K-THEORY WITH FIBRED CONTROL

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Abstract. Controlled methods proved to be very effective in the study of assembly maps in algebraic K-theory. For the questions concerning surjectivity of these maps in the context of geometries that have no immanent nonpositive curvature, the setting needs to be enlarged to bounded G-theory with fibred control. We set up the natural framework for this theory. As should be expected, the G-theory has better excision properties. The main result illustrates a fact that has no analogue in bounded K-theory. It is one of the major technical tools used in our work on the Borel Conjecture about topological rigidity of aspherical manifolds. The general framework we develop will be useful for other applications to coarse bundles and stacks.

Contents

1. Introduction

Bounded G-theory is an imperfect name for the theory designed to generalize the bounded K-theory constructed by E.K. Pedersen and C. Weibel in [13]. Bounded

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K-theory, or more generally *controlled algebra*, has been used extensively in the most effective work on the Novikov, the Borel, and the Farell–Jones conjectures about assembly maps in algebraic K-theory. The two-fold generaliation we study in this paper appears in the authors' work [7] on surjectivity of the integral assembly maps.

Bounded K-theory associates a nonconnective spectrum $K^{-\infty}(M, R)$ to a proper metric space M and a ring R. When M is bounded in diameter, the spectrum is equivalent to the nonconnective K-theory of the ring $K^{-\infty}(R)$. Similarly, bounded G-theory is a spectrum $G^{-\infty}(M, R)$ associated to M and a noetherian ring R. When M is bounded, the spectrum is equivalent to the G-theory of the ring $G^{-\infty}(R)$. This fact motivates the convenient name and notation. Now in some way the name may be misleading.

Suppose a finitely generated group Γ is given the word metric relative to a chosen generating set. It is a proper metric space, so there is the corresponding K-theory spectrum $K^{-\infty}(\Gamma, R)$. The left multiplication action of Γ on itself induces an action on $K^{-\infty}(\Gamma, R)$. There is a better version of such spectrum that has the desired equivariant properties; we will suppress the distinction in this introduction. In particular, the fixed point spectrum $K^{-\infty}(\Gamma, R)^{\Gamma}$ is equivalent to the K-theory of the group ring $K^{-\infty}(R[\Gamma])$ whenever Γ has no torsion. It is natural to treat the fixed point spectrum $G^{-\infty}(\Gamma, R)^{\Gamma}$ as a candidate for the G-theory of the (not necessarily noetherian) group ring $R[\Gamma]$. There is no hard convention for the choice of an exact structure on finitely generated $R[\Gamma]$-modules when $R[\Gamma]$ is not noetherian. The most common choice consists of simply the short exact sequences of finitely generated $R[\Gamma]$-modules. We feel justified to use the notation $G^{-\infty}(R[\Gamma])$ for $G^{-\infty}(\Gamma, R)^{\Gamma}$. However, we know from [4] that in most common geometric situations this spectrum is much closer to $K^{-\infty}(R[\Gamma])$ than to the common G-theory of $R[\Gamma]$.

The reason bounded G-theory is useful is in its better excision properties than those of bounded K-theory. These excision properties survive in situations where the traditional Karoubi filtration techniques, which are successful for $K^{-\infty}(M, R)$ and its many variants, are no longer available. The Karoubi filtrations proved sufficient for the study of the Novikov conjecture and the Farrell–Jones conjectures, so the main immediate use of the new theory is in the study of surjectivity of the assembly map in geometric circumstances that go beyond groups with immanent nonpositive curvature properties.

Our main application of the (fibrewise) excision theorem for bounded G-theory, stated in Theorem 4.2.6, is not available in bounded K-theory. If its analogue was true for bounded K-theory then our argument in [7] would show that the Loday assembly map is an isomorphism for all rings R and all geometrically finite groups of finite asymptotic dimension. There are known counterexamples to this statement. Instead, the main theorem of our paper [7] puts an additional condition on the ring R requiring it to have finite homological dimension. Under this condition, we know from [4] that $K^{-\infty}(R[\Gamma])$ and $G^{-\infty}(R[\Gamma])$ are weakly equivalent.

The Fibrewise Excision Theorem 4.2.6 that is the main result of the second half of this paper is stated in the context of the fibred version of controlled G-theory. This is a generalization of controlled algebra which we believe is useful beyond our application. It is most immediately the natural theory to consider for bundles on manifolds, or more generally stacks, or even more generally coarse analogues of these that have already appeared in the literature [21].

2. Fibred Bounded K-theory

2.1. **Review of Bounded K-theory.** We will start by reviewing the construction of bounded K-theory over proper metric spaces. Bounded K-theory was introduced by Pedersen [12] and Pedersen–Weibel [13]. In this paper we use a refined version of controlled G-theory that is required for our applications. We will work through explicit details that do not allow direct reference to [ ] where controlled G-theory was first defined. For the convenience of the reader, we will also include details that reappear in the fibred context in more intricate forms.

Recall that a *proper metric space* is a metric space where closed bounded subsets are compact. Fix an associative ring R with unity.

2.1.1. **Definition.** Let $\mathcal{C}(M, R)$ be the additive category of *geometric R-modules* associated to a proper metric space M and an associative ring R defined in Pedersen [12], Pedersen–Weibel [13]. The objects are functions $F \colon M \to \mathbf{Free}_{fg}(R)$ which are locally finite assignments of free finitely generated R-modules F_m to points m of M. The local finiteness condition requires precisely that for any bounded subset $S \subset M$ the restriction of F to S has finitely many nonzero values. Let d be the distance function in M. The morphisms in $\mathcal{C}(M, R)$ are the R-linear homomorphisms

$$\phi \colon \bigoplus_{m \in M} F_m \longrightarrow \bigoplus_{n \in M} G_n$$

with the property that the components $F_m \to G_n$ are zero for $d(m, n) > D$ for some fixed real number $D = D(\phi) \geq 0$.

The associated bounded K-theory spectrum is denoted by $K(M, R)$, or $K(M)$ when the choice of ring R is implicit, and called the *bounded K-theory* of M.

2.1.2. **Remark.** There is an ambiguity present in Definition 2.1.1. It is possible that the same finitely generated free module is assigned to different points in M. To repair the issue one should think of the objects in $\mathcal{C}(M, R)$ as triples $(\mathcal{F}, F, \iota)$. Here $\mathcal{F}$ is a free R-module, not necessarily finitely generated. The locally finite function $F \colon M \to \mathbf{Free}_{fg}(R)$ is to the category of finitely generated free R-modules, as before. The ι is an isomorphism $\iota \colon \bigoplus_{m \in M} F(m) \to \mathcal{F}$.

To simplify the notation, we may refer to $(\mathcal{F}, F, \iota)$ as simply F. When we do so, F stands for the module $\mathcal{F}$ together with the *stalks* F_m which are the values of the function $F(m)$ or equivalently the submodules $\iota(f(m))$ of $\mathcal{F}$.

Another description of the bounded category given in [3]. Let $\mathcal{B}(M, R)$ be the category with objects which are triples $(\mathcal{F}, B, \phi)$, where $\mathcal{F}$ is a free R-module, B is a free basis of $\mathcal{F}$, and $\phi \colon B \to M$ is a labeling map. The assumptions on ϕ are that (1) its image is locally finite, (2) it is cofinite, in the sense that its point-preimages are finite. The morphisms in $\mathcal{B}(M, R)$ between $(\mathcal{F}, B, \phi)$ and $(\mathcal{G}, C, \psi)$ are R-linear homomorphisms $f \colon \mathcal{F} \to \mathcal{G}$ such that there is a number $d \geq 0$ with the property that, when for any $b \in B$ the image $f(b)$ is written as a (finite) sum $\sum_{c \in C} k_c c$, the subset $\{\psi(c)\}$ is contained entirely inside the metric ball $\phi(b)[d]$ of radius d about $\phi(b)$. There is an additive equivalence between $\mathcal{B}(M, R)$ and $\mathcal{C}(M, R)$. To see that, define F_m as the R-submodule of $\mathcal{F}$ freely generated by $\phi^{-1}(m)$. It is clear that bounded homomorphisms are also bounded in $\mathcal{C}(M, R)$. Given F, $(\mathcal{F}, B, \phi)$ is constructed by selecting a basis in each F_x and defining B to be the union of these bases. The map ϕ sends a basis element b to m if $b \in F_m$.

Assuming this translation, we will feel free to quote facts about $\mathcal{C}(M,R)$ from any of these sources.

A map between proper metric spaces is *proper* if the preimage of a bounded subset of Y is a bounded subset of X. A map $f\colon M_1 \to M_2$ is called *eventually Lipschitz* if, for some number $k \geq 0$ and large enough $s \geq 0$, $d_X(x_1,x_2) \leq s$ implies $d_Y(f(x_1),f(x_2)) \leq ks$.

According to [14], the construction $K(M,R)$ becomes a functor in the first variable on the category of proper metric spaces and proper eventually Lipschitz maps.

2.1.3. **Definition.** On objects of $\mathcal{C}(M_1,R)$,

$$(f_* F)_y = \bigoplus_{z \in f^{-1}(y)} F(z).$$

The proper condition ensures that f_*F is an object of $\mathcal{C}(M_2,R)$. The eventually Lipschitz condition ensures that morphisms are mapped to morphisms of $\mathcal{C}(M_2,R)$.

Examples of proper eventually Lipschitz maps of immediate interest to us are inclusions of metric spaces.

The main result of Pedersen–Weibel [13] is the delooping theorem which can be stated as follows.

2.1.4. **Theorem** (Nonconnective Delooping of Bounded K-theory)**.** *Given a proper metric space M and the standard Euclidean metric on the real line $\mathbb{R}$, the natural inclusion $M \to M \times \mathbb{R}$ induces isomorphisms $K_n(M) \simeq K_{n-1}(M \times \mathbb{R})$ for all integers $n > 1$. If one defines the spectrum*

$$K^{-\infty}(M,R) \;=\; \underset{\overrightarrow{k}}{\operatorname{hocolim}}\, \Omega^k K(M \times \mathbb{R}^k),$$

then the stable homotopy groups of $K^{-\infty}(\text{point},R)$ coincide with the algebraic K-groups of R in positive dimensions and with the Bass negative K-theory of R in negative dimensions.

When we refer to the nonconnective K-theory of the ring R, we will mean this spectrum $K^{-\infty}(R) = K^{-\infty}(\text{point},R)$.

Suppose U is a subset of M. Let $\mathcal{C}(M,R)_{<U}$ denote the full subcategory of $\mathcal{C}(M,R)$ on the objects F with $F_m = 0$ for all points $m \in M$ with $d(m,U) \leq D$ for some fixed number $D > 0$ specific to F. This is an additive subcategory of $\mathcal{C}(M,R)$ with the associated K-theory spectrum $K^{-\infty}(M,R)_{<U}$. Similarly, if U and V are a pair of subsets of M, then there is the full additive subcategory $\mathcal{C}(M,R)_{<U,V}$ of F with $F_m = 0$ for all m with $d(m,U) \leq D_1$ and $d(m,V) \leq D_2$ for some numbers $D_1, D_2 > 0$. It is easy to see that $\mathcal{C}(M,R)_{<U}$ is in fact equivalent to $\mathcal{C}(U,R)$.

The following theorem is the basic computational device in bounded K-theory.

2.1.5. **Theorem** (Bounded Excision [3])**.** *Given a proper metric space M and a pair of subsets U, V of M, there is a homotopy pushout diagram*

$$\begin{tikzcd}
K^{-\infty}(M)_{<U,V} \arrow[r] \arrow[d] & K^{-\infty}(U) \arrow[d] \\
K^{-\infty}(V) \arrow[r] & K^{-\infty}(M)
\end{tikzcd}$$

In order to be able to restate the Bounded Excision Theorem in a more intrinsic form, we need to restrict to a special class of coverings.

2.1.6. **Definition.** A pair of subsets S, T of a proper metric space M is called *coarsely antithetic* if S and T are proper metric subspaces with the subspace metric and for each number $K > 0$ there is a number $K' > 0$ so that

$$S[K] \cap T[K] \subset (S \cap T)[K'].$$

Examples of coarsely antithetic pairs include any two nonvacuously intersecting closed subsets of a simplicial tree as well as complementary closed half-spaces in a Euclidean space.

2.1.7. **Corollary.** *If U and V is a coarsely antithetic pair of subsets of M which form a cover of M, then the commutative square*

$$\begin{array}{ccc} K^{-\infty}(U \cap V) & \longrightarrow & K^{-\infty}(U) \\ \downarrow & & \downarrow \\ K^{-\infty}(V) & \longrightarrow & K^{-\infty}(M) \end{array}$$

is a homotopy pushout.

Given a proper metric space X with a free left Γ-action by isometries, one has the equivariant K-theory K^{Γ} associated to X and a ring R from [3]. We wish to construct a version of the equivariant bounded K-theory which applies to actions by coarse equivalences and extends to the equivariant bounded G-theory.

Suppose we are given X with a free left Γ-action by isometries. There is a natural action of Γ on the geometric modules $\mathcal{C}(X, R)$ and therefore on $K(X, R)$. A different equivariant bounded K-theory with useful fixed point spectra is constructed as follows.

2.1.8. **Definition.** Let $\boldsymbol{E\Gamma}$ be the category with the object set Γ and the unique morphism $\mu\colon \gamma_1 \to \gamma_2$ for any pair γ_1, $\gamma_2 \in \Gamma$. There is a left Γ-action on $\boldsymbol{E\Gamma}$ induced by the left multiplication in Γ.

If $\mathcal{C}$ is a small category with left Γ-action, then the functor category $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is a category with the left Γ-action given on objects by

$$\gamma(F)(\gamma') = \gamma F(\gamma^{-1}\gamma')$$

and

$$\gamma(F)(\mu) = \gamma F(\gamma^{-1}\mu).$$

It is always nonequivariantly equivalent to $\mathcal{C}$. The subcategory of equivariant functors and equivariant natural transformations in $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is the fixed subcategory $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})^{\Gamma}$ known as the *lax limit* of the action of Γ.

According to Thomason [18], the objects of $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})^{\Gamma}$ can be thought of as pairs (F, ψ) where $F \in \mathcal{C}$ and ψ is a function on Γ with $\psi(\gamma) \in \mathrm{Hom}(F, \gamma F)$ such that

$$\psi(1) = 1 \quad \text{and} \quad \psi(\gamma_1\gamma_2) = \gamma_1\psi(\gamma_2)\psi(\gamma_1).$$

These conditions imply that $\psi(\gamma)$ is always an isomorphism. The set of morphisms $(F,\psi) \to (F',\psi')$ consists of the morphisms $\phi\colon F \to F'$ in $\mathcal{C}$ such that the squares

$$\begin{tikzcd} F \arrow[r, "\psi(\gamma)"] \arrow[d, "\phi"'] & \gamma F \arrow[d, "\gamma\phi"] \\ F' \arrow[r, "\psi'(\gamma)"] & \gamma F' \end{tikzcd}$$

commute for all $\gamma \in \Gamma$.

In order to specialize to the case of $\mathcal{C} = \mathcal{C}(X,R)$, notice that $\mathcal{C}(X,R)$ contains the family of isomorphisms ϕ such that ϕ and ϕ^{-1} are bounded by 0. We will express this property by saying that the filtration of ϕ is 0 and writing $\mathrm{fil}(\phi) = 0$. The full subcategory of functors $\theta\colon \boldsymbol{E\Gamma} \to \mathcal{C}(X,R)$ such that $\mathrm{fil}\,\theta(f) = 0$ for all f is invariant under the Γ-action.

2.1.9. *Notation.* We will use the notation

$$\mathcal{C}^{\Gamma}(X,R) = \mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C}(X,R))$$

for the equivariant category as described in Definition 2.1.8.

2.1.10. **Definition.** Let $\mathcal{C}^{\Gamma,0}(X,R)$ be the equivariant full subcategory of $\mathcal{C}^{\Gamma}(X,R)$ on the functors sending all morphisms of $\boldsymbol{E\Gamma}$ to filtration 0 maps. We define $K^{\Gamma,0}(X,R)$ to be the nonconnective delooping of the K-theory of the symmetric monoidal category $\mathcal{C}^{\Gamma,0}(X,R)$.

The fixed points of the induced Γ-action on $K^{\Gamma,0}(X,R)$ is the nonconnective delooping of the K-theory of $\mathcal{C}^{\Gamma,0}(X,R)^{\Gamma}$. This is the full subcategory of $\mathcal{C}^{\Gamma}(X,R)^{\Gamma}$ on the objects (F,ψ) with $\mathrm{fil}\,\psi(\gamma) = 0$ for all $\gamma \in \Gamma$.

One of the main properties of the functor $K^{\Gamma,0}$ is the following.

2.1.11. **Theorem** (Corollary VI.8 of [3])**.** *If X is a proper metric space and Γ acts on X freely, properly discontinuously, cocompactly by isometries, there are weak equivalences*

$$K^{\Gamma,0}(X,R)^{\Gamma} \;\simeq\; K^{-\infty}(X/\Gamma, R[\Gamma]) \;\simeq\; K^{-\infty}(R[\Gamma]).$$

This theorem applies in two specific cases of interest to us. One case is of the fundamental group Γ of an aspherical manifold acting on the universal cover X by covering transformations. The second is of Γ acting on itself, as a word-length metric space, by left multiplication.

2.1.12. **Remark.** The theory $K^{\Gamma,0}(X,R)$ may very well differ from $K^{\Gamma}(X,R)$. According to Theorem 2.1.11, the fixed point category of the original bounded equivariant theory is, for example, the category of free $R[\Gamma]$-modules when $X = \Gamma$ with the word-length metric. However, $\mathcal{C}^{\Gamma}(\Gamma,R)^{\Gamma}$ will include the R-module with a single basis element and equipped with trivial Γ-action, which is not free.

2.2. **Definitions and Basic Properties.** When M is the product of two proper metric spaces $X \times Y$ with the metric

$$d_{\max}((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\},$$

one has the bounded category of geometric R-modules $\mathcal{C}(X \times Y, R)$ of Pedersen–Weibel as in Definition 2.1.1. The isomorphism category has a symmetric monoidal structure, and the associated K-theory spectrum is $K(X \times Y, R)$.

To describe another category associated to $X \times Y$, fix a point x_0 in X. The new category $\mathcal{C}_X(Y)$ has the same objects as $\mathcal{C}(X \times Y, R)$ but a weaker control condition on the morphisms.

2.2.1. *Notation.* For a subset $S \subset X$ and a real number $R \geq 0$, $S[D]$ stands for the metric D-enlargement $\{x \in X \mid d(x, S) \leq D\}$. In this notation, the metric ball of radius R centered at x is $\{x\}[R]$ or simply $x[R]$.

For any function $f\colon [0, +\infty) \to [0, +\infty)$ and a real number $D \geq 0$, define

$$N(D, f)(x, y) = x[D] \times y[f(d(x, x_0))],$$

the (D, f)-neighborhood of (x, y) in $X \times Y$. A homomorphism $\phi\colon F \to G$ is called (D, f)-*bounded* if the components $F_{(x,y)} \to G_{(x',y')}$ are zero maps for (x', y') outside of the (D, f)-neighborhood of (x, y). These are the morphisms of $\mathcal{C}_X(Y)$.

The K-theory spectrum $K_X(Y)$ is the spectrum associated to the isomorphism category of $\mathcal{C}_X(Y)$.

2.2.2. **Remark.** The definition of the category $\mathcal{C}_X(Y)$ is independent of the choice of x_0 in X.

2.2.3. **Remark.** It follows from Example 1.2.2 of Pedersen–Weibel [13] that in general $\mathcal{C}_X(Y)$ is not isomorphic to $\mathcal{C}(X \times Y, R)$.

The point is that the proper generality of that work, explained in [14], included the setting of a general additive category $\mathcal{A}$ embedded in a cocomplete additive category, generalizing the free finitely generated R-modules in the category of all free R-modules. All of the excision results of Pedersen and Weibel hold in the context of $\mathcal{C}(X, \mathcal{A})$. The category $\mathcal{C}_X(Y)$ is simply the category $\mathcal{C}(X, \mathcal{A})$, where $\mathcal{A} = \mathcal{C}(Y, R)$.

The difference between $\mathcal{C}_X(Y)$ and $\mathcal{C}(X \times Y, R)$ is made to disappear in [13] by making $\mathcal{C}(Y, R)$ "remember the filtration" of morphisms when viewed as a filtered additive category with

$$\mathrm{Hom}_D(F, G) = \{\phi \in \mathrm{Hom}(F, G) \mid \mathrm{fil}(\phi) \leq D\}.$$

Identifying a small category with its set of morphisms, one can think of the bounded category as

$$\mathcal{C}(Y, R) = \underset{\overrightarrow{D \in \mathbb{R}}}{\mathrm{colim}}\, \mathcal{C}_D(Y, R),$$

where $\mathcal{C}_D(Y, R) = \mathrm{Hom}_D(\mathcal{C}(Y, R))$ is the collection of all $\mathrm{Hom}_D(F, G)$. Now we have

$$\mathcal{C}(X \times Y, R) = \underset{\overrightarrow{D \in \mathbb{R}}}{\mathrm{colim}}\, \mathcal{C}(X, \mathcal{C}_D(Y, R)).$$

2.2.4. **Remark.** There is certainly an exact embedding $\iota\colon \mathcal{C}(X \times Y, R) \to \mathcal{C}_X(Y)$ inducing the map of K-theory spectra $K(\iota)\colon K(X \times Y, R) \to K_X(Y)$.

Let us recall the notion of Karoubi filtration in additive categories. The details can be found in Cardenas–Pedersen [2].

2.2.5. **Definition.** An additive category $\mathcal{C}$ is *Karoubi filtered* by a full subcategory $\mathcal{A}$ if every object C has a family of decompositions $\{C = E_\alpha \oplus D_\alpha\}$ with $E_\alpha \in \mathcal{A}$ and $D_\alpha \in \mathcal{C}$, called a *Karoubi filtration* of C, satisfying the following properties.

- For each C, there is a partial order on Karoubi decompositions such that $E_\alpha \oplus D_\alpha \leq E_\beta \oplus D_\beta$ whenever $D_\beta \subset D_\alpha$ and $E_\alpha \subset E_\beta$.

