalence*

$$\Sigma^{n+k} F(\widetilde{M}, \mathcal{S})^{\Gamma} \;\simeq\; h^{lf}(\widetilde{N}; \mathcal{S})^{\Gamma}.$$

*In fact, the canonical map $\mathcal{S}^{\Gamma} \to \mathcal{S}^{h\Gamma}$ can be identified with a map*

$$\mathcal{S}^{\Gamma} \longrightarrow \Omega^{n+k}\, h^{lf}(\widetilde{N}; \mathcal{S})^{\Gamma}.$$

Applying this theorem to the spectrum $\mathcal{S} = K^{-\infty}(\widetilde{M}, R)$ with the action of $\Gamma$ induced from the standard free isometric action on $\widetilde{M}$, we get a weak homotopy equivalence

$$\Sigma^{n+k} F(\widetilde{M}, K^{-\infty}(\widetilde{M}, R))^{\Gamma} \;\simeq\; h^{lf}(\widetilde{N}; K^{-\infty}(\widetilde{M}, R))^{\Gamma}.$$

The composition

$$B^{\Gamma} = \Sigma^{n+k}\beta^{\Gamma} \colon \Sigma^{n+k}\, K^{-\infty}(\widetilde{M}, R)^{\Gamma} \longrightarrow h^{lf}(\widetilde{N}; K^{-\infty}(\widetilde{M}, R))^{\Gamma}$$

can be followed with the map of fixed point spectra

$$A^{\Gamma} \colon\; h^{lf}(\widetilde{N}; K^{-\infty}(\widetilde{M}, R))^{\Gamma} \longrightarrow K^{-\infty}(\widetilde{M} \times \widetilde{N}, R)^{\Gamma}$$

induced from the equivariant twisted controlled assembly map

$$A \colon\; h^{lf}(\widetilde{N}; K^{-\infty}(\widetilde{M}, R)) \longrightarrow K^{-\infty}(\widetilde{M} \times \widetilde{N}, R)$$

which is defined in section 2.4. Now one has the composition

$$A^{\Gamma} \circ B^{\Gamma} \colon \Sigma^{n+k}\, K^{-\infty}(\widetilde{M}, R)^{\Gamma} \longrightarrow K^{-\infty}(\widetilde{M} \times \widetilde{N}, R)^{\Gamma}.$$

One way to prove the Main Theorem would be to compute $K^{-\infty}(\widetilde{M} \times \widetilde{N}, R)^{\Gamma}$ and prove that $A^{\Gamma} \circ B^{\Gamma}$ is a weak equivalence, making the first map in the composition a weak split injection. This is hard if possible. Instead, we construct another target spectrum called $\mathcal{T}$ with a map

$$e^{\Gamma} \colon\; K^{-\infty}(\widetilde{M} \times \widetilde{N}, R)^{\Gamma} \longrightarrow \mathcal{T}\,.$$

A computation will show that

$$\mathcal{T} \;\simeq\; \Sigma^{n+k}\, K^{-\infty}(R[\Gamma]) \;\simeq\; \Sigma^{n+k}\, K^{-\infty}(\widetilde{M})^{\Gamma},$$

when $R$ has finite homological dimension, and the composition

$$\begin{aligned} e^{\Gamma} \circ A^{\Gamma} \circ B^{\Gamma} \colon\; \Sigma^{n+k}\, K^{-\infty}(\widetilde{M}, R)^{\Gamma} &\longrightarrow K^{-\infty}(\widetilde{M} \times \widetilde{N}, R)^{\Gamma} \\ &\xrightarrow{\;e^{\Gamma}\;} \Sigma^{n+k}\, K^{-\infty}(\widetilde{M})^{\Gamma}. \end{aligned}$$

is a weak equivalence. This will complete the proof of the Main Theorem.

**Remark 1.2.2.** In order to give a rough idea of the properties of the target $\mathcal{T}$, it is instructive to compare this sequence of maps to the analogue where the action of $\Gamma$ on $\widetilde{N}$ is trivial. Notice that there are maps

$$B_0^{\Gamma} \colon \Sigma^{n+k}\, K^{-\infty}(\widetilde{M}, R)^{\Gamma} \longrightarrow h^{lf}(\widetilde{N}_0; K^{-\infty}(\widetilde{M}, R))^{\Gamma}$$

and

$$A_0 \colon\; h^{lf}(\widetilde{N}_0; K^{-\infty}(\widetilde{M}, R)) \longrightarrow K^{-\infty}(\widetilde{M} \times \widetilde{N}_0, R),$$

similar to $B^{\Gamma}$ and $A$, where we use the zero subscript to indicate the trivial action of $\Gamma$ on $\widetilde{N}$.

One can easily see using standard tools that

$$K^{-\infty}(\widetilde{M} \times \widetilde{N}_0, R)^{\Gamma} \simeq K^{-\infty}(M \times \widetilde{N}_0, R) \simeq K^{-\infty}(\widetilde{N}_0, R) \simeq \Sigma^k K^{-\infty}(\widetilde{M}, R).$$

Unfortunately there is no natural functorial way to compare the two twisted assemblies. We will construct an ambient exact category $\mathbf{W}^{\Gamma}$ where both $\mathcal{C}(\widetilde{M} \times \widetilde{N}, R)^{\Gamma}$ and $\mathcal{C}(\widetilde{M} \times \widetilde{N}_0, R)^{\Gamma}$ embed as exact subcategories. The target spectrum $\mathcal{T}$ will be defined as the nonconnective $K$-theory spectrum $K^{-\infty}(\mathbf{W}^{\Gamma})$. On the $K$-theory level, the functors induced by the two inclusions, $e^{\Gamma} \colon K^{-\infty}(\widetilde{M} \times \widetilde{N}, R)^{\Gamma} \to \mathcal{T}$ and $e_0^{\Gamma} \colon K^{-\infty}(\widetilde{M} \times \widetilde{N}_0, R)^{\Gamma} \to \mathcal{T}$, fit in the commutative diagram

$$\begin{tikzcd}
 & h^{lf}(\widetilde{N}; K^{-\infty}(\widetilde{M}))^{\Gamma} \arrow[r, "A^{\Gamma}"] & K^{-\infty}(\widetilde{M} \times \widetilde{N})^{\Gamma} \arrow[dr, bend left, "e^{\Gamma}"] & \\
\Sigma^{n+k} K^{-\infty}(\widetilde{M})^{\Gamma} \arrow[ur, bend left, "B^{\Gamma}"] \arrow[dr, bend right, "B_0^{\Gamma}"'] & & & \mathcal{T} \\
 & h^{lf}(\widetilde{N}_0; K^{-\infty}(\widetilde{M}))^{\Gamma} \arrow[r, "A_0^{\Gamma}"] & K^{-\infty}(\widetilde{M} \times \widetilde{N}_0)^{\Gamma} \arrow[ur, bend right, "e_0^{\Gamma}"'] &
\end{tikzcd}$$

The proper setting for the definition of the category $\mathbf{W}^{\Gamma}$ and for the computation of its $K$-theory is within the boundedly controlled $G$-theory. It is defined and developed in sections 2.6, 2.7, and 9. Theorem 5.2.7 will

(1) establish the weak equivalence $\mathcal{T} \simeq \Sigma^{n+k} K^{-\infty}(\widetilde{M})^{\Gamma}$, when $R$ has finite homological dimension, and
(2) verify that the composition $e^{\Gamma} \circ A^{\Gamma} \circ B^{\Gamma}$, as well as the composition $e_0^{\Gamma} \circ A_0^{\Gamma} \circ B_0^{\Gamma}$, is a weak equivalence.

## 2. Preliminaries (Part 1)

2.1. **Elements of Coarse Geometry.** Let $X$ and $Y$ be proper metric spaces with metric functions $d_X$ and $d_Y$. This means, in particular, that closed bounded subsets of $X$ and $Y$ are compact.

We recall and elaborate on some generalizations of isometries from Pedersen–Weibel [40].

**Definition 2.1.1.** A map $f \colon X \to Y$ of proper metric spaces is *proper* if $f^{-1}(S)$ is a bounded subset of $X$ for each bounded subset $S$ of $Y$. The map $f$ is *eventually continuous* if there is a real positive function $l$ such that

$$d_X(x_1, x_2) \leq r \implies d_Y(f(x_1), f(x_2)) \leq l(r). \tag{1}$$

We say $f$ is a *coarse map* if it is proper and eventually continuous.

If $\mathcal{B}_d(X)$ stands for the collection of subsets of $X$ with diameter bounded by $d$, condition (1) is equivalent to

$$T \in \mathcal{B}_r(X) \implies f(T) \in \mathcal{B}_{l(r)}(Y). \tag{2}$$

For example, all *bounded* functions $f \colon X \to X$ with $d_X(x, f(x)) \leq D$, for all $x \in X$ and a fixed $D \geq 0$, are coarse.

The map $f$ is a *coarse equivalence* if there is a coarse map $g \colon Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded maps.

Of course, an isometry is a coarse equivalence. Also, a function bounded by $D$ is a coarse equivalence using $l(r) = r + 2D$ for both the function and its coarse inverse.

Recall that the path metric in $X$ is obtained as the infimum of length of paths joining points in $X$ over all rectifiable curves. A metric space $X$ is called a *path metric space* if the metric and the path metric in $X$ coincide. Classical examples of path metric spaces are compact Riemannian manifolds.

We will treat the group $\Gamma$ equipped with a finite generating set as a metric space. The following definition makes this precise.

**Definition 2.1.2.** The *word-length metric* $d = d_\Omega$ on a finitely generated group $\Gamma$ with a fixed generating set $\Omega$ closed under inverses is the path metric induced from the condition that $d(\gamma, \gamma\omega) = 1$ whenever $\gamma \in \Gamma$ and $\omega \in \Omega$.

The word-length metric makes $\Gamma$ a proper metric space with a free action by $\Gamma$ via left multiplication. It is well-known that varying $\Omega$ only changes $\Gamma$ to a coarsely equivalent metric space.

Next we make precise the relation between the word-length metric on a discrete group $\Gamma$ and a metric space $X$ where $\Gamma$ acts cocompactly by isometries. The following fact is known as "Milnor's lemma".

**Theorem 2.1.3** (Shvarts, Milnor)**.** *Suppose $X$ is a path metric space and $\Gamma$ is a group acting properly and cocompactly by isometries on $X$. Then $\Gamma$ is coarsely equivalent to $X$.*

*Proof.* The coarse equivalence is given by the map $\gamma \mapsto \gamma x_0$ for any fixed base point $x_0$ of $X$. □

**Corollary 2.1.4.** *If $M$ is a compact manifold with the fundamental group $\Gamma = \pi_1(M)$, the inclusion of any orbit of $\Gamma$ in the universal cover $\widetilde{M}$ is a coarse equivalence for any choice of the generating set of $\Gamma$.*

2.2. **Left-bounded Metrics.**

**Definition 2.2.1.** Let $Z$ be any metric space with a free left $\Gamma$-action by isometries. We assume that the action is properly discontinuous, that is, that for fixed points $z$ and $z'$, the infimum over $\gamma \in \Gamma$ of the distances $d(z, \gamma z')$ is attained. Then we define the orbit space metric on $\Gamma \backslash Z$ by

$$d_{\Gamma \backslash Z}([z], [z']) = \inf_{\gamma \in \Gamma} d(z, \gamma z').$$

**Lemma 2.2.2.** *$d_{\Gamma \backslash Z}$ is a metric on $\Gamma \backslash Z$.*

*Proof.* It is well-known that $d_{\Gamma \backslash Z}$ is a pseudometric. The fact that $\Gamma$ acts by isometries makes it a metric. The triangle inequality follows directly from the triangle inequality for $d$. Symmetry follows from $d(z, \gamma z') = d(\gamma^{-1} z, z') = d(z', \gamma^{-1} z)$. Finally, $d_{\Gamma \backslash Z}([z], [z']) = 0$ gives $d(z, \gamma z') = 0$ for some $\gamma \in \Gamma$, so $d(\gamma' z, \gamma' \gamma z') = 0$ for all $\gamma' \in \Gamma$, and so $[z] = [z']$. □

Now suppose $X$ is some metric space with left $\Gamma$-action by isometries.

**Definition 2.2.3.** Define

$$X^{bdd} = X \times_\Gamma \Gamma$$

where the right-hand copy of $\Gamma$ denotes $\Gamma$ regarded as a metric space with the word-length metric associated to a finite generating set, the group $\Gamma$ acts by isometries on the metric space $\Gamma$ via left multiplication, and $X \times_\Gamma \Gamma$ denotes the orbit metric space associated to the diagonal left $\Gamma$-action on $X \times \Gamma$. We will denote the orbit metric by $d^{bdd}$.

The natural left action of $\Gamma$ on $X^{bdd}$ is given by $\gamma[x,e] = [\gamma x, e]$.

**Definition 2.2.4.** A left action of $\Gamma$ on a metric space $X$ is *bounded* if for each element $\gamma \in \Gamma$ there is a number $B_\gamma \geq 0$ such that $d(x, \gamma x) \leq B_\gamma$ for all $x \in X$.

**Lemma 2.2.5.** *If the left action of $\Gamma$ on a metric space $X$ is bounded, and $B \colon \Gamma \to [0,\infty)$ is a function as above, then there is a real function $B_* \colon [0,\infty) \to [0,\infty)$ such that $|\gamma| \leq s$ implies $B_\gamma \leq B_*(s)$.*

*Proof.* One simply takes $B_*(s) = \max\{B_\gamma \mid |\gamma| \leq s\}$. □

**Proposition 2.2.6.** *The natural action of $\Gamma$ on $X^{bdd}$ is bounded.*

*Proof.* If $|\gamma| = d_\Gamma(e,\gamma)$ is the norm in $\Gamma$, we choose $B_\gamma = |\gamma|$. Now

$$\begin{aligned} d^{bdd}([x,e],[\gamma x,e]) &= \inf_{\gamma' \in \Gamma} d^\times((x,e),\gamma'(\gamma x,e)) \\ &\leq d^\times((x,e),\gamma^{-1}(\gamma x,e)) \\ &= d^\times((x,e),(x,\gamma^{-1})) = d_\Gamma(e,\gamma^{-1}) = |\gamma^{-1}| = |\gamma|, \end{aligned}$$

where $d^\times$ stands for the max metric on the product $X \times \Gamma$. □

**Definition 2.2.7.** Let $b \colon X \to X^{bdd}$ be the natural map given by $b(x) = [x,e]$ in the orbit space $X \times_\Gamma \Gamma$.

**Proposition 2.2.8.** *The map $b \colon X \to X^{bdd}$ is a coarse map.*

*Proof.* Suppose $d^{bdd}([x_1,e],[x_2,e]) \leq D$, then $d^\times((x_1,e),(\gamma x_2,\gamma)) \leq D$ for some $\gamma \in \Gamma$, so $d(x_1,\gamma x_2) \leq D$ and $|\gamma| \leq D$. Since the left action of $\Gamma$ on $X^{bdd}$ is bounded, there is a function $B_*$ guaranteed by Lemma 2.2.5. Now

$$d(x_1,x_2) \leq d(x_1,\gamma x_2) + d(x_2,\gamma x_2) \leq D + B_*(D).$$

This verifies that $b$ is proper. It is clearly distance reducing, therefore eventually continuous with $l(r) = r$. □

If we think of $X^{bdd}$ as the set $X$ with the metric induced from the bijection $b$, the map $b$ becomes the coarse identity map between the metric space $X$ with a left action of $\Gamma$ and the metric space $X^{bdd}$ where the action is made bounded.

**Proposition 2.2.9.** *Suppose that $J$, $K$, and $L$ are metric spaces with left $\Gamma$-actions by isometries. Suppose further that $f \colon K \to L$ is an equivariant coarse equivalence. Then the natural map*

$$J \times_\Gamma f \colon J \times_\Gamma K \longrightarrow J \times_\Gamma L$$

*is a coarse equivalence.*

**Corollary 2.2.10.** *Let $i \colon \Gamma \hookrightarrow \widetilde{M}$ be the inclusion of an orbit as in Corollary* 2.1.4. *For any metric space $J$ with left $\Gamma$-action by isometries, the canonical inclusion*

$$J \times_\Gamma i \colon J \times_\Gamma \Gamma \hookrightarrow J \times_\Gamma \widetilde{M}$$

*is a coarse equivalence.*

We now consider the situation of relevance to us. We will first analyze the locally finite homology of a particular situation. Let $M$ be a closed $n$-dimensional manifold $K(\Gamma,1)$-space. Embed $M$ in $\mathbb{R}^{n+k}$ for some $k$ and let $N$ denote an open tubular neighborhood of $M$, which is homeomorphic to the total space of the normal bundle to the embedding. Let $\widetilde{N}$ denote its universal covering space. Of course, $\pi_1(N)=\Gamma$. If we now construct $\widetilde{N}\times_\Gamma M$, where the action of $\Gamma$ on $M$ is the trivial action, we obtain the projection $\widetilde{N}\times_\Gamma M\to N$. The fiber of this bundle over a point $\nu\in N$ is $D_\nu\times\Gamma\times_\Gamma\widetilde{N}\cong D_\nu\times M$, where $D_\nu$ is the fiber of the bundle projection $N\to M$. We now wish to analyze the locally finite homology of $\widetilde{N}$.

By an $\mathbb{R}^n$-bundle, we will mean a fiber bundle whose fibers are homeomorphic to $\mathbb{R}^n$ and whose structure group is the group of homeomorphisms of $\mathbb{R}^n$. We suppose we are given an $\mathbb{R}^n$-bundle $p\colon X\to B$ with base space a compact closed piecewise linear manifold $B$. Of course, $B$ may be regarded as a finite simplicial complex, which we refer to as $W$. We consider the partially ordered set $\mathcal{P}=\mathcal{P}(W)$ of simplices of $W$, where the ordering is by face inclusions, and regard the partially ordered set as a category.

A locally compact space is a Hausdorff topological space where each point has a compact neighborhood. Let $\mathbf{Top}^{lc}$ denote the category of locally compact spaces and proper maps. We have the functor

$$\Phi\colon\mathcal{P}\longrightarrow\mathbf{Top}^{lc}$$

given by $\Phi(\sigma)=p^{-1}(\sigma)$. Locally finite homology with arbitrary coefficient spectrum $\mathcal{S}$ is functorial on $\mathbf{Top}^{lc}$, and we obtain therefore a functor

$$\Phi^{lf}_{\mathcal{S}}\colon\mathcal{P}\longrightarrow\mathbf{Spectra}$$

given by $\Phi^{lf}_{\mathcal{S}}=h^{lf}(p^{-1}(\sigma),\mathcal{S})$. Excision properties of $h^{lf}$ as given in [9] show that

$$h^{lf}(X,\mathcal{S})=\underset{\overrightarrow{\mathcal{P}}}{\operatorname{hocolim}}\;\Phi^{lf}_{\mathcal{S}}.$$

Next, suppose that the bundle $p$ admits a section $s\colon B\to X$. Then there is a second functor $\Psi\colon\mathcal{P}\to\mathbf{Spectra}$ defined by

$$\Psi(\sigma)=\underset{\overrightarrow{C\in\mathcal{C}}}{\operatorname{hocolim}}\;h^{lf}(C,\mathcal{S})$$

where $\mathcal{C}$ denotes the partially ordered set of closed subsets of $p^{-1}(\sigma)$ which do not intersect $s(\sigma)\subset p^{-1}(\sigma)$. From the definitions, it is clear that there is a natural transformation $\Phi^{lf}_{\mathcal{S}}\to\Psi$ of functors on $\mathcal{P}$, and it is clear that it is a weak equivalence of functors.

**Corollary 2.2.11.** *We have*

$$h^{lf}(X,\mathcal{S})=\underset{\overrightarrow{\mathcal{P}}}{\operatorname{hocolim}}\;\Psi.$$

*In other words, the locally finite homology of $X$ can be computed as the locally finite homology of a small neighborhood of the zero section $s(B)$.*

The bundle $\widetilde{N}\times_\Gamma\widetilde{M}$ can be regarded as the fibrewise external product of two $\mathbb{R}^n$-bundles over $M$, one the normal bundle to the embedded copy of $M$, and the second the bundle $\widetilde{M}\times_\Gamma\widetilde{M}\to M$ where the projection is on the first factor. The fibre is of course $\widetilde{M}$. It is standard that a small neighborhood of the zero section of this $\mathbb{R}^n$-bundle is homeomorphic to the tangent bundle of $M$. It is therefore clear that a small neighborhood of the zero section in $\widetilde{N} \times_\Gamma \widetilde{M}$ is homeomorphic (as $\mathbb{R}^n$-bundles) to a small neighborhood of the zero section in the Whitney sum of the tangent bundle to $M$ with its normal bundle in the given embedding. We now have the following.

**Proposition 2.2.12.** *The bundle $\widetilde{N} \times_\Gamma \widetilde{M}$ is equivalent to the Thom complex of the Whitney sum of the normal and tangent bundles to the embedding of $M$ in a Euclidean space. Since this bundle is trivial, being the pullback of the (trivial) tangent bundle to the ambient Euclidean space, we find that*

$$h^{lf}(\widetilde{N} \times_\Gamma \widetilde{M}, \mathcal{S}) \cong \Sigma^{n+k} M_+ \wedge \mathcal{S}$$

*where $n+k$ is the dimension of the Euclidean space in which we embedded the $n$-dimensional manifold $M$, and where $M_+$ denotes $M$ with a disjoint base point added.*

We also consider $h^{lf}(\widetilde{N}, \mathcal{S})$. By an argument similar to the one above, but using the $\mathbb{R}^k$-bundle $\widetilde{N} \to \widetilde{M}$, we now have the following results.

**Proposition 2.2.13.** *Let $\mathcal{S}$ be an arbitrary coefficient spectrum. Then*

(1) $h^{lf}(\widetilde{N}, \mathcal{S}) \simeq \Sigma^{n+k}\mathcal{S}$, *and*
(2) *the natural inclusion $\widetilde{N} \hookrightarrow \widetilde{N} \times_\Gamma \widetilde{M}$ induces the map*
$$\mathrm{id} \wedge j \colon \Sigma^{n+k}\mathcal{S} \cong \Sigma^{n+k}\mathcal{S} \wedge S^0 \longrightarrow \Sigma^{n+k} M_+ \wedge \mathcal{S},$$
*where $j \colon S^0 \hookrightarrow M_+$ is the evident inclusion.*

2.3. **Cone Construction $TX$.** For any metric space $X$, the new space $TX$ will be related to the cone construction.

**Definition 2.3.1.** Start with any set $Z$, let $S \subset Z \times Z$ denote any symmetric and reflexive subset with the property that

- for any $z$, $z'$, there are elements $z_0$, $z_1$, ... , $z_n$ so that $z_0 = z$, $z_n = z_0$, and $(z_i, z_{i+1}) \in S$.

Let $\rho \colon S \to \mathbb{R}$ be any function for which the following properties hold:

- $\rho(z_1, z_2) = \rho(z_2, z_1)$ for all $(z_1, z_2) \in S$,
- $\rho(z_1, z_2) = 0$ if and only if $z_1 = z_2$.

Given such $S$ and $\rho$, we may define a metric $d$ on $Z$ to be the largest metric $D$ for which $D(z_1, z_2) \le \rho(z_1, z_2)$ for all $(z_1, z_2) \in S$. This means that $d$ is given by

$$d(z_1, z_2) = \inf_{n, \{z_0, z_1, \ldots, z_n\}} \sum_{i=0}^{n} \rho(z_i, z_{i+1}).$$

**Definition 2.3.2.** Let $k \ge 1$ be a real number. We define a metric space $T_k X$ by first declaring that the underlying set is $X \times \mathbb{R}$. Next, we define $S$ to be the set consisting of pairs of the form $((x,r),(x,r'))$ or of the form $((x,r),(x',r))$. We then define $\rho$ on $S$ by

$$\rho((x,r),(x,r')) = |r - r'|,$$

and

$$\rho((x,r),(x',r)) = \begin{cases} d(x,x'), & \text{if } r \le 1; \\ r d(x,x'), & \text{if } 1 \le r \le k; \\ k d(x,x'), & \text{if } k \le r. \end{cases}$$

Since $\rho$ clearly satisfies the hypotheses of the above definition, we set the metric on $T_k X$ to be $d_k$.

One can also define $TX$ as the set $X \times \mathbb{R}$ with the metric $d$ given as the colimit of $d_k$. Explicitly, $TX$ can be identified with the metric space $T_\infty X$ constructed as above. For that construction, one would use $\rho$ on $S$ defined by

$$\rho((x,r),(x,r')) = |r - r'|,$$

and

$$\rho((x,r),(x',r)) = \begin{cases} d(x,x'), & \text{if } r \leq 1; \\ rd(x,x'), & \text{if } 1 \leq r. \end{cases}$$

We also extend the definition to pairs of metric spaces $(X,Y)$, where $Y$ is given the restriction of the metric on $X$.

Applying bounded $K$-theory to this construction gives predictable results when applied to familiar subspaces. The following facts follow from the proof of the main theorem of Pedersen–Weibel [40].

**Proposition 2.3.3.** *If $X$ is a compact subset of a Euclidean space, one obtains $K^{-\infty}(TX,R) = X_+ \wedge K^{-\infty}(R)$.*

**Proposition 2.3.4.** $K^{-\infty}(T\mathbb{R}^n,R) \simeq \Sigma\, K^{-\infty}(\mathbb{R}^n,R) \simeq \Sigma^{n+1}\, K^{-\infty}(R)$.

Proofs of these statements can be found in section 8.

2.4. **Twisted Assembly in Algebraic *K*-theory.** The bounded $K$-theory assembly map

$$A(X,\mathcal{A})\colon\ h^{lf}(X;K^{-\infty}(\mathcal{A})) \longrightarrow K^{-\infty}(X,\mathcal{A}),$$

for a proper metric space $X$ and a small additive category $\mathcal{A}$, can be defined as follows.

Let ${}^b S_k X$ be the collection of all locally finite families $\mathcal{F}$ of singular $k$-simplices in $X$ which are uniformly bounded, in the sense that each family possesses a number $N$ such that the diameter of the image $\mathrm{im}(\sigma)$ is bounded from above by $N$ for all simplices $\sigma \in \mathcal{F}$.

For any spectrum $\mathcal{S}$, the theory ${}^b h^{lf}(X;\mathcal{S})$ is the realization of the simplicial spectrum

$$k \mapsto \underset{C \in {}^b S_k X}{\underrightarrow{\mathrm{hocolim}}}\ h^{lf}(C,\mathcal{S}).$$

There is a similar theory $J^h(X;\mathcal{A})$ obtained as the realization of the simplicial spectrum

$$k \mapsto \underset{C \in {}^b S_k X}{\underrightarrow{\mathrm{hocolim}}}\ K^{-\infty}(C,\mathcal{A})$$

by viewing $C$ as a discrete metric space.

Recall that a map between metric spaces $\phi\colon (M_1,d_1) \to (M_2,d_2)$ is *eventually continuous* if there is a real function $g$ such that $d_2(\phi(x),\phi(y)) \leq g(d_1(x,y))$ for all pairs of points $x$, $y$ in $M_1$.

Since [10] is written in appropriate generality, in terms of general additive category coefficients, Corollary III.14 of [9] gives a weak homotopy equivalence

$$\eta\colon\ {}^b h^{lf}(X;K^{-\infty}(\mathcal{A})) \longrightarrow J^h(X;\mathcal{A})$$

of functors from proper locally compact metric spaces and proper eventually continuous maps to spectra.

We next define a natural transformation

$$\ell\colon\ J^h(X;\mathcal{A}) \longrightarrow K^{-\infty}(X,\mathcal{A}).$$

In the case $\mathcal{A}$ is the category of finitely generated free $R$-modules, this transformation is defined as part of the proof of Proposition III.20 of [9]. Since the definition is entirely in terms of maps between singular simplices in $X$, the construction can be generalized to give $\ell$ as above, for any additive category $\mathcal{A}$. For convenience of the reader, we present the necessary details.

Let $\mathcal{D}$ be any collection of singular $n$-simplices of $X$ and $\zeta$ be any point of the standard $n$-simplex. Define a function $\vartheta_\zeta\colon \mathcal{D} \to X$ by $\vartheta_\zeta(\sigma) = \sigma(\zeta)$. Since $\mathcal{D}$ is viewed as a discrete metric space, if $\mathcal{D}$ is locally finite then $\vartheta_\zeta$ is proper and eventually continuous, so we have the induced functor $\mathcal{C}(\mathcal{D},\mathcal{A}) \to \mathcal{C}(X,\mathcal{A})$ given by

$$\bigoplus_{d\in\mathcal{D}} F_d \longrightarrow \bigoplus_{x\in X}\ \bigoplus_{\vartheta_\zeta(d)=x} F_d$$

which is the identity for each $d \in \mathcal{D}$. Therefore, there is the induced map of spectra

$$K(\vartheta_\zeta,\mathcal{A})\colon K(\mathcal{D},\mathcal{A}) \longrightarrow K(X,\mathcal{A}).$$

Suppose further that $\mathcal{D} \in {}^bS_kX$ and that $N$ is the number required to exist for $\mathcal{D}$ in ${}^bS_kX$. If $\zeta$ and $\theta$ are both points in the standard $n$-simplex, we have a symmetric monoidal natural transformation $N^\theta_\zeta\colon K(\vartheta_\zeta,\mathcal{A}) \to K(\vartheta_\theta,\mathcal{A})$ induced from the functors which are identities on objects in the cocompletion of $\mathcal{A}$. Each of those identity morphisms are isomorphisms in $\mathcal{C}(X,\mathcal{A})$ because they and their inverses are bounded by $N$.

Recall that the standard $n$-simplex can be viewed as the nerve of the ordered set $\underline{n} = \{0,1,\dots,n\}$, with the natural order, viewed as a category. Let $\mathcal{D} \in {}^bS_nX$. We define a functor

$$l(\mathcal{D},n)\colon i\mathcal{C}(\mathcal{D},\mathcal{A}) \times \underline{n} \longrightarrow i\mathcal{C}(X,\mathcal{A})$$

as follows. On objects, $(l(\mathcal{D},n)F)_x = \bigoplus_{\vartheta(i)=x} F_d$, where $i$ denotes the vertex of $\Delta^n = N.\,\underline{n}$ corresponding to $i$. On morphisms, $l(\mathcal{D},n)$ is defined by the requirement that the restriction to the subcategory $i\mathcal{C}(\mathcal{D},\mathcal{A}) \times j$ is the functor induced by $\theta_j$, and that $(\mathrm{id}\times(i \le j))(F)$ is sent to $N^j_i(F)$. This is compatible with the inclusion of elements in ${}^bS_nX$, so we obtain a functor

$$\underset{\mathcal{D}\in{}^bS_nX}{\underrightarrow{\operatorname{colim}}}\ i\mathcal{C}(\mathcal{D},\mathcal{A}) \times \underline{n} \longrightarrow i\mathcal{C}(X,\mathcal{A}),$$

and therefore a map

$$\underset{\mathcal{D}\in{}^bS_nX}{\underrightarrow{\operatorname{hocolim}}}\ N.\,i\mathcal{C}(\mathcal{D},\mathcal{A}) \times \Delta^n \longrightarrow N.\,i\mathcal{C}(X,\mathcal{A}).$$

If $\mathcal{M}$ is a symmetric monoidal category, let the $t$-th space in $\mathrm{Spt}(\mathcal{M})$ be denoted by $\mathrm{Spt}_t(\mathcal{M})$, and let $\sigma_t\colon S^1 \wedge \mathrm{Spt}_t(\mathcal{M}) \to \mathrm{Spt}_{t+1}(\mathcal{M})$ be the structure map for $\mathrm{Spt}(\mathcal{M})$. The fact that the natural transformations $N^j_i$ are symmetric monoidal shows in particular that we ontain maps

$$\Lambda_t\colon\ \underset{\mathcal{D}\in{}^bS_nX}{\underrightarrow{\operatorname{hocolim}}}\ \mathrm{Spt}_t(i\mathcal{C}(\mathcal{D},\mathcal{A})) \times \Delta^n \longrightarrow \mathrm{Spt}_t(i\mathcal{C}(X,\mathcal{A})),$$

so that the diagrams

$$\begin{array}{ccc} \underset{\overrightarrow{\mathcal{D}\in{}^bS_nX}}{\mathrm{hocolim}}\,(S^1\wedge \mathrm{Spt}_t(i\mathcal{C}(\mathcal{D},\mathcal{A})))\times\Delta^n & \longrightarrow & S^1\wedge \mathrm{Spt}_t(i\mathcal{C}(X,\mathcal{A})) \\ \Big\downarrow{\scriptstyle \sigma_t\times\mathrm{id}} & & \Big\downarrow{\scriptstyle \sigma_t} \\ \underset{\overrightarrow{\mathcal{D}\in{}^bS_nX}}{\mathrm{hocolim}}\,\mathrm{Spt}_{t+1}(i\mathcal{C}(\mathcal{D},\mathcal{A}))\times\Delta^n & \xrightarrow{\Lambda_{t+1}} & \mathrm{Spt}_{t+1}(i\mathcal{C}(X,\mathcal{A})) \end{array}$$

commute. Further, for each $t$ we obtain a map

$$\left| \underline{k}\mapsto \underset{\overrightarrow{\mathcal{D}\in{}^bS_nX}}{\mathrm{hocolim}}\,\mathrm{Spt}_t(i\mathcal{C}(\mathcal{D},\mathcal{A}))\right| \longrightarrow \mathrm{Spt}_t(i\mathcal{C}(X,\mathcal{A}))$$

respecting the structure maps in $\mathrm{Spt}_t$. This gives a map

$$\ell\colon {}^cJ^h(X;\mathcal{A})\longrightarrow K(X,\mathcal{A})$$

where ${}^cJ^h(X;\mathcal{A})$ stands for the realization of the simplicial spectrum

$$k\mapsto \underset{\overrightarrow{C\in{}^bS_kX}}{\mathrm{hocolim}}\,K(C,\mathcal{A}).$$

We note that $\ell$ is natural in $X$ and is compatible with delooping. Therefore it generalizes to the homotopy natural transformation

$$\ell\colon\ J^h(X;\mathcal{A})\longrightarrow K^{-\infty}(X,\mathcal{A}).$$

Recall also that there is an equivalence of spectra

$${}^bh^{lf}(X;K^{-\infty}(\mathcal{A}))\longrightarrow h^{lf}(X;K^{-\infty}(\mathcal{A})),$$

for any proper metric space $X$, by Corollary II.21 of [9].

**Definition 2.4.1.** The composition

$$A(X,\mathcal{A})\colon\ h^{lf}(X;K^{-\infty}(\mathcal{A}))\longrightarrow K^{-\infty}(X,\mathcal{A})$$

of $\ell$ and $\eta$ is a homotopy natural transformation.

*Notation* 2.4.2. Given proper metric spaces $M$, $N$, and a ring $R$, specializing to $\mathcal{A}=\mathcal{C}(M,R)$, gives the twisted assembly map

$$A(N,M)\colon\ h^{lf}(N;K^{-\infty}(M,R))\longrightarrow K^{-\infty}(N,\mathcal{C}(M,R)).$$

Note that the twisted assembly reduces to the homotopy natural transformation from Proposition 3.20 in [9] when $M$ is a point. In this case

$$A(X,R)=A(X,\mathcal{C}(\mathrm{point},R))$$

is the equivariant controlled assembly map from section 1.1.

Next note that the bounded $K$-theory spectrum $K^{-\infty}(M,R)$ can be viewed as the homotopy colimit of a family of nonconnective spectra

$$K^{-\infty}(M,R)=\underset{\overrightarrow{d}}{\mathrm{hocolim}}\,K[d](M,R),$$

where $K[d](M,R)$ is the spectrum associated with a $\Gamma$-space given by the subspace of the nerve of the category with bounded isomorphisms as morphisms, for which a simplex is included if and only if all the maps which make up the simplex and all the composites which are computed to obtain iterated face maps are bounded by $d$ in $M$.

**Definition 2.4.3.** Let

$$A^{\times}(N, M)\colon \varinjlim_{d} h^{lf}(N, K[d](M, R)) \longrightarrow K^{-\infty}(N \times M, R)$$

be the map induced from

$$A_d(N \times M)\colon h^{lf}(N, K[d](M, R)) \longrightarrow K^{-\infty}(N \times M, R).$$

Clearly $A^{\times}(N, M)$ agrees with $A(N, R)$ when $M$ is a point.

The exact embedding

$$i\colon \mathcal{C}(M \times N, R) \longrightarrow \mathcal{C}(M, \mathcal{C}(N, R))$$

induces the map of $K$-theory spectra

$$i_*\colon K^{-\infty}(M \times N, R) \longrightarrow K^{-\infty}(M, \mathcal{C}(N, R))$$

which, in general, is not an equivalence.

**Definition 2.4.4.** The composition of $i_*$ with $A^{\times}$ gives the *fibred assembly map*

$$A^{\mathrm{fib}}(N, M)\colon \varinjlim_{d} h^{lf}(N, K[d](M, R)) \longrightarrow K^{-\infty}(M, \mathcal{C}(N, R)).$$

2.5. **Equivariant Fibred $K$-theory.** Given a proper metric space $X$ with a free left $\Gamma$-action by isometries, one has the equivariant $K$-theory $K^{\Gamma}$ associated to $X$ and a ring $R$ from [9]. We wish to construct a version of the equivariant bounded $K$-theory which applies to actions by coarse equivalences and extends to the equivariant bounded $G$-theory.

Suppose we are given $X$ with a free left $\Gamma$-action by isometries. There is a natural action of $\Gamma$ on the geometric modules $\mathcal{C}(X, R)$ and therefore on $K(X, R)$. A different equivariant bounded $K$-theory with useful fixed point spectra is constructed as follows.

**Definition 2.5.1.** Let $\boldsymbol{E\Gamma}$ be the category with the object set $\Gamma$ and the unique morphism $\mu\colon \gamma_1 \to \gamma_2$ for any pair $\gamma_1$, $\gamma_2 \in \Gamma$. There is a left $\Gamma$-action on $\boldsymbol{E\Gamma}$ induced by the left multiplication in $\Gamma$.

If $\mathcal{C}$ is a small category with left $\Gamma$-action, then the functor category $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is a category with the left $\Gamma$-action given on objects by

$$\gamma(F)(\gamma') = \gamma F(\gamma^{-1}\gamma')$$

and

$$\gamma(F)(\mu) = \gamma F(\gamma^{-1}\mu).$$

It is always nonequivariantly equivalent to $\mathcal{C}$. The subcategory of equivariant functors and equivariant natural transformations in $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is the fixed subcategory $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})^{\Gamma}$ known as the *lax limit* of the action of $\Gamma$.

According to Thomason [44], the objects of $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})^{\Gamma}$ can be thought of as pairs $(F, \psi)$ where $F \in \mathcal{C}$ and $\psi$ is a function on $\Gamma$ with $\psi(\gamma) \in \mathrm{Hom}(F, \gamma F)$ such that

$$\psi(1) = 1 \quad \text{and} \quad \psi(\gamma_1\gamma_2) = \gamma_1\psi(\gamma_2)\psi(\gamma_1).$$

These conditions imply that $\psi(\gamma)$ is always an isomorphism. The set of morphisms $(F,\psi) \to (F',\psi')$ consists of the morphisms $\phi\colon F \to F'$ in $\mathcal{C}$ such that the squares

$$\begin{tikzcd} F \arrow[r, "\psi(\gamma)"] \arrow[d, "\phi"'] & \gamma F \arrow[d, "\gamma\phi"] \\ F' \arrow[r, "\psi'(\gamma)"] & \gamma F' \end{tikzcd}$$

commute for all $\gamma \in \Gamma$.

In order to specialize to the case of $\mathcal{C} = \mathcal{C}(X,R)$, notice that $\mathcal{C}(X,R)$ contains the family of isomorphisms $\phi$ such that $\phi$ and $\phi^{-1}$ are bounded by 0. We will express this property by saying that the filtration of $\phi$ is 0 and writing $\mathrm{fil}(\phi) = 0$. The full subcategory of functors $\theta\colon \boldsymbol{E\Gamma} \to \mathcal{C}(X,R)$ such that $\mathrm{fil}\,\theta(f) = 0$ for all $f$ is invariant under the $\Gamma$-action.

*Notation* 2.5.2. We will use the notation

$$\mathcal{C}^{\Gamma}(X,R) = \mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C}(X,R))$$

for the equivariant category as described in Definition 2.5.1.

**Definition 2.5.3.** Let $\mathcal{C}^{\Gamma,0}(X,R)$ be the equivariant full subcategory of $\mathcal{C}^{\Gamma}(X,R)$ on the functors sending all morphisms of $\boldsymbol{E\Gamma}$ to filtration 0 maps. We define $K^{\Gamma,0}(X,R)$ to be the nonconnective delooping of the $K$-theory of the symmetric monoidal category $\mathcal{C}^{\Gamma,0}(X,R)$.

The fixed points of the induced $\Gamma$-action on $K^{\Gamma,0}(X,R)$ is the nonconnective delooping of the $K$-theory of $\mathcal{C}^{\Gamma,0}(X,R)^{\Gamma}$. This is the full subcategory of $\mathcal{C}^{\Gamma}(X,R)^{\Gamma}$ on the objects $(F,\psi)$ with $\mathrm{fil}\,\psi(\gamma) = 0$ for all $\gamma \in \Gamma$.

One of the main properties of the functor $K^{\Gamma,0}$ is the following.

**Theorem 2.5.4** (Corollary VI.8 of [9])**.** *If $X$ is a proper metric space and $\Gamma$ acts on $X$ freely, properly discontinuously, cocompactly by isometries, there are weak equivalences*

$$K^{\Gamma,0}(X,R)^{\Gamma} \simeq K^{-\infty}(X/\Gamma, R[\Gamma]) \simeq K^{-\infty}(R[\Gamma]).$$

This theorem applies in two specific cases of interest to us: when $\Gamma$ is the fundamental group of a closed aspherical manifold $M$ acting on the universal cover $X$ by covering transformations, and when $\Gamma$ acts on itself, as a word-length metric space, by left multiplication.

**Remark 2.5.5.** The theory $K^{\Gamma,0}(X,R)$ may very well differ from $K^{\Gamma}(X,R)$. According to Theorem 2.5.4, the fixed point category of the original bounded equivariant theory is, for example, the category of free $R[\Gamma]$-modules when $X = \Gamma$ with the word-length metric. However, $\mathcal{C}^{\Gamma}(\Gamma,R)^{\Gamma}$ will include the $R$-module with a single basis element and equipped with trivial $\Gamma$-action, which is not free.

Now suppose we are given a metric space $Y$ with a free left $\Gamma$-action by coarse equivalences and observe that the equivariant theory $K^{\Gamma}$ still applies in this case.

**Definition 2.5.6** (Coarse Equivariant Theories)**.** We associate several new equivariant theories on metric spaces with $\Gamma$-action, both by isometries and coarse equivalences. The theory $K^{\Gamma}_i$ is defined only for metric spaces with actions by isometries, while $K^{\Gamma}_c$ and $K^{\Gamma}_p$ only for metric spaces with coarse actions.

(1) $k_i^\Gamma(Y)$ is defined to be the $K$-theory of $\mathcal{C}_i^\Gamma(Y) = \mathcal{C}^{\Gamma,0}(\Gamma \times Y, R)$, where $\Gamma$ is regarded as a word-length metric space with isometric $\Gamma$-action given by left multiplication, and $\Gamma \times Y$ is given the product metric and the product isometric action.
(2) $k_c^\Gamma(Y)$ is defined for any metric space $Y$ equipped with a $\Gamma$-action by coarse equivalences. It is the $K$-theory spectrum attached to a symmetric monoidal category $\mathcal{C}_c^\Gamma(Y)$ with $\Gamma$-action whose objects are given by functors

$$\theta\colon\ \boldsymbol{E\Gamma} \longrightarrow \mathcal{C}(\Gamma \times Y, R)$$

such that the morphisms $\theta(f)$ have the property that, in addition to being bounded as maps of modules with bases labelled in $\Gamma \times Y$, they are of degree zero when projected into $\Gamma$.
(3) $k_p^\Gamma(Y)$ is defined for any metric space with action by coarse equivalences. Again, this spectrum is attached to a symmetric monoidal category $\mathcal{C}_p^\Gamma(Y)$ whose objects are functors

$$\theta\colon\ \boldsymbol{E\Gamma} \longrightarrow \mathcal{C}_\Gamma(Y) = \mathcal{C}(\Gamma, \mathcal{C}(Y, R))$$

with the additional condition that the morphisms $\theta(f)$ are bounded by zero but only as homomorphisms between $R$-modules parametrized over $\Gamma$.

Now the nonconnective equivariant $K$-theory spectra $K_i^\Gamma$, $K_c^\Gamma$, $K_p^\Gamma$ are the nonconnective deloopings of $k_i^\Gamma$, $k_c^\Gamma$, $k_p^\Gamma$. For details we refer to section 8.2.

We note that for metric spaces with isometric $\Gamma$-actions, there is a natural transformation $K_i^\Gamma(Y) \to K_c^\Gamma(Y)$, and similarly for metric spaces with action by coarse equivalences there is a natural transformation $K_c^\Gamma(Y) \to K_p^\Gamma(Y)$.

We point out an instructive property of $K_p^\Gamma$ that is not explicitly used in the paper.

**Proposition 2.5.7.** *Let $Y$ be a metric space with a $\Gamma$-action by coarse equivalences. Suppose further that the action is bounded in the sense that for every $\gamma \in \Gamma$ there is a bound $r_\gamma$ so that for every $y \in Y$, $d(y, \gamma y) \leq r_\gamma$. Let $Y_0$ denote the metric space $Y$ equipped with the trivial $\Gamma$-action. Then there is an equivariant equivalence of equivariant spectra $K_p^\Gamma(Y) = K_p^\Gamma(Y_0)$.*

One relation between the three equivariant fiberwise theories is through the observation that in all three cases, when $Y$ is a single point space, $\mathcal{C}_i^\Gamma(\text{point})$, $\mathcal{C}_c^\Gamma(\text{point})$, and $\mathcal{C}_p^\Gamma(\text{point})$ can be identified with $\mathcal{C}^{\Gamma,0}(\Gamma, R)$.

**Definition 2.5.8.** Given left $\Gamma$-actions by isometries on metric spaces $X$ and $Y$, there are evident diagonal actions induced on the categories $\mathcal{C}(X \times Y, R)$ and $\mathcal{C}(X, \mathcal{C}(Y, R))$. The equivariant embedding induces the equivariant functor

$$i^\Gamma \colon \mathcal{C}^\Gamma(X \times Y, R) \longrightarrow \mathcal{C}^\Gamma(X, \mathcal{C}(Y, R)).$$

It is clear that one gets an equivariant version of the fibred assembly

$$A_\Gamma^{\text{fib}}\colon\ \underset{\overrightarrow{d}}{\operatorname{hocolim}}\ h^{lf}(Y, K^\Gamma[d](X, R)) \xrightarrow{A^\times} K^\Gamma(X \times Y, R) \xrightarrow{i_*^\Gamma} K^\Gamma(X, \mathcal{C}(Y, R)).$$

Similarly, for the $K$-theory of subcategories with filtration zero control along the factor $X$, we get the fibred assembly map

$$A^{\mathrm{fib}}_{\Gamma,0} \colon \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; h^{lf}(Y, K^{\Gamma,0}[d](X,R)) \xrightarrow{A^{\times}} K^{\Gamma,0}(X \times Y, R)$$
$$\xrightarrow{i^{\Gamma}_{*}} K^{\Gamma,0}(X, \mathcal{C}(Y,R)).$$

Since the group $\Gamma$ with the word-length metric is commensurable with $\widetilde{M}$ where it acts cocompactly, we can apply the above construction to the universal covers of both the manifold $M$ and its normal bundle $N$.

**Definition 2.5.9.** Let

$$\widetilde{A} \colon \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; h^{lf}(\widetilde{N}, K^{\Gamma,0}[d](\widetilde{M},R)) \longrightarrow K^{\Gamma}_{i}(\widetilde{N})$$

be the resulting assembly map.

The duality theory from section 7 now gives us the map

$$\underset{\overrightarrow{d}}{\mathrm{hocolim}}\; \Sigma^{n+k} F(\widetilde{M}, K[d](\widetilde{M},R)) \cong \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; h^{lf}(\widetilde{N}, K[d](\widetilde{M},R)) \xrightarrow{\widetilde{A}} K^{\Gamma}_{i}(\widetilde{N})$$

which yields the induced map on fixed point spectra

$$\widetilde{A}^{\Gamma} \colon \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; \Sigma^{n+k} K^{\Gamma,0}[d](\widetilde{M},R)^{h\Gamma} \longrightarrow K^{\Gamma}_{i}(\widetilde{N})^{\Gamma}.$$

**Lemma 2.5.10.** *Since $\Gamma$ has a finite classifying space, there is a natural equivalence*

$$\Delta \colon K^{\Gamma,0}(\widetilde{M},R)^{h\Gamma} \xrightarrow{\simeq} \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; K^{\Gamma,0}[d](\widetilde{M},R)^{h\Gamma}.$$

*Proof.* The homotopy inverse limit is in this case a finite limit, which will commute past a filtered colimit. □

**Definition 2.5.11.** We can now define the map

$$\eta \colon \Sigma^{n+k} K^{\Gamma,0}(\widetilde{M},R)^{h\Gamma} \longrightarrow K^{\Gamma}_{i}(\widetilde{N})^{\Gamma}$$

as the composition of $\Delta$ with $\widetilde{A}^{\Gamma}$.

The following variant of $\eta$ will be used in section 5.1.

**Definition 2.5.12.** The action of $\Gamma$ on $\widetilde{N}$ induces the isometric action on $T\widetilde{N}$. Theorem 8.3.9 gives a weak equivalence from $\Sigma K^{\Gamma}_{i}(\widetilde{N})^{\Gamma}$ to the fixed point spectrum $K^{\Gamma}_{i}(T\widetilde{N})^{\Gamma}$. The map

$$T\eta \colon \Sigma^{n+k+1} K^{\Gamma,0}(\widetilde{M},R)^{h\Gamma} \longrightarrow K^{\Gamma}_{i}(T\widetilde{N})^{\Gamma}$$

is the composition of $\Sigma\eta$ and this equivalence.

2.6. **Equivariant Fibred $G$-theory.** The proof of the Main Theorem will require the use of bounded $G$-theory which is defined for a proper metric space $M$ and a noetherian ring $R$ and relates to bounded $K$-theory in ways that are similar to the relationship between $G$-theory and $K$-theory of rings.

In this section we provide enough details of the theory to state the excision theorem that is needed in the course of the proof. The complete proof of the excision theorem is given in Part 3.

To motivate the basic construction first notice that, given a geometric module $F$ in $\mathcal{C}(M,R)$, to every subset $S \subset M$ there corresponds a direct summand which is the free submodule

$$F(S) = \bigoplus_{m \in S} F_m.$$

In this context we say an element $x \in F$ is *supported* on a subset $S$ if $x \in F(S)$. The restriction to bounded homomorphisms can be described entirely in terms of these submodules.

We generalize this as follows.

**Definition 2.6.1.** The objects of the new category $\mathbf{B}(M,R)$ are left $R$-modules $F$ filtered by the subsets of $X$ in the sense that they are functors from subsets of $X$ to submodules of $F$ ordered by inclusion so that the value on the whole space $M$ is the whole object $F$ and the empty set to 0. These functors will need to satisfy certain conditions in order for the bounded $K$-theory to have good localization properties. We will specify them shortly. By abuse of notation we usually denote the functor by the same letter $F$.

The morphisms are the $R$-homomorphisms $\phi\colon F_1 \to F_2$ for which there exists a number $D \geq 0$ such that the image $\phi(F_1(S))$ is contained in the submodule $F_2(S[D])$ for all subsets $S \subset M$. Here $S[D]$ stands for the metric $D$-enlargement of $S$ in $M$.

The filtrations can be subject to several constraints that we explain next. The following conditions may be imposed on a given filtered object:

- $F$ is *locally finite*, that is, $F(V)$ is a finitely generated submodule of $F$ whenever $V$ is a bounded subset of $M$,
- $F$ is *lean*, that is, there is a number $D \geq 0$ such that for every subset $S$ of $M$
$$F(S) \subset \sum_{x \in S} F(x[D]),$$
where $x[D]$ is the metric ball of radius $D$ centered at $x$,
- $F$ is *insular*, that is, there is a number $d \geq 0$ such that
$$F(T) \cap F(U) \subset F(T[d] \cap U[d])$$
for every pair of subsets $T$, $U$ of $M$.

If $T$ is a subset of $X$, the module $F_T = F(T)$ can be viewed as a filtered module with the induced filtration $F_T(S) = F(T) \cap F(S)$. Consider the family of filtered modules $F$ which are *strict* in the sense that $F(S)$ is locally finite, lean, and insular for each subset $S$ of $X$. The objects of $\mathbf{B}(M,R)$ are those filtered modules that are isomorphic to strict objects

In this context, we say a subobject $G \subset F$ is *supported on a subset* $S \subset M$ if $G$ factors through the subobject $F(S)$.

In order to describe a new, nonsplit Quillen exact structure on the additive category $\mathbf{B}(M,R)$, we define an additional property a boundedly controlled homomorphism $\phi\colon F_1 \to F_2$ in $\mathbf{B}(M,R)$ may or may not have.

**Definition 2.6.2.** We call $\phi$ *boundedly bicontrolled* if in addition to containments

$$\phi(F_1(S)) \subset F_2(S[D])$$

as above, there are containments

$$\phi(F_1) \cap F_2(S) \subset \phi F_1(S[D])$$

for all subsets $S$ of $M$.

The admissible monomorphisms consist of boundedly bicontrolled injections of modules with cokernels in $\mathbf{B}(M,R)$. The admissible epimorphisms are the boundedly bicontrolled surjections with kernels in $\mathbf{B}(M,R)$. The exact sequences in $\mathbf{B}(M,R)$ are the short exact sequences of boundedly bicontrolled injections and boundedly bicontrolled surjections. Notice that split injections and surjections are boundedly bicontrolled, so $\mathcal{C}(M,R)$ is an exact subcategory of $\mathbf{B}(M,R)$. This theory is functorial in the space variable $M$ with respect to coarse maps.

Two properties of filtered modules are weakenings of leanness and insularity.

**Definition 2.6.3.** An $M$-filtered $R$-module $F$ is *split* or *$D'$-split* if there is a number $D' \geq 0$ such that we have

$$F(S) \subset F(T[D']) + F(U[D'])$$

whenever a subset $S$ of $M$ is written as a union $T \cup U$. A $D$-lean filtered module is clearly $D$-split.

An $M$-filtered module $F$ is *separated* or *$d'$-separated* if there is a number $d' \geq 0$ such that

$$F(S) \cap F(U) = 0$$

whenever $S[d'] \cap U = \emptyset$ for a pair of subsets $S$, $U$ of $M$. A $d$-insular module is clearly $2d$-separated.

The following fact is immediate.

**Lemma 2.6.4.** *An $M$-filtered module $F$ is separated if and only if there is a number $d' \geq 0$ such that*

$$F(S) \cap F(\complement S[d']) = 0$$

*for any subset $S$ of $M$. Here $\complement S[d']$ stands for the complement of $S[d']$ in $M$.*

Now suppose we are given two proper metric spaces $X$ and $Y$. We will describe a fibred analogue $\mathbf{G}_X(Y)$ of the category $\mathbf{B}(M,R)$. The complete details of the construction and proofs of properties of $\mathbf{G}_X(Y)$ are postponed to 9.2 and subsequent sections.

**Definition 2.6.5.** An *$(X,Y)$-filtration* of an $R$-module $F$ is a functor from the power set of the product $X \times Y$ to the partially ordered family of $R$-submodules of $F$. The associated $X$-filtered $R$-module $F_X$ is given by

$$F_X(S) = F(S \times Y).$$

Similarly, for each subset $S \subset X$, one has the $Y$-filtered $R$-module $F^S$ given by

$$F^S(T) = F(S \times T).$$

Given a subset $U$ of $X \times Y$, a number $K \geq 0$ and a function $k\colon X \to [0, +\infty)$, we define the subset

$$U[K,k] = \{(x,y) \in X \times Y \mid \text{there is } (x',y) \in U[k] \text{ with } d(x,x') \leq K\}.$$

The following conditions may be imposed on $(X,Y)$-filtered modules:

- $F(S \times T)$ is a finitely generated submodule whenever both subsets $S$ and $T$ are bounded,
- $F$ is *split* if there is a number $D' \geq 0$ and a monotone function $\Delta'\colon [0,+\infty) \to [0,+\infty)$ so that
$$F(U_1 \cup U_2) \subset F(U_1[D', \Delta'_{x_0}]) + F(U_2[D', \Delta'_{x_0}])$$
for each pair of subsets $U_1$ and $U_2$ of $X \times Y$,
- $F$ is *lean/split* if there is a number $D \geq 0$ and a monotone function $\Delta'\colon [0,+\infty) \to [0,+\infty)$ so that
  - the $X$-filtered module $F_X$ is $D$-lean, while
  - the $(X,Y)$-filtered module $F$ is $(D,\Delta')$-split,
- $F$ is *insular* if there is a number $d \geq 0$ and a monotone function $\delta\colon [0,+\infty) \to [0,+\infty)$ so that
$$F(U_1) \cap F(U_2) \subset F\big(U_1[d,\delta_{x_0}] \cap U_2[d,\delta_{x_0}]\big)$$
for each pair of subsets $U_1$ and $U_2$ of $X \times Y$.

An $R$-linear homomorphism $f\colon F \to G$ between $(X,Y)$-filtered modules is *boundedly controlled* if there is a number $b \geq 0$ and a function $\theta\colon \mathcal{B}(X) \to [0,+\infty)$ such that

$$fF(S,T) \subset G(S[b], T[\theta(S)]) \qquad (\dagger)$$

for all pairs of bounded subsets $S \subset X$, $T \subset Y$.

The objects of the category $\mathbf{B}_X(Y)$ are defined as the locally finite, lean/split, insular $(X,Y)$-filtered $R$-modules. The morphisms are the boundedly controlled homomorphisms.

Given an object $F$ of $\mathbf{B}_X(Y)$, a *grading* of $F$ is a functor

$$\mathcal{F}\colon \mathcal{P}(X,Y) \longrightarrow \mathcal{I}(F)$$

with the properties

(1) if $\mathcal{F}(\mathcal{C})$ is given the standard filtration, it is an object of $\mathbf{B}_X(Y)$,
(2) there is an enlargement data $(K,k)$ such that
$$F(C) \subset \mathcal{F}(C) \subset F(C[K, k_{x_0}]),$$
for all subsets $C$ of $(X,Y)$.

We define $\mathbf{G}_X(Y)$ as the full subcategory of $\mathbf{B}_X(Y)$ on graded filtered modules. There is a slight difference between this definition and the partial $Y$-gradings defined in 9.4.4 which suffice for all purposes in this paper. We disregard the distinction in this part and refer to section 9.4 for complete details.

A homomorphism $f\colon F \to G$ is *boundedly bicontrolled* if in addition to ($\dagger$) one also has

$$fF \cap G(S,T) \subset fF(S[b], T[\theta(S)]).$$

The exact sequences in $\mathbf{G}_X(Y)$ are the short exact sequences of boundedly bicontrolled injections and boundedly bicontrolled surjections. This gives an exact structure for $\mathbf{G}_X(Y)$.

Notice that $\mathcal{C}_X(Y)$ is again an exact subcategory of $\mathbf{G}_X(Y)$. In particular, when $X$ is the group $\Gamma$ with a word-length metric, we have the exact inclusion of $\mathcal{C}_\Gamma(Y)$ in $\mathbf{G}_\Gamma(Y)$.

Given a proper metric space $Y$ with a left $\Gamma$-action by either isometries or coarse equivalences, there are $G$-theory analogues of equivariant theories $K_i^\Gamma$, $K_c^\Gamma$, $K_p^\Gamma$ from Definition 2.5.6. We are specifically interested in the analogue of $K_p^\Gamma$.

**Definition 2.6.6** (Coarse Equivariant Theories, continued)**.** Suppose $\Gamma$ acts on $Y$ by coarse equivalences.

(4) $g_p^\Gamma(Y)$ is defined as the $K$-theory spectrum of the exact category $\mathbf{G}_p^\Gamma(Y)$ of functors
$$\theta\colon \boldsymbol{E\Gamma} \longrightarrow \mathbf{G}_\Gamma(Y)$$
such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of $\Gamma$-filtered modules.

The nonconnective equivariant $G$-theory spectrum $G_p^\Gamma(Y)$ is a nonconnective delooping of $g_p^\Gamma(Y)$. For details we refer to sections 9.7 and 9.8.

The natural inclusions $\mathcal{C}_p^\Gamma(Y) \to \mathbf{G}_p^\Gamma(Y)$ induce maps between diagrams involved in delooping $k_p^\Gamma(Y)$ and $g_p^\Gamma(Y)$, so we obtain the induced equivariant maps between the colimits
$$\kappa\colon K_p^\Gamma(Y) \longrightarrow G_p^\Gamma(Y).$$

**Definition 2.6.7.** The induced map of fixed points
$$\kappa^\Gamma\colon K_p^\Gamma(Y)^\Gamma \longrightarrow G_p^\Gamma(Y)^\Gamma$$
is the *fibred Cartan map*.

Let $Y'$ be a subset of $Y$.

**Definition 2.6.8.** $\mathbf{G}_\Gamma(Y)_{<Y'}$ denotes the full subcategory of $\mathbf{G}_\Gamma(Y)$ on objects $F$ such that there is a number $k \geq 0$ and an order preserving function $\lambda\colon \mathcal{B}(\Gamma) \to [0,+\infty)$ such that
$$F(S) \subset F(S[k])(C[\lambda(S)])$$
for each bounded subset $S \subset \Gamma$. This is a right filtering Grothendieck subcategory. In particular, there is a quotient exact category $\mathbf{G}_\Gamma(Y)/\,\mathbf{G}_\Gamma(Y)_{<Y'}$ which we denote by $\mathbf{G}_\Gamma(Y,Y')$. For details we refer to section 9.5.

**Definition 2.6.9.** Given a proper metric space $Y$ with a left $\Gamma$-action, a subset $Y'$ is called *coarsely* $\Gamma$-*invariant* or simply *coarsely invariant* if for each element $\gamma$ of $\Gamma$ there is a number $t(\gamma)$ with
$$\gamma \cdot Y' \subset Y'[t(\gamma)].$$
The subset $Y'$ is further $\Gamma$-*invariant* if the function $t(\gamma) = 0$.

The following is now clear.

**Proposition 2.6.10.** *If $Y'$ is coarsely invariant, the subcategory $\mathbf{G}_\Gamma(Y)_{<Y'}$ is invariant under the action of $\Gamma$ on $\mathbf{G}_\Gamma(Y)$, so there is a left $\Gamma$-action on the quotient $\mathbf{G}_\Gamma(Y)/\,\mathbf{G}_\Gamma(Y)_{<Y'}$, and one obtains the equivariant relative theory $G_p^\Gamma(Y,Y')$.*

2.7. **Excision Theorems.** The following theorem is the basic computational device in bounded $K$-theory.

**Theorem 2.7.1** (Bounded Excision [9, 8]). *For every pair of subsets $U_1$ and $U_2$ of $Y$, using the notation from Theorem 1.1.6, the commutative square*

$$\begin{tikzcd} K^{-\infty}(Y)_{<U_1,U_2} \arrow[r] \arrow[d] & K^{-\infty}(Y)_{<U_1} \arrow[d] \\ K^{-\infty}(Y)_{<U_2} \arrow[r] & K^{-\infty}(Y) \end{tikzcd}$$

*is a homotopy pushout.*

In order to be able to restate the Bounded Excision Theorem in a more intrinsic form, we need to restrict to a special class of coverings.

**Definition 2.7.2.** A pair of subsets $S$, $T$ of a proper metric space $X$ is called *coarsely antithetic* if $S$ and $T$ are proper metric subspaces with the subspace metric and for each number $K > 0$ there is a number $K_1 > 0$ so that

$$S[K] \cap T[K] \subset (S \cap T)[K_1].$$

Examples of coarsely antithetic pairs include any two nonvacuously intersecting closed subsets of a simplicial tree as well as complementary closed half-spaces in a Euclidean space.

**Theorem 2.7.3.** *If $U_1$ and $U_2$ is a coarsely antithetic pair of subsets of $Y$ which form a cover of $Y$, then the commutative square*

$$\begin{tikzcd} K^{-\infty}(U_1 \cap U_2) \arrow[r] \arrow[d] & K^{-\infty}(U_1) \arrow[d] \\ K^{-\infty}(U_2) \arrow[r] & K^{-\infty}(Y) \end{tikzcd}$$

*is a homotopy pushout.*

We now develop some excision results for the fixed points of specific actions of $\Gamma$ on $Y$ for the theories $K_i^{\Gamma}(Y)$ and $G_p^{\Gamma}(Y)$.

**Theorem 2.7.4.** *Suppose the action of $\Gamma$ on $Y$ is trivial. If $U_1$ and $U_2$ is a coarsely antithetic pair of subsets of $Y$ which form a cover of $Y$, then the commutative square*

$$\begin{tikzcd} K_i^{\Gamma}(U_1 \cap U_2)^{\Gamma} \arrow[r] \arrow[d] & K_i^{\Gamma}(U_1)^{\Gamma} \arrow[d] \\ K_i^{\Gamma}(U_2)^{\Gamma} \arrow[r] & K_i^{\Gamma}(Y)^{\Gamma} \end{tikzcd}$$

*is a homotopy pushout.*

The proof is given in section 8, as Theorem 8.3.5.

We will require relative versions of the excision theorems. In the presence of a Karoubi filtration, one has the notion of Karoubi quotient summarized in Definition 8.2.1 and applied to bounded categories in Definition 8.3.1.

**Definition 2.7.5.** The quotient map of categories induces the equivariant map

$$K_i^\Gamma(Y) \longrightarrow K_i^\Gamma(Y, Y')$$

and the map of fixed points

$$K_i^\Gamma(Y)^\Gamma \longrightarrow K_i^\Gamma(Y, Y')^\Gamma.$$

More generally, if $Y''$ is another coarsely invariant subset of $Y$, then the intersection $Y'' \cap Y'$ is coarsely invariant in both $Y$ and $Y'$, there is an equivariant map

$$K_i^\Gamma(Y'', Y'' \cap Y') \longrightarrow K_i^\Gamma(Y, Y')$$

and the map of fixed points

$$K_i^\Gamma(Y'', Y'' \cap Y')^\Gamma \longrightarrow K_i^\Gamma(Y, Y')^\Gamma.$$

We also obtain the spectra $K_c^\Gamma(Y, Y')$, and $K_p^\Gamma(Y, Y')$, and equivariant maps just as above.

Here is the first relative equivariant analogue of Theorem 2.7.4.

**Theorem 2.7.6.** *Suppose the action of $\Gamma$ on $Y$ is trivial. If $Y'$ is a coarsely invariant subset of $Y$ and if $\{C_1, C_2\}$ is a covering of $Y$ by two coarsely invariant subsets such that the three subsets, $Y'$, $C_1$ and $C_2$, are pairwise coarsely antithetic, then the commutative square*

$$\begin{tikzcd}
K_i^\Gamma(C_1 \cap C_2, Y' \cap C_1 \cap C_2)^\Gamma \arrow[r] \arrow[d] & K_i^\Gamma(C_1, Y' \cap C_1)^\Gamma \arrow[d] \\
K_i^\Gamma(C_2, Y' \cap C_2)^\Gamma \arrow[r] & K_i^\Gamma(Y, Y')^\Gamma
\end{tikzcd}$$

*induced by inclusions of pairs is a homotopy pushout.*

In order to extend the equivariant excision theorems to appropriate nontrivial actions, we need to develop constructions related to coverings of $Y$.

**Definition 2.7.7.** A pair of subsets $A$, $B$ in a proper metric space $X$ are called *coarsely equivalent* if there are numbers $d_{A,B}$, $d_{B,A}$ with $A \subset B[d_{A,B}]$ and $B \subset A[d_{B,A}]$. It is clear this is an equivalence relation among subsets. We will use notation $A \parallel B$ for this equivalence.

A family of subsets $\mathcal{A}$ is called *coarsely saturated* if it is maximal with respect to this equivalence relation. Given a subset $A$, let $\mathcal{S}(A)$ be the smallest boundedly saturated family containing $A$.

If $\mathcal{A}$ is a coarsely saturated family, define $K(\mathcal{A})$ to be

$$\underset{\overrightarrow{A \in \mathcal{A}}}{\operatorname{hocolim}}\, K(A).$$

**Proposition 2.7.8.** *If $A$ and $B$ are coarsely equivalent then $\mathcal{C}(A) \cong \mathcal{C}(B)$ and so $K(A) \simeq K(B)$. For all subsets $A$,*

$$\mathcal{C}(X)_{<A} \cong \mathcal{C}(A)$$

*and*

$$K(X)_{<A} \simeq K(\mathcal{S}(A)) \simeq K(A).$$

**Definition 2.7.9.** A collection of subsets $\mathcal{U} = \{U_i\}$ is a *coarse covering* of $X$ if $X = \bigcup S_i$ for some $S_i \in \mathcal{S}(U_i)$. Similarly, $\mathcal{U} = \{\mathcal{A}_i\}$ is a *coarse covering* by coarsely saturated families if for some (and therefore any) choice of subsets $A_i \in \mathcal{A}_i$, $\{A_i\}$ is a coarse covering in the sense above.

Recall that a pair of subsets $A$, $B$ in a proper metric space $X$ are *coarsely antithetic* if for any two numbers $d_1$ and $d_2$ there is a third number $d$ such that

$$A[d_1] \cap B[d_2] \subset (A \cap B)[d].$$

We will write $A \natural B$ to indicate that $A$ and $B$ are coarsely antithetic.

Given two subsets $A$ and $B$, define

$$\mathcal{S}(A, B) = \{A' \cap B' \mid A' \in \mathcal{S}(A), B' \in \mathcal{S}(B), A' \natural B'\}.$$

**Proposition 2.7.10.** *$\mathcal{S}(A, B)$ is a coarsely saturated family.*

*Proof.* Suppose $A_1$, $A_1'$ and $A_2$, $A_2'$ are two coarsely antithetic pairs, and $A_1 \subset A_2[d_{12}]$, $A_1' \subset A_2'[d_{12}']$ for some $d_{12}$ and $d_{12}'$. Then

$$A_1 \cap A_1' \subset A_2[d_{12}] \cap A_2'[d_{12}'] \subset (A_2 \cap A_2')[d]$$

for some $d$. $\square$

**Proposition 2.7.11.** *If $U$ and $T$ are coarsely antithetic then*

$$K(X)_{<U,T} \simeq K(\mathcal{S}(U \cap T)) \simeq K(U \cap T).$$

There is the obvious generalization of the constructions and propositions to the case of a finite number of subsets of $X$.

**Definition 2.7.12.** We write $A_1 \natural A_2 \natural \ldots \natural A_k$ if for arbitrary $d_i$ there is a number $d$ so that

$$A_1[d_1] \cap A_2[d_2] \cap \ldots \cap A_k[d_k] \subset (A_1 \cap A_2 \cap \ldots \cap A_k)[d]$$

and define

$$\mathcal{S}(A_1, A_2, \ldots, A_k) = \{A_1' \cap A_2' \cap \ldots \cap A_k' \mid A_i' \in \mathcal{S}(A_i), A_1 \natural A_2 \natural \ldots \natural A_k\}.$$

Equivalently, identifying any coarsely saturated family $\mathcal{A}$ with $\mathcal{S}(A)$ for $A \in \mathcal{A}$, one has $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$.

We will refer to $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ as the coarse intersection of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$. A coarse covering $\mathcal{U}$ is *closed under coarse intersections* if all coarse intersections $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ are nonempty and are contained in $\mathcal{U}$. If $\mathcal{U}$ is a given coarse covering, the smallest coarse covering that is closed under coarse intersections and contains $\mathcal{U}$ will be called the *closure* of $\mathcal{U}$ under coarse intersections.

The inclusions induce the diagrams of spectra $\{K(Y)_{<A}\}$ and $\{K_i^\Gamma(Y, Y')^\Gamma_{<A}\}$ for representatives $A$ in $\mathcal{A} \in \mathcal{U}$.

**Definition 2.7.13.** Suppose $\mathcal{U}$ is a coarse covering of $Y$ closed under coarse intersections. We define the homotopy pushouts

$$\mathcal{K}(Y; \mathcal{U}) = \underset{\overrightarrow{\mathcal{A} \in \mathcal{U}}}{\operatorname{hocolim}}\ K(Y)_{<A}$$

and

$$\mathcal{K}_i^\Gamma(Y, Y'; \mathcal{U})^\Gamma = \underset{\overrightarrow{\mathcal{A} \in \mathcal{U}}}{\operatorname{hocolim}}\ K_i^\Gamma(Y, Y')^\Gamma_{<A}.$$

The following result is equivalent to the Bounded Excision Theorem.

**Theorem 2.7.14.** *If $\mathcal{U}$ is a coarse covering of $Y$ closed under coarse intersections, then there is a weak equivalence*

$$\mathcal{K}(Y;\mathcal{U}) \simeq K(Y).$$

*Proof.* Apply Theorem 2.7.1 inductively to the sets in $\mathcal{U}$. $\square$

Now suppose there is a free action of $\Gamma$ on $Y$ by isometries and $\mathcal{U}$ is a coarse covering of $Y$ closed under coarse intersections.

**Definition 2.7.15.** The action is *$\mathcal{U}$-bounded* if all coarse families $\{\mathcal{A}_i\}$ in $\mathcal{U}$ are closed under the action. In this case one has the $K$-theory spectra $K^\Gamma(\mathcal{A}_i) = K^\Gamma(Y;\mathcal{A}_i)$ and the fixed point spectra $K^\Gamma(\mathcal{A}_i)^\Gamma = K^\Gamma(Y;\mathcal{A}_i)^\Gamma$.

An action is $\mathcal{U}$-bounded for any coarse covering $\mathcal{U}$ if the action is by bounded coarse equivalences. The trivial action is an instance of such action.

**Theorem 2.7.16.** *If the action of $\Gamma$ on $Y$ is trivial, then*

$$\mathcal{K}^\Gamma(Y;\mathcal{U})^\Gamma = \underset{\overrightarrow{\mathcal{A}\in\mathcal{U}}}{\operatorname{hocolim}}\ K^\Gamma(\mathcal{A})^\Gamma \simeq K^\Gamma(Y)^\Gamma$$

*and*

$$\mathcal{K}^\Gamma_i(Y,Y';\mathcal{U})^\Gamma = \underset{\overrightarrow{\mathcal{A}\in\mathcal{U}}}{\operatorname{hocolim}}\ K^\Gamma_i(\mathcal{A})^\Gamma \simeq K^\Gamma_i(Y,Y')^\Gamma.$$

Similar constructions are available in $G$-theory. The quotient map of exact categories induces the equivariant map

$$G^\Gamma_p(Y) \longrightarrow G^\Gamma_p(Y,Y')$$

and the map of fixed points

$$G^\Gamma_p(Y)^\Gamma \longrightarrow G^\Gamma_p(Y,Y')^\Gamma.$$

If $Y''$ is a coarsely invariant subset of $Y$, there is an equivariant map

$$G^\Gamma_p(Y'',Y''\cap Y') \longrightarrow G^\Gamma_p(Y,Y')$$

and the map of fixed points

$$G^\Gamma_p(Y'',Y''\cap Y')^\Gamma \longrightarrow G^\Gamma_p(Y,Y')^\Gamma.$$

We can define two types of homotopy pushouts:

$$\mathcal{G}^\Gamma_p(Y,Y')^\Gamma_{<\mathcal{U}} = \underset{\overrightarrow{A\in\mathcal{U}}}{\operatorname{hocolim}}\ G^\Gamma_p(Y,Y')^\Gamma_{<A}$$

for an arbitrary action of $\Gamma$ by bounded coarse equivalences, and

$$\mathcal{G}^\Gamma_p(Y,Y';\mathcal{U})^\Gamma = \underset{\overrightarrow{A\in\mathcal{U}}}{\operatorname{hocolim}}\ G^\Gamma_p(A,A\cap Y')^\Gamma$$

for the trivial action. In case of the trivial action, we get the same excision result as in $K$-theory. However, bounded $G$-theory possesses excision properties also with respect to some more general, not necessarily trivial actions.

The main theorem of section 9, which is Theorem 9.9.10, will establish the following.

**Theorem 2.7.17.** *Suppose the action of $\Gamma$ on $Y$ is by bounded coarse equivalences. Given a finite coarse covering $\mathcal{U}$ of $Y$ such that the family of all subsets $U \in \mathcal{U}$ and $Y'$ together are pairwise coarsely antithetic, there is a weak equivalence*

$$G^{\Gamma}_{p}(Y,Y')^{\Gamma} \simeq \mathcal{G}^{\Gamma}_{p}(Y,Y')^{\Gamma}_{<\mathcal{U}}.$$

*If the action of $\Gamma$ is in fact trivial, then there is a weak equivalence*

$$G^{\Gamma}_{p}(Y,Y')^{\Gamma} \simeq \mathcal{G}^{\Gamma}_{p}(Y,Y';\mathcal{U})^{\Gamma}.$$

2.8. **Cartan Equivalence.** If the ring $\Lambda$ is a noetherian ring then the category of finitely generated $\Lambda$-modules $\mathbf{Modf}(\Lambda)$ is an abelian category, and the $G$-theory of $\Lambda$ is the Quillen $K$-theory of $\mathbf{Modf}(\Lambda)$. However, there are few group rings $R[\Gamma]$ which are noetherian. For non-noetherian rings, the family of all short exact sequences still serves as a canonical choice for an exact structure in $\mathbf{Modf}(\Lambda)$. In either case, the split exact sequences are exact.

The term *Cartan map* usually refers to the map from $K$-theory of the ring, which is the $K$-theory of the split exact structure in the additive category of finitely generated free $\Lambda$-modules $\mathbf{Freef}(\Lambda)$, to its $G$-theory induced by the inclusion of split exact additive structure in $\mathbf{Modf}(\Lambda)$ as above. For regular noetherian rings this map is known to be an equivalence.

Given a finitely generated group $\Gamma$ and a noetherian ring $R$, there is another choice for exact structure in $\mathbf{Modf}(R[\Gamma])$ which we define in section 10 and call $\mathbf{B}(R[\Gamma])$. We refer to its $K$-theory spectrum as $G^{-\infty}(R[\Gamma])$. There are exact inclusions of categories

$$\mathbf{Freef}(R[\Gamma]) \longrightarrow \mathbf{B}(R[\Gamma]) \longrightarrow \mathbf{Modf}(R[\Gamma]).$$

The following theorem is a summary of main results of section 10.

**Theorem 2.8.1.** *Let $G^{\Gamma,0}(\Gamma,R)$ be the equivariant spectrum $G^{\Gamma}_{p}(\mathrm{point})$ which is a special case of Definition 2.6.6. There is a weak equivalence*

$$G^{-\infty}(R[\Gamma]) \simeq G^{\Gamma,0}(\Gamma,R)^{\Gamma}$$

*when $\Gamma$ is a finitely generated torsion-free group and $R$ is a noetherian ring. If, in addition, $\Gamma$ has finite asymptotic dimension and $R$ has finite homological dimension then there is a weak equivalence*

$$G^{-\infty}(R[\Gamma]) \simeq K^{\Gamma,0}(\Gamma,R)^{\Gamma}.$$

*Proof.* For the first statement, see Corollary 10.1.13. The second statement is part (5) of Theorem 10.2.2. □

## 3. A Sketch of the Proof

This description of the proof serves to isolate the main idea that would be harder to parse otherwise. It might be helpful to refer to the Abbreviated Diagram in section 3.4 as a general guide to the argument. We will return to verifying the claims made here in sections 4 and 5.

3.1. **Review of the Strategy.** We start by recapitulating the course of the proof in terms of the prerequisites already assembled in section 2.

As explained in section 1.2, the object of study is the canonical map

$$\rho \colon K^{\Gamma,0}(\widetilde{M}, R)^{\Gamma} \longrightarrow K^{\Gamma,0}(\widetilde{M}, R)^{h\Gamma}.$$

Here $K^{\Gamma,0}(\widetilde{M}, R)$ is the nonconnective equivariant bounded $K$-theory of the universal cover of a smooth closed aspherical manifold $M$ with fundamental group $\Gamma$. Our goal is to show that $\rho$ is a split injection.

We accomplish this goal by constructing a nonconnective spectrum $\mathcal{S}$ and a map of spectra

$$f \colon K^{\Gamma,0}(\widetilde{M}, R)^{\Gamma} \longrightarrow \mathcal{S}$$

such that

- $f$ factors through the map $\rho$, and
- $f$ is a weak equivalence.

The combination of these properties shows that $\rho$ is an injection split by the second map in the factorization of $f$.

Such $\mathcal{S}$ would then play the role of the delooping $\Omega^{n+k+1}\,\mathcal{T}$ of the spectrum $\mathcal{T}$ proposed in the proof of the Main Theorem in section 1.2.

3.2. **Construction of $\mathcal{S}$.** We proceed to define the category $\mathbf{W}^{\Gamma}$ announced in the sketch of the proof of the Main Theorem in section 1.2, specifically in Remark 1.2.2. The construction of this category and formal properties resemble the category of fixed objects $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$ while the action is not uniquely specified.

First, we recapitulate the main points in the construction of $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$ and the spectrum $G^{\Gamma}_p(Y, Y')^{\Gamma}$ in a revisionist way that can be generalized.

Suppose $\Gamma$ acts on the proper metric space $Y$ by coarse equivalences and $Y'$ is a coarsely invariant subspace. The objects of $\mathbf{G}^{\Gamma}_p(Y, Y')$ are the functors $\theta \colon \boldsymbol{E\Gamma} \to \mathbf{G}_{\Gamma}(Y, Y')$ such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of $\Gamma$-filtered modules. Recall that this category has the left action by $\Gamma$ induced from the diagonal action on $\Gamma \times Y$. This is the category used in the description of $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$. So, again, a fixed object in $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$ is determined by an object $F$ of $\mathbf{G}_{\Gamma}(Y, Y')$ and isomorphisms $\psi(\gamma) \colon F \to \gamma F$ which are of filtration 0 when projected to $\Gamma$. We will exploit the fact that this category and its exact structure can be described independently from the construction of the equivariant functor category $\mathbf{G}^{\Gamma}_p(Y, Y')$. The spectrum $g^{\Gamma}_p(Y, Y')^{\Gamma}$ can be defined as the $K$-theory spectrum of $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$.

Let $\mathbb{R}^k$ be the Euclidean space with the trivial action of $\Gamma$. Then the product $\Gamma \times \mathbb{R}^k$ has the $\Gamma$-action defined by $\gamma(\gamma', x) = (\gamma\gamma', x)$. By using the diagonal actions on $\Gamma \times \mathbb{R}^k \times Y$, one obtains the equivariant categories $\mathbf{G}_{\Gamma\times\mathbb{R}^k}(Y, Y')$ and also $\mathbf{G}^{\Gamma,k}_p(Y, Y')$ where the objects are the functors $\theta \colon \boldsymbol{E\Gamma} \to \mathbf{G}_{\Gamma\times\mathbb{R}^k}(Y, Y')$ such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of $(\Gamma \times \mathbb{R}^k)$-filtered modules. The $K$-theory of $\mathbf{G}^{\Gamma,k}_p(Y, Y')^{\Gamma}$ is denoted by $g^{\Gamma,k}_p(Y, Y')^{\Gamma}$. Now the nonconnective delooping of $g^{\Gamma}_p(Y, Y')^{\Gamma}$ can be constructed as

$$G^{\Gamma}_p(Y, Y')^{\Gamma} \;=\; \underset{\substack{\longrightarrow \\ k>0}}{\operatorname{hocolim}}\; \Omega^k g^{\Gamma,k}_p(Y, Y')^{\Gamma}.$$

The following is the construction of $\mathbf{W}^{\Gamma}(Y,Y')$ and the corresponding nonconnective spectrum $W^{\Gamma}(Y,Y')$. The details are mostly straightforward generalizations of the summary above. It should be helpful to point out that the category $\mathbf{W}^{\Gamma}(Y,Y')$ itself is not a lax limit with respect to an action of $\Gamma$. However lax limits such as $\mathbf{G}^{\Gamma}_p(Y,Y')^{\Gamma}$ are going to be exact subcategories of $\mathbf{W}^{\Gamma}(Y,Y')$.

**Definition 3.2.1.** The category $\mathbf{W}^{\Gamma}(Y,Y')$ has objects which are sets of data $(\{F_\gamma\},\alpha,\{\psi_\gamma\})$ where

- $F_\gamma$ is an object of $\mathbf{G}_{\Gamma}(Y,Y')$ for each $\gamma$ in $\Gamma$,
- $\alpha$ is an action of $\Gamma$ on $Y$ by bounded coarse equivalences,
- $\psi_\gamma$ is an isomorphism $F_e \to F_\gamma$ induced from $\alpha_\gamma$,
- $\psi_\gamma$ has filtration 0 when viewed as a morphism in $\mathbf{U}(\Gamma,\mathbf{U}(Y))$,
- $\psi_e = \mathrm{id}$,
- $\psi_{\gamma_1\gamma_2} = \gamma_1\psi_{\gamma_2}\psi_{\gamma_1}$.

The morphisms $(\{F_\gamma\},\alpha,\{\psi_\gamma\}) \to (\{F'_\gamma\},\alpha',\{\psi'_\gamma\})$ are collections $\{\phi_\gamma\}$, where each $\phi_\gamma$ is a morphism $F_\gamma \to F'_\gamma$ in $\mathbf{G}_{\Gamma}(Y,Y')$, such that the squares

$$\begin{tikzcd} F_e \arrow[r, "\psi_\gamma"] \arrow[d, "\phi_e"'] & F_\gamma \arrow[d, "\phi_\gamma"] \\ F'_e \arrow[r, "\psi'_\gamma"] & F'_\gamma \end{tikzcd}$$

commute for all $\gamma \in \Gamma$.

The exact structure on $\mathbf{W}^{\Gamma}(Y,Y')$ is induced from that on $\mathbf{G}_{\Gamma}(Y,Y')$, described in 2.6.5. The details can be found in section 9.11. For any action $\alpha$ on $(Y,Y')$ by bounded coarse equivalences, the lax limit category $\mathbf{G}^{\Gamma}_p(Y,Y')^{\Gamma}_{\alpha}$ is an exact subcategory of $\mathbf{W}^{\Gamma}(Y,Y')$. This embedding $E_\alpha$ is realized by sending the object $(F,\psi)$ of $\mathbf{G}_{\Gamma}(Y,Y')^{\Gamma}_{\alpha}$ to $(\{\alpha_\gamma F\},\alpha,\{\psi(\gamma)\})$. On the morphisms, $E_\alpha(\phi) = \{\alpha_\gamma\phi\}$.

In particular, we have the embedding

$$E\colon\ \mathbf{G}^{\Gamma}_p((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^{\Gamma} \longrightarrow \mathbf{W}^{\Gamma}((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd}).$$

Let $W^{\Gamma}(Y,Y')$ be the nonconnective delooping of the $K$-theory of $\mathbf{W}^{\Gamma}(Y,Y')$.

**Definition 3.2.2.** Then we have the map of spectra

$$\varepsilon\colon G^{\Gamma}_p((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^{\Gamma} \longrightarrow W^{\Gamma}((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})$$

induced by $E$.

There is the familiar construction of the full subcategory $\mathbf{W}^{\Gamma}(Y,Y')_{<C}$ associated to any subset $C$ of $Y$. It will be shown that $\mathbf{W}^{\Gamma}(Y,Y')_{<C}$ is a Grothendieck subcategory of $\mathbf{W}^{\Gamma}(Y,Y')$, and there is a quotient exact category $\mathbf{W}^{\Gamma}(Y,Y')_{>C}$.

Suppose $\mathcal{U}$ is a finite coarse covering of $Y$. We define the homotopy colimit

$$\mathcal{W}^{\Gamma}(Y,Y')_{<\mathcal{U}} \;=\; \underset{\overrightarrow{U_i\in\mathcal{U}}}{\mathrm{hocolim}}\; W^{\Gamma}(Y,Y')_{<U_i}.$$

The following excision result is part of Corollary 9.11.8.

**Theorem 3.2.3.** *Consider the actions of $\Gamma$ on $Y$ is by bounded coarse equivalences. Suppose $\mathcal{U}$ is a finite coarse covering of $Y$ such that the family of all subsets $U$ in $\mathcal{U}$ together with $Y'$ are pairwise coarsely antithetic. Then the natural map*

$$\delta \colon \mathcal{W}^{\Gamma}(Y, Y')_{<\mathcal{U}} \longrightarrow W^{\Gamma}(Y, Y'),$$

*induced by inclusions, is a weak equivalence.*

We can finally describe the categories that play the roles of $\mathbf{W}^{\Gamma}$, $\mathcal{T}$, and $\mathcal{S}$.

**Definition 3.2.4.** Recall that the compact manifold $M$ is embedded in a Euclidean space $\mathbb{R}^{n+k}$, and we let $N$ denote a small tubular neighborhood of $M$. Choose a closed Euclidean ball $B$ in $N$. If $\widetilde{N}$ denotes the universal covering of $N$, let $\widehat{B}$ be an arbitrary lift of $B$ in $\widetilde{N}$ expressed by a homeomorphism $\sigma \colon B \to \widehat{B}$.

Using the bounded metric from section 2.2 and the cone construction from section 2.3, we define the category

$$\mathbf{W}^{\Gamma} = \mathbf{W}^{\Gamma}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{>\complement T\widehat{B}}$$

and the spectrum

$$\mathcal{T} = W^{\Gamma}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{>\complement T\widehat{B}}.$$

Therefore

$$\mathcal{S} = \Omega^{n+k+1} W^{\Gamma}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{>\complement T\widehat{B}}.$$

*Notation* 3.2.5. We will need to localize these constructions further to certain subsets of $T\widetilde{N}$. In order to clear the plate for further subscripts, we will use the following notation for the quotient category

$$\mathbf{W}^{\Gamma} = \mathbf{W}^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}),$$

and for the spectrum

$$\mathcal{T} = W^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}).$$

3.3. **The Core of the Proof.** We plan to construct a map of spectra

$$\phi \colon K^{\Gamma,0}(\widetilde{M}, R)^{h\Gamma} \longrightarrow \mathcal{S} = \Omega^{n+k+1} W^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

in section 5.1 and prove further in section 5 that the composition $f = \phi \circ \rho$ is a weak equivalence. In fact, $f$ will be identified with a different map that is ultimately computable.

The core of the proof is the essential idea behind the computation explained in this section in a sequence of steps. We refer to Figures 1–2 in section 3.4 for the assembly of the steps into a commutative diagram.

**Step 1.** Choose a metric ball $B$ in $\mathbb{R}^{n+k}$ centered at $c$ with radius $R$ and contained entirely in $N$. Let $h \colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ be the linear map $h(x) = c + Rx$, so $h$ restricts to a linear homeomorphism $h \colon D^{n+k} \to B$ from the unit disk $D^{n+k} = 0[1]$ onto the chosen ball $B$.

*Notation* 3.3.1. We identify the following subsets of $\mathbb{R}^{n+k}$:

$$\begin{aligned}
&E_i^+ = \{(x_1, \ldots, x_{n+k}) \mid x_l = 0 \text{ for all } l > i,\ x_i \geq 0\},\\
&E_i^- = \{(x_1, \ldots, x_{n+k}) \mid x_l = 0 \text{ for all } l > i,\ x_i \leq 0\},\\
&E_i = E_i^- \cup E_i^+, \text{ for } 1 \leq i \leq n+k, \text{ and}\\
&E_0 = E_0^- = E_0^+ = \{(0, \ldots, 0)\}.
\end{aligned}$$

**Definition 3.3.2.** The subsets $E_i^*$ form a coarsely antithetic covering of $\mathbb{R}^{n+k}$. It is easy to see that $TE_i^*$ form a coarsely antithetic covering of $T\mathbb{R}^{n+k}$. Therefore, we obtain a coarsely antithetic covering $\mathcal{E}$ of $T\mathbb{R}^{n+k}$ by the subsets $Th(E_i^*)$ . Notice that this covering is closed under coarse intersections since it includes the subsets $Th(E_i)$ for $0 \le i < n+k$, where $E_i$ are the intersections $E_{i+1}^- \cap E_{i+1}^+$. We define

$$\mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma \;=\; \underset{E_i^* \in \mathcal{E}}{\underrightarrow{\text{hocolim}}}\; K_i^\Gamma(Th(E_i^*))^\Gamma$$

where the structure maps in the diagram are induced by inclusions of subsets.

**Proposition 3.3.3.** *There is a weak equivalence*

$$\Sigma K_i^\Gamma(T\mathbb{R}^{d-1})^\Gamma \longrightarrow K_i^\Gamma(T\mathbb{R}^d)^\Gamma$$

*for each* $0 < d \le n+k$.

*Proof.* This follows from Theorem 2.7.6 applied to the coarsely antithetic covering of $T\mathbb{R}^d$ by $T(\mathbb{R}^{d-1} \times [0,+\infty))$ and $T(\mathbb{R}^{d-1} \times (-\infty,0])$. For either one of the two covering sets $C$, the category $\mathcal{C}^{\Gamma,0}(\Gamma \times C)^\Gamma$ is flasque, so $K_i^\Gamma(C)^\Gamma$ is contractible. $\square$

**Corollary 3.3.4.** *There is a weak equivalence*

$$\alpha_0''\colon \Sigma^{n+k+1} K^{-\infty}(R[\Gamma]) \longrightarrow \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma.$$

*Proof.* Use iterations of Proposition 3.3.3 and Proposition 2.3.4. $\square$

**Proposition 3.3.5.** *The natural map*

$$\alpha_0'''\colon \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma \longrightarrow K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma$$

*is a weak equivalence.*

*Proof.* Apply Theorem 2.7.16 to $\mathcal{E}$. $\square$

There are also related to $E_i^\pm$ subsets

$$\begin{aligned} &D_0 = D_0^- = D_0^+ = \{(0,\dots,0)\},\\ &D_i^\pm = E_i^\pm \cap D^{n+k}, \text{ for } 1 \le i \le n+k, \text{ and}\\ &D_i = D_i^- \cup D_i^+ = D_{i+1}^- \cap D_{i+1}^+. \end{aligned}$$

The images of $D_i^*$ under $h$ will be called $B_i^*$.

**Definition 3.3.6.** Let $\mathcal{V}_1$ be the coarse antithetic covering of $TB$ by $TB_i^*$. There is a spectrum defined as the homotopy pushout

$$\mathcal{K}_i^\Gamma(TB,T\partial B;\mathcal{V}_1)^\Gamma = \underset{\mathcal{V}_1}{\underrightarrow{\text{hocolim}}}\; K_i^\Gamma(TB_i^*,T(B_i^*\cap\partial B))^\Gamma.$$

In section 5.5 we define forget control equivalences

$$\psi_1^{i,*}\colon K_i^\Gamma(Th(E_i^*))^\Gamma \longrightarrow K_i^\Gamma(TB_i^*,T(B_i^*\cap\partial B))^\Gamma$$

There are two natural maps

$$\xi\colon \mathcal{K}_i^\Gamma(TB,T\partial B;\mathcal{V}_1)^\Gamma \longrightarrow K_i^\Gamma(TN,T\partial N)^\Gamma$$

induced by inclusions and

$$\psi_1\colon \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma \longrightarrow \mathcal{K}_i^\Gamma(TB,T\partial B;\mathcal{V}_1)^\Gamma$$

induced from $\psi_1^{i,*}$. Since all $\psi_1^{i,*}$ are equivalences, $\psi_1$ is an equivalence. It is part of the commutative square

$$\begin{tikzcd} K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma \arrow[r, "\alpha_1"] & K_i^\Gamma(TN, T\partial N)^\Gamma \\ \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma \arrow[u, "\simeq", "\alpha_0'''"'] \arrow[r, "\psi_1", "\simeq"'] & \mathcal{K}_i^\Gamma(TB, T\partial B;\mathcal{V}_1)^\Gamma \arrow[u, "\xi"'] \end{tikzcd}$$

The map $\alpha_1$ in the diagram is induced by the quotient map combined with excision. In particular, we now have the equivalence

$$\psi_1 \circ \alpha_0'' \colon \Sigma^{n+k+1} K^{-\infty}(R[\Gamma]) \xrightarrow{\simeq} \mathcal{K}_i^\Gamma(TB, T\partial B;\mathcal{V}_1)^\Gamma.$$

**Step 2.** The map of spectra

$$\varepsilon\colon G_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

from Definition 3.2.2 can be composed with

$$q\colon W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \longrightarrow W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

induced by the quotient map of categories.

There is a parametrized transfer map associated to a free properly discontinuous action on a proper metric space pair. It is systematically developed in section 4.1. In our situation it provides a map

$$\Lambda\colon K_i^\Gamma(TN, T\partial N)^\Gamma \longrightarrow K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma.$$

This construction allows to view the map $\Sigma^{n+k+1} f$ as the composition

$$\begin{aligned} \mathcal{K}_i^\Gamma(TB, T\partial B;\mathcal{V}_1)^\Gamma &\xrightarrow{\xi} K_i^\Gamma(TN, T\partial N)^\Gamma \xrightarrow{\Lambda} K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \\ &\xrightarrow{\varepsilon} W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \xrightarrow{q} W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}). \end{aligned}$$

The next step finally gives the interpretation of $\Sigma^{n+k+1} f$ alluded to in the first paragraph of this section 3.3.

**Step 3.** Consider the following metric subspaces of $(T\widetilde{N})^{bdd}$:

$$\begin{aligned} &V = (T\widehat{B})^{bdd},\ V' = (T\partial\widehat{B})^{bdd}, \\ &V_i^{\pm} = \big(T\sigma h(D_i^{\pm})\big)^{bdd},\ \text{ for } 0 \le i \le n+k, \text{ and} \\ &V_i = (T\sigma h(D_i))^{bdd}. \end{aligned}$$

**Definition 3.3.7.** Let $\mathcal{U}_3$ be the coarse antithetic covering of $T\widehat{B}$ by $V_i^*$. There is a homotopy pushout

$$\mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd};\mathcal{U}_3) = \underset{\overrightarrow{\mathcal{U}_3}}{\operatorname{hocolim}}\, W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<V_i^* \cup \complement T\widehat{B}}.$$

From Theorem 3.2.3 we have a weak equivalence

$$\delta\colon \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd};\mathcal{U}_3) \longrightarrow W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}).$$

The parametrized transfers

$$\lambda_i^*\colon K_i^\Gamma(TB, T\partial B)^\Gamma_{<B_i^*} \longrightarrow W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<V_i^*}$$

compatible with $q \circ \varepsilon \circ \Lambda$ induce the natural map of the homotopy pushouts

$$\mu \colon \mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \longrightarrow \mathcal{W}_{T\widehat{B}}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_3).$$

Now we have the commutative diagram $q \circ \varepsilon \circ \Lambda \circ \xi = \delta \circ \mu$.

**Step 4.** The map $\delta \circ \mu$ can be computed as follows. Each of the maps $\lambda_i^*$ is either a map between contractible spectra for $* = \pm$ or, because of the excision available at each end, the suspension of the analogous map

$$\lambda_{i-1} \colon K_i^\Gamma(TB, T\partial B)^\Gamma_{<B^*_{i-1}} \longrightarrow W_{T\widehat{B}}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<V_{i-1}},$$

for the empty index $*$. This shows that $\delta \circ \mu$ is the $(n+k+1)$-fold suspension of the "core" map

$$K_i^\Gamma(TB_0, T(B_0 \cap \partial B))^\Gamma \longrightarrow W_{T\widehat{B}}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<V_0}$$

which is the Cartan map

$$\begin{aligned} u \colon\ K^{-\infty}(R[\Gamma]) = K_i^\Gamma(\text{point})^\Gamma \longrightarrow W_{T\widehat{B}}^\Gamma&((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<V_0} \\ &= G_p^\Gamma(\text{point})^\Gamma = G^{-\infty}(R[\Gamma]). \end{aligned}$$

This gives simultaneous computations of

(1) the spectrum $\mathcal{S} = W_{T\widehat{B}}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$ as $\Sigma^{n+k+1} G^{-\infty}(R[\Gamma])$ and
(2) the map $\Sigma^{n+k+1} f$ as the $(n+k+1)$-fold suspension of the Cartan map.

**Step 5.** We finally refer to the Cartan equivalence from Theorem 2.8.1 for groups $\Gamma$ with finite asymptotic dimension and rings $R$ of finite homological dimension, which verifies that the Cartan map $u$ is a weak equivalence. This shows that $f$ is a weak equivalence.

3.4. **The Abbreviated Diagram.** The Abbreviated Diagram collects the maps we defined so far into one commutative diagram. It is presented in two parts which should make two separate points. Part (a) in Figure 1 illustrates that the composition of maps

$$\Sigma^{n+k+1}\, K^{-\infty}(R[\Gamma]) \longrightarrow W_{T\widehat{B}}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

factors through the homotopy fixed point spectrum $\Sigma^{n+k+1} K^{\Gamma,0}(\widetilde{M}, R)^{h\Gamma}$. Part (b) in Figure 2 shows that the same composition can be computed up to equivalence as the suspension map described in Step 4 above.

$$\begin{tikzcd}
 & \Sigma^{n+k+1} K^{\Gamma,0}(\widetilde{M}, R)^{\Gamma} \arrow[r, bend left, "\Sigma^{n+k+1}\rho"] \arrow[dd, "\simeq"] \arrow[drr, bend left, "\Sigma^{n+k+1} f"] & \Sigma^{n+k+1} K^{\Gamma,0}(\widetilde{M}, R)^{h\Gamma} \arrow[dr, "\Sigma^{n+k+1}\phi"] & \\
\Sigma^{n+k+1} K^{-\infty}(R[\Gamma]) \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & & W^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \\
 & K^{\Gamma}_{i}(T\mathbb{R}^{n+k})^{\Gamma} \arrow[r, "\alpha_1"] & K^{\Gamma}_{i}(TN, T\partial N)^{\Gamma} \arrow[ur, "q\circ\varepsilon\circ\Lambda"'] &
\end{tikzcd}$$

FIGURE 1. The Abbreviated Diagram. Part (a).

$$
\begin{tikzcd}
& K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma \arrow[r, "\alpha_1"] & K_i^\Gamma(TN, T\partial N)^\Gamma \arrow[r, "q\circ\varepsilon\circ\Lambda"] & W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \arrow[dr, bend left, "\simeq"] & \\
\Sigma^{n+k+1} K^{-\infty}(R[\Gamma]) \arrow[ur, bend left, "\simeq"] \arrow[dr, bend right, "\alpha_0''", "\simeq"'] & & & & \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]) \\
& \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma \arrow[uu, "\alpha_0''", "\simeq"'] \arrow[r, "\psi_1", "\simeq"'] & \mathcal{K}_i^\Gamma(TB, T\partial B;\mathcal{V}_1)^\Gamma \arrow[uu, "\xi"] \arrow[r, "\mu"] & \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd};\mathcal{U}_3)^\Gamma \arrow[uu, "\delta"] \arrow[ur, bend right, "\simeq"'] &
\end{tikzcd}
$$

Figure 2. The Abbreviated Diagram. Part (b).

## 4. Preliminaries (Part 2)

4.1. **Parametrized Transfer Map.** Let $Y$ be a proper metric space. Suppose $\Gamma$ is a finitely generated group which acts freely, properly discontinuously on $Y$. In this case, one has the orbit space metric on $\Gamma\backslash Y$ as in Definition 2.2.1.

Let $s\colon \Gamma\backslash Y \to Y$ be a section of the orbit space projection $p\colon Y \to \Gamma\backslash Y$. We will require $s$ to be a coarse map. For completeness, we provide an option for such map whenever $\Gamma\backslash Y$ is a finite complex.

**Lemma 4.1.1.** *Suppose $\Gamma\backslash Y$ is a finite simplicial complex $K$, then there is a section $s\colon \Gamma\backslash Y \to Y$ which is a coarse map.*

*Proof.* The complex $K$ is the union of finitely many open simplices which we denote $S_1, \ldots, S_t$. Choose arbitrary base points $b_i$ in $S_i$. Since each $S_i$ is contractible, an arbitrary map $s\colon \{b_1, \ldots, b_t\} \to Y$ extends uniquely to a section $s\colon K \to Y$. □

Let $d$ be the standard simplicial metric in $Y$ which gives the simplicial orbit metric $d_{\Gamma\backslash Y}$ in $K$.

**Definition 4.1.2.** The image of $s$, $F \subset Y$, is a bounded fundamental domain for the action of $\Gamma$, since it is a union of finitely many simplices.

Let $\epsilon \geq 0$ and let $\Omega_\epsilon$ be the subset of $\Gamma$ given by

$$\Omega_\epsilon = \{\gamma \in \Gamma \mid d(F, \gamma F) \leq \epsilon\}.$$

Since the action of $\Gamma$ is properly discontinuous, the set $\Omega_\epsilon$ is finite. Let $D_\epsilon$ be the maximal norm of an element in $\Omega_\epsilon$.

**Proposition 4.1.3.** *If $d_{\Gamma\backslash Y}(z_1, z_2) \leq \epsilon$ then there is an element $\omega$ of $\Omega_\epsilon$ such that $d(s(z_1), \omega s(z_2)) \leq \epsilon$.*

*Proof.* Let $\omega$ be the element of $\Gamma$ such that

$$d_{\Gamma\backslash Y}(z_1, z_2) = d(s(z_1), \Gamma s(z_2)) = d(s(z_1), \omega s(z_2)).$$

Then clearly, if $d_{\Gamma\backslash Y}(z_1, z_2) \leq \epsilon$ then $\omega \in \Omega_\epsilon$. □

Recall the construction of $T_k X$ from Definition 2.3.2. A choice of $s$ determines a section $T_k s\colon T_k(\Gamma\backslash Y) \to T_k Y$.

**Proposition 4.1.4.** *$T_k s$ is a coarse map for all $k$.*

*Proof.* Explicitly, $(T_k s)(z, r) = (s(z), r)$, so if $s$ is bounded by $D$ then $T_k s$ is bounded by $kD$. □

We now consider the situation where $\Gamma$ acts on $(TY)^{bdd}$ via $\gamma(y, r) = (\gamma y, r)$, and $Tp\colon (TY)^{bdd} \to TK$ is the orbit space projection.

**Proposition 4.1.5.** *Suppose $d_{TK}((z_1, r), (z_2, r)) \leq R$ where $d_{TK}$ is the orbit metric in $TK$. Then $d_{(TY)^{bdd}}((s(z_1), r), (s(z_2), r)) \leq R + 2D_1$.*

*Proof.* Suppose $\gamma$ is an element of $\Gamma$ such that

$$d_{TK}((z_1, r), (z_2, r)) = d_{(TY)^{bdd}}((s(z_1), r), (\gamma s(z_2), r)),$$

then $\|\gamma\| = 1$ and $d_{(TY)^{bdd}}((\gamma s(z_2), r), (\gamma s(z_2), r)) \leq 2D_1$. The result follows from the triangle inequality. □

Let $Ts\colon TK \to (TY)^{bdd}$ be given by $Ts(z, r) = (s(z), r)$. It is a section of the projection $Tp$.

**Corollary 4.1.6.** *The section $Ts$ is a coarse map.*

**Definition 4.1.7.** We will denote the product metric space $\Gamma \times (TX)^{bdd}$ by $T_\Gamma(X)$. Notice that when the action of $\Gamma$ on $X$ is trivial then $(TX)^{bdd} = TX$, and so $T_\Gamma(X) = \Gamma \times TX$.

Given an isometric action of $\Gamma$ on $X$, there is the induced action on $\Gamma \times X \times \mathbb{R}$ given by $\gamma(\gamma', x, r) = (\gamma\gamma', \gamma x, r)$. This induces an isometric action on $T_\Gamma(X)$.

Recall that we are given a free, properly discontinuous isometric action of $\Gamma$ on a proper metric space $Y$. Now we have the associated metric spaces $T_\Gamma(\Gamma\backslash Y)$ and $T_\Gamma(Y)$ with isometric actions by $\Gamma$.

Suppose $\Gamma\backslash Y$ is finite, then we can choose a coarse section $s\colon \Gamma\backslash Y \to Y$ of the projection $p\colon Y \to \Gamma\backslash Y$ as in Lemma 4.1.1.

The sections $Ts\colon T(\Gamma\backslash Y) \to T(Y)$ assemble to give a map

$$T_\Gamma s\colon T_\Gamma(\Gamma\backslash Y) \longrightarrow T_\Gamma(Y)$$

given by

$$Ts(\gamma, z, r) = (\gamma, Ts(z), r).$$

**Proposition 4.1.8.** *The map $T_\Gamma s$ is a coarse map.*

*Proof.* Products of coarse maps are coarse. $\square$

**Remark 4.1.9.** We note that if the action of $\Gamma$ on $T_\Gamma Y$ is via $\gamma'(\gamma, y, r) = (\gamma, \gamma' y, r)$ then $T_\Gamma(\Gamma\backslash Y)$ is precisely the orbit space with the orbit space metric. The projection $T_\Gamma p\colon T_\Gamma(Y) \to T_\Gamma(\Gamma\backslash Y)$ is the orbit space projection, and the map $Ts$ is a section of this projection.

This is certainly different from the situation of main interest to us. We are interested in the diagonal action of $\Gamma$ on $T_\Gamma Y$ given by $\gamma'(\gamma, y, r) = (\gamma'\gamma, \gamma' y, r)$. We now construct and examine a section $s_\Delta\colon \Gamma\backslash T_\Gamma Y \to T_\Gamma Y$.

**Definition 4.1.10.** (1) As soon as a section $s\colon \Gamma\backslash Y \to Y$ is chosen, there is a well-defined function $t\colon Y \to \Gamma$ determined by $t(y)\, y = s([y])$.

(2) The diagonal action of $\Gamma$ on $T_\Gamma Y$ gives the orbit space projection

$$p_\Delta\colon T_\Gamma(Y) \longrightarrow \Gamma\backslash T_\Gamma Y$$

which endows $\Gamma\backslash T_\Gamma Y$ with the orbit space metric $d_\Delta$. We define a map

$$s_\Delta\colon \Gamma\backslash T_\Gamma Y \longrightarrow T_\Gamma(Y)$$

by $s_\Delta([\gamma, y, r]) = (t(y)\,\gamma, s([y]), r)$. Here $[y]$ stands for the class $p(y)$ and $[\gamma, y, r]$ for $p_\Delta(\gamma, y, r)$.

*Notation* 4.1.11. Given a subset $S \subset \Gamma$, let $T_S X$ denote the metric subspace $S \times X \times \mathbb{R}$ of $T_\Gamma X$.

**Proposition 4.1.12.** *The map $s_\Delta$ is a section of $p_\Delta$. It is not necessarily a coarse map. However the restriction of $s_\Delta$ to each $T_S(\Gamma\backslash Y)$ for a bounded subset $S \subset \Gamma$ is coarse.*

*Proof.* By the defining property of $t(y)$,

$$p_\Delta(\gamma, y, r) = p_\Delta(\gamma, t(y)^{-1} s([y]), r) = p_\Delta(t(y)\gamma, s([y]), r).$$

So $p_\Delta s_\Delta([\gamma, y, r]) = p_\Delta(\gamma, y, r) = [\gamma, y, r]$. A bound on the norm of $\gamma$ in $S$ gives a linear coefficient to exhibit the restriction of $s_\Delta$ to $T_S(\Gamma\backslash Y)$ as a coarse map. In general, the bound for the action of $\gamma$ on $Y$ grows indefinitely. $\square$

The function
$$l\colon T_\Gamma(\Gamma\backslash Y) \longrightarrow \Gamma\backslash T_\Gamma Y$$
is given by $l(\gamma, z, r) = [\gamma, s(z), r]$.

**Proposition 4.1.13.** *$l$ is a coarse bijection.*

*Proof.* The function $l$ is proper and distance nonincreasing, therefore coarse. The inverse $l^{-1}\colon \Gamma\backslash T_\Gamma Y \to T_\Gamma(\Gamma\backslash Y)$ is given by $l^{-1}([\gamma, y, r]) = (t(y)\gamma, [y], r)$. $\square$

**Definition 4.1.14.** We define a function $u\colon T_\Gamma Y \to T_\Gamma Y$ by
$$u(\gamma, y, r) = (t(y)^{-1}\gamma, t(y)^{-1}s([y]), r).$$

Clearly, $u$ is the identity on $\mathrm{im}(Ts)$ since $y$ in $\mathrm{im}(s)$ gives $t(y) = 1$ and $s([y]) = y$.

The following diagram shows the relationship between the geometric maps we have defined.

$$\begin{tikzcd}
T_\Gamma Y \arrow[d, shift left, "p_\Delta"] & T_\Gamma Y \arrow[l, "u"'] \arrow[d, shift left, "T_\Gamma p"] \\
\Gamma\backslash T_\Gamma Y \arrow[u, shift left, "s_\Delta"] \arrow[r, shift left, "l^{-1}"] & T_\Gamma(\Gamma\backslash Y) \arrow[u, shift left, "T_\Gamma s"] \arrow[l, shift left, "l"]
\end{tikzcd}$$

Let $B(d)$ stand for the subset of all elements $\gamma$ in $\Gamma$ with $\|\gamma\| \le d$.

**Definition 4.1.15.** Let $X$ and $Y$ be arbitrary metric spaces. Given a function $f\colon T_\Gamma X \to T_\Gamma Y$, we say $f$ is *$p$-bounded* if there is a function $\lambda\colon [0,\infty) \to [0,\infty)$ such that for all $d$ the restriction of $f$ to $T_{B(d)}X$ is a coarse map bounded by $\lambda(d)$.

It is clear that compositions of $p$-bounded maps are $p$-bounded.

**Proposition 4.1.16.** (1) *The function $u$ is $p$-bounded.*

(2) *The function $l^{-1}$ that was defined in Proposition* 4.1.13 *is $p$-bounded.*

**Definition 4.1.17.** We define $s\colon T_\Gamma(\Gamma\backslash Y) \to T_\Gamma Y$ as the composition $u \circ Ts$. It is therefore $p$-bounded.

**Proposition 4.1.18.** *We have $s_\Delta = u \circ Ts \circ l^{-1}$. Therefore $s_\Delta = s \circ l^{-1}$ and is $p$-bounded.*

*Proof.* Since $l^{-1}([\gamma, y, r]) = (t(y)\gamma, [y], r)$, we have
$$T(s)\, l^{-1}([\gamma, y, r]) = (t(y)\gamma, s([y]), r).$$
Then also $u\,T(s)\,l^{-1}([\gamma, y, r]) = (t(y)\gamma, s([y]), r)$ because $t(s[y])^{-1} = 1$. $\square$

This shows that the diagram shown above is commutative.

We are ready to define the parametrized transfer.

**Definition 4.1.19.** Suppose $\Gamma$ acts on $Y$ by isometries. The *parametrized transfer map*
$$P\colon K^\Gamma_c(T(\Gamma\backslash Y))^\Gamma \longrightarrow K^\Gamma_p((TY)^{bdd})^\Gamma$$
is obtained from the $p$-bounded section $s\colon T_\Gamma(\Gamma\backslash Y) \to T_\Gamma(Y)$ via the induced additive functor
$$\varrho\colon \mathcal{C}^\Gamma_c(T(\Gamma\backslash Y))^\Gamma \longrightarrow \mathcal{C}^\Gamma_p((TY)^{bdd})^\Gamma.$$

Here the action of $\Gamma$ on the set $T_\Gamma(\Gamma\backslash Y) = \Gamma \times \Gamma\backslash Y \times \mathbb{R}$ is given by $\gamma(\gamma', z, r) = (\gamma\gamma', z, r)$, and the action on $T_\Gamma Y = \Gamma \times Y \times \mathbb{R}$ by $\gamma(\gamma', y, r) = (\gamma\gamma', \gamma y, r)$.

In order to describe how the map $s$ induces the functor $\varrho$, let us reiterate the interpretation in Definition 2.5.1 of an object of $\mathcal{C}_c^\Gamma(T(\Gamma\backslash Y))^\Gamma$. It is a pair $(F, \psi)$ where $F$ is an object of $\mathcal{C}(T_\Gamma(\Gamma\backslash Y))$ and $\psi$ is a function on $\Gamma$ with $\psi_\gamma \in \mathrm{Hom}(F, \gamma F)$ in $\mathcal{C}(T_\Gamma(\Gamma\backslash Y))$ such that $\psi_\gamma$ is of filtration 0 when projected to $\Gamma$, $\psi_e = \mathrm{id}$, and $\psi_{\gamma_1\gamma_2} = \gamma_1\psi_{\gamma_2}\psi_{\gamma_1}$.

Given $(F, \psi)$ in $\mathcal{C}_c^\Gamma(T(\Gamma\backslash Y))^\Gamma$, we define $(\varrho F, \varrho\psi)$ in $\mathcal{C}_p^\Gamma((TY)^{bdd})^\Gamma$. Using the notation $[y]$ for the orbit of $y \in Y$, we set $(\varrho F)_{(\gamma,y,r)} = F_{(\gamma,[y],r)}$. The morphisms $(\varrho\psi)_\gamma \in \mathrm{Hom}(\varrho F, \gamma\varrho F)$ are defined by making the component

$$((\varrho\psi)_\gamma)_{(\gamma_1,y_1,r_1),(\gamma_2,y_2,r_2)} \colon (\varrho F)_{(\gamma_1,y_1,r_1)} \longrightarrow (\gamma\varrho F)_{(\gamma_2,y_2,r_2)},$$

where $(\varrho F)_{(\gamma_1,y_1,r_1)} = F_{(\gamma_1,[y_1],r_1)}$ and

$$(\gamma\varrho F)_{(\gamma_2,y_2,r_2)} = (\varrho F)_{(\gamma^{-1}\gamma_2,\gamma^{-1}y_2,r_2)} = F_{(\gamma^{-1}\gamma_2,[\gamma^{-1}y_2],r_2)} = F_{(\gamma^{-1}\gamma_2,[y_2],r_2)},$$

the 0 homomorphism unless

$$t(y_1)\, t(y_2)^{-1} = \gamma_1^{-1}\gamma_2.$$

In the latter case, the component $((\varrho\psi)_\gamma)_{(\gamma_1,y_1,r_1),(\gamma_2,y_2,r_2)}$ is identified with

$$(\psi_\gamma)_{(\gamma_1,[y_1],r_1),(\gamma_2,[y_2],r_2)} \colon F_{(\gamma_1,[y_1],r_1)} \longrightarrow (\gamma F)_{(\gamma_2,[y_2],r_2)} = F_{(\gamma^{-1}\gamma_2,[y_2],r_2)}.$$

The point is that in this case there is an element $\gamma' \in \Gamma$ such that $t(y_1) = \gamma_1^{-1}(\gamma')^{-1}$ and $t(y_2) = \gamma_2^{-1}(\gamma')^{-1}$, so $y_1 = \gamma'\gamma_1 s([y_1])$ and $y_2 = \gamma'\gamma_2 s([y_2])$.

This shows that $(\varrho\psi)_\gamma$ is a morphism of $\mathcal{C}_p^\Gamma((TY)^{bdd})^\Gamma$ if $\psi_\gamma$ is a morphism of $\mathcal{C}(T_\Gamma(\Gamma\backslash Y))$. Also $(\varrho\psi)_e = \psi_e = \mathrm{id}$, and clearly $(\varrho\psi)_{\gamma_1\gamma_2} = \gamma_1(\varrho\psi)_{\gamma_2}(\varrho\psi)_{\gamma_1}$.

Given a morphism $\phi\colon (F,\psi) \to (F',\psi')$ in $\mathcal{C}_c^\Gamma(T(\Gamma\backslash Y))^\Gamma$ specified by $\phi\colon F \to F'$ in $\mathcal{C}(T_\Gamma(\Gamma\backslash Y))$, then $\varrho\phi\colon \varrho F \to \varrho F'$ is defined by

$$(\varrho\phi)_{(\gamma_1,y_1,r_1),(\gamma_2,y_2,r_2)} = \begin{cases} 0 & \text{if } t(y_1)\, t(y_2)^{-1} \neq \gamma_1^{-1}\gamma_2, \\ \phi_{(\gamma_1,[y_1],r_1),(\gamma_2,[y_2],r_2)} & \text{otherwise.} \end{cases}$$

It is a morphism of $\mathcal{C}_p((TY)^{bdd})$ because the section $s$ is $p$-bounded and the action of $\Gamma$ is bounded.

Let $N$ be the normal closed disk bundle . Then the discussion in this section applies to $Y = \widetilde{N}$ and $\Gamma\backslash Y = N$.

**Definition 4.1.20.** We obtain a parametrized transfer map

$$P\colon K_c^\Gamma(TN)^\Gamma \longrightarrow K_p^\Gamma((T\widetilde{N})^{bdd})^\Gamma.$$

It is clear that there is also a relative version of the map

$$\Lambda\colon K_c^\Gamma(TN, T\partial N)^\Gamma \longrightarrow K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma.$$

4.2. **Additional Constructions.** We will use the terms and notation introduced in section 2.7.

Given a left free, properly discontinuous action of $\Gamma$ on $Y$, let $\pi\colon Y \to \Gamma\backslash Y$ be the projection onto the orbit space. We assume that orbit space is given the orbit metric $d(x,y) = \inf_{\gamma\in\Gamma} d_Y(\gamma x, y)$. When the action of $\Gamma$ on $X$ is by isometries, $\pi^{-1}(T)$ is in $\mathcal{S}(\pi^{-1}(U))$ for all $T$ in $\mathcal{S}(U)$.

**Definition 4.2.1.** If $\mathcal{U}$ is a coarse covering of $\Gamma\backslash X$, define
$$\pi^*\mathcal{U} = \{\mathcal{S}(\pi^{-1}(U)) \,|\, U \in \mathcal{U}\},$$
which is a coarse covering of $X$.

It easy to see that if $\mathcal{U}$ is closed under coarse intersections, the same is true about $\pi^*\mathcal{U}$.