- Every map $A \to C$ factors as $A \to E_\alpha \to E_\alpha \oplus D_\alpha = C$ for some α.
- Every map $C \to A$ factors as $C = E_\alpha \oplus D_\alpha \to E_\alpha \to A$ for some α.
- For each pair of objects C and C' with the corresponding filtrations $\{E_\alpha \oplus D_\alpha\}$ and $\{E'_\alpha \oplus D'_\alpha\}$, the filtration of $C \oplus C'$ is the family $\{C \oplus C' = (E_\alpha \oplus E'_\alpha) \oplus (D_\alpha \oplus D'_\alpha)\}$.

A morphism $f \colon C \to D$ is *$\mathcal{A}$-zero* if f factors through an object of $\mathcal{A}$. Define the *Karoubi quotient* $\mathcal{C}/\mathcal{A}$ to be the additive category with the same objects as $\mathcal{C}$ and morphism sets $\mathrm{Hom}_{\mathcal{C}/\mathcal{A}}(C, D) = \mathrm{Hom}(C, D)/\{\mathcal{A}\text{–zero morphisms}\}$.

The following is the main theorem of Cardenas–Pedersen [2].

2.2.6. **Theorem** (Fibration Theorem)**.** *Suppose $\mathcal{C}$ is an $\mathcal{A}$-filtered category, then there is a homotopy fibration*

$$K(\mathcal{A}^{\wedge K}) \longrightarrow K(\mathcal{C}) \longrightarrow K(\mathcal{C}/\mathcal{A}).$$

Here $\mathcal{A}^{\wedge K}$ is a certain subcategory of the idempotent completion of $\mathcal{A}$ with the same positive K-theory as $\mathcal{A}$.

We will apply this theorem to a variety of bounded categories.

2.2.7. *Notation.* Let

$$\begin{aligned} \mathcal{C}_k &= \mathcal{C}_X(Y \times \mathbb{R}^k), \\ \mathcal{C}_k^+ &= \mathcal{C}_X(Y \times \mathbb{R}^{k-1} \times [0, +\infty)), \\ \mathcal{C}_k^- &= \mathcal{C}_X(Y \times \mathbb{R}^{k-1} \times (-\infty, 0]). \end{aligned}$$

We will also use the notation

$$\begin{aligned} \mathcal{C}_k^{<+} &= \underset{D \geq 0}{\underrightarrow{\mathrm{colim}}}\ \mathcal{C}_X(Y \times \mathbb{R}^{k-1} \times [-D, +\infty)), \\ \mathcal{C}_k^{<-} &= \underset{D \geq 0}{\underrightarrow{\mathrm{colim}}}\ \mathcal{C}_X(Y \times \mathbb{R}^{k-1} \times (-\infty, D]), \\ \mathcal{C}_k^{<0} &= \underset{D \geq 0}{\underrightarrow{\mathrm{colim}}}\ \mathcal{C}_X(Y \times \mathbb{R}^{k-1} \times [-D, D]). \end{aligned}$$

It is easy to see that $\mathcal{C}_k$ is $\mathcal{C}_k^{<-}$-filtered and that $\mathcal{C}_k^{<+}$ is $\mathcal{C}_k^{<0}$-filtered. There are also isomorphisms $\mathcal{C}_k^{<0} \cong \mathcal{C}_{k-1}$, $\mathcal{C}_k^{<-} \cong \mathcal{C}_k^-$, and $\mathcal{C}_k/\mathcal{C}_k^{<-} \cong \mathcal{C}_k^{<+}/\mathcal{C}_k^{<0}$. By Theorem 2.2.6, the commutative diagram

$$\begin{tikzcd} K((\mathcal{C}_k^{<0})^{\wedge K}) \arrow[r] \arrow[d] & K(\mathcal{C}_k^{<+}) \arrow[r] \arrow[d] & K(\mathcal{C}_k^{<+}/\mathcal{C}_k^{<0}) \arrow[d, "\cong"] \\ K((\mathcal{C}_k^{<-})^{\wedge K}) \arrow[r] & K(\mathcal{C}_k) \arrow[r] & K(\mathcal{C}_k/\mathcal{C}_k^{<-}) \end{tikzcd}$$

where all maps are induced by inclusions, is in fact a map of homotopy fibrations. The categories $\mathcal{C}_k^{<+}$ and $\mathcal{C}_k^{<-}$ are *flasque*, that is, possess an endofunctor Sh such that $\mathrm{Sh}(F) \cong F \oplus \mathrm{Sh}(F)$, which can be seen by the usual Eilenberg swindle argument. Therefore $K(\mathcal{C}_k^{<+})$ and $K(\mathcal{C}_k^{<-})$ are contractible. This gives a map $K(\mathcal{C}_{k-1}) \to \Omega K(\mathcal{C}_k)$ which induces isomorphisms of K-groups in positive dimensions.

2.2.8. **Definition.** We define the nonconnective spectrum

$$K_X^{-\infty}(Y) \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\, \Omega^k K(\mathcal{C}_k).$$

If Y is the single point space then the delooping $K_X^{-\infty}(\text{point})$ is clearly equivalent to the nonconnective delooping $K^{-\infty}(X, R)$ of Pedersen–Weibel, reviewed in Theorem 2.1.4, via the map

$$K(\iota)\colon\ K^{-\infty}(X \times \text{point}, R) \longrightarrow K_X^{-\infty}(\text{point}).$$

2.3. **Elements of Coarse Geometry, Part 1.** Let X and Y be proper metric spaces with metric functions d_X and d_Y. This means, in particular, that closed bounded subsets of X and Y are compact.

2.3.1. **Definition.** A map $f\colon X \to Y$ of proper metric spaces is *eventually continuous* if there is a real positive function l such that

$$d_X(x_1, x_2) \le r \implies d_Y(f(x_1), f(x_2)) \le l(r). \tag{1}$$

This is the same concept as *bornologous* maps in Roe [16].

A map $f\colon X \to Y$ of proper metric spaces is *proper* if $f^{-1}(S)$ is a bounded subset of X for each bounded subset S of Y.

We say f is a *coarse map* if it is proper and eventually continuous.

If $\mathcal{B}_d(X)$ stands for the collection of subsets of X with diameter bounded by d, condition (1) is equivalent to

$$T \in \mathcal{B}_r(X) \implies f(T) \in \mathcal{B}_{l(r)}(Y). \tag{2}$$

For example, all *bounded* functions $f\colon X \to X$ with $d_X(x, f(x)) \le D$, for all $x \in X$ and a fixed $D \ge 0$, are coarse.

The map f is a *coarse equivalence* if there is a coarse map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded maps.

2.3.2. **Example.** Any *bounded* function $f\colon X \to X$, with $d_X(x, f(x)) \le D$ for all $x \in X$ and a fixed $D \ge 0$, is coarse. In fact, it is a coarse equivalence using $l(r) = r + 2D$ for both f and its coarse inverse.

2.3.3. **Example.** The isometric embedding of a metric subspace is a coarse map. An isometry, which is a bijective isometric map, is a coarse equivalence. An isometric embedding onto a subspace that has the property that its bounded enlargement is the whole target metric space is also a coarse equivalence.

2.3.4. **Example.** A map $f\colon X \to Y$ is called *eventually Lipschitz* in [14] if, for some number $k \ge 0$ and large enough $s \ge 0$, $d_X(x, x') \le s$ implies $d_Y(f(x), f(x')) \le ks$. A proper eventually Lipschitz map is coarse.

2.3.5. **Example.** Quasi-isometries are coarse maps [16].

2.3.6. **Theorem.** *Coarse maps between proper metric spaces induce additive functors between bounded categories.*

Proof. Let $f\colon X \to Y$ be coarse. We want to induce an additive functor $f_*\colon \mathcal{C}(X, R) \to \mathcal{C}(Y, R)$. On objects, the functor is induced by the assignment

$$(f_* F)_y = \bigoplus_{x \in f^{-1}(y)} F_x.$$

Since f is proper, $f^{-1}(y)$ is a bounded set for all y in Y. So the direct sum in the formula is finite, and $(f_*F)_y$ is a finitely generated free R-module. If $S \subset Y$ is a bounded subset then $f^{-1}(S)$ is bounded. There are finitely many $F_z \neq 0$ for $z \in f^{-1}(S)$ and therefore finitely many $(f_*F)_y \neq 0$ for $y \in S$. This shows f_*F is locally finite.

Notice that

$$f_*F = \bigoplus_{y \in Y} (f_*F)_y = \bigoplus_{y \in Y} \bigoplus_{z \in f^{-1}(y)} F_z = F.$$

Suppose we are given a morphism $\phi \colon F \to G$ in $\mathcal{C}(X, R)$. Interpreting f_*F and f_*G as the same R-modules, as in the formula above, we define $f_*\phi \colon f_*F \to f_*G$ equal to ϕ. We must check that $f_*\phi$ is bounded. Suppose ϕ is bounded by D, and f is l-coarse. We claim that $f_*\phi$ is bounded by $l(D)$. Indeed, if $d_Y(y, y') > l(D)$ then $d_X(x, x') > D$ for all x, $x' \in X$ such that $f(x) = y$ and $f(x') = y'$. So all components $\phi_{x,x'} = 0$, therefore all components $(f_*\phi)_{y,y'} = 0$. □

Recall that the path metric in X is obtained as the infimum of length of paths joining points in X over all rectifiable curves. A metric space X is called a *path metric space* if the metric and the path metric in X coincide. Classical examples of path metric spaces are compact Riemannian manifolds.

We will treat the group Γ equipped with a finite generating set as a metric space. The following definition makes this precise.

2.3.7. **Definition.** The *word-length metric* $d = d_\Omega$ on a finitely generated group Γ with a fixed generating set Ω closed under inverses is the path metric induced from the condition that $d(\gamma, \gamma\omega) = 1$ whenever $\gamma \in \Gamma$ and $\omega \in \Omega$.

The word-length metric makes Γ a proper metric space with a free action by Γ via left multiplication. It is well-known that varying Ω only changes Γ to a coarsely equivalent metric space.

Next we make precise the relation between the word-length metric on a discrete group Γ and a metric space X where Γ acts cocompactly by isometries. The following fact is known as "Milnor's lemma".

2.3.8. **Theorem** (Shvarts, Milnor)**.** *Suppose X is a path metric space and Γ is a group acting properly and cocompactly by isometries on X. Then Γ is coarsely equivalent to X.*

Proof. The coarse equivalence is given by the map $\gamma \mapsto \gamma x_0$ for any fixed base point x_0 of X. □

2.3.9. **Corollary.** *If M is a compact manifold with the fundamental group $\Gamma = \pi_1(M)$, the inclusion of any orbit of Γ in the universal cover $\widetilde{M}$ is a coarse equivalence for any choice of the generating set of Γ.*

2.4. **Equivariant Fibred *K*-theory.** Now suppose we are given a metric space Y with a free left Γ-action by coarse equivalences and observe that the equivariant theory K^Γ still applies in this case.

2.4.1. **Definition** (Coarse Equivariant Theories)**.** We associate several new equivariant theories on metric spaces with Γ-action, both by isometries and coarse equivalences. The theory K^Γ_i is defined only for metric spaces with actions by isometries, while K^Γ_c and K^Γ_p only for metric spaces with coarse actions.

(1) $k^{\Gamma}_i(Y)$ is defined to be the K-theory of $\mathcal{C}^{\Gamma}_i(Y) = \mathcal{C}^{\Gamma,0}(\Gamma \times Y, R)$, where Γ is regarded as a word-length metric space with isometric Γ-action given by left multiplication, and $\Gamma \times Y$ is given the product metric and the product isometric action.
(2) $k^{\Gamma}_c(Y)$ is defined for any metric space Y equipped with a Γ-action by coarse equivalences. It is the K-theory spectrum attached to a symmetric monoidal category $\mathcal{C}^{\Gamma}_c(Y)$ with Γ-action whose objects are given by functors

$$\theta\colon\ \boldsymbol{E\Gamma} \longrightarrow \mathcal{C}(\Gamma \times Y, R)$$

such that the morphisms $\theta(f)$ have the property that, in addition to being bounded as maps of modules with bases labelled in $\Gamma \times Y$, they are of degree zero when projected into Γ.
(3) $k^{\Gamma}_p(Y)$ is defined for any metric space with action by coarse equivalences. Again, this spectrum is attached to a symmetric monoidal category $\mathcal{C}^{\Gamma}_p(Y)$ whose objects are functors

$$\theta\colon\ \boldsymbol{E\Gamma} \longrightarrow \mathcal{C}_{\Gamma}(Y) = \mathcal{C}(\Gamma, \mathcal{C}(Y, R))$$

with the additional condition that the morphisms $\theta(f)$ are bounded by 0 but only as homomorphisms between R-modules parametrized over Γ.

Now the nonconnective equivariant K-theory spectra K^{Γ}_i, K^{Γ}_c, K^{Γ}_p are the nonconnective deloopings of k^{Γ}_i, k^{Γ}_c, k^{Γ}_p.

For example, if we define

$$\mathcal{C}^{\Gamma}_{i,k} = \mathcal{C}^{\Gamma,0}(\Gamma \times \mathbb{R}^k \times Y, R),$$

where Γ acts on the product $\Gamma \times \mathbb{R}^k \times Y$ according to $\gamma(\gamma', x, y) = (\gamma\gamma', x, \gamma(y))$, and

$$\mathcal{C}^{\Gamma,+}_{i,k} = \mathcal{C}^{\Gamma,0}(\Gamma \times [0, +\infty) \times Y, R), \text{ etc.}$$

then the delooping construction in Definition 2.2.8 can be applied verbatim. The same is true for the theory k^{Γ}_c. Similarly, one can use the Γ-action on $\Gamma \times \mathbb{R}^k$ given by $\gamma(\gamma', x) = (\gamma\gamma', x)$ and define $\mathcal{C}^{\Gamma}_{p,k}$ as the symmetric monoidal category of functors

$$\theta\colon\ \boldsymbol{E\Gamma} \to \mathcal{C}(\Gamma \times \mathbb{R}^k, \mathcal{C}(Y, R))$$

such that the morphisms $\theta(f)$ are bounded by 0 as R-linear homomorphisms over $\Gamma \times \mathbb{R}^k$. There are obvious analogues of the categories $\mathcal{C}^{\Gamma,+}_{p,k}$, etc. If $*$ is any of the three subscripts i, c, or p, and Y is equipped with actions by Γ via respectively isometries or coarse equivalences, we obtain equivariant maps

$$K(\mathcal{C}^{\Gamma}_{*,k-1}(Y)) \longrightarrow \Omega K(\mathcal{C}^{\Gamma}_{*,k}(Y)).$$

2.4.2. **Definition.** Let $*$ be any of the three subscripts i, c, or p. We define

$$k^{\Gamma}_{*,k}(Y) = K(\mathcal{C}^{\Gamma}_{*,k}(Y))$$

and the nonconnective equivariant spectra

$$K^{\Gamma}_{*}(Y) \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k k^{\Gamma}_{*,k}(Y).$$

The same construction gives for the fixed points

$$K_*^\Gamma(Y)^\Gamma = \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k k_{*,k}^\Gamma(Y)^\Gamma.$$

Given left Γ-actions by isometries on metric spaces X and Y, there are evident diagonal actions induced on the categories $\mathcal{C}(X \times Y, R)$ and $\mathcal{C}(X, \mathcal{C}(Y, R))$. The equivariant embedding induces the equivariant functor

$$i^\Gamma \colon \mathcal{C}^\Gamma(X \times Y, R) \longrightarrow \mathcal{C}^\Gamma(X, \mathcal{C}(Y, R)).$$

So for metric spaces with isometric Γ-actions, there is a natural transformation $K_i^\Gamma(Y) \to K_c^\Gamma(Y)$, and similarly for metric spaces with action by coarse equivalences there is a natural transformation $K_c^\Gamma(Y) \to K_p^\Gamma(Y)$.

We point out an instructive property of K_p^Γ that is not needed in this paper.

2.4.3. **Proposition.** *Let Y be a metric space with a Γ-action by coarse equivalences. Suppose further that the action is bounded in the sense that for every $\gamma \in \Gamma$ there is a bound r_γ so that for every $y \in Y$, $d(y, \gamma y) \leq r_\gamma$. Let Y_0 denote the metric space Y equipped with the trivial Γ-action. Then there is an equivariant equivalence of equivariant spectra $K_p^\Gamma(Y) = K_p^\Gamma(Y_0)$.*

One basic relation between the three equivariant fiberwise theories is through the observation that in all three cases, when Y is a single point space, $\mathcal{C}_i^\Gamma(\text{point})$, $\mathcal{C}_c^\Gamma(\text{point})$, and $\mathcal{C}_p^\Gamma(\text{point})$ can be identified with $\mathcal{C}^{\Gamma,0}(\Gamma, R)$.

2.5. **Fibrewise Localization and Excision Theorems.** Suppose Y' is a subspace of Y. Recall that $\mathcal{C}(Y)_{<Y'}$ is the full subcategory of $\mathcal{C}(Y)$ on objects supported in a bounded neighborhood of Y'.

2.5.1. **Definition.** Since $\mathcal{C}(Y)$ is $\mathcal{C}(Y)_{<Y'}$-filtered, we obtain the Karoubi quotient $\mathcal{C}(Y)/\mathcal{C}(Y)_{<Y'}$ which will be denoted by $\mathcal{C}(Y, Y')$.

Using the notation from Definition 2.4.1, let $\mathcal{C}_i^\Gamma(Y)_{<Y'}$ be the full subcategory of $\mathcal{C}_i^\Gamma(Y)$ on objects θ such that the support of each $\theta(\gamma)$ is contained in a bounded neighborhood of $\Gamma \times Y'$. One similarly defines the subcategories $\mathcal{C}_c^\Gamma(Y)_{<Y'}$ and $\mathcal{C}_p^\Gamma(Y)_{<Y'}$ of $\mathcal{C}_c^\Gamma(Y)$ and $\mathcal{C}_p^\Gamma(Y)$. In all of these cases, the subcategories give Karoubi filtrations and therefore Karoubi quotients $\mathcal{C}_*^\Gamma(Y, Y')$. It is clear that the actions of Γ extend to the quotients in each case. Taking K-theory of the equivariant symmetric monoidal categories gives the Γ-equivariant spectra $k_*^\Gamma(Y, Y')$.

One can now construct the parametrized versions of the relative module categories $\mathcal{C}_{*,k}^\Gamma(Y, Y')$, their K-theory spectra $k_{*,k}^\Gamma(Y, Y')$, and the resulting deloopings. Thus we obtain the nonconnective Γ-equivarint spectra

$$K_*^\Gamma(Y, Y') \overset{\text{def}}{=} \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k k_{*,k}^\Gamma(Y, Y')$$

for $* = i$, c, p.

The Fibration Theorem 2.2.6 can be applied as follows.

2.5.2. **Corollary.** *There is a homotopy fibration*

$$K^{-\infty}(Y') \longrightarrow K^{-\infty}(Y) \longrightarrow K^{-\infty}(Y, Y')$$

In the case of the trivial action of Γ on Y, there is a homotopy fibration

$$K_i^\Gamma(Y')^\Gamma \longrightarrow K_i^\Gamma(Y)^\Gamma \longrightarrow K_i^\Gamma(Y, Y')^\Gamma.$$

Proof. The inclusion of the subspace Y' induces isomorphisms of categories $\mathcal{C}(Y') \cong \mathcal{C}(Y)_{<Y'}$ and $\mathcal{C}_i^\Gamma(Y')^\Gamma \cong \mathcal{C}_i^\Gamma(Y)^\Gamma_{<Y'}$. For the second fibration, one should observe that $\mathcal{C}_i^\Gamma(Y)^\Gamma$ is $\mathcal{C}_i^\Gamma(Y)^\Gamma_{<Y'}$-filtered. □

2.5.3. **Remark.** It is important to note here that when the index i is changed to either c or p, the last sentence is no longer true. This is the source for the need to develop controlled G-theory where there is a similar homotopy fibration for G_p^Γ but derived without use of Karoubi filtrations.

Next we recapitulate and generalize the intrinsic version of the Bounded Excision theorem using the idea of Cardenas–Pedersen [2].

2.5.4. **Theorem** (Bounded Excision). *If U_1 and U_2 are a coarsely antithetic pair of subsets of Y which form a cover of Y, then the commutative square*

$$\begin{tikzcd} K^{-\infty}(U_1 \cap U_2) \arrow[r] \arrow[d] & K^{-\infty}(U_1) \arrow[d] \\ K^{-\infty}(U_2) \arrow[r] & K^{-\infty}(Y) \end{tikzcd}$$

is a homotopy pushout. If the action of Γ on Y is trivial, then

$$\begin{tikzcd} K_i^\Gamma(U_1 \cap U_2)^\Gamma \arrow[r] \arrow[d] & K_i^\Gamma(U_1)^\Gamma \arrow[d] \\ K_i^\Gamma(U_2)^\Gamma \arrow[r] & K_i^\Gamma(Y)^\Gamma \end{tikzcd}$$

is a homotopy pushout.

Proof. In view of the isomorphisms $\mathcal{C}(U_1, U_1 \cap U_2) \cong \mathcal{C}(Y, U_2)$ and $\mathcal{C}_i^\Gamma(U_1, U_1 \cap U_2) \cong \mathcal{C}_i^\Gamma(Y, U_2)$, we have the weak equivalences

$$K^{-\infty}(U_1, U_1 \cap U_2) \ \simeq \ K^{-\infty}(Y, U_2)$$

and

$$K_i^\Gamma(U_1, U_1 \cap U_2)^\Gamma \ \simeq \ K_i^\Gamma(Y, U_2)^\Gamma.$$

Similar to the use of the Fibration Theorem in the construction of nonconnective deloopings $K^{-\infty}(Y)$ and $K_i^\Gamma(Y)^\Gamma$, we have a map of homotopy fibrations

$$\begin{tikzcd} K^{-\infty}(U_1 \cap U_2) \arrow[r] \arrow[d] & K^{-\infty}(U_1) \arrow[r] \arrow[d] & K^{-\infty}(U_1, U_1 \cap U_2) \arrow[d, "\simeq"] \\ K^{-\infty}(U_1) \arrow[r] & K^{-\infty}(Y) \arrow[r] & K^{-\infty}(Y, U_1) \end{tikzcd}$$

This gives the first homotopy pushout. Similarly, the map of fibrations

$$\begin{tikzcd} K_i^\Gamma(U_1 \cap U_2)^\Gamma \arrow[r] \arrow[d] & K_i^\Gamma(U_1)^\Gamma \arrow[r] \arrow[d] & K_i^\Gamma(U_1, U_1 \cap U_2)^\Gamma \arrow[d, "\simeq"] \\ K_i^\Gamma(U_2)^\Gamma \arrow[r] & K_i^\Gamma(Y)^\Gamma \arrow[r] & K_i^\Gamma(Y, U_2)^\Gamma \end{tikzcd}$$

gives the second homotopy pushout. □

We will require relative versions of the excision theorems.