Let $A$ be a subset of $\Gamma\backslash Y$. Given a choice of a splitting of $\pi$, one has the usual functor $\mathcal{C}(A) \to \mathcal{C}(\pi^{-1}A)^\Gamma$ of bounded categories of $R$-modules. This can be interpreted as a functor $\mathcal{C}(A)^\Gamma \to \mathcal{C}(\pi^{-1}A)^\Gamma$ where the action of $\Gamma$ on $A$ is trivial, so we have a map
$$\pi_A^*\colon K^\Gamma(A)^\Gamma \longrightarrow K^\Gamma(\pi^{-1}A)^\Gamma.$$
This map induces $\pi_A^*\colon K^\Gamma(\mathcal{S}(A))^\Gamma \to K^\Gamma(\mathcal{S}(\pi^{-1}A))^\Gamma$. Consider the homotopy colimits
$$\mathcal{K}^\Gamma(\Gamma\backslash Y;\mathcal{U})^\Gamma = \underset{\overrightarrow{A\in\mathcal{U}}}{\operatorname{hocolim}}\, K^\Gamma(\mathcal{S}(A))^\Gamma,$$
$$\mathcal{K}^\Gamma(Y;\pi^*\mathcal{U})^\Gamma = \underset{\overrightarrow{A\in\mathcal{U}}}{\operatorname{hocolim}}\, K^\Gamma(\mathcal{S}(\pi^{-1}A))^\Gamma.$$

**Definition 4.2.2.** The maps $\pi_A^*$ induce the natural maps
$$\pi^*\colon \mathcal{K}^\Gamma(\Gamma\backslash Y;\mathcal{U})^\Gamma \longrightarrow \mathcal{K}^\Gamma(Y;\pi^*\mathcal{U})^\Gamma$$
and
$$\alpha\colon \mathcal{K}_i^\Gamma(\Gamma\backslash Y;\mathcal{U})^\Gamma \longrightarrow \mathcal{K}_i^\Gamma(Y;\pi^*\mathcal{U})^\Gamma.$$

Clearly these maps are natural with respect to lattice morphisms of coarse coverings: if $\mathcal{U} \subset \mathcal{U}'$ then the square
$$\begin{array}{ccc}
\mathcal{K}_i^\Gamma(\Gamma\backslash Y;\mathcal{U})^\Gamma & \xrightarrow{\alpha} & \mathcal{K}_i^\Gamma(Y;\pi^*\mathcal{U})^\Gamma \\
\downarrow & & \downarrow \\
K_i^\Gamma(\Gamma\backslash Y;\mathcal{U}')^\Gamma & \xrightarrow{\alpha'} & K_i^\Gamma(Y;\pi^*\mathcal{U}')^\Gamma
\end{array}$$
commutes up to homotopy.

When $\mathcal{U}' = \{Y\}$, the bottom map is the transfer
$$\overline{\alpha}\colon \mathcal{K}_i^\Gamma(\Gamma\backslash Y)^\Gamma \longrightarrow \mathcal{K}_i^\Gamma(Y)^\Gamma.$$

All of the above can be directly relativized.

**Definition 4.2.3.** Suppose $Y'$ is a metric subspace of $Y$ invariant under the action of $\Gamma$. The maps
$$\pi_A^*\colon K^\Gamma(A, A\cap\pi(Y'))^\Gamma \longrightarrow K^\Gamma(\pi^{-1}A, \pi^{-1}A\cap Y')^\Gamma,$$
where the the action of $\Gamma$ on $A$ is trivial, induce the natural maps
$$\pi^*\colon \mathcal{K}^\Gamma(\Gamma\backslash Y, \pi(Y');\mathcal{U})^\Gamma \longrightarrow \mathcal{K}^\Gamma(Y, Y';\pi^*\mathcal{U})^\Gamma$$
and
$$\alpha\colon \mathcal{K}_i^\Gamma(\Gamma\backslash Y, \pi(Y');\mathcal{U})^\Gamma \longrightarrow \mathcal{K}_i^\Gamma(Y, Y';\pi^*\mathcal{U})^\Gamma.$$

**Remark 4.2.4.** Whenever $\mathcal{U}$ is a covering of $Y$ by coarsely antithetic subsets $U_1$ and $U_2$, then there is a fully faithful embedding

$$\mathcal{C}_i^\Gamma(U_1 \cap U_2, Y' \cap U_1 \cap U_2)^\Gamma \longrightarrow \mathcal{C}_i^\Gamma(Y, Y')^\Gamma_{<U_1,U_2}$$

which is essentially onto, cf. Lemma 9.5.15. The induced weak equivalences

$$K_i^\Gamma(U_i, Y' \cap U_i)^\Gamma \simeq K_i^\Gamma(Y, Y')^\Gamma_{<U_i},$$
$$K_i^\Gamma(U_1 \cap U_2, Y' \cap U_1 \cap U_2)^\Gamma \simeq K_i^\Gamma(Y, Y')^\Gamma_{<U_1,U_2}$$

give the weak equivalence

$$\mathcal{K}_i^\Gamma(Y, Y'; \mathcal{U})^\Gamma \simeq \underset{\overrightarrow{U \in \overline{\mathcal{U}}}}{\operatorname{hocolim}}\ K_i^\Gamma(U, Y' \cap U)^\Gamma,$$

where $\overline{\mathcal{U}}$ is the covering by pairwise coarsely antithetic subsets $U_1$, $U_2$, and $U_1 \cap U_2$.

More generally, if $\mathcal{U}$ is a finite covering of $Y$ by pairwise coarsely antithetic subsets $U_i$ then there is a weak equivalence

$$\mathcal{K}_i^\Gamma(Y, Y'; \mathcal{U})^\Gamma \simeq \underset{\overrightarrow{U_i \in \mathcal{U}}}{\operatorname{hocolim}}\ K_i^\Gamma(U_i, Y' \cap U_i)^\Gamma.$$

## 5. Conclusion of the Proof

We define the map

$$\phi\colon K^{\Gamma,0}(\widetilde{M}, R)^{h\Gamma} \longrightarrow \mathcal{S} = W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

in section 5.1. The proof of the fact that the composition $f = \phi \circ \rho$ is a weak equivalence is completed in section 5.5. All of the constructions and maps can be seen assembled in Figures 3–5 in section 5.6.

5.1. **Construction of the Splitting Map.** For convenience of notation, we construct and study the suspension of the promised map, $\psi = \Sigma^{n+k+1}\phi$. Of course, conclusions such as being a weak homotopy equivalence or being a weak split surjection hold equally for maps of spectra and their suspensions.

We assume that we are given a smooth closed $n$-dimensional manifold $M$ which is a $K(\Gamma, 1)$-space, so that the universal covering $\widetilde{M}$ of $M$ is contractible. Embed $M$ in a Euclidean space $\mathbb{R}^{n+k}$, and let $N$ denote a small tubular neighborhood of $M$, so $N$ is homeomorphic to the unit $k$-disk bundle in the normal bundle to the embedding $e\colon M \hookrightarrow \mathbb{R}^{n+k}$. In fact, for a sufficiently small neighborhood, the exponential map gives a retraction $p\colon N \to M$, which is a bundle projection, and we may assume that we are given a bundle preserving projection from $N$ to the $k$-disk bundle in the normal bundle to $M$. We will abuse notation by referring to the $k$-disk bundle as $N$.

Let $\widetilde{N}$ denote the universal covering of $N$. The splitting map $\psi$ referred to above will now be the composite of the following maps.

($\beta_1$) The first map

$$\beta_1\colon \Sigma^{n+k+1}K^{\Gamma,0}(\widetilde{M}, R)^{h\Gamma} \longrightarrow K_i^\Gamma(T\widetilde{N})^\Gamma.$$

is the map $T\eta$ constructed in Definition 2.5.12. It is the composition of the suspension

$$\Sigma\eta\colon \Sigma^{n+k+1}K^{\Gamma,0}(\widetilde{M}, R)^{h\Gamma} \longrightarrow \Sigma K_i^\Gamma(\widetilde{N})^\Gamma$$

of $\eta$ from Definition 2.5.11 and the weak equivalence

$$\sigma\colon \Sigma K_i^\Gamma(\widetilde{N})^\Gamma \longrightarrow K_i^\Gamma(T\widetilde{N})^\Gamma$$

from Theorem 8.3.9.

($\beta_2$) We define

$$\beta_2\colon K_i^\Gamma(T\widetilde{N})^\Gamma \longrightarrow K_i^\Gamma(T\widetilde{N}, T\partial\widetilde{N})^\Gamma$$

as the map induced from the exact quotient map of the controlled categories. It can be viewed as obtained by relaxing control in $\mathcal{C}^{\Gamma,0}(\Gamma \times T\widetilde{N})^\Gamma$.

($\beta_3$) The map

$$\beta_3\colon K_i^\Gamma(T\widetilde{N}, T\partial\widetilde{N})^\Gamma \longrightarrow K_c^\Gamma(T\widetilde{N}, T\partial\widetilde{N})^\Gamma$$

is induced by the natural transformation $K_i^\Gamma \to K_c^\Gamma$.

($\beta_4$) The map

$$\beta_4\colon K_c^\Gamma(T\widetilde{N}, T\partial\widetilde{N})^\Gamma \longrightarrow K_c^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma$$

is induced by the natural map $b\colon T\widetilde{N} \to (T\widetilde{N})^{bdd}$. We verified in Proposition 2.2.8 that $b$ is a coarse map.

($\beta_5$) The map

$$\beta_5\colon K_c^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma$$

is induced by the natural transformation $K_c^\Gamma \to K_p^\Gamma$.

*Note*: For the construction of the next map, the ring $R$ is required to be noetherian.

($\beta_6$) The map

$$\beta_6\colon K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow G_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma$$

is induced by the natural Cartan transformation $\kappa^\Gamma\colon K_p^\Gamma \to G_p^\Gamma$ from Definition 2.6.7.

($\beta_7$) The map

$$\beta_7\colon G_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

is the map of spectra $\varepsilon$ from Definition 3.2.2 induced by the exact embedding

$$E\colon \mathbf{G}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow \mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}).$$

($\beta_8$) Recall from Definition 3.2.4 and Notation 3.2.5 that for a given choice of a lift $\sigma\colon B \to \widehat{B}$ onto a connected component of $\widetilde{B} = \pi^{-1}(B)$, there is the exact quotient category

$$\mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) = \mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{>\mathtt{C}T\widehat{B}}.$$

The map

$$\beta_8\colon W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \longrightarrow W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

is the map $q$ in the Abbreviated Diagram in section 3.4, defined as the map in nonconnective $K$-theory induced by this exact quotient.

**Remark 5.1.1.** One can similarly define the map

$$q\colon G_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow G_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma_{>\complement T\widehat{B}}$$

induced from the corresponding quotient map between exact categories. Using the same notational convention, we get a commutative square

$$\begin{tikzcd}
G_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \arrow[r, "q"] \arrow[d, "\beta_7"'] & G^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \arrow[d, "\varepsilon"] \\
W^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd}) \arrow[r, "\beta_8"] & W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})
\end{tikzcd}$$

**Definition 5.1.2.** We define $\psi$ as the composition

$$\beta_8\circ\beta_7\circ\beta_6\circ\beta_5\circ\beta_4\circ\beta_3\circ\beta_2\circ\beta_1\colon$$
$$\Sigma^{n+k+1}K^{\Gamma,0}(\widetilde{M},R)^{h\Gamma}\longrightarrow W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd}).$$

By the discussion in section 3.1, the Main Theorem now follows from the following statement.

**Theorem 5.1.3.** *The composition $\mathcal{F}$ of the suspension $\Sigma^{n+k+1}\rho$ of the canonical map $\rho\colon S^\Gamma\to S^{h\Gamma}$ applied to $S=K^{\Gamma,0}(\widetilde{M},R)$ and the map $\psi$ from Definition* 5.1.2 *is a weak equivalence.*

As a preparation for the proof of Theorem 5.1.3, we will perform several reductions in order to express the map $\mathcal{F}$ in a convenient form.

5.2. **First Reduction.** Consider the following new sequence of maps.

($\alpha_0$) This is a map

$$\alpha_0\colon \Sigma^{n+k+1}K^{\Gamma,0}(\widetilde{M},R)^\Gamma\longrightarrow K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma,$$

where the action of $\Gamma$ on $T\mathbb{R}^{n+k}$ is trivial.

It is the composition of two maps. The first is the weak equivalence

$$\alpha_0'\colon \Sigma^{n+k+1}K^{\Gamma,0}(\widetilde{M},R)^\Gamma \xrightarrow{\ \simeq\ } \Sigma^{n+k+1}\,K^{-\infty}(R[\Gamma])$$

which is the $(n+k+1)$-fold suspension of the weak equivalence $K^{\Gamma,0}(\widetilde{M},R)^\Gamma\simeq K^{-\infty}(R[\Gamma])$ from Theorem 2.5.4.

The second map

$$\Sigma^{n+k+1}\,K^{-\infty}(R[\Gamma])\longrightarrow K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma$$

is defined as follows.

Recall that given a vector $c$ in $\mathbb{R}^{n+k}$ and a positive constant $R$, we let $h\colon\mathbb{R}^{n+k}\to\mathbb{R}^{n+k}$ be the linear homeomorphism given by $h(x)=c+Rx$.

We refer to Notation 3.3.1 for a decomposition of $\mathbb{R}^{n+k}$ into subsets $E_i^+$, $E_i^-$, $E_i=E_i^-\cup E_i^+$ for $1\le i\le n+k$, and $E_0=E_0^-=E_0^+=\{(0,\dots,0)\}$. The subsets $E_i^*$ form a coarsely antithetic covering of $\mathbb{R}^{n+k}$, so $TE_i^*$ form a coarsely antithetic covering of $T\mathbb{R}^{n+k}$. Therefore, we obtain a coarsely antithetic covering $\mathcal{E}$ of $T\mathbb{R}^{n+k}$ by the subsets $Th(E_i^*)$ closed under coarse intersections. From Definition 3.3.2 we have the spectrum

$$\mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma = \underset{\substack{\longrightarrow\\ E_i^*\in\mathcal{E}}}{\operatorname{hocolim}}\ K_i^\Gamma(Th(E_i^*))^\Gamma$$

where the structure maps in the diagram are induced by inclusions of subsets.

Since Proposition 3.3.3 gives weak equivalences

$$\Sigma K_i^\Gamma(T\mathbb{R}^{d-1})^\Gamma \longrightarrow K_i^\Gamma(T\mathbb{R}^d)^\Gamma$$

for each $0 < d \leq n+k$, by Corollary 3.3.4 there is a weak equivalence

$$\alpha_0''\colon \Sigma^{n+k+1} K^{-\infty}(R[\Gamma]) \longrightarrow \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma.$$

Applying Theorem 2.7.16 to $\mathcal{E}$, we see that the natural map

$$\alpha_0'''\colon \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k};\mathcal{E})^\Gamma \longrightarrow K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma$$

is a weak equivalence.

**Corollary 5.2.1.** $\alpha_0 = \alpha_0''' \circ \alpha_0'' \circ \alpha_0'$ *is a weak equivalence.*

($\alpha_1$) Define

$$\alpha_1\colon K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma \longrightarrow K_i^\Gamma(TN, T\partial N)^\Gamma$$

as the composition of relaxing the control via

$$K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma \longrightarrow K_i^\Gamma(T\mathbb{R}^{n+k}, T\mathbb{R}^{n+k} - T(\operatorname{int} N))^\Gamma$$

and the equivalence

$$K_i^\Gamma(T\mathbb{R}^{n+k}, T\mathbb{R}^{n+k} - T(\operatorname{int} N))^\Gamma \simeq K_i^\Gamma(TN, T\partial N)^\Gamma$$

from Theorem 9.10.1. Here all $\Gamma$-actions on $T\mathbb{R}^{n+k}$, $TN$, $T\partial N$ are trivial.

($\alpha_2$) The map

$$\alpha_2\colon K_i^\Gamma(TN, T\partial N)^\Gamma \longrightarrow K_c^\Gamma(TN, T\partial N)^\Gamma$$

is induced by the natural transformation $K_i^\Gamma \to K_c^\Gamma$.

($\alpha_3$) Let

$$\alpha_3\colon K_c^\Gamma(TN, T\partial N)^\Gamma \longrightarrow K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma$$

be the parametrized transfer $\Lambda$ associated to the projection $\widetilde{N} \to N$ described in Definition 4.1.20. Equivalently, $\alpha_3$ is induced by an inclusion of the fixed point spectra.

**Proposition 5.2.2.** *The map* $\beta_5 \circ \beta_4 \circ \beta_3 \circ \beta_2 \circ \beta_1$ *factors the composite* $\alpha_3 \circ \alpha_2 \circ \alpha_1$ *in the sense that there is a homotopy commutative diagram*

$$\begin{tikzcd}
\Sigma^{n+k+1} K^{\Gamma,0}(\widetilde{M}, R)^\Gamma \arrow[r, "\beta_1 \circ \Sigma^{n+k+1}\rho"] \arrow[d, "\alpha_0", "\simeq"'] & K_i^\Gamma(T\widetilde{N})^\Gamma \arrow[d, "\beta_5\circ\beta_4\circ\beta_3\circ\beta_2"] \\
K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma \arrow[r, "\alpha_3\circ\alpha_2\circ\alpha_1"] & K_p^\Gamma(T\widetilde{N}^{bdd}, T\partial\widetilde{N}^{bdd})^\Gamma
\end{tikzcd}$$

*Proof.* Let us view $\Gamma$ as a metric space with the chosen word metric equipped with the free left multiplication action on itself. Then up to homotopy we can identify the fixed points

$$K^{\Gamma,0}(\widetilde{M}, R)^\Gamma \simeq K^{\Gamma,0}(\Gamma, R)^\Gamma.$$

It is easy to give the intuitive idea behind the proof using the description of objects of $\mathcal{C}^{\Gamma,0}(\Gamma, R)^\Gamma$ in the style of [14]. Since $\mathcal{C}^{\Gamma,0}(\Gamma, R)^\Gamma$ is isomorphic to the category of finitely generated free $R[\Gamma]$-modules, an object $A$ can be represented by assigning a copy of a free finitely generated $R$-module $A_\gamma$ to each $\gamma$ and observing that the collection is invariant under the permutation action of $\Gamma$ induced from left multiplication action on itself. Up to suspension, the horizontal map

$$\beta_1 \circ \Sigma^{n+k+1}\rho \colon \Sigma^{n+k+1} K^{\Gamma,0}(\Gamma, R)^\Gamma \longrightarrow K_i^\Gamma(T\widetilde{N})^\Gamma$$

is induced by sending $A$ to the same collection of $R$-modules associated to points in the orbit of $\Gamma$ for the diagonal action on $\Gamma \times T\widetilde{N}$. The correspondence is given by $A_\gamma = A_{(\gamma,\gamma x)}$ for $x \in T\widetilde{N}$. In fact, this describes the entire map

$$\Sigma^{n+k+1} K^{\Gamma,0}(\Gamma, R)^\Gamma \longrightarrow K_p^\Gamma(T\widetilde{N}^{bdd}, T\partial\widetilde{N}^{bdd})^\Gamma$$

factoring through $K_i^\Gamma(T\widetilde{N})^\Gamma$. On the other hand,

$$\alpha_0 \colon \Sigma^{n+k+1} K^{\Gamma,0}(\Gamma, R)^\Gamma \longrightarrow K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma$$

is induced by sending $A$ to the same collection of $R$-modules associated to points in the orbit of $\Gamma$ for the action on $\Gamma \times T\mathbb{R}^{n+k}$ by the identification $A_\gamma = A_{\gamma,0}$. By arrangement $0 \in N$, so we have described

$$\alpha_2 \circ \alpha_1 \circ \alpha_0 \colon \Sigma^{n+k+1} K^{\Gamma,0}(\Gamma, R)^\Gamma \longrightarrow K_c^\Gamma(TN, T\partial N)^\Gamma.$$

Finally, by inspection of the definition of the transfer

$$\alpha_3 \colon K_c^\Gamma(TN, T\partial N)^\Gamma \longrightarrow K_p^\Gamma(T\widetilde{N}^{bdd}, T\partial\widetilde{N}^{bdd})^\Gamma$$

in Definition 4.1.20, one sees that the composition factoring through the lower left corner sends $A$ to the same object. This argument can be easily made precise by the reader using the explicit description of the fixed point categories as in Definition 4.1.19. $\square$

($\gamma$) *Note*: The ring $R$ in the remainder of this section is assumed to be a regular noetherian ring and have finite homological dimension.

**Proposition 5.2.3.** *There is a weak equivalence*

$$\gamma \colon W_{T\widehat{B}}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \longrightarrow \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]).$$

The proof will require a construction of several specific subsets of $T\widetilde{N}$.

We may assume that the ball $B$ is a metric ball in $\mathbb{R}^{n+k}$ centered at $c$ with radius $R$. Let $h \colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ be the linear map $h(x) = c + Rx$, so $h$ restricts to a linear homeomorphism $h \colon D^{n+k} \to B$ from the unit disk $D^{n+k} = 0[1]$ onto the chosen ball $B \subset N$.

*Notation* 5.2.4. Referring to the subsets $E_i^\pm$ from Notation 3.3.1, we recall some related subsets of $\mathbb{R}^{n+k}$ defined in the course of section 3.3:

$$\begin{aligned} &D_0 = D_0^- = D_0^+ = \{(0, \ldots, 0)\}, \\ &D_i^\pm = E_i^\pm \cap D^{n+k}, \text{ for } 1 \le i \le n+k, \text{ and} \\ &D_i = D_i^- \cup D_i^+ = D_{i+1}^- \cap D_{i+1}^+. \end{aligned}$$

The images of $D_i^*$ under the linear homeomorphism $h$ will be denoted by $B_i^*$.

We now define the following collection of metric subspaces of $V = T\widehat{B}$:

$$\begin{aligned} &V' = T\partial\widehat{B} \text{ and} \\ &V_i^* = T\sigma(B_i^*) \text{ for } 0 \le i \le n+k. \end{aligned}$$

The subsets $\{V_i^*\}$ can be thought of as a coarsely antithetic covering of $T\widehat{B}$. These can be extended to a coarsely antithetic covering of $\overline{V} = (T\widetilde{N})^{bdd}$:

$$\overline{V'} = T\partial\widehat{B} \cup \complement T\widehat{B} \text{ and}$$
$$\overline{V_i^*} = T\sigma(B_i^*) \cup \complement T\widehat{B} \quad \text{for } 0 \leq i \leq n+k.$$

*Proof of Proposition* 5.2.3. Let $C_1$ and $C_2$ be any two of the subsets $\overline{V_i^*}$. The category $\mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_1,C_2}$ defined as

$$\mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_1} \cap \mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_2}$$

is a Grothendieck subcategory of $\mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$.

Applying Theorem 2.7.17 to $\overline{V}$, $\overline{V'}$, $C_1$, $C_2$, we obtain a homotopy pushout square

$$\begin{tikzcd}
W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_1,C_2} \arrow[r] \arrow[d] & W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_1} \arrow[d] \\
W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_2} \arrow[r] & W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})
\end{tikzcd}$$

Here $W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_i}$ are contractible by Lemma 9.10.2. If $C_1 = \overline{V_i^*}$ and $C_2 = \overline{V_j^*}$ with $j \leq i$, so that $C_2 \subset C_1$, then $W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_1,C_2}$ is also contractible. If $C_1 = \overline{V_i^\pm}$ and $C_2 = \overline{V_i^\mp}$ then

$$W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C_1,C_2} = W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<\overline{V_{i-1}}}.$$

This gives a weak equivalence

$$\Sigma\, W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<\overline{V_{i-1}}} \simeq W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<\overline{V_i}}.$$

Let $\overline{\mathcal{V}}$ be the coarse antithetic cover of $\overline{V} = (T\widetilde{N})^{bdd}$ by the subsets $\overline{V_i^*}$. This computation gives the weak equivalences

$$\begin{aligned}
&\underset{\overline{\mathcal{V}}}{\underrightarrow{\mathrm{hocolim}}}\; W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<\overline{V_i^*}} \\
&\qquad \simeq \Sigma^{n+k} W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<\overline{V_0}} \simeq \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]).
\end{aligned}$$

On the other hand, by Excision Theorem 2.7.17,

$$\underset{\overline{\mathcal{V}}}{\underrightarrow{\mathrm{hocolim}}}\; W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<\overline{V_i^*}} \simeq W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}).$$

So we obtain a map

$$\gamma \colon W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \longrightarrow \Sigma^{n+k+1} G^{-\infty}(R[\Gamma])$$

which is a weak equivalence. □

**Corollary 5.2.5.** *There is a weak equivalence*

$$W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N}))^{bdd}) \longrightarrow \Sigma^{n+k+1} K^{-\infty}(R[\Gamma]).$$

*Proof.* This is a consequence of the equivalence $G^{-\infty}(R[\Gamma]) \simeq K^{-\infty}(R[\Gamma])$ given by Theorem 2.8.1. □

**Definition 5.2.6.** Let $\mathcal{L}$ be the composition

$$\gamma \circ \beta_8 \circ \beta_7 \circ \beta_6 \circ \alpha_3 \circ \alpha_2 \circ \alpha_1 \circ \alpha_0 \colon \Sigma^{n+k+1} K^{\Gamma,0}(\widetilde{M}, R)^{\Gamma} \longrightarrow \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]).$$

In order to prove Theorem 5.1.3, it now suffices to prove the following theorem.

**Theorem 5.2.7.** *The map $\mathcal{L}$ is a weak equivalence.*

5.3. **Second Reduction.** We will define a sequence of maps between homotopy pushouts of diagrams of spectra associated to certain coverings. The maps will be related to the maps that make up $\mathcal{L}$.

We begin with the trivial action of $\Gamma$ on $T\mathbb{R}^{n+k}$. Recall from Definition 3.3.2 the complete coarsely antithetic covering $\mathcal{E}$ of $T\mathbb{R}^{n+k}$ by sets $Th(E_i^*)$ for $0 \le i \le n+k$ and $* = -$, $+$, or blank. So each set in this covering is indexed by $(i, *)$. This covering is ordered by inclusion thus forming a category.

All diagrams in this section, and all coverings we consider, will be indexed by this same ordered category.

There is the corresponding homotopy pushout

$$\mathcal{K}_i^{\Gamma}(T\mathbb{R}^{n+k}; \mathcal{E})^{\Gamma} = \underset{\overrightarrow{\mathcal{E}}}{\operatorname{hocolim}}\ K_i^{\Gamma}(Th(E_i^*))^{\Gamma}.$$

By Theorem 2.7.1, the natural map

$$\alpha_0''' \colon \mathcal{K}_i^{\Gamma}(T\mathbb{R}^{n+k}; \mathcal{E})^{\Gamma} \longrightarrow K_i^{\Gamma}(T\mathbb{R}^{n+k})^{\Gamma}$$

is an equivalence.

*Notation* 5.3.1. Let the subspaces $N_i^*$ of $N$ be given by $N_i^* = N \cap h(E_i^*)$. Then $TN_i^*$ are coarsely antithetic subsets of $TN$.

**Definition 5.3.2.** We define $\mathcal{U}_1$ as the covering of $TN$ by $TN_i^*$.

There are a homotopy pushout

$$\mathcal{K}_i^{\Gamma}(TN, T\partial N; \mathcal{U}_1)^{\Gamma} = \underset{\overrightarrow{\mathcal{U}_1}}{\operatorname{hocolim}}\ K_i^{\Gamma}(TN_i^*, T(N_i^* \cap \partial N))^{\Gamma},$$

where the action of $\Gamma$ on $N_i^*$ is the trivial action, and the natural map

$$\pi_1 \colon \mathcal{K}_i^{\Gamma}(TN, T\partial N; \mathcal{U}_1)^{\Gamma} \longrightarrow K_i^{\Gamma}(TN, T\partial N)^{\Gamma}.$$

It is a weak equivalence by the same Theorem 2.7.1.

($\omega_1$) This is the canonical map

$$\omega_1 \colon \mathcal{K}_i^{\Gamma}(T\mathbb{R}^{n+k}; \mathcal{E})^{\Gamma} \longrightarrow \mathcal{K}_i^{\Gamma}(TN, T\partial N; \mathcal{U}_1)^{\Gamma}$$

induced by relaxing control

$$\omega_1^{i,*} \colon \mathcal{K}_i^{\Gamma}(Th(E_i^*))^{\Gamma} \longrightarrow K_i^{\Gamma}(Th(E_i^*), Th(E_i^* - \operatorname{int} N))^{\Gamma} \longrightarrow K_i^{\Gamma}(TN_i^*, T(N_i^* \cap \partial N))^{\Gamma}$$

on each level indexed by $(i, *)$. The map $\omega_1$ is part of the commutative square

$$\begin{tikzcd}
\mathcal{K}_i^{\Gamma}(T\mathbb{R}^{n+k}; \mathcal{E})^{\Gamma} \arrow[r, "\omega_1"] \arrow[d, "\alpha_0'''"', "\simeq"] & \mathcal{K}_i^{\Gamma}(TN, T\partial N; \mathcal{U}_1)^{\Gamma} \arrow[d, "\pi_1"'] \\
K_i^{\Gamma}(T\mathbb{R}^{n+k}) \arrow[r, "\alpha_1"] & K_i^{\Gamma}(TN, T\partial N)^{\Gamma}
\end{tikzcd}$$

($\psi_1$) Consider also the metric ball $B$ in the interior of $N$. The maps

$$\omega_1^{i,*}\colon K_i^\Gamma(Th(E_i^*))^\Gamma \longrightarrow K_i^\Gamma(TN_i^*, T\partial N_i^*)^\Gamma$$

were constructed by relaxing control. Relaxing control further, we have

$$\begin{aligned}\psi_1^{i,*}\colon K_i^\Gamma(Th(E_i^*))^\Gamma &\longrightarrow K_i^\Gamma(Th(E_i^*), Th(E_i^* - \operatorname{int} B))^\Gamma\\ &\longrightarrow K_i^\Gamma(TB_i^*, T(B_i^* \cap \partial B))^\Gamma\end{aligned}$$

By Theorem 9.10.4, each $\psi_1^{i,*}$ is an equivalence. There are also inclusion induced maps

$$\xi_1^{i,*}\colon K_i^\Gamma(TB_i^*, T(B_i^* \cap \partial B))^\Gamma \longrightarrow K_i^\Gamma(TN_i^*, T(N_i^* \cap \partial N))^\Gamma.$$

**Definition 5.3.3.** Let $\mathcal{V}_1$ be the coarse antithetic covering of $TB$ by $TB_i^*$.

There is a homotopy pushout

$$\mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma = \underset{\overrightarrow{\mathcal{V}_1}}{\operatorname{hocolim}}\ K_i^\Gamma(TB_i^*, T(B_i^* \cap \partial B))^\Gamma$$

and two natural maps

$$\xi_1\colon \mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \longrightarrow \mathcal{K}_i^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma$$

induced from $\xi_1^{i,*}$ and

$$\psi_1\colon \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k}; \mathcal{E})^\Gamma \longrightarrow \mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma$$

induced from $\psi_1^{i,*}$. Since all $\psi_1^{i,*}$ are equivalences, $\psi_1$ is an equivalence. It is part of the commutative triangle

$$\begin{tikzcd}
\mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k}; \mathcal{E})^\Gamma \arrow[r, "\psi_1", "\simeq"'] \arrow[d, "="'] & \mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[d, "\xi_1"'] \\
\mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k}; \mathcal{E})^\Gamma \arrow[r, "\omega_1"] & \mathcal{K}_i^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma
\end{tikzcd}$$

($\omega_2$, $\psi_2$) The natural transformation $K_i^\Gamma \to K_c^\Gamma$ induces the commutative squares

$$\begin{tikzcd}
\mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[r, "\psi_2"] \arrow[d, "\xi_1"'] & \mathcal{K}_c^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[d, "\xi_2"'] \\
\mathcal{K}_i^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma \arrow[r, "\omega_2"] \arrow[d, "\pi_1"'] & \mathcal{K}_c^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma \arrow[d, "\pi_2"'] \\
K_i^\Gamma(TN, T\partial N)^\Gamma \arrow[r, "\alpha_2"] & K_c^\Gamma(TN, T\partial N)^\Gamma
\end{tikzcd}$$

where we define

$$\mathcal{K}_c^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma = \underset{\overrightarrow{\mathcal{U}_1}}{\operatorname{hocolim}}\ K_c^\Gamma(TN_i^*, T(N_i^* \cap \partial N))^\Gamma,$$

and similarly $\mathcal{K}_c^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma$.

**Definition 5.3.4.** Let $\pi\colon \widetilde{N} \to N$ be the orbit space projection for the free properly discontinuous action of $\Gamma$ on $\widetilde{N}$ by isometries.

We denote by $\widetilde{N}_i^*$ the subsets $\pi^{-1}(N_i^*)$ of $\widetilde{N}$. Then $\partial\widetilde{N}_i^*$ are the subsets $\pi^{-1}(\partial N \cap N_i^*)$. Let $\mathcal{U}_2$ be the covering of $T\widetilde{N}$ by $T\widetilde{N}_i^*$. Since $\mathcal{U}_1$ is a coarse antithetic covering of $TN$, the covering $\mathcal{U}_2$ of $T\widetilde{N}$ is a coarsely antithetic covering.

If $\widetilde{B}_i^*$ are the subsets $\pi^{-1}(B_i^*)$ of $\widetilde{N}$, we also define $\mathcal{V}_2$ to be the coarsely antithetic covering of $T\widetilde{B}$ by $T\widetilde{B}_i^*$.

Notice that under the map $b\colon T\widetilde{N} \to (T\widetilde{N})^{bdd}$ from Proposition 2.2.8 the subsets $T\widetilde{N}_i^*$ and $T\partial\widetilde{N}_i^*$ map onto the metric subspaces of $(T\widetilde{N})^{bdd}$ which are isometric to $(T\widetilde{N}_i^*)^{bdd}$ and $(T\partial\widetilde{N}_i^*)^{bdd}$ respectively. We will use the notation $\mathcal{U}_2^{bdd}$ to indicate that the homotopy colimit is taken over the covering $\mathcal{U}_2$ viewed as a covering by metric subspaces of $(T\widetilde{N})^{bdd}$.

Now there is a homotopy pushout

$$\mathcal{K}_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd};\mathcal{U}_2^{bdd})^\Gamma = \underset{\overrightarrow{\mathcal{U}_2}}{\text{hocolim}}\ K_p^\Gamma((T\widetilde{N}_i^*)^{bdd},(T\partial\widetilde{N}_i^*)^{bdd})^\Gamma.$$

Since $\mathcal{V}_1$ is a coarse antithetic covering of $TB$, the subsets $(T\widetilde{B}_i^*)^{bdd}$ form a coarse antithetic covering of $(T\widetilde{B})^{bdd}$. There is a similar homotopy pushout

$$\mathcal{K}_p^\Gamma((T\widetilde{B})^{bdd},(T\partial\widetilde{B})^{bdd};\mathcal{V}_2^{bdd})^\Gamma = \underset{\overrightarrow{\mathcal{V}_2}}{\text{hocolim}}\ K_p^\Gamma((T\widetilde{B}_i^*)^{bdd},(T\partial\widetilde{B}_i^*)^{bdd})^\Gamma.$$

($\omega_3$, $\psi_3$) Notice that $\mathcal{U}_2$ is exactly the covering referred to as $\pi^*\mathcal{U}_1$ in Definition 4.2.3. So we have the map of homotopy pushouts

$$\omega_3 = \pi^*\colon \mathcal{K}_c^\Gamma(TN,T\partial N;\mathcal{U}_1)^\Gamma \longrightarrow \mathcal{K}_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd};\mathcal{U}_2^{bdd})^\Gamma.$$

Now there is a diagram of natural maps

$$\begin{tikzcd}
\mathcal{K}_c^\Gamma(TN,T\partial N;\mathcal{U}_1)^\Gamma \arrow[r, "\omega_3"] \arrow[d, "\pi_2"'] & \mathcal{K}_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd};\mathcal{U}_2^{bdd})^\Gamma \arrow[d, "\pi_3"'] \\
K_c^\Gamma(TN,T\partial N)^\Gamma \arrow[r, "\alpha_3"] & K_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma
\end{tikzcd}$$

**Theorem 5.3.5.** *This is a commutative square.*

*Proof.* Since $\partial N_i^*$ is the subset $N_i^* \cap \partial N$, the statement follows from the fact that we have commutative squares

$$\begin{tikzcd}
K_c^\Gamma(TN_i^*,T\partial N_i^*)^\Gamma \arrow[r, "\Lambda_i^*"] \arrow[d] & K_p^\Gamma((T\widetilde{N}_i^*)^{bdd},(T\partial\widetilde{N}_i^*)^{bdd})^\Gamma \arrow[d] \\
K_c^\Gamma(TN,T\partial N)^\Gamma \arrow[r, "\alpha_3"] & K_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma
\end{tikzcd}$$

for all values of $i$ and $*$. The maps $\Lambda_i^*$ in each square are induced by

$$P_i^*\colon K_c^\Gamma(TN_i^*)^\Gamma \longrightarrow K_p^\Gamma((T\widetilde{N}_i^*)^{bdd})^\Gamma,$$

which are the parametrized transfer maps from Definition 4.1.19 applied to $Y = \widetilde{N}_i^*$ and $\Gamma\backslash Y = N_i^*$. So we are reduced to checking commutativity of the square of additive functors

$$\begin{tikzcd}
\mathcal{C}_c^\Gamma(TN_i^*, T\partial N_i^*)^\Gamma \arrow[r, "\varrho_i^*"] \arrow[d] & \mathcal{C}_p^\Gamma((T\widetilde{N}_i^*)^{bdd}, (T\partial \widetilde{N}_i^*)^{bdd})^\Gamma \arrow[d] \\
\mathcal{C}_c^\Gamma(TN, T\partial N)^\Gamma \arrow[r, "\varrho"] & \mathcal{C}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd})^\Gamma
\end{tikzcd}$$

where the vertical functors are induced by inclusions of metric subspaces. This is clear from inspection of the construction of $\varrho$ in Definition 4.1.19. $\square$

For each index $(i, *)$ there is a map

$$\psi_3^{i,*} \colon K_c^\Gamma(TB_i^*, T\partial B_i^*)^\Gamma \longrightarrow K_p^\Gamma((T\widetilde{B}_i^*)^{bdd}, (T\partial \widetilde{B}_i^*)^{bdd})^\Gamma,$$

where the action of $\Gamma$ on $\widetilde{B}_i^*$ is by restrictions of deck transformations from the natural action of $\Gamma$ on $\widetilde{N}$.

Let

$$\psi_3 \colon \mathcal{K}_c^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \longrightarrow \mathcal{K}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial \widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma.$$

be the map induced from the map of coverings $\pi^* \colon \mathcal{V}_1 \to \pi^*\mathcal{V}_1 = \mathcal{V}_2$.

There is a commutative square of natural maps

$$\begin{tikzcd}
\mathcal{K}_c^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[r, "\psi_3"] \arrow[d, "\xi_2"'] & \mathcal{K}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial \widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \arrow[d, "\xi_6"'] \\
\mathcal{K}_c^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma \arrow[r, "\omega_3"] & \mathcal{K}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd}; \mathcal{U}_2^{bdd})^\Gamma
\end{tikzcd}$$

where $\xi_6$ is induced by inclusions of $\widetilde{B}_i^*$ in $\widetilde{N}_i^*$.

($\omega_6$, $\psi_6$) The natural Cartan transformation $K_p^\Gamma \to G_p^\Gamma$ induces the commutative diagram

$$\begin{tikzcd}
\mathcal{K}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial \widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \arrow[r, "\psi_6"] \arrow[d, "\xi_5"'] & \mathcal{G}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial \widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \arrow[d, "\xi_6"'] \\
\mathcal{K}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd}; \mathcal{U}_2^{bdd})^\Gamma \arrow[r, "\omega_6"] \arrow[d, "\pi_3"'] & \mathcal{G}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd}; \mathcal{U}_2^{bdd})^\Gamma \arrow[d, "\pi_6"'] \\
K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd})^\Gamma \arrow[r, "\beta_6"] & G_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd})^\Gamma
\end{tikzcd}$$

where

$$\mathcal{G}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd}; \mathcal{U}_2^{bdd})^\Gamma = \underset{\overrightarrow{\mathcal{U}_2}}{\mathrm{hocolim}}\; G_p^\Gamma((T\widetilde{N}_i^*)^{bdd}, (T\partial \widetilde{N}_i^*)^{bdd})^\Gamma,$$

and $\mathcal{G}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial \widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma$ is defined similarly.

($\omega_7$, $\psi_7$) If we define

$$\mathcal{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial \widetilde{N})^{bdd}; \mathcal{U}_2^{bdd}) = \underset{\overrightarrow{\mathcal{U}_2}}{\mathrm{hocolim}}\; W^\Gamma((T\widetilde{N}_i^*)^{bdd}, (T\partial \widetilde{N}_i^*)^{bdd}),$$

and similarly $\mathcal{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})$, then the exact inclusion of categories $\mathbf{B}_p^\Gamma \to \mathbf{W}^\Gamma$ induces the commutative squares

$$\begin{tikzcd}
\mathcal{G}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \arrow[r, "\psi_7"] \arrow[d, "\xi_6"'] & \mathcal{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd}) \arrow[d, "\xi_7"'] \\
\mathcal{G}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2^{bdd})^\Gamma \arrow[r, "\omega_7"] \arrow[d, "\pi_6"'] & \mathcal{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2^{bdd}) \arrow[d, "\pi_7"'] \\
G_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \arrow[r, "\beta_7"] & W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})
\end{tikzcd}$$

($\omega_8$, $\psi_8$) The subsets $T\complement\widehat{B} \cap T\widetilde{N}_i^*$ of $T\widetilde{N}_i^*$ give the corresponding Grothendieck subcategories of $\mathbf{W}^\Gamma((T\widetilde{N}_i^*)^{bdd}, (T\partial\widetilde{N}_i^*)^{bdd})$.

**Definition 5.3.6.** Let $\widehat{B}_i^*$ be the subsets $\widehat{B} \cap \widetilde{N}_i^*$ of $\widetilde{N}_i^*$.

The resulting exact quotients $\mathbf{W}^\Gamma_{T\widehat{B}_i^*}((T\widetilde{N}_i^*)^{bdd}, (T\partial\widetilde{N}_i^*)^{bdd})$ are indexed by the same partially ordered set. We define

$$\mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2^{bdd}) = \underset{\overrightarrow{\mathcal{U}_2}}{\operatorname{hocolim}} \; W^\Gamma_{T\widehat{B}_i^*}((T\widetilde{N}_i^*)^{bdd}, (T\partial\widetilde{N}_i^*)^{bdd})$$

and similarly $\mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})$. Now the quotient maps induce the commutative diagram

$$\begin{tikzcd}
\mathcal{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd}) \arrow[r, "\psi_8"] \arrow[d, "\xi_7"'] & \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd}) \arrow[d, "\xi_8"'] \\
\mathcal{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2^{bdd}) \arrow[r, "\omega_8"] \arrow[d, "\pi_7"'] & \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2^{bdd}) \arrow[d, "\pi_8"'] \\
W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \arrow[r, "\beta_8"] & \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})
\end{tikzcd}$$

We conclude with the following interpretation.

**Corollary 5.3.7.** *The map $\mathcal{L}$ from Definition* 5.2.6 *coincides with the map*

$$\Omega = \gamma \circ \pi_8 \circ \omega_8 \circ \omega_7 \circ \omega_6 \circ \omega_3 \circ \omega_2 \circ \omega_1 \circ \alpha_0'' \circ \alpha_0'$$

*and further with the map*

$$\Psi = \gamma \circ \pi_8 \circ \xi_8 \circ \psi_8 \circ \psi_7 \circ \psi_6 \circ \psi_3 \circ \psi_2 \circ \psi_1 \circ \alpha_0'' \circ \alpha_0'.$$

It remains to argue that $\Psi$ is an equivalence.

5.4. **Third Reduction.** To motivate further reduction of the map $\Psi$ in this section, we observe the following. The components $\psi_*$ of $\Psi$ are colimits of maps between diagrams indexed by the coverings $\mathcal{E}$, $\mathcal{U}_1$, and $\mathcal{U}_2$. The indexing set for all coverings is the same set of indices $\{(i,*)\}$, where $0 \leq i \leq n+k$ and $* = -$, $+$, or blank. The maps between the diagrams are identities on the indexing sets. We would like to argue that

(1) if one fixes an index $(i,*)$ then the composition

$$\omega^*_{8,i} \circ \omega^*_{7,i} \circ \omega^*_{6,i} \circ \omega^*_{3,i} \circ \omega^*_{2,i} \circ \omega^*_{1,i} \colon K^\Gamma_i(Th(E^*_i))^\Gamma \longrightarrow W^\Gamma_{T\widehat{B}^*_i}((T\widetilde{B}^*_i)^{bdd}, (T\partial\widetilde{B}^*_i)^{bdd})$$

is an equivalence for all $i$ and $*$, and

(2) depending on the index, the two spectra are either both contractible or both weakly equivalent to $\Sigma^{i+1}G^{-\infty}(R[\Gamma])$.

To prove both points we construct another sequence of coverings with the same indexing set, the associated $K$-theoretic homotopy colimits, and a sequence of natural maps between them. On each level $(i,*)$ the composition will be homotopic to the composition in (1). This time, on each level, all of the component maps will be weak equivalences. This will show that all natural induced maps between new colimits are equivalences, and so the map $\Psi$ is indeed an equivalence.

Consider the trivial action of $\Gamma$ on $\widetilde{N}$, that is the action by bounded coarse equivalences which are identity maps. The induced trivial action of $\Gamma$ on $(T\widetilde{N})^{bdd}$, including the subspaces $(T\widetilde{N}^*_i)^{bdd}$ and $(T\widetilde{B}^*_i)^{bdd}$, will be indicated by the 0 subscript. The equivalences of fixed point spectra

$$G^\Gamma_p((T\widetilde{N}^*_i)^{bdd}, (T\partial\widetilde{N}^*_i)^{bdd})^\Gamma \longrightarrow G^\Gamma_p((T\widetilde{N}^*_i)^{bdd}_0, (T\partial\widetilde{N}^*_i)^{bdd}_0)^\Gamma$$

from Theorem 9.10.5 induce the equivalence

$$\mathcal{G}^\Gamma_p((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2^{bdd})^\Gamma \longrightarrow \mathcal{G}^\Gamma_p((T\widetilde{N})^{bdd}_0, (T\partial\widetilde{N})^{bdd}_0; \mathcal{U}_2^{bdd})^\Gamma.$$

The same can be said about the equivalences

$$\zeta^*_i \colon G^\Gamma_p((T\widetilde{B}^*_i)^{bdd}, (T\partial\widetilde{B}^*_i)^{bdd})^\Gamma \longrightarrow G^\Gamma_p((T\widetilde{B}^*_i)^{bdd}_0, (T\partial\widetilde{B}^*_i)^{bdd}_0)^\Gamma,$$

however the target spectra are much easier to compute. According to Remark 9.10.6 we also have analogues in $K$-theory, so there is the evident commutative square

$$\begin{tikzcd}
\mathcal{K}^\Gamma_p((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \arrow[r, "\psi_6"] \arrow[d, "\zeta_3"'] & \mathcal{G}^\Gamma_p((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \arrow[d, "\zeta_6"'] \\
\mathcal{K}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}_2^{bdd})^\Gamma \arrow[r, "\psi'_6"] & \mathcal{G}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}_2^{bdd})^\Gamma
\end{tikzcd}$$

Recall that $B$ is a chosen metric ball in the interior of $N$, $h$ is a linear homeomorphism which maps the standard unit ball $D^{n+k}$ onto $B$, and $\widehat{B}$ is an isometric lift of $B$ to $\widetilde{N}$. The map $\sigma \colon B \to \widehat{B}$ is the isometry. From the definition of the parametrized transfer $\psi_3$ associated to the section $\sigma$ it is clear that we obtain a commutative triangle

$$\begin{tikzcd}
 & \mathcal{K}^\Gamma_p((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \arrow[dd, "\zeta_3"] \\
\mathcal{K}^\Gamma_c(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[ur, "\psi_3"] \arrow[dr, "\psi'_3"'] & \\
 & \mathcal{K}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}_2^{bdd})^\Gamma
\end{tikzcd}$$

where $\psi_3'$ is induced by the isometric inclusion $\sigma$. The inclusion of the fixed point category with respect to the trivial action gives

$$\psi_7' \colon \mathcal{G}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \longrightarrow \mathcal{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd}).$$

By the discussion above using Theorem 9.10.5, the natural transformation between the compositions of the exact functors

$$\tau \colon \mathcal{C}_c^\Gamma(TB_i^*, T\partial B_i^*)^\Gamma \longrightarrow \mathcal{C}_p^\Gamma((T\widetilde{B}_i^*)^{bdd}, (T\partial\widetilde{B}_i^*)^{bdd})^\Gamma$$

and

$$\tau' \colon \mathcal{C}_c^\Gamma(TB_i^*, T\partial B_i^*)^\Gamma \longrightarrow \mathcal{C}_p^\Gamma((T\widetilde{B}_i^*)_0^{bdd}, (T\partial\widetilde{B}_i^*)_0^{bdd})^\Gamma$$

which induce $\psi_3$ and $\psi_3'$, followed by the inclusions of $\mathcal{C}_p^\Gamma((T\widetilde{B}_i^*)^{bdd}, (T\partial\widetilde{B}_i^*)^{bdd})^\Gamma$ and $\mathcal{C}_p^\Gamma((T\widetilde{B}_i^*)_0^{bdd}, (T\partial\widetilde{B}_i^*)_0^{bdd})^\Gamma$ in the category $\mathbf{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd})$ is a natural equivalence. Now we see that

$$\psi_7 \circ \psi_6 \circ \psi_3 = \psi_7' \circ \psi_6' \circ \psi_3'.$$

*Notation* 5.4.1. We will use several families of subsets of $\widetilde{B}$:

$$\begin{aligned} &\widehat{B}_i^* = \sigma h(D_i^*), \quad \partial\widehat{B}_i^* = \partial\widehat{B} \cap \widehat{B}_i^* \\ &\ddot{B}_i^* = \widehat{B}_i^* \cup (\widetilde{B} - \widehat{B}), \quad \partial\ddot{B}_i^* = \partial\widehat{B}_i^* \cup (\widetilde{B} - \widehat{B}). \end{aligned}$$

**Definition 5.4.2.** Let $\mathcal{V}_3^{bdd}$ be the covering of $(T\widetilde{B})^{bdd}$ by the metric subspaces $T\ddot{B}_i^*$. Let $\mathcal{Y}$ be the covering of the metric subspace $T\widehat{B}$ of $(T\widetilde{N})^{bdd}$ by the metric subspaces $T\widehat{B}_i^*$. Both are coarse antithetic coverings.

We define the homotopy colimits

$$\mathcal{K}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_3^{bdd})^\Gamma \ = \ \underset{\overrightarrow{\mathcal{V}_3}}{\operatorname{hocolim}} \ K_p^\Gamma((T\ddot{B}_i^*)^{bdd}, (T\partial\ddot{B}_i^*)^\Gamma$$

and

$$\mathcal{K}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \ = \ \underset{\overrightarrow{\mathcal{Y}}}{\operatorname{hocolim}} \ K_p^\Gamma((T\widehat{B}_i^*)^{bdd}, (T\partial\widehat{B}_i^*)^\Gamma$$

and their evident counterparts $\mathcal{G}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_3^{bdd})^\Gamma$, $\mathcal{G}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma$.

The inclusions of metric spaces $\widetilde{B}_i^*$ in $\ddot{B}_i^*$, for each index $(i,*)$, induce maps in both theories that give the commutative square

$$\begin{array}{ccc} \mathcal{K}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_2^{bdd})^\Gamma & \xrightarrow{\psi_6'} & \mathcal{G}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \\ \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_6} \\ \mathcal{K}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_3^{bdd})^\Gamma & \xrightarrow{\psi_6''} & \mathcal{G}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_3^{bdd})^\Gamma \end{array}$$

Relaxing control and applying the Excision Theorem 2.7.16 gives the weak equivalence

$$r_3 \colon \mathcal{K}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_3^{bdd})^\Gamma \longrightarrow \mathcal{K}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma$$

There is an analogue of this map in $G$-theory.

**Proposition 5.4.3.** *There is a relax control map*
$$r_6 \colon \mathcal{G}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}^{bdd}_3)^\Gamma \longrightarrow \mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma$$
*that completes the commutative square*
$$\begin{tikzcd} \mathcal{K}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}^{bdd}_3)^\Gamma \arrow[r, "\psi''_6"] \arrow[d, "r_3"] & \mathcal{G}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}^{bdd}_3)^\Gamma \arrow[d, "r_6"] \\ \mathcal{K}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \arrow[r, "\psi'''_6"] & \mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \end{tikzcd}$$

*Proof.* First notice that Definition 2.6.8 and Proposition 2.6.10 can be promoted to the fixed points of the relative fibred $G$-theory as follows. Given any subset $C$ of $Y$, where $\Gamma$ acts by bounded coarse equivalences, there is the full subcategory $\mathbf{G}^\Gamma_p(Y, Y')_{<C}$ of $\mathbf{G}^\Gamma_p(Y, Y')$ invariant under the action. Therefore for the fixed objects there is the full subcategory $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_{<C}$ of $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma$ on the objects $(F, \psi)$ such that there is a number $k \geq 0$ and a function $\lambda \colon \mathcal{B}(Y) \to [0, +\infty)$ with
$$F(S) \subset F(S[k])(C[\lambda(S)])$$
for all bounded subsets $S$ of $\Gamma$. This is a right filtering Grothendieck subcategory by Proposition 9.9.2. We get the resulting exact quotient $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_{>C}$ as in 9.9.3 and its nonconnective $K$-theory $G^\Gamma_p(Y, Y')^\Gamma_{>C}$.

When $Y' \subset C$ and $C$ is invariant under the action of $\Gamma$, we have an isomorphism $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_{>C} \cong \mathbf{G}^\Gamma_p(Y, C)^\Gamma$, so
$$G^\Gamma_p(Y, Y')^\Gamma_{>C} \simeq G^\Gamma_p(Y, C)^\Gamma.$$
Suppose $Y$ is covered by mutually antithetic subsets $C$ and $U$, and $Y' \subset C$. Then applying the Excision Theorem 9.9.5 to this covering we get the map of two fibration sequences
$$\begin{tikzcd} G^\Gamma_p(Y, Y')^\Gamma_{<U,C} \arrow[r] \arrow[d] & G^\Gamma_p(Y, Y')^\Gamma_{<U} \arrow[r] \arrow[d] & G^\Gamma_p(Y, Y')^\Gamma_{<U,>C} \arrow[d] \\ G^\Gamma_p(Y, Y')^\Gamma_{<C} \arrow[r] & G^\Gamma_p(Y, Y')^\Gamma \arrow[r] & G^\Gamma_p(Y, Y')^\Gamma_{>C} \end{tikzcd}$$
where the square on the left is a homotopy pushout and the rightmost map is a weak equivalence. We get the composition
$$G^\Gamma_p(Y, Y')^\Gamma \longrightarrow G^\Gamma_p(Y, Y')^\Gamma_{>C} \xrightarrow{\simeq} G^\Gamma_p(Y, Y')^\Gamma_{<U,>C} \xrightarrow{\simeq} G^\Gamma_p(Y, C)^\Gamma_{<U}$$
If the action of $\Gamma$ is in fact trivial, we can extent the composition by the weak equivalence
$$G^\Gamma_p(Y, C)^\Gamma_{<U} \xrightarrow{\simeq} G^\Gamma_p(U_0, C \cap U_0)^\Gamma$$
according to Lemma 9.9.7 and Theorem 9.10.5.

This construction can be applied to the spaces $Y = (T\ddot{B}^*_i)^{bdd}_0$, $Y' = (T\partial\ddot{B}^*_i)^{bdd}_0$, and $C = T\ddot{B}^*_i \setminus T\operatorname{int}\widehat{B}$, $U = \widehat{B}^*_i$ in the case of the trivial action. The entire composition gives
$$r^*_{6,i} \colon \mathcal{G}^\Gamma_p((T\ddot{B}^*_i)^{bdd}_0, (T\partial\ddot{B}^*_i)^{bdd}_0)^\Gamma \longrightarrow \mathcal{G}^\Gamma_p(T\widehat{B}^*_i, T\partial\widehat{B}^*_i)^\Gamma.$$
Passing to homotopy colimits gives
$$\mathcal{G}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}^{bdd}_3)^\Gamma \longrightarrow \mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma$$

which is the required map. □

**Definition 5.4.4.** We define $\delta_3 = r_3 \circ i_3$ and $\delta_6 = r_6 \circ i_6$.

Now it is clear that there is a commutative diagram

$$\begin{tikzcd}
 & \mathcal{K}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}_2^{bdd})^\Gamma \arrow[d, "\delta_3"] \\
\mathcal{K}^\Gamma_c(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[ur, "\psi'_3"] \arrow[dr, "\psi'''_3"'] & \\
 & \mathcal{K}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma
\end{tikzcd}$$

where $\psi'''_3$ is induced from the isometry $\sigma\colon B \to \widehat{B}$. On the other hand, the inclusions of categories associated to the trivial actions in $\mathbf{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd})$ commute with the forget control retraction, so the triangle

$$\begin{tikzcd}
\mathcal{G}^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0; \mathcal{V}_2^{bdd})^\Gamma \arrow[dr, "\psi_7"] \arrow[dd, "\delta_6"'] & \\
 & \mathcal{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma \\
\mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \arrow[ur, "\psi'''_7"] &
\end{tikzcd}$$

also commutes.

**Definition 5.4.5.** We will need to use two additional maps from $\mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma$. One is the composition

$$\psi'''_8 = \psi_8 \circ \psi'''_7 \colon \mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \longrightarrow \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2^{bdd})^\Gamma$$

The other is the natural excision map

$$\varsigma\colon \mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \longrightarrow G^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma.$$

The map $\varsigma$ is a weak equivalence by the Excision Theorem 2.7.17. On the other hand, $\mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma$ can be computed as follows.

**Lemma 5.4.6.** *There is a weak equivalence*

$$G^\Gamma_p(T\widehat{B}_i, T\widehat{B}_i \cap T\partial\widehat{B})^\Gamma \longrightarrow \Sigma G^\Gamma_p(T\widehat{B}_{i-1}, T\widehat{B}_{i-1} \cap T\partial\widehat{B})^\Gamma$$

*for each index* $0 < i \leq n+k$.

*Proof.* Let $C_1$ and $C_2$ be any two of the subsets $T\widehat{B}^\pm_i$. The category

$$\mathbf{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma_{<C_1,C_2} = \mathbf{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma_{<C_1} \cap \mathbf{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma_{<C_2}$$

is a Grothendieck subcategory of $\mathbf{G}^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma$. According to Lemma 9.9.7, its $K$-theory can be computed as

$$G^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma_{<C_1,C_2} = G^\Gamma_p(C_1 \cap C_2, C_1 \cap C_2 \cap T\partial\widehat{B})^\Gamma.$$

Applying Theorem 2.7.17 to $T\widehat{B}$, $T\partial\widehat{B}$, $C_1$, $C_2$, we obtain a homotopy pushout square

$$\begin{tikzcd}
G_p^\Gamma(C_1 \cap C_2, C_1 \cap C_2 \cap T\partial\widehat{B})^\Gamma \arrow[r] \arrow[d] & G_p^\Gamma(C_1, C_1 \cap T\partial\widehat{B})^\Gamma \arrow[d] \\
G_p^\Gamma(C_2, C_2 \cap T\partial\widehat{B})^\Gamma \arrow[r] & G_p^\Gamma(C_1 \cup C_2, (C_1 \cup C_2) \cap T\partial\widehat{B})^\Gamma
\end{tikzcd}$$

Here $G_p^\Gamma(C_i, C_i \cap T\partial\widehat{B})^\Gamma$ are contractible by Lemma 9.10.2. If $C_1 = T\widehat{B}_i^\pm$ and $C_2 = T\widehat{B}_j^\pm$ with $j \le i$, so that $C_2 \subset C_1$, then $G_p^\Gamma(C_1 \cap C_2, C_1 \cap C_2 \cap T\partial\widehat{B})^\Gamma$ are also contractible. If $C_1 = T\widehat{B}_i^\pm$ and $C_2 = T\widehat{B}_i^\mp$ then

$$G_p^\Gamma(C_1 \cap C_2, C_1 \cap C_2 \cap T\partial\widehat{B})^\Gamma = G_p^\Gamma(T\widehat{B}_{i-1}, T\widehat{B}_{i-1} \cap T\partial\widehat{B})^\Gamma.$$

The induced map

$$G_p^\Gamma(T\widehat{B}_i, T\widehat{B}_i \cap T\partial\widehat{B})^\Gamma \longrightarrow \Sigma G_p^\Gamma(T\widehat{B}_{i-1}, T\widehat{B}_{i-1} \cap T\partial\widehat{B})^\Gamma$$

is the required weak equivalence. □

**Corollary 5.4.7.** *There is a weak equivalence*

$$\mathcal{G}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \longrightarrow \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]).$$

*Proof.* The sequence of weak equivalences

$$\mathcal{G}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \xrightarrow{\simeq} \Sigma^{n+k-1} G_p^\Gamma(T\widehat{B}_1, T\partial\widehat{B}_1)^\Gamma$$

can be obtained inductively from Lemma 5.4.6, and eventually

$$G_p^\Gamma(T\widehat{B}_1, T\partial\widehat{B}_1)^\Gamma \xrightarrow{\simeq} \Sigma G_p^\Gamma(T\widehat{B}_0)^\Gamma \xrightarrow{\simeq} \Sigma^2 G^{-\infty}(R[\Gamma]).$$

The composition gives the needed equivalence. □

**Definition 5.4.8.** Let $\psi_9'''$ be the equivalence from Corollary 5.4.7.

There is clearly a commutative triangle

$$\begin{tikzcd}
\mathcal{G}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \arrow[dr, "\psi_9'''"] \arrow[dd, "\psi_8'''"'] & \\
& \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]) \\
\mathcal{W}_{T\widehat{B}}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2)^\Gamma \arrow[ur] &
\end{tikzcd}$$

We have constructed the composition of maps

$$\Psi''' = \psi_9''' \circ \psi_6''' \circ \psi_3''' \circ \psi_2 \circ \psi_1 \circ \alpha_0'' \circ \alpha_0'$$

which agrees with the map $\Psi$ from Corollary 5.3.7. We are ready to argue that $\Psi'''$ is a weak equivalence.

5.5. **Finale.** We have established that $\psi_1 \circ \alpha_0'' \circ \alpha_0'$ and $\psi_9'''$ are weak equivalences. It remains to examine

$$\psi_6''' \circ \psi_3''' \circ \psi_2 \colon \mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \longrightarrow \mathcal{G}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma.$$

From the computations at the ends, this map is the $(n+k+1)$-fold suspension of the Cartan weak equivalence $K^{-\infty}(R[\Gamma]) \to G^{-\infty}(R[\Gamma])$ from Theorem 2.8.1 and part (5) of Theorem 10.2.2. We conclude that $\psi_6''' \circ \psi_3''' \circ \psi_2$ and, therefore, $\Psi'''$ is an equivalence.

By Theorem 5.2.7, the composition $\phi$ is a weak equivalence. It follows that the canonical map $\rho$ from Theorem 5.1.3 is a split injection, in conjunction with being a splitting of an injection. This verifies the following theorem.

**Theorem 5.5.1.** *Suppose $\Gamma$ is the fundamental group of a closed aspherical manifold and the canonical map*

$$\rho(\Gamma, R)\colon\ K^{-\infty}(\Gamma, R)^\Gamma \longrightarrow K^{-\infty}(\Gamma, R)^{h\Gamma}$$

*is a splitting of the integral $K$-theoretic assembly map*

$$a(\Gamma, R)\colon h(B\Gamma, K^{-\infty}(R)) \longrightarrow K^{-\infty}(R[\Gamma]).$$

*If $\Gamma$ is weakly regular noetherian and $R$ has finite homological dimension, then $a(\Gamma, R)$ is, in fact, an equivalence.*

Since the Novikov Conjecture in $K$-theory for groups of finite asymptotic dimension is verified in [12] by showing that $\rho(\Gamma, R)$ is a splitting of $a(\Gamma, R)$ in that case, this theorem gives the Main Theorem from the Introduction.

**Theorem 5.5.2** (Main Theorem)**.** *Suppose $\Gamma$ is the fundamental group of a closed aspherical manifold. If $\Gamma$ has finite asymptotic dimension and $R$ has finite homological dimension, then the assembly map $a(\Gamma, R)$ is an equivalence.*

5.6. **The Complete Diagram.** Figures 3 through 5 show the cumulative commutative diagram showing the relations between maps constructed in the course of the proof of Theorem 5.5.1.

$$
\begin{tikzcd}
 & & & & K_i^\Gamma(T\widetilde{N}, T\partial\widetilde{N})^\Gamma \arrow[r, "\beta_3"] & {} \\
 & \Sigma^{n+k+1}K^{\Gamma,0}(\widetilde{M},R)^{h\Gamma} \arrow[rr, "\beta_1 = T\eta"] \arrow[dr, "\Sigma\eta"'] & & K_i^\Gamma(T\widetilde{N})^\Gamma \arrow[ur, "\beta_2"] & & \\
 \Sigma^{n+k+1}K^{\Gamma,0}(\widetilde{M},R)^\Gamma \arrow[ur, "\Sigma^{n+k+1}\rho"] \arrow[dr, "\alpha_0", "\simeq"'] \arrow[dddd, "\alpha_0'", "\simeq"'] & & \Sigma K_i^\Gamma(\widetilde{N})^\Gamma \arrow[ur, "\sigma", "\simeq"'] & & K_c^\Gamma(TN, T\partial N)^\Gamma \arrow[r, "\alpha_3"] & {} \\
 & K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma \arrow[rr, "\alpha_1"] \arrow[dr] & & K_i^\Gamma(TN, T\partial N)^\Gamma \arrow[ur, "\alpha_2"] & & \\
 & & K_i^\Gamma(T\mathbb{R}^{n+k}, T\mathbb{R}^{n+k} - \operatorname{int} TN)^\Gamma \arrow[ur] & & \mathcal{K}_c^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma \arrow[uu, "\pi_2"'] \arrow[r, "\omega_3"] & {} \\
 & \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k}; \mathcal{E})^\Gamma \arrow[uu, "\alpha_0'''", "\simeq"'] \arrow[rr, "\omega_1"] \arrow[dr] \arrow[drr, "\psi_1", "\simeq"'] \arrow[dddr] & & \mathcal{K}_i^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma \arrow[uu, "\pi_1"'] \arrow[ur, "\omega_2"] & & \\
 \Sigma^{n+k+1}K^{-\infty}(R[\Gamma]) \arrow[ur, "\alpha_0''", "\simeq"'] & & \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k}, T\mathbb{R}^{n+k} - \operatorname{int} TB; \mathcal{E})^\Gamma \arrow[ur] & & \mathcal{K}_c^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[uu, "\xi_2"'] \arrow[r, "\psi_3"] & {} \\
 & & & \mathcal{K}_i^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[uu, "\xi_1"'] \arrow[ur, "\psi_2"] & & \\
 & & \mathcal{K}_i^\Gamma(T\mathbb{R}^{n+k}, T\mathbb{R}^{n+k} - \operatorname{int} TB; \mathcal{E})^\Gamma \arrow[ur] & & &
\end{tikzcd}
$$

FIGURE 3. The Complete Diagram. Part 1 of 3.

$$
\begin{tikzcd}
{} \arrow[r, "\beta_3"] & K_c^\Gamma(T\widetilde{N}, T\partial\widetilde{N})^\Gamma \arrow[r, "\beta_4"] & K_c^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \arrow[dr, bend left, "\beta_5"] & & & \\
 & & & K_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \arrow[r, "\beta_6"] & G_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \arrow[r, "\beta_7"] & {} \\
{} \arrow[r, "\alpha_2"] & K_c^\Gamma(TN, T\partial N)^\Gamma \arrow[urr, bend left, "\alpha_3"] & & & & \\
 & & & \mathcal{K}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2)^\Gamma \arrow[uu, "\pi_3"] \arrow[r, "\omega_6"] & \mathcal{G}_p^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2)^\Gamma \arrow[uu, "\pi_6"] \arrow[r, "\omega_7"] & {} \\
{} \arrow[r, "\omega_2"] & \mathcal{K}_c^\Gamma(TN, T\partial N; \mathcal{U}_1)^\Gamma \arrow[uu, "\pi_2"] \arrow[urr, bend left, "\omega_3"] & & & & \\
 & & & \mathcal{K}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd.}; \mathcal{V}_2)^\Gamma \arrow[uu, "\xi_3"] \arrow[r, "\psi_6"] & \mathcal{G}_p^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd.}; \mathcal{V}_2)^\Gamma \arrow[uu, "\xi_6"] \arrow[r, "\psi_7"] & {} \\
{} \arrow[r, "\psi_2"] & \mathcal{K}_c^\Gamma(TB, T\partial B; \mathcal{V}_1)^\Gamma \arrow[uu, "\xi_2"] \arrow[urr, bend left, "\psi_3"] \arrow[drr, bend right, "\psi_3'"] \arrow[ddrr, bend right, "\psi_3''"] & & & & \\
 & & & \mathcal{K}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd.}; \mathcal{V}_2)^\Gamma \arrow[d, "\delta_3"] \arrow[r, "\psi_6'"] & \mathcal{G}_p^\Gamma((T\widetilde{B})_0^{bdd}, (T\partial\widetilde{B})_0^{bdd}; \mathcal{V}_2)^\Gamma \arrow[d, "\delta_6"] \arrow[r, "\psi_7'"] & {} \\
 & & & \mathcal{K}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \arrow[r, "\psi_6''"] & \mathcal{G}_p^\Gamma(T\widehat{B}, T\partial\widehat{B}; \mathcal{Y})^\Gamma \arrow[r, "\psi_7''"] & {}
\end{tikzcd}
$$

FIGURE 4. The Complete Diagram. Part 2 of 3.

$$
\begin{tikzcd}
{} \arrow[r, "\beta_7"] & W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \arrow[r, "\beta_8"] & W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \arrow[r] \arrow[ddr, "\gamma"] & \Sigma^{n+k} W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<T\widehat{B}_0} \arrow[dd, bend left] & \\
{} \arrow[r, "\omega_7"] & \mathcal{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2) \arrow[r, "\omega_8"] \arrow[u, "\pi_7"] & \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}; \mathcal{U}_2) \arrow[u, "\pi_8"] \arrow[ur, bend right] & & \\
{} \arrow[r, "\psi_7"] & \mathcal{W}^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2) \arrow[r, "\psi_8"] \arrow[u, "\xi_7"] & \mathcal{W}^\Gamma_{T\widehat{B}}((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}; \mathcal{V}_2) \arrow[u, "\xi_8"] \arrow[rr] & & \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]) \\
{} \arrow[ur, bend right, "\psi_7'"] \arrow[r, "\psi_6'''"] & \mathcal{G}^\Gamma_p(T\widehat{B}, T\partial\widetilde{B}; \mathcal{Y})^\Gamma \arrow[u, "\psi_7'''"] \arrow[ur, "\psi_8'''"] \arrow[uuurr, bend right] \arrow[urrr, bend right, "\psi_9'''", "\simeq"'] \arrow[dr, "\varsigma", "\simeq"'] & & & \\
& & G^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma \arrow[uurr, bend right] & &
\end{tikzcd}
$$

FIGURE 5. The Complete Diagram. Part 3 of 3.

5.7. **Concluding Remarks.**

**Remark 5.7.1.** There is a useful point of view of the Main Theorem. One of the goals in our paper [13] was to define and study the assembly map in $G$-theory using the methods that were successful in $K$-theory [9, 12]. This is a map of the type

$$A_G \colon B\Gamma_+ \wedge G^{-\infty}(R) \longrightarrow G^{-\infty}(R[\Gamma]).$$

Here, since we assume $R$ is a noetherian ring, $G^{-\infty}(R)$ can be any of the nonconnective $G$-theory spectra of $R$ described in section 2.8.

The standard exact structure on finitely generated $R[\Gamma]$-modules has all injective and surjective $R[\Gamma]$-homomorphisms with finitely generated cokernels and kernels as admissible morphisms so that the exact sequences are the usual short exact sequences. It can be seen from Remark 2.23 in Lück [35] that with the classical $G$-theory of $R[\Gamma]$ as target the map $A_G$ would not be injective even in the case when $R$ is the ring of complex number $\mathbb{C}$ and $\Gamma$ is the free group on two generators $F_2$. Lück's computation shows that the class $[\mathbb{C}]$ is zero in $G_0(\mathbb{C}[\mathbb{Z}])$. Similarly, the theory $G^{-\infty}(R[\Gamma])$ presented in section 10 is still not excisive enough to allow a splitting argument similar to [9, 12]. It is shown in [13] that full restriction to *strictly insular* objects in $\mathbf{B}(R[\Gamma])$ instead of just *insular* suffices to prove that $A_G$ is split injective.

This shows that there is a possibility to avoid reference to $K$-theoretic assembly map in the proof of the Main Theorem in section 5.1 and instead prove a $G$-theoretic theorem.

**Theorem A.** *If $\Gamma$ is a geometrically finite group and $R$ is a noetherian ring then the canonical map*

$$\rho_G \colon G^{-\infty}(R[\Gamma]) = G^{-\infty}(\Gamma, R)^{\Gamma} \longrightarrow G^{-\infty}(\Gamma, R)^{h\Gamma}$$

*is a split injection. This map can be viewed as a splitting of the $G$-theoretic assembly map $a_G$. Therefore, $a_G$ is an equivalence.*

Theorem A can indeed be proved. From [12] we know that the conditions of this theorem are satisfied by groups of finite asymptotic dimension.

The connection with the $K$-theoretic result in this paper is established through the result contained in Theorem 10.2.2 and [11] which shows that groups of finite asymptotic dimension are weakly regular noetherian, that is for any regular noetherian ring $R$ of finite homological dimension all $R[\Gamma]$-modules in $\mathbf{B}(R[\Gamma])$ have finite homological dimension. From Quillen's Resolution Theorem one has the following result.

**Theorem B.** *If $\Gamma$ is a weakly regular noetherian group and $R$ has finite homological dimension, the Cartan map*

$$\kappa \colon\ K^{-\infty}(R[\Gamma]) \longrightarrow G^{-\infty}(R[\Gamma])$$

*is a weak equivalence.*

The combination of Theorems A and B and the commutative square

$$\begin{tikzcd}
h(B\Gamma, K^{-\infty}(R)) \arrow[r, "a_K"] \arrow[d, "\simeq"] & K^{-\infty}(R[\Gamma]) \arrow[d, "\kappa", "\simeq"'] \\
h(B\Gamma, G^{-\infty}(R)) \arrow[r, "a_G", "\simeq"'] & G^{-\infty}(R[\Gamma])
\end{tikzcd}$$

proves our Main Theorem.

The value in this separation into two theorems is that Theorem A is essentially geometric while Theorem B is combinatorial.

**Remark 5.7.2.** This is a good place to clarify one misconception that we had to address since the appearance of first drafts of [13]. It has to do with the notion of bounded *bi*control introduced in [13] as opposed to bounded control. A map between lean, strictly insular, locally finite $R$-modules can be bounded without being bicontrolled.

Let us go back to W. Lück's example mentioned in Remark 5.7.1. The fact that $[\mathbb{C}] = 0$ in $G_0(\mathbb{C}[\mathbb{Z}])$ follows from the exact sequence

$$0 \longrightarrow \mathbb{C}[\mathbb{Z}] \xrightarrow{s-\mathrm{id}} \mathbb{C}[\mathbb{Z}] \longrightarrow \mathbb{C} \longrightarrow 0$$

where $s$ a generator of $\mathbb{Z}$. Let's take $s = 1$ for instance. By contrast, this sequence is not exact in $\mathbf{B}(\mathbb{C}[\mathbb{Z}])$ because $s - \mathrm{id}$ is bounded but not boundedly bicontrolled, which is required of admissible monomorphisms. Indeed, in the category $\mathbf{B}(\mathbb{C}[\mathbb{Z}])$ the group ring $\mathbb{C}[\mathbb{Z}]$ is a free geometric $\mathbb{C}$-module $F$ parametrized over $\mathbb{Z}$ with $\mathbb{Z}$-filtration $F(S) = \bigoplus_{n \in S} \mathbb{C}$. Given any integral interval $[a, b]$ in $\mathbb{Z}$, the image of $1_a + \ldots + 1_b$ is $-1_a + 1_{b+1}$. As $s - 1$ is injective and

$$-1_a + 1_{b+1} \in \mathbb{C}[\mathbb{Z}](\{a\} \cup \{b + 1\}),$$

we see that if $-1_a + 1_{b+1} \in (s - 1)F(T)$ for $T \subset \mathbb{Z}$ then $[a, b] \subset T$. Since $b - a$ can be chosen arbitrarily large, this shows that $s - 1$ is not bicontrolled.

**Remark 5.7.3.** The fixed points of coarse actions in the proof of Theorem 5.2.7 are represented by categories of parametrized modules that are no longer free, cf. Remark 2.5.5. Therefore the Karoubi filtration techniques are no longer sufficient for localization. This leads naturally to definition of controlled $G$-theory and localization with respect to Grothendieck subcategories.

**Remark 5.7.4.** It has been the experience with the methods employed here that an application to $K$-theoretic assembly maps can be modified to yield a similar result in algebraic $L$-theory. It is well-known that such theorem combined with vanishing of the Whitehead group from this paper would give a proof of topological rigidity of closed manifolds whose fundamental group has finite asymptotic dimension.

# Part 2. Proper $S$-duality

## 6. Homotopy Theoretic Preliminaries

### 6.1. **Homotopy Limits and Colimits.**

We will be performing a number of constructions involving homotopy limits and colimits. Let $\mathcal{C}$ be a category, and let $F \colon \mathcal{C} \to \mathbf{s.Sets}$ be a functor into the category of simplicial sets. Then recall that Bousfield and Kan [7] define the notions of homotopy colimit and limit

$$\underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}\, F \quad \text{and} \quad \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\, F$$

as the total spaces of simplicial (respectively cosimplicial) objects in the category of simplicial sets. If $F \colon \mathcal{C} \to \mathbf{Top}$, where $\mathbf{Top}$ denotes the category of topological spaces, we define the homotopy colimits and limits by the formulas

$$\underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}\, F = |\underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}\, S.\circ F|, \quad \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\, F = |\underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\, S.\circ F|,$$

where $|\_\_|$ denotes geometric realization and where $S.\colon$ **Top** $\to$ **s.Sets** denotes the singular complex functor.

**Proposition 6.1.1.** *Let $F$, $G\colon \mathcal{C} \to$ **s.Sets** (or **Top**) be functors, and suppose that $N\colon F \to G$ is a natural transformation. Then $N$ induces maps*

$$N_*\colon \underset{\overrightarrow{\mathcal{C}}}{\text{hocolim}}\ F \longrightarrow \underset{\overrightarrow{\mathcal{C}}}{\text{hocolim}}\ G,$$
$$N_*\colon \underset{\overleftarrow{\mathcal{C}}}{\text{holim}}\ F \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\text{holim}}\ G$$

*Suppose $F\colon \mathcal{D} \to$ **s.Sets** (or **Top**) is a functor, and that $f\colon \mathcal{C} \to \mathcal{D}$ is a functor. Then $f$ induces a natural map*

$$f^*\colon \underset{\overleftarrow{\mathcal{D}}}{\text{holim}}\ F \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\text{holim}}\ F \circ f.$$

It will be useful to have criteria which guarantee that these maps are weak equivalences.

**Proposition 6.1.2.** *Let $F$, $G\colon \mathcal{C} \to$ **s.Sets** (or **Top**), and suppose $N\colon F \to G$ is a natural transformation, so that $N(c)\colon F(c) \to G(c)$ is a weak equivalence for every object $c$ of $\mathcal{C}$. Then the maps of homotopy limits and colimits induced by $N$ as in Proposition* 6.1.1 *are weak equivalences.*

In order to study the maps $f^*$, we will need some terminology. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let $x$ be an object in $\mathcal{D}$. Then the category $f{\downarrow}x$ has as objects the collection of pairs $(z, \theta)$, where $z$ is an object in $\mathcal{C}$ and $\theta\colon f(z) \to x$ is a morphism in $\mathcal{D}$. A morphism from $(z, \theta)$ to $(z', \theta')$ in $f{\downarrow}x$ is a morphism $\varphi\colon z \to z'$ so that $\theta' \circ \varphi = \theta$. We also recall the nerve construction, which associates to any category a simplicial set, in such a way that functors induce maps of nerves and natural transformations induce homotopies between maps.

**Proposition 6.1.3.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F\colon \mathcal{D} \to$ **s.Sets** (or **Top**) and $f\colon \mathcal{C} \to \mathcal{D}$ be functors. If $F$ takes values in **s.Sets**, we assume that $F(x)$ is a Kan complex for every object $x$ in $\mathcal{D}$. Suppose further that for every object $x$ in $\mathcal{D}$, the nerve of the category $f{\downarrow}x$ is weakly contractible. Then*

$$f^*\colon \underset{\overleftarrow{\mathcal{D}}}{\text{holim}}\ F \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\text{holim}}\ F \circ f$$

*is also a weak equivalence.*

In view of the preceding proposition, it will also be useful to have criteria which guarantee the weak contractibility of the nerve of a category. Here is a very useful one.

**Definition 6.1.4.** We say a category $\mathcal{C}$ is *left filtering* if the following two conditions hold.

- For every pair of objects $x$ and $y$ in $\mathcal{C}$, there is an object $z$ in $\mathcal{C}$ and there are morphisms $u : z \to x$ and $v\colon z \to y$ in $\mathcal{C}$.
- If $u$, $v\colon x \to y$ are morphisms in $\mathcal{C}$, then there is an object $z$ in $\mathcal{C}$ and a morphism $w\colon z \to x$ so that $uw = vw$

These conditions hold, in particular, when $\mathcal{C}$ has an initial object.

**Proposition 6.1.5.** *The nerve of a left filtering category is weakly contractible.*

6.2. **Spectra, Module Spectra, and Pairings of Spectra.** We recall the notions of cofibre (or equivalently, fibre) sequences of spectra, as well as maps of spectra, homotopy groups of spectra, and weak equivalences of spectra.

When we speak of spectra, we will be referring to the $\Omega$-spectra, that is, families $\{X_i, \sigma_i\}_{i\geq 0}$, where $X_i$ is a based topological space and where $\sigma_i \colon X_i \to \Omega X_{i+1}$ is a based homotopy equivalence. A prespectrum will be a family of this type but without the requirement that the map $\sigma_i$ be a homotopy equivalence. Let $\mathcal{S} = \{S_i, \sigma_i\}_{i\geq 0}$ be a spectrum, and let $X$ be a based topological space. We can then form the prespectrum $X \wedge \mathcal{S} = \{X \wedge S_i, \mathrm{id} \wedge \sigma_i\}$. There is a canonical construction which constructs for any prespectrum an associated spectrum, in a way which is functorial and which produces a weakly equivalent spectrum when applied to a spectrum. We apply this construction to $X \wedge \mathcal{S}$ to obtain a spectrum which is denoted by $h(X, \mathcal{S})$ and refer to as the homology spectrum of $X$ with coefficients in $\mathcal{S}$.

**Proposition 6.2.1.** *The functor $X \to \pi_*(h(X, \mathcal{S}))$ is a generalized homology theory in $X$. In particular, cofibre sequences induce fibre sequences of spectra and hence long exact sequences of homotopy groups. Further, if $X$ and $Y$ are based spaces, we have a natural transformation*

$$h(X, h(Y, \mathcal{S})) \longrightarrow h(X \wedge Y, \mathcal{S})$$

*which is an equivalence when $X = S^0$.*

Again, let $X$ be a based space and $\mathcal{S} = \{S_i, \sigma_i\}$ be a spectrum. Then we define the function spectrum $F(X, \mathcal{S})$ to be the family $\{F(X, S_i), F(\mathrm{id}_X, \sigma_i)\}$. Here, $F(X, S_i)$ denotes the space of based continuous maps from $X$ to $S_i$, in the compact open topology. Now $F(\underline{\quad}, \mathcal{S})$ is a contravariant functor from based spaces to spectra. It is referred to as the cohomology spectrum of $X$ with coefficients in $\mathcal{S}$.

**Proposition 6.2.2.** *The functor $X \to \pi_*(F(X, \mathcal{S}))$ is a generalized cohomology theory in $X$. In particular, cofibration sequences in $X$ induce fibre sequences on $F(X, \mathcal{S})$ and hence long exact sequences on $\pi_*(F(X, \mathcal{S}))$. There is also a natural transformation*

$$\nu \colon h(F(X, Y), \mathcal{S}) \longrightarrow F(X, h(Y, \mathcal{S}))$$

*where $X$ and $Y$ are based spaces and $F(X, Y)$ denotes the space of based maps from $X$ to $Y$ equipped with the compact open topology. This transformation is a weak equivalence of spectra when $X = S^0$. In addition, the function spectrum construction has the property that the "adjoint" natural transformation*

$$F(h(X, h(Y, \mathcal{S})), T) \longrightarrow F(h(Y, \mathcal{S}), F(X, T))$$

*is a weak equivalence of spectra for all based spaces $X$ and $Y$ and all spectra $\mathcal{S}$.*

The homotopy colimit and limit constructions introduced above also apply to spectra.

**Definition 6.2.3.** Let $F \colon \mathcal{C} \to$ **Spectra** be a functor, so we have functors $F_i \colon \mathcal{C} \to$ **Top**$_*$ (the category of based topological spaces and based maps) and natural transformations $\Sigma_i \colon F_i \to \Omega F_{i+1}$ so that for each object $c$ in $\mathcal{C}$, $\{F_i(c), \Sigma_i(c)\}$ forms a spectrum. We can then construct the homotopy limits $\operatorname{holim} F_i$ and the maps

$$s_i \colon \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, F_i \xrightarrow{\Sigma_i(c)_*} \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, \Omega F_{i+1} \xrightarrow{\simeq} \Omega \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\, F_{i+1}$$

so that $\{\underleftarrow{\operatorname{holim}}_{\mathcal{C}} F_i, s_i\}_{i\geq 0}$ is now a spectrum, which we write as $\underleftarrow{\operatorname{holim}}_{\mathcal{C}} F$ and refer to as the homotopy limit spectrum. Similarly, we can construct the spaces $\operatorname{hocolim} F_i$, and the maps

$$s_i\colon \underrightarrow{\operatorname{hocolim}}_{\mathcal{C}} F_i \xrightarrow{\Sigma_{i*}} \underrightarrow{\operatorname{hocolim}}_{\mathcal{C}} \Omega F_i \longrightarrow \Omega \underrightarrow{\operatorname{hocolim}}_{\mathcal{C}} F_i.$$

Now $\{\underrightarrow{\operatorname{hocolim}}_{\mathcal{C}} F_i, s_i\}_{i\geq 0}$ is a prespectrum. The spectrum associated functorially to this prespectrum will be referred to as the homotopy colimit spectrum of $F$, and will be written $\underrightarrow{\operatorname{hocolim}}_{\mathcal{C}} F$.

**Proposition 6.2.4.** *The spectrum homotopy colimits and limits have exactly the same functoriality properties as the corresponding space level constructions. Moreover, the criteria in Propositions* 6.1.2 *and* 6.1.3 *apply to the spectrum level constructions. In addition, consider a sequence $F \to G \to H$ of spectrum valued functors on $\mathcal{C}$ where the arrows are natural transformations. We say the sequence is a cofibre (or fibre) sequence if it is a cofibre sequence for any given object of $\mathcal{C}$. Cofibre sequences of functors induce cofibre sequences of homotopy limits or colimits.*

Finally, we recall that a simplicial object in **Spectra** is called a *simplicial spectrum.* One can form the geometric realization of a simplicial spectrum by first applying the realization construction to each space in the spectrum, to obtain a prespectrum, and then functorially associating to that prespectrum a spectrum.

**Proposition 6.2.5.** *Let $S.$ and $T.$ be simplicial spectra, and let $f\colon S. \to T.$ be a map of simplicial spectra so that $f_k\colon S_k \to T_k$ is a weak equivalence for each $k \geq 0$. Then the induced map $|S.| \to |T.|$ is a weak equivalence of spectra.*

## 7. Locally Finite Homology and Proper $S$-duality

7.1. **Review of $S$-duality for Compact Manifolds with Boundary.** $S$-duality tells us that if $X$ is a based finite complex and $\mathcal{S}$ is a spectrum, then the function spectrum $F(X, \mathcal{S})$ can be described as an appropriate desuspension of $h(Z, \mathcal{S})$ for some based finite complex $Z$. When $X$ is a compact smooth manifold with boundary, one obtains a particularly nice description of the complex $Z$ in terms of the Thom complex of the normal bundle to a smooth embedding of $X$ in a Euclidean space. We will eventually generalize this result to noncompact manifolds, but first we want to state the result for compact manifolds with boundary in a form we can use in later proofs.

Let $(X^n, \partial X^n)$ be a compact smooth manifold with boundary, and let $\xi \to X$ be a vector bundle equipped with a trivialization from $\xi \oplus \tau_X$ to $\varepsilon^m$, where $m = n + \dim(\xi)$, and where $\tau_X$ denotes the tangent bundle to the manifold $X$. Equip $\xi$ and $X$ with a Riemannian metric, and let $D(\xi)$ and $S(\xi)$ denote the unit disk and sphere bundles in $\xi$, respectively. Let $s\colon X \to D(\xi)$ denote the zero section, and let $e\colon X \to D(\xi) \times X$ be the embedding $s \times \mathrm{id}$, with image denoted by $\Sigma$. Equip $D(\xi) \times X$ with the product Riemannian metric. For $\varepsilon < 1$ and sufficiently small, $N_\varepsilon(\Sigma)$, the closed $\varepsilon$-neighborhood of $\Sigma$, is canonically diffeomorphic to the total space of a disk bundle in $\xi \oplus \tau_X$. This means that if we let $\partial N_\varepsilon(\Sigma)$ be the set of points a distance exactly $\varepsilon$ from $\Sigma$, we get a composite

$$\lambda\colon \frac{D(\xi)\times X}{D(\xi)\times X-\operatorname{int} N_\varepsilon(\Sigma)} \simeq \frac{N_\varepsilon(\Sigma)}{\partial N_\varepsilon(\Sigma)} \xrightarrow{\alpha} \frac{D(\xi\oplus\tau_X)}{S(\xi\oplus\tau_X)} \xrightarrow{\beta} \frac{D(\varepsilon^m)}{S(\varepsilon^m)} \xrightarrow{\gamma} S^m$$

where $\alpha$ is the identification from the Tubular Neighborhood Theorem, $\beta$ is induced by the trivialization of $\xi\oplus\tau_X$, and $\gamma$ is the projection

$$\frac{D(\varepsilon^m)}{S(\varepsilon^m)} \simeq \frac{X\times D^m}{X\times S^{m-1}} \longrightarrow \frac{D^m}{S^{m-1}}.$$

Now consider the composite

$$D(\xi)\times(X-N_\varepsilon(\partial X)) \longrightarrow \frac{D(\xi)\times X}{D(\xi)\times X-\operatorname{int} N_\varepsilon(\Sigma)} \xrightarrow{\lambda} S^m.$$

This composite has an adjoint map

$$\varphi\colon D(\xi)\longrightarrow F((X-N_\varepsilon(\partial X))_+, S^m),$$

where the subscript $+$ denotes the addition of a disjoint basepoint. (This has the effect of producing the "unbased" mapping space.) We claim that the restriction of $\varphi$ to $S(\xi)\cup D(\xi|\partial X)$ is trivial. To see this, we note first that points in $S(\xi)$ are at a distance one from the image of $s$, which means that in the product $D(\xi)\times X$, points of $S(\xi)\times X$ are a distance one from $\Sigma$. Since $\varepsilon<1$, we have that

$$S(\xi)\times X\cap N_\varepsilon(\Sigma)=\emptyset,$$

so

$$S(\xi)\subset D(\xi)\times(X-\operatorname{int} N_\varepsilon(\Sigma)),$$

and hence $S(\xi)\times X$ maps trivially under $\lambda$. From the definition of the adjoint, it follows that $S(\xi)$ maps trivially under $\varphi$. To examine where $D(\xi|\partial X)$ is mapped under $\varphi$, we claim that

$$D(\xi|\partial X)\times(X-\operatorname{int} N_\varepsilon(\partial X))\cap N_\varepsilon(\Sigma)=\emptyset.$$

We note that any point in $\Sigma\cap D(\xi|\partial X)$ has second coordinate in $\partial X$. Hence it is a distance at least $\varepsilon$ from $D(\xi|\partial X)\times(X-\operatorname{int} N_\varepsilon(\partial X))$. This means that

$$D(\xi|\partial X)\times(X-\operatorname{int} N_\varepsilon(\partial X))\subset D(\xi)\times X-\operatorname{int} N_\varepsilon(\Sigma),$$

so $(\xi|\partial X)\times(X-\operatorname{int} N_\varepsilon(\partial X))$ maps trivially under $\lambda$. The definition of the adjoint now tells us that $D(\xi|\partial X)$ maps trivially under $\varphi$. The conclusion is that we obtain a map

$$\frac{D(\xi)}{S(\xi)\cup D(\xi|\partial X)} \longrightarrow F((X-N_\varepsilon(\partial X))_+, S^m).$$

Since $\varepsilon$ is chosen so that the inclusion $X-N_\varepsilon(\partial X)\subset X$ is a homotopy equivalence, we in fact have determined a map

$$\overline{\varphi}\colon \frac{D(\xi)}{S(\xi)\cup D(\xi|\partial X)} \longrightarrow F(X_+, S^m).$$

Now, let $\mathcal{S}=\{S_i,\sigma_i\}_{i\geq 0}$ be a spectrum. We define $\widehat{\varphi}$ to be the composite

$$h\left(\frac{D(\xi)}{S(\xi)\cup D(\xi|\partial X)};\mathcal{S}\right) \xrightarrow{h(\overline{\varphi},\mathcal{S})} h(F(X_+,S^m);\mathcal{S}) \xrightarrow{\nu} F(X_+, h(S^m,\mathcal{S}))$$

Standard Spanier–Whitehead duality theory shows easily that this composite is a weak equivalence of spectra.

7.2. **Locally Finite Homology with Coefficients in a Spectrum.** An important goal in this paper is the development of a version of the duality theory which holds for noncompact manifolds. A key ingredient in that version is locally finite homology, which we will now develop in a form which is suitable for proving a duality theorem. Locally finite homology was also developed in [9], in a form which is suitable for defining assembly maps. We will prove later that these two versions agree, at least for locally finite simplicial complexes.

For any space $X$, let $C(X)$ denote the partially ordered set of closed subsets $U$ of $X$ so that $X-U$ is relatively compact, i.e. so that the closure of $X-U$ is compact. Note that if $f\colon X\to Y$ is a proper map, then we have a functor $f^*\colon C(Y)\to C(X)$ given on objects by $f^*U=f^{-1}U$. The behavior of $f^*$ on morphisms is forced since there is at most one morphism between any pair of objects in $C(X)$. We must still show that $f^{-1}U\in C(X)$. This means we must show that $X-f^{-1}U$ is relatively compact, i.e. that the closure $\mathrm{cl}(X-f^{-1}U)$ is compact. But $X-f^{-1}U=f^{-1}(Y-U)$, so

$$\mathrm{cl}(X-f^{-1}U)=\mathrm{cl}\,f^{-1}(Y-U)\subset f^{-1}(\mathrm{cl}(Y-U)).$$

Since $U\in C(Y)$, $\mathrm{cl}(Y-U)$ is compact, and since $f$ is proper, so is $f^{-1}(\mathrm{cl}(Y-U))$. But now $\mathrm{cl}\,f^{-1}(Y-U)$ is a closed subset of the compact set $f^{-1}(\mathrm{cl}(Y-U))$, hence it too is compact. We have now defined $f^*$.

**Definition 7.2.1.** Let $\mathcal{S}$ be a spectrum, and $X$ a topological space. Then we define a functor $\Phi^X_{\mathcal{S}}\colon C(X)\to\textbf{Spectra}$ by $\Phi^X_{\mathcal{S}}=h(X/U,\mathcal{S})$. We now define

$$h^{lf}(X,\mathcal{S})=\underset{\overleftarrow{C(X)}}{\mathrm{holim}}\ \Phi^X_{\mathcal{S}}.$$

This assignment of a spectrum to a space is functorial for proper maps. The induced map $h^{lf}(f,\mathcal{S})$ is defined to be the composite

$$\underset{\overleftarrow{C(X)}}{\mathrm{holim}}\ \Phi^X_{\mathcal{S}}\xrightarrow{(a)}\underset{\overleftarrow{C(Y)}}{\mathrm{holim}}\ \Phi^X_{\mathcal{S}}\circ f^*\xrightarrow{(b)}\underset{\overleftarrow{C(Y)}}{\mathrm{holim}}\ \Phi^Y_{\mathcal{S}}$$

The map (a) is the usual pullback map of homotopy inverse limits along a functor of the parameter categories. The map (b) is $N_*$, where $N\colon\ \Phi^X_{\mathcal{S}}\circ f^*\to\Phi^Y_{\mathcal{S}}$ is a natural transformation which we describe as follows: $\Phi^X_{\mathcal{S}}\circ f^*(U)$ is given by $h(X/f^{-1}U,\mathcal{S})$, and $\Phi^Y_{\mathcal{S}}(U)$ is given by $h(Y/U,\mathcal{S})$, while $N(U)$ is the map of spectra $h(\alpha,\mathcal{S})$, where $\alpha\colon X/f^{-1}U\to Y/U$ is the evident map induced by $f$.

Locally finite homology admits a second form of functoriality. Suppose that we have an open embedding $i\colon X\hookrightarrow E$, where $E$ (and therefore also $X$) is a Hausdorff space.

**Proposition 7.2.2.** *Suppose $U\subset X$ is a relatively compact open set. Then $U$ is relatively compact when viewed as a subset of $E$ as well.*

*Proof.* Let $\mathrm{cl}_X(U)$ and $\mathrm{cl}_E(U)$ denote the closures of $U$ in $X$ and $E$, respectively. We claim $\mathrm{cl}_X(U)=\mathrm{cl}_E(U)$. For, $\mathrm{cl}_X(U)$ is compact by hypothesis, hence it is already closed as a subset of $E$, so $\mathrm{cl}_E(U)\subset\mathrm{cl}_X(U)$, which is the required result. □

So, if $U\subset X$ is a closed subset with relatively compact complement, then $U\cup(X-E)$ has open and relatively compact complement in $E$. This means that we obtain a functor $i_*\colon C(X) \to C(E)$, and consequently a composite map

$$\underset{\overleftarrow{U\in C(E)}}{\operatorname{holim}}\ h(E/U,\mathcal{S}) \longrightarrow \underset{\overleftarrow{U\in C(X)}}{\operatorname{holim}}\ h(E/U\cup E-X,\mathcal{S}) \simeq \underset{\overleftarrow{U\in C(X)}}{\operatorname{holim}}\ h(X/U,\mathcal{S})$$

which we also call $i_*\colon\ h^{lf}(E,\mathcal{S}) \to h^{lf}(X,\mathcal{S})$.

7.3. **Construction of the $S$-duality Map in the Noncompact Case.** Earlier we reviewed the $S$-duality theory for smooth manifolds with boundary. The $S$-duality equivalence had as its domain the homology of the boundary of the disk bundle of the stable normal bundle to the manifold $X$ with coefficients in a spectrum $\mathcal{S}$, and as its target the function spectrum of maps from $X$ into a suspension of $\mathcal{S}$. For non-compact manifolds, some changes have to be made. There are two avenues, one involves modifying the function spectrum by replacing it by "cohomology with compact supports of $X$ with coefficients in $\mathcal{S}$", and the other involves replacing the homology spectrum with locally finite homology. We will carry out the second of these two options. Also, since we will only need the case of framed manifolds, which is somewhat simpler to develop, we will only develop the theory in that case.

We begin by considering a construction on framed smooth manifolds. Let $M^m$ be a framed smooth manifold, let $\tau = \tau_M$ denote its tangent bundle, let $D = D(\tau)$ and $S = S(\tau)$ denote closed unit disk and sphere bundles in $\tau$ associated to some positive definite metric on $\tau$. This metric of course is a Riemannian metric on $M$. Let $\Delta\colon M \to M \times M$ be the diagonal inclusion, and equip $M \times M$ with the product of two copies of the metric we have given to $M$. Let $N$ be the closure of any tubular neighborhood of the image of $\Delta$ obtained by applying the exponential map to the tangent space of $M \times M$ along $\Delta$. Since the neighborhood is defined using the exponential map, we are now given a specific diffeomorphism of pairs $\theta\colon (N, \partial N) \to (D, S)$ so that the composite

$$M \xrightarrow{\Delta} N \xrightarrow{\theta} D \xrightarrow{\pi} M$$

is the identity, and so that $\theta \circ \Delta$ identifies $M$ with the image of the zero section in $D$. We now define a map $\lambda\colon M \times M \to S^m$ as follows. Since $M$ is framed, we are given a specific identification

$$\varepsilon\colon (D, S) \longrightarrow (M \times D^m, M \times S^{m-1})$$

where $D^m$ and $S^{m-1}$ denote the unit disk and unit sphere in $m$-dimensional Euclidean space. We now have a projection

$$p\colon M \times D^m/M \times S^{m-1} \longrightarrow D^m/S^{m-1} \simeq S^m.$$

The map $\lambda$ is defined to be the composite

$$M \times M \longrightarrow \frac{M \times M}{M \times M - \operatorname{int} N} \xrightarrow{\simeq} \frac{N}{\partial N} \xrightarrow{\theta} \frac{D}{S} \simeq \frac{M \times D^m}{M \times S^{m-1}} \xrightarrow{p} S^m$$

Since $S^m$ is described as $D^m/S^{m-1}$, we also define $S^m(\varepsilon)$ to be $D^m_\varepsilon/S^{m-1}_\varepsilon$, where $D^m_\varepsilon$ denotes the closed ball of radius $\varepsilon$ within $D^m$, and $S^{m-1}_\varepsilon$ is the sphere of radius $\varepsilon$. We have an evident collapse map $S^m \to S^m(\varepsilon)$, and more generally if $1 \geq \varepsilon \geq \varepsilon'$ we have a collapse map $S^m(\varepsilon) \to S^m(\varepsilon')$. Of course, $S^m = S^m(1)$. We also let

$N(\varepsilon) = \lambda^{-1}(D^m_\varepsilon)$, and we have commutative diagrams

$$\begin{tikzcd}
\dfrac{M \times M}{M \times M - \operatorname{int} N(\varepsilon)} \arrow[r, "\lambda_\varepsilon"] \arrow[d, "\text{collapse}"] & S^m(\varepsilon) \arrow[d] \\
\dfrac{M \times M}{M \times M - \operatorname{int} N(\varepsilon')} \arrow[r, "\lambda_{\varepsilon'}"] & S^m(\varepsilon')
\end{tikzcd}$$

where $\lambda_\varepsilon$ is the composite

$$\frac{M \times M}{M \times M - \operatorname{int} N(\varepsilon)} \longrightarrow \frac{D(\varepsilon)}{S(\varepsilon)} \longrightarrow \frac{M \times D^m_\varepsilon}{M \times S^{m-1}_\varepsilon} \longrightarrow \frac{D^m_\varepsilon}{S^{m-1}_\varepsilon}.$$

We now define a category $C_M$ associated to this situation. The objects in $C_M$ are pairs $(U, V)$, where $U \subset M$ is a closed subset so that $M - U$ is relatively compact, and where $V$ is a compact subset of $M$, so that $U \times V \subset M \times M - \operatorname{int} N$. There is a unique morphism from $(U, V)$ to $(U', V')$ if and only if $U \subset U'$ and $V \supseteq V'$. For any topological space $Y$, let $K(Y)$ denote the category whose objects are the compact subsets of $Y$, and where there is a unique morphism from $U$ to $V$ in $K(Y)$ if and only if $U \subset V$. For any spectrum $S$, we define a functor

$$\Psi^Y_S \colon K(Y)^{\mathrm{op}} \longrightarrow \mathbf{Spectra}$$

on objects by setting $\Psi^Y_S(U) = F(U_+, S)$, where $F(-, S)$ was defined above. For any $U \subset V$, $\Psi^Y_S(U \subset V)$ is defined to be the restriction map $F(V, S) \to F(U, S)$. Note that we have a natural map

$$\eta_Y \colon F(Y, S) \longrightarrow \underset{\overleftarrow{K(Y)^{\mathrm{op}}}}{\operatorname{holim}}\ \Psi^Y_S,$$

induced by the restriction maps $F(Y, S) \to F(U, S)$ for any $U \in K(Y)$.

**Proposition 7.3.1.** *Let $Y$ be a connected smooth manifold. Then $\eta_Y$ is a weak equivalence of spectra.*

*Proof.* As follows from the definition of homotopy inverse limits,

$$\underset{\overleftarrow{K(Y)^{\mathrm{op}}}}{\operatorname{holim}}\ \Psi^Y_S = F\left(\underset{\overrightarrow{U \in K(Y)}}{\operatorname{hocolim}}\ U, S\right).$$

The map $\eta_Y$ is also induced by the natural map

$$\underset{\overrightarrow{U \in K(Y)}}{\operatorname{hocolim}}\ U \longrightarrow Y$$

induced by the inclusions $U \to Y$. Consequently, for spaces $Y$ so that this map is a homotopy equivalence, the result will hold. But, if $Y$ is a smooth manifold, it admits a metric in which all balls are compact. One readily checks that for a point $y \in Y$, the evident map

$$\underset{\overrightarrow{n}}{\operatorname{hocolim}}\ B_n(y) \longrightarrow \underset{\overrightarrow{K(Y)}}{\operatorname{hocolim}}\ \Psi^Y_S$$

is a homotopy equivalence, and that the map

$$\underset{\overrightarrow{n}}{\operatorname{hocolim}}\ B_n(y) \longrightarrow Y$$

is also a homotopy equivalence. This gives the required result. $\square$

We have two functors $\pi_1 \colon C_M \to C(M)$ and $\pi_2 \colon C_M \to K(M)^{\mathrm{op}}$ given by $\pi_1(U,V) = U$ and $\pi_2(U,V) = V$. We obtain standard pullback maps of homotopy inverse limits

$$j_1 \colon h^{lf}(M,\mathcal{S}) \xrightarrow{\simeq} \underset{\overleftarrow{C(M)}}{\mathrm{holim}}\ \Phi^M_{\mathcal{S}} \xrightarrow{\pi_1^*} \underset{\overleftarrow{C_M}}{\mathrm{holim}}\ \Phi^M_{\mathcal{S}} \circ \pi_1^*$$

and

$$j_2 \colon \underset{\overleftarrow{K(M)^{\mathrm{op}}}}{\mathrm{holim}}\ \Psi^M_{\mathcal{S}} \longrightarrow \underset{\overleftarrow{C_M}}{\mathrm{holim}}\ \Psi^M_{\mathcal{S}} \circ \pi_2^*$$

**Proposition 7.3.2.** *Both $j_1$ and $j_2$ are weak equivalences of spectra.*

*Proof.* We use the criterion in 6.1.3, which means that we must now show that the nerves of the categories $\pi_1{\downarrow}U$ and $\pi_2{\downarrow}V$ are weakly contractible for all $U \in C(M)$ and $V \in K(M)^{\mathrm{op}}$. But this is immediate, since $\pi_1{\downarrow}U$ and $\pi_2{\downarrow}V$ are left filtering partially ordered sets. We conclude that

$$\underset{\overleftarrow{C_M}}{\mathrm{holim}}\ \Phi^M_{\mathcal{S}} \circ \pi_1^* \simeq h^{lf}(M,\mathcal{S})$$

and

$$\underset{\overleftarrow{C_M}}{\mathrm{holim}}\ \Psi^M_{\mathcal{S}} \circ \pi_2^* \simeq F(M,\mathcal{S})$$

are weak equivalences. $\square$

We now define a natural transformation $N_M$ from $\Phi^M_{\mathcal{S}} \circ \pi_1^*$ to $\Psi^M_{\mathcal{S}} \circ \pi_2^*$.

**Definition 7.3.3.** For any object $(U,V)$ in $C_M$, we define a map of spectra

$$N_M(U,V) \colon \Phi^M_{\mathcal{S}} \circ \pi_1^*(U,V) \longrightarrow \Psi^M_{h(S^m,\mathcal{S})} \circ \pi_2^*(U,V),$$

where $M$ is a framed manifold of dimension $m$, as follows. Since

$$\Phi^M_{\mathcal{S}} \circ \pi_1^*(U,V) = h(M/U,\mathcal{S})$$

and

$$\Psi^M_{h(S^m,\mathcal{S})} \circ \pi_2^*(U,V) = F(V_+, h(S^m,\mathcal{S}),$$

we will need to define a map from

$$h(M/U,\mathcal{S}) \longrightarrow F(V_+, h(S^m,\mathcal{S})).$$

By Proposition 6.2.2 this is equivalent to constructing a map of spectra

$$h(V_+, h(M/U,\mathcal{S})) \longrightarrow h(S^m,\mathcal{S}),$$

where as before $V_+$ denotes $V$ with a disjoint basepont added. Since by Proposition 6.2.1,

$$h(V_+, h(M/U,\mathcal{S})) \simeq h(V_+ \wedge M/U,\mathcal{S}),$$

we must produce a map from

$$h(V_+ \wedge M/U,\mathcal{S}) = h(V \times M/V \times U,\mathcal{S})$$

to $h(S^m,\mathcal{S})$. In order to do this, it suffices to specify a map

$$\mu \colon V \times M/V \times U \longrightarrow S^m.$$

As a map from $V \times M$ to $S^m$, $\mu$ is the restriction of $\lambda$ to $V \times M$. Since $V \times U$ lies in the complement of $\operatorname{int} N$, by the definition of $C_M$, $V \times U$ goes to the base point under $\lambda$, and so the restriction of $\lambda$ to $V \times M$ induces a map from $V \times M/V \times U$ to $S^m$, which is $\mu$. One readily checks that we get a natural transformation $N_M$ this way. The induced map $N_*$ on homotopy inverse limits gives us a map

$$N_{M*}\colon\ h^{lf}(M,\mathcal{S}) \longrightarrow F(M, h(S^m,\mathcal{S})) = F(M,\Sigma^m\mathcal{S}).$$

The map $N_{M*}$ has a useful relationship with the "wrong way" maps $i_*$ we constructed earlier.

**Proposition 7.3.4.** *Let $M^m$ be a smooth framed manifold, and suppose $i\colon M' \subset M$ is an open embedding of a submanifold. Then $M'$ is then equipped with a framing by restriction of the framing on $M$, and the diagram*

$$\begin{tikzcd} h^{lf}(M,\mathcal{S}) \arrow[r, "N_{M*}"] \arrow[d, "i_*"] & F(M,\Sigma^m\mathcal{S}) \arrow[d, "\text{restriction}"] \\ h^{lf}(M',\mathcal{S}) \arrow[r, "N_{M*}"] & F(M',\Sigma^m\mathcal{S}) \end{tikzcd}$$

*commutes.*

*Proof.* Straightforward. □

We now prove the main result of this section.

**Theorem 7.3.5.** *The map $N_{M*}$ is a weak equivalence of spectra for a framed smooth manifold $M$.*

*Proof.* We first outline the strategy of the proof. We will construct additional left filtered categories $\widetilde{C}_M$ and $\mathcal{E}$, together with inclusion functors $i\colon C_M \to \widetilde{C}_M$ and $j\colon \mathcal{E} \to \widetilde{C}_M$. Further, we have functors $\widetilde{\pi}_1\colon \widetilde{C}_M \to C(M)$ and $\pi_1^E\colon \mathcal{E} \to C(M)$ so that the diagram

(A)
$$\begin{tikzcd} C_M \arrow[r, "i"] \arrow[dr, "\pi_1"'] & \widetilde{C}_M \arrow[d, "\widetilde{\pi}_1"] & \mathcal{E} \arrow[l, "j"'] \arrow[dl, "\pi_1^E"] \\ & C(M) & \end{tikzcd}$$

commutes. Further, we will construct a category $\widetilde{K}(M)^{\mathrm{op}}$, for which there is an inclusion $h\colon K(M)^{\mathrm{op}} \hookrightarrow \widetilde{K}(M)^{\mathrm{op}}$, and so that there are functors $\widetilde{\pi}_2\colon \widetilde{C}_M \to \widetilde{K}(M)^{\mathrm{op}}$ and $\pi_2^E\colon \widetilde{C}_M \to \widetilde{K}(M)^{\mathrm{op}}$ so that the diagram

(B)
$$\begin{tikzcd} C_M \arrow[r, "i"] \arrow[d, "\pi_2"'] & \widetilde{C}_M \arrow[d, "\widetilde{\pi}_2"] & \mathcal{E} \arrow[l, "j"'] \arrow[d, "\pi_2^E"] \\ K(M)^{\mathrm{op}} \arrow[r, "h"] & \widetilde{K}(M)^{\mathrm{op}} & \widetilde{K}(M)^{\mathrm{op}} \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

commutes. Also, there is a functor

$$\widetilde{\Psi}^M_{\mathcal{S}}\colon \widetilde{K}(M)^{\mathrm{op}} \longrightarrow \mathbf{Spectra}$$

so that $\widetilde{\Psi}^M_{\mathcal{S}} \circ h = \Psi^M_{\mathcal{S}}$. In addition, there are natural transformations

$$\widetilde{N}\colon \Phi^M_{\mathcal{S}} \circ \widetilde{\pi}_2 \longrightarrow \widetilde{\Psi}^M_{\mathcal{S}} \circ \widetilde{\pi}_2$$

and

$$N^E\colon \Phi^M_{\mathcal{S}} \circ \pi^E_2 \longrightarrow \widetilde{\Psi}^M_{\mathcal{S}} \circ \pi^E_2$$

so that for any object $x$ in $C_M$, $i(N(x)) = \widetilde{N}(I(x))$, and for any object $e$ in $\mathcal{E}$, $j(N^E(e)) = \widetilde{N}(j(e))$. All these data now give us a commutative diagram of spectra

$$\text{(C)}\qquad \begin{array}{ccccc} \underset{\overleftarrow{\mathcal{C}_M}}{\text{holim}}\ \Phi^M_{\mathcal{S}} \circ \pi_1 & \xleftarrow{i^*} & \underset{\overleftarrow{\widetilde{\mathcal{C}}_M}}{\text{holim}}\ \Phi^M_{\mathcal{S}} \circ \widetilde{\pi}_1 & \xrightarrow{j^*} & \underset{\overleftarrow{\mathcal{E}}}{\text{holim}}\ \Phi^M_{\mathcal{S}} \circ \pi^E_1 \\ \downarrow{\scriptstyle N_{M*}} & & \downarrow{\scriptstyle \widetilde{N}_*} & & \downarrow{\scriptstyle (N^E)_*} \\ \underset{\overleftarrow{\mathcal{C}_M}}{\text{holim}}\ \Psi^M_{\mathcal{S}} \circ \pi_2 & \xleftarrow{i^*} & \underset{\overleftarrow{\widetilde{\mathcal{C}}_M}}{\text{holim}}\ \widetilde{\Psi}^M_{\mathcal{S}} \circ \widetilde{\pi}_2 & \xrightarrow{j^*} & \underset{\overleftarrow{\mathcal{E}}}{\text{holim}}\ \widetilde{\Psi}^M_{\mathcal{S}} \circ \pi^E_2 \end{array}$$

where the horizontal maps are all restriction or pullback maps of homotopy inverse limits. We would like to show that $N_{M*}$ is a weak equivalence of spectra. The strategy will be to prove that all the horizontal arrows and $(N^E)_*$ are weak equivalences, which will clearly establish the result.

We will now define all the categories, functors, and natural transformations involved in the diagram.

**$\widetilde{K}(M)^{\text{op}}$:** the objects of this category are pairs $(\varepsilon, V)$, where $\varepsilon$ is a real number, with $0 < \varepsilon \leq 1$, and $V$ is a compact subset of $M$. The set of objects admits a partial ordering, given by $(\varepsilon, V) \leq (\varepsilon', V')$ if and only if $\varepsilon \geq \varepsilon'$ and $V \supseteq V'$.

**$h$:** this functor is determined by its behavior on objects, which is given by $h(V) = (1, V)$.

**$\widetilde{\Psi}^M_{\mathcal{S}}$:** this is a functor from $\widetilde{K}(M)^{\text{op}}$ to spectra. It is given on objects by

$$\widetilde{\Psi}^M_{\mathcal{S}}(\varepsilon, V) = F(V_+, h(S^m(\varepsilon), \mathcal{S}).$$

On morphisms, $\widetilde{\Psi}^M_{\mathcal{S}}((\varepsilon, V) \leq \varepsilon', V'))$ is the composite

$$F(V_+, h(S^m(\varepsilon), \mathcal{S}) \xrightarrow{(i)} F(V'_+, h(S^m(\varepsilon), \mathcal{S}) \xrightarrow{(ii)} F(V'_+, h(S^m(\varepsilon'), \mathcal{S})$$

where $(i)$ is the restriction map along the inclusion $V'_+ \hookrightarrow V_+$ and where $(ii)$ is induced by the collapse $S^m(\varepsilon) \to S^m(\varepsilon')$.

**$\widetilde{C}_M$:** the objects of $\widetilde{C}_M$ are triples $(\varepsilon, U, V)$, where $\varepsilon \in (0, 1]$, $U \subset M$ has relatively compact complement, $V \subset M$ is compact, and

$$U \times V \cap \operatorname{int} N(\varepsilon) = \emptyset.$$

The morphisms are specified by a partial ordering $\leq$, given by setting $(\varepsilon, U, V) \leq (\varepsilon', U', V')$ if and only if $\varepsilon \geq \varepsilon'$, $U \subset U'$, and $V \supseteq V'$.

**$\widetilde{\pi}_1$:** $\widetilde{\pi}_1(\varepsilon, U, V) = U$.

**$\widetilde{\pi}_2$:** $\widetilde{\pi}_2(\varepsilon, U, V) = (\varepsilon, V)$.

**$\widetilde{N}$:** this is a natural transformation from the functor $\Phi^M_{\mathcal{S}} \circ \widetilde{\pi}_1$ to $\widetilde{\Psi}^M_{\mathcal{S}} \circ \widetilde{\pi}_2$. Now

$$\Phi^M_{\mathcal{S}} \circ \widetilde{\pi}_1(\varepsilon, U, V) = h(M/U, \mathcal{S}),$$

and

$$\widetilde{\Psi}^M_{\mathcal{S}}(\varepsilon, U, V) = F(V_+, h(S^m(\varepsilon), \mathcal{S}).$$

We must therefore produce a map

$$\theta\colon h(M/U, \mathcal{S}) \longrightarrow F(V_+, h(S^m(\varepsilon), S).$$

We first define the map $\varphi\colon M \times V/U \times V \to S^m(\varepsilon)$ by requiring that it be induced by the composite

$$M \times V \longrightarrow M \times M \longrightarrow \frac{M \times M}{M \times M - \operatorname{int} N} \xrightarrow{\lambda_\varepsilon} S^m(\varepsilon)$$

Since $U \times V \subset M \times M - \operatorname{int} N(\varepsilon)$, this map induces a map

$$\varphi\colon M \times V/U \times V \longrightarrow S^m(\varepsilon).$$

Now, $M \times V/U \times V \simeq M/U \wedge V_+$. Using $F$ to denote a function space rather than a function spectrum, we obtain the adjoint map

$$\operatorname{ad}(\varphi)\colon M/U \longrightarrow F(V_+, S^m(\varepsilon)).$$

Now $\theta$ is defined to be the composite

$$h(M/U, \mathcal{S}) \xrightarrow{h(\operatorname{ad}(\varphi),\mathcal{S})} h(F(V_+, S^m(\varepsilon)), \mathcal{S}) \xrightarrow{\nu} F(V_+, h(S^m(\varepsilon), \mathcal{S})$$

where $\nu$ was defined earlier. $\theta$ is readily checked to be a natural transformation and defines $\widetilde{N}$.

**$\mathcal{E}$:** this is the category associated to a partial ordering on the set of half integers. This partial ordering is specified by the requirements that $n \leq n+\frac{1}{2}$ and $n+1 \leq n+\frac{1}{2}$, for $n$ an integer, and that all other pairs of distinct elements are incomparable. A picture of a piece of this category is

$$\begin{tikzcd}[column sep=small]
 & \frac{1}{2} & & \frac{3}{2} & & \frac{5}{2} & & \frac{7}{2} & \\
0 \arrow[ur] & & 1 \arrow[ul] \arrow[ur] & & 2 \arrow[ul] \arrow[ur] & & 3 \arrow[ul] \arrow[ur] & & 4 \arrow[ul]
\end{tikzcd}$$

**$j$:** this is a functor from $\mathcal{E}$ to $\widetilde{\mathcal{C}}_M$. To define it, we first note that as a consequence of the Whitney Embedding Theorem, there is a proper smooth map $f\colon M \to [0, +\infty)$. From Sard's Theorem, it follows that there is a sequence of values $\{R_n\}_{n\geq 0}$, with $R_n \leq R_{n+1}$, and $\lim_{n\to\infty} R_n = +\infty$, so that $f^{-1}([0, R_n])$ is a compact manifold of $M$ of dimension $m-1$. Consequently, $f^{-1}([0, R_n])$ is a compact manifold with boundary, which we'll denote by $(M_n, \partial M_n)$, and

$$M = \bigcup_{n\geq 0} M_n.$$

Because of the existence of a collared neighborhood of $\partial M_n$ in each case, there is a number $\varepsilon_n$, with $0 < \varepsilon_n \leq 1$, so that the inclusion $M_n - N_{\varepsilon_n}(\partial M_n) \to M_n$ is a homotopy equivalence, and so that

$$\partial M_n \times (M_n - N_{\varepsilon_n}(\partial M_n)) \cap \operatorname{int} N(\varepsilon_n) = \emptyset.$$

We can clearly choose the numbers $\varepsilon_n$ so that $\varepsilon_{n+1} \leq \varepsilon_n$ and so that

$$M_n - N_{\varepsilon_n}(\partial M_n) \subset M_{n+1} - N_{\varepsilon_{n+1}}(\partial M_{n+1}).$$

We are now in a position to define $j$; it suffices to define it on objects, by setting

$$j(n) = (\varepsilon_n, M - \operatorname{int} M_n, M_n - N_{\varepsilon_n}(\partial M_n)),$$
$$j(n+\tfrac{1}{2}) = (\varepsilon_{n+1}, M - \operatorname{int} M_n, M_n - N_{\varepsilon_n}(\partial M_n)).$$

It is now easy to see that $j$ defines a functor; $j(n \to n+\frac{1}{2})$ is the arrow $j(n) \to j(n+\frac{1}{2})$ corresponding to the fact that $\varepsilon_{n+1} \le \varepsilon_n$, and $j(n+1 \to n+\frac{1}{2})$ is the arrow $j(n+1) \to j(n+\frac{1}{2})$ which exists since

$$\begin{aligned} \operatorname{int} M_{n+1} &\supseteq \operatorname{int} M_n, \\ M_{n+1} - N_{\varepsilon_{n+1}}(\partial M_{n+1}) &\supseteq M_n - N_{\varepsilon_n}(\partial M_n). \end{aligned}$$

**$\pi_1^E$:** it is defined to be $\widetilde{\pi}_1 \circ i$. So, on objects, we get

$$\begin{aligned} \pi_1^E(n) &= h(M - \operatorname{int} M_n, \mathcal{S}), \\ \pi_1^E(n+\tfrac{1}{2}) &= h(M - \operatorname{int} M_n, \mathcal{S}). \end{aligned}$$

**$\pi_2^E$:** it is defined to be $\widetilde{\pi}_2 \circ i$. So, on objects,

$$\begin{aligned} \pi_2^E &= (\varepsilon_n, M_n - \operatorname{int} N_{\varepsilon_n}(\partial M_n)), \\ \pi_2^E(n+\tfrac{1}{2}) &= (\varepsilon_{n+1}, M_n - \operatorname{int} N_{\varepsilon_n}(\partial M_n)). \end{aligned}$$

**$N^E$:** this is the natural transformation determined by the requirement that

$$j(N^E(e)) = \widetilde{N}(j(e)).$$

Its definition is therefore identical to that given in the definition of $\widetilde{N}$, except that it is restricted to the objects in the image of $j$.