2.5.5. **Definition.** The quotient map of categories induces the equivariant map
$$K_i^\Gamma(Y) \longrightarrow K_i^\Gamma(Y, Y')$$
and the map of fixed points
$$K_i^\Gamma(Y)^\Gamma \longrightarrow K_i^\Gamma(Y, Y')^\Gamma.$$
More generally, if Y'' is another coarsely invariant subset of Y, then the intersection $Y'' \cap Y'$ is coarsely invariant in both Y and Y', there is an equivariant map
$$K_i^\Gamma(Y'', Y'' \cap Y') \longrightarrow K_i^\Gamma(Y, Y')$$
and the map of fixed points
$$K_i^\Gamma(Y'', Y'' \cap Y')^\Gamma \longrightarrow K_i^\Gamma(Y, Y')^\Gamma.$$

We also obtain the spectra $K_c^\Gamma(Y, Y')$, and $K_p^\Gamma(Y, Y')$, and equivariant maps just as above.

2.5.6. **Theorem** (Relative Bounded Excision). *Suppose U_1, U_2, and Y' are three pairwise coarsely antithetic subsets of Y such that U_1 and U_2 form a cover of Y. Assuming the trivial action of Γ on Y, the commutative square*
$$\begin{tikzcd} K_i^\Gamma(U_1 \cap U_2, Y' \cap U_1 \cap U_2)^\Gamma \arrow[r] \arrow[d] & K_i^\Gamma(U_1, Y' \cap U_1)^\Gamma \arrow[d] \\ K_i^\Gamma(U_2, Y' \cap U_2)^\Gamma \arrow[r] & K_i^\Gamma(Y, Y')^\Gamma \end{tikzcd}$$
induced by inclusions of pairs is a homotopy pushout.

Proof. This follows from the fact that whenever C is a subset of Y which is coarsely antithetic to Y', the category $\mathcal{C}_i^\Gamma(Y, Y')^\Gamma$ is $\mathcal{C}_i^\Gamma(Y, Y')^\Gamma_{<C}$-filtered and $\mathcal{C}_i^\Gamma(Y, Y')^\Gamma_{<C}$ is isomorphic to $\mathcal{C}_i^\Gamma(C, Y' \cap C)^\Gamma$. The details are left to the reader. □

2.6. **Elements of Coarse Geometry, Part 2.**

2.6.1. **Definition.** Let Z be any metric space with a free left Γ-action by isometries. We assume that the action is properly discontinuous, that is, that for fixed points z and z', the infimum over $\gamma \in \Gamma$ of the distances $d(z, \gamma z')$ is attained. Then we define the orbit space metric on $\Gamma \backslash Z$ by
$$d_{\Gamma \backslash Z}([z], [z']) = \inf_{\gamma \in \Gamma} d(z, \gamma z').$$

2.6.2. **Lemma.** *$d_{\Gamma \backslash Z}$ is a metric on $\Gamma \backslash Z$.*

Proof. It is well-known that $d_{\Gamma \backslash Z}$ is a pseudometric. The fact that Γ acts by isometries makes it a metric. The triangle inequality follows directly from the triangle inequality for d. Symmetry follows from $d(z, \gamma z') = d(\gamma^{-1} z, z') = d(z', \gamma^{-1} z)$. Finally, $d_{\Gamma \backslash Z}([z], [z']) = 0$ gives $d(z, \gamma z') = 0$ for some $\gamma \in \Gamma$, so $d(\gamma' z, \gamma' \gamma z') = 0$ for all $\gamma' \in \Gamma$, and so $[z] = [z']$. □

Now suppose X is some metric space with left Γ-action by isometries.

2.6.3. **Definition.** Define
$$X^{bdd} = X \times_\Gamma \Gamma$$
where the right-hand copy of Γ denotes Γ regarded as a metric space with the word-length metric associated to a finite generating set, the group Γ acts by isometries

on the metric space Γ via left multiplication, and $X \times_\Gamma \Gamma$ denotes the orbit metric space associated to the diagonal left Γ-action on $X \times \Gamma$. We will denote the orbit metric by d^{bdd}.

The natural left action of Γ on X^{bdd} is given by $\gamma[x,e] = [\gamma x, e]$.

2.6.4. **Definition.** A left action of Γ on a metric space X is *bounded* if for each element $\gamma \in \Gamma$ there is a number $B_\gamma \geq 0$ such that $d(x, \gamma x) \leq B_\gamma$ for all $x \in X$.

2.6.5. **Lemma.** *If the left action of* Γ *on a metric space* X *is bounded, and* $B \colon \Gamma \to [0,\infty)$ *is a function as above, then there is a real function* $B_* \colon [0,\infty) \to [0,\infty)$ *such that* $|\gamma| \leq s$ *implies* $B_\gamma \leq B_*(s)$.

Proof. One simply takes $B_*(s) = \max\{B_\gamma \mid |\gamma| \leq s\}$. □

2.6.6. **Proposition.** *The natural action of* Γ *on* X^{bdd} *is bounded.*

Proof. If $|\gamma| = d_\Gamma(e,\gamma)$ is the norm in Γ, we choose $B_\gamma = |\gamma|$. Now

$$\begin{aligned} d^{bdd}([x,e],[\gamma x,e]) &= \inf_{\gamma' \in \Gamma} d^\times((x,e), \gamma'(\gamma x, e)) \\ &\leq d^\times((x,e), \gamma^{-1}(\gamma x, e)) \\ &= d^\times((x,e),(x,\gamma^{-1})) = d_\Gamma(e,\gamma^{-1}) = |\gamma^{-1}| = |\gamma|, \end{aligned}$$

where $d^\times$ stands for the max metric on the product $X \times \Gamma$. □

2.6.7. **Definition.** Let $b \colon X \to X^{bdd}$ be the natural map given by $b(x) = [x,e]$ in the orbit space $X \times_\Gamma \Gamma$.

2.6.8. **Proposition.** *The map* $b \colon X \to X^{bdd}$ *is a coarse map.*

Proof. Suppose $d^{bdd}([x_1,e],[x_2,e]) \leq D$, then $d^\times((x_1,e),(\gamma x_2,\gamma)) \leq D$ for some $\gamma \in \Gamma$, so $d(x_1,\gamma x_2) \leq D$ and $|\gamma| \leq D$. Since the left action of Γ on X^{bdd} is bounded, there is a function B_* guaranteed by Lemma 2.6.5. Now

$$d(x_1,x_2) \leq d(x_1,\gamma x_2) + d(x_2,\gamma x_2) \leq D + B_*(D).$$

This verifies that b is proper. It is clearly distance reducing, therefore eventually continuous with $l(r) = r$. □

If we think of X^{bdd} as the set X with the metric induced from the bijection b, the map b becomes the coarse identity map between the metric space X with a left action of Γ and the metric space X^{bdd} where the action is made bounded.

For any metric space X, the new space TX will be related to the cone construction.

2.6.9. **Definition.** Start with any set Z, let $S \subset Z \times Z$ denote any symmetric and reflexive subset with the property that

- for any z, z', there are elements z_0, z_1, ... , z_n so that $z_0 = z$, $z_n = z_0$, and $(z_i, z_{i+1}) \in S$.

Let $\rho \colon S \to \mathbb{R}$ be any function for which the following properties hold:

- $\rho(z_1,z_2) = \rho(z_2,z_1)$ for all $(z_1,z_2) \in S$,
- $\rho(z_1,z_2) = 0$ if and only if $z_1 = z_2$.

Given such S and ρ, we may define a metric d on Z to be the largest metric D for which $D(z_1, z_2) \leq \rho(z_1, z_2)$ for all $(z_1, z_2) \in S$. This means that d is given by

$$d(z_1, z_2) = \inf_{n, \{z_0, z_1, \ldots, z_n\}} \sum_{i=0}^{n} \rho(z_i, z_{i+1}).$$

2.6.10. **Definition.** We define a metric space TX by first declaring that the underlying set is $X \times \mathbb{R}$. Next, we define S to be the set consisting of pairs of the form $((x, r), (x, r'))$ or of the form $((x, r), (x', r))$. We then define ρ on S by

$$\rho((x, r), (x, r')) = |r - r'|,$$

and

$$\rho((x, r), (x', r)) = \begin{cases} d(x, x'), & \text{if } r \leq 1; \\ rd(x, x'), & \text{if } 1 \leq r. \end{cases}$$

Since ρ clearly satisfies the hypotheses of the above definition, we set the metric on TX to be d. We also extend the definition to pairs of metric spaces (X, Y), where Y is given the restriction of the metric on X.

Applying bounded K-theory to this construction gives predictable results when applied to familiar subspaces.

2.6.11. **Proposition.** $K^{-\infty}(T\mathbb{R}^n, R) \simeq \Sigma\, K^{-\infty}(\mathbb{R}^n, R) \simeq \Sigma^{n+1}\, K^{-\infty}(R)$.

Proof. The covering of $T\mathbb{R}^n$ by $T\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0] \times \mathbb{R}$ and $T\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, +\infty) \times \mathbb{R}$ is a covering by a coarsely antithetic pair of subsets. Both $\mathcal{C}(T\mathbb{R}^n_-)$ and $\mathcal{C}(T\mathbb{R}^n_+)$ are flasque, and $\mathcal{C}(T\mathbb{R}^n_- \cap T\mathbb{R}^n_+)$ is isomorphic to $\mathcal{C}(T\mathbb{R}^{n-1})$. Using iterated bounded excision, we have the equivalence

$$K^{-\infty}(T\mathbb{R}^n, R) \simeq \Sigma^n\, K^{-\infty}(\mathbb{R}, R) \;\simeq\; \Sigma^{n+1}\, K^{-\infty}(R).$$

The equivalence

$$K^{-\infty}(\mathbb{R}^n, R) \;\simeq\; \Sigma^n\, K^{-\infty}(R)$$

follows from a similar bounded excision argument. □

2.6.12. **Proposition.** *For the trivial actions of* Γ *on* $\mathbb{R}^{n-1}$ *and* $\mathbb{R}^n$*, there is a weak equivalence*

$$\Sigma K^\Gamma_i (T\mathbb{R}^{n-1})^\Gamma \;\simeq\; K^\Gamma_i (T\mathbb{R}^n)^\Gamma.$$

Proof. The argument is along the same lines as that in proof of Proposition 2.6.11, using the second homotopy pushout square from Theorem 2.5.6. □

2.6.13. **Corollary.** *For the trivial action of* Γ *on* $\mathbb{R}^n$*, there is a weak equivalence*

$$K^\Gamma_i (T\mathbb{R}^n)^\Gamma \;\simeq\; \Sigma^{n+1}\, K^{-\infty}(R[\Gamma]).$$

The next result applies to a nontrivial action of Γ but in the direction transverse to the previous theorems.

2.6.14. **Theorem.** *Suppose* Γ *acts on* $\widetilde{N}$ *by deck transformations, with the compact quotient* N*. Then there is a weak equivalence*

$$\sigma \colon \Sigma K^\Gamma_i (\widetilde{N})^\Gamma \;\simeq\; K^\Gamma_i (T\widetilde{N})^\Gamma.$$

Proof. Clearly, the subcategories $\mathcal{C}_- = \mathcal{C}^{\Gamma,0}(\Gamma \times (-\infty, 0] \times \widetilde{N})$ and $\mathcal{C}_+ = \mathcal{C}^{\Gamma,0}(\Gamma \times [0, +\infty) \times \widetilde{N})$ are invariant under the induced action of Γ on $\mathcal{C} = \mathcal{C}^{\Gamma,0}(\Gamma \times T\widetilde{N})$. The lax limit category $\mathcal{C}^\Gamma$ is Karoubi filtered by both $\mathcal{C}^\Gamma_-$ and $\mathcal{C}^\Gamma_+$. Notice that both of these subcategories are flasque, and their intersection is $\mathcal{C}^{\Gamma,0}(\Gamma \times \widetilde{N})^\Gamma$. □

2.7. **Elementary Equivariant Excision Theorems.** In order to extend the excision theorems to fibred G-theory and appropriate nontrivial actions further in the paper, we need to develop constructions related to coverings of Y.

2.7.1. **Definition.** Two subsets A, B in a proper metric space X are called *coarsely equivalent* if there are numbers $d_{A,B}$, $d_{B,A}$ with $A \subset B[d_{A,B}]$ and $B \subset A[d_{B,A}]$. It is clear this is an equivalence relation among subsets. We will use notation $A \parallel B$ for this equivalence.

A family of subsets $\mathcal{A}$ is called *coarsely saturated* if it is maximal with respect to this equivalence relation. Given a subset A, let $\mathcal{S}(A)$ be the smallest boundedly saturated family containing A.

If $\mathcal{A}$ is a coarsely saturated family, define $K(\mathcal{A})$ to be

$$\underset{\overrightarrow{A \in \mathcal{A}}}{\operatorname{hocolim}} \, K(A).$$

2.7.2. **Proposition.** *If A and B are coarsely equivalent then $\mathcal{C}(A) \cong \mathcal{C}(B)$ and so $K(A) \simeq K(B)$. For all subsets A,*

$$\mathcal{C}(X)_{<A} \cong \mathcal{C}(A)$$

and

$$K(X)_{<A} \simeq K(\mathcal{S}(A)) \simeq K(A).$$

2.7.3. **Definition.** A collection of subsets $\mathcal{U} = \{U_i\}$ is a *coarse covering* of X if $X = \bigcup S_i$ for some $S_i \in \mathcal{S}(U_i)$. Similarly, $\mathcal{U} = \{\mathcal{A}_i\}$ is a *coarse covering* by coarsely saturated families if for some (and therefore any) choice of subsets $A_i \in \mathcal{A}_i$, $\{A_i\}$ is a coarse covering in the sense above.

Recall that a pair of subsets A, B in a proper metric space X are *coarsely antithetic* if for any two numbers d_1 and d_2 there is a third number d such that

$$A[d_1] \cap B[d_2] \subset (A \cap B)[d].$$

We will write $A \natural B$ to indicate that A and B are coarsely antithetic.

Given two subsets A and B, define

$$\mathcal{S}(A, B) = \{A' \cap B' \mid A' \in \mathcal{S}(A), B' \in \mathcal{S}(B), A' \natural B'\}.$$

2.7.4. **Proposition.** *$\mathcal{S}(A, B)$ is a coarsely saturated family.*

Proof. Suppose A_1, A'_1 and A_2, A'_2 are two coarsely antithetic pairs, and $A_1 \subset A_2[d_{12}]$, $A'_1 \subset A'_2[d'_{12}]$ for some d_{12} and d'_{12}. Then

$$A_1 \cap A'_1 \subset A_2[d_{12}] \cap A'_2[d'_{12}] \subset (A_2 \cap A'_2)[d]$$

for some d. $\square$

2.7.5. **Proposition.** *If U and T are coarsely antithetic then*

$$K(X)_{<U,T} \simeq K(\mathcal{S}(U \cap T)) \simeq K(U \cap T).$$

There is the obvious generalization of the constructions and propositions to the case of a finite number of subsets of X.

2.7.6. **Definition.** We write $A_1 \natural A_2 \natural \ldots \natural A_k$ if for arbitrary d_i there is a number d so that

$$A_1[d_1] \cap A_2[d_2] \cap \ldots \cap A_k[d_k] \subset (A_1 \cap A_2 \cap \ldots \cap A_k)[d]$$

and define

$$\mathcal{S}(A_1, A_2, \ldots, A_k) = \{A_1' \cap A_2' \cap \ldots \cap A_k' \,|\, A_i' \in \mathcal{S}(A_i), A_1 \natural A_2 \natural \ldots \natural A_k\}.$$

Equivalently, identifying any coarsely saturated family $\mathcal{A}$ with $\mathcal{S}(A)$ for $A \in \mathcal{A}$, one has $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$.

We will refer to $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ as the coarse intersection of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$. A coarse covering $\mathcal{U}$ is *closed under coarse intersections* if all coarse intersections $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ are nonempty and are contained in $\mathcal{U}$. If $\mathcal{U}$ is a given coarse covering, the smallest coarse covering that is closed under coarse intersections and contains $\mathcal{U}$ will be called the *closure* of $\mathcal{U}$ under coarse intersections.

The inclusions induce the diagrams of spectra $\{K(Y)_{<A}\}$ and $\{K_i^\Gamma(Y, Y')^\Gamma_{<A}\}$ for representatives A in $\mathcal{A} \in \mathcal{U}$.

2.7.7. **Definition.** Suppose $\mathcal{U}$ is a coarse covering of Y closed under coarse intersections. We define the homotopy pushouts

$$\mathcal{K}(Y;\mathcal{U}) = \underset{\overrightarrow{\mathcal{A}\in\mathcal{U}}}{\mathrm{hocolim}}\; K(Y)_{<A}$$

and

$$\mathcal{K}_i^\Gamma(Y, Y';\mathcal{U})^\Gamma = \underset{\overrightarrow{\mathcal{A}\in\mathcal{U}}}{\mathrm{hocolim}}\; K_i^\Gamma(Y, Y')^\Gamma_{<A}.$$

The following result is equivalent to the Bounded Excision Theorem 2.1.5.

2.7.8. **Theorem.** *If $\mathcal{U}$ is a finite coarse covering of Y closed under coarse intersections, then there is a weak equivalence*

$$\mathcal{K}(Y;\mathcal{U}) \simeq K(Y).$$

Proof. Apply the Theorem 2.1.5 inductively to the sets in $\mathcal{U}$. $\square$

Now suppose there is an action of Γ on Y by isometries and $\mathcal{U}$ is a coarse covering of Y closed under coarse intersections.

2.7.9. **Definition.** The action is $\mathcal{U}$-*bounded* if all coarse families $\{\mathcal{A}_i\}$ in $\mathcal{U}$ are closed under the action. In this case one has the K-theory spectra $K^\Gamma(\mathcal{A}_i) = K^\Gamma(Y;\mathcal{A}_i)$ and the fixed point spectra $K^\Gamma(\mathcal{A}_i)^\Gamma = K^\Gamma(Y;\mathcal{A}_i)^\Gamma$.

An action is $\mathcal{U}$-bounded for any coarse covering $\mathcal{U}$ if the action is by bounded coarse equivalences. The trivial action is an instance of such action.

2.7.10. **Theorem.** *If the action of Γ on Y is trivial, then*

$$\mathcal{K}^\Gamma(Y;\mathcal{U})^\Gamma \simeq \underset{\overrightarrow{\mathcal{A}\in\mathcal{U}}}{\mathrm{hocolim}}\; K^\Gamma(\mathcal{A})^\Gamma \simeq K^\Gamma(Y)^\Gamma$$

and

$$\mathcal{K}_i^\Gamma(Y, Y';\mathcal{U})^\Gamma \simeq \underset{\overrightarrow{\mathcal{A}\in\mathcal{U}}}{\mathrm{hocolim}}\; K_i^\Gamma(\mathcal{A})^\Gamma \simeq K_i^\Gamma(Y, Y')^\Gamma.$$

Proof. Apply the Theorem 2.5.6 inductively to the sets in $\mathcal{U}$. $\square$

3. Fibred Bounded G-theory

3.1. **Bounded G-theory.** Bounded G-theory is a variant of the Pedersen–Weibel bounded K-theory made applicable to certain nonsplit exact structures. We will review and augment some material from [6] in the form best fit for the fibred theory.

This section is dedicated to the absolute theory, while our main interest is in the general fibred version. The restriction to a single metric space allows for more concise arguments which will be reused in the fibred situation.

The variation of the basic construction of bounded K-theory is based on the following observation. Given an object F in $\mathcal{C}(X,R)$, to every subset $S \subset X$ one can associate a direct sum $F(S)$ generated by those $F(x)$ with $x \in S$. Now the restriction from arbitrary R-linear homomorphisms to the bounded ones can be described entirely in terms of these subobjects.

To motivate the basic construction first notice that, given a geometric module F in $\mathcal{C}(M,R)$, to every subset $S \subset M$ there corresponds a direct summand which is the free submodule

$$F(S) = \bigoplus_{m \in S} F_m.$$

In this context we say an element $x \in F$ is *supported* on a subset S if $x \in F(S)$. The restriction to bounded homomorphisms can be described entirely in terms of these submodules.

We generalize this using the language of Quillen exact categories.