We have now defined all the terms which are necessary for constructing the diagram. Let's first prove that the upper horizontal arrows in diagram (C) are weak equivalences. Since all the spectrum valued functors in the top row are pullbacks of $\Phi_{\mathcal{S}}^M$, we have a commutative diagram

$$\begin{tikzcd}[column sep=small]
\underset{\overleftarrow{C_M}}{\operatorname{holim}}\ \Phi_{\mathcal{S}}^M \circ \pi_1 & & \underset{\overleftarrow{\widetilde{S}_M}}{\operatorname{holim}}\ \Phi_{\mathcal{S}}^M \circ \widetilde{\pi}_1 \arrow[ll, "i^*"'] \arrow[rr, "j^*"] & & \underset{\overleftarrow{\mathcal{E}}}{\operatorname{holim}}\ \Phi_{\mathcal{S}}^M \circ \pi_1^E \\
& & \underset{\overleftarrow{C(M)}}{\operatorname{holim}}\ \Phi_{\mathcal{S}}^M \arrow[ull, "(a)"] \arrow[u, "(b)"'] \arrow[urr, "(c)"'] & &
\end{tikzcd}$$

If we can show that arrows (a), (b), and (c) are weak equivalences, then $i^*$ and $j^*$ will be equivalences, which is what we want to show. From the criterion in 6.1.3, it will suffice to show that all the categories $\pi_1{\downarrow}x$, $\widetilde{\pi}_1{\downarrow}x$, and $\pi_1^E$ have weakly contractible nerves. But it is easy to check that all the categories $\pi_1{\downarrow}x$ are left filtering, so their nerves are weakly contractible.

To see that (b) is an equivalence, we must show that the categories $\widetilde{\pi}_1{\downarrow}x$ have contractible nerves for all objects $x$ in $C(M)$. Now, let $\widetilde{C}_M[\varepsilon]$ denote the full subcategory on objects of the form $(\delta, U, V)$, with $1 \ge \delta \ge \varepsilon$, and let $\widetilde{C}_M\langle\varepsilon\rangle$ denote the full subcategory on objects of the form $(\varepsilon, U, V)$. We have inclusions

$$\widetilde{C}_M\langle\varepsilon\rangle \hookrightarrow \widetilde{C}[\varepsilon] \hookrightarrow \widetilde{C}_M.$$

Let $\widetilde{\pi}_1[\varepsilon]$ and $\widetilde{\pi}\langle\varepsilon\rangle$ denote the restrictions of $\widetilde{\pi}_1$ to $\widetilde{C}_M[\varepsilon]$ and $\widetilde{C}\langle\varepsilon\rangle$, respectively. It is clear that $\widetilde{\pi}_1{\downarrow}x \simeq \bigcup_\varepsilon \widetilde{\pi}_1[\varepsilon]{\downarrow}x$, so to prove that $\widetilde{\pi}_1{\downarrow}x$ has weakly contractible nerve, it suffices to show that $\widetilde{\pi}_1[\varepsilon]{\downarrow}x$ does for each $\varepsilon$. On the other hand, the assignment $(\delta, U, V) \to (\varepsilon, U, V)$ for any $\delta \ge \varepsilon$ defines a functor $\alpha$ from $\widetilde{C}_M[\varepsilon]$ to

$\widetilde{C}_M\langle\varepsilon\rangle$, with $\widetilde{\pi}_1 \circ \alpha = \widetilde{\pi}_1$. This means that $\alpha$ induces a functor $\alpha{\downarrow}x$ from $\widetilde{\pi}_1[\varepsilon]{\downarrow}x$ to $\widetilde{\pi}_1\langle\varepsilon\rangle{\downarrow}x$. Moreover, there is an evident morphism

$$N(\delta, U, V)\colon (\delta, U, V) \longrightarrow (\varepsilon, U, V)$$

whenever $\delta \geq \varepsilon$, hence we have a natural transformation from the identity functor on $\widetilde{\pi}_1[\varepsilon]{\downarrow}x$ to $\alpha{\downarrow}x$. Since functors induce maps of nerves and natural transformations induce homotopies, it is clear that the inclusion $\widetilde{\pi}_1\langle\varepsilon\rangle{\downarrow}x \hookrightarrow \widetilde{\pi}_1[\varepsilon]{\downarrow}x$ induces a weak equivalence on nerves. But one sees readily that $\widetilde{\pi}_1\langle\varepsilon\rangle{\downarrow}x$ is left filtering, and hence its nerve is contractible.

To see that the nerve of $\pi_1^E{\downarrow}x$ is weakly contractible, we proceed like this. $x$ corresponds to a closed set $W$ in $M$ with relatively compact complement. Let $n$ be the smallest integer so that $M - \operatorname{int} M_n \subset W$. Then $\pi_1^E{\downarrow}x$ is clearly identified with the full subcategory of $\mathcal{E}$ on the half integers which are greater than or equal to $n$, which we denote by $\mathcal{E}_{\leq n}$. But the nerve of $\mathcal{E}_{\leq n}$ can be identified with the simplicial set corresponding to the triangulation of $[n, +\infty)$, whose vertices are the half integers greater than or equal to $n$, and where the partial ordering on the vertex set declares that numbers of the form $n + \frac{1}{2}$ are greater than any integers. Since $[n, +\infty)$ is contractible, we are done, i.e., we have shown that the upper horizontal arrows in diagram (C) are all weak equivalences.

In order to prove that the lower horizontal arrows in diagram (C) are weak equivalences, we consider the diagram

$$\begin{tikzcd}
\underleftarrow{\mathrm{holim}}_{C_M}\, \Psi_{\mathcal{S}}^M \circ \pi_2 & \underleftarrow{\mathrm{holim}}_{\widetilde{C}_M}\, \widetilde{\Psi}_{\mathcal{S}}^M \circ \widetilde{\pi}_2 \arrow[l, "i^*"'] \arrow[r, "j^*"] & \underleftarrow{\mathrm{holim}}_{\mathcal{E}}\, \widetilde{\Psi}_{\mathcal{S}}^M \circ \pi_2^E \\
\underleftarrow{\mathrm{holim}}_{K(M)^{\mathrm{op}}}\, \Psi_{\mathcal{S}}^M \arrow[u, "(a)"] & \underleftarrow{\mathrm{holim}}_{\widetilde{K}(M)^{\mathrm{op}}}\, \widetilde{\Psi}_{\mathcal{S}}^M \arrow[l, "h^*"] \arrow[r, "id"] \arrow[u, "(b)"] & \underleftarrow{\mathrm{holim}}_{\widetilde{K}(M)^{\mathrm{op}}}\, \widetilde{\Psi}_{\mathcal{S}}^M \arrow[u, "(c)"]
\end{tikzcd}$$

where (a) denotes pullback along $\pi_2$, (b) pullback along $\widetilde{\pi}_2$, and (c) pullback along $\pi_2^E$. To verify that (a) is an equivalence, we must verify that $\pi_2{\downarrow}x$ has weakly contractible nerve for all objects $x$ in $K(M)^{\mathrm{op}}$. But one readily sees that $\pi_2{\downarrow}x$ is left filtering, hence the result. To show that (b) and (c) are equivalences, one must verify that $\widetilde{\pi}_2{\downarrow}x$ and $\pi_2^E{\downarrow}x$ have weakly contractible nerves for all objects $x$ in $\widetilde{K}(M)^{\mathrm{op}}$. But here too the categories in question are left filtering, so all the horizontal arrows in diagram (C) are in fact equivalences. To complete the proof of the theorem, it remains to show that

$$(N^E)^*\colon \underleftarrow{\mathrm{holim}}_{\mathcal{E}}\, \Phi_{\mathcal{S}}^M \circ \pi_1^E \longrightarrow \underleftarrow{\mathrm{holim}}_{\mathcal{E}}\, \widetilde{\Psi}_{\mathcal{S}}^M \circ \pi_2^E$$

is a weak equivalence. To prove this, it will suffice by 6.1.2 to prove that

$$N^E(l)\colon \Phi_{\mathcal{S}}^M \circ \pi_1^E(l) \longrightarrow \widetilde{\Psi}_{\mathcal{S}}^M \circ \pi_2^E(l)$$

is a weak equivalence of spectra for all half integers $l$. Now

$$\Phi_{\mathcal{S}}^M \circ \pi_1^E(l) \simeq h(M/M - \operatorname{int} M, \mathcal{S})$$

and

$$\widetilde{\Psi}_{\mathcal{S}}^M \circ \pi_2^E(l) = F(M_l - N_{\varepsilon_l}(\partial M_l), h(S(\varepsilon_m), \mathcal{S}).$$

One readily checks that $N^E(l)$ is compatible with the map

$$\hat{\varphi}\colon h(M_l/\partial M_l, \mathcal{S}) \longrightarrow F(M_l - N_{\varepsilon_l}(\partial M_l), S^m)$$

defined earlier, in the sense that the diagrams

$$\begin{array}{ccc} h(M/M-\operatorname{int} M_l, \mathcal{S}) & \xrightarrow{N^E(l)} & F(M_l - N_{\varepsilon_l}(\partial M_l), h(S^m(\varepsilon_l), \mathcal{S})) \\ \uparrow & & \uparrow \\ h(M_l/\partial M_l, \mathcal{S}) & \xrightarrow{\widehat{\varphi}} & F(M_l - N_{\varepsilon_l}(\partial M_l), h(S^m, \mathcal{S})) \end{array}$$

commute and the vertical arrows are weak equivalences. Since $\hat{\varphi}$ is a weak equivalence, so is $N^E(l)$. This completes the proof. $\square$

7.4. **Equivariant Theory.** Suppose a group $\Gamma$ acts on a manifold $X$. Then we have a natural action of $\Gamma$ on the category $C(X)$. Moreover, suppose that $\Gamma$ also acts on the spectrum $\mathcal{S}$. Then we obtain an extension of the functor $\Phi^X_{\mathcal{S}}\colon C(X) \to$ **Spectra** to a functor

$$\overline{\Phi}^X_{\mathcal{S}}\colon \Gamma \wr C(X) \longrightarrow \textbf{Spectra},$$

where $\Gamma \wr C(X)$ is the Grothendieck construction, as in [9]. This extension is defined as follows. Recall that the objects of $\Gamma \wr C(X)$ are simply the objects of $C(X)$, and that a morphism in $\Gamma \wr C(X)$ from $U$ to $V$ is a pair $(g, \varphi)$, where $g \in \Gamma$, and $\varphi\colon U \to gV$ is a morphism in $C(X)$. The extension is given on objects by

$$\overline{\Phi}^X_{\mathcal{S}}(U) = \Phi^X_{\mathcal{S}}(U) = h(M/U, \mathcal{S}),$$

and on morphisms $(g, \varphi)$, with $\varphi\colon U \to gV$ in $C(X)$, by setting $\overline{\Phi}^X_{\mathcal{S}}(g, \varphi)$ equal to the composite

$$h(M/U, \mathcal{S}) \xrightarrow{h(\varphi, \mathrm{id}_{\mathcal{S}})} h(M/gV, \mathcal{S}) \xrightarrow{h(g^{-1}, \mathrm{id}_{\mathcal{S}})} h(M/V, \mathcal{S}) \xrightarrow{h(\mathrm{id}_{M/V}, g)} h(M/V, \mathcal{S})$$

It is easy to check that this action agrees with the action on $h^{lf}(X, S)$ given on $g \in \Gamma$ by

$$h^{lf}(X, \mathcal{S}) \xrightarrow{h^{lf}(g, \mathrm{id}_{\mathcal{S}})} h^{lf}(X, \mathcal{S}) \xrightarrow{h^{lf}(\mathrm{id}_X, g)} h^{lf}(X, \mathcal{S})$$

In the case where $\Gamma$ acts freely and proper discontinuously on $X$, the fixed point spectrum $h^{lf}(X, \mathcal{S})^\Gamma$ is referred to as the "twisted locally finite homology" spectrum of $X/\Gamma$ with coefficients in the $\Gamma$-spectrum $\mathcal{S}$.

Next, let us consider the $\Gamma$-action on $C(X)$.

**Proposition 7.4.1.** *Suppose $\Gamma$ is a torsion-free group acting freely and proper discontinuously on the manifold $X$. Then $\Gamma$ acts freely on the objects of $S(X)$.*

*Proof.* Since $\Gamma$ acts proper discontinuously, the stabilizer of any compact subset must be a finite subgroup of $\Gamma$. Since $\Gamma$ is torsion-free, this subgroup must be trivial, which gives the result, since the stabilizer of any object $U$ in $S(X)$ is contained in the stabilizer of the compact subset $\mathrm{cl}(X - U)$. $\square$

**Corollary 7.4.2.** *Suppose $\Gamma$ is as in Proposition 7.4.1, and that $\mathcal{S}$ is a spectrum with $\Gamma$-action. Then for the associated $\Gamma$-action on $h^{lf}(X, \mathcal{S})$, we have that the natural map $h^{lf}(X, \mathcal{S})^\Gamma \to h^{lf}(X, \mathcal{S})^{h\Gamma}$ is a weak equivalence of spectra.*

*Proof.* This is just I.18 of [9]. $\square$

We also study the map from fixed point set to homotopy fixed set for the spectra $F(X, \mathcal{S})$. Suppose $\Gamma$ acts on $X$, and on the spectrum $\mathcal{S}$. Then we obtain an action of $\Gamma$ on $F(X, \mathcal{S})$, by $(gf)(x) = g \cdot f(g^{-1}x)$.

**Proposition 7.4.3.** *Suppose $\Gamma$ acts freely on the manifold $X$, that $X/\Gamma$ has the homotopy type of a CW complex, and that $\Gamma$ acts of the spectrum $\mathcal{S}$. Then the natural map $F(X, \mathcal{S})^{\Gamma} \to F(X, \mathcal{S})^{h\Gamma}$ is a weak equivalence of spectra.*

*Proof.* Since $X/\Gamma$ has the homotopy type of a CW-complex, $X$ has the homotopy type of a $\Gamma$-CW complex. By using the skeletal filtration of the equivalent $\Gamma$-CW complex, we can reduce to the case where $X = \Gamma$, viewed as a discrete space. In this case, it is clear that both $F(\Gamma, \mathcal{S})^{\Gamma}$and $F(\Gamma, \mathcal{S})^{h\Gamma}$ are equivalent to $\mathcal{S}$ which gives the result. □

Next, we consider the equivariance of the Spanier–Whitehead duality map. Let $M^m$ be an equivariantly framed smooth manifold with proper discontinuous and free $\Gamma$-action. Then inside $M \times M$ (equipped with diagonal $\Gamma$-action), the neighborhood $N$ used to construct the duality map may be chosen to be $\Gamma$-invariant. Moreover, because of the equivariance of the framing on $M$, the diffeomorphism $(N, \partial N) \to (M \times D^m, M \times S^{m-1})$ may be chosen to be equivariant, where the action on $M \times D^m$ is the product of the original action on $M$ with the trivial action on $D^m$. We can follow this by the projection on the second factor to get the equivariant map of pairs $(N, \partial N) \to (D^m, S^{m-1})$, which induces a map

$$N/\partial N \longrightarrow D^m/S^{m-1} \simeq S^m.$$

Composing with the equivariant map

$$M \times M \longrightarrow M \times M/M \times M - \operatorname{int} N \simeq N/\partial N$$

we get that the map $\lambda\colon M \times M \to S^m$ is equivariant, where $M \times M$ is equipped with the diagonal $\Gamma$-action.

We must now study the equivariance properties of the map $N_*$, where $N$ is the natural transformation constructed above. First, we note that it is clear from the $\Gamma$-invariance of the neighborhood $N$ that $N(gU, gV) = gN(U, V)$. Next, consdier the category $K(M)^{\mathrm{op}}$; its objects are the compact subsets of $M$.

**Proposition 7.4.4.** *$\Gamma$ acts freely on the objects of $K(M)^{\mathrm{op}}$.*

*Proof.* The proof is identical to that for 7.4.1. □

Next, it is clear that $\widetilde{C}_M$ inherits the diagonal $\Gamma$-action on $C(M) \times K(M)^{\mathrm{op}}$, under the inclusion

$$\pi_1 \times \pi_2 \colon \widetilde{C}_M \longrightarrow C(M) \times K(M)^{\mathrm{op}},$$

so $\pi_1$ and $\pi_2$ are equivariant and the action of $\Gamma$ on the objects of $\widetilde{C}_M$ is free. Suppose now that $\mathcal{S}$ is a spectrum with $\Gamma$-action. Then, as above, we have an extension

$$\overline{\Phi}^M_{\mathcal{S}} : \Gamma \wr C(M) \longrightarrow \mathbf{Spectra}$$

of $\Phi^M_{\mathcal{S}}$, where again $\Gamma \wr C(M)$ denotes the Grothendieck construction. Also, we have an extension

$$\overline{\Psi}^M_{\mathcal{S}} : \Gamma \wr K(M)^{\mathrm{op}} \longrightarrow \mathbf{Spectra}$$

of $\Psi^M_{\mathcal{S}}$ which is defined as follows. On objects,

$$\overline{\Psi}^M_{\mathcal{S}}(U) = \Psi^M_{\mathcal{S}}(U) = F(U_+, h(S^m, \mathcal{S}));$$

recall that objects in $\Gamma \wr K(M)^{\mathrm{op}}$ are just objects in $K(M)^{\mathrm{op}}$. Morphisms in $K(M)^{\mathrm{op}}$ are given by pairs $(g, \varphi)$, where $g \in \Gamma$, and $\varphi \colon U \to gV$ is a morphism in $K(M)^{\mathrm{op}}$. $\overline{\Psi}^M_{\mathcal{S}}(g, \varphi)$ is defined to be the composite

$$F(U_+, h(S^m, \mathcal{S})) \xrightarrow{F(\varphi, h(S^m, \mathcal{S}))} F(gV_+, h(S^m, \mathcal{S})) \xrightarrow{F(g, h(S^m, \mathcal{S}))}$$
$$F(V_+, h(S^m, \mathcal{S})) \xrightarrow{F(\mathrm{id}, h(S^m, \mathcal{S}))} F(V_+, h(S^m, \mathcal{S}))$$

It is clear that $\overline{\Psi}^M_{\mathcal{S}}$ extends $\Psi^M_{\mathcal{S}}$.

We now have two spectrum valued functors on $\Gamma \wr \widetilde{C}_M$, namely $\overline{\Phi}^M_{\mathcal{S}} \circ (\Gamma \wr \pi_1)$ and $\overline{\Psi}^M_{\mathcal{S}} \circ (\Gamma \wr \pi_2)$. We will now construct a natural transformation

$$\overline{N} \colon \overline{\Phi}^M_{\mathcal{S}} \circ (\Gamma \wr \pi_1) \longrightarrow \overline{\Psi}^M_{\mathcal{S}} \circ (\Gamma \wr \pi_2)$$

by letting $\overline{N}(U, V) = N(U, V)$. Recall that the objects of $\Gamma \wr \widetilde{C}_M$ are just the objects of $\widetilde{C}_M$ itself. To check that this is a natural transformation, we observe that every morphism $\alpha = (g, \varphi)$ in $\Gamma \wr \widetilde{C}_M$, where $g \in \Gamma$ and $\varphi \colon x \to gy$ is a morphism in $\widetilde{C}_M$, can be decomposed as $\alpha_1 \circ \alpha_2$, where $\alpha_2 = (e, \varphi)$ and $\alpha_1 = (g, \mathrm{id}_{gV})$. Note that $\alpha_2$ is a morphism from $x$ to $gy$ and $\alpha_1$ is a morphism from $gy$ to $y$ in $\Gamma \wr \widetilde{C}_M$. $\alpha_2$ is a morphism in the subcategory $\widetilde{C}_M \subset \Gamma \wr \widetilde{C}_M$, so the diagrams

$$\begin{tikzcd}
\overline{\Phi}^M_{\mathcal{S}} \circ \pi_1(x) \arrow[r, "\overline{N}(x)"] \arrow[d, "\overline{\Phi}^M_{\mathcal{S}} \circ \pi_1(\alpha_2)"'] & \overline{\Psi}^M_{\mathcal{S}} \circ \pi_2(x) \arrow[d, "\overline{\Psi}^M_{\mathcal{S}} \circ \pi_2(\alpha_2)"] \\
\overline{\Phi}^M_{\mathcal{S}} \circ \pi_1(gy) \arrow[r, "\overline{N}(gy)"] & \overline{\Psi}^M_{\mathcal{S}} \circ \pi_2(gy)
\end{tikzcd}$$

commute, since $N$ is a natural transformation on $\widetilde{C}_M$ and $\overline{N}$ restricts to $N$. It remains, therefore, to check that the diagrams involving $\alpha_1$ commute. This means that for every $(U, V)$ in $\widetilde{C}_M$, the diagram

$$\begin{tikzcd}
h(M/gU, \mathcal{S}) \arrow[r, "\overline{N}(gU, gV)"] \arrow[d, "h(g^{-1}, g)"'] & F(gV_+, \mathcal{S}) \arrow[d, "F(g, g)"] \\
h(M/U, \mathcal{S}) \arrow[r, "\overline{N}(U, V)"] & F(V_+, \mathcal{S})
\end{tikzcd}$$

commutes. That this is the case reduces to the verification that the diagram

$$\begin{tikzcd}
\dfrac{M \times gV}{gU \times gV} \arrow[r, "\text{(a)}"] \arrow[d, "(g^{-1}, g^{-1})"'] & S^m \\
\dfrac{M \times V}{U \times V} \arrow[ur, "\text{(b)}"'] &
\end{tikzcd}$$

commutes, where (a) and (b) are induced by the composite

$$M \times M \longrightarrow \frac{M \times M}{M \times M - \operatorname{int} N} \longrightarrow S^m.$$

But this is clear, since the map $M \times M \to S^m$ is equivariant when $M \times M$ is equipped with the diagonal action and the action on $S^m$ is trivial. We now have the following general statement concerning the equivariance of maps induced by natural transformations.

**Proposition 7.4.5.** *Let $\mathcal{C}$ be a category with action by a group $\Gamma$. Suppose we have functors $\alpha$, $\beta\colon \Gamma \wr \mathcal{C} \to$ **Spectra**, and hence group actions on $\underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\ \alpha$ and $\underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\ \beta$. If $N$ is a natural transformation on $\Gamma \wr \mathcal{C}$ from $\alpha$ to $\beta$, then the map*

$$(N|\mathcal{C})_*\colon \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\ \alpha \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}}\ \beta$$

*induced by the natural transformation $N|\mathcal{C}$ is equivariant.*

*Proof.* This is a straightforward verification. □

**Corollary 7.4.6.** *With $M$, $\Gamma$, and $\mathcal{S}$ as above, the Spanier–Whitehead duality map*

$$N_*\colon\ h^{lf}(X, \mathcal{S}) \longrightarrow F(X_+, h(S^m, \mathcal{S}))$$

*is equivariant.*

**Corollary 7.4.7.** *The induced maps*

$$N_*^\Gamma\colon\ h^{lf}(X, S)^\Gamma \longrightarrow F(X_+, h(S^m, \mathcal{S})^\Gamma$$

*are weak equivalences of spectra.*

**Corollary 7.4.8.** *Let $\Gamma$ be a group and suppose $M^m$ is an equivariantly framed smooth contractible manifold with free, properly discontinuous group action, so that $M/\Gamma$ has the homotopy type of a finite dimensional CW complex. Suppose further that that $\mathcal{S}$ is a $\Gamma$-spectrum. Then the $m$-fold suspension of the homotopy fixed spectrum $\mathcal{S}^{h\Gamma}$ is weakly equivalent to the twisted locally finite homology spectrum $h^{lf}(M, \mathcal{S})^\Gamma$*

*Proof.* This is immediate since the homotopy fixed spectrum $\mathcal{S}^{h\Gamma}$ is defined to be the function spectrum $F(M_+, \mathcal{S})^\Gamma$ for any free contractible $\Gamma$-space $M$. □

**Remark 7.4.9.** For any group $\Gamma$ with a classifying space $B\Gamma$ which is a finite complex, we can construct such an $M$. For, embed $B\Gamma$ in a Euclidean space $E^n$, and take a regular neighborhood $N$ of $B\Gamma$. Since $N$ is an open submanifold of $E^n$, it is framed, and hence the universal cover $\widetilde{N}$ is equivariantly framed. $\widetilde{N}$ will do as a choice for $N$. Of course, in the case where $B\Gamma$ is a closed smooth manifold, we can take $M$ to be the universal cover of the total space of the normal bundle to an embedding of $B\Gamma$ in Euclidean space.

We recall that for any spectrum with $\Gamma$-action $\mathcal{S}$, we have an inclusion

$$\eta\colon \mathcal{S}^\Gamma \longrightarrow \mathcal{S}^{h\Gamma} = F(E\Gamma, \mathcal{S})^\Gamma,$$

induced by the map $E\Gamma \to$ point. When $B\Gamma$ is a finite complex, we found that we are able to rewrite $\Sigma^n \mathcal{S}^{h\Gamma}$ as twisted locally finite homology $h^{lf}(M^n, \mathcal{S})^\Gamma$, where $M$

is an equivariantly framed contractible, smooth manifold with $\Gamma$-action. We now wish to express the map $\eta$ as a map

$$\eta \colon \Sigma^n \mathcal{S}^\Gamma \longrightarrow h^{lf}(M, \mathcal{S})^\Gamma.$$

In order to do this, we will first have to recall the simplicial version of locally finite homology introduced in [9], and compare it with the version we have defined here. Let's call the theory developed in [9] $\check{h}^{lf}$. We first give the definition of $\check{h}^{lf}$ on discrete sets. For any set, let $F(X)$ denote the category of subsets of $X$ with finite complement. Then we define

$$\check{h}^{lf}(X, \mathcal{S}) \overset{\text{def}}{=} \underset{U \in F(X)}{\underleftarrow{\operatorname{holim}}} \; h(X/U, \mathcal{S}).$$

Note that this definition actually agrees with our current definition of $h^{lf}$, since $F(X) = C(X)$ in this case. So $\check{h}^{lf}(\_\_, \mathcal{S})$ is functorial on the category of discrete sets and proper maps (proper in this situation means that inverse images of finite sets are finite), and agrees with $h^{lf}$ on discrete sets.

We say a simplicial set $X.$ is *locally finite* if all face and degeneracy maps are proper. For such a simplicial set, we can apply $\check{h}^{lf}(\_\_, \mathcal{S})$ levelwise, and define $\check{h}^{lf}(X, \mathcal{S})$ to be the total spectrum of the resulting simplicial spectrum. We will now construct a natural transformation

$$\check{h}^{lf}(X, \mathcal{S}) \longrightarrow h^{lf}(|X.|, \mathcal{S})$$

on the category of locally finite simplicial sets and proper maps of such. We first note that for each $k$, we have a proper map $\Delta^k \times X_k \to |X.|$, to which we can apply $h^{lf}$ and get a map

$$i_k \colon \, h^{lf}(\Delta^k \times X_k, \mathcal{S}) \longrightarrow h^{lf}(|X.|, \mathcal{S}).$$

Secondly, we can compose this map with the map

$$h(\Delta^k_+, h^{lf}(X_k, \mathcal{S}) \longrightarrow h^{lf}(\Delta^k \times X_k, \mathcal{S})$$

to get a map

$$j_k \colon h(\Delta^k_+, h^{lf}(X_k, \mathcal{S})) \longrightarrow h^{lf}(|X.|, \mathcal{S}).$$

It is readily checked that the map

$$\bigvee_{k=0}^{\infty} j_k \colon \; \bigvee_{k=0}^{\infty} h(\Delta^k_+, h^{lf}(X_k, \mathcal{S}) \longrightarrow h^{lf}(|X.|, \mathcal{S})$$

respects the equivalence relation defining the geometric realization of the simplicial spectrum $\{k \mapsto h^{lf}(X_k, \mathcal{S})\}$, so we get a map from that realization to $h^{lf}(X, \mathcal{S})$. Since $\check{h}^{lf}$ and $h^{lf}$ agree on discrete sets, we find that we have a map

$$\varepsilon \colon \check{h}^{lf}(X, \mathcal{S}) \longrightarrow h^{lf}(|X.|, \mathcal{S}),$$

which is readily checked to be a natural transformation.

**Proposition 7.4.10.** *When $X.$ is finite dimensional, $\varepsilon$ is a weak equivalence.*

*Proof.* We have already seen that $\varepsilon$ is a weak equivalence on discrete sets, of 0-dimensional simplicial sets. We will determine the relative terms for the inclusion $X^{(k)} \to X^{(k+1)}$ for each of the two functors. If we can show that they are equivalent, we'll be done by induction.

Since $\check{h}^{lf}(X.,\mathcal{S})$ is the realization of the simplicial spectrum $\{k \mapsto \check{h}^{lf}(X_k,\mathcal{S})\}$, it is clear that we have a cofibre sequence of spectra

$$\check{h}^{lf}(X.,\mathcal{S})^{(k-1)} \longrightarrow \check{h}^{lf}(X.,\mathcal{S})^{(k)} \longrightarrow \Sigma^k\, h^{lf}(X_k,\mathcal{S})$$

It is a straightforward verification that $\check{h}^{lf}(X^{(k)},\mathcal{S}) \simeq \check{h}^{lf}(X.,S^{(k)}$, so we have a cofibre sequence

$$\check{h}^{lf}(X^{(k-1)},\mathcal{S}) \longrightarrow \check{h}^{lf}(X^{(k)},\mathcal{S}) \longrightarrow \Sigma^k\, h^{lf}(X_k,\mathcal{S}).$$

We show that the relative term of the inclusion

$$h^{lf}(|X^{(k-1)}|,\mathcal{S}) \longrightarrow h^{lf}(|X^{(k)}|,\mathcal{S})$$

is also equivalent to $\Sigma^k\, h^{lf}(X_k,\mathcal{S})$. To see this, we first consider the subcategory $C^{\mathrm{simp}}(|X.|) \subset C(|X.|)$, consisting of those $U \in C(|X.|)$ which are the realization of subsimplicial set of $X.$. One readily checks that the inclusion $C^{\mathrm{simp}}(|X.|) \to C(|X.|)$ satisfy the hypotheses of 6.1.3, and therefore that the pullback map

$$h^{lf}(|X.|,\mathcal{S}) \simeq \underset{\overleftarrow{C(|X.|)}}{\operatorname{holim}}\ \Phi_{\mathcal{S}}^{|X|} \longrightarrow \underset{\overleftarrow{C^{\mathrm{simp}}(X.)}}{\operatorname{holim}}\ \Phi_{\mathcal{S}}^{|X|}\,|C^{\mathrm{simp}}(X.)$$

is a weak equivalence for any locally finite simplicial set $X.$. It therefore suffices to study the relative terms for the second construction, based on $C^{\mathrm{simp}}$. For any $U \in C^{\mathrm{simp}}(X)$, we have a cofibre sequence of spectra

$$h(X^{(k-1)}/U^{(k-1)},\mathcal{S}) \longrightarrow h(X^{(k)}/U^{(k)},\mathcal{S}) \longrightarrow \Sigma^k h(X_k/U_k,\mathcal{S}).$$

In fact, this is a cofibre sequence of functors on $C^{\mathrm{simp}}(X.)$. By Proposition 6.2.4, we get a cofibre sequence of spectra when we take homotopy inverse limits over $C^{\mathrm{simp}}(X.)$. Further, the two left hand terms give $h^{lf}(X^{(k-1)},\mathcal{S})$ and $h^{lf}(X^{(k)},\mathcal{S})$, respectively, so we obtain a cofibre sequence of spectra

$$h^{lf}(X^{(k-1)},\mathcal{S}) \longrightarrow h^{lf}(X^{(k)},\mathcal{S}) \longrightarrow \underset{\overleftarrow{U\in C^{\mathrm{simp}}(X.)}}{\operatorname{holim}}\ \Sigma^k h(X_k,U_k,\mathcal{S})$$

It is clear that the right hand limit is equivalent to

$$\Sigma^k \left( \underset{\overleftarrow{U\in C^{\mathrm{simp}}(X.)}}{\operatorname{holim}}\ h(X_k/U_k,\mathcal{S}) \right).$$

We therefore want to show that

$$\underset{\overleftarrow{U\in C^{\mathrm{simp}}(X.)}}{\operatorname{holim}}\ h(X_k/U_k,\mathcal{S}) \simeq h^{lf}(X_k,\mathcal{S}).$$

But we have a functor $C^{\mathrm{simp}}(X.) \to C(X_k)$ given by $U. \to U_k$ and hence a pullback map

$$h^{lf}(X_k,\mathcal{S}) \simeq \underset{\overleftarrow{U\in C(X_k)}}{\operatorname{holim}}\ h(X_k/U_k,\mathcal{S}) \longrightarrow \underset{\overleftarrow{U\in C^{\mathrm{simp}}(X)}}{\operatorname{holim}}\ h(X_k/U_k,\mathcal{S}).$$

This pullback map is an equivalence because the functor $C^{\mathrm{simp}}(X.) \to C(X_k)$ clearly satisfies the hypotheses of Proposition 6.1.3. Consequently, the two relative terms agree. We leave it to the reader to check that the induced map on relative terms in fact is an equivalence. $\square$

The main conclusion we wish to draw from this analysis is the following identification of $h^{lf}(X,\mathcal{S})^{\Gamma}$.

**Proposition 7.4.11.** *Let $X.$ be a locally finite simplicial set, with free $\Gamma$-action. Then we have a natural weak equivalence*

$$h^{lf}(|X.|, \mathcal{S})^\Gamma \longrightarrow h^{lf}(|X.|/\Gamma, \mathcal{S}).$$

*Proof.* In [9], it is proved that this result holds for $\check{h}^{lf}$; the previous proposition now gives the required result. □

We are now in a position to identify the inclusion $\mathcal{S}^\Gamma \to \mathcal{S}^{h\Gamma}$ in terms of the twisted locally finite homology construction.

**Theorem 7.4.12.** *Let $\mathcal{S}$ be any spectrum with $\Gamma$-action. Suppose $\Gamma$ is a group whose classifying space is a finite complex $B$, and let $B$ be embedded in a Euclidean space $E^n$. Let $i\colon B \to E^n$ be the embedding. Let $N$ be a regular open neighborhood of $B$. Then the composite*

$$\Sigma^n \mathcal{S}^\Gamma \simeq h^{lf}(E^n, \mathcal{S}^\Gamma) \xrightarrow{i_*} h^{lf}(N, \mathcal{S}^\Gamma) \xrightarrow{\simeq} h^{lf}(\widetilde{N}, \mathcal{S}^\Gamma)^\Gamma \longrightarrow h^{lf}(\widetilde{N}, \mathcal{S})^\Gamma$$

*where $i_*$ is the "wrong way" map from section 7.2, the second arrow comes from the equivalence*

$$h^{lf}(N, \mathcal{S}^\Gamma) = h^{lf}(\widetilde{N}/\Gamma, \mathcal{S}^\Gamma) \simeq h^{lf}(\widetilde{N}, \mathcal{S}^\Gamma)^\Gamma,$$

*and the third is induced by the inclusion $\mathcal{S}^\Gamma \hookrightarrow \mathcal{S}$, is compatible with the inclusion $\Sigma^n \mathcal{S}^\Gamma \to \Sigma^n \mathcal{S}^{h\Gamma}$ when $\Sigma^n \mathcal{S}^{h\Gamma}$ is identified with $h^{lf}(\widetilde{N}, \mathcal{S})^\Gamma$.*

# Part 3. Bounded *K*-theory

## 8. Fibred Bounded *K*-theory

8.1. **Definitions and Basic Properties.** When $M$ is the product of two proper metric spaces $X \times Y$ with the metric

$$d_{\max}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\},$$

one has the bounded category of geometric $R$-modules $\mathcal{C}(X \times Y, R)$ of Pedersen–Weibel, see Definition 1.1.2. The associated $K$-theory spectrum is $K(X \times Y, R)$.

To describe another category associated to $X \times Y$, fix a point $x_0$ in $X$. The new category $\mathcal{C}_X(Y)$ has the same objects as $\mathcal{C}(X \times Y, R)$ but a weaker control condition on the morphisms.

*Notation* 8.1.1. For a subset $S \subset X$ and a real number $R \geq 0$, $S[D]$ stands for the metric $D$-enlargement $\{x \in X \mid d(x, S) \leq D\}$. In this notation, the metric ball of radius $R$ centered at $x$ is $\{x\}[R]$ or simply $x[R]$.

For any function $f\colon [0, +\infty) \to [0, +\infty)$ and a real number $D \geq 0$, define

$$N(D, f)(x, y) = x[D] \times y[f(d(x, x_0))],$$

the $(D, f)$-neighborhood of $(x, y)$ in $X \times Y$. A homomorphism $\phi\colon F \to G$ is called $(D, f)$-*bounded* if the components $F_{(x,y)} \to G_{(x',y')}$ are zero maps for $(x', y')$ outside of the $(D, f)$-neighborhood of $(x, y)$. These are the morphisms of $\mathcal{C}_X(Y)$.

The $K$-theory spectrum $K_X(Y)$ is the spectrum associated to the isomorphism category of $\mathcal{C}_X(Y)$.

**Remark 8.1.2.** The definition of the category $\mathcal{C}_X(Y)$ is independent of the choice of $x_0$ in $X$.

**Remark 8.1.3.** It follows from Example 1.2.2 of Pedersen–Weibel [39] that in general $\mathcal{C}_X(Y)$ is not isomorphic to $\mathcal{C}(X \times Y, R)$.

The point is that the proper generality of that work, explained in [40], included a general additive category $\mathcal{A}$ embedded in a cocomplete additive category, generalizing the free finitely generated $R$-modules in the category of all free $R$-modules. All of the excision results of Pedersen–Weibel hold in the context of $\mathcal{C}(X, \mathcal{A})$. The category $\mathcal{C}_X(Y)$ is simply the category $\mathcal{C}(X, \mathcal{A})$, where $\mathcal{A} = \mathcal{C}(Y, R)$.

The difference between $\mathcal{C}_X(Y)$ and $\mathcal{C}(X \times Y, R)$ is made to disappear in [39] by making $\mathcal{C}(Y, R)$ "remember the filtration" of morphisms when viewed as a filtered additive category with

$$\mathrm{Hom}_D(F, G) = \{\phi \in \mathrm{Hom}(F, G) \mid \mathrm{fil}(\phi) \leq D\}.$$

Identifying a small category with its set of morphisms, one can think of the bounded category as

$$\mathcal{C}(Y, R) = \underset{\substack{\longrightarrow \\ D \in \mathbb{R}}}{\mathrm{colim}}\, \mathcal{C}_D(Y, R)$$

where $\mathcal{C}_D(Y, R) = \mathrm{Hom}_D(\mathcal{C}(Y, R))$ is the collection of all $\mathrm{Hom}_D(F, G)$. Now we have

$$\mathcal{C}(X \times Y, R) = \underset{\substack{\longrightarrow \\ D \in \mathbb{R}}}{\mathrm{colim}}\, \mathcal{C}(X, \mathcal{C}_D(Y, R)).$$

**Remark 8.1.4.** There is certainly an exact embedding $\iota \colon \mathcal{C}(X \times Y, R) \to \mathcal{C}_X(Y)$ inducing the map of $K$-theory spectra $K(\iota) \colon K(X \times Y, R) \to K_X(Y)$.

8.2. **Equivariance, Fixed Points, Nonconnective Delooping.** Let us recall the notion of Karoubi filtration in additive categories. The details can be found in Cardenas–Pedersen [8].

**Definition 8.2.1.** An additive category $\mathcal{C}$ is *Karoubi filtered* by a full subcategory $\mathcal{A}$ if every object $C$ has a family of decompositions $\{C = E_\alpha \oplus D_\alpha\}$ with $E_\alpha \in \mathcal{A}$ and $D_\alpha \in \mathcal{C}$, called a *Karoubi filtration* of $C$, satisfying the following properties.

- For each $C$, there is a partial order on Karoubi decompositions such that $E_\alpha \oplus D_\alpha \leq E_\beta \oplus D_\beta$ whenever $D_\beta \subset D_\alpha$ and $E_\alpha \subset E_\beta$.
- Every map $A \to C$ factors as $A \to E_\alpha \to E_\alpha \oplus D_\alpha = C$ for some $\alpha$.
- Every map $C \to A$ factors as $C = E_\alpha \oplus D_\alpha \to E_\alpha \to A$ for some $\alpha$.
- For each pair of objects $C$ and $C'$ with the corresponding filtrations $\{E_\alpha \oplus D_\alpha\}$ and $\{E'_\alpha \oplus D'_\alpha\}$, the filtration of $C \oplus C'$ is the family $\{C \oplus C' = (E_\alpha \oplus E'_\alpha) \oplus (D_\alpha \oplus D'_\alpha)\}$.

A morphism $f \colon C \to D$ is $\mathcal{A}$-*zero* if $f$ factors through an object of $\mathcal{A}$. Define the *Karoubi quotient* $\mathcal{C}/\mathcal{A}$ to be the additive category with the same objects as $\mathcal{C}$ and morphism sets $\mathrm{Hom}_{\mathcal{C}/\mathcal{A}}(C, D) = \mathrm{Hom}(C, D)/\{\mathcal{A}\text{–zero morphisms}\}$.

The following is the main theorem of Cardenas–Pedersen [8].

**Theorem 8.2.2** (Fibration Theorem)**.** *Suppose $\mathcal{C}$ is an $\mathcal{A}$-filtered category, then there is a homotopy fibration*

$$K(\mathcal{A}^{\wedge K}) \longrightarrow K(\mathcal{C}) \longrightarrow K(\mathcal{C}/\mathcal{A}).$$

*Here $\mathcal{A}^{\wedge K}$ is a certain subcategory of the idempotent completion of $\mathcal{A}$ with the same positive $K$-theory as $\mathcal{A}$.*

We will apply this theorem to a variety of bounded categories.

*Notation* 8.2.3. Let

$$\begin{aligned}
\mathcal{C}_k &= \mathcal{C}_{X\times\mathbb{R}^k}(Y),\\
\mathcal{C}_k^+ &= \mathcal{C}_{X\times\mathbb{R}^{k-1}\times[0,+\infty)}(Y),\\
\mathcal{C}_k^- &= \mathcal{C}_{X\times\mathbb{R}^{k-1}\times(-\infty,0]}(Y).
\end{aligned}$$

We will also use the notation

$$\begin{aligned}
\mathcal{C}_k^{<+} &= \underrightarrow{\operatorname{colim}}_{D\geq 0}\, \mathcal{C}_{X\times\mathbb{R}^{k-1}\times[-D,+\infty)}(Y),\\
\mathcal{C}_k^{<-} &= \underrightarrow{\operatorname{colim}}_{D\geq 0}\, \mathcal{C}_{X\times\mathbb{R}^{k-1}\times(-\infty,D]}(Y),\\
\mathcal{C}_k^{<0} &= \underrightarrow{\operatorname{colim}}_{D\geq 0}\, \mathcal{C}_{X\times\mathbb{R}^{k-1}\times[-D,D]}(Y).
\end{aligned}$$

It is easy to see that $\mathcal{C}_k$ is $\mathcal{C}_k^{<-}$-filtered and that $\mathcal{C}_k^{<+}$ is $\mathcal{C}_k^{<0}$-filtered. There are also isomorphisms $\mathcal{C}_k^{<0} \cong \mathcal{C}_{k-1}$, $\mathcal{C}_k^{<-} \cong \mathcal{C}_k^-$, and $\mathcal{C}_k/\mathcal{C}_k^{<-} \cong \mathcal{C}_k^{<+}/\mathcal{C}_k^{<0}$. By Theorem 8.2.2, the commutative diagram

$$\begin{tikzcd}
K((\mathcal{C}_k^{<0})^{\wedge K}) \arrow[r] \arrow[d] & K(\mathcal{C}_k^{<+}) \arrow[r] \arrow[d] & K(\mathcal{C}_k^{<+}/\mathcal{C}_k^{<0}) \arrow[d, "\cong"] \\
K((\mathcal{C}_k^{<-})^{\wedge K}) \arrow[r] & K(\mathcal{C}_k) \arrow[r] & K(\mathcal{C}_k/\mathcal{C}_k^{<-})
\end{tikzcd}$$

where all maps are induced by inclusion, is in fact a map of homotopy fibrations. The categories $\mathcal{C}_k^{<+}$ and $\mathcal{C}_k^{<-}$ are *flasque*, that is, possess an endofunctor Sh such that $\mathrm{Sh}(F) \cong F \oplus \mathrm{Sh}(F)$, which can be seen by the usual Eilenberg swindle argument. Therefore $K(\mathcal{C}_k^{<+})$ and $K(\mathcal{C}_k^{<-})$ are contractible. This gives a map $K(\mathcal{C}_{k-1}) \to \Omega K(\mathcal{C}_k)$ which induces isomorphisms of $K$-groups in positive dimensions.

**Definition 8.2.4.** We define the nonconnective spectrum

$$K_X^{-\infty}(Y) \overset{\text{def}}{=} \underrightarrow{\operatorname{hocolim}}_{k>0}\, \Omega^k K(\mathcal{C}_k).$$

If $Y$ is the single point space then the delooping $K_X^{-\infty}(\text{point})$ is clearly equivalent to $K^{-\infty}(X,R)$ of Pedersen–Weibel, reviewed in Theorem 1.1.5, via

$$K(\iota)\colon\ K^{-\infty}(X\times \text{point}, R) \longrightarrow K_X^{-\infty}(\text{point}).$$

The $K$-theory of equivariant categories from Definition 2.5.6 has nonconnective deloopings which are constructed in the same fashion.

For example, if we define

$$\mathcal{C}_{i,k}^{\Gamma} = \mathcal{C}^{\Gamma,0}(\Gamma\times\mathbb{R}^k\times Y, R),$$

where $\Gamma$ acts on the product $\Gamma\times\mathbb{R}^k\times Y$ according to $\gamma(\gamma',x,y) = (\gamma\gamma',x,\gamma(y))$, and

$$\mathcal{C}_{i,k}^{\Gamma,+} = \mathcal{C}^{\Gamma,0}(\Gamma\times[0,+\infty)\times Y, R), \text{ etc.}$$

then the delooping construction in Definition 8.2.4 can be applied verbatim. The same is true for the theory $k_c^{\Gamma}$. Similarly, one can use the $\Gamma$-action on $\Gamma\times\mathbb{R}^k$ given by $\gamma(\gamma',x) = (\gamma\gamma',x)$ and define $\mathcal{C}_{p,k}^{\Gamma}$ as the symmetric monoidal category of functors $\theta\colon\ \boldsymbol{E\Gamma} \to \mathcal{C}(\Gamma\times\mathbb{R}^k, \mathcal{C}(Y,R))$ such that the morphisms $\theta(f)$ are bounded by 0 as $R$-linear homomorphisms over $\Gamma \times \mathbb{R}^k$. There are obvious analogues of the categories $\mathcal{C}^{\Gamma,+}_{p,k}$, etc. If $*$ is any of the three subscripts $i$, $c$, or $p$, and $Y$ is equipped with actions by $\Gamma$ via respectively isometries or coarse equivalences, we obtain equivariant maps

$$K(\mathcal{C}^{\Gamma}_{*,k-1}(Y)) \longrightarrow \Omega K(\mathcal{C}^{\Gamma}_{*,k}(Y)).$$

**Definition 8.2.5.** Let $*$ be any of the three subscripts $i$, $c$, or $p$. We define

$$k^{\Gamma}_{*,k}(Y) = K(\mathcal{C}^{\Gamma}_{*,k}(Y))$$

and the nonconnective equivariant spectra

$$K^{\Gamma}_{*}(Y) \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k k^{\Gamma}_{*,k}(Y).$$

These are the theories referenced in Definition 2.5.6. The same construction gives for the fixed points

$$K^{\Gamma}_{*}(Y)^{\Gamma} = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k k^{\Gamma}_{*,k}(Y)^{\Gamma}.$$

8.3. **Fibrewise Localization and Excision.** Suppose $Y'$ is a subspace of $Y$. Recall that $\mathcal{C}(Y)_{<Y'}$ is the full subcategory of $\mathcal{C}(Y)$ on objects supported in a bounded neighborhood of $Y'$.

**Definition 8.3.1.** Since $\mathcal{C}(Y)$ is $\mathcal{C}(Y)_{<Y'}$-filtered, we obtain the Karoubi quotient $\mathcal{C}(Y)/\mathcal{C}(Y)_{<Y'}$ which will be denoted by $\mathcal{C}(Y,Y')$.

Using the notation from Definition 2.5.6, let $\mathcal{C}^{\Gamma}_{i}(Y)_{<Y'}$ be the full subcategory of $\mathcal{C}^{\Gamma}_{i}(Y)$ on objects $\theta$ such that the support of each $\theta(\gamma)$ is contained in a bounded neighborhood of $\Gamma \times Y'$. One similarly defines the subcategories $\mathcal{C}^{\Gamma}_{c}(Y)_{<Y'}$ and $\mathcal{C}^{\Gamma}_{p}(Y)_{<Y'}$ of $\mathcal{C}^{\Gamma}_{c}(Y)$ and $\mathcal{C}^{\Gamma}_{p}(Y)$. In all of these cases, the subcategories give Karoubi filtrations and therefore Karoubi quotients $\mathcal{C}^{\Gamma}_{*}(Y,Y')$. It is clear that the actions of $\Gamma$ extend to the quotients in each case. Taking $K$-theory of the equivariant symmetric monoidal categories gives the $\Gamma$-equivariant spectra $k^{\Gamma}_{*}(Y,Y')$.

One can now construct the parametrized versions of the relative module categories $\mathcal{C}^{\Gamma}_{*,k}(Y,Y')$, their $K$-theory spectra $k^{\Gamma}_{*,k}(Y,Y')$, and the resulting deloopings. Thus we obtain the nonconnective $\Gamma$-equivarint spectra

$$K^{\Gamma}_{*}(Y,Y') \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k k^{\Gamma}_{*,k}(Y,Y')$$

for $* = i$, $c$, $p$.

The Fibration Theorem 8.2.2 can be applied as follows.

**Corollary 8.3.2.** *There is a homotopy fibration*

$$K^{-\infty}(Y') \longrightarrow K^{-\infty}(Y) \longrightarrow K^{-\infty}(Y,Y')$$

*In the case we consider the trivial action of* $\Gamma$ *on* $Y$*, there is a homotopy fibration*

$$K^{\Gamma}_{i}(Y')^{\Gamma} \longrightarrow K^{\Gamma}_{i}(Y)^{\Gamma} \longrightarrow K^{\Gamma}_{i}(Y,Y')^{\Gamma}.$$

*Proof.* The inclusion of the subspace $Y'$ induces isomorphisms of categories $\mathcal{C}(Y') \cong \mathcal{C}(Y)_{<Y'}$ and $\mathcal{C}^{\Gamma}_{i}(Y')^{\Gamma} \cong \mathcal{C}^{\Gamma}_{i}(Y)^{\Gamma}_{<Y'}$. For the second fibration, one should observe that $\mathcal{C}^{\Gamma}_{i}(Y)^{\Gamma}$ is $\mathcal{C}^{\Gamma}_{i}(Y)^{\Gamma}_{<Y'}$-filtered. $\square$

**Remark 8.3.3.** It is important to note here that when the index $i$ is changed to either $c$ or $p$, the last sentence is no longer true. This is the source for the need to develop controlled $G$-theory where there is a similar homotopy fibration for $G^\Gamma_p$ but derived without use of Karoubi filtrations.

**Theorem 8.3.4** (Bounded Excision)**.** *If $U_1$ and $U_2$ are a coarsely antithetic pair of subsets of $Y$ which form a cover of $Y$, then the commutative square*

$$\begin{tikzcd} K^{-\infty}(U_1 \cap U_2) \arrow[r] \arrow[d] & K^{-\infty}(U_1) \arrow[d] \\ K^{-\infty}(U_2) \arrow[r] & K^{-\infty}(Y) \end{tikzcd}$$

*is a homotopy pushout. If the action of $\Gamma$ on $Y$ is trivial, then*

$$\begin{tikzcd} K^\Gamma_i(U_1 \cap U_2)^\Gamma \arrow[r] \arrow[d] & K^\Gamma_i(U_1)^\Gamma \arrow[d] \\ K^\Gamma_i(U_2)^\Gamma \arrow[r] & K^\Gamma_i(Y)^\Gamma \end{tikzcd}$$

*is a homotopy pushout.*

*Proof.* In view of the isomorphisms $\mathcal{C}(U_1, U_1 \cap U_2) \cong \mathcal{C}(Y, U_2)$ and $\mathcal{C}^\Gamma_i(U_1, U_1 \cap U_2) \cong \mathcal{C}^\Gamma_i(Y, U_2)$, we have the weak equivalences

$$K^{-\infty}(U_1, U_1 \cap U_2) \ \simeq \ K^{-\infty}(Y, U_2)$$

and

$$K^\Gamma_i(U_1, U_1 \cap U_2)^\Gamma \ \simeq \ K^\Gamma_i(Y, U_2)^\Gamma.$$

Similar to the use of the Fibration Theorem in the construction of nonconnective deloopings $K^{-\infty}(Y)$ and $K^\Gamma_i(Y)^\Gamma$, we have two maps of homotopy fibrations:

$$\begin{tikzcd} K^{-\infty}(U_1 \cap U_2) \arrow[r] \arrow[d] & K^{-\infty}(U_1) \arrow[r] \arrow[d] & K^{-\infty}(U_1, U_1 \cap U_2) \arrow[d, "\simeq"] \\ K^{-\infty}(U_1) \arrow[r] & K^{-\infty}(Y) \arrow[r] & K^{-\infty}(Y, U_1) \end{tikzcd}$$

which gives the first homotopy pushout, and

$$\begin{tikzcd} K^\Gamma_i(U_1 \cap U_2)^\Gamma \arrow[r] \arrow[d] & K^\Gamma_i(U_1)^\Gamma \arrow[r] \arrow[d] & K^\Gamma_i(U_1, U_1 \cap U_2)^\Gamma \arrow[d, "\simeq"] \\ K^\Gamma_i(U_2)^\Gamma \arrow[r] & K^\Gamma_i(Y)^\Gamma \arrow[r] & K^\Gamma_i(Y, U_2)^\Gamma \end{tikzcd}$$

which gives the second homotopy pushout. □

**Theorem 8.3.5** (Bounded Excision, a.k.a. Theorem 2.7.6)**.** *Suppose $U_1$, $U_2$, and $Y'$ are three pairwise coarsely antithetic subsets of $Y$ such that $U_1$ and $U_2$ form a cover of $Y$. Assuming the trivial action of $\Gamma$ on $Y$, the commutative square*

$$\begin{tikzcd} K^\Gamma_i(U_1 \cap U_2, Y' \cap U_1 \cap U_2)^\Gamma \arrow[r] \arrow[d] & K^\Gamma_i(U_1, Y' \cap U_1)^\Gamma \arrow[d] \\ K^\Gamma_i(U_2, Y' \cap U_2)^\Gamma \arrow[r] & K^\Gamma_i(Y, Y')^\Gamma \end{tikzcd}$$

*is a homotopy pushout.*

*Proof.* This follows from the fact that whenever $C$ is a subset of $Y$ which is coarsely antithetic to $Y'$, the category $\mathcal{C}_i^\Gamma(Y,Y')^\Gamma$ is $\mathcal{C}_i^\Gamma(Y,Y')^\Gamma_{<C}$-filtered and $\mathcal{C}_i^\Gamma(Y,Y')^\Gamma_{<C}$ is isomorphic to $\mathcal{C}_i^\Gamma(C, Y'\cap C)^\Gamma$. The details are left to the reader. □

We end this section with proofs of Propositions 2.3.4 and 3.3.3 that were used in the proof of the Main Theorem.

**Proposition 8.3.6** (a.k.a. Proposition 2.3.4)**.** $K^{-\infty}(T\mathbb{R}^n, R) \simeq \Sigma\, K^{-\infty}(\mathbb{R}^n, R) \simeq \Sigma^{n+1}\, K^{-\infty}(R)$.

*Proof.* The covering of $T\mathbb{R}^n$ by $T\mathbb{R}^n_- = \mathbb{R}^{n-1}\times(-\infty,0]\times\mathbb{R}$ and $T\mathbb{R}^n_+ = \mathbb{R}^{n-1}\times[0,+\infty)\times\mathbb{R}$ is a covering by a coarsely antithetic pair of subsets. Both $\mathcal{C}(T\mathbb{R}^n_-)$ and $\mathcal{C}(T\mathbb{R}^n_+)$ are flasque, and $\mathcal{C}(T\mathbb{R}^n_-\cap T\mathbb{R}^n_+)$ is isomorphic to $\mathcal{C}(T\mathbb{R}^{n-1})$. Using iterated bounded excision, we have the equivalence

$$K^{-\infty}(T\mathbb{R}^n, R) \simeq \Sigma^n\, K^{-\infty}(\mathbb{R}, R) \,\simeq\, \Sigma^{n+1}\, K^{-\infty}(R).$$

The equivalence

$$K^{-\infty}(\mathbb{R}^n, R) \,\simeq\, \Sigma^n\, K^{-\infty}(R)$$

follows from a similar bounded excision argument. □

**Proposition 8.3.7** (a.k.a. Proposition 3.3.3)**.** *For the trivial actions of $\Gamma$ on $\mathbb{R}^{n+k-1}$ and $\mathbb{R}^{n+k}$, there is a weak equivalence*

$$\Sigma K_i^\Gamma(T\mathbb{R}^{n+k-1})^\Gamma \,\simeq\, K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma.$$

*Proof.* The argument is along the same lines as that in proof of Proposition 8.3.6, using the second homotopy pushout square from Theorem 8.3.5. □

**Corollary 8.3.8.** *For the trivial action of $\Gamma$ on $\mathbb{R}^{n+k}$, there is a weak equivalence*

$$K_i^\Gamma(T\mathbb{R}^{n+k})^\Gamma \,\simeq\, \Sigma^{n+k+1}\, K^{-\infty}(R[\Gamma]).$$

The next result applies to a nontrivial action of $\Gamma$ but in the direction transverse to the previous theorems.

**Theorem 8.3.9.** *Suppose $\Gamma$ acts on $\widetilde{N}$ by deck transformations, with the compact quotient $N$. Then there is a weak equivalence*

$$\sigma\colon \Sigma K_i^\Gamma(\widetilde{N})^\Gamma \,\simeq\, K_i^\Gamma(T\widetilde{N})^\Gamma.$$

*Proof.* Clearly, the subcategories $\mathcal{C}_- = \mathcal{C}^{\Gamma,0}(\Gamma\times(-\infty,0]\times\widetilde{N})$ and $\mathcal{C}_+ = \mathcal{C}^{\Gamma,0}(\Gamma\times[0,+\infty)\times\widetilde{N})$ are invariant under the induced action of $\Gamma$ on $\mathcal{C} = \mathcal{C}^{\Gamma,0}(\Gamma\times T\widetilde{N})$. The lax limit category $\mathcal{C}^\Gamma$ is Karoubi filtered by both $\mathcal{C}_-^\Gamma$ and $\mathcal{C}_+^\Gamma$. Notice that both of these subcategories are flasque, and their intersection is $\mathcal{C}^{\Gamma,0}(\Gamma\times\widetilde{N})^\Gamma$. □

## 9. Algebraic $G$-theory with Bounded Control

9.1. **Bounded $G$-theory.** Boundedly controlled $G$-theory [13] is a variant of the Pedersen–Weibel bounded $K$-theory applicable to some more general, nonsplit exact structures. We will review and augment some material in the form best fit for the fibred situation.

This subsection is dedicated to an exposition of the absolute theory, while our main interest is in the more general fibred version. The restriction to a single metric space allows for more concise arguments which will be reused in the fibred situation.

The variation of the basic construction of bounded $K$-theory is based on the following observation. Given an object $F$ in $\mathcal{C}(X,R)$, to every subset $S \subset X$ one can associate a direct sum $F(S)$ generated by those $F(x)$ with $x \in S$. Now the restriction from arbitrary $R$-linear homomorphisms to the bounded ones can be described entirely in terms of these subobjects.

The description of admissible exact sequences will require the language of Quillen exact categories.

**Definition 9.1.1.** Let $\mathbf{C}$ be an additive category. Suppose $\mathbf{C}$ has two classes of morphisms $\mathbf{m}(\mathbf{C})$, called *admissible monomorphisms*, and $\mathbf{e}(\mathbf{C})$, called *admissible epimorphisms*, and a class $\mathcal{E}$ of *exact* sequences, or extensions, of the form

$$C^{\boldsymbol{\cdot}}\colon \quad C' \xrightarrow{i} C \xrightarrow{j} C''$$

with $i \in \mathbf{m}(\mathbf{C})$ and $j \in \mathbf{e}(\mathbf{C})$ which satisfy the three axioms:

a) any sequence in $\mathbf{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$; the canonical sequence

$$C' \xrightarrow{\mathrm{incl}_1} C' \oplus C'' \xrightarrow{\mathrm{proj}_2} C''$$

is in $\mathcal{E}$; for any sequence $C^{\boldsymbol{\cdot}}$, $i$ is a kernel of $j$ and $j$ is a cokernel of $i$ in $\mathbf{C}$,

b) both classes $\mathbf{m}(\mathbf{C})$ and $\mathbf{e}(\mathbf{C})$ are subcategories of $\mathbf{C}$; $\mathbf{e}(\mathbf{C})$ is closed under base-changes along arbitrary morphisms in $\mathbf{C}$ in the sense that for every exact sequence $C' \to C \to C''$ and any morphism $f\colon D'' \to C''$ in $\mathbf{C}$, there is a pullback commutative diagram

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "="'] & D \arrow[r, "j'"] \arrow[d, "f'"] & D'' \arrow[d, "f"] \\ C' \arrow[r] & C \arrow[r, "j"] & C'' \end{tikzcd}$$

where $j'\colon D \to D''$ is an admissible epimorphism; $\mathbf{m}(\mathbf{C})$ is closed under cobase-changes along arbitrary morphisms in $\mathbf{C}$ in the (dual) sense that for every exact sequence $C' \to C \to C''$ and any morphism $g\colon C' \to D'$ in $\mathbf{C}$, there is a pushout diagram

$$\begin{tikzcd} C' \arrow[r, "i"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "g'"'] & C'' \arrow[d, "="] \\ D' \arrow[r, "i'"] & D \arrow[r] & C'' \end{tikzcd}$$

where $i'\colon D' \to D$ is an admissible monomorphism,

c) if $f\colon C \to C''$ is a morphism with a kernel in $\mathbf{C}$, and there is a morphism $D \to C$ so that the composition $D \to C \to C''$ is an admissible epimorphism, then $f$ is an admissible epimorphism; dually for admissible monomorphisms.

According to Keller [32], axiom (c) follows from the other two.

Recall that an *abelian* category is an additive category with kernels and cokernels such that every morphism $f$ is *balanced*, that is, the canonical map from the coimage $\mathrm{coim}(f) = \mathrm{coker}(\ker f)$ to the image $\mathrm{im}(f) = \ker(\mathrm{coker}\, f)$ is an isomorphism.

**Definition 9.1.2.** If a category has kernels and cokernels for all morphisms, and the canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is always monic and epic but not necessarily invertible, we say the category is *pseudoabelian*.

A pseudoabelian category is *exact pseudoabelian* if it has the canonical exact structure where all kernels and cokernels are respectively admissible monomorphisms and admissible epimorphisms.

Recall also that a category is called *cocomplete* if it contains colimits of arbitrary small diagrams, cf. Mac Lane [36], chapter V.

A full subcategory $\mathbf{H}$ of an exact category $\mathbf{C}$ is said to be *closed under extensions* in $\mathbf{C}$ if $\mathbf{H}$ contains the zero object and for any exact sequence $C' \to C \to C''$ in $\mathbf{C}$, if $C'$ and $C''$ are isomorphic to objects from $\mathbf{H}$ then so is $C$. A subcategory closed under extensions in $\mathbf{C}$ inherits the exact structure from $\mathbf{C}$.

Let $X$ be a proper metric space and let $R$ be a noetherian ring.

*Notation* 9.1.3. We will use the notation $\mathcal{P}(X)$ for the power set of $X$ partially ordered by inclusion and viewed as a category. Let $\mathcal{B}(X)$ be the subcategory of bounded subsets, $\mathcal{B}_D(X)$ be the subcategory of subsets with diameter bounded by $D \geq 0$, and let $\mathbf{Mod}(R)$ denote the category of left $R$-modules. If $F$ is a left $R$-module, let $\mathcal{I}(F)$ denote the family of all $R$-submodules of $F$ partially ordered by inclusion.

**Definition 9.1.4.** An *$X$-filtered* $R$-module is a module $F$ together with a functor $\mathcal{P}(X) \to \mathcal{I}(F)$ from the power set of $X$ to the family of $R$-submodules of $F$, both ordered by inclusion, such that the value on $X$ is $F$. It will be most convenient to think of $F$ as the functor above and use notation $F(S)$ for the value of the functor on $S$. We will call $F$ *reduced* if $F(\emptyset) = 0$.

An $R$-homomorphism $f \colon F \to G$ of $X$-filtered modules is *boundedly controlled* if there is a fixed number $b \geq 0$ such that the image $f(F(S))$ is a submodule of $G(S[b])$ for all subsets $S$ of $X$.

The objects of the category $\mathbf{U}(X, R)$ are the reduced $X$-filtered $R$-modules and the morphisms are the boundedly controlled homomorphisms.

**Remark 9.1.5.** If $X$ is unbounded, $\mathbf{U}(X, R)$ is not a balanced category and therefore not an abelian category. For an explicit description of a boundedly controlled morphism in $\mathbf{U}(\mathbb{Z}, R)$ which is an isomorphism of left $R$-modules but whose inverse is not boundedly controlled, we refer to Example 1.5 of [39].

We will show that $\mathbf{U}(X, R)$ is an exact pseudoabelian category. The exact structure in $\mathbf{U}(X, R)$ involves an additional property a boundedly controlled morphism may or may not have.

**Definition 9.1.6.** A morphism $f \colon F \to G$ in $\mathbf{U}(X, R)$ is called *boundedly bicontrolled* if, for some fixed $b \geq 0$, in addition to inclusions of submodules

$$f(F(S)) \subset G(S[b]),$$

there are inclusions

$$f(F) \cap G(S) \subset fF(S[b])$$

for all subsets $S \subset X$. In this case we will say that $f$ has filtration degree $b$ and write $\mathrm{fil}(f) \leq b$.

**Lemma 9.1.7.** *Let $f_1 \colon F \to G$, $f_2 \colon G \to H$ be in $\mathbf{U}(X, R)$ and $f_3 = f_2 f_1$.*

(1) *If $f_1$, $f_2$ are boundedly bicontrolled and either $f_1 \colon F(X) \to G(X)$ is an epimorphism or $f_2 \colon G(X) \to H(X)$ is a monomorphism, then $f_3$ is also boundedly bicontrolled.*
(2) *If $f_1$, $f_3$ are boundedly bicontrolled and $f_1$ is an epimorphism then $f_2$ is also boundedly bicontrolled; if $f_3$ is only boundedly controlled then $f_2$ is also boundedly controlled.*
(3) *If $f_2$, $f_3$ are boundedly bicontrolled and $f_2$ is a monomorphism then $f_1$ is also boundedly bicontrolled; if $f_3$ is only boundedly controlled then $f_1$ is also boundedly controlled.*

*Proof.* Suppose $\mathrm{fil}(f_i) \leq b$ and $\mathrm{fil}(f_j) \leq b'$ for $\{i,j\} \subset \{1,2,3\}$, then in fact $\mathrm{fil}(f_{6-i-j}) \leq b + b'$ in each of the three cases. For example, there are factorizations

$$f_2 G(S) \subset f_2 f_1 F(S[b]) = f_3 F(S[b]) \subset H(S[b+b'])$$
$$f_2 G(X) \cap H(S) \subset f_3 F(S[b']) = f_2 f_1 F(S[b']) \subset f_2 G(S[b+b'])$$

which verify part 2 with $i = 1$, $j = 3$. $\square$

**Definition 9.1.8.** Let the *admissible monomorphisms* in $\mathbf{U}(X,R)$ be the boundedly bicontrolled homomorphisms $m \colon F_1 \to F_2$ such that the induced $F_1(X) \to F_2(X)$ is a monomorphism. Let the *admissible epimorphisms* be the boundedly bicontrolled homomorphisms $e \colon F_1 \to F_2$ such that $F_1(X) \to F_2(X)$ is an epimorphism.

The class $\mathcal{E}$ of *exact sequences* consists of the sequences

$$F^{\cdot} \colon \quad F' \xrightarrow{\ i\ } F \xrightarrow{\ j\ } F'',$$

where $i$ is an admissible monomorphism, $j$ is an admissible epimorphism, and $\mathrm{im}(i) = \ker(j)$.

**Theorem 9.1.9.** $\mathbf{U}(X,R)$ *is a cocomplete exact pseudoabelian category.*

*Proof.* The additive properties are inherited from $\mathbf{Mod}(R)$. In particular, the biproduct is given by the filtration-wise operation

$$(F \oplus G)(S) = F(S) \oplus G(S).$$

For any boundedly controlled homomorphism $f \colon F \to G$, the kernel of $f$ in $\mathbf{Mod}(R)$ has the standard $X$-filtration $K$ where

$$K(S) = \ker(f) \cap F(S)$$

which gives the kernel of $f$ in $\mathbf{U}(X,R)$. The canonical monomorphism $\kappa \colon K \to F$ has filtration degree 0. It follows from part 3 of Lemma 9.1.7 that $K$ has the universal properties of the kernel in $\mathbf{U}(X,R)$.

Similarly, let $I$ be the standard $X$-filtration of the image of $f$ in $\mathbf{Mod}(R)$ by

$$I(S) = \mathrm{im}(f) \cap G(S).$$

If we define $C(S) = G(S)/I(S)$ for all $S \subset X$ then clearly $C(X)$ is the cokernel of $f$ in $\mathbf{Mod}(R)$. There is an $X$-filtered module $C_X$ associated to $C$ given by

$$C_X(S) = \mathrm{im}\{C(S) \to C(X)\}.$$

The canonical homomorphism $\sigma \colon G(X) \to C(X)$ gives a filtration 0 morphism $\sigma \colon G \to C_X$ since

$$\mathrm{im}(\sigma \circ \{G(S) \to G(X)\}) = \mathrm{im}\{C(S) \to C(X)\} = C_X(S).$$

The universal cokernel properties of $C_X$ and $\sigma$ in $\mathbf{U}(X,R)$ follow from part 2 of Lemma 9.1.7.

The preceding combined with the fact that $\mathbf{Mod}(R)$ is cocomplete shows that $\mathbf{U}(X,R)$ is cocomplete, cf. Mac Lane [36], section V.2.

It follows from Lemma 9.1.7 that any exact sequence $F^{\cdot}$ isomorphic to some short exact sequence in $\mathcal{E}$ is also in $\mathcal{E}$, that

$$F' \xrightarrow{[\mathrm{id},0]} F' \oplus F'' \xrightarrow{[0,\mathrm{id}]^T} F''$$

is in $\mathcal{E}$, and that $i = \ker(j)$, $j = \operatorname{coker}(i)$ in any sequence $F^{\cdot}$ in $\mathcal{E}$.

The collections of admissible monomorphisms and epimorphisms are closed under composition by part 1 of Lemma 9.1.7. Given $F^{\cdot}$ in $\mathcal{E}$ and any $f\colon A \to F''$ in $\mathbf{U}(X,R)$, there is a base change diagram

$$\begin{array}{ccccc} F' & \longrightarrow & F \times_f A & \xrightarrow{j'} & A \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ F' & \longrightarrow & F & \xrightarrow{j} & F'' \end{array}$$

where $F \times_f A$ is the kernel of the epimorphism

$$j \operatorname{pr}_1 - f \operatorname{pr}_2 \colon F \oplus A \to F''.$$

If $m\colon F \times_f A \to F \oplus A$ is the inclusion of the kernel then $f' = \operatorname{pr}_1 m$, $j' = \operatorname{pr}_2 m$. The $X$-filtration is given by

$$(F \times_f A)(S) = F \times_f A \cap (F(S) \times A(S)),$$

so that $j'$ is boundedly controlled and has the same kernel as $j$. In fact,

$$\operatorname{im}(j') \cap A(S) \subset j'(F \times_f A)(S[b(f) + b(j)])$$

since $fA(S) \subset F''(S[b(f)])$, so $j'$ is boundedly bicontrolled of filtration degree $b(f) + b(j)$. Given an admissible submodule $E \subset F \times A$, the restriction $j'|E$ is the pullback of the admissible epimorphism $f(E) \to fj'(E)$. This shows that the class of admissible epimorphisms is closed under base change by arbitrary morphisms in $\mathbf{U}(X,R)$. The proof of closure under cobase changes by admissible monomorphisms is similar. □

**Definition 9.1.10.** Let $F$ be an $X$-filtered $R$-module.

- $F$ is called *lean* or *$D$-lean* if there is a number $D \geq 0$ such that
$$F(S) \subset \sum_{x \in S} F(x[D])$$
for every subset $S$ of $X$.
- $F$ is called *insular* or *$d$-insular* if there is a number $d \geq 0$ such that
$$F(S) \cap F(U) \subset F(S[d] \cap U[d])$$
for every pair of subsets $S$, $U$ of $X$.

**Proposition 9.1.11.** *The properties of being lean and insular are preserved under isomorphisms in* $\mathbf{U}(X,R)$.

*Proof.* If $f\colon F_1 \to F_2$ is an isomorphism with $\operatorname{fil}(f) \leq b$, and $F_1$ is $D$-lean and $d$-insular, then $F_2$ is $(D+b)$-lean and $(d+2b)$-insular. □

There is a property of filtered modules which is a consequence of leanness.

**Definition 9.1.12.** An $X$-filtered $R$-module $F$ is called *split* or *$D'$-split* if there is a number $D' \geq 0$ such that we have

$$F(S) \subset F(T[D']) + F(U[D'])$$

whenever a subset $S$ of $X$ is written as a union $T \cup U$.

**Proposition 9.1.13.** *A $D$-lean filtered module is $D$-split.*

*Proof.* We have

$$F(T \cup U) \subset \sum_{x \in T} F(x[D]) + \sum_{x \in U} F(x[D]) \subset F(T[D]) + F(U[D])$$

since in general $\sum_{x \in S} F(x[D]) \subset F(S[D])$. $\square$

**Proposition 9.1.14.** *The property of being split is preserved under isomorphisms in $\mathbf{U}(X, R)$.*

*Proof.* If $f \colon F_1 \to F_2$ is an isomorphism with $\mathrm{fil}(f) \leq b$, and $F_1$ is $D'$-split, then $F_2$ is $(D' + b)$-split. $\square$

**Lemma 9.1.15.** (1) *Lean objects are closed under exact extensions.*
(2) *Split objects are closed under exact extensions.*
(3) *Insular objects are closed under exact extensions.*

*Proof.* Let

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in $\mathbf{U}(X, R)$ and let $b \geq 0$ be a common filtration degree for $f$ and $g$.

(1) Suppose both $E'$ and $E''$ are $D$-lean. For an arbitrary subset $S \subset X$,

$$gE(S) \subset E''(S[b]),$$

so

$$gE(S) \subset \sum_{x \in S[b]} E''(x[D]).$$

For each $x \in S[b]$,

$$E''(x[D]) \subset gE(x[b + D]),$$

so

$$E(S) \subset \sum_{x \in S[b]} E(x[2b + D]) + \sum_{x \in S[b]} fE'(x[2b + D]).$$

Therefore

$$E(S) \subset \sum_{x \in S[b]} E(x[3b + D]) \subset \sum_{x \in S} E(x[4b + D]),$$

and $E$ is $(4b + D)$-lean.

(2) Suppose both $E'$ and $E''$ are $D'$-split. We have

$$gE(T \cup U) \subset E''(T[b] \cup U[b]),$$

because in general $(T \cup U)[b] \subset T[b] \cup U[b]$. So

$$\begin{aligned} &g(T \cup U) \\ &\subset E''(T[b + D']) + E''(U[b + D']) \\ &\subset gE(T[2b + D']) + gE(U[2b + D']). \end{aligned}$$

If $z \in E(T \cup U)$ then we can write $g(z) = g(z_1) + g(z_2)$ where $z_1 \in E(T[2b + D'])$ and $z_2 \in E(U[2b + D'])$. Since $z - z_1 - z_2$ is an element of $\ker(g) \cap E(T[2b + D'] \cup U[2b + D'])$, we have an element

$$\begin{aligned} k &\in E'(T[3b + D'] \cup U[3b + D']) \\ &\subset E'(T[3b + 2D']) + E'(U[3b + 2D']) \end{aligned}$$

such that

$$z = f(k) + z_1 + z_2 \in E(T[4b + 2D']) + E(U[4b + 2D']).$$

So $E$ is $(4b + 2D')$-split.

(3) Assuming that both $E'$ and $E''$ are $d$-insular, for any pair of subsets $T$ and $U$ of $X$,

$$\begin{aligned} &g(E(T) \cap E(U)) \\ \subset\ &E''(T[b]) \cap E''(U[b]) \\ \subset\ &E''(T[b + d] \cap U[b + d]). \end{aligned}$$

Now we have

$$\begin{aligned} &E(T) \cap E(U) \\ \subset\ &E(T[2b + d] \cap U[2b + d]) + fE' \cap E(T[2b + d]) \cap E(U[2b + d]) \\ \subset\ &E(T[2b + d] \cap U[2b + d]) + f(E'(T[3b + d]) \cap E'(U[3b + d]) \\ \subset\ &E(T[2b + d] \cap U[2b + d]) + fE'(T[3b + 2d] \cap U[3b + 2d]) \\ \subset\ &E(T[4b + 2d] \cap U[4b + 2d]). \end{aligned}$$

This shows that $E$ is $(4b + 2d)$-insular. □

**Lemma 9.1.16.** *Let*

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

*be an exact sequence in* $\mathbf{U}(X, R)$*.*

(1) *If the object $E$ is lean then $E''$ is lean.*
(2) *If $E$ is split then $E''$ is split.*
(3) *If $E$ is insular then $E'$ is insular.*
(4) *If $E$ is insular and $E'$ is lean then $E''$ is insular.*
(5) *If $E$ is insular and $E'$ is split then $E''$ is insular.*
(6) *If $E$ is split and $E''$ is insular then $E'$ is split.*

*Suppose $X$ has finite asymptotic dimension, then*

(7) *if $E$ is lean and $E''$ is insular then $E'$ is lean.*

**Corollary 9.1.17.** *If $E$ is split and insular then $E''$ is insular if and only if $E'$ is split.*

*Proof.* Let $b \geq 0$ be a common filtration degree for $f$ and $g$. If $E$ is $D$-lean, $D'$-split, or $d$-insular, it is easy to show that $E''$ is $(D + 2b)$-lean or $(D' + 2b)$-split and $E'$ is $(d + 2b)$-insular respectively, which verifies (1), (2), and (3).

(4) Suppose $E'$ is $D$-lean and $E$ is $d$-insular. For any pair of subsets $T$, $U \subset X$,

$$E''(T) \cap E''(U) \subset gE(T[b]) \cap gE(U[b]).$$

Given $z \in E''(T) \cap E''(U)$, let $y' \in E(T[b])$ and $y'' \in E(U[b])$ so that $g(y') = g(y'') = z$. Now

$$k = y' - y'' \in \big(E(T[b]) + E(U[b])\big) \cap \ker(g),$$

so there is $\overline{k} \in E'(T[2b]) + E'(U[2b]) \subset E'(T[2b] \cup U[2b])$ with $f(\overline{k}) = k$. Since $E'$ is $D$-lean,

$$\text{(A)} \qquad \overline{k} \in \sum_{x \in T[2b] \cup U[2b]} E'(x[D]) = \sum_{x \in T[2b]} E'(x[D]) + \sum_{y \in U[2b]} E'(y[D]).$$

Hence,

$$\text{(B)} \qquad \overline{k} \in E'(T[2b + D]) + E'(U[2b + D]).$$

This allows us to write $\overline{k} = \overline{k}_1 + \overline{k}_2$, where $\overline{k}_1 \in E'(T[2b+D])$ and $\overline{k}_2 \in E'(U[2b+D])$. Now $k = f\overline{k}_1 + f\overline{k}_2$. Notice that

$$y' = y'' + k = y'' + f\overline{k}_1 + f\overline{k}_2.$$

So

$$y = y' - f\overline{k}_1 = y'' + f\overline{k}_2$$

has the property

$$y \in E(T[3b + D]) \cap E(U[3b + D]) \subset E(T[3b + D + d] \cap U[3b + D + d]),$$

and $g(y) = z$. Hence

$$z \in E''(T[4b + D + d] \cap U[4b + D + d]).$$

We conclude that $E''$ is $(4b + D + d)$-insular.

(5) Showing that $E''$ is $(4b + D' + d)$-insular if $E'$ is $D'$-split is entirely similar to the proof of part (4). Equation (A) in that proof is the only step that uses $D$-leanness of $E'$. The consequence in Equation (B) follows, in fact, directly from the assumption that $E'$ is $D$-split.

(6) We now address the converse. Suppose $E$ is $D'$-split and $E''$ is $d$-insular. Given $z \in E'(T \cup U)$, we have $f(z) \in E(T[b] \cup U[b])$. Now $f(z) \in E(T[b + D']) + E(U[b + D'])$, so we can write accordingly $f(z) = y_1 + y_2$. Now $f(z) \in \ker(g)$, because $g(y_1) + g(y_2) = 0$. Since $E''$ is $d$-insular,

$$g(y_1) = -g(y_2) \in E''(T[2b + D' + d] \cap U[2b + D' + d]),$$

so we are able to find

$$y \in E(T[3b + D' + d] \cap U[3b + D' + d])$$

such that $g(y) = g(y_1) = -g(y_2)$, because generally $(S \cap P)[b] \subset S[b] \cap P[b]$. Thus

$$f(z) = y_1 + y_2 = (y_1 - y) + (y_2 + y)$$

and

$$y_1 - y \in E(T[3b + D' + d]), \quad y_2 + y \in E(U[3b + D' + d]).$$

Let $z_1 = f^{-1}(y_1 - y)$ and $z_2 = f^{-1}(y_2 + y)$, and we have $z = z_1 + z_2$ such that

$$z_1 \in E'(T[4b + D' + d]), \quad z_2 \in E'(U[4b + D' + d]),$$

so $E'$ is $(4b + D' + d)$-split.

(7) This is a consequence of Lemma 10.2.5 which is a recasting of the main theorem from [11]. We postpone the proof to section 10.2. $\square$

**Definition 9.1.18.** We define $\mathbf{L}(X, R)$ as the full subcategory of $\mathbf{U}(X, R)$ on objects that are lean and insular with the induced exact structure. Let $\mathbf{S}(X, R)$ be the full subcategory of $\mathbf{U}(X, R)$ on objects that are split and insular.

**Theorem 9.1.19.** *$\mathbf{L}(X, R)$ and $\mathbf{S}(X, R)$ are closed under extensions in $\mathbf{U}(X, R)$.*

*Proof.* The first fact follows from parts (1) and (3) of Lemma 9.1.15, the second from (2) and (3). □

It is known that a subcategory closed under extensions inherits the exact structure from the ambient category, and is, in fact, an exact subcategory.

**Corollary 9.1.20.** $\mathbf{L}(X,R)$ *and* $\mathbf{S}(X,R)$ *are exact subcategories of* $\mathbf{U}(X,R)$*. Therefore we have exact inclusions*

$$\mathbf{L}(X,R) \longrightarrow \mathbf{S}(X,R) \longrightarrow \mathbf{U}(X,R).$$

**Definition 9.1.21.** An $X$-filtered $R$-module $F$ is *locally finitely generated* if $F(S)$ is a finitely generated $R$-module for every bounded subset $S \subset X$.

**Definition 9.1.22.** We define the category $\mathbf{BL}(X,R)$ as the full subcategory of $\mathbf{L}(X,R)$ on the locally finitely generated objects. We also define the companion category $\mathbf{BS}(X,R)$ as as the full subcategory of $\mathbf{S}(X,R)$ on the locally finitely generated objects.

**Theorem 9.1.23.** *The category* $\mathbf{BL}(X,R)$ *is closed under extensions in* $\mathbf{L}(X,R)$*. Similarly, the category* $\mathbf{BS}(X,R)$ *is closed under extensions in* $\mathbf{S}(X,R)$*.*

*Proof.* If $f\colon F \to G$ is an isomorphism with $\mathrm{fil}(f) \leq b$ and $G$ is locally finitely generated, then $F(U)$ are finitely generated submodules of $G(U[b])$ for all bounded $U$, since $R$ is a noetherian ring. Suppose

$$F' \xrightarrow{f} F \xrightarrow{g} F''$$

is an exact sequence and let $b \geq 0$ be a common filtration degree for both $f$ and $g$. Assume that $F'$ and $F''$ are locally finitely generated. For every bounded subset $U \subset X$ the restriction $g\colon F(U) \to gF(U)$ is an epimorphism onto a submodule of the finitely generated $R$-module $F''(U[b])$. The kernel of $g|F(U)$ is a submodule of $F'(U[b])$, which is also finitely generated. So the extension $F(U)$ is finitely generated. □

**Corollary 9.1.24.** $\mathbf{BL}(X,R)$ *and* $\mathbf{BS}(X,R)$ *are exact categories. The additive category* $\mathcal{C}(X,R)$ *of geometric $R$-modules with the split exact structure is an exact subcategory of* $\mathbf{BL}(X,R)$*. Therefore, we have a sequence of exact inclusions*

$$\mathcal{C}(X,R) \longrightarrow \mathbf{BL}(X,R) \longrightarrow \mathbf{BS}(X,R) \longrightarrow \mathbf{U}(X,R).$$

Recall that a morphism $e\colon F \to F$ is an idempotent if $e^2 = e$. Categories in which every idempotent is the projection onto a direct summand of $F$ are called *idempotent complete.*

**Proposition 9.1.25.** *A pseudoabelian category is idempotent complete.*

*Proof.* The proof is exactly the same as for an abelian category: if $e$ is an idempotent then its kernel is split by $1-e$. □

**Corollary 9.1.26.** $\mathbf{BL}(X,R)$ *and* $\mathbf{BS}(X,R)$ *are idempotent complete.*

*Proof.* Since the restriction of an idempotent $e$ to the image of $e$ is the identity, every idempotent is boundedly bicontrolled of filtration 0. It follows easily that the splitting of $e$ in $\mathbf{Mod}(R)$ is in fact a splitting in $\mathbf{BL}(X,R)$ or $\mathbf{BS}(X,R)$. □

Finally, we need to address (the lack of) inheritance features in filtered modules. First, we recall the following definition from [13].

**Definition 9.1.27.** An $X$-filtered object $F$ is called *strict* if there exists an order preserving function $\ell\colon \mathcal{P}(X) \to [0, +\infty)$ such that for every $S \subset X$ the submodule $F(S)$ is $\ell_S$-lean and $\ell_S$-insular with respect to the standard $X$-filtration $F(S)(T) = F(S) \cap F(T)$.

It is important to note that this property is not preserved under isomorphisms, so the subcategory of strict objects is not essentially full in $\mathbf{BL}(X, R)$.

**Definition 9.1.28.** The *bounded category* $\mathbf{B}(X, R)$ was defined in [13] as the full subcategory of $\mathbf{BL}(X, R)$ on objects isomorphic to strict objects.

A consequence of strictness (or being isomorphic to a strict object) is the following feature. Given a filtered module $F$ in $\mathbf{B}(X, R)$, a *grading* of $F$ is a functor $\widetilde{F}\colon \mathcal{P}(X) \to \mathcal{I}(F)$ from the power set of $X$ to the submodules of $F$ such that

(1) each $\widetilde{F}(S)$ is an object of $\mathbf{BL}(X, R)$ when given the standard filtration,
(2) there is a number $K \geq 0$ such that

$$F(S) \subset \widetilde{F}(S) \subset F(S[K])$$

for all subsets $S$ of $X$.

Clearly, each $\widetilde{F}(S)$ is an object of $\mathbf{B}(X, R)_{<S}$. Also a strict object has a grading by $\widetilde{F}(S) = F(S)$ with $K = 0$.

We note for the interested reader that the theory in [13], including the excision theorems, could be alternately developed for graded modules in place of strict filtered modules. We do not require such theory in this paper. Instead, we develop a similar but weaker notion of gradings in $\mathbf{BS}(X, R)$.

**Definition 9.1.29.** Given a filtered module $F$ in $\mathbf{BS}(X, R)$, a *grading* of $F$ is a functor $\mathcal{F}\colon \mathcal{P}(X) \to \mathcal{I}(F)$ such that

(1) each $\mathcal{F}(S)$ is an object of $\mathbf{BS}(X, R)$ when given the standard filtration,
(2) there is a number $K \geq 0$ such that

$$F(S) \subset \mathcal{F}(S) \subset F(S[K])$$

for all subsets $S$ of $X$.

We will say that a filtered module $F$ is *graded* if it has a grading.

**Proposition 9.1.30.** *The graded objects are closed under isomorphisms.*

*Proof.* If $f\colon F \to F'$ is an isomorphism and $F$ has a grading $\mathcal{F}$, a grading for $F'$ is given by $\mathcal{F}'(C) = f\mathcal{F}(C[K + b])$, where $b$ is a filtration bound for $f$. □

**Definition 9.1.31.** We define $\mathbf{G}(X, R)$ as the full subcategory of $\mathbf{BS}(X, R)$ on the locally finitely generated graded filtered modules.

**Proposition 9.1.32.** $\mathbf{G}(X, R)$ *is closed under extensions in* $\mathbf{BS}(X, R)$*. Therefore* $\mathbf{G}(X, R)$ *is an exact subcategory of* $\mathbf{BS}(X, R)$*.*

*Proof.* Given an exact sequence in $\mathbf{BS}(X, R)$

$$F \xrightarrow{f} G \xrightarrow{g} H,$$

let $b \geq 0$ be a common filtration degree for both $f$ and $g$ as boundedly bicontrolled maps and assume that $F$ and $H$ are graded modules in $\mathbf{G}(X, R)$ with the associated functors $\mathcal{F}$ and $\mathcal{H}$.

To define a grading for $G$, consider a subset $S$ and suppose $\mathcal{H}(S[b])$ is $D$-split and $d$-insular. The induced epimorphism

$$g \colon G(S[2b]) \cap g^{-1}\mathcal{H}(S[b]) \longrightarrow \mathcal{H}(S[b])$$

extends to another epimorphism

$$g' \colon f\mathcal{F}(S[3b]) + G(S[2b]) \cap g^{-1}\mathcal{H}(S[b]) \longrightarrow \mathcal{H}(S[b])$$

with $\ker(g') = \mathcal{F}(S[3b])$. Without loss of generality, suppose $\mathcal{F}(S[3b])$ is $D$-split and $d$-insular. We define

$$\mathcal{G}(S) = f\mathcal{F}(S[3b]) + G(S[2b]) \cap g^{-1}\mathcal{H}(S[b]).$$

From parts (2) and (3) of Lemma 9.1.15, $\mathcal{G}(S)$ is $(4b+2d)$-split and $(4b+2d)$-insular. Since $G(S) \subset g^{-1}\mathcal{H}(S[b])$, we have $G(S) \subset \mathcal{G}(S)$. On the other hand, if the grading $\mathcal{F}$ has characteristic number $K \geq 0$ then $\mathcal{G}(S) \subset G(S[4b+K])$. The last fact together with Theorem 9.1.23 shows that $\mathcal{G}(S)$ is finitely generated. □

**Corollary 9.1.33.** *There is a commutative diagram of exact inclusions and exact forgetful functors*

$$\begin{tikzcd}
 & \mathbf{BL}(X,R) \arrow[r] & \mathbf{BS}(X,R) \\
\mathcal{C}(X,R) \arrow[ur, bend left] \arrow[dr, bend right] & & \\
 & \mathbf{B}(X,R) \arrow[uu] \arrow[r] & \mathbf{G}(X,R) \arrow[uu]
\end{tikzcd}$$

The advantage of working with the category $\mathbf{G}(X,R)$ is that one can easily localize to geometrically defined subobjects.

**Lemma 9.1.34.** *Suppose $G$ is a graded $X$-filtered module with a grading $\mathcal{G}$. Let $F$ be a submodule which is split with respect to the standard filtration. Then $\mathcal{F}(S) = F \cap \mathcal{G}(S)$ is a grading of $F$.*

*Proof.* Of course, $F(S) = F \cap G(S) \subset F \cap \mathcal{G}(S) = \mathcal{F}(S)$. On the other hand, there is $d \geq 0$ such that $\mathcal{G}(S) \subset G(S[d])$, so $\mathcal{F}(S) \subset F \cap G(S[d]) = F(S[d])$.

Consider the inclusion $i \colon F \to G$, and take the quotient $q \colon G \to H$. Both $F$ and $G$ are split and insular, so $H$ is split and insular by parts (2) and (4) of Lemma 9.1.16, with respect to the quotient filtration. We define $\mathcal{H}(S)$ as the image $q\mathcal{G}(S)$ and give $\mathcal{H}(S)$ the standard filtration in $H$. Then $\mathcal{H}(S)$ is split as the image of a split $\mathcal{G}(S)$ and insular since $H$ is insular. Now the kernel of $q| \colon \mathcal{G}(S) \to \mathcal{H}(S)$, which is $F \cap \mathcal{G}(S)$, is split by part (6) of Lemma 9.1.16. Since $F$ is insular, $\mathcal{F}(S)$ is also insular. This shows that $\mathcal{F}(S)$ gives a grading for $F$. □

This result can be promoted to the following statement.

**Proposition 9.1.35.** *Suppose $F$ is the kernel of a boundedly bicontrolled epimorphism $g \colon G \to H$ in $\mathbf{BS}(X,R)$. If $G$ is graded and $F$ is split then both $H$ and $F$ are graded.*

*Proof.* The grading for $H$ is given by $\mathcal{H}(S) = g\mathcal{G}(S[b])$, where $b$ is a chosen bicontrol bound for $g$. Each $\mathcal{H}(S)$ is split and insular as in the proof of Lemma 9.1.34. The inclusions $H(S) \subset gG(S[b]) \subset g\mathcal{G}(S[b]) = \mathcal{H}(S)$ and $g\mathcal{G}(S[b]) \subset gG(S[b+K]) \subset H(S[2b+K])$ show that $\mathcal{H}$ is a grading. The same argument as in Lemma 9.1.34 shows that $\mathcal{F}(S) = F \cap \mathcal{G}(S[b])$ gives a grading for $F$. □

We will use the following convention. When $d \leq 0$, the notation $S[d]$ will stand for the subset $S \setminus (X \setminus S)[-d]$.

**Corollary 9.1.36.** *Given an object $F$ in $\mathbf{G}(X, R)$ and a subset $S$ of $X$, there is a number $K \geq 0$ and an admissible subobject $i \colon F_S \to F$ in $\mathbf{G}(X, R)$ with the property that $F_S \subset F(S[K])$. Moreover, the quotient $q \colon F \to H$ of the inclusion has the property that $H(X) = H((X \setminus S)[2D'])$, where $D'$ is a splitting constant for $G$.*

*Proof.* For the first statement, choose $F_S = \mathcal{F}(S)$ with the grading defined in Lemma 9.1.34 and apply Proposition 9.1.35. The second statement is shown as follows. By part (2) of Lemma 9.1.16, since $\mathrm{fil}(q) = 0$, if $G$ is $D'$-split then $H$ is $D'$-split. Let $T = S[-D']$, then $T[D'] \subset S$, so $H(T[D']) = qF(T[D']) \subset qF(S) \subset qF_S = 0$. Using the decomposition $X = T \cup (X \setminus T)$ we can write $H(X) = H(T[D']) + H((X \setminus T)[D']) = H((X \setminus T)[D']) = H((X \setminus S)[2D'])$. □

The last three results can be summarized as follows.

**Corollary 9.1.37.** *Given a graded object $F$ in $\mathbf{G}(X, R)$ and a subset $S$ of $X$, we assume that $F$ is $D'$-split and $d$-insular and is graded by $\mathcal{F}$. The submodule $\mathcal{F}(S)$ has these properties:*

1. *$\mathcal{F}(S)$ is graded by $\mathcal{F}_S(T) = \mathcal{F}(S) \cap \mathcal{F}(T)$,*
2. *$F(S) \subset \mathcal{F}(S) \subset F(S[K])$ for some fixed number $K \geq 0$,*
3. *if $q \colon F \to H$ is the quotient of the inclusion $i \colon \mathcal{F}(S) \to H$ and $D'$ is the splitting constant for $F$, then $H$ is supported on $(X \setminus S)[2D']$,*
4. *$H(S[-2D' - 2d]) = 0$.*

*Proof.* Properties (1), (2), (3) are consequences of the last three results. (4) follows from the fact that a $d$-insular filtered module is $2d$-separated, in the sense that for any pair of subsets $S$ and $T$ such that $S[2d] \cap T = \emptyset$ we have $S[d] \cap T[d] = \emptyset$ so $F(S) \cap F(T) = 0$. Now $H(S[-2D' - 2d]) \cap H((X \setminus S)[2D']) = 0$, but $H((X \setminus S)[2D']) = H(X)$ so $H(S[-2D' - 2d]) = 0$. □

9.2. **Fibred Bounded $G$-theory.** Suppose $X$ and $Y$ are two proper metric spaces and $R$ is a noetherian ring. The product $X \times Y$ is given the *product metric*

$$d((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\}.$$

Of course, there is the exact category $\mathbf{L}(X \times Y, R)$ and the associated bounded category $\mathbf{BL}(X \times Y, R)$.

We now wish to construct a larger *fibred bounded category* $\mathbf{B}_X(Y)$ which will extend $\mathcal{C}_X(Y)$ similarly to the extension of $\mathcal{C}(X, R)$ by $\mathbf{BL}(X, R)$.

**Definition 9.2.1.** Given an $R$-module $F$, an $(X, Y)$-*filtration* of $F$ is a functor

$$\phi_F \colon \mathcal{P}(X \times Y) \longrightarrow \mathcal{I}(F)$$

from the power set of the product metric space to the partially ordered family of $R$-submodules of $F(X \times Y)$. Whenever $F$ is given a filtration, and there is no ambiguity, we will denote the values $\phi_F(U)$ by $F(U)$. We assume that $F$ is *reduced* in the sense that the value on the empty subset is 0.

The associated $X$-filtered $R$-module $F_X$ is given by

$$F_X(S) = F(S \times Y).$$

Similarly, for each subset $S \subset X$, one has the $Y$-filtered $R$-module $F^S$ given by

$$F^S(T) = F(S \times T).$$

In particular, $F^X(T) = F(X \times T)$.

We will use the following notation generalizing enlargements in a metric space.

*Notation* 9.2.2. Given a subset $U$ of $X \times Y$ and a function $k\colon X \to [0, +\infty)$, let

$$U[k] = \{(x, y) \in X \times Y \mid \text{there is } (x, y') \in U \text{ with } d(y, y') \leq k(x)\}.$$

If in addition we are given a number $K \geq 0$ then

$$U[K, k] = \{(x, y) \in X \times Y \mid \text{there is } (x', y) \in U[k] \text{ with } d(x, x') \leq K\}.$$

So $U[k] = U[0, k]$. Notice that if $U$ is a single point $(x, y)$ then

$$U[K, k] = x[K] \times y[k(x)] = (x, y)[K, 0] \times (x, y)[0, k(x)].$$

More generally, one can equivalently write

$$U[K, k] = \bigcup_{(x,y)\in U} x[K] \times y[k(x)].$$

If $U$ is a product set $S \times T$, it is convenient to use the notation $(S, T)[K, k]$ for $(S \times T)[K, k]$.

**Definition 9.2.3.** We will refer to the pair $(K, k)$ in the notation $U[K, k]$ as the *enlargement data.*

It is clear that when $Y =$ point, $U[K, k] = U[K]$ for any function $k$ under the identification $X \times Y = X$.

*Notation* 9.2.4. Let $x_0$ be a chosen fixed point in $X$. Given a monotone function $h\colon [0, +\infty) \to [0, +\infty)$, there is a function $h_{x_0}\colon X \to [0, +\infty)$ defined by

$$h_{x_0}(x) = h(d_X(x_0, x)).$$

**Definition 9.2.5.** Given two $(X, Y)$-filtered modules $F$ and $G$, an $R$-homomorphism $f\colon F(X \times Y) \to G(X \times Y)$ is *boundedly controlled* if there are a number $b \geq 0$ and a monotone function $\theta\colon [0, +\infty) \to [0, +\infty)$ such that

$$fF(U) \subset G(U[b, \theta_{x_0}]) \tag{$\dagger$}$$

for all subsets $U \subset X \times Y$ and some choice of $x_0 \in X$.

It is clear that the condition is independent of the choice of $x_0$.

**Definition 9.2.6.** An $(X, Y)$-filtered module $F$ is called

- *lean* or $(D, \Delta)$-*lean* if there is a number $D \geq 0$ and a monotone function $\Delta\colon [0, +\infty) \to [0, +\infty)$ so that

$$\begin{aligned} F(U) &\subset \sum_{(x,y)\in U} F(x[D] \times y[\Delta_{x_0}(x)]) \\ &= \sum_{(x,y)\in U} F((x, y)[D, \Delta_{x_0}]) \end{aligned}$$

for any subset $U$ of $X \times Y$,

- *split* or $(D', \Delta')$-*split* if there is a number $D' \geq 0$ and a monotone function $\Delta' \colon [0, +\infty) \to [0, +\infty)$ so that
$$F(U_1 \cup U_2) \subset F(U_1[D', \Delta'_{x_0}]) + F(U_2[D', \Delta'_{x_0}])$$
for each pair of subsets $U_1$ and $U_2$ of $X \times Y$,
- *lean/split* or $(D, \Delta')$-*lean/split* if there is a number $D \geq 0$ and a monotone function $\Delta' \colon [0, +\infty) \to [0, +\infty)$ so that
  - the $X$-filtered module $F_X$ is $D$-lean, while
  - the $(X, Y)$-filtered module $F$ is $(D, \Delta')$-split,
- *insular* or $(d, \delta)$-*insular* if there is a number $d \geq 0$ and a monotone function $\delta \colon [0, +\infty) \to [0, +\infty)$ so that
$$F(U_1) \cap F(U_2) \subset F\big(U_1[d, \delta_{x_0}] \cap U_2[d, \delta_{x_0}]\big)$$
for each pair of subsets $U_1$ and $U_2$ of $X \times Y$.

**Remark 9.2.7.** Suppose $F$ is an $(X, Y)$-filtered $R$-module.

(1) If $F$ is $(D, \Delta)$-lean then the corresponding $X$-filtered module $F_X$ is $D$-lean.
(2) Similarly, if $F$ is $(d, \delta)$-insular then $F_X$ is $d$-insular.
(3) If $F$ is $(D, \Delta)$-lean then it is $(D, \Delta)$-lean/split and, further, $(D, \Delta)$-split.
(4) An $(X, Y)$-filtered module $F$ which is lean/split and insular can be thought of as an object $F_X$ of $\mathbf{L}(X, R)$.

**Proposition 9.2.8.** *If $f \colon F \to G$ is boundedly controlled then it satisfies the following two conditions:*

(1) *$f$ is bounded by some $b \geq 0$ when viewed as a morphism $F_X \to G_X$ in $\mathbf{U}(X, R)$, and*
(2) *for each bounded subset $S \subset X$, the restriction $f \colon F_X(S) \longrightarrow G_X(S[b])$ is bounded when viewed as a morphism $F^S \to G^{S[b]}$ of $Y$-filtered modules in $\mathbf{U}(Y, R)$.*

*Proof.* If $f \colon F \to G$ is $(b, \theta)$-controlled then for any subset $S \subset X$ we have $fF_X(S) \subset G((S, Y)[b, \theta_{x_0}]) \subset G(S[b], Y) = G_X(S[b])$. So $f \colon F_X \to G_X$ is bounded by $b$. Now for a given bounded subset $S \subset X$, let us define $\theta_S = \sup_{x \in S} \theta_{x_0}(x)$. Then $fF_X(S)(T) = fF(S, T) \subset G(S[b], T[\theta_S]) = G_X(S[b])(T[\theta_S])$ verifying that $f| \colon F^S \to G^{S[b]}$ is bounded by $\theta_S$. $\square$

**Remark 9.2.9.** The converse to this fact is only true when $F$ is lean but not necessarily when $F$ is lean/split.

**Definition 9.2.10.** There are several related bounded categories of $(X, Y)$-filtered $R$-modules.

- $\mathbf{U}_X(Y)$ has objects that are arbitrary $(X, Y)$-filtered $R$-modules, the morphisms are the boundedly controlled homomorphisms.
- $\mathbf{LS}_X(Y)$ is the full subcategory of $\mathbf{U}_X(Y)$ on objects $F$ that are lean/split and insular,
- $\mathbf{B}_X(Y)$ is the full subcategory of $\mathbf{LS}_X(Y)$ on objects $F$ such that $F(U)$ is a finitely generated submodule whenever $U \subset X \times Y$ is bounded. Equivalently, the subcategory $\mathbf{B}_X(Y)$ is full on objects $F$ such that all $Y$-filtered

modules $F^S$ associated to bounded subsets $S \subset X$ are locally finitely generated.

**Definition 9.2.11.** A morphism $f\colon F \to G$ in $\mathbf{U}_X(Y)$ is *boundedly bicontrolled* if there is filtration data $b \leq 0$ and $\theta\colon [0,+\infty) \to [0,+\infty)$ as in Definition 9.2.5, and in addition to (†) one also has the containments

$$fF \cap G(U) \subset fF(U[b,\theta_{x_0}]).$$

In this case, we will use the notation $\mathrm{fil}(f) \leq (b,\theta)$.

**Definition 9.2.12.** Let the *admissible monomorphisms* in $\mathbf{U}_X(Y)$ be the boundedly bicontrolled homomorphisms $m\colon F_1 \to F_2$ such that the module homomorphism $F_1(X \times Y) \to F_2(X \times Y)$ is a monomorphism. Let the *admissible epimorphisms* be the boundedly bicontrolled homomorphisms $e\colon F_1 \to F_2$ such that $F_1(X \times Y) \to F_2(X \times Y)$ is an epimorphism. The class of *exact sequences* consists of the sequences

$$F^{\cdot}\colon\ F' \xrightarrow{i} F \xrightarrow{j} F'',$$

where $i$ is an admissible monomorphism, $j$ is an admissible epimorphism, and $\mathrm{im}(i) = \ker(j)$.

One can argue as in [13] that the admissible monomorphisms are precisely the morphisms isomorphic in $\mathbf{U}_X(Y)$ to filtration-wise monomorphisms and the admissible epimorphisms are the morphisms isomorphic to filtration-wise epimorphisms.

**Proposition 9.2.13.** *Assume that $\mathbf{U}_X(Y)$ is given the class of exact sequences as in Definition* 9.2.12*.*

(1) *$\mathbf{U}_X(Y)$ is a cocomplete exact pseudoabelian category.*
(2) *The lean/split objects are closed under extensions.*
(3) *The insular objects are closed under extensions.*
*Suppose*

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

*is an exact sequence in $\mathbf{U}_X(Y)$.*
(4) *If the object $E$ is lean/split then $E''$ is lean/split.*
(5) *If $E$ is insular then $E'$ is insular.*
(6) *If $E$ is lean/split and insular then $E''$ is insular if and only if $E'$ is lean/split.*

*Proof.* (1) can be checked directly. An alternative is to use the iterative idea that $\mathbf{U}_X(Y)$ can be viewed as $\mathbf{U}(X, \mathbf{U}(Y,R))$ and the observation that the cocomplete exact pseudoabelian category $\mathbf{U}(Y,R)$ can be substituted for $\mathbf{Mod}(R)$ in the proof of Theorem 9.1.9 and related constructions.

Other parts of the Theorem are proved by adapting the proofs of Lemmas 9.1.15 and 9.1.16. To illustrate, let

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in $\mathbf{U}_X(Y)$. Suppose that $(b,\theta)$ is common filtration data for $f$ and $g$ and both $E'$ and $E''$ are $(D,\Delta')$-lean/split. For the first statement of part (2), notice that $E_X$ is $(4b+D)$-lean by part (1) of Lemma 9.1.15, so we need to verify that split objects are closed under extensions. Consider two subsets $U_1$ and $U_2$ of $X \times Y$. Then

$$\begin{aligned} gE(U) &\subset E''((U_1 \cup U_2)[b, \theta_{x_0}]) \\ &= E''(U_1[b, \theta_{x_0}] \cup U_2[b, \theta_{x_0}]) \\ &\subset E''(U_1[b + D, \theta_{x_0} + \Delta'_{x_0}]) + E''(U_2[b + D, \theta_{x_0} + \Delta'_{x_0}]). \end{aligned}$$

Therefore

$$\begin{aligned} E(U) \subset\ & E(U_1[2b + D, 2\theta_{x_0} + \Delta'_{x_0}]) + E(U_2[2b + D, 2\theta_{x_0} + \Delta'_{x_0}]) \\ &+ fE'(U_1[3b + 2D, 3\theta_{x_0} + 2\Delta'_{x_0}]) + fE'(U_2[3b + 2D, 3\theta_{x_0} + 2\Delta'_{x_0}]) \\ &\subset E(U_1[4b + 2D, 4\theta_{x_0} + 2\Delta'_{x_0}]) + E(U_2[4b + 2D, 4\theta_{x_0} + 2\Delta'_{x_0}]), \end{aligned}$$

showing that $E$ is $(4b + 2D, 4\theta + 2\Delta')$-lean/split. □

**Proposition 9.2.14.** $\mathbf{LS}_X(Y)$ *is closed under extensions in* $\mathbf{U}_X(Y)$*.*

*Proof.* This follows from parts (2) and (3) of Proposition 9.2.13. □

**Proposition 9.2.15.** $\mathbf{B}_X(Y)$ *is closed under extensions in* $\mathbf{LS}_X(Y)$*. Therefore,* $\mathbf{B}_X(Y)$ *is an exact category, and the inclusion*

$$e \colon \mathcal{C}_X(Y) \longrightarrow \mathbf{B}_X(Y)$$

*is an exact embedding.*

*Proof.* Suppose $f \colon F \to G$ is an isomorphism with $\mathrm{fil}(f) \leq (b, \theta)$ and $G$ is locally finitely generated, then $F(U)$ is a finite generated submodule of $G(U[b, \theta])$ for any bounded subset $U \subset X \times Y$ since $R$ is noetherian. If

$$F' \xrightarrow{f} F \xrightarrow{g} F''$$

is an exact sequence in $\mathbf{LS}_X(Y)$, $F'$ and $F''$ are locally finitely generated, and $(b, \theta)$ is common filtration data for $f$ and $g$, then $gF(U)$ is a finitely generated submodule of $F''(U[b, \theta])$ for any bounded subset $U$. The kernel of the restriction of $g$ to $F(U)$ is a finitely generated submodule of $F'(U[b, \theta])$, so the extension $F(U)$ is finitely generated. □

**Remark 9.2.16.** There is also the exact embedding

$$\iota \colon\ \mathbf{B}(X \times Y, R) \longrightarrow \mathbf{B}_X(Y)$$

which is given by the identity on objects.

The same comments as in the case of geometric bounded categories of Pedersen–Weibel apply: the morphism sets in the image of $\iota$ are in general properly smaller than in $\mathbf{B}_X(Y)$. This time, however, $\iota$ is also proper on objects. For example, the lean objects in $\mathbf{BL}(X \times Y, R)$ are generated by the submodules $f(S \times T)$ where the diameters of $S$ and $T$ are uniformly bounded from above. This is different from the weaker condition in $\mathbf{B}_X(Y)$.

9.3. **Support and Fibrewise Support.** Suppose $X$ and $Y$ are two proper metric spaces, $Z$ is a subset of $X$, and $C$ is a subset of $Y$.

**Definition 9.3.1.** An object $F$ of $\mathbf{U}(X,R)$ is *supported near* $Z$ if there is a number $d \geq 0$ such that

$$F(X) \subset F(Z[d]).$$

The objects supported near $Z$ form the full subcategory $\mathbf{U}(X,R)_{<Z}$. One can readily check that $\mathbf{U}(X,R)_{<Z}$ is closed under exact extensions in $\mathbf{U}(X,R)$, so $\mathbf{U}(X,R)_{<Z}$ is an exact subcategory.

For each $Z \subset X$ there is the corresponding intersection

$$\mathbf{B}(X,R)_{<Z} = \mathbf{B}(X,R) \cap \mathbf{U}(X,R)_{<Z}$$

which is an exact subcategory of $\mathbf{B}(X,R)$.

There are two complementary ways to introduce support in $\mathbf{B}_X(Y)$.

(1) Let $\mathbf{B}_{<Z}(Y)$ be the full subcategory of $\mathbf{B}_X(Y)$ on objects $F$ supported near $Z$ viewed as objects $F_X$ in $\mathbf{U}(X,R)$. In other words, $F$ is an object of $\mathbf{B}_{<Z}(Y)$ if

$$F_X \subset F_X(Z[d]) = F(Z[d] \times Y)$$

for some number $d \geq 0$.

(2) Let $\mathbf{B}_X(Y)_{<C}$ be the full subcategory of $\mathbf{B}_X(Y)$ on objects $F$ such that

$$F(X,Y) \subset F\big((X,C)[r,\rho_{x_0}]\big)$$

for some number $r \geq 0$ and an order preserving function $\rho\colon [0,+\infty) \to [0,+\infty)$.

The notion (1) of support is the straightforward generalization of support for geometric modules and was exploited in [13]. We will summarize the results of that theory referring to [13] for proofs.

**Definition 9.3.2.** A *Grothendieck subcategory* of an exact category is a subcategory which is closed under exact extensions and closed under passage to admissible subobjects and admissible quotients.

**Theorem 9.3.3.** *The category $\mathbf{B}_{<Z}(Y)$ is a Grothendieck subcategory of $\mathbf{B}_X(Y)$. In particular, it is exact. We will denote its $K$-theory by $G_{<Z}(Y)$. There is an exact quotient category $\mathbf{B}_{X,Z}(Y)$ whose $K$-theory we denote by $G_{X,Z}(Y)$. The inclusion of the subcategory and the quotient functor induce a sequence of maps*

$$G_{<Z}(Y) \longrightarrow G_X(Y) \longrightarrow G_{X,Z}(Y)$$

*which is a homotopy fibration.*

*Suppose $Z_1$ and $Z_2$ are a covering of $X$. Let $G_{<Z_1,Z_2}(Y)$ denote the $K$-theory of the intersection $\mathbf{B}_{<Z_1,Z_2}(Y)$ of the subcategories $\mathbf{B}_{<Z_1}(Y)$ and $\mathbf{B}_{<Z_2}(Y)$ in $\mathbf{B}_X(Y)$. There is a map of homotopy fibrations*

$$\begin{tikzcd}
G_{<Z_1,Z_2}(Y) \arrow[r] \arrow[d] & G_{<Z_1}(Y) \arrow[r] \arrow[d] & G_{Z_1,Z_2}(Y) \arrow[d] \\
G_{<Z_2}(Y) \arrow[r] & G_X(Y) \arrow[r] & G_{X,Z_2}(Y)
\end{tikzcd}$$

*If $Z_1$ and $Z_2$ are mutually antithetic then there is a map of homotopy fibrations*

$$\begin{tikzcd} G_{Z_1\cap Z_2}(Y) \arrow[r] \arrow[d] & G_{Z_1}(Y) \arrow[r] \arrow[d] & G_{Z_1,Z_2}(Y) \arrow[d] \\ G_{Z_2}(Y) \arrow[r] & G_X(Y) \arrow[r] & G_{X,Z_2}(Y) \end{tikzcd}$$

*In both diagrams the leftmost vertical arrow is a weak equivalence.*

*Applying the last statement to the subsets $X \times [0,+\infty)$, $X \times (-\infty, 0]$ of $X \times \mathbb{R}$ and using the fact that $G_{X\times[0,+\infty)}(Y)$ is contractible for all $X$ and $Y$, one obtains a map $G_X(Y) \to \Omega G_{X\times\mathbb{R}}(Y)$ which induces isomorphisms of homotopy groups in positive dimensions. Iterations of this construction give weak equivalences*

$$\Omega^k G_{X\times\mathbb{R}^k}(Y) \longrightarrow \Omega^{k+1} G_{X\times\mathbb{R}^{k+1}}(Y)$$

*for $k \geq 2$. This allows to define the nonconnective spectrum*

$$\widehat{G}_X(Y) \ = \ \underset{\overrightarrow{k}}{\mathrm{hocolim}}\ \Omega^k G_{X\times\mathbb{R}^k}(Y).$$

*Now there is a homotopy pushout*

$$\begin{tikzcd} \widehat{G}_{<Z_1,Z_2}(Y) \arrow[r] \arrow[d] & \widehat{G}_{<Z_1}(Y) \arrow[d] \\ \widehat{G}_{<Z_2}(Y) \arrow[r] & \widehat{G}_X(Y) \end{tikzcd}$$

*for a covering of $X$ by $Z_1$ and $Z_2$.*

We now explore the latter version (2) of support from Definition 9.3.1.

**Proposition 9.3.4.** *Suppose $F$ is a $(D,\Delta)$-lean/split object of $\mathbf{B}_X(Y)$. The following are equivalent statements.*

(1) *$F$ is an object of $\mathbf{B}_X(Y)_{<C}$.*

(2) *There is a number $k \geq 0$ and an order preserving function $\lambda \colon \mathcal{B}(X) \to [0,+\infty)$ such that*
$$F^S \subset F^{S[k]}(C[\lambda(S)])$$
*for all bounded subsets $S \subset X$.*

(3) *There is a number $k \geq 0$ and a monotone function $\Lambda \colon [0,+\infty) \to [0,+\infty)$ such that*
$$F^{x[D]} \subset F^{x[D+k]}(C[\Lambda_{x_0}(x)])$$
*for all $x \in X$.*

*Proof.* (2) $\Longleftrightarrow$ (3): If $F$ satisfies (2) then
$$F^{x[D]} \subset F^{x[D+k]}\big(C[\lambda(x[D])]\big).$$
It suffices to define $\Lambda$ such that
$$\lambda(x[D]) \leq \Lambda_{x_0}(x) = \Lambda(d(x_0,x)).$$
Since $x[D] \leq x_0[d(x_0,x)+D]$ and $\lambda$ is order preserving, one can take
$$\Lambda(r) = \lambda\big(x_0[r+D]\big).$$

In the opposite direction, given a bounded subset $S \subset X$

$$\begin{aligned} F^S &\subset \sum_{x\in S} F^{x[D]} \\ &\subset \sum_{x\in S} F^{x[D+k]}(C[\Lambda_{x_0}(x)]) \\ &\subset F^{x[D+k]}(C[\lambda(S)]) \end{aligned}$$

when $\lambda(S) = \sup\{\Lambda_{x_0}(x) \mid x \in S\}$.

(1) $\Longleftrightarrow$ (3): If $F$ is in $\mathbf{B}_X(Y)_{<C}$ then $F^{x[D]} \subset F\big((X,C)[r,\rho]\big)$, so

$$F^{x[D]} \subset F^{x[D]} \cap F\big((X,C)[r,\rho]\big).$$

If $F$ is $(d,\delta)$-insular then

$$\begin{aligned} F^{x[D]} &\subset F\big((x,C)[D+r+d,\rho+\delta]\big) \\ &\subset F^{x[D+d+r]}(C[\Lambda_{x_0}(x)]) \end{aligned}$$

for $\Lambda(a) = \sup\{(\delta+\rho)(z) \mid d(x_0,y) \le a + D + d + r\}$.

In the opposite direction, we have

$$\begin{aligned} F &\subset \sum_{x\in X} F^{x[D]} \\ &\subset \sum_{x\in X} F^{x[D+k]}(C[\Lambda_{x_0}(x)]) \\ &\subset F\big((X,C)[D+k,\Lambda_{x_0}]\big) \end{aligned}$$

for an object $F$ of $\mathbf{B}_X(Y)$ satisfying (3). $\square$

**Proposition 9.3.5.** $\mathbf{B}_X(Y)_{<C}$ *is a Grothendieck subcategory of* $\mathbf{B}_X(Y)$.

*Proof.* First we show closure under exact extensions. Let

$$F \xrightarrow{f} G \xrightarrow{g} H$$

be an exact sequence in $\mathbf{B}_X(Y)$. Let $(b,\theta)$ be common set of filtration data for $f$ and $g$ and let all objects be $(D,\Delta)$-lean/split. We assume that $F$ and $H$ are objects of $\mathbf{B}_X(Y)_{<C}$, so there is a number $r \ge 0$ and a monotone function $\rho\colon [0,+\infty) \to [0,+\infty)$ such that at the same time $F(X,Y) = F\big((X,C)[r,\rho_{x_0}]\big)\big)$ and $H(X,Y) = H\big((X,C)[r,\rho_{x_0}]\big)\big)$ for some choice of a base point $x_0$ in $X$. Therefore

$$fF(X,Y) = fF\big((X,C)[r,\rho_{x_0}]\big)\big) \subset G\big((X,C)[r+b,\rho_{x_0}+\theta_{x_0}]\big)\big).$$

In particular, the image $I = \mathrm{im}(f)$ with the standard filtration $I^S(T) = I \cap G^S(T)$ is an object of $\mathbf{B}_X(Y)_{<C}$. Now

$$H(X,Y) = gG(X,Y) \cap H\big((X,C)[r,\rho_{x_0}]\big)\big) \subset gG\big((X,C)[r+b,\rho_{x_0}+\theta_{x_0}]\big)\big).$$

Let $L = G\big((X,C)[r+b,\rho_{x_0}+\theta_{x_0}]\big)$ viewed as a subobject of $G$ with the standard filtration. Since $G = I + L$ for any submodule $L$ with $g(L) = H$, we have

$$G(X,Y) = G\big((X,C)[r+b,\rho_{x_0}+\theta_{x_0}]\big)\big),$$

so $G$ is an object of $\mathbf{B}_X(Y)_{<C}$.

Suppose $f\colon F \to G$ is an admissible monomorphism in $\mathbf{B}_X(Y)$, which is a boundedly bicontrolled monic with $\mathrm{fil}(f) \le (b,\theta)$, $F$ is $(D',\Delta')$-lean/split, $G$ is $(D,\Delta)$-lean/split for some $D \ge D' + b$, and $G$ is $(d,\delta)$-insular.