3.1.1. **Definition.** Let $\mathbf{C}$ be an additive category. Suppose $\mathbf{C}$ has two classes of morphisms $\mathbf{m}(\mathbf{C})$, called *admissible monomorphisms*, and $\mathbf{e}(\mathbf{C})$, called *admissible epimorphisms*, and a class $\mathcal{E}$ of *exact* sequences, or extensions, of the form

$$C^{\cdot}\colon \quad C' \xrightarrow{i} C \xrightarrow{j} C''$$

with $i \in \mathbf{m}(\mathbf{C})$ and $j \in \mathbf{e}(\mathbf{C})$ which satisfy the three axioms:

a) any sequence in $\mathbf{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$; the canonical sequence

$$C' \xrightarrow{\mathrm{incl}_1} C' \oplus C'' \xrightarrow{\mathrm{proj}_2} C''$$

is in $\mathcal{E}$; for any sequence $C^{\cdot}$, i is a kernel of j and j is a cokernel of i in $\mathbf{C}$,

b) both classes $\mathbf{m}(\mathbf{C})$ and $\mathbf{e}(\mathbf{C})$ are subcategories of $\mathbf{C}$; $\mathbf{e}(\mathbf{C})$ is closed under base-changes along arbitrary morphisms in $\mathbf{C}$ in the sense that for every exact sequence $C' \to C \to C''$ and any morphism $f\colon D'' \to C''$ in $\mathbf{C}$, there is a pullback commutative diagram

$$\begin{array}{ccccc} C' & \longrightarrow & D & \xrightarrow{j'} & D'' \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ C' & \longrightarrow & C & \xrightarrow{j} & C'' \end{array}$$

where $j'\colon D \to D''$ is an admissible epimorphism; $\mathbf{m}(\mathbf{C})$ is closed under cobase-changes along arbitrary morphisms in $\mathbf{C}$ in the (dual) sense that for every exact sequence $C' \to C \to C''$ and any morphism $g\colon C' \to D'$ in $\mathbf{C}$, there is a pushout diagram

$$\begin{tikzcd} C' \arrow[r, "i"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "g'"'] & C'' \arrow[d, "="] \\ D' \arrow[r, "i'"] & D \arrow[r] & C'' \end{tikzcd}$$

where $i' \colon D' \to D$ is an admissible monomorphism,

c) if $f \colon C \to C''$ is a morphism with a kernel in $\mathbf{C}$, and there is a morphism $D \to C$ so that the composition $D \to C \to C''$ is an admissible epimorphism, then f is an admissible epimorphism; dually for admissible monomorphisms.

According to Keller [9], axiom (c) follows from the other two.

Recall that an *abelian* category is an additive category with kernels and cokernels such that every morphism f is *balanced*, that is, the canonical map from the coimage $\mathrm{coim}(f) = \mathrm{coker}(\ker f)$ to the image $\mathrm{im}(f) = \ker(\mathrm{coker}\, f)$ is an isomorphism.

3.1.2. **Definition.** If a category has kernels and cokernels for all morphisms, and the canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is always monic and epic but not necessarily invertible, we say the category is *pseudoabelian*.

Recall also that a category is called *cocomplete* if it contains colimits of arbitrary small diagrams, cf. Mac Lane [11], chapter V.

Let X be a proper metric space and let R be a noetherian ring.

3.1.3. *Notation.* We will use the notation $\mathcal{P}(X)$ for the power set of X partially ordered by inclusion and viewed as a category. Let $\mathcal{B}(X)$ be the subcategory of bounded subsets, $\mathcal{B}_D(X)$ be the subcategory of subsets with diameter bounded by $D \geq 0$, and let $\mathbf{Mod}(R)$ denote the category of left R-modules. If F is a left R-module, let $\mathcal{I}(F)$ denote the family of all R-submodules of F partially ordered by inclusion.

3.1.4. **Definition.** An X*-filtered* R-module is a module F together with a functor $\mathcal{P}(X) \to \mathcal{I}(F)$ from the power set of X to the family of R-submodules of F, both ordered by inclusion, such that the value on X is F. It will be most convenient to think of F as the functor above and use notation $F(S)$ for the value of the functor on S. We will call F *reduced* if $F(\emptyset) = 0$.

An R-homomorphism $f \colon F \to G$ of X-filtered modules is *boundedly controlled* if there is a fixed number $b \geq 0$ such that the image $f(F(S))$ is a submodule of $G(S[b])$ for all subsets S of X.

The objects of the category $\mathbf{U}(X,R)$ are the reduced X-filtered R-modules and the morphisms are the boundedly controlled homomorphisms.

3.1.5. **Remark.** If X is unbounded, $\mathbf{U}(X,R)$ is not a balanced category and therefore not an abelian category. For an explicit description of a boundedly controlled morphism in $\mathbf{U}(\mathbb{Z},R)$ which is an isomorphism of left R-modules but whose inverse is not boundedly controlled, we refer to Example 1.5 of [13].

We will show that $\mathbf{U}(X,R)$ is a pseudoabelian category. It turns out that the kernels and cokernels in $\mathbf{U}(X,R)$ can be characterized using an additional property a boundedly controlled morphism may or may not have.

3.1.6. **Definition.** A morphism $f\colon F \to G$ in $\mathbf{U}(X,R)$ is called *boundedly bicontrolled* if, for some fixed $b \geq 0$, in addition to inclusions of submodules

$$f(F(S)) \subset G(S[b]),$$

there are inclusions

$$f(F) \cap G(S) \subset fF(S[b])$$

for all subsets $S \subset X$. In this case we will say that f has filtration degree b and write $\mathrm{fil}(f) \leq b$.

3.1.7. **Lemma.** *Let $f_1\colon F \to G$, $f_2\colon G \to H$ be in $\mathbf{U}(X,R)$ and $f_3 = f_2 f_1$.*

(1) *If f_1, f_2 are boundedly bicontrolled and either $f_1\colon F(X) \to G(X)$ is an epimorphism or $f_2\colon G(X) \to H(X)$ is a monomorphism, then f_3 is also boundedly bicontrolled.*
(2) *If f_1, f_3 are boundedly bicontrolled and f_1 is an epimorphism then f_2 is also boundedly bicontrolled; if f_3 is only boundedly controlled then f_2 is also boundedly controlled.*
(3) *If f_2, f_3 are boundedly bicontrolled and f_2 is a monomorphism then f_1 is also boundedly bicontrolled; if f_3 is only boundedly controlled then f_1 is also boundedly controlled.*

Proof. Suppose $\mathrm{fil}(f_i) \leq b$ and $\mathrm{fil}(f_j) \leq b'$ for $\{i,j\} \subset \{1,2,3\}$, then in fact $\mathrm{fil}(f_{6-i-j}) \leq b + b'$ in each of the three cases. For example, there are factorizations

$$f_2G(S) \subset f_2f_1F(S[b]) = f_3F(S[b]) \subset H(S[b+b'])$$
$$f_2G(X) \cap H(S) \subset f_3F(S[b']) = f_2f_1F(S[b']) \subset f_2G(S[b+b'])$$

which verify part 2 with $i = 1$, $j = 3$. $\square$

3.1.8. **Definition.** We define the *admissible monomorphisms* in $\mathbf{U}(X,R)$ be the boundedly bicontrolled homomorphisms $m\colon F_1 \to F_2$ such that the map $F_1(X) \to F_2(X)$ is a monomorphism. We define the *admissible epimorphisms* be the boundedly bicontrolled homomorphisms $e\colon F_1 \to F_2$ such that $F_1(X) \to F_2(X)$ is an epimorphism.

Let the class $\mathcal{E}$ of *exact sequences* consist of the sequences

$$F^{\boldsymbol{\cdot}}\colon \quad F' \xrightarrow{i} F \xrightarrow{j} F'',$$

where i is an admissible monomorphism, j is an admissible epimorphism, and $\mathrm{im}(i) = \ker(j)$.

3.1.9. **Theorem.** $\mathbf{U}(X,R)$ *is a cocomplete pseudoabelian category. The class of exact sequences $\mathcal{E}$ gives an exact structure on $\mathbf{U}(X,R)$.*

Proof. The additive properties are inherited from $\mathbf{Mod}(R)$. In particular, the biproduct is given by the filtration-wise operation

$$(F \oplus G)(S) = F(S) \oplus G(S).$$

For any boundedly controlled homomorphism $f\colon F \to G$, the kernel of f in $\mathbf{Mod}(R)$ has the standard X-filtration K where

$$K(S) = \ker(f) \cap F(S)$$

which gives the kernel of f in $\mathbf{U}(X,R)$. The canonical monomorphism $\kappa\colon K \to F$ has filtration degree 0. It follows from part 3 of Lemma 3.1.7 that K has the universal properties of the kernel in $\mathbf{U}(X,R)$.

Similarly, let I be the standard X-filtration of the image of f in $\mathbf{Mod}(R)$ by

$$I(S) = \mathrm{im}(f) \cap G(S).$$

If we define $C(S) = G(S)/I(S)$ for all $S \subset X$ then clearly $C(X)$ is the cokernel of f in $\mathbf{Mod}(R)$. There is an X-filtered module C_X associated to C given by

$$C_X(S) = \mathrm{im}\{C(S) \to C(X)\}.$$

The canonical homomorphism $\sigma \colon G(X) \to C(X)$ gives a filtration 0 morphism $\sigma \colon G \to C_X$ since

$$\mathrm{im}(\sigma \circ \{G(S) \to G(X)\}) = \mathrm{im}\{C(S) \to C(X)\} = C_X(S).$$

The universal cokernel properties of C_X and σ in $\mathbf{U}(X,R)$ follow from part 2 of Lemma 3.1.7.

The preceding combined with the fact that $\mathbf{Mod}(R)$ is cocomplete shows that $\mathbf{U}(X,R)$ is cocomplete, cf. Mac Lane [11], section V.2.

It follows from Lemma 3.1.7 that any exact sequence $F^{\cdot}$ isomorphic to some short exact sequence in $\mathcal{E}$ is also in $\mathcal{E}$, that

$$F' \xrightarrow{[\mathrm{id},0]} F' \oplus F'' \xrightarrow{[0,\mathrm{id}]^T} F''$$

is in $\mathcal{E}$, and that $i = \ker(j)$, $j = \mathrm{coker}(i)$ in any sequence $F^{\cdot}$ in $\mathcal{E}$.

The collections of admissible monomorphisms and epimorphisms are closed under composition by part 1 of Lemma 3.1.7. Given $F^{\cdot}$ in $\mathcal{E}$ and any $f \colon A \to F''$ in $\mathbf{U}(X,R)$, there is a base change diagram

$$\begin{tikzcd}
F' \arrow[r] \arrow[d, "="'] & F \times_f A \arrow[r, "j'"] \arrow[d, "f'"] & A \arrow[d, "f"] \\
F' \arrow[r] & F \arrow[r, "j"] & F''
\end{tikzcd}$$

where $F \times_f A$ is the kernel of the epimorphism

$$j \,\mathrm{pr}_1 - f \,\mathrm{pr}_2 \colon F \oplus A \to F''.$$

If $m \colon F \times_f A \to F \oplus A$ is the inclusion of the kernel then $f' = \mathrm{pr}_1 m$, $j' = \mathrm{pr}_2 m$. The X-filtration is given by

$$(F \times_f A)(S) = F \times_f A \cap (F(S) \times A(S)),$$

so that j' is boundedly controlled and has the same kernel as j. In fact,

$$\mathrm{im}(j') \cap A(S) \subset j' (F \times_f A) (S[b(f) + b(j)])$$

since $fA(S) \subset F''(S[b(f)])$, so j' is boundedly bicontrolled of filtration degree $b(f) + b(j)$. Given an admissible submodule $E \subset F \times A$, the restriction $j'|E$ is the pullback of the admissible epimorphism $f(E) \to fj'(E)$. This shows that the class of admissible epimorphisms is closed under base change by arbitrary morphisms in $\mathbf{U}(X,R)$. The proof of closure under cobase changes by admissible monomorphisms is similar. □

Another characterization of admissible morphisms is given as follows.

3.1.10. **Proposition.** *The exact structure $\mathcal{E}$ in $\mathbf{U}(X,R)$ consists of sequences isomorphic to those*

$$E^{\cdot} \colon \quad E' \xrightarrow{i} E \xrightarrow{j} E''$$

which possess restrictions

$$E^{\boldsymbol{\cdot}}(S)\colon \quad E'(S) \xrightarrow{i} E(S) \xrightarrow{j} E''(S)$$

for all subsets $S \subset X$, and each $E^{\boldsymbol{\cdot}}(S)$ is an exact sequence in $\mathbf{Mod}(R)$.

Proof. Compare to Proposition 2.14 in [6]. □

3.1.11. **Definition.** Let F be an X-filtered R-module.

- F is called *lean* or *D-lean* if there is a number $D \geq 0$ such that
$$F(S) \subset \sum_{x \in S} F(x[D])$$
for every subset S of X.
- F is called *insular* or *d-insular* if there is a number $d \geq 0$ such that
$$F(S) \cap F(U) \subset F(S[d] \cap U[d])$$
for every pair of subsets S, U of X.

3.1.12. **Proposition.** *The properties of being lean and insular are preserved under isomorphisms in* $\mathbf{U}(X, R)$.

Proof. If $f\colon F_1 \to F_2$ is an isomorphism with $\mathrm{fil}(f) \leq b$, and F_1 is D-lean and d-insular, then F_2 is $(D+b)$-lean and $(d+2b)$-insular. □

There is a property of filtered modules which is a consequence of leanness.

3.1.13. **Definition.** An X-filtered R-module F is called *split* or *D'-split* if there is a number $D' \geq 0$ such that we have
$$F(S) \subset F(T[D']) + F(U[D'])$$
whenever a subset S of X is written as a union $T \cup U$.

3.1.14. **Proposition.** *A D-lean filtered module is D-split.*

Proof. We have
$$F(T \cup U) \subset \sum_{x \in T} F(x[D]) + \sum_{x \in U} F(x[D]) \subset F(T[D]) + F(U[D])$$
since in general $\sum_{x \in S} F(x[D]) \subset F(S[D])$. □

3.1.15. **Proposition.** *The property of being split is preserved under isomorphisms in* $\mathbf{U}(X, R)$.

Proof. If $f\colon F_1 \to F_2$ is an isomorphism with $\mathrm{fil}(f) \leq b$, and F_1 is D'-split, then F_2 is $(D'+b)$-split. □

3.1.16. **Lemma.** (1) *Lean objects are closed under exact extensions.*
(2) *Split objects are closed under exact extensions.*
(3) *Insular objects are closed under exact extensions.*

Proof. Let
$$E' \xrightarrow{f} E \xrightarrow{g} E''$$
be an exact sequence in $\mathbf{U}(X, R)$ and let $b \geq 0$ be a common filtration degree for f and g.

(1) Suppose both E' and E'' are D-lean. For an arbitrary subset $S \subset X$,

$$gE(S) \subset E''(S[b]),$$

so

$$gE(S) \subset \sum_{x \in S[b]} E''(x[D]).$$

For each $x \in S[b]$,

$$E''(x[D]) \subset gE(x[b+D]),$$

so

$$E(S) \subset \sum_{x \in S[b]} E(x[2b+D]) + \sum_{x \in S[b]} fE'(x[2b+D]).$$

Therefore

$$E(S) \subset \sum_{x \in S[b]} E(x[3b+D]) \subset \sum_{x \in S} E(x[4b+D]),$$

and E is $(4b+D)$-lean.

(2) Suppose both E' and E'' are D'-split. We have

$$gE(T \cup U) \subset E''(T[b] \cup U[b]),$$

because in general $(T \cup U)[b] \subset T[b] \cup U[b]$. So

$$\begin{aligned} &g(T \cup U) \\ &\subset E''(T[b+D']) + E''(U[b+D']) \\ &\subset gE(T[2b+D']) + gE(U[2b+D']). \end{aligned}$$

If $z \in E(T \cup U)$ then we can write $g(z) = g(z_1) + g(z_2)$ where $z_1 \in E(T[2b+D'])$ and $z_2 \in E(U[2b+D'])$. Since $z - z_1 - z_2$ is an element of $\ker(g) \cap E(T[2b+D'] \cup U[2b+D'])$, we have an element

$$\begin{aligned} k &\in E'(T[3b+D'] \cup U[3b+D']) \\ &\subset E'(T[3b+2D']) + E'(U[3b+2D']) \end{aligned}$$

such that

$$z = f(k) + z_1 + z_2 \in E(T[4b+2D']) + E(U[4b+2D']).$$

So E is $(4b+2D')$-split.

(3) Assuming that both E' and E'' are d-insular, for any pair of subsets T and U of X,

$$\begin{aligned} &g(E(T) \cap E(U)) \\ &\subset E''(T[b]) \cap E''(U[b]) \\ &\subset E''(T[b+d] \cap U[b+d]). \end{aligned}$$

Now we have

$$\begin{aligned} &E(T) \cap E(U) \\ &\subset E(T[2b+d] \cap U[2b+d]) + fE' \cap E(T[2b+d]) \cap E(U[2b+d]) \\ &\subset E(T[2b+d] \cap U[2b+d]) + f(E'(T[3b+d]) \cap E'(U[3b+d]) \\ &\subset E(T[2b+d] \cap U[2b+d]) + fE'(T[3b+2d] \cap U[3b+2d]) \\ &\subset E(T[4b+2d] \cap U[4b+2d]). \end{aligned}$$

This shows that E is $(4b+2d)$-insular. $\square$

3.1.17. **Lemma.** *Let*

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in $\mathbf{U}(X, R)$.

(1) *If the object* E *is lean then* E'' *is lean.*
(2) *If* E *is split then* E'' *is split.*
(3) *If* E *is insular then* E' *is insular.*
(4) *If* E *is insular and* E' *is lean then* E'' *is insular.*
(5) *If* E *is insular and* E' *is split then* E'' *is insular.*
(6) *If* E *is split and* E'' *is insular then* E' *is split.*

Proof. Let $b \geq 0$ be a common filtration degree for f and g. If E is D-lean, D'-split, or d-insular, it is easy to show that E'' is $(D+2b)$-lean or $(D'+2b)$-split and E' is $(d+2b)$-insular respectively, which verifies (1), (2), and (3).

(4) Suppose E' is D-lean and E is d-insular. For any pair of subsets T, $U \subset X$,

$$E''(T) \cap E''(U) \subset gE(T[b]) \cap gE(U[b]).$$

Given $z \in E''(T) \cap E''(U)$, let $y' \in E(T[b])$ and $y'' \in E(U[b])$ so that $g(y') = g(y'') = z$. Now

$$k = y' - y'' \in \big(E(T[b]) + E(U[b])\big) \cap \ker(g),$$

so there is $\overline{k} \in E'(T[2b]) + E'(U[2b]) \subset E'(T[2b] \cup U[2b])$ with $f(\overline{k}) = k$. Since E' is D-lean,

$$(\natural) \qquad \overline{k} \in \sum_{x \in T[2b] \cup U[2b]} E'(x[D]) = \sum_{x \in T[2b]} E'(x[D]) + \sum_{y \in U[2b]} E'(y[D]).$$

Hence,

$$(\sharp) \qquad \overline{k} \in E'(T[2b+D]) + E'(U[2b+D]).$$

This allows us to write $\overline{k} = \overline{k}_1 + \overline{k}_2$, where $\overline{k}_1 \in E'(T[2b+D])$ and $\overline{k}_2 \in E'(U[2b+D])$. Now $k = f\overline{k}_1 + f\overline{k}_2$. Notice that

$$y' = y'' + k = y'' + f\overline{k}_1 + f\overline{k}_2.$$

So

$$y = y' - f\overline{k}_1 = y'' + f\overline{k}_2$$

has the property

$$y \in E(T[3b+D]) \cap E(U[3b+D]) \subset E(T[3b+D+d] \cap U[3b+D+d]),$$

and $g(y) = z$. Hence

$$z \in E''(T[4b+D+d] \cap U[4b+D+d]).$$

We conclude that E'' is $(4b+D+d)$-insular.

(5) Showing that E'' is $(4b+D'+d)$-insular if E' is D'-split is entirely similar to the proof of part (4). Equation ($\natural$) in that proof is the only step that uses D-leanness of E'. The consequence in Equation ($\sharp$) follows, in fact, directly from the assumption that E' is D-split.

(6) We now address the converse. Suppose E is D'-split and E'' is d-insular. Given $z \in E'(T \cup U)$, we have $f(z) \in E(T[b] \cup U[b])$. Now $f(z) \in E(T[b+D']) + E(U[b+D'])$, so we can write accordingly $f(z) = y_1 + y_2$. Now $f(z) \in \ker(g)$, because $g(y_1) + g(y_2) = 0$. Since E'' is d-insular,

$$g(y_1) = -g(y_2) \in E''(T[2b+D'+d] \cap U[2b+D'+d]),$$

so we are able to find

$$y \in E(T[3b + D' + d] \cap U[3b + D' + d])$$

such that $g(y) = g(y_1) = -g(y_2)$, because generally $(S \cap P)[b] \subset S[b] \cap P[b]$. Thus

$$f(z) = y_1 + y_2 = (y_1 - y) + (y_2 + y)$$

and

$$y_1 - y \in E(T[3b + D' + d]), \quad y_2 + y \in E(U[3b + D' + d]).$$

Let $z_1 = f^{-1}(y_1 - y)$ and $z_2 = f^{-1}(y_2 + y)$, and we have $z = z_1 + z_2$ such that

$$z_1 \in E'(T[4b + D' + d]), \quad z_2 \in E'(U[4b + D' + d]),$$

so E' is $(4b + D' + d)$-split. □

3.1.18. **Corollary.** *Let*

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in $\mathbf{U}(X, R)$*. If* E *is split and insular then* E'' *is insular if and only if* E' *is split.*

Proof. This fact is the combination of parts (5) and (6) of the Lemma. □

3.1.19. **Remark.** The last Corollary is in contrast with the absence of the analogous fact if one substitutes the lean property for the split property. However, the analogue is true in the presence of certain geometric assumptions on the metric space X. For example, suppose X has finite asymptotic dimension. Then from the main theorem of [4], we have the following counterpart to part (6) of the Lemma: if E is lean and E'' is insular then E' is lean. This fact is not needed in this paper. Here, the excision properties of the theory rely only on the properties of the cokernels. For the applications in [5], the properties of the kernels become crucial, and the geometric conditions need to be imposed.

3.1.20. **Definition.** We define $\mathbf{L}(X, R)$ as the full subcategory of $\mathbf{U}(X, R)$ on objects that are lean and insular with the induced exact structure. Similarly, $\mathbf{S}(X, R)$ is the full subcategory of $\mathbf{U}(X, R)$ on objects that are split and insular.

Exactness of $\mathbf{L}(X, R)$ and $\mathbf{S}(X, R)$ can be induced from $\mathbf{U}(X, R)$.

3.1.21. **Definition.** A full subcategory $\mathbf{H}$ of an exact category $\mathbf{C}$ is said to be *closed under extensions* in $\mathbf{C}$ if $\mathbf{H}$ contains the zero object and for any exact sequence $C' \to C \to C''$ in $\mathbf{C}$, if C' and C'' are isomorphic to objects from $\mathbf{H}$ then so is C.

It is known that a subcategory closed under extensions in $\mathbf{C}$ inherits the exact structure from $\mathbf{C}$.

3.1.22. **Theorem.** $\mathbf{L}(X, R)$ *and* $\mathbf{S}(X, R)$ *are closed under extensions in* $\mathbf{U}(X, R)$*.*

Proof. The first fact follows from parts (1) and (3) of Lemma 3.1.16, the second from (2) and (3). □

3.1.23. **Corollary.** $\mathbf{L}(X, R)$ *and* $\mathbf{S}(X, R)$ *are exact subcategories of* $\mathbf{U}(X, R)$*. Therefore, we have a sequence of exact inclusions*

$$\mathbf{L}(X, R) \longrightarrow \mathbf{S}(X, R) \longrightarrow \mathbf{U}(X, R).$$

3.1.24. **Definition.** An X-filtered R-module F is *locally finitely generated* if $F(S)$ is a finitely generated R-module for every bounded subset $S \subset X$.

3.1.25. **Definition.** The category $\mathbf{BL}(X,R)$ is the full subcategory of $\mathbf{L}(X,R)$ on the locally finitely generated objects. Similarly, the companion category $\mathbf{BS}(X,R)$ is the full subcategory of $\mathbf{S}(X,R)$ on the locally finitely generated objects.

3.1.26. **Theorem.** *The category $\mathbf{BL}(X,R)$ is closed under extensions in $\mathbf{L}(X,R)$. Similarly, the category $\mathbf{BS}(X,R)$ is closed under extensions in $\mathbf{S}(X,R)$.*

Proof. If $f\colon F \to G$ is an isomorphism with $\mathrm{fil}(f) \leq b$ and G is locally finitely generated, then $F(U)$ are finitely generated submodules of $G(U[b])$ for all bounded U, since R is a noetherian ring. Suppose

$$F' \xrightarrow{f} F \xrightarrow{g} F''$$

is an exact sequence and let $b \geq 0$ be a common filtration degree for both f and g. Assume that F' and F'' are locally finitely generated. For every bounded subset $U \subset X$ the restriction $g\colon F(U) \to gF(U)$ is an epimorphism onto a submodule of the finitely generated R-module $F''(U[b])$. The kernel of $g|F(U)$ is a submodule of $F'(U[b])$, which is also finitely generated. So the extension $F(U)$ is finitely generated. □

3.1.27. **Corollary.** *$\mathbf{BL}(X,R)$ and $\mathbf{BS}(X,R)$ are exact categories. The additive category $\mathcal{C}(X,R)$ of geometric R-modules with the split exact structure is an exact subcategory of $\mathbf{BL}(X,R)$. There is a sequence of exact inclusions*

$$\mathcal{C}(X,R) \longrightarrow \mathbf{BL}(X,R) \longrightarrow \mathbf{BS}(X,R) \longrightarrow \mathbf{U}(X,R).$$

Recall that a morphism $e\colon F \to F$ is an idempotent if $e^2 = e$. Categories in which every idempotent is the projection onto a direct summand of F are called *idempotent complete.*

3.1.28. **Proposition.** *A pseudoabelian category is idempotent complete.*

Proof. The proof is exactly the same as for an abelian category: if e is an idempotent then its kernel is split by $1 - e$. □

3.1.29. **Corollary.** *$\mathbf{BL}(X,R)$ and $\mathbf{BS}(X,R)$ are idempotent complete.*

Proof. Since the restriction of an idempotent e to the image of e is the identity, every idempotent is boundedly bicontrolled of filtration 0. It follows easily that the splitting of e in $\mathbf{Mod}(R)$ is in fact a splitting in $\mathbf{BL}(X,R)$ or $\mathbf{BS}(X,R)$. □

Finally, we need to address (the lack of) inheritance features in filtered modules. First, we recall the following definition from [6].

3.1.30. **Definition.** An X-filtered object F is called *strict* if there exists an order preserving function $\ell\colon \mathcal{P}(X) \to [0,+\infty)$ such that for every $S \subset X$ the submodule $F(S)$ is ℓ_S-lean and ℓ_S-insular with respect to the standard X-filtration $F(S)(T) = F(S) \cap F(T)$.

It is important to note that this property is not preserved under isomorphisms, so the subcategory of strict objects is not essentially full in $\mathbf{BL}(X,R)$.

3.1.31. **Definition.** The *bounded category* $\mathbf{B}(X,R)$ was defined in [6] as the full subcategory of $\mathbf{BL}(X,R)$ on objects isomorphic to strict objects.

A consequence of strictness (or being isomorphic to a strict object) is the following feature. Given a filtered module F in $\mathbf{B}(X,R)$, a *lean grading* of F is a functor $\widetilde{F}\colon \mathcal{P}(X) \to \mathcal{I}(F)$ from the power set of X to the submodules of F such that

(1) each $\widetilde{F}(S)$ is an object of $\mathbf{BL}(X,R)$ when given the standard filtration,
(2) there is a number $K \geq 0$ such that
$$F(S) \subset \widetilde{F}(S) \subset F(S[K])$$
for all subsets S of X.

Clearly, each $\widetilde{F}(S)$ is an object of $\mathbf{B}(X,R)_{<S}$. Also a strict object has a grading by $\widetilde{F}(S) = F(S)$ with $K = 0$.

We note for the interested reader that the theory in [6], including the excision theorems, could be alternately developed for lean graded modules in place of strict filtered modules. We do not require such theory in this paper. Instead, we develop a similar but weaker notion of gradings in $\mathbf{BS}(X,R)$.

3.1.32. **Definition.** Given a filtered module F in $\mathbf{BS}(X,R)$, a *grading* of F is a functor $\mathcal{F} \colon \mathcal{P}(X) \to \mathcal{I}(F)$ such that

(1) each $\mathcal{F}(S)$ is an object of $\mathbf{BS}(X,R)$ when given the standard filtration,
(2) there is a number $K \geq 0$ such that
$$F(S) \subset \mathcal{F}(S) \subset F(S[K])$$
for all subsets S of X.

We will say that a filtered module F is *graded* if it has a grading.

3.1.33. **Proposition.** *The graded objects are closed under isomorphisms.*

Proof. If $f \colon F \to F'$ is an isomorphism and F has a grading $\mathcal{F}$, a grading for F' is given by $\mathcal{F}'(C) = f\mathcal{F}(C[K+b])$, where b is a filtration bound for f. □

3.1.34. **Definition.** We define $\mathbf{G}(X,R)$ as the full subcategory of $\mathbf{BS}(X,R)$ on the locally finitely generated graded filtered modules.

3.1.35. **Proposition.** $\mathbf{G}(X,R)$ *is closed under extensions in* $\mathbf{BS}(X,R)$*. Therefore* $\mathbf{G}(X,R)$ *is an exact subcategory of* $\mathbf{BS}(X,R)$*.*

Proof. Given an exact sequence in $\mathbf{BS}(X,R)$
$$F \xrightarrow{f} G \xrightarrow{g} H,$$
let $b \geq 0$ be a common filtration degree for both f and g as boundedly bicontrolled maps, and assume that F and H are graded modules in $\mathbf{G}(X,R)$ with the associated functors $\mathcal{F}$ and $\mathcal{H}$.

To define a grading for G, consider a subset S and suppose $\mathcal{H}(S[b])$ is D-split and d-insular. The induced epimorphism
$$g \colon G(S[2b]) \cap g^{-1}\mathcal{H}(S[b]) \longrightarrow \mathcal{H}(S[b])$$
extends to another epimorphism
$$g' \colon f\mathcal{F}(S[3b]) + G(S[2b]) \cap g^{-1}\mathcal{H}(S[b]) \longrightarrow \mathcal{H}(S[b])$$
with $\ker(g') = \mathcal{F}(S[3b])$. Without loss of generality, suppose $\mathcal{F}(S[3b])$ is D-split and d-insular. We define
$$\mathcal{G}(S) = f\mathcal{F}(S[3b]) + G(S[2b]) \cap g^{-1}\mathcal{H}(S[b]).$$
From parts (2) and (3) of Lemma 3.1.16, the module $\mathcal{G}(S)$ with the standard filtration is $(4b+2d)$-split and $(4b+2d)$-insular. Since $G(S) \subset g^{-1}\mathcal{H}(S[b])$, we have $G(S) \subset \mathcal{G}(S)$. On the other hand, if the grading $\mathcal{F}$ has characteristic number $K \geq 0$

then $\mathcal{G}(S) \subset G(S[4b+K])$. The last fact together with Theorem 3.1.26 shows that $\mathcal{G}(S)$ is finitely generated. $\square$

3.1.36. **Corollary.** *There is a commutative diagram of exact inclusions and exact forgetful functors*

$$\begin{tikzcd} & \mathbf{BL}(X,R) \arrow[r] & \mathbf{BS}(X,R) \\ \mathcal{C}(X,R) \arrow[ur, bend left] \arrow[dr, bend right] & & \\ & \mathbf{B}(X,R) \arrow[r] \arrow[uu] & \mathbf{G}(X,R) \arrow[uu] \end{tikzcd}$$

The advantage of working with the category $\mathbf{G}(X,R)$ is that one can readily localize to geometrically defined subobjects.

3.1.37. **Lemma.** *Suppose G is a graded X-filtered module with a grading $\mathcal{G}$. Let F be a submodule which is split with respect to the standard filtration. Then $\mathcal{F}(S) = F \cap \mathcal{G}(S)$ is a grading of F.*

Proof. Of course, $F(S) = F \cap G(S) \subset F \cap \mathcal{G}(S) = \mathcal{F}(S)$. On the other hand, there is $d \geq 0$ such that $\mathcal{G}(S) \subset G(S[d])$, so $\mathcal{F}(S) \subset F \cap G(S[d]) = F(S[d])$.

Consider the inclusion of modules $i\colon F \to G$, and take the quotient $q\colon G \to H$. Both F and G are split and insular, so H is split and insular by parts (2) and (4) of Lemma 3.1.17, with respect to the quotient filtration. We define $\mathcal{H}(S)$ as the partial image $q\mathcal{G}(S)$ and give $\mathcal{H}(S)$ the standard filtration in H. Then $\mathcal{H}(S)$ is split as the image of a split $\mathcal{G}(S)$ and insular since H is insular. Now the kernel of the epimorphism $q|\colon \mathcal{G}(S) \to \mathcal{H}(S)$, which is $F \cap \mathcal{G}(S)$, is split by part (6) of Lemma 3.1.17. Since F is insular, $\mathcal{F}(S)$ is also insular. This shows that $\mathcal{F}(S)$ gives a grading for F. $\square$

This result can be promoted to the following statement.

3.1.38. **Proposition.** *Suppose F is the kernel of a boundedly bicontrolled epimorphism $g\colon G \to H$ in $\mathbf{BS}(X,R)$. If G is graded and F is split then both H and F are graded.*

Proof. The grading for H is given by $\mathcal{H}(S) = g\mathcal{G}(S[b])$, where b is a chosen bicontrol bound for g. Each $\mathcal{H}(S)$ is split and insular as in the proof of Lemma 3.1.37. The inclusions $H(S) \subset gG(S[b]) \subset g\mathcal{G}(S[b]) = \mathcal{H}(S)$ and $g\mathcal{G}(S[b]) \subset gG(S[b+K]) \subset H(S[2b+K])$ show that $\mathcal{H}$ is a grading. The same argument as in Lemma 3.1.37 shows that $\mathcal{F}(S) = F \cap \mathcal{G}(S[b])$ gives a grading for F. $\square$

We will use the following convention. When $d \leq 0$, the notation $S[d]$ will stand for the subset $S \setminus (X \setminus S)[-d]$.

3.1.39. **Corollary.** *Given an object F in $\mathbf{G}(X,R)$ and a subset S of X, there is a number $K \geq 0$ and an admissible subobject $i\colon F_S \to F$ in $\mathbf{G}(X,R)$ with the property that $F_S \subset F(S[K])$. Moreover, the cokernel $q\colon F \to H$ has the property that $H(X) = H((X \setminus S)[2D'])$, where D' is a splitting constant for F.*

Proof. For the first statement, choose $F_S = \mathcal{F}(S)$ with the grading defined in Lemma 3.1.37 and apply Proposition 3.1.38. The second statement is shown as follows. By part (2) of Lemma 3.1.17, since $\mathrm{fil}(q) = 0$, if F is D'-split then H is D'-split. Let $T = S[-D']$, then $T[D'] \subset S$, so $H(T[D']) = qF(T[D']) \subset qF(S) \subset qF_S = 0$. Using the decomposition $X = T \cup (X \setminus T)$ we can write $H(X) = H(T[D']) + H((X \setminus T)[D']) = H((X \setminus T)[D']) = H((X \setminus S)[2D'])$. $\square$

The last three results can be summarized as follows.

3.1.40. **Corollary.** *Given a graded object F in $\mathbf{G}(X,R)$ and a subset S of X, we assume that F is D'-split and d-insular and is graded by $\mathcal{F}$. The submodules $\mathcal{F}(S)$ have the following properties:*

1. *each $\mathcal{F}(S)$ is graded by $\mathcal{F}_S(T) = \mathcal{F}(S) \cap \mathcal{F}(T)$;*
2. *$F(S) \subset \mathcal{F}(S) \subset F(S[K])$ for some fixed number $K \geq 0$;*
3. *suppose $q \colon F \to H$ is the cokernel of the inclusion $i \colon \mathcal{F}(S) \to H$, then H is supported on $(X \setminus S)[2D']$;*
4. *$H(S[-2D' - 2d]) = 0$.*

Proof. Properties (1), (2), (3) are consequences of the last three results. (4) follows from the fact that a d-insular filtered module is $2d$-separated, in the sense that for any pair of subsets S and T such that $S[2d] \cap T = \emptyset$ we have $S[d] \cap T[d] = \emptyset$ so $F(S) \cap F(T) = 0$. Now $H(S[-2D' - 2d]) \cap H((X \setminus S)[2D']) = 0$, but $H((X \setminus S)[2D']) = H(X)$, thus $H(S[-2D' - 2d]) = 0$. $\square$

3.2. **Fibred Bounded G-theory.** Suppose X and Y are two proper metric spaces and R is a noetherian ring. The product $X \times Y$ is given the *product metric*

$$d((x,y),(x',y')) = \max\{d(x,x'), d(y,y')\}.$$

Of course, there is the exact category $\mathbf{L}(X \times Y, R)$ and the associated bounded category $\mathbf{BL}(X \times Y, R)$. We now wish to construct a larger *fibred bounded category* $\mathbf{B}_X(Y)$ which will extend $\mathcal{C}_X(Y)$ similarly to the extension of $\mathcal{C}(X,R)$ by $\mathbf{BL}(X,R)$. The result will in fact have a mix of features from $\mathbf{BL}(X,R)$ and $\mathbf{BS}(Y,R)$.

3.2.1. **Definition.** Given an R-module F, an (X,Y)-*filtration* of F is a functor

$$\phi_F \colon \mathcal{P}(X \times Y) \longrightarrow \mathcal{I}(F)$$

from the power set of the product metric space to the partially ordered family of R-submodules of $F(X \times Y)$. Whenever F is given a filtration, and there is no ambiguity, we will denote the values $\phi_F(U)$ by $F(U)$. We assume that F is *reduced* in the sense that the value on the empty subset is 0.

The associated X-filtered R-module F_X is given by

$$F_X(S) = F(S \times Y).$$

Similarly, for each subset $S \subset X$, one has the Y-filtered R-module F^S given by

$$F^S(T) = F(S \times T).$$

In particular, $F^X(T) = F(X \times T)$.

We will use the following notation generalizing enlargements in a metric space.

3.2.2. *Notation.* Given a subset U of $X \times Y$ and a function $k \colon X \to [0, +\infty)$, let

$$U[k] = \{(x,y) \in X \times Y \mid \text{there is } (x,y') \in U \text{ with } d(y,y') \leq k(x)\}.$$

If in addition we are given a number $K \geq 0$ then

$$U[K,k] = \{(x,y) \in X \times Y \mid \text{there is } (x',y) \in U[k] \text{ with } d(x,x') \leq K\}.$$

So $U[k] = U[0,k]$. Notice that if U is a single point (x,y) then

$$U[K,k] = x[K] \times y[k(x)] = (x,y)[K,0] \times (x,y)[0,k(x)].$$

More generally, one can equivalently write

$$U[K,k] = \bigcup_{(x,y)\in U} x[K] \times y[k(x)].$$

If U is a product set $S \times T$, it will be convenient to use the notation $(S,T)[K,k]$ in place of $(S \times T)[K,k]$. More generally, because the roles of the factors are very different when working with (X,Y)-filtrations, we will use the notation (X,Y) for the product metric space so that the order of the factors is unambiguous. Similarly, we will use the notation (S,T) for the product subset $S \times T$ in (X,Y).

3.2.3. **Definition.** We will refer to the pair (K,k) in the notation $U[K,k]$ as the *enlargement data.*

It is clear that when Y = point, $U[K,k] = U[K]$ for any function k under the identification $X \times Y = X$.

3.2.4. *Notation.* Let x_0 be a chosen fixed point in X. Given a monotone function $h\colon [0,+\infty) \to [0,+\infty)$, there is a function $h_{x_0}\colon X \to [0,+\infty)$ defined by

$$h_{x_0}(x) = h(d_X(x_0,x)).$$

3.2.5. **Definition.** Given two (X,Y)-filtered modules F and G, an R-homomorphism $f\colon F(X\times Y) \to G(X \times Y)$ is *boundedly controlled* if there are a number $b \geq 0$ and a monotone function $\theta\colon [0,+\infty) \to [0,+\infty)$ such that

$$fF(U) \subset G(U[b,\theta_{x_0}]) \tag{$\dagger$}$$

for all subsets $U \subset X \times Y$ and some choice of $x_0 \in X$.

It is clear that the condition is independent of the choice of x_0.

3.2.6. **Definition.** An (X,Y)-filtered module F is called

- *lean* or (D,Δ)-*lean* if there is a number $D \geq 0$ and a monotone function $\Delta\colon [0,+\infty) \to [0,+\infty)$ so that
$$\begin{aligned} F(U) &\subset \sum_{(x,y)\in U} F(x[D] \times y[\Delta_{x_0}(x)]) \\ &= \sum_{(x,y)\in U} F((x,y)[D,\Delta_{x_0}]) \end{aligned}$$
for any subset U of $X \times Y$,
- *split* or (D',Δ')-*split* if there is a number $D' \geq 0$ and a monotone function $\Delta'\colon [0,+\infty) \to [0,+\infty)$ so that
$$F(U_1 \cup U_2) \subset F(U_1[D',\Delta'_{x_0}]) + F(U_2[D',\Delta'_{x_0}])$$
for each pair of subsets U_1 and U_2 of $X \times Y$,
- *lean/split* or (D,Δ')-*lean/split* if there is a number $D \geq 0$ and a monotone function $\Delta'\colon [0,+\infty) \to [0,+\infty)$ so that

– the X-filtered module F_X is D-lean, while
– the (X,Y)-filtered module F is (D,Δ')-split,

- *insular* or (d,δ)-*insular* if there is a number $d \geq 0$ and a monotone function $\delta\colon [0,+\infty) \to [0,+\infty)$ so that
$$F(U_1) \cap F(U_2) \subset F\big(U_1[d,\delta_{x_0}] \cap U_2[d,\delta_{x_0}]\big)$$
for each pair of subsets U_1 and U_2 of $X \times Y$.

3.2.7. **Remark.** Suppose F is an (X,Y)-filtered R-module.

(1) If F is (D,Δ)-lean then the corresponding X-filtered module F_X is D-lean.
(2) Similarly, if F is (d,δ)-insular then F_X is d-insular.
(3) If F is (D,Δ)-lean then it is (D,Δ)-lean/split and, further, (D,Δ)-split.
(4) An (X,Y)-filtered module F which is lean/split and insular can be thought of as an object F_X of $\mathbf{L}(X,R)$.

3.2.8. **Proposition.** *Suppose $f\colon F \to G$ is boundedly controlled. Then*

(1) *f is bounded when viewed as a morphism $: F_X \to G_X$ in $\mathbf{U}(X,R)$, and*
(2) *for each bounded subset $S \subset X$, the restriction $f\colon F_X(S) \to G_X(S[b])$ is bounded when viewed as a morphism $F^S \to G^{S[b]}$ of Y-filtered modules in $\mathbf{U}(Y,R)$.*

Proof. If $f\colon F \to G$ is (b,θ)-controlled then for any subset $S \subset X$ we have $fF_X(S) \subset G((S,Y)[b,\theta_{x_0}]) \subset G(S[b],Y) = G_X(S[b])$. So $f\colon F_X \to G_X$ is bounded by b. Now for a given bounded subset $S \subset X$, let us define $\theta_S = \sup_{x \in S} \theta_{x_0}(x)$. Then $fF_X(S)(T) = fF(S,T) \subset G(S[b],T[\theta_S]) = G_X(S[b])(T[\theta_S])$ verifying that $f|\colon F^S \to G^{S[b]}$ is bounded by θ_S. □

3.2.9. **Remark.** The converse to part (2) is only true when F is lean but not necessarily when F is lean/split.

3.2.10. **Definition.** There are several nested categories of (X,Y)-filtered modules.

- $\mathbf{U}_X(Y)$ has objects that are arbitrary (X,Y)-filtered R-modules, the morphisms are the boundedly controlled homomorphisms.
- $\mathbf{LS}_X(Y)$ is the full subcategory of $\mathbf{U}_X(Y)$ on objects F that are lean/split and insular,
- $\mathbf{B}_X(Y)$ is the full subcategory of $\mathbf{LS}_X(Y)$ on objects F such that $F(U)$ is a finitely generated submodule whenever $U \subset X \times Y$ is bounded. Equivalently, the subcategory $\mathbf{B}_X(Y)$ is full on objects F such that all Y-filtered modules F^S associated to bounded subsets $S \subset X$ are locally finitely generated.

3.2.11. **Definition.** A morphism $f\colon F \to G$ in $\mathbf{U}_X(Y)$ is *boundedly bicontrolled* if there is filtration data $b \leq 0$ and $\theta\colon [0,+\infty) \to [0,+\infty)$ as in Definition 3.2.5, and in addition to (†) one also has the containments
$$fF \cap G(U) \subset fF(U[b,\theta_{x_0}]).$$
In this case, we will use the notation $\mathrm{fil}(f) \leq (b,\theta)$.