If $G$ is an object of $\mathbf{B}_X(Y)_{<C}$, according to Proposition 9.3.4,

$$G^S \subset G^{S[k]}(C[\lambda(S)])$$

for some number $k \geq 0$, an order preserving function $\lambda\colon \mathcal{B}(X) \to [0,+\infty)$, and all bounded subsets $S \subset X$. Then

$$fF^{x[D']} \subset G^{x[D'+b]} \subset G^{x[D'+b+k]}(C[\lambda(x[D'+b])]) \subset G^{x[D+k]}(C[\lambda(x[D])]),$$

using the fact that $\lambda$ is order preserving. Since

$$G^{x[D+k]}(Y - C[\lambda(x[D+k]) + \Delta(x[D]) + \theta(x[D']) + 2\delta(x[D+k])]) = 0,$$

we have

$$F^{x[D']}(Y - C[\lambda(x[D+k]) + \Delta(x[D]) + 2\theta(x[D']) + 2\delta(x[D+k])]) = 0.$$

Therefore

$$F^{x[D']} \subset F^{x[D']}(C[\lambda(x[D]) + \Delta(x[D]) + \Delta'(x[D]) + 2\theta(x[D']) + 2\delta(x[D+k])],$$

so $F$, which is generated by $F^{x[D']}$, is also an object of $\mathbf{B}_X(Y)_{<C}$.

On the other hand, let $g\colon G \to H$ be an admissible quotient with $\mathrm{fil}(g) \leq (b,\theta)$ and suppose $G$ is an object of $\mathbf{B}_X(Y)_{<C}$ so that there is a number $r \geq 0$ and a monotone function $\rho\colon [0,+\infty) \to [0,+\infty)$ such that

$$G(X,Y) = G\big((X,C)[r,\rho_{x_0}]\big).$$

This implies that

$$H(X,Y) = gG(X,Y) \subset H\big((X,C)[r+b,\rho_{x_0}+\theta_{x_0}]\big),$$

so $H$ is also in $\mathbf{B}_X(Y)_{<C}$. □

9.4. **Gradings of Filtered Modules.** Introduction of gradings of the objects of $\mathbf{B}_X(Y)$ is a generalization of the gradings from Definition 9.1.29.

*Notation* 9.4.1. Since the roles of the factors in the product $X \times Y$ are very different when working with the category $\mathbf{B}_X(Y)$, we will use the notation $(X,Y)$ for the product metric space so that the order of the factors is unambiguous.

**Definition 9.4.2.** Given an object $F$ of $\mathbf{B}_X(Y)$, a *grading* of $F$ is a functor

$$\mathcal{F}\colon \mathcal{P}(X,Y) \longrightarrow \mathcal{I}(F)$$

with the properties

(1) if $\mathcal{F}(\mathcal{C})$ is given the standard filtration, it is an object of $\mathbf{B}_X(Y)$,
(2) there is an enlargement data $(K,k)$ such that

$$F(C) \subset \mathcal{F}(C) \subset F(C[K,k_{x_0}]),$$

for all subsets $C$ of $(X,Y)$.

**Remark 9.4.3.** If $C = (X,S)$ then $\mathcal{F}(C)$ is an object of $\mathbf{B}_X(Y)_{<S}$.

We are concerned with localizations to a specific type of subspaces of $(X,Y)$. This makes the following partial gradings sufficient and easier to work with.

**Definition 9.4.4.** Let $\mathcal{M}^{\geq 0}$ be the set of all monotone functions $\delta\colon [0,+\infty) \to [0,+\infty)$. Let $\mathcal{P}_X(Y)$ be the subcategory of $\mathcal{P}(X,Y)$ consisting of all subsets described as $(X,C)[D,\delta_{x_0}]$ for some $C \subset Y$, $D \geq 0$, and $\delta \in \mathcal{M}^{\geq 0}$.

Given an object $F$ of $\mathbf{B}_X(Y)$, a $Y$-*grading* of $F$ is a functor

$$\mathcal{F}\colon \mathcal{P}_X(Y) \longrightarrow \mathcal{I}(F)$$

with the properties:

(1) the submodule $\mathcal{F}((X,C)[D,\delta_{x_0}])$ with the standard filtration is an object of $\mathbf{B}_X(Y)$,
(2) there is an enlargement data $(K,k)$ such that

$$F((X,C)[D,\delta_{x_0}]) \subset \mathcal{F}((X,C)[D,\delta_{x_0}]) \subset F((X,C)[D+K,\delta_{x_0}+k_{x_0}]),$$

for all subsets in $\mathcal{P}_X(Y)$.

Since $U[D+K,\delta_{x_0}+k_{x_0}] = U[D,\delta_{x_0}][K,k_{x_0}]$ for general subsets $U$, the third, largest submodule is independent of the choice of $D$, $\delta_{x_0}$.

**Definition 9.4.5.** We define $\mathbf{G}_X(Y)$ as the full subcategory of $\mathbf{B}_X(Y)$ on $Y$-graded filtered modules.

**Proposition 9.4.6.** *The graded objects are closed under isomorphisms. The subcategory $\mathbf{G}_X(Y)$ is closed under extensions in $\mathbf{B}_X(Y)$. Therefore, $\mathbf{G}_X(Y)$ is an exact subcategory of $\mathbf{B}_X(Y)$.*

*Proof.* The argument closely follows those for Propositions 9.1.30 and 9.1.32. □

As with the category $\mathbf{G}(X,R)$, the advantage of working with $\mathbf{G}_X(Y)$ as opposed to $\mathbf{B}_X(Y)$ is that we are able to localize to the grading subobjects associated to subsets from the family $\mathcal{P}_X(Y)$. Since that is the only case we use in this paper, we specialize to $X$ with finite asymptotic dimension in order to use part (7) of Lemma 9.1.16 when needed.

**Lemma 9.4.7.** *Let $F$ be a submodule of a $Y$-filtered module $G$ in $\mathbf{G}_X(Y)$ which is lean/split with respect to the standard filtration. Then $\mathcal{F}(U) = F \cap \mathcal{G}(U)$ is a $Y$-grading of $F$.*

*Proof.* As in Lemma 9.1.34, the proof is easily reduced to checking that $\mathcal{F}(U)$ is an object of $\mathbf{B}_X(Y)$ for each subset $U \in \mathcal{P}_X(Y)$. Suppose $i\colon F \to G$ is the inclusion and $q\colon G \to H$ is the quotient of $i$. Since $F$ is insular by part (5) of Proposition 9.2.13, both $F$ and $G$ are lean/split and insular. Thus $H$ is lean/split and insular by parts (4) and (6) of 9.2.13. Let $\mathcal{H}(U) = q\mathcal{G}(U)$ with the standard filtration in $H$. Then $\mathcal{H}(U)$ is lean/split by part (4) and insular as a submodule of insular $H$. The kernel $\mathcal{F}(U)$ of the filtration $(0,0)$ map $q|\colon \mathcal{G}(U) \to \mathcal{H}(U)$ is lean/split by part (6) of 9.2.13 and is insular as a submodule of insular $F$. Locally finite generation of $\mathcal{F}(U)$ follows from that of $\mathcal{G}(U)$. □

**Corollary 9.4.8.** *Suppose $F$ is the kernel of a boundedly bicontrolled epimorphism $g\colon G \to H$ in $\mathbf{B}_X(Y)$. If $G$ is $Y$-graded and $F$ is lean/split then both $H$ and $F$ are $Y$-graded.*

*Proof.* Suggested by the proof of Proposition 9.1.35, the $Y$-grading for $H$ is given by $\mathcal{H}(U) = g\mathcal{G}(U[b,\theta])$, where $(b,\theta)$ is a chosen filtration data for $g$. The argument for Lemma 9.4.7 shows that $\mathcal{F}(U) = F \cap \mathcal{G}(U[b,\theta])$ gives a $Y$-grading for $F$. □

**Corollary 9.4.9.** *Given an object $F$ in $\mathbf{G}_X(Y)$ and a subset $U$ from the family $\mathcal{P}_X(Y)$, there is a set of enlargement data $(K,k)$ and an admissible subobject $i\colon F_U \to F$ in $\mathbf{G}_X(Y)$ with the property that $F_U \subset F(U[K,k])$. If $G$ is $(D,\Delta')$-lean/split then the quotient $q\colon F \to H$ of the inclusion has the property that $H(X) = H((X \setminus U)[2D, 2\Delta'])$.*

*Proof.* See the proof of Corollary 9.1.36. □

Now we have a summary similar to Corollary 9.1.37.

**Corollary 9.4.10.** *Given a graded object $F$ in $\mathbf{G}_X(Y)$ and a subset $U$ from the family $\mathcal{P}_X(Y)$, we assume that $F$ is $(D,\Delta')$-split and $(d,\delta)$-insular and is graded by $\mathcal{F}$. The submodule $\mathcal{F}(U)$ has these properties:*

1. *$\mathcal{F}(U)$ is graded by $\mathcal{F}_U(T) = \mathcal{F}(U) \cap \mathcal{F}(T)$,*
2. *$F(U) \subset \mathcal{F}(U) \subset F(U[K,k])$ for some fixed number enlargement data $(K,k)$,*
3. *if $q\colon F \to H$ is the quotient of the inclusion $i\colon \mathcal{F}(U) \to F$ and $F$ is $(D,\Delta')$-lean/split, then $H$ is supported on $(X \setminus U)[2D, 2\Delta']$,*
4. *$H(U[-2D-2d, -2\Delta'-2\delta]) = 0$.*

In preparation for statements of fibred localization and excision theorems in terms of $\mathbf{G}_X(Y)$, we need to generalize some notions from section 2.7. Using the terminology from that section, suppose we are given a finite coarse antithetic covering $\mathcal{U}$ of $Y$ closed under coarse intersections and of cardinality $s$. The coarsely saturated families $\mathcal{A}_i$ which are members of $\mathcal{U}$ are partially ordered by inclusion. In fact, the union of the families $\mathcal{A}_i$ forms the set $\mathcal{A}$ closed under intersections.

**Definition 9.4.11.** Two subsets $A$, $B$ of $(X,Y)$ are called *coarsely equivalent* if there is a set of enlargement data $(K,k)$ such that $A \subset B[K, k_{x_0}]$ and $B \subset A[K, k_{x_0}]$. It is an equivalence relation among subsets. We will again use notation $A \parallel B$ for this equivalence. As we demonstrated before, this is a generalization of the notation from Definition 2.7.7. It should be clear from the context which notion is meant by this notation.

A family of subsets $\mathcal{A}$ is called *coarsely saturated* if it is maximal with respect to this equivalence relation. Given a subset $A$, let $\mathcal{S}(A)$ be the smallest boundedly saturated family containing $A$.

A collection of subsets $\mathcal{U} = \{U_i\}$ is a *coarse covering* of $(X,Y)$ if $(X,Y) = \bigcup S_i$ for some $S_i \in \mathcal{S}(U_i)$. Similarly, $\mathcal{U} = \{\mathcal{A}_i\}$ is a *coarse covering* by coarsely saturated families if for some (and therefore any) choice of subsets $A_i \in \mathcal{A}_i$, $\{A_i\}$ is a coarse covering in the sense above.

We will say that a pair of subsets $A$, $B$ of $(X,Y)$ are *coarsely antithetic* if for any two sets of enlargement data $(D_1,d_1)$ and $(D_2,d_2)$ there is a third set $(D,d)$ such that
$$A[D_1,(d_1)_{x_0}] \cap B[D_2,(d_2)_{x_0}] \subset (A \cap B)[D, d_{x_0}].$$
We will write $A \natural B$ to indicate that $A$ and $B$ are coarsely antithetic.

Given two subsets $A$ and $B$, we define
$$\mathcal{S}(A,B) = \{A' \cap B' \mid A' \in \mathcal{S}(A), B' \in \mathcal{S}(B), A' \natural B'\}.$$
It is easy to see that $\mathcal{S}(A,B)$ is a coarsely saturated family, cf. Proposition 2.7.10.

There is the straightforward generalization to the case of a finite number of subsets of $(X,Y)$. Again, we write $A_1 \natural A_2 \natural \ldots \natural A_k$ if for arbitrary sets of data

$(D_i, d_i)$ there is a set of enlargement data $(D, d)$ so that

$$A_1[D_1, (d_1)_{x_0}] \cap A_2[D_2, (d_2)_{x_0}] \cap \ldots \cap A_k[D_k, (d_k)_{x_0}] \\ \subset (A_1 \cap A_2 \cap \ldots \cap A_k)[D, d_{x_0}]$$

and define

$$\mathcal{S}(A_1, A_2, \ldots, A_k) = \{A'_1 \cap A'_2 \cap \ldots \cap A'_k \mid A'_i \in \mathcal{S}(A_i), A_1 \natural A_2 \natural \ldots \natural A_k\}.$$

Identifying any coarsely saturated family $\mathcal{A}$ with $\mathcal{S}(A)$ for $A \in \mathcal{A}$, one has the coarse saturated family $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$. We will refer to $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ as the *coarse intersection* of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$. A coarse covering $\mathcal{U}$ is *closed under coarse intersections* if all coarse intersections $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ are nonempty and are contained in $\mathcal{U}$.

**Proposition 9.4.12.** *If $\mathcal{U}$ is a coarse antithetic covering of $Y$ then $(X,\mathcal{U})$ consisting of subsets $(X, U)$, $U \in \mathcal{U}$, is a coarse antithetic covering of $(X, Y)$. If $\mathcal{U}$ is closed under coarse intersections, $(X,\mathcal{U})$ is closed under coarse intersections.*

*Proof.* Left to the reader. □

9.5. **Localization and Fibrewise Localization.**

**Definition 9.5.1.** A class of morphisms $\Sigma$ in an additive category $\mathbf{A}$ *admits a calculus of right fractions* if

(1) the identity of each object is in $\Sigma$,
(2) $\Sigma$ is closed under composition,
(3) each diagram $F \xrightarrow{f} G \xleftarrow{s} G'$ with $s \in \Sigma$ can be completed to a commutative square

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ F & \xrightarrow{f} & G \end{array}$$

with $t \in \Sigma$, and
(4) if $f$ is a morphism in $\mathbf{A}$ and $s \in \Sigma$ such that $sf = 0$ then there exists $t \in \Sigma$ such that $ft = 0$.

In this case there is a construction of the *localization* $\mathbf{A}[\Sigma^{-1}]$ which has the same objects as $\mathbf{A}$. The morphism sets $\mathrm{Hom}(F, G)$ in $\mathbf{A}[\Sigma^{-1}]$ consist of equivalence classes of diagrams

$$(s, f)\colon \quad F \xleftarrow{s} F' \xrightarrow{f} G$$

with the equivalence relation generated by $(s_1, f_1) \sim (s_2, f_2)$ if there is a map $h\colon F'_1 \to F'_2$ so that $f_1 = f_2 h$ and $s_1 = s_2 h$. Let $(s|f)$ denote the equivalence class of $(s, f)$. The composition of morphisms in $\mathbf{A}[\Sigma^{-1}]$ is defined by

$$(s|f) \circ (t|g) = (st'|gf')$$

where $g'$ and $s'$ fit in the commutative square

$$\begin{array}{ccc} F'' & \xrightarrow{f'} & G' \\ \downarrow{\scriptstyle t'} & & \downarrow{\scriptstyle t} \\ F & \xrightarrow{f} & G \end{array}$$

from axiom 3.

**Proposition 9.5.2.** *The localization* $\mathbf{A}[\Sigma^{-1}]$ *is a category. The morphisms of the form* $(\mathrm{id}\,|s)$ *where* $s \in \Sigma$ *are isomorphisms in* $\mathbf{A}[\Sigma^{-1}]$. *The rule* $P_\Sigma(f) = (\mathrm{id}\,|f)$ *gives a functor* $P_\Sigma \colon \mathbf{A} \to \mathbf{A}[\Sigma^{-1}]$ *which is universal among the functors making the morphisms* $\Sigma$ *invertible.*

*Proof.* The proofs of these facts can be found in Chapter I of [26]. The inverse of $(\mathrm{id}\,|s)$ is $(s|\,\mathrm{id})$. □

We have seen that for a given subset $C$ of $Y$, the category $\mathbf{B}_X(Y)_{<C}$ is a Grothendieck subcategory of $\mathbf{B}_X(Y)$. Clearly, restriction to $Y$-gradings in $\mathbf{B}_X(Y)_{<C}$ gives a full exact subcategory $\mathbf{G}_X(Y)_{<C}$ which is a Grothendieck subcategory of $\mathbf{G}_X(Y)$. The following shorthand notation is convenient when the choice of $C$ is clear.

*Notation* 9.5.3. The category $\mathbf{G}$ is the exact subcategory of $Y$-graded objects in $\mathbf{B}_X(Y)$. When the choice of the subset $C \subset Y$ is understood, we will use notation $\mathbf{C}$ for the Grothendieck subcategory $\mathbf{G}_X(Y)_{<C}$ of $\mathbf{G}$.

**Definition 9.5.4.** Define the class of *weak equivalences* $\Sigma(C)$ in $\mathbf{B}$ to consist of all finite compositions of admissible monomorphisms with cokernels in $\mathbf{C}$ and admissible epimorphisms with kernels in $\mathbf{C}$.

We will show that the class $\Sigma(C)$ admits calculus of right fractions.

**Definition 9.5.5.** A Grothendieck subcategory $\mathbf{C}$ of an exact category $\mathbf{G}$ is *right filtering* if each morphism $f \colon F_1 \to F_2$ in $\mathbf{G}$, where $F_2$ is an object of $\mathbf{C}$, factors through an admissible epimorphism $e \colon F_1 \to \overline{F}_2$, where $\overline{F}_2$ is in $\mathbf{C}$.

**Lemma 9.5.6.** *The Grothendieck subcategory* $\mathbf{C} = \mathbf{G}_X(Y)_{<C}$ *of* $\mathbf{G} = \mathbf{G}_X(Y)$ *is right filtering.*

*Proof.* For a morphism between filtered $(X,Y)$-modules as in Definition 9.5.5, we assume that both $F_1$ and $F_2$ are $(D,\Delta')$-lean/split and $(d,\delta)$-insular. Suppose $f \colon F_1 \to F_2$ is bounded by $(b,\theta)$ and let $r \geq 0$ and $\rho \colon [0,+\infty) \to [0,+\infty)$ be a monotone function such that
$$F_2(X,Y) \subset F_2((X,C)[r,\rho_{x_0}]).$$
Now for any characteristic set of data $(K,k)$ for the grading $\mathcal{F}_1$ and any subset $R$ we have
$$f\mathcal{F}(R) \subset fF_1(R[K,k_{x_0}]) \subset F_2(R[K+b,k_{x_0}+\theta_{x_0}]).$$
By part (4) of Corollary 9.4.10, $F_2(R[K+b,k_{x_0}+\theta_{x_0}]) \cap F_2((X,C)[r,\rho_{x_0}]) = 0$ for any $R$ such that
$$R[K+b+2D+2d, k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}] \cap (X,C)[r,\rho_{x_0}] = \emptyset.$$
If we choose
$$R = (X,Y) \setminus (X,C)[K+b+2D+2d+r, k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}+\rho_{x_0}]$$
and define $E = \mathcal{F}_1(R)$, then $fE = 0$. Let $\overline{F}_2$ be the cokernel of the inclusion $E \to F_1$. Then $\overline{F}_2$ is lean/split and insular and has a grading given by $\overline{\mathcal{F}}_2(S) = q\mathcal{F}_1(S[b,\theta_{x_0}])$. Since
$$\overline{F}_2(X,Y) \subset \overline{F}_2((X,C)[K+b+2D+2d+r, k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}+\rho_{x_0}]),$$

the quotient $\overline{F}_2$ is in $\mathbf{C}$, and $f$ factors as $F_1 \to \overline{F}_2 \to F_2$ in the right square in the map of exact sequences

$$\begin{array}{ccccc} E & \longrightarrow & F_1 & \xrightarrow{j'} & \overline{F}_2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle =} & & \downarrow \\ K & \xrightarrow{k} & F_1 & \xrightarrow{f} & F_2 \end{array}$$

as required. ☐

**Corollary 9.5.7.** *The class $\Sigma(C)$ admits calculus of right fractions.*

*Proof.* This follows from Lemma 9.5.6, see Lemma 1.13 of [43]. ☐

**Definition 9.5.8.** The category $\mathbf{G}/\mathbf{C}$ is the localization $\mathbf{G}[\Sigma(C)^{-1}]$.

It is clear that the quotient $\mathbf{G}/\mathbf{C}$ is an additive category, and $P_{\Sigma(C)}$ is an additive functor. In fact, we have the following.

**Theorem 9.5.9.** *The short sequences in $\mathbf{G}/\mathbf{C}$ which are isomorphic to images of exact sequences from $\mathbf{G}$ form a Quillen exact structures.*

This follows from Proposition 1.16 of Schlichting [43]. Since $\mathbf{C}$ is right filtering by Lemma 9.5.6, it remains to check that $\mathbf{C}$ right s-filtering in $\mathbf{G}$ in the following sense.

**Definition 9.5.10.** A subcategory $\mathbf{C}$ of an exact category $\mathbf{G}$ is *right s-filtering* if given an admissible monomorphism $f\colon F_1 \to F_2$ with $F_1$ in $\mathbf{C}$, there exist $E$ in $\mathbf{C}$ and an admissible epimorphism $e\colon F_2 \to E$ such that the composition $ef$ is an admissible monomorphism.

*Proof of Theorem* 9.5.9. Suppose that $F_1$ and $F_2$ have the same properties as in the proof of Lemma 9.5.6, and $\mathrm{fil}(f) \le (b, \theta)$. Since $F_1$ is in $\mathbf{C}$, there are $r \ge 0$ and a monotone function $\rho\colon [0, +\infty) \to [0, +\infty)$ such that

$$F_1(X, Y) \subset F_1((X, C)[r, \rho_{x_0}]).$$

Then let $F_2' = \mathcal{F}_2(T)$ where

$$T = (X, Y) \setminus (X, C)(X, C)[K + b + 2D + 2d + r, k_{x_0} + \theta_{x_0} + 2\Delta'_{x_0} + 2\delta_{x_0} + \rho_{x_0}].$$

Define $E$ as the cokernel of the inclusion $F_2' \to F_2$ and let $e\colon F_2 \to E$ be the quotient map. The composition $ef$ is an admissible monomorphism with $\mathrm{fil}(ef) = \mathrm{fil}(f) \le (b, \theta)$. ☐

*Notation* 9.5.11. If $C$ is a subset of $Y$ as before, $\mathbf{G}_X(Y, C)$ will stand for the exact category $\mathbf{G}/\mathbf{C}$ and $G_X(Y, C)$ for its Quillen $K$-theory.

Our main tool in proving excision theorems will be the following localization sequence.

**Theorem 9.5.12** (Theorem 2.1 of Schlichting [43])**.** *Let $\mathbf{Z}$ be an idempotent complete right s-filtering subcategory of an exact category $\mathbf{E}$. Then the sequence of exact categories $\mathbf{Z} \to \mathbf{E} \to \mathbf{E}/\mathbf{Z}$ induces a homotopy fibration of Quillen $K$-theory spectra*

$$K(\mathbf{Z}) \longrightarrow K(\mathbf{E}) \longrightarrow K(\mathbf{E}/\mathbf{Z}).$$

**Corollary 9.5.13.** *There is a homotopy fibration*

$$G_X(Y)_{<C} \longrightarrow G_X(Y) \longrightarrow G_X(Y,C).$$

*Therefore, there is a homotopy fibration*

$$\widehat{G}_X(Y)_{<C} \longrightarrow \widehat{G}_X(Y) \longrightarrow \widehat{G}_X(Y,C).$$

**Theorem 9.5.14** (Localization)**.** *There is a homotopy fibration*

$$\widehat{G}_X(C) \longrightarrow \widehat{G}_X(Y) \longrightarrow \widehat{G}_X(Y,C).$$

Theorem 9.5.14 follows directly from Corollary 9.5.13 as soon as we show that $\widehat{G}_X(C)$ and $\widehat{G}_X(Y)_{<C}$ are weakly equivalent.

Recall that the *essential full image* of a functor $F\colon \mathbf{C} \to \mathbf{D}$ is the full subcategory of $\mathbf{D}$ whose objects are those $D$ that are isomorphic to $F(C)$ for some $C$ from $\mathbf{C}$.

**Lemma 9.5.15.** *Given a pair of proper metric spaces $C \subset Y$, there is a fully faithful embedding $\epsilon\colon \mathbf{G}_X(C) \to \mathbf{G}_X(Y)$. The Grothendieck subcategory $\mathbf{G}_X(Y)_{<C}$ is the essential full image of $\mathbf{G}_X(C)$ in $\mathbf{G}_X(Y)$.*

*Proof.* Suppose $F$ is an object of $\mathbf{G}_X(C)$. The embedding $\epsilon$ is given by $\epsilon(F)(U) = F((X,C) \cap U)$, $\epsilon(\mathcal{F})(S) = \mathcal{F}((X,C) \cap U)$. It is clear that $\epsilon(F)$ is in $\mathbf{G}_X(Y)_{<C}$.

To show that $\mathbf{G}_X(Y)_{<C}$ is the essential full image, for an object $G$ of $\mathbf{G}_X(Y)_{<C}$ assume that $G \subset G((X,C)[r,\rho_{x_0}])$ for some number $r \geq 0$ and a monotone function $\rho\colon [0,+\infty) \to [0,+\infty)$. Choose any set function $\tau\colon (X,C)[r,\rho_{x_0}] \to X \times C$ with the properties

(1) $\tau(x,y) = (x, \tau_x(y))$,
(2) $d(y, \tau_x(y)) \leq \rho_{x_0} + r$ for all $x$ in $X$,
(3) $\tau | X \times C = \mathrm{id}$.

Then the $Y$-filtered module $E$ associated to $G$ given by $E(S) = G(\tau^{-1}(S))$ with the grading $\mathcal{E}(U') = \mathcal{G}(\tau^{-1}(U'))$ is an object of $\mathbf{G}_X(C)$. Indeed, if $\mathcal{G}(\tau^{-1}(U'))$ is $(D,\Delta')$-lean/split and $(d,\delta)$-insular then $\mathcal{E}(U')$ is $(D+r, \Delta'+\rho)$-lean/split and $(d+r, \delta+\rho)$-insular. The identity map is an isomorphism in $\mathbf{G}_X(Y)$ with $\mathrm{fil}(\mathrm{id}) \leq (2r, 2\rho + 2r)$. □

**Corollary 9.5.16.** *The inclusion $C \subset Y$ induces a weak equivalence*

$$\widehat{G}_X(C) \longrightarrow \widehat{G}_X(Y)_{<C}.$$

Finally, there is a relative version of the preceding results. Here $C_1$ and $C_2$ are two subsets of $Y$. Let $\mathbf{C}_{12} = \mathbf{G}_X(Y)_{<C_1,C_2}$ denote the intersection of subcategories $\mathbf{C}_1 \cap \mathbf{C}_2 = \mathbf{G}_X(Y)_{<C_1} \cap \mathbf{G}_X(Y)_{<C_2}$.

**Theorem 9.5.17.** $\mathbf{C}_{12}$ *is a Grothendieck subcategory of* $\mathbf{C}_1$*. It is right filtering and right s-filtering.*

*Proof.* Left to the reader. □

**Corollary 9.5.18** (Relative Localization)**.** *There are homotopy fibrations*

$$\widehat{G}_X(Y)_{<C_1,C_2} \longrightarrow \widehat{G}_X(Y)_{<C_1} \longrightarrow K(\mathbf{C}_1/\mathbf{C}_{12}).$$

9.6. **$K$-theory of Categories of Bounded Chain Complexes.** The proof of controlled excision in the boundedly controlled $G$-theory requires the context of Waldhausen $K$-theory of categories of bounded chain complexes.

**Definition 9.6.1** (Waldhausen Categories)**.** A *Waldhausen category* is a category $\mathbf{D}$ with a zero object 0 together with two chosen subcategories of *cofibrations* $\mathrm{co}(\mathbf{D})$ and *weak equivalences* $\boldsymbol{w}(\mathbf{D})$ satisfying the four axioms:

(1) every isomorphism in $\mathbf{D}$ is in both $\mathrm{co}(\mathbf{D})$ and $\boldsymbol{w}(\mathbf{D})$,
(2) every map $0 \to D$ in $\mathbf{D}$ is in $\mathrm{co}(\mathbf{D})$,
(3) if $A \to B \in \mathrm{co}(\mathbf{D})$ and $A \to C \in \mathbf{D}$ then the pushout $B \cup_A C$ exists in $\mathbf{D}$, and the canonical map $C \to B \cup_A C$ is in $\mathrm{co}(\mathbf{D})$,
(4) ("gluing lemma") given a commutative diagram

$$\begin{tikzcd} B \arrow[d] & A \arrow[l, hook', "a"'] \arrow[r] \arrow[d] & C \arrow[d] \\ B' & A' \arrow[l, hook', "a'"'] \arrow[r] & C' \end{tikzcd}$$

in $\mathbf{D}$, where the morphisms $a$ and $a'$ are in $\mathrm{co}(\mathbf{D})$ and the vertical maps are in $\boldsymbol{w}(\mathbf{D})$, the induced map $B \cup_A C \to B' \cup_{A'} C'$ is also in $\boldsymbol{w}(\mathbf{D})$.

A Waldhausen category $\mathbf{D}$ with weak equivalences $\boldsymbol{w}(\mathbf{D})$ is often denoted by $\boldsymbol{w}\mathbf{D}$ as a reminder of the choice. A functor between Waldhausen categories is exact if it preserves the chosen zero objects, cofibrations, weak equivalences, and cobase changes.

A Waldhausen category may or may not satisfy the following additional axioms.

**Saturation axiom 9.6.2.** Given two morphisms $\phi\colon F \to G$ and $\psi\colon G \to H$ in $\mathbf{D}$, if any two of $\phi$, $\psi$, or $\psi\phi$, are in $\boldsymbol{w}(\mathbf{D})$ then so is the third.

**Extension axiom 9.6.3.** Given a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, "\psi"] & H \arrow[d, "\mu"] \\ F' \arrow[r] & G' \arrow[r] & H' \end{tikzcd}$$

with exact rows, if both $\phi$ and $\mu$ are in $\boldsymbol{w}(\mathbf{D})$ then so is $\psi$.

A *cylinder functor* on $\mathbf{D}$ is a functor $C$ from the category of morphisms $f\colon F \to G$ in $\mathbf{D}$ to $\mathbf{D}$ together with three natural transformations $j_1\colon F \to C(f)$, $j_2\colon G \to C(f)$, and $p\colon C(f) \to G$ such that $pj_2 = \mathrm{id}_G$ and $pj_1 = f$ for all $f$, and which has a number of properties listed in point 1.3.1 of [45] which will be rather automatic for the functors we construct later.

**Cylinder axiom 9.6.4.** A cylinder functor $C$ satisfies this axiom if for all morphisms $f\colon F \to G$ the required map $p$ is in $\boldsymbol{w}(\mathbf{D})$.

Let $\mathbf{D}$ be a small Waldhausen category with respect to two categories of weak equivalences $\boldsymbol{v}(\mathbf{D}) \subset \boldsymbol{w}(\mathbf{D})$ with a cylinder functor $T$ both for $\boldsymbol{v}\mathbf{D}$ and for $\boldsymbol{w}\mathbf{D}$ satisfying the cylinder axiom for $\boldsymbol{w}\mathbf{D}$. Suppose also that $\boldsymbol{w}(\mathbf{D})$ satisfies the extension and saturation axioms. Define $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$ to be the full subcategory of $\boldsymbol{v}\mathbf{D}$ whose objects are $F$ such that $0 \to F \in \boldsymbol{w}(\mathbf{D})$. Then $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$ is a small Waldhausen category with cofibrations $\mathrm{co}(\mathbf{D}^{\boldsymbol{w}}) = \mathrm{co}(\mathbf{D}) \cap \mathbf{D}^{\boldsymbol{w}}$ and weak equivalences $\boldsymbol{v}(\mathbf{D}^{\boldsymbol{w}}) = \boldsymbol{v}(\mathbf{D}) \cap \mathbf{D}^{\boldsymbol{w}}$.

The cylinder functor $T$ for $\boldsymbol{v}\mathbf{D}$ induces a cylinder functor for $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$. If $T$ satisfies the cylinder axiom then the induced functor does so too.

**Theorem 9.6.5** (Approximation Theorem)**.** *Let $E\colon \mathbf{D}_1 \to \mathbf{D}_2$ be an exact functor between two small saturated Waldhausen categories. It induces a map of $K$-theory spectra*

$$K(E)\colon K(\mathbf{D}_1) \longrightarrow K(\mathbf{D}_2).$$

*Assume that $\mathbf{D}_1$ has a cylinder functor satisfying the cylinder axiom. If $E$ satisfies two conditions:*

(1) *a morphism $f \in \mathbf{D}_1$ is in $\boldsymbol{w}(\mathbf{D}_1)$ if and only if $E(f) \in \mathbf{D}_2$ is in $\boldsymbol{w}(\mathbf{D}_2)$,*
(2) *for any object $D_1 \in \mathbf{D}_1$ and any morphism $g\colon E(D_1) \to D_2$ in $\mathbf{D}_2$, there is an object $D'_1 \in \mathbf{D}_1$, a morphism $f\colon D_1 \to D'_1$ in $\mathbf{D}_1$, and a weak equivalence $g'\colon E(D'_1) \to D_2 \in \boldsymbol{w}(\mathbf{D}_2)$ such that $g = g'E(f)$,*

*then $K(E)$ is a homotopy equivalence.*

*Proof.* This is Theorem 1.6.7 of [47]. The presence of the cylinder functor with the cylinder axiom allows to make condition (2) weaker than that of Waldhausen, see point 1.9.1 in [45]. □

**Definition 9.6.6.** In any additive category, a sequence of morphisms

$$E^{\cdot}\colon \quad 0 \longrightarrow E^1 \xrightarrow{d_1} E^2 \xrightarrow{d_2} \;\ldots\; \xrightarrow{d_{n-1}} E^n \longrightarrow 0$$

is called a *(bounded) chain complex* if the compositions $d_{i+1}d_i$ are the zero maps for all $i = 1,\ldots, n-1$. A *chain map* $f\colon F^{\cdot} \to E^{\cdot}$ is a collection of morphisms $f^i\colon F^i \to E^i$ such that $f^i d_i = d_i f^i$. A chain map $f$ is *null-homotopic* if there are morphisms $s_i\colon F^{i+1} \to E^i$ such that $f = ds+sd$. Two chain maps $f, g\colon F^{\cdot} \to E^{\cdot}$ are *chain homotopic* if $f - g$ is null-homotopic. Now $f$ is a *chain homotopy equivalence* if there is a chain map $h\colon E^i \to F^i$ such that the compositions $fh$ and $hf$ are chain homotopic to the respective identity maps.

The Waldhausen structures on categories of bounded chain complexes are based on homotopy equivalence as a weakening of the notion of isomorphism of chain complexes.

**Definition 9.6.7.** A sequence of maps in an exact category is called *acyclic* if it is assembled out of short exact sequences in the sense that each map factors as the composition of the cokernel of the preceding map and the kernel of the succeeding map.

It is known that the class of acyclic complexes in an exact category is closed under isomorphisms in the homotopy category if and only if the category is idempotent complete, which is also equivalent to the property that each contractible chain complex is acyclic, cf. [33, sec. 11].

**Definition 9.6.8.** Given an exact category $\mathbf{E}$, there is a standard choice for the Waldhausen structure on the category $\mathbf{E}'$ of bounded chain complexes in $\mathbf{E}$ where the degree-wise admissible monomorphisms are the cofibrations and the chain maps whose mapping cones are homotopy equivalent to acyclic complexes are the weak equivalences $\boldsymbol{v}(\mathbf{E}')$.

**Proposition 9.6.9.** *The category $\boldsymbol{v}\mathbf{E}'$ is a Waldhausen category satisfying the extension and saturation axioms and has cylinder functor satisfying the cylinder axiom.*

*Proof.* The pushouts along cofibrations in $\mathbf{E}'$ are the complexes of pushouts in each degree. All standard Waldhausen axioms including the gluing lemma are clearly satisfied. The saturation and the extension axioms are also clear. The cylinder functor $C$ for $\boldsymbol{v}\mathbf{E}'$ is defined using the canonical homotopy pushout as in point 1.1.2 in Thomason–Trobaugh [45]. Given a chain map $f\colon F \to G$, $C(f)$ is the canonical homotopy pushout of $f$ and the identity $\mathrm{id}\colon F \to F$. With this construction, the map $p\colon C(f) \to G$ is a chain homotopy equivalence, so the cylinder axiom is also satisfied. □

**Definition 9.6.10.** There are three choices for the Waldhausen structure on the category of bounded chain complexes $\mathbf{G}' = \mathbf{G}'_X(Y)$. One is $\boldsymbol{v}\mathbf{G}'$ as in Definition 9.6.8. Given a subset $C \subset Y$, another choice for the weak equivalences $\boldsymbol{w}(\mathbf{G}')$ is the chain maps whose mapping cones are homotopy equivalent to acyclic complexes in the quotient $\mathbf{G}/\mathbf{C}$.

**Corollary 9.6.11.** *The categories $\boldsymbol{v}\mathbf{G}'$ and $\boldsymbol{w}\mathbf{G}'$ are Waldhausen categories satisfying the extension and saturation axioms and have cylinder functors satisfying the cylinder axiom.*

*Proof.* All axioms and constructions, including the cylinder functor, for $\boldsymbol{w}\mathbf{G}'$ are inherited from $\boldsymbol{v}\mathbf{G}'$. □

The $K$-theory functor from the category of small Waldhausen categories $\mathbf{D}$ and exact functors to the category of connective spectra is defined in terms of $S.$-construction as in Waldhausen [47]. It extends to simplicial categories $\mathbf{D}$ with cofibrations and weak equivalences and inductively delivers the connective spectrum $n \mapsto |\boldsymbol{w}S^{(n)}_{.}\,\mathbf{D}\,|$. We obtain the functor assigning to $\mathbf{D}$ the connective $\Omega$-spectrum

$$K(\mathbf{D}) = \Omega^\infty |\boldsymbol{w}S^{(\infty)}_{.}\,\mathbf{D}\,| = \underset{n\geq 1}{\underrightarrow{\operatorname{colim}}}\ \Omega^n |\boldsymbol{w}S^{(n)}_{.}\,\mathbf{D}\,|$$

representing the Waldhausen algebraic $K$-theory of $\mathbf{D}$. For example, if $\mathbf{D}$ is the additive category of free finitely generated $R$-modules with the canonical Waldhausen structure, then the stable homotopy groups of $K(\mathbf{D})$ are the usual $K$-groups of the ring $R$. In fact, there is a general identification of the two theories. Recall that for any exact category $\mathbf{E}$, the category $\mathbf{E}'$ of bounded chain complexes has the Waldhausen structure $\boldsymbol{v}\mathbf{E}'$ as in Definition 9.6.8.

**Theorem 9.6.12.** *The Quillen $K$-theory of an exact category $\mathbf{E}$ is equivalent to the Waldhausen $K$-theory of $\boldsymbol{v}\mathbf{E}'$.*

*Proof.* The proof is based on repeated applications of the Additivity Theorem, cf. Thomason's Theorem 1.11.7 [45]. Thomason's proof of his Theorem 1.11.7 can be repeated verbatim here. It is in fact simpler in this case since condition 1.11.3.1 is not required. □

9.7. **Fibrewise Excision Theorems.** The computational tools from nonconnective bounded $K$-theory, the Bounded Excision Theorems of [39, 40, 8], can be adapted to $\mathbf{G}_X(Y)_{<\mathcal{U}}$.

Suppose $Y_1$ and $Y_2$ are mutually antithetic subsets of a proper metric space $Y$, and $Y = Y_1 \cup Y_2$. Consider the coarse covering $\mathcal{U}$ of $Y$ by $\mathcal{S}(Y_1)$, $\mathcal{S}(Y_2)$, and

$\mathcal{S}(Y_1, Y_2)$. We use the notation $\mathbf{G} = \mathbf{G}_X(Y)_{<\mathcal{U}}$, $\mathbf{G}_i = \mathbf{G}_X(Y)_{<Y_i}$ for $i = 1$ or $2$, and $\mathbf{G}_{12}$ for the intersection $\mathbf{G}_1 \cap \mathbf{G}_2$. There is a commutative diagram

$$(\natural) \qquad \begin{array}{ccccc} K(\mathbf{G}_{12}) & \longrightarrow & K(\mathbf{G}_1) & \longrightarrow & K(\mathbf{G}_1/\mathbf{G}_{12}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle K(I)} \\ K(\mathbf{G}_2) & \longrightarrow & K(\mathbf{G}) & \longrightarrow & K(\mathbf{G}/\mathbf{G}_2) \end{array}$$

where the rows are homotopy fibrations from Theorem 9.5.12 and $I \colon \mathbf{G}_1/\mathbf{G}_{12} \to \mathbf{G}/\mathbf{G}_2$ is the functor induced from the exact inclusion $I \colon \mathbf{G}_1 \to \mathbf{G}$. We observe that $I$ is not necessarily full and, therefore, not an isomorphism of categories.

**Proposition 9.7.1.** *$K(\boldsymbol{w}\mathbf{G}') \simeq K(\mathbf{G}/\mathbf{C})$.*

*Proof.* This follows from Lemma 2.3 in [43] as part of the proof of Theorem 9.5.12 where $K(\boldsymbol{w}\mathbf{G}')$ from Waldhausen's Fibration Theorem is identified with the Quillen $K$-theory spectrum $K(\mathbf{G}/\mathbf{C})$. □

**Lemma 9.7.2.** *If $f^{\cdot} \colon F^{\cdot} \to G^{\cdot}$ is a degreewise admissible monomorphism with cokernel in $\mathbf{C}$ then $f^{\cdot}$ is a weak equivalence in $\boldsymbol{w}\mathbf{G}'$.*

*Proof.* The mapping cone $Cf^{\cdot}$ is quasi-isomorphic to the cokernel of $f^{\cdot}$, by Lemma 11.6 of [33], which is zero in $\mathbf{G}/\mathbf{C}$. □

The exact inclusion $I$ induces the exact functor $\boldsymbol{w}\mathbf{G}'_1 \to \boldsymbol{w}\mathbf{G}'$.

**Lemma 9.7.3.** *The map $K(\boldsymbol{w}\mathbf{G}'_1) \to K(\boldsymbol{w}\mathbf{G}')$ is a weak equivalence.*

*Proof.* Applying the Approximation Theorem, the first condition is clear. To check the second condition, consider

$$F^{\cdot}\colon \quad 0 \longrightarrow F^1 \xrightarrow{\ \phi_1\ } F^2 \xrightarrow{\ \phi_2\ } \ \dots\ \xrightarrow{\ \phi_{n-1}\ } F^n \longrightarrow 0$$

in $\mathbf{G}_1$ and a chain map $g \colon F^{\cdot} \to G^{\cdot}$ for some complex

$$G^{\cdot}\colon \quad 0 \longrightarrow G^1 \xrightarrow{\ \psi_1\ } G^2 \xrightarrow{\ \psi_2\ } \ \dots\ \xrightarrow{\ \psi_{n-1}\ } G^n \longrightarrow 0$$

in $\mathbf{G}$. Suppose all $F^i$ and $G^i$ are $(D, \Delta')$-lean/split and $(d, \delta)$-insular. Also assume that there is a fixed number $r \geq 0$ and a monotone function $\rho \colon [0, +\infty) \to [0, +\infty)$ such that

$$F^i(X, Y) \subset F^i((X, C)[r, \rho_{x_0}])$$

holds for all $0 \leq i \leq n$. If the pair $(b, \theta)$ serves as bounded control data for all $\phi_i$, $\psi_i$, and $g_i$, we define the submodule

$$F'^i = \mathcal{G}^i((X, Y_1)[r + 3ib, \rho_{x_0} + 3i\theta_{x_0}])$$

and define $\xi_i \colon F'^i \to F'^{i+1}$ to be the restrictions of $\psi_i$ to $F'^i$. This gives a chain subcomplex $(F'^i, \xi_i)$ of $(G^i, \psi_i)$ in $\mathbf{G}$ with the inclusion $i \colon F'^i \to G^i$. Notice that we have the induced chain map $\overline{g} \colon F^{\cdot} \to F'^{\cdot}$ in $\mathbf{G}_1$ so that $g = iI(\overline{g})$.

We will argue that $C^{\cdot} = \operatorname{coker}(i)$ is in $\mathbf{G}_2$. Given that, $i$ is a weak equivalence by Lemma 9.7.2. Since

$$F'^i \subset G^i((X, Y_1)[r + 3ib + K, \rho_{x_0} + 3i\theta_{x_0} + k_{x_0}]),$$

each $C^i$ is supported on

$$\begin{aligned} (X, Y \setminus Y_1)[2D + 2d - r - 3ib - K, 2\Delta'_{x_0} + 2\delta_{x_0} - \rho_{x_0} - 3i\theta_{x_0} - k_{x_0}] \\ \subset (X, Y_2)[2D + 2d, 2\Delta'_{x_0} + 2\delta_{x_0}], \end{aligned}$$

cf. Lemma 9.5.6. So the complex $C^i$ is indeed in $\mathbf{G}_2$. $\square$

Let $\mathbb{R}$, $[0,+\infty)$, and $(-\infty,0]$ denote the metric spaces of reals, nonnegative reals, and nonpositive reals with the restriction of the usual metric on the real line $\mathbb{R}$. Then we have the following instance of commutative diagram ($\natural$)

$$\begin{tikzcd} \widehat{G}_X(Y) \arrow[r] \arrow[d] & \widehat{G}_X(Y \times [0,+\infty)) \arrow[r] \arrow[d] & K(\mathbf{G}_1/\mathbf{G}_{12}) \arrow[d, "K(I)"] \\ \widehat{G}_X(Y \times (-\infty,0]) \arrow[r] & \widehat{G}_X(Y \times \mathbb{R}) \arrow[r] & K(\mathbf{G}/\mathbf{G}_2) \end{tikzcd}$$

**Lemma 9.7.4.** *The spectra $\widehat{G}_X(Y \times \mathbb{R}^{\ge 0})$ and $\widehat{G}_X(Y \times \mathbb{R}^{\le 0})$ are contractible.*

*Proof.* This follows from the fact that these controlled categories are flasque, that is, the usual shift functor $T$ in the positive (respectively negative) direction along $\mathbb{R}^{\ge 0}$ (respectively $\mathbb{R}^{\le 0}$) interpreted in the obvious way is an exact endofunctor, and there is a natural equivalence $1 \oplus \pm T \cong \pm T$. Contractibility follows from the Additivity Theorem, cf. Pedersen–Weibel [39]. $\square$

In view of Lemma 9.7.3, we obtain a map $\widehat{G}_X(Y) \to \Omega\widehat{G}_X(Y \times \mathbb{R})$ which induces isomorphisms of $K$-groups in positive dimensions. Iterations give weak equivalences

$$\Omega^k \widehat{G}_X(Y \times \mathbb{R}^k) \longrightarrow \Omega^{k+1}\widehat{G}_X(Y \times \mathbb{R}^{k+1})$$

for $k \ge 2$.

**Definition 9.7.5.** The *nonconnective fibred bounded $G$-theory* over the pair $(X,Y)$ is the spectrum

$$G_X^{-\infty}(Y) \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k \widehat{G}_X(Y \times \mathbb{R}^k).$$

**Remark 9.7.6.** Since $\mathbf{BL}(X,R)$ can be identified with $\mathbf{B}_X(\text{point})$, this definition recovers the nonconnective $G$-theory of $X$:

$$G^{-\infty}(X,R) = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^{k+l}\widehat{G}_X(\mathbb{R}^k).$$

**Remark 9.7.7.** In view of Theorem 9.3.3, there are maps

$$\Omega^{k+l} G_{X \times \mathbb{R}^k}(Y \times \mathbb{R}^l) \longrightarrow \Omega^{k+k'+l+l'} G_{X \times \mathbb{R}^{k+k'}}(Y \times \mathbb{R}^{l+l'})$$

which induce isomorphisms of homotopy groups for $k \ge 2$, $l \ge 2$, and all $k'$, $l' \ge 0$. Now the homotopy colimit in Definition 9.7.5 can be interpreted as

$$G_X^{-\infty}(Y) = \underset{\overrightarrow{k,l>0}}{\operatorname{hocolim}}\ \Omega^{k+l} G_{X \times \mathbb{R}^k}(Y \times \mathbb{R}^l).$$

**Definition 9.7.8.** Define

$$G_X^{-\infty}(Y)_{<C} \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k \widehat{G}_X(Y \times \mathbb{R}^k)_{<C \times \mathbb{R}^k},$$

where $\mathbf{G}_X(Y \times \mathbb{R}^k)_{<C \times \mathbb{R}^k}$ is a Grothendieck subcategory of $\mathbf{G}_X(Y \times \mathbb{R}^k)$. Using the methods above, one easily obtains the weak equivalence

$$G_X^{-\infty}(Y)_{<C} \simeq G_X^{-\infty}(C).$$

We also define

$$G_X^{-\infty}(Y)_{<C_1,C_2} \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k \widehat{G}_X(Y \times \mathbb{R}^k)_{<C_1\times\mathbb{R}^k,\, C_2\times\mathbb{R}^k}.$$

**Theorem 9.7.9** (Bounded Fiberwise Excision)**.** *Suppose $Y_1$ and $Y_2$ are mutually antithetic subsets of a metric space $Y$, and $Y = Y_1 \cup Y_2$. There is a homotopy pushout diagram of spectra*

$$\begin{tikzcd} G_X^{-\infty}(Y)_{<Y_1,Y_2} \arrow[r] \arrow[d] & G_X^{-\infty}(Y)_{<Y_1} \arrow[d] \\ G_X^{-\infty}(Y)_{<Y_2} \arrow[r] & G_X^{-\infty}(Y) \end{tikzcd}$$

*where the maps of spectra are induced from the exact inclusions.*

*Proof.* Let us write $S^k\,\mathbf{G}$ for $\mathbf{G}_X(Y\times\mathbb{R}^k)$ whenever $\mathbf{G}$ is the fibred bounded category for a pair $(X,Y)$. If $C$ represents a family of coarsely equivalent subsets in a coarse covering $\mathcal{U}$ of $Y$, consider the fibration

$$G_X(C) \longrightarrow G_X(Y) \longrightarrow K(\mathbf{G}/\mathbf{C})$$

from Theorem 9.5.14. Notice that there is a map

$$K(\mathbf{G}/\mathbf{C}) \longrightarrow \Omega K(S\,\mathbf{G}/S\,\mathbf{C})$$

which is a weak equivalence in positive dimensions by the Five Lemma. If one defines

$$G_X^{-\infty}(Y,C) = K^{-\infty}(\mathbf{G}/\mathbf{C}) = \underset{\overrightarrow{k}}{\operatorname{hocolim}}\ \Omega^k K(S^k\,\mathbf{G}/S^k\,\mathbf{C}),$$

there is an induced fibration

$$G_X^{-\infty}(C) \longrightarrow G_X^{-\infty}(Y) \longrightarrow G_X^{-\infty}(Y,C).$$

The theorem follows from the commutative diagram

$$\begin{tikzcd} G_X^{-\infty}(Y)_{<Y_1,Y_2} \arrow[r] \arrow[d] & G_X^{-\infty}(Y)_{<Y_1} \arrow[r] \arrow[d] & K^{-\infty}(\mathbf{G}_1/\mathbf{G}_{12}) \arrow[d] \\ G_X^{-\infty}(Y)_{<Y_2} \arrow[r] & G_X^{-\infty}(Y) \arrow[r] & K^{-\infty}(\mathbf{G}/\mathbf{G}_2) \end{tikzcd}$$

and the fact that $K^{-\infty}(\mathbf{G}_1/\mathbf{G}_{12}) \to K^{-\infty}(\mathbf{G}/\mathbf{G}_2)$ is a weak equivalence. □

We will require a relative version of bounded $G$-theory and the corresponding Fiberwise Excision Theorem.

**Definition 9.7.10.** Let $Y' \in \mathcal{A}$ for a coarse covering $\mathcal{U}$ of $Y$. Let $\mathbf{G} = \mathbf{G}_X(Y)_{<\mathcal{U}}$ and $\mathbf{Y}' = \mathbf{G}_X(Y)_{<Y'}$. The category $\mathbf{G}_X(Y,Y')$ is the quotient category $\mathbf{G}\,/\,\mathbf{Y}'$.

**Proposition 9.7.11.** *Given a proper subset $U$ of $Y'$, there is a weak equivalence*

$$G_X^{-\infty}(Y,Y') \simeq G_X^{-\infty}(Y-U, Y'-U).$$

*Proof.* Consider the setup of Theorem 9.7.9 with $Y_1 = Y - U$ and $Y_2 = Y'$, then Lemma 9.7.3 shows that the map

$$\frac{\mathbf{G}_X(Y)_{<(Y-U)}}{\mathbf{G}_X(Y)_{<(Y-U)} \cap \mathbf{G}_X(Y)_{<Y'}} \longrightarrow \frac{\mathbf{G}_X(Y)}{\mathbf{G}_X(Y)_{<Y'}}$$

induces a weak equivalence on the level of $K$-theory. Notice that, since $U$ is a subset of $Y'$,

$$\mathbf{G}_X(Y)_{<(Y-U)} \cap \mathbf{G}_X(Y)_{<Y'} = \mathbf{G}_X(Y)_{<(Y'-U)}.$$

Now the maps of quotients

$$\frac{\mathbf{G}_X(Y)}{\mathbf{G}_X(Y')} \longrightarrow \frac{\mathbf{G}_X(Y)}{\mathbf{G}_X(Y)_{<Y'}}$$

and

$$\frac{\mathbf{G}_X(Y)_{<(Y-U)}}{\mathbf{G}_X(Y)_{<(Y'-U)}} \longleftarrow \frac{\mathbf{G}_X(Y-U)}{\mathbf{G}_X(Y'-U)}$$

induced by fully faithful embeddings also induce weak equivalences. Their composition gives the required equivalence. $\square$

One can easily show that the theory developed in this section can be relativized to give the following excision theorem.

**Theorem 9.7.12** (Relative Bounded Excision)**.** *If $Y$ is the union of two mutually antithetic subsets $U_1$ and $U_2$, there is a homotopy pushout diagram of spectra*

$$\begin{tikzcd}
G_X^{-\infty}(Y,Y')_{<U_1,U_2} \arrow[r] \arrow[d] & G_X^{-\infty}(Y,Y')_{<U_1} \arrow[d] \\
G_X^{-\infty}(Y,Y')_{<U_2} \arrow[r] & G_X^{-\infty}(Y,Y')
\end{tikzcd}$$

*where the maps of spectra are induced from the exact inclusions. In fact, there is a homotopy pushout*

$$\begin{tikzcd}
G_X^{-\infty}(U_1 \cap U_2, U_1 \cap U_2 \cap Y') \arrow[r] \arrow[d] & G_X^{-\infty}(U_1, U_1 \cap Y') \arrow[d] \\
G_X^{-\infty}(U_2, U_2 \cap Y') \arrow[r] & G_X^{-\infty}(Y,Y')
\end{tikzcd}$$

9.8. **Functoriality, Equivariant Theories.** Recall from Definition 2.1.1 that we call a map $f\colon X \to Y$ between proper metric spaces a *coarse map* if it is proper and there is a real positive function $l$ such that

$$T \in \mathcal{B}_r(X) \implies f(T) \in \mathcal{B}_{l(r)}(Y),$$

where $\mathcal{B}_d(X)$ denotes the collection of subsets of $X$ with diameter bounded by $d$.

The map $f$ is a *coarse equivalence* if there is a coarse map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded maps.

**Proposition 9.8.1.** *A coarse map $f\colon X \to Y$ induces a functor*

$$f_*\colon \mathbf{U}(X,R) \longrightarrow \mathbf{U}(Y,R).$$

*If $f$ is a coarse equivalence then $f_*$ is an equivalence of categories. In this case, one also obtains a functor*

$$f_*\colon \mathbf{BL}(X,R) \longrightarrow \mathbf{BL}(Y,R).$$

*Proof.* Define $f_*(F)(S) = F(f^{-1}(S))$. $\square$

**Corollary 9.8.2.** *Consider the category of proper metric spaces $X$ and coarse equivalences. Then $\mathbf{BL}(X,R)$ is a covariant functor in the space variable to small exact categories and exact functors. Composing with the covariant functor $G^{-\infty}$ gives the spectrum-valued functor $G^{-\infty}(X,R)$.*

*Proof.* The containment $f^{-1}(S)[b] \subset f^{-1}(S[kb])$ shows that if $f \in \mathbf{B}(X,R)$ is boundedly bicontrolled with $\mathrm{fil}(f) \leq b$ then $f_f\phi$ is boundedly bicontrolled with $\mathrm{fil}(f_* f) \leq kD$. $\square$

A subset $W$ of a metric space $X$ is *boundedly dense* or *commensurable* if $W[d] = X$ for some $d \geq 0$.

**Proposition 9.8.3.** *For a commensurable metric subspace $W$ of $X$, there is a natural exact equivalence of categories $\mathbf{BL}(W,R) \to \mathbf{BL}(X,R)$ and the induced weak homotopy equivalence $G^{-\infty}(W,R) \simeq G^{-\infty}(X,R)$.*

*Proof.* Any surjective coarse equivalence $f\colon X \to Y$ induces two functors on filtered modules. One is the contravariant $f^*\colon \mathbf{BL}(Y,R) \to \mathbf{BL}(X,R)$ given by $f^*F(S) = F(f(S))$. The other is the covariant functor $f_*$ as in Proposition 9.8.1, so that $f^*f_* = \mathrm{id}$. Even when $f$ is not surjective, there is the endofunctor $\omega = f^{-1}f$ of $\mathcal{P}(X)$ which induces an endofunctor $\omega_*$ of $\mathbf{BL}(X,R)$. If $f\colon X \to X$ is a bounded function, there is always an isomorphism $\omega_*(F) \cong F$ induced by the identity on $F(X)$. This shows that $f_*F \cong F$ for all $F \in \mathbf{BL}(X,R)$.

Now if $W \subset X$ is commensurable, there is a bounded surjection $f\colon X \to W$, so $f$ induces a natural transformation $\eta\colon \mathrm{id} \to f_*$ where all $\eta(F)$ are isomorphisms. $\square$

**Corollary 9.8.4.** *If $X$ is a bounded metric space then the natural equivalence*

$$\mathbf{BL}(X,R) \cong \mathbf{BL}(\text{point},R) = \mathbf{Modf}(R)$$

*induces a weak equivalence $G^{-\infty}(X,R) \simeq G^{-\infty}(R)$.*

Given a geometric action of $\Gamma$ on a metric space $X$, there are natural actions of $\Gamma$ on $\mathbf{U}(X,R)$ and $\mathbf{B}(X,R)$ induced from the action on the power set $\mathcal{P}(X)$.

*Notation* 9.8.5. The nonconnective $K$-theory of $\mathcal{C}^\Gamma = \mathbf{BL}^\Gamma(X,R)$ associated to $\mathcal{C} = \mathbf{BL}(X,R)$ is denoted by $G^\Gamma(X,R)$. The nonconnective $K$-theory of the full subcategory $\mathcal{C}^{\Gamma,0} = \mathbf{BL}^{\Gamma,0}(X,R)$ on functors with values in filtration 0 morphisms is $G^{\Gamma,0}(X,R)$.

**Proposition 9.8.6.** *The fixed point category $\mathbf{BL}^{\Gamma,0}(X,R)^\Gamma$ is exact.*

*Proof.* The exact structure is inherited from $\mathbf{BL}(X,R)$ in the sense that a morphism $\phi\colon (F,\psi) \to (F',\psi')$ is an admissible monomorphism or epimorphism if the map $\phi\colon F \to F'$ is an admissible monomorphism or epimorphism in $\mathbf{BL}(X,R)$, respectively. The fact that this is an exact structure follows from the proofs of Theorems 9.1.9, 9.1.19, and 9.1.23 by observing that all constructions in those proofs produce equivariant objects and morphisms. $\square$

Given a metric space $Y$ with a left $\Gamma$-action by isometries or coarse equivalences, there are all three $G$-theory analogues of equivariant theories $K^\Gamma_i$, $K^\Gamma_c$, $K^\Gamma_p$ from Definition 2.5.6.

The analogue of $K^\Gamma_p$ is described in Definition 2.6.6. Suppose $\Gamma$ acts on $Y$ by coarse equivalences. Recall that $g^\Gamma_p(Y)$ is the Quillen $K$-theory spectrum of the exact category $\mathbf{G}^\Gamma_p(Y)$ of functors

$$\theta\colon \boldsymbol{E\Gamma} \longrightarrow \mathbf{G}_\Gamma(Y)$$

such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of $\Gamma$-filtered modules.

The equivariant $G$-theory $g_p^\Gamma(Y)$ has a nonconnective delooping $G_p^\Gamma(Y)$ constructed similar to the equivariant $K$-theory. Thus there are maps

$$g_p^\Gamma(Y \times \mathbb{R}^{k-1}) \to \Omega g_p^\Gamma(Y \times \mathbb{R}^k),$$

where the action of $\Gamma$ on $\mathbb{R}^k$ is trivial, and the nonconnective equivariant theory is defined as the colimit

$$G_p^\Gamma(Y) \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k g_p^\Gamma(Y \times \mathbb{R}^k).$$

Similarly for the fixed points

$$G_p^\Gamma(Y)^\Gamma = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k g_p^\Gamma(Y \times \mathbb{R}^k)^\Gamma.$$

9.9. **Equivariant Localization and Excision.** We will finally develop localization and excision results with respect to specific subsets and systems of subsets of the variable $Y$ in $G_p^\Gamma(Y)^\Gamma$.

First recall from Definition 2.6.9 that whenever a proper metric space $Y$ possesses a left $\Gamma$-action by isometries, we call a subset $Y'$ *coarsely invariant* if there is a function $t\colon \Gamma \to \mathbb{R}$ such that

$$\gamma \cdot Y' \subset Y'[t(\gamma)].$$

In this case, the Grothendieck subcategory $\mathbf{G}_p^\Gamma(Y)_{<Y'}$ is invariant under the action of $\Gamma$, so there is a well-defined left $\Gamma$-action on the quotient $\mathbf{G}_p^\Gamma(Y, Y')$, and we obtain the equivariant relative theory $G_p^\Gamma(Y, Y')$.