3.2.12. **Definition.** Let the *admissible monomorphisms* in $\mathbf{U}_X(Y)$ be the boundedly bicontrolled homomorphisms $m\colon F_1 \to F_2$ such that the module homomorphism $F_1(X \times Y) \to F_2(X \times Y)$ is a monomorphism. Let the *admissible epimorphisms* be the boundedly bicontrolled homomorphisms $e\colon F_1 \to F_2$ such that

$F_1(X \times Y) \to F_2(X \times Y)$ is an epimorphism. The class $\mathcal{E}$ of *exact sequences* consists of the sequences

$$F^{\cdot}\colon\ F' \xrightarrow{i} F \xrightarrow{j} F'',$$

where i is an admissible monomorphism, j is an admissible epimorphism, and $\mathrm{im}(i) = \ker(j)$.

One can argue as in [6] that the admissible monomorphisms are precisely the morphisms isomorphic in $\mathbf{U}_X(Y)$ to the filtration-wise monomorphisms and the admissible epimorphisms are those morphisms isomorphic to the filtration-wise epimorphisms.

3.2.13. **Proposition.** *Assume that $\mathbf{U}_X(Y)$ is given the class of exact sequences $\mathcal{E}$.*

(1) *$\mathbf{U}_X(Y)$ is a cocomplete exact pseudoabelian category.*
(2) *The lean/split objects are closed under extensions.*
(3) *The insular objects are closed under extensions.*

Suppose

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

is an exact sequence in $\mathbf{U}_X(Y)$.

(4) *If the object E is lean/split then E'' is lean/split.*
(5) *If E is insular then E' is insular.*
(6) *Suppose E is insular then E'' is insular if E' is lean/split.*

Proof. (1) can be checked directly. An alternative is to use the iterative idea that $\mathbf{U}_X(Y)$ can be viewed as $\mathbf{U}(X, \mathbf{U}(Y,R))$ and the observation that the cocomplete exact pseudoabelian category $\mathbf{U}(Y,R)$ can be substituted for $\mathbf{Mod}(R)$ in the proof of Theorem 3.1.9 and related constructions.

Other parts of the Theorem are proved by adapting the proofs of Lemmas 3.1.16 and 3.1.17. To illustrate, let

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in $\mathbf{U}_X(Y)$. Suppose that (b,θ) is common filtration data for f and g and both E' and E'' are (D,Δ')-lean/split. For the first statement of part (2), notice that E_X is $(4b+D)$-lean by part (1) of Lemma 3.1.16, so we need to verify that split objects are closed under extensions. Consider two subsets U_1 and U_2 of $X \times Y$. Then

$$\begin{aligned} gE(U) &\subset E''((U_1 \cup U_2)[b,\theta_{x_0}]) \\ &= E''(U_1[b,\theta_{x_0}] \cup U_2[b,\theta_{x_0}]) \\ &\subset E''(U_1[b+D,\theta_{x_0}+\Delta'_{x_0}]) + E''(U_2[b+D,\theta_{x_0}+\Delta'_{x_0}]). \end{aligned}$$

Therefore

$$\begin{aligned} E(U) \subset\ & E(U_1[2b+D, 2\theta_{x_0}+\Delta'_{x_0}]) + E(U_2[2b+D, 2\theta_{x_0}+\Delta'_{x_0}]) \\ &+ fE'(U_1[3b+2D, 3\theta_{x_0}+2\Delta'_{x_0}]) + fE'(U_2[3b+2D, 3\theta_{x_0}+2\Delta'_{x_0}]) \\ &\subset E(U_1[4b+2D, 4\theta_{x_0}+2\Delta'_{x_0}]) + E(U_2[4b+2D, 4\theta_{x_0}+2\Delta'_{x_0}]), \end{aligned}$$

showing that E is $(4b+2D, 4\theta+2\Delta')$-lean/split. □

3.2.14. **Proposition.** *$\mathbf{LS}_X(Y)$ is closed under extensions in $\mathbf{U}_X(Y)$.*

Proof. This follows from parts (2) and (3) of Proposition 3.2.13. □

3.2.15. **Proposition.** *$\mathbf{B}_X(Y)$ is closed under extensions in $\mathbf{LS}_X(Y)$. Therefore, $\mathbf{B}_X(Y)$ is an exact category, and the inclusion $e\colon \mathcal{C}_X(Y) \to \mathbf{B}_X(Y)$ is an exact embedding.*

Proof. Suppose $f\colon F \to G$ is an isomorphism with $\mathrm{fil}(f) \leq (b, \theta)$ and G is locally finitely generated, then $F(U)$ is a finite generated submodule of $G(U[b, \theta])$ for any bounded subset $U \subset X \times Y$ since R is noetherian. If

$$F' \xrightarrow{f} F \xrightarrow{g} F''$$

is an exact sequence in $\mathbf{LS}_X(Y)$, F' and F'' are locally finitely generated, and (b, θ) is common filtration data for f and g, then $gF(U)$ is a finitely generated submodule of $F''(U[b, \theta])$ for any bounded subset U. The kernel of the restriction of g to $F(U)$ is a finitely generated submodule of $F'(U[b, \theta])$, so the extension $F(U)$ is finitely generated. □

3.2.16. **Remark.** There is an exact embedding $\iota\colon\ \mathbf{B}(X \times Y, R) \to \mathbf{B}_X(Y)$ which is given by the identity on objects. The same comments as in the case of geometric bounded categories of Pedersen–Weibel apply: the morphism sets in the image of ι are in general properly smaller than in $\mathbf{B}_X(Y)$. This time, however, ι is also proper on objects. For example, the lean objects in $\mathbf{BL}(X \times Y, R)$ are generated by the submodules $f(S \times T)$ where the diameters of S and T are uniformly bounded from above. This is different from the weaker condition in $\mathbf{B}_X(Y)$.

Suppose X is a proper metric space and Z is a subset of X. There are localization and fibration theorems for controlled G-theory developed in [6]. We will generalize some of those results to the fibred setting.

3.2.17. **Definition.** An object F of $\mathbf{U}(X, R)$ is *supported near* Z if there is a number $d \geq 0$ such that

$$F(X) \subset F(Z[d]).$$

The objects supported near Z form the full subcategory $\mathbf{U}(X, R)_{<Z}$.

One can readily check that $\mathbf{U}(X, R)_{<Z}$ is closed under exact extensions in $\mathbf{U}(X, R)$, so $\mathbf{U}(X, R)_{<Z}$ is an exact subcategory. The intersection $\mathbf{B}(X, R)_{<Z} = \mathbf{B}(X, R) \cap \mathbf{U}(X, R)_{<Z}$ is an exact subcategory of $\mathbf{B}(X, R)$.

There are two complementary ways to introduce support in $\mathbf{B}_X(Y)$.

(1) Let $\mathbf{B}_{<Z}(Y)$ be the full subcategory of $\mathbf{B}_X(Y)$ on objects F supported near Z viewed as objects F_X in $\mathbf{U}(X, R)$. In other words, F is an object of $\mathbf{B}_{<Z}(Y)$ if

$$F_X \subset F_X(Z[d]) = F(Z[d] \times Y)$$

for some number $d \geq 0$.

(2) Let $\mathbf{B}_X(Y)_{<C}$ be the full subcategory of $\mathbf{B}_X(Y)$ on objects F such that

$$F(X, Y) \subset F\big((X, C)[r, \rho_{x_0}]\big)$$

for some number $r \geq 0$ and an order preserving function $\rho\colon [0, +\infty) \to [0, +\infty)$.

The notion (1) of support is a straightforward generalization of support for geometric modules and was exploited in [6]. We now explore the latter version (2) of support.

3.2.18. **Proposition.** *Suppose F is a (D,Δ)-lean/split object of $\mathbf{B}_X(Y)$. The following are equivalent statements.*

(1) *F is an object of $\mathbf{B}_X(Y)_{<C}$.*
(2) *There is a number $k \geq 0$ and an order preserving function $\lambda\colon \mathcal{B}(X) \to [0,+\infty)$ such that*
$$F^S \subset F^{S[k]}(C[\lambda(S)])$$
for all bounded subsets $S \subset X$.
(3) *There is a number $k \geq 0$ and a monotone function $\Lambda\colon [0,+\infty) \to [0,+\infty)$ such that*
$$F^{x[D]} \subset F^{x[D+k]}(C[\Lambda_{x_0}(x)])$$
for all $x \in X$.

Proof. (2) $\Longleftrightarrow$ (3): If F satisfies (2) then
$$F^{x[D]} \subset F^{x[D+k]}\big(C[\lambda(x[D])]\big).$$
It suffices to define Λ such that
$$\lambda(x[D]) \leq \Lambda_{x_0}(x) = \Lambda(d(x_0,x)).$$
Since $x[D] \leq x_0[d(x_0,x)+D]$ and λ is order preserving, one can take
$$\Lambda(r) = \lambda\big(x_0[r+D]\big).$$
In the opposite direction, given a bounded subset $S \subset X$,
$$F^S \subset \sum_{x\in S} F^{x[D]} \subset \sum_{x\in S} F^{x[D+k]}(C[\Lambda_{x_0}(x)]) \subset \ F^{x[D+k]}(C[\lambda(S)])$$
when $\lambda(S) = \sup\{\Lambda_{x_0}(x) \mid x \in S\}$.

(1) $\Longleftrightarrow$ (3): If F is in $\mathbf{B}_X(Y)_{<C}$ then $F^{x[D]} \subset F\big((X,C)[r,\rho]\big)$, so
$$F^{x[D]} \subset F^{x[D]} \cap F\big((X,C)[r,\rho]\big).$$
If F is (d,δ)-insular then
$$F^{x[D]} \subset \ F\big((x,C)[D+r+d,\rho+\delta]\big) \subset \ F^{x[D+d+r]}(C[\Lambda_{x_0}(x)])$$
for $\Lambda(a) = \sup\{(\delta+\rho)(z) \mid d(x_0,y) \leq a+D+d+r\}$.

In the opposite direction, we have
$$F \subset \sum_{x\in X} F^{x[D]} \subset \sum_{x\in X} F^{x[D+k]}(C[\Lambda_{x_0}(x)]) \subset \ F\big((X,C)[D+k,\Lambda_{x_0}]\big)$$
for an object F of $\mathbf{B}_X(Y)$ satisfying (3). $\square$

3.2.19. **Definition.** A *Grothendieck subcategory* of an exact category is a subcategory which is closed under exact extensions and closed under passage to admissible subobjects and admissible quotients.

3.2.20. **Proposition.** *$\mathbf{B}_X(Y)_{<C}$ is a Grothendieck subcategory of $\mathbf{B}_X(Y)$.*

Proof. First we show closure under exact extensions. Let
$$F \xrightarrow{f} G \xrightarrow{g} H$$
be an exact sequence in $\mathbf{B}_X(Y)$. Let (b,θ) be common set of filtration data for f and g and let all objects be (D,Δ)-lean/split. We assume that F and H are objects of $\mathbf{B}_X(Y)_{<C}$, so there is a number $r \geq 0$ and a monotone function $\rho\colon [0,+\infty) \to$

$[0, +\infty)$ such that at the same time $F(X, Y) = F\big((X, C)[r, \rho_{x_0}]\big)$ and $H(X, Y) = H\big((X, C)[r, \rho_{x_0}]\big)$ for some choice of a base point x_0 in X. Therefore

$$fF(X, Y) = fF\big((X, C)[r, \rho_{x_0}]\big) \subset G\big((X, C)[r + b, \rho_{x_0} + \theta_{x_0}]\big).$$

In particular, the image $I = \operatorname{im}(f)$ with the standard filtration $I^S(T) = I \cap G^S(T)$ is an object of $\mathbf{B}_X(Y)_{<C}$. Now

$$H(X, Y) = gG(X, Y) \cap H\big((X, C)[r, \rho_{x_0}]\big) \subset gG\big((X, C)[r + b, \rho_{x_0} + \theta_{x_0}]\big).$$

Let $L = G\big((X, C)[r + b, \rho_{x_0} + \theta_{x_0}]\big)$ viewed as a subobject of G with the standard filtration. Since $G = I + L$ for any submodule L with $g(L) = H$, we have

$$G(X, Y) = G\big((X, C)[r + b, \rho_{x_0} + \theta_{x_0}]\big),$$

so G is an object of $\mathbf{B}_X(Y)_{<C}$.

Suppose $f\colon F \to G$ is an admissible monomorphism in $\mathbf{B}_X(Y)$, which is a boundedly bicontrolled monic with $\operatorname{fil}(f) \leq (b, \theta)$, F is (D', Δ')-lean/split, G is (D, Δ)-lean/split for some $D \geq D' + b$, and G is (d, δ)-insular.

If G is an object of $\mathbf{B}_X(Y)_{<C}$, according to Proposition 3.2.18,

$$G^S \subset G^{S[k]}(C[\lambda(S)])$$

for some number $k \geq 0$, an order preserving function $\lambda\colon \mathcal{B}(X) \to [0, +\infty)$, and all bounded subsets $S \subset X$. Then

$$fF^{x[D']} \subset G^{x[D'+b]} \subset G^{x[D'+b+k]}(C[\lambda(x[D' + b])]) \subset G^{x[D+k]}(C[\lambda(x[D])]),$$

using the fact that λ is order preserving. Since

$$G^{x[D+k]}(Y - C[\lambda(x[D + k]) + \Delta(x[D]) + \theta(x[D']) + 2\delta(x[D + k])]) = 0,$$

we have

$$F^{x[D']}(Y - C[\lambda(x[D + k]) + \Delta(x[D]) + 2\theta(x[D']) + 2\delta(x[D + k])]) = 0.$$

Therefore

$$F^{x[D']} \subset F^{x[D']}(C[\lambda(x[D]) + \Delta(x[D]) + \Delta'(x[D]) + 2\theta(x[D']) + 2\delta(x[D + k])],$$

so F, which is generated by $F^{x[D']}$, is also an object of $\mathbf{B}_X(Y)_{<C}$.

On the other hand, let $g\colon G \to H$ be an admissible quotient with $\operatorname{fil}(g) \leq (b, \theta)$ and suppose G is an object of $\mathbf{B}_X(Y)_{<C}$ so that there is a number $r \geq 0$ and a monotone function $\rho\colon [0, +\infty) \to [0, +\infty)$ such that

$$G(X, Y) = G\big((X, C)[r, \rho_{x_0}]\big).$$

This implies that

$$H(X, Y) = gG(X, Y) \subset H\big((X, C)[r + b, \rho_{x_0} + \theta_{x_0}]\big),$$

so H is also in $\mathbf{B}_X(Y)_{<C}$. $\square$

3.3. **Fibrewise Gradings and Localization.** The gradings from Definition 3.1.32 can be generalized to gradings of objects from $\mathbf{B}_X(Y)$.

3.3.1. **Definition.** Given an object F of $\mathbf{B}_X(Y)$, a *grading* of F is a functor

$$\mathcal{F}\colon \mathcal{P}(X,Y) \longrightarrow \mathcal{I}(F)$$

with the following properties:

(1) if $\mathcal{F}(\mathcal{C})$ is given the standard filtration, it is an object of $\mathbf{B}_X(Y)$,
(2) there is an enlargement data (K,k) such that

$$F(C) \subset \mathcal{F}(C) \subset F(C[K,k_{x_0}]),$$

for all subsets C of (X,Y).

3.3.2. **Remark.** If $C = (X,S)$ then $\mathcal{F}(C)$ is an object of $\mathbf{B}_X(Y)_{<S}$.

We are concerned with localizations to a specific type of subspaces of (X,Y). This makes the following partial gradings sufficient and easier to work with.

3.3.3. **Definition.** Let $\mathcal{M}^{\geq 0}$ be the set of all monotone functions $\delta\colon [0,+\infty) \to [0,+\infty)$. Let $\mathcal{P}_X(Y)$ be the subcategory of $\mathcal{P}(X,Y)$ consisting of all subsets described as $(X,C)[D,\delta_{x_0}]$ for some choices of a subset $C \subset Y$, a number $D \geq 0$, and a function $\delta \in \mathcal{M}^{\geq 0}$.

Given an object F of $\mathbf{B}_X(Y)$, a Y-*grading* of F is a functor

$$\mathcal{F}\colon \mathcal{P}_X(Y) \longrightarrow \mathcal{I}(F)$$

with the following properties:

(1) the submodule $\mathcal{F}((X,C)[D,\delta_{x_0}])$ with the standard filtration is an object of $\mathbf{B}_X(Y)$,
(2) there is an enlargement data (K,k) such that

$$F((X,C)[D,\delta_{x_0}]) \subset \mathcal{F}((X,C)[D,\delta_{x_0}]) \subset F((X,C)[D+K,\delta_{x_0}+k_{x_0}]),$$

for all subsets in $\mathcal{P}_X(Y)$.

Since $U[D+K,\delta_{x_0}+k_{x_0}] = U[D,\delta_{x_0}][K,k_{x_0}]$ for general subsets U, the third, largest submodule is independent of the choice of D, δ_{x_0}.

3.3.4. **Definition.** We say that an object F of $\mathbf{B}_X(Y)$ is Y-*graded* if there is Y-grading of F. We define $\mathbf{G}_X(Y)$ as the full subcategory of $\mathbf{B}_X(Y)$ on Y-graded filtered modules.

3.3.5. **Proposition.** *The Y-graded objects in $\mathbf{B}_X(Y)$ are closed under isomorphisms. The subcategory $\mathbf{G}_X(Y)$ is closed under extensions in $\mathbf{B}_X(Y)$. Therefore, $\mathbf{G}_X(Y)$ is an exact subcategory of $\mathbf{B}_X(Y)$.*

Proof. The argument closely follows those for Propositions 3.1.33 and 3.1.35. The details are straightforward and are left to the reader. □

As with the category $\mathbf{G}(X,R)$, the advantage of working with $\mathbf{G}_X(Y)$ as opposed to $\mathbf{B}_X(Y)$ is that we are able to localize to the grading subobjects associated to subsets from the family $\mathcal{P}_X(Y)$ defined in 3.3.3.

3.3.6. **Lemma.** *Let F be a submodule of a Y-filtered module G in $\mathbf{G}_X(Y)$ which is lean/split with respect to the standard filtration. Then $\mathcal{F}(U) = F \cap \mathcal{G}(U)$ is a Y-grading of F.*

Proof. As in Lemma 3.1.37, the proof is easily reduced to checking that $\mathcal{F}(U)$ is an object of $\mathbf{B}_X(Y)$ for each subset $U \in \mathcal{P}_X(Y)$. Suppose $i\colon F \to G$ is the inclusion and $q\colon G \to H$ is the quotient of i. Since F is insular by part (5) of Proposition 3.2.13, both F and G are lean/split and insular. Thus H is lean/split and insular by parts (4) and (6) of 3.2.13. Let $\mathcal{H}(U) = q\mathcal{G}(U)$ with the standard filtration in H. Then $\mathcal{H}(U)$ is lean/split by part (4) and insular as a submodule of insular H. The kernel $\mathcal{F}(U)$ of the filtration $(0,0)$ map $q|\colon \mathcal{G}(U) \to \mathcal{H}(U)$ is lean/split by part (6) of 3.2.13 and is insular as a submodule of insular F. Locally finite generation of $\mathcal{F}(U)$ follows from that of $\mathcal{G}(U)$. □

3.3.7. **Corollary.** *Suppose F is the kernel of a boundedly bicontrolled epimorphism $g\colon G \to H$ in $\mathbf{B}_X(Y)$. If G is Y-graded and F is lean/split then both H and F are Y-graded.*

Proof. Suggested by the proof of Proposition 3.1.38, the Y-grading for H is given by $\mathcal{H}(U) = g\mathcal{G}(U[b,\theta])$, where (b,θ) is a chosen filtration data for g. The argument for Lemma 3.3.6 shows that $\mathcal{F}(U) = F \cap \mathcal{G}(U[b,\theta])$ gives a Y-grading for F. □

3.3.8. **Corollary.** *Given an object F in $\mathbf{G}_X(Y)$ and a subset U from the family $\mathcal{P}_X(Y)$, there is a set of enlargement data (K,k) and an admissible subobject $i\colon F_U \to F$ in $\mathbf{G}_X(Y)$ with the property that $F_U \subset F(U[K,k])$. If G is (D,Δ')-lean/split then the quotient $q\colon F \to H$ of the inclusion has the property that $H(X) = H((X \setminus U)[2D,2\Delta'])$.*

Proof. See the proof of Corollary 3.1.39. □

Now we have a summary similar to Corollary 3.1.40.

3.3.9. **Corollary.** *Given a graded object F in $\mathbf{G}_X(Y)$ and a subset U from the family $\mathcal{P}_X(Y)$, we assume that F is (D,Δ')-split and (d,δ)-insular and is graded by $\mathcal{F}$. The submodule $\mathcal{F}(U)$ has these properties:*

1. *$\mathcal{F}(U)$ is graded by $\mathcal{F}_U(T) = \mathcal{F}(U) \cap \mathcal{F}(T)$,*
2. *$F(U) \subset \mathcal{F}(U) \subset F(U[K,k])$ for some fixed enlargement data (K,k),*
3. *if $q\colon F \to H$ is the quotient of the inclusion $i\colon \mathcal{F}(U) \to F$ and F is (D,Δ')-lean/split, then H is supported on $(X \setminus U)[2D,2\Delta']$,*
4. *$H(U[-2D-2d, -2\Delta'-2\delta]) = 0$.*

We will use the localization theorem of Schlichting [17] for Grothendieck subcategories of exact categories. These techniques require the Grothendieck subcategory to satisfy some additional assumptions that we verify next.

3.3.10. **Definition.** A class of morphisms Σ in an additive category $\mathbf{A}$ *admits a calculus of right fractions* if

1. the identity of each object is in Σ,
2. Σ is closed under composition,
3. each diagram $F \xrightarrow{f} G \xleftarrow{s} G'$ with $s \in \Sigma$ can be completed to a commutative square

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s} \\ F & \xrightarrow{f} & G \end{array}$$

with $t \in \Sigma$, and

(4) if f is a morphism in $\mathbf{A}$ and $s \in \Sigma$ such that $sf = 0$ then there exists $t \in \Sigma$ such that $ft = 0$.

In this case there is a construction of the *localization* $\mathbf{A}[\Sigma^{-1}]$ which has the same objects as $\mathbf{A}$. The morphism sets $\mathrm{Hom}(F, G)$ in $\mathbf{A}[\Sigma^{-1}]$ consist of equivalence classes of diagrams

$$(s, f)\colon \quad F \xleftarrow{s} F' \xrightarrow{f} G$$

with the equivalence relation generated by $(s_1, f_1) \sim (s_2, f_2)$ if there is a map $h\colon F_1' \to F_2'$ so that $f_1 = f_2 h$ and $s_1 = s_2 h$. Let $(s|f)$ denote the equivalence class of (s, f). The composition of morphisms in $\mathbf{A}[\Sigma^{-1}]$ is defined by

$$(s|f) \circ (t|g) = (st'|gf')$$

where g' and s' fit in the commutative square

$$\begin{tikzcd} F'' \arrow[r, "f'"] \arrow[d, "t'"] & G' \arrow[d, "t"] \\ F \arrow[r, "f"] & G \end{tikzcd}$$

from axiom 3.

3.3.11. **Proposition.** *The localization $\mathbf{A}[\Sigma^{-1}]$ is a category. The morphisms of the form $(\mathrm{id}\,|s)$ where $s \in \Sigma$ are isomorphisms in $\mathbf{A}[\Sigma^{-1}]$. The rule $P_\Sigma(f) = (\mathrm{id}\,|f)$ gives a functor $P_\Sigma\colon \mathbf{A} \to \mathbf{A}[\Sigma^{-1}]$ which is universal among the functors making the morphisms Σ invertible.*

Proof. The proofs of these facts can be found in Chapter I of [8]. The inverse of $(\mathrm{id}\,|s)$ is $(s|\,\mathrm{id})$. $\square$

We have seen that for a given subset C of Y, the category $\mathbf{B}_X(Y)_{<C}$ is a Grothendieck subcategory of $\mathbf{B}_X(Y)$. Clearly, restriction to Y-gradings in $\mathbf{B}_X(Y)_{<C}$ gives a full exact subcategory $\mathbf{G}_X(Y)_{<C}$ which is a Grothendieck subcategory of $\mathbf{G}_X(Y)$. The following shorthand notation is convenient when the choice of C is clear.

3.3.12. *Notation.* The category $\mathbf{G}$ is the exact subcategory of Y-graded objects in $\mathbf{B}_X(Y)$. When the choice of the subset $C \subset Y$ is understood, we will use notation $\mathbf{C}$ for the Grothendieck subcategory $\mathbf{G}_X(Y)_{<C}$ of $\mathbf{G}$.

3.3.13. **Definition.** Define the class of *weak equivalences* $\Sigma(C)$ in $\mathbf{B}$ to consist of all finite compositions of admissible monomorphisms with cokernels in $\mathbf{C}$ and admissible epimorphisms with kernels in $\mathbf{C}$.

We will show that the class $\Sigma(C)$ admits calculus of right fractions.

3.3.14. **Definition.** A Grothendieck subcategory $\mathbf{C}$ of an exact category $\mathbf{G}$ is *right filtering* if each morphism $f\colon F_1 \to F_2$ in $\mathbf{G}$, where F_2 is an object of $\mathbf{C}$, factors through an admissible epimorphism $e\colon F_1 \to \overline{F}_2$, where $\overline{F}_2$ is in $\mathbf{C}$.

3.3.15. **Lemma.** *The Grothendieck subcategory $\mathbf{C} = \mathbf{G}_X(Y)_{<C}$ of $\mathbf{G} = \mathbf{G}_X(Y)$ is right filtering.*

Proof. For a morphism between filtered (X,Y)-modules as in Definition 3.3.14, we assume that both F_1 and F_2 are (D,Δ')-lean/split and (d,δ)-insular. Suppose $f\colon F_1 \to F_2$ is bounded by (b,θ) and let $r \geq 0$ and $\rho\colon [0,+\infty) \to [0,+\infty)$ be a monotone function such that

$$F_2(X,Y) \subset F_2((X,C)[r,\rho_{x_0}]).$$

Now for any characteristic set of data (K,k) for the grading $\mathcal{F}_1$ and any subset R we have

$$f\mathcal{F}(R) \subset fF_1(R[K,k_{x_0}]) \subset F_2(R[K+b,k_{x_0}+\theta_{x_0}]).$$

By part (4) of Corollary 3.3.9, $F_2(R[K+b,k_{x_0}+\theta_{x_0}]) \cap F_2((X,C)[r,\rho_{x_0}]) = 0$ for any R such that

$$R[K+b+2D+2d, k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}] \cap (X,C)[r,\rho_{x_0}] = \emptyset.$$

If we choose

$$R = (X,Y) \setminus (X,C)[K+b+2D+2d+r, k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}+\rho_{x_0}]$$

and define $E = \mathcal{F}_1(R)$, then $fE = 0$. Let $\overline{F}_2$ be the cokernel of the inclusion $E \to F_1$. Then $\overline{F}_2$ is lean/split and insular and has a grading given by $\overline{\mathcal{F}}_2(S) = q\mathcal{F}_1(S[b,\theta_{x_0}])$. Since

$$\overline{F}_2(X,Y) \subset \overline{F}_2((X,C)[K+b+2D+2d+r, k_{x_0}+\theta_{x_0}+2\Delta'_{x_0}+2\delta_{x_0}+\rho_{x_0}]),$$

the quotient $\overline{F}_2$ is in $\mathbf{C}$, and f factors as $F_1 \to \overline{F}_2 \to F_2$ in the right square in the map of exact sequences

$$\begin{tikzcd} E \arrow[r] \arrow[d, "i"'] & F_1 \arrow[r, "j'"] \arrow[d, "="] & \overline{F}_2 \arrow[d] \\ K \arrow[r, "k"] & F_1 \arrow[r, "f"] & F_2 \end{tikzcd}$$

as required. □

3.3.16. **Corollary.** *The class $\Sigma(C)$ admits calculus of right fractions.*

Proof. This follows from Lemma 3.3.15, see Lemma 1.13 of [17]. □

3.3.17. **Definition.** The category $\mathbf{G}/\mathbf{C}$ is the localization $\mathbf{G}[\Sigma(C)^{-1}]$.

It is clear that the quotient $\mathbf{G}/\mathbf{C}$ is an additive category, and $P_{\Sigma(C)}$ is an additive functor. In fact, we have the following.

3.3.18. **Theorem.** *The short sequences in $\mathbf{G}/\mathbf{C}$ which are isomorphic to images of exact sequences from $\mathbf{G}$ form a Quillen exact structures.*

This will follow from Proposition 1.16 of Schlichting [17]. Since $\mathbf{C}$ is right filtering by Lemma 3.3.15, it remains to check that $\mathbf{C}$ right s-filtering in $\mathbf{G}$ in the following sense.

3.3.19. **Definition.** A subcategory $\mathbf{C}$ of an exact category $\mathbf{G}$ is *right s-filtering* if given an admissible monomorphism $f\colon F_1 \to F_2$ with F_1 in $\mathbf{C}$, there exist E in $\mathbf{C}$ and an admissible epimorphism $e\colon F_2 \to E$ such that the composition ef is an admissible monomorphism.

Proof of Theorem 3.3.18. Suppose that F_1 and F_2 have the same properties as in the proof of Lemma 3.3.15, and $\mathrm{fil}(f) \le (b, \theta)$. Since F_1 is in $\mathbf{C}$, there are $r \ge 0$ and a monotone function $\rho\colon [0, +\infty) \to [0, +\infty)$ such that

$$F_1(X, Y) \subset F_1\left((X, C)[r, \rho_{x_0}]\right).$$

Then let $F_2' = \mathcal{F}_2(T)$ where

$$T = (X, Y) \setminus (X, C)[K + b + 2D + 2d + r, k_{x_0} + \theta_{x_0} + 2\Delta'_{x_0} + 2\delta_{x_0} + \rho_{x_0}].$$

Define E as the cokernel of the inclusion $F_2' \to F_2$ and let $e\colon F_2 \to E$ be the quotient map. The composition ef is an admissible monomorphism with $\mathrm{fil}(ef) = \mathrm{fil}(f) \le (b, \theta)$. □

3.3.20. *Notation.* If C is a subset of Y as before, $\mathbf{G}_X(Y, C)$ will stand for the exact category $\mathbf{G}/\mathbf{C}$ and $G_X(Y, C)$ for its Quillen K-theory.

The main tool in proving controlled excision theorems will be the following localization sequence.

3.3.21. **Theorem** (Theorem 2.1 of Schlichting [17])**.** *Let $\mathbf{Z}$ be an idempotent complete right s-filtering subcategory of an exact category $\mathbf{E}$. Then the sequence of exact categories $\mathbf{Z} \to \mathbf{E} \to \mathbf{E}/\mathbf{Z}$ induces a homotopy fibration of Quillen K-theory spectra*

$$K(\mathbf{Z}) \longrightarrow K(\mathbf{E}) \longrightarrow K(\mathbf{E}/\mathbf{Z}).$$

3.3.22. **Corollary.** *There is a homotopy fibration*

$$G_X(Y)_{<C} \longrightarrow G_X(Y) \longrightarrow G_X(Y, C).$$

There is a more intrinsic formulation of the same fact.

3.3.23. **Theorem** (Localization)**.** *There is a homotopy fibration*

$$G_X(C) \longrightarrow G_X(Y) \longrightarrow G_X(Y, C).$$

Theorem 3.3.23 follows directly from Corollary 3.3.22 as soon as we show that $G_X(C)$ and $G_X(Y)_{<C}$ are weakly equivalent.

Recall that the *essential full image* of a functor $F\colon \mathbf{C} \to \mathbf{D}$ is the full subcategory of $\mathbf{D}$ whose objects are those D that are isomorphic to $F(C)$ for some C from $\mathbf{C}$.

3.3.24. **Lemma.** *Given a pair of proper metric spaces $C \subset Y$, there is a fully faithful embedding $\epsilon\colon \mathbf{G}_X(C) \to \mathbf{G}_X(Y)$. The Grothendieck subcategory $\mathbf{G}_X(Y)_{<C}$ is the essential full image of $\mathbf{G}_X(C)$ in $\mathbf{G}_X(Y)$. Therefore, the inclusion $C \subset Y$ induces a weak equivalence*

$$G_X(C) \longrightarrow G_X(Y)_{<C}.$$

Proof. Suppose F is an object of $\mathbf{G}_X(C)$. The embedding ϵ is given by $\epsilon(F)(U) = F((X, C) \cap U)$, $\epsilon(\mathcal{F})(S) = \mathcal{F}((X, C) \cap U)$. It is clear that $\epsilon(F)$ is in $\mathbf{G}_X(Y)_{<C}$.

To show that $\mathbf{G}_X(Y)_{<C}$ is the essential full image, for an object G of $\mathbf{G}_X(Y)_{<C}$ assume that $G \subset G((X, C)[r, \rho_{x_0}])$ for some number $r \ge 0$ and a monotone function $\rho\colon [0, +\infty) \to [0, +\infty)$. Choose any set function $\tau\colon (X, C)[r, \rho_{x_0}] \to X \times C$ with the properties

1. $\tau(x, y) = (x, \tau_x(y))$,
2. $d(y, \tau_x(y)) \le \rho_{x_0} + r$ for all x in X,
3. $\tau | X \times C = \mathrm{id}$.

Then the Y-filtered module E associated to G given by $E(S) = G(\tau^{-1}(S))$ with the grading $\mathcal{E}(U') = \mathcal{G}(\tau^{-1}(U'))$ is an object of $\mathbf{G}_X(C)$. Indeed, if $\mathcal{G}(\tau^{-1}(U'))$ is (D, Δ')-lean/split and (d, δ)-insular then $\mathcal{E}(U')$ is $(D + r, \Delta' + \rho)$-lean/split and $(d + r, \delta + \rho)$-insular. The identity map is an isomorphism in $\mathbf{G}_X(Y)$ with $\mathrm{fil}(\mathrm{id}) \leq (2r, 2\rho + 2r)$. $\square$

3.4. ***K*-theoretic Preliminaries.** The proof of controlled excision in the boundedly controlled G-theory requires the context of Waldhausen K-theory of categories of bounded chain complexes.

3.4.1. **Definition** (Waldhausen Categories)**.** A *Waldhausen category* is a category $\mathbf{D}$ with a zero object 0 together with two chosen subcategories of *cofibrations* $\mathrm{co}(\mathbf{D})$ and *weak equivalences* $\boldsymbol{w}(\mathbf{D})$ satisfying the four axioms:

(1) every isomorphism in $\mathbf{D}$ is in both $\mathrm{co}(\mathbf{D})$ and $\boldsymbol{w}(\mathbf{D})$,
(2) every map $0 \to D$ in $\mathbf{D}$ is in $\mathrm{co}(\mathbf{D})$,
(3) if $A \to B \in \mathrm{co}(\mathbf{D})$ and $A \to C \in \mathbf{D}$ then the pushout $B \cup_A C$ exists in $\mathbf{D}$, and the canonical map $C \to B \cup_A C$ is in $\mathrm{co}(\mathbf{D})$,
(4) ("gluing lemma") given a commutative diagram

$$\begin{tikzcd} B \arrow[d] & A \arrow[l, hook', "a"'] \arrow[r] \arrow[d] & C \arrow[d] \\ B' & A' \arrow[l, hook', "a'"'] \arrow[r] & C' \end{tikzcd}$$

in $\mathbf{D}$, where the morphisms a and a' are in $\mathrm{co}(\mathbf{D})$ and the vertical maps are in $\boldsymbol{w}(\mathbf{D})$, the induced map $B \cup_A C \to B' \cup_{A'} C'$ is also in $\boldsymbol{w}(\mathbf{D})$.

A Waldhausen category $\mathbf{D}$ with weak equivalences $\boldsymbol{w}(\mathbf{D})$ is often denoted by $\boldsymbol{w}\mathbf{D}$ as a reminder of the choice. A functor between Waldhausen categories is exact if it preserves the chosen zero objects, cofibrations, weak equivalences, and cobase changes.

A Waldhausen category may or may not satisfy the following additional axioms.

3.4.2. **Saturation axiom.** Given two morphisms $\phi \colon F \to G$ and $\psi \colon G \to H$ in $\mathbf{D}$, if any two of ϕ, ψ, or $\psi\phi$, are in $\boldsymbol{w}(\mathbf{D})$ then so is the third.

3.4.3. **Extension axiom.** Given a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, "\psi"] & H \arrow[d, "\mu"] \\ F' \arrow[r] & G' \arrow[r] & H' \end{tikzcd}$$

with exact rows, if both ϕ and μ are in $\boldsymbol{w}(\mathbf{D})$ then so is ψ.

A *cylinder functor* on $\mathbf{D}$ is a functor C from the category of morphisms $f \colon F \to G$ in $\mathbf{D}$ to $\mathbf{D}$ together with three natural transformations $j_1 \colon F \to C(f)$, $j_2 \colon G \to C(f)$, and $p \colon C(f) \to G$ such that $pj_2 = \mathrm{id}_G$ and $pj_1 = f$ for all f, and which has a number of properties listed in point 1.3.1 of [19] which will be rather automatic for the functors we construct later.

3.4.4. **Cylinder axiom.** A cylinder functor C satisfies this axiom if for all morphisms $f \colon F \to G$ the required map p is in $\boldsymbol{w}(\mathbf{D})$.

Let $\mathbf{D}$ be a small Waldhausen category with respect to two categories of weak equivalences $\boldsymbol{v}(\mathbf{D}) \subset \boldsymbol{w}(\mathbf{D})$ with a cylinder functor T both for $\boldsymbol{v}\mathbf{D}$ and for $\boldsymbol{w}\mathbf{D}$ satisfying the cylinder axiom for $\boldsymbol{w}\mathbf{D}$. Suppose also that $\boldsymbol{w}(\mathbf{D})$ satisfies the extension and saturation axioms. Define $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$ to be the full subcategory of $\boldsymbol{v}\mathbf{D}$ whose objects are F such that $0 \to F \in \boldsymbol{w}(\mathbf{D})$. Then $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$ is a small Waldhausen category with cofibrations $\mathrm{co}(\mathbf{D}^{\boldsymbol{w}}) = \mathrm{co}(\mathbf{D}) \cap \mathbf{D}^{\boldsymbol{w}}$ and weak equivalences $\boldsymbol{v}(\mathbf{D}^{\boldsymbol{w}}) = \boldsymbol{v}(\mathbf{D}) \cap \mathbf{D}^{\boldsymbol{w}}$. The cylinder functor T for $\boldsymbol{v}\mathbf{D}$ induces a cylinder functor for $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$. If T satisfies the cylinder axiom then the induced functor does so too.

3.4.5. **Theorem** (Approximation Theorem)**.** *Let $E \colon \mathbf{D}_1 \to \mathbf{D}_2$ be an exact functor between two small saturated Waldhausen categories. It induces a map of K-theory spectra*

$$K(E) \colon K(\mathbf{D}_1) \longrightarrow K(\mathbf{D}_2).$$

Assume that $\mathbf{D}_1$ has a cylinder functor satisfying the cylinder axiom. If E satisfies two conditions:

(1) *a morphism $f \in \mathbf{D}_1$ is in $\boldsymbol{w}(\mathbf{D}_1)$ if and only if $E(f) \in \mathbf{D}_2$ is in $\boldsymbol{w}(\mathbf{D}_2)$,*
(2) *for any object $D_1 \in \mathbf{D}_1$ and any morphism $g \colon E(D_1) \to D_2$ in $\mathbf{D}_2$, there is an object $D_1' \in \mathbf{D}_1$, a morphism $f \colon D_1 \to D_1'$ in $\mathbf{D}_1$, and a weak equivalence $g' \colon E(D_1') \to D_2 \in \boldsymbol{w}(\mathbf{D}_2)$ such that $g = g'E(f)$,*

then $K(E)$ is a homotopy equivalence.

Proof. This is Theorem 1.6.7 of [20]. The presence of the cylinder functor with the cylinder axiom allows to make condition (2) weaker than that of Waldhausen, see point 1.9.1 in [19]. □

3.4.6. **Definition.** In any additive category, a sequence of morphisms

$$E^{\cdot} \colon \quad 0 \longrightarrow E^1 \xrightarrow{d_1} E^2 \xrightarrow{d_2} \ \ldots \ \xrightarrow{d_{n-1}} E^n \longrightarrow 0$$

is called a *(bounded) chain complex* if the compositions $d_{i+1}d_i$ are the zero maps for all $i = 1, \ldots, n-1$. A *chain map* $f \colon F^{\cdot} \to E^{\cdot}$ is a collection of morphisms $f^i \colon F^i \to E^i$ such that $f^i d_i = d_i f^i$. A chain map f is *null-homotopic* if there are morphisms $s_i \colon F^{i+1} \to E^i$ such that $f = ds + sd$. Two chain maps $f, g \colon F^{\cdot} \to E^{\cdot}$ are *chain homotopic* if $f - g$ is null-homotopic. Now f is a *chain homotopy equivalence* if there is a chain map $h \colon E^i \to F^i$ such that the compositions fh and hf are chain homotopic to the respective identity maps.

The Waldhausen structures on categories of bounded chain complexes are based on homotopy equivalence as a weakening of the notion of isomorphism of chain complexes.

3.4.7. **Definition.** A sequence of maps in an exact category is called *acyclic* if it is assembled out of short exact sequences in the sense that each map factors as the composition of the cokernel of the preceding map and the kernel of the succeeding map.

It is known that the class of acyclic complexes in an exact category is closed under isomorphisms in the homotopy category if and only if the category is idempotent complete, which is also equivalent to the property that each contractible chain complex is acyclic, cf. [10, sec. 11].

3.4.8. **Definition.** Given an exact category $\mathbf{E}$, there is a standard choice for the Waldhausen structure on the category $\mathbf{E}'$ of bounded chain complexes in $\mathbf{E}$ where the degree-wise admissible monomorphisms are the cofibrations and the chain maps whose mapping cones are homotopy equivalent to acyclic complexes are the weak equivalences $\boldsymbol{v}(\mathbf{E}')$.

3.4.9. **Proposition.** *The category $\boldsymbol{v}\mathbf{E}'$ is a Waldhausen category satisfying the extension and saturation axioms and has cylinder functor satisfying the cylinder axiom.*

Proof. The pushouts along cofibrations in $\mathbf{E}'$ are the complexes of pushouts in each degree. All standard Waldhausen axioms including the gluing lemma are clearly satisfied. The saturation and the extension axioms are also clear. The cylinder functor C for $\boldsymbol{v}\mathbf{E}'$ is defined using the canonical homotopy pushout as in point 1.1.2 in Thomason–Trobaugh [19]. Given a chain map $f \colon F \to G$, $C(f)$ is the canonical homotopy pushout of f and the identity $\mathrm{id} \colon F \to F$. With this construction, the map $p \colon C(f) \to G$ is a chain homotopy equivalence, so the cylinder axiom is also satisfied. $\square$

3.4.10. **Definition.** There are three choices for the Waldhausen structure on the category of bounded chain complexes $\mathbf{G}' = \mathbf{G}'_X(Y)$. One is $\boldsymbol{v}\mathbf{G}'$ as in Definition 3.4.8. Given a subset $C \subset Y$, another choice for the weak equivalences $\boldsymbol{w}(\mathbf{G}')$ is the chain maps whose mapping cones are homotopy equivalent to acyclic complexes in the quotient $\mathbf{G}/\mathbf{C}$.