The quotient map of categories induces the equivariant map

$$G_p^\Gamma(Y) \longrightarrow G_p^\Gamma(Y, Y')$$

and the map of fixed point spectra

$$G_p^\Gamma(Y)^\Gamma \longrightarrow G_p^\Gamma(Y, Y')^\Gamma.$$

More generally, if $Y''$ is another coarsely invariant subset of $Y$ that is coarsely antithetic to $Y'$, then the intersection $Y'' \cap Y'$ is coarsely invariant in both $Y$ and $Y'$. Therefore, there is an equivariant map

$$G_p^\Gamma(Y'', Y'' \cap Y') \longrightarrow G_p^\Gamma(Y, Y')$$

and the map of fixed points

$$G_p^\Gamma(Y'', Y'' \cap Y')^\Gamma \longrightarrow G_p^\Gamma(Y, Y')^\Gamma.$$

Suppose $C$ is a coarsely invariant subset of $Y$.

**Definition 9.9.1.** Define $\mathbf{G}_p^\Gamma(Y)^\Gamma_{<C}$ to be the full subcategory of $\mathbf{G}_p^\Gamma(Y)^\Gamma$ on the objects $F$ with $F(S)$ contained in $\mathbf{G}(Y, R)_{<C}$ for all bounded subsets $S \subset \Gamma$. Similarly, $\mathbf{G}_p^\Gamma(Y, Y')^\Gamma_{<C}$ is the full subcategory of $\mathbf{G}_p^\Gamma(Y, Y')^\Gamma$ on objects in $\mathbf{G}_p^\Gamma(Y)^\Gamma_{<C}$.

**Proposition 9.9.2.** *$\mathbf{G}_p^\Gamma(Y)^\Gamma_{<C}$ and $\mathbf{G}_p^\Gamma(Y, Y')^\Gamma_{<C}$ are Grothendieck subcategories of respectively $\mathbf{G}_p^\Gamma(Y)^\Gamma$ and $\mathbf{G}_p^\Gamma(Y, Y')^\Gamma$ and are idempotent complete.*

*Proof.* Since $\psi(\gamma)\colon F \to \gamma F$ are isomorphisms, all $\gamma F$ are objects of $\mathbf{G}_p(Y)_{<C}$ if $F$ is in $\mathbf{G}_p(Y)_{<C}$. In fact, if $\text{fil}(\psi(\gamma)) \le (b_\gamma, \theta_\gamma)$ and

$$F(X, Y) \subset F((X, C)[K, k_{x_0}])$$

then

$$(\gamma F)(X,Y) \subset F((X,C)[K+b_\gamma, k_{x_0} + (\theta_\gamma)_{x_0}]).$$

The rest of the proof is similar to that of Proposition 9.3.5. The details are left to the reader. □

*Notation* 9.9.3. Let $\mathbf{G}^\Gamma_p(Y,Y')^\Gamma_{>C}$ denote the quotient of the embedding

$$\mathbf{G}^\Gamma_p(Y,Y')^\Gamma_{<C} \longrightarrow \mathbf{G}^\Gamma_p(Y,Y')^\Gamma$$

and $G^\Gamma_p(Y,Y')^\Gamma_{>C}$ be the corresponding nonconnective $K$-theory spectrum.

**Theorem 9.9.4.** *For a pair of coarsely invariant mutually antithetic subsets $Y'$ and $C$ of $Y$, there is a homotopy fibration*

$$G^\Gamma_p(Y,Y')^\Gamma_{<C} \longrightarrow G^\Gamma_p(Y,Y')^\Gamma \longrightarrow G^\Gamma_p(Y,Y')^\Gamma_{>C}.$$

*Proof.* The proof is by application of the localization theorem. □

Suppose $C_1$ and $C_2$ are two coarsely invariant mutually antithetic subsets of $Y$ such that $Y = C_1 \cup C_2$ and let

$$\mathbf{G}^\Gamma_p(Y,Y')^\Gamma_{<C_1,C_2} = \mathbf{G}^\Gamma_p(Y,Y')^\Gamma_{<C_1} \cap \mathbf{G}^\Gamma_p(Y,Y')^\Gamma_{<C_2}$$

be the intersection of two full subcategories in $\mathbf{G}^\Gamma_p(Y,Y')^\Gamma$. Since the action is by bounded coarse equivalences, if $F \in \mathbf{G}^\Gamma_p(Y,Y')_{<C_i}$ then $\gamma F \in \mathbf{G}^\Gamma_p(Y,Y')_{<C_i}$ for all $\gamma \in \Gamma$ and $i = 1, 2$. Also $\mathbf{G}^\Gamma_p(Y,Y')^\Gamma_{<C_1,C_2}$ clearly closed under extensions so it is an exact category.

**Theorem 9.9.5.** *The commutative square*

$$\begin{tikzcd}
G^\Gamma_p(Y,Y')^\Gamma_{<C_1,C_2} \arrow[r] \arrow[d] & G^\Gamma_p(Y,Y')^\Gamma_{<C_1} \arrow[d] \\
G^\Gamma_p(Y,Y')^\Gamma_{<C_2} \arrow[r] & G^\Gamma_p(Y,Y')^\Gamma
\end{tikzcd}$$

*induced by inclusions is a homotopy pushout of spectra.*

*Proof.* The square is the left square in the map of two fibration sequences

$$\begin{tikzcd}
G^\Gamma_p(Y,Y')^\Gamma_{<C_1,C_2} \arrow[r] \arrow[d] & G^\Gamma_p(Y,Y')^\Gamma_{<C_1} \arrow[r] \arrow[d] & G^\Gamma_p(Y,Y')^\Gamma_{<C_1,>C_2} \arrow[d] \\
G^\Gamma_p(Y,Y')^\Gamma_{<C_2} \arrow[r] & G^\Gamma_p(Y,Y')^\Gamma \arrow[r] & G^\Gamma_p(Y,Y')^\Gamma_{>C_2}
\end{tikzcd}$$

The rightmost vertical map is an equivalence by the Approximation Theorem. □

**Definition 9.9.6.** Recall that we call a pair of subsets $S$ and $T$ of a metric space $X$ *coarsely antithetic* if each subset is a proper metric space with respect to the induced metric and for each number $K > 0$ there is $L > 0$ so that

$$S[K] \cap T[K] \subset (S \cap T)[L].$$

We will say that three subsets $S$, $T$, and $V$ form a *coarsely antithetic triple* if they are pairwise coarsely antithetic.

The following are examples of antithetic triples:

(1) half-spaces $\mathbb{R}^{n-1} \times [0, +\infty)$, $\mathbb{R}^{n-1} \times (-\infty, 0]$, and the line $\{0\} \times (-\infty, +\infty)$ in $\mathbb{R}^n = \mathbb{R}^{n-1} \times (-\infty, +\infty)$,

(2) $T(D^n_+)$, $T(D^n_-)$, and $T(\partial D^n)$ where $T$ is the construction from section 2.3, $D^n$ is the unit disk in $\mathbb{R}^n$, and $D^n_\pm = D^n \cap \mathbb{R}^{n-1} \times [0, \pm\infty)$.

**Lemma 9.9.7.** *Suppose $Y'$ is a subset of $Y$ and $\{C_1, C_2\}$ is a covering such that all three subsets are coarsely invariant and form a coarsely antithetic triple. If the action of $\Gamma$ on $(Y, Y')$ is trivial then there is a weak equivalence*

$$G^\Gamma_p(Y, Y')^\Gamma_{<C_1,C_2} \simeq G^\Gamma_p(C_1 \cap C_2, C_1 \cap C_2 \cap Y')^\Gamma.$$

*Proof.* Apply Lemma 9.5.15 to each isomorphic copy $\gamma F$ of $F$ in $(F, \psi)$ expressed as a lax limit. □

**Theorem 9.9.8.** *Suppose $Y'$ is a subset of $Y$ and $\{C_1, C_2\}$ is a covering such that all three subsets are coarsely invariant and form a coarsely antithetic triple. Suppose the action of $\Gamma$ on $Y$ is trivial. Then the commutative square*

$$\begin{tikzcd}
G^\Gamma_p(C_1 \cap C_2, Y' \cap C_1 \cap C_2)^\Gamma \arrow[r] \arrow[d] & G^\Gamma_p(C_1, Y' \cap C_1)^\Gamma \arrow[d] \\
G^\Gamma_p(C_2, Y' \cap C_2)^\Gamma \arrow[r] & G^\Gamma_p(Y, Y')^\Gamma
\end{tikzcd}$$

*is a homotopy pushout.*

*Proof.* This follows from Theorem 9.9.5, using Lemma 9.5.15 and Corollary 9.5.16 to verify that the maps

$$G^\Gamma_p(C_1 \cap C_2, Y' \cap C_1 \cap C_2)^\Gamma \longrightarrow G^\Gamma_p(Y, Y')^\Gamma_{<C_1,C_2}$$

and

$$G^\Gamma_p(C_i, Y' \cap C_i)^\Gamma \longrightarrow G^\Gamma_p(Y, Y')^\Gamma_{<C_i}$$

induced by inclusions are weak equivalences. □

**Definition 9.9.9.** Given a finite coarse covering $\mathcal{U}$ of $Y$, there are two kinds of homotopy colimit constructions. For any action of $\Gamma$ by bounded coarse equivalences, one has

$$G^\Gamma_p(Y, Y')^\Gamma_{<\mathcal{U}} = \underset{\overrightarrow{U_i \in \mathcal{U}}}{\mathrm{hocolim}}\; G^\Gamma_p(Y, Y')^\Gamma_{<U_i}.$$

If the action trivial, one also has

$$\mathcal{G}^\Gamma_p(Y, Y'; \mathcal{U})^\Gamma = \underset{\overrightarrow{U_i \in \mathcal{U}}}{\mathrm{hocolim}}\; G^\Gamma_p(U_i, Y' \cap U_i)^\Gamma.$$

Inductive applications of Theorems 9.9.5 and 9.9.8 give

**Theorem 9.9.10.** *Suppose the action of $\Gamma$ on $(Y, Y')$ is by bounded coarse equivalences and $\mathcal{U}$ is a finite coarse covering of $Y$ such that all subsets $U \in \mathcal{U}$ and $Y'$ are pairwise coarsely antithetic. Then there is a weak equivalence*

$$G^\Gamma_p(Y, Y')^\Gamma \simeq G^\Gamma_p(Y, Y')^\Gamma_{<\mathcal{U}}.$$

*If the action of $\Gamma$ is in fact trivial, then there is a weak equivalence*

$$G^\Gamma_p(Y, Y')^\Gamma \simeq \mathcal{G}^\Gamma_p(Y, Y'; \mathcal{U})^\Gamma.$$

9.10. **Some Applications.** The first application simply summarizes the excision properties in an alternative way.

**Theorem 9.10.1.** *Suppose the action of $\Gamma$ on $(Y, Y')$ is by bounded coarse equivalences, and a proper subset $U$ of $Y$ is coarsely invariant under the action of $\Gamma$. If $U$, $Y - U$, and $Y'$ form a coarse covering by pairwise antithetic subsets, there is an equivalence*

$$G_p^\Gamma(Y, Y')^\Gamma \simeq G_p^\Gamma(Y - U, Y' - U)^\Gamma.$$

*If the action of $\Gamma$ on $Y$ is trivial, then there is a similar equivalence in $K$-theory:*

$$K_i^\Gamma(Y, Y')^\Gamma \simeq K_i^\Gamma(Y - U, Y' - U)^\Gamma.$$

*Proof.* The fact in $G$-theory follows from Proposition 9.7.11. The $K$-theory analogues require Karoubi filtration arguments similar to those above. These are left to the reader. □

The next result is a computation of $G_p^\Gamma(TD, T\partial D)^\Gamma$ for the standard $r$-disk $D$ and the trivial $\Gamma$-action. We have the following subsets of $\mathbb{R}^r$:

$$\begin{aligned}
E_i^+ &= \{(x_1, \ldots, x_r) \mid x_l = 0 \text{ for all } l > i,\ x_i \geq 0\},\\
E_i^- &= \{(x_1, \ldots, x_r) \mid x_l = 0 \text{ for all } l > i,\ x_i \leq 0\},\\
E_i &= E_i^- \cup E_i^+, \text{ for } 1 \leq i \leq r,\\
&= E_{i+1}^- \cap E_{i+1}^+, \text{ for } 1 \leq i < r,\\
E_0 &= E_0^- = E_0^+ = D_0 = D_0^- = D_0^+ = \{(0, \ldots, 0)\},\\
D_i^\pm &= E_i^\pm \cap D^r,\\
D_i &= D_i^- \cup D_i^+, \text{ for } 1 \leq i \leq r,\\
&= D_{i+1}^- \cap D_{i+1}^+, \text{ for } 1 \leq i < r.
\end{aligned}$$

The subsets $TD_i^*$ form a coarsely antithetic covering of $TD$.

**Lemma 9.10.2.** *$G_p^\Gamma(TD_i^\pm, T(D_i^\pm \cap \partial D))^\Gamma$ is contractible for all $1 \leq i \leq r$.*

*Proof.* The localization sequence

$$G_p^\Gamma(T(E_i^\pm - \operatorname{int} D))^\Gamma \longrightarrow G_p^\Gamma(TE_i^\pm)^\Gamma \longrightarrow G_p^\Gamma(TE_i^\pm, T(E_i^\pm - \operatorname{int} D))^\Gamma,$$

with the first two terms contractible by the Eilenberg swindle, shows that the spectrum $G_p^\Gamma(TE_i^\pm, T(E_i^\pm - \operatorname{int} D))^\Gamma$ is contractible. Applying Theorem 9.10.1 with $U = E_i^\pm - D$, we conclude that $G_p^\Gamma(TD_i^\pm, T(D_i^\pm \cap \partial D))^\Gamma$ is contractible. □

**Lemma 9.10.3.** *There are weak equivalences*

$$G_p^\Gamma(TD_i, T\partial D_i)^\Gamma \ \simeq\ \Sigma G_p^\Gamma(TD_{i-1}, T\partial D_{i-1})^\Gamma$$

*for all $1 \leq i \leq r$.*

*Proof.* Since $TD_{i-1} = TD_i^- \cap TD_i^+$, applying Theorem 9.9.8 gives a homotopy pushout

$$\begin{tikzcd}
G_p^\Gamma(TD_{i-1}, T\partial D_{i-1})^\Gamma \arrow[r] \arrow[d] & G_p^\Gamma(TD_i^+, T(D_i^+ \cap \partial D))^\Gamma \arrow[d] \\
G_p^\Gamma(TD_i^-, T(D_i^- \cap \partial D))^\Gamma \arrow[r] & G_p^\Gamma(TD_i, T\partial D_i)^\Gamma
\end{tikzcd}$$

where $G^\Gamma_p(TD^-_i, T(D^-_i \cap \partial D))^\Gamma$ and $G^\Gamma_p(TD^+_i, T(D^+_i \cap \partial D))^\Gamma$ are contractible by Lemma 9.10.2. This gives the result. $\square$

**Theorem 9.10.4.** *If the action of $\Gamma$ on $Y$ is trivial, there is a weak equivalence*

$$G^\Gamma_p(TD, T\partial D)^\Gamma \longrightarrow \Sigma^r K^{-\infty}(R[\Gamma]).$$

*Proof.* The subsets $TD^*_i$ form a coarse antithetic covering of $D$. Applying Theorem 9.9.10 and using Lemmas 9.10.2 and 9.10.3 gives the result. $\square$

We used the following important property of the theory $G^\Gamma_p$ in the construction of maps $\beta_7$ and $\omega_7$ in section 5.

**Theorem 9.10.5.** *Suppose $\Gamma$ acts on a metric space $Y$ by bounded coarse equivalences, and $Y'$ is an invariant metric subspace. Let $Y_0$ and $Y'_0$ be the same metric spaces but with $\Gamma$ acting trivially. Then there is a weak equivalence*

$$\varepsilon\colon G^\Gamma_p(Y, Y')^\Gamma \xrightarrow{\simeq} G^\Gamma_p(Y_0, Y'_0)^\Gamma.$$

*Proof.* The category $\mathbf{G}^\Gamma_p(Y, Y')$ has the left action by $\Gamma$ induced from the diagonal action on $\Gamma \times Y$. Recall from section 2.5 that an object of $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma$ is determined by an object $F$ of $\mathbf{G}_\Gamma(Y, Y')$ and isomorphisms $\psi(\gamma)\colon F \to \gamma F$ which are of filtration 0 when projected to $\Gamma$.

Given two objects $(F, \{\psi(\gamma)\})$ and $(G, \{\phi(\gamma)\})$, a morphism $\lambda\colon (F, \{\psi(\gamma)\}) \to (G, \{\phi(\gamma)\})$ is given by a morphism $\lambda\colon F \to G$ in $\mathbf{G}_\Gamma(Y, Y')$ such that the collection of morphisms $\gamma\lambda\colon \gamma F \to \gamma G$ satisfies

$$\psi(\gamma) \circ \lambda = \gamma\lambda \circ \phi(\gamma)$$

for all $\gamma$ in $\Gamma$.

Given $(F, \{\psi(\gamma)\})$, define $(F_0, \{\psi_0(\gamma)\})$ by $F_0 = F$ and $\psi_0(\gamma) = \mathrm{id}_F$ for all $\gamma \in \Gamma$. Then $\psi(\gamma)^{-1}$ give a natural isomorphism $Z_F$ from $F$ to $F_0$ and induce an equivalence

$$\zeta\colon G^\Gamma_p(Y, Y')^\Gamma \simeq G^\Gamma_p(Y_0, Y'_0)^\Gamma.$$

Of course, the bound for the isomorphism $\psi(\gamma)^{-1}$ can vary with $\gamma$. $\square$

**Remark 9.10.6.** The same fact is true for the theory $K^\Gamma_p$. More precisely, under the same hypotheses there is a weak equivalence

$$\zeta\colon K^\Gamma_p(Y, Y')^\Gamma \xrightarrow{\simeq} K^\Gamma_p(Y_0, Y'_0)^\Gamma.$$

The theory $K^\Gamma_p$ is insufficient for our purposes compared to $G^\Gamma_p$ in that it lacks the Excision Theorem 9.9.10.

9.11. **Construction and Properties of the Target.** The final goal of the section is to define the category $\mathbf{W}^\Gamma$ announced in the proof of the Main Theorem in section 1.2 and show its required properties. It is also part of the diagram in section 5.6. The construction of this category and formal properties resemble the category of fixed objects $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma$ while the action is not uniquely specified. The description of this category in section 3.2 contained some motivation for the construction.

**Definition 9.11.1.** The category $\mathbf{W}^\Gamma(Y, Y')$ has objects which are sets of data $(\{F_\gamma\}, \alpha, \{\psi_\gamma\})$ where

- $F_\gamma$ is an object of $\mathbf{G}_\Gamma(Y, Y')$ for each $\gamma$ in $\Gamma$,
- $\alpha$ is an action of $\Gamma$ on $Y$ by bounded coarse equivalences,

- $\psi_\gamma$ is an isomorphism $F_e \to F_\gamma$ induced from $\alpha_\gamma$,
- $\psi_\gamma$ has filtration 0 when viewed as a morphism in $\mathbf{U}(\Gamma, \mathbf{U}(Y))$,
- $\psi_e = \mathrm{id}$,
- $\psi_{\gamma_1\gamma_2} = \gamma_1\psi_{\gamma_2}\psi_{\gamma_1}$.

The morphisms $(\{F_\gamma\}, \alpha, \{\psi_\gamma\}) \to (\{F'_\gamma\}, \alpha', \{\psi'_\gamma\})$ are collections $\{\phi_\gamma\}$, where each $\phi_\gamma$ is a morphism $F_\gamma \to F'_\gamma$ in $\mathbf{G}_\Gamma(Y, Y')$, such that the squares

$$\begin{tikzcd} F_e \arrow[r, "\psi_\gamma"] \arrow[d, "\phi_e"'] & F_\gamma \arrow[d, "\phi_\gamma"] \\ F'_e \arrow[r, "\psi'_\gamma"] & F'_\gamma \end{tikzcd}$$

commute for all $\gamma \in \Gamma$.

In order to describe the exact structure on $\mathbf{W}^\Gamma(Y, Y')$, let us consider the case of a single action $\alpha$ on a proper metric space $Y$ by bounded coarse equivalences. Then a subspace $Y'$ is a coarsely invariant subspace. The objects of $\mathbf{G}^\Gamma_p(Y, Y')_\alpha$ are the functors $\theta\colon \boldsymbol{E\Gamma} \to \mathbf{G}_\Gamma(Y, Y')$ such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of $\Gamma$-filtered modules. This category has a left action by $\Gamma$ induced from the diagonal action on $\Gamma \times Y$. A fixed object in $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_\alpha$ can be described as a collection of the following data: an object $F$ of $\mathbf{G}_\Gamma(Y, Y')$ and isomorphisms $\psi(\gamma)\colon F \to \alpha(\gamma)F$ which are of filtration 0 when projected to the factor $\Gamma$.

The exact structure on $\mathbf{W}^\Gamma(Y, Y')$ is induced from that on $\mathbf{G}_\Gamma(Y, Y')$. First we observe that for any action $\alpha$ on $(Y, Y')$ by bounded coarse equivalences, the category $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_\alpha$ is a subcategory of $\mathbf{W}^\Gamma(Y, Y')$. This embedding is realized by sending the object $(F, \psi)$ of $\mathbf{G}_\Gamma(Y, Y')^\Gamma_\alpha$ to $(\{\alpha_\gamma F\}, \alpha, \{\psi(\gamma)\})$. On the morphisms, $E_\alpha(\phi) = \{\alpha_\gamma \phi\}$.

A morphism $\phi$ in $\mathbf{W}^\Gamma(Y, Y')$ is an admissible monomorphism if $\phi_e\colon F \to F'$ is an admissible monomorphism in $\mathbf{G}_\Gamma(Y, Y')$. This of course implies that all structure maps $\phi_\gamma$ are admissible monomorphisms. Similarly, a morphism $\phi$ in $\mathbf{W}^\Gamma(Y, Y')$ is an admissible epimorphism if $\phi_e\colon F \to F'$ is an admissible epimorphism. It is clear that for each bounded coarse action $\alpha$ the inclusion map $E_\alpha$ of $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_\alpha$ in $\mathbf{W}^\Gamma(Y, Y')$ is exact.

**Definition 9.11.2.** The spectrum $w^\Gamma(Y, Y')$ is defined as the $K$-theory spectrum of $\mathbf{W}^\Gamma(Y, Y')$. If $g^\Gamma_p(Y, Y')^\Gamma_\alpha$ stands for the $K$-theory of $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_\alpha$ then we get the induced map of spectra

$$\varepsilon_\alpha\colon g^\Gamma_p(Y, Y')^\Gamma_\alpha \longrightarrow w^\Gamma(Y, Y').$$

Let $\mathbb{R}^k$ be the Euclidean space with the trivial action of $\Gamma$. Then the product $\Gamma \times \mathbb{R}^k$ has the $\Gamma$-action defined by $\gamma(\gamma', x) = (\gamma\gamma', x)$. By using the diagonal actions on $\Gamma \times \mathbb{R}^k \times Y$, one obtains the equivariant categories $\mathbf{G}_{\Gamma\times\mathbb{R}^k}(Y, Y')_\alpha$ and also $\mathbf{G}^{\Gamma,k}_p(Y, Y')_\alpha$ where the objects are the functors $\theta\colon \boldsymbol{E\Gamma} \to \mathbf{G}_{\Gamma\times\mathbb{R}^k}(Y, Y')$ such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of $(\Gamma\times\mathbb{R}^k)$-filtered modules. So there is the category of the fixed objects $\mathbf{G}^{\Gamma,k}_p(Y, Y')^\Gamma_\alpha$. Similarly, there are categories $\mathbf{W}^{\Gamma,k}(Y, Y')$ and the evident exact inclusions

$$E^k\colon \mathbf{G}^{\Gamma,k}_p(Y, Y')^\Gamma_\alpha \longrightarrow \mathbf{W}^{\Gamma,k}(Y, Y').$$

*Notation* 9.11.3. The $K$-theory of $\mathbf{G}_p^{\Gamma,k}(Y,Y')_\alpha^\Gamma$ will be denoted by $g_p^{\Gamma,k}(Y,Y')_\alpha^\Gamma$. The $K$-theory of $\mathbf{W}^{\Gamma,k}(Y,Y')$ will be denoted by $w^{\Gamma,k}(Y,Y')$.

Now the nonconnective delooping of $g_p^\Gamma(Y,Y')^\Gamma$ can be constructed as

$$G_p^\Gamma(Y,Y')_\alpha^\Gamma \;=\; \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\; \Omega^k g_p^{\Gamma,k}(Y,Y')_\alpha^\Gamma.$$

This agrees with the definition in section 2.6 when the action $\alpha$ of $\Gamma$ is the natural free action on the metric space $Y=(T\widetilde{N})^{bdd}$ associated to the universal cover of the normal bundle $\widetilde{N}$ of an embedding of the closed aspherical manifold $B\Gamma$.

**Definition 9.11.4.** We define the nonconnective delooping of $w^\Gamma(Y,Y')$ as

$$W^\Gamma(Y,Y') \;=\; \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\; \Omega^k w^{\Gamma,k}(Y,Y').$$

The exact inclusions $E^k$ induce the map of nonconnective spectra

$$\varepsilon_\alpha\colon G_p^\Gamma(Y,Y')_\alpha^\Gamma \longrightarrow W^\Gamma(Y,Y').$$

In particular, we have the embedding

$$E\colon\ \mathbf{G}_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow \mathbf{W}^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})$$

associated to the natural free action on $\widetilde{N}$ and the map of spectra

$$\varepsilon\colon G_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow W^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})$$

induced by $E$.

The map

$$\varepsilon\colon K_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow W^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})$$

that appears in the Abbreviated Diagram in section 3.4 is the latter map precomposed with the Cartan map

$$\beta_6\colon K_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \longrightarrow G_p^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma.$$

In order to establish commutativity up to homotopy of some parts of the Complete Diagram in section 5.6, we need to consider other embeddings $E_\alpha$ for actions of special importance.

**Definition 9.11.5.** Let

$$E_0\colon\ \mathbf{G}_p^\Gamma((T\widetilde{N})_0^{bdd},(T\partial\widetilde{N})_0^{bdd})^\Gamma \longrightarrow \mathbf{W}^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})$$

denote the embedding for the trivial action on $(T\widetilde{N})^{bdd}$. Similarly, one has

$$E_0\colon\ \mathbf{G}_p^\Gamma((T\widetilde{B})_0^{bdd},(T\partial\widetilde{B})_0^{bdd})^\Gamma \longrightarrow \mathbf{W}^\Gamma((T\widetilde{B})^{bdd},(T\partial\widetilde{B})^{bdd})$$

for the trivial action on $T\widehat{B}$, but also

$$E\colon\ \mathbf{G}_p^\Gamma((T\widetilde{B})^{bdd},(T\partial\widetilde{B})^{bdd})^\Gamma \longrightarrow \mathbf{W}^\Gamma((T\widetilde{B})^{bdd},(T\partial\widetilde{B})^{bdd})$$

induced from the restriction of the natural action of $\Gamma$ by covering transformations. We will use the same notation $\varepsilon$ and $\varepsilon_0$ for the induced maps on nonconnective spectra, it should be clear which map is meant from the context.

The following is a corollary to the proof of Theorem 9.10.5.

**Corollary 9.11.6.** *Consider actions of $\Gamma$ on a metric space $Y$ by bounded coarse equivalences and let $Y'$ be a metric subspace. Let $Y_0$ and $Y'_0$ be the same metric spaces with the trivial action of $\Gamma$. Then there are a natural transformation from the identity functor on $\mathbf{W}^\Gamma(Y,Y')$ to a functor*

$$Z\colon \mathbf{W}^\Gamma(Y,Y') \longrightarrow \mathbf{G}^\Gamma_p(Y_0,Y'_0)^\Gamma,$$

*where each $Z(F)$ is an isomorphism and the induced weak equivalence*

$$\zeta\colon W^\Gamma(Y,Y') \xrightarrow{\ \simeq\ } G^\Gamma_p(Y_0,Y'_0)^\Gamma.$$

*In particular, the maps $\varepsilon$ and $\varepsilon_0$ in Definition* 9.11.5 *are weak equivalences.*

The following homotopy commutative diagram is implicit in the third part of the Complete Diagram in section 5.6.

$$\begin{tikzcd}
G^\Gamma_p((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd})^\Gamma \arrow[rr, "\varepsilon"] \arrow[dr] & & W^\Gamma((T\widetilde{N})^{bdd},(T\partial\widetilde{N})^{bdd}) \arrow[dl, bend left, "\zeta"] \\
& G^\Gamma_p((T\widetilde{N})^{bdd}_0,(T\partial\widetilde{N})^{bdd}_0)^\Gamma \arrow[ur, bend left, "\varepsilon_0"] & \\
G^\Gamma_p((T\widetilde{B})^{bdd},(T\partial\widetilde{B})^{bdd})^\Gamma \arrow[uu, "\xi"] \arrow[rr, "\varepsilon"] \arrow[dr] & & W^\Gamma((T\widetilde{B})^{bdd},(T\partial\widetilde{B})^{bdd}) \arrow[uu, "\xi"'] \arrow[dl, bend left, "\zeta"] \\
& G^\Gamma_p((T\widetilde{B})^{bdd}_0,(T\partial\widetilde{B})^{bdd}_0)^\Gamma \arrow[uu, "\xi_0"] \arrow[ur, bend left, "\varepsilon_0"] \arrow[d, bend left, "\delta"] & \\
& G^\Gamma_p(T\widehat{B},T\partial\widehat{B})^\Gamma \arrow[u, bend left, "\iota_0"] \arrow[uur, bend right, "\iota"'] &
\end{tikzcd}$$

Now the development of localization and excision results for $W^\Gamma(Y,Y')$ is a straightforward extrapolation of those for $G^\Gamma_p(Y,Y')^\Gamma$. There is the familiar construction of the full subcategory $\mathbf{W}^\Gamma(Y,Y')_{<C}$ associated to any subset $C$ of $Y$ which is a Grothendieck subcategory of $\mathbf{W}^\Gamma(Y,Y')$, and there is an exact quotient category $\mathbf{W}^\Gamma(Y,Y')_{>C}$ obtained by *localizing away from $C$*.

Suppose $\mathcal{U}$ is a finite coarse covering of $Y$. We define the homotopy colimit

$$\mathcal{W}^\Gamma(Y,Y')_{<\mathcal{U}} = \underset{\overrightarrow{U_i\in\mathcal{U}}}{\operatorname{hocolim}}\, W^\Gamma(Y,Y')_{<U_i}.$$

The following excision result is proved along the lines of Theorem 2.7.17.

**Theorem 9.11.7.** *Suppose the action of $\Gamma$ on $Y$ is by bounded coarse equivalences. If $\mathcal{U}$ is a finite coarse covering of $Y$ such that the family of all subsets $U$ in $\mathcal{U}$ together with $Y'$ are pairwise coarsely antithetic, then the natural map induced by inclusions*

$$\delta\colon \mathcal{W}^\Gamma(Y,Y')_{<\mathcal{U}} \longrightarrow W^\Gamma(Y,Y')$$

*is a weak equivalence.*

Recall that we defined the category announced in section 1.2 as

$$\mathbf{W}^\Gamma = \mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{>\complement T\widehat{B}}$$

so that $\mathcal{S} = \Omega^{n+k+1}\,\mathbf{W}^\Gamma$. We also introduced in Notation 3.2.5 the compact notation $\mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$ for $\mathbf{W}^\Gamma$. The quotient map induces the map of spectra

$$\beta_8\colon W^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \longrightarrow W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}).$$

By localizing away from the complement of $T\widehat{B}$ in $(T\widetilde{B})^{bdd}$, same conventions can be used to define $\mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd})$ and the map

$$\psi_8\colon W^\Gamma((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}) \longrightarrow W^\Gamma_{T\widehat{B}}((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}).$$

Composition with $\varepsilon$ and $\varepsilon_0$ produces the following extension of the above diagram, also commutative up to homotopy.

$$\begin{tikzcd}
G^\Gamma_p((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})^\Gamma \arrow[rr, "\beta_8\circ\varepsilon"] \arrow[dr] & & W^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}) \\
& G^\Gamma_p((T\widetilde{N})^{bdd}_0, (T\partial\widetilde{N})^{bdd}_0)^\Gamma \arrow[ur, "\beta_8\circ\varepsilon_0"] & \\
G^\Gamma_p((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd})^\Gamma \arrow[uu, "\xi"] \arrow[rr, "\psi_8\circ\varepsilon"] \arrow[dr] & & W^\Gamma_{T\widehat{B}}((T\widetilde{B})^{bdd}, (T\partial\widetilde{B})^{bdd}) \arrow[uu, "\xi"'] \\
& G^\Gamma_p((T\widetilde{B})^{bdd}_0, (T\partial\widetilde{B})^{bdd}_0)^\Gamma \arrow[uu, "\xi_0"] \arrow[ur, "\psi_8\circ\varepsilon_0"] \arrow[d, bend left, "\delta"] & \\
& G^\Gamma_p(T\widehat{B}, T\partial\widehat{B})^\Gamma \arrow[u, bend left, "\iota_0"] \arrow[uur, bend right, "\iota"'] &
\end{tikzcd}$$

Note that $\iota$ is still induced by a faithful inclusion of categories.

Suppose $C' \subset C''$ form a subset triple with $T\widehat{B}$, then we have the inclusion $C' \cup \complement T\widehat{B} \subset C'' \cup \complement T\widehat{B}$. We will use the notation $\mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C^*}$ for the localization of $\mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$ to each $C^* \cup \complement T\widehat{B}$. Then the homotopy fibrations we have been constructing in proofs of excision theorems in $G$-theory promoted to this setting can be written as

$$\mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C'} \longrightarrow \mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C''} \longrightarrow \mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{>C', <C''}$$

Whenever $C' = \emptyset$, we use the compact notation as before, so the quotient category in the localization sequence gives the spectrum

$$\mathbf{W}^\Gamma((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{>C', <C''} = \mathbf{W}^\Gamma_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<C''}$$

The proofs of Theorem 2.7.17 or Theorem 9.11.7 applied verbatim give the following excision result.

**Corollary 9.11.8.** *If $\mathcal{C}$ is a finite coarse covering of $T\widehat{B}$ such that the family of all subsets $C$ in $\mathcal{C}$ together with $T\partial\widehat{B}$ are pairwise coarsely antithetic, then the natural map*

$$\delta\colon \mathcal{W}^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<\mathcal{C}} \longrightarrow W^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})$$

*is a weak equivalence.*

*Applying this to the covering $\mathcal{C} = \{T\widehat{B}^{*}_{i}\}$, gives a weak equivalence*

$$\delta\colon \mathcal{W}^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd})_{<T\widehat{B}^{*}_{i}} \longrightarrow W^{\Gamma}_{T\widehat{B}}((T\widetilde{N})^{bdd}, (T\partial\widetilde{N})^{bdd}).$$

## 10. Bounded $G$-theory of a Group

If $\Gamma$ is viewed as a word-length metric space with respect to a fixed finite set of generators, the objects of the category $\mathbf{BL}^{\Gamma,0}(\Gamma, R)^{\Gamma}$ are finitely generated $R[\Gamma]$-modules. The conventional exact structure in the category $\mathbf{Modf}(R[\Gamma])$ of finitely generated $R[\Gamma]$-modules for a noetherian ring $R$ consists of respectively injective module homomorphisms and surjective homomorphisms with finitely generated kernels. When the group ring $R[\Gamma]$ is itself noetherian, so that $\mathbf{Modf}(R[\Gamma])$ is an abelian category, this coincides with the conventional choice of all injections for admissible monomorphisms and all surjections for admissible epimorphisms. However, all known groups $\Gamma$ with this property are polycyclic-by-finite, and there is a conjecture of Philip Hall that only polycyclic-by-finite groups have noetherian (integral) group rings.

We are going to define a new exact structure on a subcategory $\mathbf{B}(R[\Gamma])$ of $\mathbf{Modf}(R[\Gamma])$ and relate it to the exact category $\mathbf{BL}^{\Gamma,0}(\Gamma, R)^{\Gamma}$ and to the additive category $\mathcal{C}^{\Gamma,0}(\Gamma, R)^{\Gamma}$.

### 10.1. Definitions and Basic Properties.

**Definition 10.1.1.** Given a finitely generated $R[\Gamma]$-module $F$, fix a finite generating set $\Sigma$ and define a $\Gamma$-filtration of the $R$-module $F$ by $F(S) = \langle S\Sigma \rangle_R$, the $R$-submodule of $F$ generated by $S\Sigma$. Let $s(F, \Sigma)$ stand for the resulting $\Gamma$-filtered $R$-module.

Conversely, if $F$ is a $\Gamma$-filtered $R$-module which is $\Gamma$-*equivariant*, in the sense that $f(\gamma S) = \gamma f(S)$ for all $\gamma \in \Gamma$ and $S \subset \Gamma$, then $F$ has the obvious $R[\Gamma]$-module structure.

**Lemma 10.1.2.** *Every $R[\Gamma]$-homomorphism $\phi\colon F \to G$ between finitely generated $R[\Gamma]$-modules is bounded as an $R$-homomorphism between the filtered $R$-modules $s(F, \Sigma_F) \to s(G, \Sigma_G)$ with respect to any choice of the finite generating sets $\Sigma_F$ and $\Sigma_G$.*

*Proof.* Recall that the word-length metric on $\Gamma$ is the path metric induced from the condition that $d(\gamma, \omega\gamma) = 1$ whenever $\gamma \in \Gamma$ and $\omega \in \Omega$. Consider $x \in F(S) = \langle S\Sigma_F \rangle_R$, then

$$x = \sum_{s,\sigma} r_{s,\sigma} s\sigma$$

for a finite collection of pairs $s \in S$, $\sigma \in \Sigma_F$. Since $F(\{e\}) = \langle \Sigma_F \rangle_R$ for the identity element $e$ in $\Gamma$, there is a number $d \geq 0$ such that $\phi F(\{e\}) \subset G(e[d])$. Therefore,

$$\phi(x) = \sum_{s,\sigma} r_{s,\sigma} \phi(s\sigma) = \sum_{s,\sigma} r_{s,\sigma} s\phi(\sigma) \subset \sum_{s \in S} sG(e[d]) \subset G(S[d])$$

because the left translation action by any element $s \in S$ on $e[d]$ in $\Gamma$ is an isometry onto $s[d]$. $\square$

**Corollary 10.1.3.** *Given a finitely generated $R[\Gamma]$-module $F$ and two choices of finite generating sets $\Sigma_1$ and $\Sigma_2$, the filtered $R$-modules $s(F, \Sigma_1)$ and $s(F, \Sigma_2)$ are isomorphic as $\Gamma$-filtered $R$-modules.*

*Proof.* The identity map and its inverse are boundedly controlled as maps between $s(F, \Sigma_1)$ and $s(F, \Sigma_2)$ by Lemma 10.1.2. $\square$

**Corollary 10.1.4.** *Finitely generated $R[\Gamma]$-modules $F$ with filtrations $s(F, \Sigma)$, with respect to arbitrary finite generating sets $\Sigma$, are locally finite and lean. If $s(F, \Sigma)$ is insular and $\Sigma'$ is another finite generating set then $s(F, \Sigma')$ is also insular.*

*Proof.* For a finite subset $S$, the submodule $F(S)$ is generated by the finite set $S\Sigma$. Since $F(x) = \langle x\Sigma \rangle_R$,

$$F(S) = \sum_{x \in \Sigma} \langle x\Sigma \rangle_R = \langle S\Sigma \rangle_R,$$

so $s(F, \Sigma)$ is 0-lean. The second claim follows from Corollary 10.1.3. $\square$

**Definition 10.1.5.** Let $\mathbf{B}(R[\Gamma])$ be the full subcategory of $\mathbf{Modf}(R[\Gamma])$ on $R$-modules $F$ which are lean and insular as filtered modules $s(F, \Sigma)$ with respect to some choice of the finite generating set $\Sigma$. We will refer to objects of $\mathbf{B}(R[\Gamma])$ as *properly generated* $R[\Gamma]$-modules.

Define $\mathbf{B}_\times(R[\Gamma])$ to be the category of pairs $(F, \Sigma)$ with $F$ in $\mathbf{B}(R[\Gamma])$ and $\Sigma$ a finite generating set for $F$. The morphisms are the $R[\Gamma]$-homomorphisms between the modules.

Lemma 10.1.2 shows that the map

$$s\colon \mathbf{B}_\times(R[\Gamma]) \longrightarrow \mathbf{B}(\Gamma, R)$$

described in Definition 10.1.1 is a functor. In fact, it defines a functor

$$s_\Gamma\colon \mathbf{B}_\times(R[\Gamma]) \longrightarrow \mathbf{B}^{\Gamma,0}(\Gamma, R)^\Gamma$$

by interpreting $s_\Gamma(F, \Sigma) = (F, \psi)$ with $F = s(F, \Sigma)$ and $\psi(\gamma)\colon F \to \gamma F$ induced from $s\sigma \mapsto \gamma^{-1} s\sigma$. Since $(\gamma F)(S) = \langle \gamma^{-1}(S)\Sigma \rangle_R$, it follows that $s_\Gamma(F, \Sigma)$ is an object of $\mathbf{B}^{\Gamma,0}(\Gamma, R)^\Gamma$, and $s$ sends all $R[\Gamma]$-homomorphisms to $\Gamma$-equivariant homomorphisms.

**Lemma 10.1.6.** *Let $F \in \mathbf{B}^{\Gamma,0}(\Gamma, R)^\Gamma$ and let $\Sigma$ be a finite generating set for the $R[\Gamma]$-module $F$. Then the identity homomorphism* $\mathrm{id}\colon s_\Gamma(F, \Sigma) \to F$ *is bounded with respect to the induced and the original filtrations of $F$.*

*Proof.* If $\Sigma$ is contained in $F(e[d])$, where $e$ is the identity element in $\Gamma$, then $\gamma\Sigma \subset F(\gamma[d])$ for all $\gamma \in \Gamma$, and

$$s(F, \Sigma)(S) = \langle S\Sigma \rangle_R \subset F(S[d])$$

for all subsets $S \subset \Gamma$. $\square$

Both functors $s$ and $s_\Gamma$ are additive with respect to the obvious additive structure in $\mathbf{B}_\times(R[\Gamma])$ where $(F,\Sigma_F)\oplus(G,\Sigma_G)=(F\oplus G,\Sigma_F\times\Sigma_G)$.

**Definition 10.1.7.** Let the admissible monomorphisms $\phi\colon (F,\Sigma_F)\to(G,\Sigma_G)$ in $\mathbf{B}_\times(R[\Gamma])$ be the injections $\phi\colon F\to G$ of $R[\Gamma]$-modules $\phi$ such that $s(\phi)\colon s(F,\Sigma_F)\to s(G,\Sigma_G)$ is a boundedly bicontrolled homomorphism of $\Gamma$-filtered $R$-modules. This is equivalent to requiring that $s(\phi)$ be an admissible monomorphism in $\mathbf{B}(\Gamma,R)$. Let the admissible epimorphisms be the morphisms $\phi$ such that $s(\phi)$ are admissible epimorphisms in $\mathbf{B}(\Gamma,R)$.

**Proposition 10.1.8.** *The choice of admissible morphisms defines an exact structure on $\mathbf{B}_\times(R[\Gamma])$ such that both $s$ and $s_\Gamma$ are exact functors.*

*Proof.* When checking Quillen's axioms in $\mathbf{B}_\times(R[\Gamma])$, all required universal constructions are performed in $\mathbf{B}(R[\Gamma])$ with the canonical choices of finite generating sets. In particular, $\Sigma$ in the pushout $B\cup_A C$ is the image of the product set $\Sigma_B\times\Sigma_C$ in $B\times C$. The fact that all candidates for admissible morphisms are boundedly bicontrolled in $\mathbf{B}(\Gamma,R)$ or $\mathbf{B}^{\Gamma,0}(\Gamma,R)^\Gamma$ follows from the proof of Theorem 9.1.9. Exactness of $s$ and $s_\Gamma$ is immediate. $\square$

**Definition 10.1.9.** We give $\mathbf{B}(R[\Gamma])$ the minimal exact structure that makes the forgetful functor
$$p\colon\ \mathbf{B}_\times(R[\Gamma])\to\mathbf{B}(R[\Gamma])$$
sending $(F,\Sigma)$ to $F$ an exact functor. In other words, an $R[\Gamma]$-homomorphism $\phi\colon F\to G$ is an admissible monomorphism or epimorphism if for some choice of finite generating sets, $\phi\colon(F,\Sigma_F)\to(G,\Sigma_G)$ is respectively an admissible monomorphism or epimorphism in $\mathbf{B}_\times(R[\Gamma])$.

Corollary 10.1.3 shows that if $\phi\colon F\to G$ is boundedly bicontrolled as a map of filtered $R$-modules $s(F,\Sigma_F)\to s(G,\Sigma_G)$ then it is boundedly bicontrolled with respect to any other choice of finite generating sets, so this structure is well-defined.

*Notation* 10.1.10. The new exact category will be referred to as $\mathbf{B}(R[\Gamma])$, with the corresponding nonconnective $G$-theory spectrum $G^{-\infty}(R[\Gamma])$. We will use notation $G^{-\infty}_\times(R[\Gamma])$ for the nonconnective $G$-theory of $\mathbf{B}_\times(R[\Gamma])$.

Let $(F,\psi)$ be an object of $\mathbf{B}^{\Gamma,0}(\Gamma,R)^\Gamma$. One may think of $\gamma F\in\mathbf{B}(\Gamma,R)$, $\gamma\in\Gamma$, as the module $F$ with a new $\Gamma$-filtration. Now the $R$-module structure $\eta\colon R\to\operatorname{End}F$ induces an $R[\Gamma]$-module structure $\eta(\psi)\colon R[\Gamma]\to\operatorname{End}F$ given by
$$\sum_\gamma r_\gamma\gamma\mapsto\sum_\gamma\eta(r_\gamma)\psi(\gamma)$$
since the sums are taken over a finite subset of $\Gamma$. It is easy to see that sending $(F,\psi)$ to $F$ defines a map
$$\pi\colon\ \mathbf{B}^{\Gamma,0}(\Gamma,R)^\Gamma\longrightarrow\mathbf{B}(R[\Gamma]),$$
so that $p=\pi s_\Gamma$. Notice, however, that in general $\pi$ is not exact as the identity homomorphism in Lemma 10.1.6 is not necessarily an isomorphism.

In the rest of the section we assume $\Gamma$ is torsion-free. The exact functors $p$ and $s_\Gamma$ induce maps in nonconnective $K$-theory
$$G^{-\infty}(R[\Gamma])\xleftarrow{\ p\ }G^{-\infty}_\times(R[\Gamma])\xrightarrow{\ s_\Gamma\ }G^{\Gamma,0}(\Gamma,R)^\Gamma.$$
We claim that both of these maps are weak equivalences.

**Proposition 10.1.11.** *The functor $p$ induces a weak equivalence*

$$G^{-\infty}_{\times}(R[\Gamma]) \simeq G^{-\infty}(R[\Gamma]).$$

*Proof.* This follows from the Approximation Theorem. The two categories are saturated, and $\mathbf{B}_{\times}(R[\Gamma])'$ has a cylinder functor satisfying the cylinder axiom which is constructed as the canonical homotopy pushout with the canonical product basis, see section 1 of [45]. The first condition of the Approximation Theorem is clear. For the second condition, let $(F^{\cdot}_1, \Sigma^{\cdot}_1)$ be a complex in $\mathbf{B}_{\times}(R[\Gamma])$ and let $g\colon F^{\cdot}_1 \to F^{\cdot}_2$ be a chain map in $\mathbf{B}(R[\Gamma])'$. For each $R[\Gamma]$-module $F^i_2$ choose any finite generating set $\Sigma^i_2$, then using $f = g$ and $g' = \mathrm{id}$, we have $g = g'p(f)$. □

**Proposition 10.1.12.** *The functor $s_\Gamma$ induces a weak homotopy equivalence*

$$G^{-\infty}_{\times}(R[\Gamma]) \simeq G^{\Gamma,0}(\Gamma, R)^{\Gamma}.$$

*Proof.* The target category is again saturated and has a cylinder functor satisfying the cylinder axiom. To check condition (2) of the Approximation Theorem, let

$$E^{\cdot}\colon \quad 0 \longrightarrow (E^1, \Sigma_1) \longrightarrow (E^2, \Sigma_2) \longrightarrow \ \dots \ \longrightarrow (E^n, \Sigma_n) \longrightarrow 0$$

be a complex in $\mathbf{B}_{\times}(R[\Gamma])$,

$$(F^{\cdot}, \psi.)\colon \quad 0 \longrightarrow (F^1, \psi_1) \xrightarrow{f_1} (F^2, \psi_2) \xrightarrow{f_2} \ \dots \ \xrightarrow{f_{n-1}} (F^n, \psi_n) \longrightarrow 0$$

be a complex in $\mathbf{B}^{\Gamma,0}(\Gamma, R)^{\Gamma}$, and $g\colon s'_\Gamma(E^{\cdot}) \to (F^{\cdot}, \psi.)$ be a chain map. Each $F^i$ can be thought of as an $R[\Gamma]$-module, and there is a chain complex

$$F^{\cdot}\colon \quad 0 \longrightarrow F^1 \xrightarrow{f_1} F^2 \xrightarrow{f_2} \ \dots \ \xrightarrow{f_{n-1}} F^n \longrightarrow 0$$

in $\mathbf{Modf}(R[\Gamma])$. Choose arbitrary finite generating sets $\Omega_i$ in $F^i$ for all $1 \le i \le n$. Now

$$\pi_\Omega F^{\cdot}\colon \quad 0 \longrightarrow (F^1, \Omega_1) \xrightarrow{f_1} (F^2, \Omega_2) \xrightarrow{f_2} \ \dots \ \xrightarrow{f_{n-1}} (F^n, \Omega_n) \longrightarrow 0$$

is a chain complex in $\mathbf{B}_{\times}(R[\Gamma])$. The chain map $g$ is degree-wise an $R[\Gamma]$-homomorphism, so there is a corresponding chain map $f\colon E^{\cdot} \to \pi_\Omega F^{\cdot}$ which coincides with $g$ on modules. On the other hand, the degree-wise identity gives a chain map $g'\colon s'_\Gamma(\pi_\Omega F^{\cdot}) \to F^{\cdot}$ in $\mathbf{B}^{\Gamma,0}(\Gamma, R)^{\Gamma}$ by Lemma 10.1.6. This $g'$ is a quasi-isomorphism, as required. □

**Corollary 10.1.13.** *Let $\Gamma$ be a finitely generated torsion-free group and $R$ be a noetherian ring. There is a weak equivalence*

$$G^{\Gamma,0}(\Gamma, R)^{\Gamma} \simeq G^{-\infty}(R[\Gamma]).$$

10.2. **Weak Coherence and Finite Asymptotic Dimension.** We will need to adapt and interpret some of the results from [11] where they are stated for equivariant $\Gamma$-modules $F$ over noetherian $R$ which are locally finite, lean, and satisfy a certain weakening of the insularity property. Here we restate and prove those results in the category $\mathbf{B}(R[\Gamma])$ of properly generated modules.

**Definition 10.2.1.** An $R[\Gamma]$-module is *finitely presented* if it is the cokernel of a homomorphism, called *presentation*, between free finitely generated $R[\Gamma]$-modules. If this homomorphism is boundedly bicontrolled, we will call the presentation *admissible.*

The group ring $R[\Gamma]$ is *weakly coherent* if every $R[\Gamma]$-module with an admissible presentation has a resolution by finitely generated projective $R[\Gamma]$-modules. We call the ring $R[\Gamma]$ *weakly regular coherent* if every $R[\Gamma]$-module with an admissible presentation has finite projective dimension and *weakly regular noetherian* if every properly generated $R[\Gamma]$-module has finite projective dimension.

Recall that in general a ring $R$ is said to have *homological dimension* $\leq d$ if every left $R$-module $M$ has a resolution

$$0 \longrightarrow P_d \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where all $P_i$ are projective $R$-modules. If such number $d$ exists then $R$ has *finite homological dimension*.

**Theorem 10.2.2.** *Let $R$ be a noetherian ring and $\Gamma$ be a discrete group of finite asymptotic dimension. Then*

(1) *properly generated $R[\Gamma]$-modules have resolutions by finitely generated free modules,*
(2) *all $R[\Gamma]$-modules with admissible presentations are properly generated.*

*If, in addition, $R$ has finite homological dimension, and there exists a finite $K(\Gamma,1)$-complex, then*

(3) *every properly generated $R[\Gamma]$-module has finite projective dimension, that is, the group ring $R[\Gamma]$ is weakly regular noetherian,*
(4) *the group ring $R[\Gamma]$ is weakly regular coherent,*
(5) *the exact inclusion of bounded categories $\mathcal{C}(\Gamma,R) \to \mathbf{BL}(\Gamma,R)$ induces a weak equivalence $K^{\Gamma,0}(\Gamma,R)^\Gamma \simeq G^{\Gamma,0}(\Gamma,R)^\Gamma$.*

The proof of Theorem 10.2.2 is based on the following characterization of metric spaces of finite asymptotic dimension and a sequence of lemmas.

**Definition 10.2.3.** A map between metric spaces $\phi\colon (M_1,d_1) \to (M_2,d_2)$ is an *asymptotic* or *uniform embedding* if there are two real functions $f$ and $g$ with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ such that

$$f(d_1(x,y)) \leq d_2(\phi(x),\phi(y)) \leq g(d_1(x,y))$$

for all pairs of points $x$, $y$ in $M_1$.

**Theorem 10.2.4** (Dranishnikov [17, 18])**.** *A group $\Gamma$ has finite asymptotic dimension if and only if there is a uniform embedding of $\Gamma$ in a finite product of locally finite simplicial trees.*

**Lemma 10.2.5.** *Let $P$ be a finite product of locally finite simplicial trees, with the product simplicial metric.*

(1) *The kernel of a surjective boundedly bicontrolled homomorphism between properly generated $P$-filtered $R$-modules is properly generated.*
(2) *Every surjective $R[\Gamma]$-homomorphism between properly generated $R[\Gamma]$-modules is boundedly bicontrolled.*
(3) *The cokernel of a boundedly bicontrolled homomorphism of properly generated $P$-filtered $R$-modules is properly generated.*
(4) *If $\phi\colon M_1 \to M_2$ is an injective asymptotic embedding between proper metric spaces then the $M_2$-filtration $F_*(S) = F(\phi^{-1}(S))$ induced from an $M_1$-filtration $F$ is properly generated if and only if $F$ is properly generated.*
(5) *If $\Gamma$ has a uniform embedding in $P$ then the kernel of a surjective $R[\Gamma]$-homomorphism of properly generated $R[\Gamma]$-modules is properly generated. In particular, it is finitely generated.*

*Proof.* (1) The proof of leanness of the kernel is crucial to the whole argument; it proceeds exactly as that of Lemma 2.4 in [11]. To make this precise, we need to explain one difference in assumptions on filtered modules used in [11] and in this paper. The property used in the argument is the insularity property applied only to coarsely antithetic pairs of subsets of the metric space as in Definition 2.7.2. The insularity property in this paper is required for all pairs of subsets. It is clearly stronger than the antithetic insularity. Up to this reinterpretation, the proof of Lemma 2.4 from [11] is valid verbatim in our present setting.

(2) First, we claim that every $R[\Gamma]$-homomorphism $\phi\colon F \to G$ between properly generated $R[\Gamma]$-modules is boundedly controlled. Consider $z \in F(S)$, then $z = \sum r_i z_i$, where $z_i \in F(x_i[D])$ for some $x_i \in S$, and assuming $F$ is $D$-lean. Since $\phi$ is an $R[\Gamma]$-homomorphism, there is a number $b \geq 0$ such that $\phi(z)$ is in $g(x[D+b])$ for all $z \in f(x[D])$ and all $x \in \Gamma$. Then

$$\phi(z) = \sum r_i \phi(z_i) \in \sum g(x_i[D+b]) \subset g(S[D+b]).$$

If $\phi$ is surjective, and $y \in G(S)$, then $y = \sum r_i y_i$ with $y_i \in G(x_i[D])$, $x_i \in G(S)$, assuming $G$ is also $D$-lean. Each $G(x_i[D])$ is a finitely generated $R$-module, so there is a constant $C \geq 0$ and $z_i \in F(x_i[D+C])$ so that $\phi(z_i) = y_i$. Now $z = \sum r_i z_i$ is in $F(S[D+C])$.

(3) The image is lean by part (1) and insular by part (2) of Lemma 9.1.16. Now the cokernel is lean by part (1) and insular by part (4) of Lemma 9.1.16. Local finiteness follows from the noetherian property of $R$.

(4) Notice that the fact that $d_2(\phi(x), \phi(y)) \leq g(d_1(x,y))$ implies

$$x[D] \subset \phi^{-1}(\phi(x)[g(D)])$$

for all $D \geq 0$. Suppose $F$ is $D$-lean, then given $y \in F_*(S) = F(\phi^{-1}(S))$ and

$$y \in \sum_{x \in \phi^{-1}(S)} F(x[D]),$$

we have

$$\begin{aligned} y &\in \sum_{x \in \phi^{-1}(S)} F(\phi^{-1}(\phi(x)[g(D)])) \\ &= \sum_{x \in \phi^{-1}(S)} F_*(\phi(x)[g(D)]) \\ &\subset \sum_{z \in S} F_*(z[g(D)]). \end{aligned}$$

For insularity, notice that the fact that $f(d_1(x,y)) \leq d_2(\phi(x), \phi(y))$ implies

$$(\phi^{-1}(S))[d] \subset \phi^{-1}(S[f(d)])$$

for all $d \geq 0$. If

$$y \in F_*(S) \cap F_*(T) = F(\phi^{-1}(S)) \cap F(\phi^{-1}(T))$$

then, assuming $F$ is $d$-insular,

$$\begin{aligned} y &\in F((\phi^{-1}(S))[d] \cap (\phi^{-1}(T))[d]) \\ &\subset F(\phi^{-1}(S[f(d)]) \cap \phi^{-1}(T[f(d)])) \\ &= F(\phi^{-1}(S[f(d)] \cap T[f(d)])) \\ &= F_*(S[f(d)] \cap T[f(d)]). \end{aligned}$$

So $F_*$ is $g(D)$-lean and $f(d)$-insular. The sufficiency half of the argument is left to the reader.

(5) Suppose $i\colon \Gamma \to P$ is the given asymptotic embedding and $\phi\colon F_1 \to F_2$ is the given homomorphism between two properly generated $R[\Gamma]$-modules. Now $\phi$ can be thought of as an $R[\Gamma]$-homomorphism between $P$-filtered $R$-modules $F_{1*}$ and $F_{2*}$ defined by $F_*(S) = F(i^{-1}(S))$. By part (4), $F_*$ is properly generated if and only if $F$ is properly generated. When $\phi$ is surjective, it is boundedly bicontrolled by part (2). The rest follows from part (1). $\square$

*Proof of Theorem* 10.2.2. (1) Given a properly generated $R[\Gamma]$-module $F$, let $F_0$ be the free $R[\Gamma]$-module on the finite generating set $\Sigma$ of $F$. We can view it as a properly generated $\Gamma$-filtered $R$-module with the canonical filtration induced from the product generating set $\Gamma \times \Sigma$. Then the surjection of $R$-modules $\pi\colon F_0 \to F$ induced by the map of generating sets $\Gamma \times \Sigma \to \Gamma\Sigma$ given by $(\gamma, \sigma) \mapsto \gamma\sigma$ is boundedly bicontrolled by part (2) of Lemma 10.2.5. The kernel $K_1 = \ker(\pi)$ is properly generated by Theorem 10.2.4 and part (5) of Lemma 10.2.5. Construct a free finitely generated $R[\Gamma]$-module $F_1$ with a boundedly bicontrolled projection $\pi_1\colon F_1 \to K_1$ just as above. This shows that $F$ is finitely presented as the quotient of the composition $d_1 = i_1\pi_1$ which is boundedly bicontrolled by part (1) of Lemma 9.1.7. This construction also inductively gives a resolution

$$\ldots \longrightarrow F_i \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0$$

by finitely generated free $R[\Gamma]$-modules.

(2) follows directly from part (3) of Lemma 10.2.5.

(3) We examine the resolution of $F$ from part (1). Since $\Gamma$ has a finite $K(\Gamma, 1)$ complex, $\Gamma$ belongs to Kropholler's hierarchy LH$\mathcal{F}$. By Theorem A of [34], there is $d \geq 0$ such that the kernel $K_d$ of $F_d \to F_{d-1}$ is isomorphic to a direct summand of a polyelementary module. When $\Gamma$ is torsion-free, the elementary modules of the form $U \otimes_R R[\Gamma]$, where $U$ is projective over $R$, are themselves projective over $R[\Gamma]$. So the polyelementary modules are also projective. Now we have a finite resolution

$$0 \longrightarrow K_d \longrightarrow F_d \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0$$

where $K_d$ is a finitely generated projective $R[\Gamma]$-module.

(4) follows from parts (2) and (3).

(5) By Quillen's Resolution theorem and part (3), $G^{-\infty}(R[\Gamma])$ is weakly equivalent to $K^{-\infty}(R[\Gamma])$. The equivalence $K^{\Gamma,0}(\Gamma, R)^{\Gamma} \simeq G^{\Gamma,0}(\Gamma, R)^{\Gamma}$ follows from Corollary 10.1.13 and Theorem 2.5.4. $\square$

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