3.4.11. **Corollary.** *The categories $\boldsymbol{v}\mathbf{G}'$ and $\boldsymbol{w}\mathbf{G}'$ are Waldhausen categories satisfying the extension and saturation axioms and have cylinder functors satisfying the cylinder axiom.*

Proof. All axioms and constructions, including the cylinder functor, for $\boldsymbol{w}\mathbf{G}'$ are inherited from $\boldsymbol{v}\mathbf{G}'$. $\square$

The K-theory functor from the category of small Waldhausen categories $\mathbf{D}$ and exact functors to the category of connective spectra is defined in terms of $S.$-construction as in Waldhausen [20]. It extends to simplicial categories $\mathbf{D}$ with cofibrations and weak equivalences and inductively delivers the connective spectrum $n \mapsto |\boldsymbol{w}S^{(n)}_{.}\,\mathbf{D}\,|$. We obtain the functor assigning to $\mathbf{D}$ the connective Ω-spectrum

$$K(\mathbf{D}) = \Omega^{\infty}|\boldsymbol{w}S^{(\infty)}_{.}\,\mathbf{D}\,| = \underset{n \geq 1}{\underrightarrow{\operatorname{colim}}}\ \Omega^{n}|\boldsymbol{w}S^{(n)}_{.}\,\mathbf{D}\,|$$

representing the Waldhausen algebraic K-theory of $\mathbf{D}$. For example, if $\mathbf{D}$ is the additive category of free finitely generated R-modules with the canonical Waldhausen structure, then the stable homotopy groups of $K(\mathbf{D})$ are the usual K-groups of the ring R. In fact, there is a general identification of the two theories. Recall that for any exact category $\mathbf{E}$, the category $\mathbf{E}'$ of bounded chain complexes has the Waldhausen structure $\boldsymbol{v}\mathbf{E}'$ as in Definition 3.4.8.

3.4.12. **Theorem.** *The Quillen K-theory of an exact category $\mathbf{E}$ is equivalent to the Waldhausen K-theory of $\boldsymbol{v}\mathbf{E}'$.*

Proof. The proof is based on repeated applications of the Additivity Theorem, cf. Thomason's Theorem 1.11.7 [19]. Thomason's proof of his Theorem 1.11.7 can be repeated verbatim here. It is in fact simpler in this case since condition 1.11.3.1 is not required. □

3.5. **Nonequivariant Excision Theorems.** Bounded excision theorems of section 2.5 can be adapted to $\mathbf{G}_X(Y)$.

Suppose Y_1 and Y_2 are mutually antithetic subsets of a proper metric space Y, and $Y = Y_1 \cup Y_2$. Consider the coarse covering $\mathcal{U}$ of Y by $\mathcal{S}(Y_1)$, $\mathcal{S}(Y_2)$, and $\mathcal{S}(Y_1, Y_2)$. We use the notation $\mathbf{G} = \mathbf{G}_X(Y)_{<\mathcal{U}}$, $\mathbf{G}_i = \mathbf{G}_X(Y)_{<Y_i}$ for $i = 1$ or 2, and $\mathbf{G}_{12}$ for the intersection $\mathbf{G}_1 \cap \mathbf{G}_2$. There is a commutative diagram

$$(\natural) \qquad \begin{array}{ccccc} K(\mathbf{G}_{12}) & \longrightarrow & K(\mathbf{G}_1) & \longrightarrow & K(\mathbf{G}_1/\mathbf{G}_{12}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle K(I)} \\ K(\mathbf{G}_2) & \longrightarrow & K(\mathbf{G}) & \longrightarrow & K(\mathbf{G}/\mathbf{G}_2) \end{array}$$

where the rows are homotopy fibrations from Theorem 3.3.21 and $I \colon \mathbf{G}_1/\mathbf{G}_{12} \to \mathbf{G}/\mathbf{G}_2$ is the functor induced from the exact inclusion $I \colon \mathbf{G}_1 \to \mathbf{G}$. We observe that I is not necessarily full and, therefore, not an isomorphism of categories.

3.5.1. **Proposition.** *$K(\boldsymbol{w}\mathbf{G}') \simeq K(\mathbf{G}/\mathbf{C})$.*

Proof. This follows from Lemma 2.3 in [17] as part of the proof of Theorem 3.3.21 where $K(\boldsymbol{w}\mathbf{G}')$ from Waldhausen's Fibration Theorem is identified with the Quillen K-theory spectrum $K(\mathbf{G}/\mathbf{C})$. □

3.5.2. **Lemma.** *If $f^{\cdot} \colon F^{\cdot} \to G^{\cdot}$ is a degreewise admissible monomorphism with cokernel in $\mathbf{C}$ then $f^{\cdot}$ is a weak equivalence in $\mathbf{w}\mathbf{G}'$.*

Proof. The mapping cone $Cf^{\cdot}$ is quasi-isomorphic to the cokernel of $f^{\cdot}$, by Lemma 11.6 of [10], which is zero in $\mathbf{G}/\mathbf{C}$. □

The exact inclusion I induces the exact functor $\boldsymbol{w}\mathbf{G}'_1 \to \boldsymbol{w}\mathbf{G}'$.

3.5.3. **Lemma.** *The map $K(\boldsymbol{w}\mathbf{G}'_1) \to K(\boldsymbol{w}\mathbf{G}')$ is a weak equivalence.*

Proof. Applying the Approximation Theorem, the first condition is clear. To check the second condition, consider

$$F^{\cdot} \colon \quad 0 \longrightarrow F^1 \xrightarrow{\phi_1} F^2 \xrightarrow{\phi_2} \ \dots \ \xrightarrow{\phi_{n-1}} F^n \longrightarrow 0$$

in $\mathbf{G}_1$ and a chain map $g \colon F^{\cdot} \to G^{\cdot}$ for some complex

$$G^{\cdot} \colon \quad 0 \longrightarrow G^1 \xrightarrow{\psi_1} G^2 \xrightarrow{\psi_2} \ \dots \ \xrightarrow{\psi_{n-1}} G^n \longrightarrow 0$$

in $\mathbf{G}$. Suppose all F^i and G^i are (D, Δ')-lean/split and (d, δ)-insular. Also assume that there is a fixed number $r \geq 0$ and a monotone function $\rho \colon [0, +\infty) \to [0, +\infty)$ such that

$$F^i(X, Y) \subset F^i((X, C)[r, \rho_{x_0}])$$

holds for all $0 \leq i \leq n$. If the pair (b, θ) serves as bounded control data for all ϕ_i, ψ_i, and g_i, we define the submodule

$$F'^i = \mathcal{G}^i((X, Y_1)[r + 3ib, \rho_{x_0} + 3i\theta_{x_0}])$$

and define $\xi_i \colon F'^i \to F'^{i+1}$ to be the restrictions of ψ_i to F'^i. This gives a chain subcomplex (F'^i, ξ_i) of (G^i, ψ_i) in $\mathbf{G}$ with the inclusion $i \colon F'^i \to G^i$. Notice that we have the induced chain map $\overline{g} \colon F^{\cdot} \to F'^{\cdot}$ in $\mathbf{G}_1$ so that $g = iI(\overline{g})$.

We will argue that $C^{\cdot} = \mathrm{coker}(i)$ is in $\mathbf{G}_2$. Given that, i is a weak equivalence by Lemma 3.5.2. Since

$$F'^i \subset G^i((X, Y_1)[r + 3ib + K, \rho_{x_0} + 3i\theta_{x_0} + k_{x_0}]),$$

each C^i is supported on

$$\begin{aligned}(X, Y \setminus Y_1)[2D + 2d - r - 3ib - K, 2\Delta'_{x_0} + 2\delta_{x_0} - \rho_{x_0} - 3i\theta_{x_0} - k_{x_0}]\\ \subset (X, Y_2)[2D + 2d, 2\Delta'_{x_0} + 2\delta_{x_0}],\end{aligned}$$

cf. Lemma 3.3.15. So the complex C^i is indeed in $\mathbf{G}_2$. □

Let $\mathbb{R}$, $\mathbb{R}^{\geq 0}$, and $\mathbb{R}^{\leq 0}$ denote the metric spaces of the reals, the nonnegative reals, and the nonpositive reals with the restriction of the usual metric on the real line $\mathbb{R}$. Then we have the following instance of commutative diagram (♮)

$$\begin{array}{ccccc} G_X(Y) & \longrightarrow & G_X(Y \times \mathbb{R}^{\geq 0}) & \longrightarrow & K(\mathbf{G}_1/\mathbf{G}_{12}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle K(I)} \\ G_X(Y \times \mathbb{R}^{\geq 0}) & \longrightarrow & G_X(Y \times \mathbb{R}) & \longrightarrow & K(\mathbf{G}/\mathbf{G}_2) \end{array}$$

3.5.4. **Lemma.** *The spectra $G_X(Y \times \mathbb{R}^{\geq 0})$ and $G_X(Y \times \mathbb{R}^{\leq 0})$ are contractible.*

Proof. This follows from the fact that these controlled categories are flasque, that is, the usual shift functor T in the positive (respectively negative) direction along $\mathbb{R}^{\geq 0}$ (respectively $\mathbb{R}^{\leq 0}$) interpreted in the obvious way is an exact endofunctor, and there is a natural equivalence $1 \oplus \pm T \cong \pm T$. Contractibility follows from the Additivity Theorem, cf. Pedersen–Weibel [13]. □

In view of Lemma 3.5.3, we obtain a map $G_X(Y) \to \Omega G_X(Y \times \mathbb{R})$ which induces isomorphisms of K-groups in positive dimensions. Iterations give weak equivalences

$$\Omega^k G_X(Y \times \mathbb{R}^k) \longrightarrow \Omega^{k+1} G_X(Y \times \mathbb{R}^{k+1})$$

for $k \geq 2$.

3.5.5. **Definition.** The *nonconnective fibred bounded G-theory* over the pair (X, Y) is the spectrum

$$G_X^{-\infty}(Y) \overset{\mathrm{def}}{=} \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k G_X(Y \times \mathbb{R}^k).$$

3.5.6. **Remark.** Since $\mathbf{BL}(X, R)$ can be identified with $\mathbf{B}_X(\text{point})$, this definition gives a nonconnective delooping of the G-theory of X:

$$G^{-\infty}(X, R) = \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k G_X(\mathbb{R}^k).$$

The subcategory $\mathbf{G}_X(Y \times \mathbb{R}^k)_{<C \times \mathbb{R}^k}$ is evidently a Grothendieck subcategory of $\mathbf{G}_X(Y \times \mathbb{R}^k)$ for any choice of the subset $C \subset Y$.

3.5.7. **Definition.** We define

$$G_X^{-\infty}(Y)_{<C} \overset{\mathrm{def}}{=} \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k G_X(Y \times \mathbb{R}^k)_{<C \times \mathbb{R}^k}.$$

Using the methods above, one easily obtains the weak equivalence

$$G_X^{-\infty}(Y)_{<C} \simeq G_X^{-\infty}(C).$$

We also define

$$G_X^{-\infty}(Y)_{<C_1,C_2} \stackrel{\text{def}}{=} \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\, \Omega^k G_X(Y \times \mathbb{R}^k)_{<C_1\times\mathbb{R}^k,\, C_2\times\mathbb{R}^k}.$$

3.5.8. **Theorem** (Fibrewise Bounded Excision)**.** *Suppose Y_1 and Y_2 are subsets of a metric space Y, and $Y = Y_1 \cup Y_2$. There is a homotopy pushout diagram of spectra*

$$\begin{tikzcd} G_X^{-\infty}(Y)_{<Y_1,Y_2} \arrow[r] \arrow[d] & G_X^{-\infty}(Y)_{<Y_1} \arrow[d] \\ G_X^{-\infty}(Y)_{<Y_2} \arrow[r] & G_X^{-\infty}(Y) \end{tikzcd}$$

where the maps of spectra are induced from the exact inclusions. If Y_1 and Y_2 are mutually antithetic subsets of Y, there is a homotopy pushout

$$\begin{tikzcd} G_X^{-\infty}(Y_1\cap Y_2) \arrow[r] \arrow[d] & G_X^{-\infty}(Y_1) \arrow[d] \\ G_X^{-\infty}(Y_2) \arrow[r] & G_X^{-\infty}(Y) \end{tikzcd}$$

Proof. Let us write $S^k\,\mathbf{G}$ for $\mathbf{G}_X(Y\times\mathbb{R}^k)$ whenever $\mathbf{G}$ is the fibred bounded category for a pair (X,Y). If C represents a family of coarsely equivalent subsets in a coarse covering $\mathcal{U}$ of Y, consider the fibration

$$G_X(C) \longrightarrow G_X(Y) \longrightarrow K(\mathbf{G}/\mathbf{C})$$

from Theorem 3.3.23. Notice that there is a map

$$K(\mathbf{G}/\mathbf{C}) \longrightarrow \Omega K(S\,\mathbf{G}/S\,\mathbf{C})$$

which is an equivalence in positive dimensions by the Five Lemma. Defining

$$G_X^{-\infty}(Y,C) = K^{-\infty}(\mathbf{G}/\mathbf{C}) = \underset{\overrightarrow{k}}{\operatorname{hocolim}}\, \Omega^k K(S^k\,\mathbf{G}/S^k\,\mathbf{C})$$

gives an induced fibration

$$G_X^{-\infty}(C) \longrightarrow G_X^{-\infty}(Y) \longrightarrow G_X^{-\infty}(Y,C).$$

The theorem follows from the commutative diagram

$$\begin{tikzcd} G_X^{-\infty}(Y)_{<Y_1,Y_2} \arrow[r] \arrow[d] & G_X^{-\infty}(Y)_{<Y_1} \arrow[r] \arrow[d] & K^{-\infty}(\mathbf{G}_1/\mathbf{G}_{12}) \arrow[d] \\ G_X^{-\infty}(Y)_{<Y_2} \arrow[r] & G_X^{-\infty}(Y) \arrow[r] & K^{-\infty}(\mathbf{G}/\mathbf{G}_2) \end{tikzcd}$$

and the fact that $K^{-\infty}(\mathbf{G}_1/\mathbf{G}_{12}) \to K^{-\infty}(\mathbf{G}/\mathbf{G}_2)$ is a weak equivalence. $\square$

Suppose we are given a finite coarse antithetic covering $\mathcal{U}$ of Y closed under coarse intersections and of cardinality s. The coarsely saturated families $\mathcal{A}_i$ which are members of $\mathcal{U}$ are partially ordered by inclusion. In fact, the union of the families $\mathcal{A}_i$ forms the set $\mathcal{A}$ closed under intersections.

3.5.9. **Definition.** Two subsets A, B of (X,Y) are called *coarsely equivalent* if there is a set of enlargement data (K,k) such that $A \subset B[K,k_{x_0}]$ and $B \subset A[K,k_{x_0}]$. It is an equivalence relation among subsets. We will again use notation $A \parallel B$ for this equivalence. As we demonstrated before, this is a generalization of the notation from Definition 2.7.1. It should be clear from the context which notion is meant by this notation.

A family of subsets $\mathcal{A}$ is called *coarsely saturated* if it is maximal with respect to this equivalence relation. Given a subset A, let $\mathcal{S}(A)$ be the smallest boundedly saturated family containing A.

A collection of subsets $\mathcal{U} = \{U_i\}$ is a *coarse covering* of (X, Y) if $(X, Y) = \bigcup S_i$ for some $S_i \in \mathcal{S}(U_i)$. Similarly, $\mathcal{U} = \{\mathcal{A}_i\}$ is a *coarse covering* by coarsely saturated families if for some (and therefore any) choice of subsets $A_i \in \mathcal{A}_i$, $\{A_i\}$ is a coarse covering in the sense above.

We will say that a pair of subsets A, B of (X, Y) are *coarsely antithetic* if for any two sets of enlargement data (D_1, d_1) and (D_2, d_2) there is a third set (D, d) such that

$$A[D_1, (d_1)_{x_0}] \cap B[D_2, (d_2)_{x_0}] \subset (A \cap B)[D, d_{x_0}].$$

We will write $A \natural B$ to indicate that A and B are coarsely antithetic.

Given two subsets A and B, we define

$$\mathcal{S}(A, B) = \{A' \cap B' \mid A' \in \mathcal{S}(A), B' \in \mathcal{S}(B), A' \natural B'\}.$$

It is easy to see that $\mathcal{S}(A, B)$ is a coarsely saturated family, cf. Proposition 2.7.4.

There is the straightforward generalization to the case of a finite number of subsets of (X, Y). Again, we write $A_1 \natural A_2 \natural \ldots \natural A_k$ if for arbitrary sets of data (D_i, d_i) there is a set of enlargement data (D, d) so that

$$A_1[D_1, (d_1)_{x_0}] \cap A_2[D_2, (d_2)_{x_0}] \cap \ldots \cap A_k[D_k, (d_k)_{x_0}] \\ \subset (A_1 \cap A_2 \cap \ldots \cap A_k)[D, d_{x_0}]$$

and define

$$\mathcal{S}(A_1, A_2, \ldots, A_k) = \{A'_1 \cap A'_2 \cap \ldots \cap A'_k \mid A'_i \in \mathcal{S}(A_i), A_1 \natural A_2 \natural \ldots \natural A_k\}.$$

Identifying any coarsely saturated family $\mathcal{A}$ with $\mathcal{S}(A)$ for $A \in \mathcal{A}$, one has the coarse saturated family $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$. We will refer to $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ as the *coarse intersection* of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$. A coarse covering $\mathcal{U}$ is *closed under coarse intersections* if all coarse intersections $\mathcal{S}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ are nonempty and are contained in $\mathcal{U}$.

3.5.10. **Proposition.** *If $\mathcal{U}$ is a coarse antithetic covering of Y then $(X, \mathcal{U})$ consisting of subsets (X, U), $U \in \mathcal{U}$, is a coarse antithetic covering of (X, Y). If $\mathcal{U}$ is closed under coarse intersections, $(X, \mathcal{U})$ is closed under coarse intersections.*

Proof. Left to the reader. $\square$

3.5.11. **Corollary.** *Suppose $\mathcal{U}$ is a finite coarse covering of Y closed under coarse intersections. We can define the homotopy pushout*

$$\mathcal{G}_X(Y; \mathcal{U}) = \underset{\overrightarrow{\mathcal{A} \in \mathcal{U}}}{\operatorname{hocolim}}\, G^{-\infty}_X(Y)_{<A}.$$

Then there is a weak equivalence

$$\mathcal{G}_X(Y; \mathcal{U}) \simeq G^{-\infty}_X(Y).$$

Proof. Apply Theorem 3.5.8 inductively to the sets in $\mathcal{U}$. $\square$

We will require a relative version of bounded G-theory and the corresponding Fiberwise Excision Theorem.

3.5.12. **Definition.** Let $Y' \in \mathcal{A}$ for a coarse covering $\mathcal{U}$ of Y. Let $\mathbf{G} = \mathbf{G}_X(Y)_{<\mathcal{U}}$ and $\mathbf{Y}' = \mathbf{G}_X(Y)_{<Y'}$. The category $\mathbf{G}_X(Y, Y')$ is the quotient category $\mathbf{G} \,/\, \mathbf{Y}'$.

It is now straightforward to define

$$G_X^{-\infty}(Y, Y') = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G_X(Y \times \mathbb{R}^k, Y' \times \mathbb{R}^k),$$

$$G_X^{-\infty}(Y, Y')_{<C} = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G_X(Y \times \mathbb{R}^k, Y' \times \mathbb{R}^k)_{<C\times\mathbb{R}^k},$$

and

$$G_X^{-\infty}(Y, Y')_{<C_1,C_2} = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G_X(Y \times \mathbb{R}^k, Y' \times \mathbb{R}^k)_{<C_1\times\mathbb{R}^k,\, C_2\times\mathbb{R}^k}.$$

3.5.13. **Proposition.** *Given a subset U of Y', there is a weak equivalence*

$$G_X^{-\infty}(Y, Y') \simeq G_X^{-\infty}(Y - U, Y' - U).$$

Proof. Consider the setup of Theorem 3.5.8 with $Y_1 = Y - U$ and $Y_2 = Y'$, then Lemma 3.5.3 shows that the map

$$\frac{\mathbf{G}_X(Y)_{<(Y-U)}}{\mathbf{G}_X(Y)_{<(Y-U)} \cap \mathbf{G}_X(Y)_{<Y'}} \longrightarrow \frac{\mathbf{G}_X(Y)}{\mathbf{G}_X(Y)_{<Y'}}$$

induces a weak equivalence on the level of K-theory. Notice that, since U is a subset of Y',

$$\mathbf{G}_X(Y)_{<(Y-U)} \cap \mathbf{G}_X(Y)_{<Y'} = \mathbf{G}_X(Y)_{<(Y'-U)}.$$

Now the maps of quotients

$$\frac{\mathbf{G}_X(Y)}{\mathbf{G}_X(Y')} \longrightarrow \frac{\mathbf{G}_X(Y)}{\mathbf{G}_X(Y)_{<Y'}}$$

and

$$\frac{\mathbf{G}_X(Y)_{<(Y-U)}}{\mathbf{G}_X(Y)_{<(Y'-U)}} \longleftarrow \frac{\mathbf{G}_X(Y-U)}{\mathbf{G}_X(Y'-U)}$$

induced by fully faithful embeddings also induce weak equivalences. Their composition gives the required equivalence. □

The theory developed in this section can be easily relativized to give the following excision theorem.

3.5.14. **Theorem** (Relative Fibred Excision)**.** *If Y is the union of two subsets U_1 and U_2, there is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc} G_X^{-\infty}(Y, Y')_{<U_1,U_2} & \longrightarrow & G_X^{-\infty}(Y, Y')_{<U_1} \\ \big\downarrow & & \big\downarrow \\ G_X^{-\infty}(Y, Y')_{<U_2} & \longrightarrow & G_X^{-\infty}(Y, Y') \end{array}$$

where the maps of spectra are induced from the exact inclusions. In fact, if Y is the union of two mutually antithetic subsets U_1 and U_2, and Y' is antithetic to both U_1 and U_2, there is a homotopy pushout

$$\begin{array}{ccc} G_X^{-\infty}(U_1 \cap U_2, U_1 \cap U_2 \cap Y') & \longrightarrow & G_X^{-\infty}(U_1, U_1 \cap Y') \\ \big\downarrow & & \big\downarrow \\ G_X^{-\infty}(U_2, U_2 \cap Y') & \longrightarrow & G_X^{-\infty}(Y, Y') \end{array}$$

More generally, we can define the homotopy pushout

$$\mathcal{G}_X(Y,Y';\mathcal{U}) = \underset{\overrightarrow{\mathcal{A}\in\mathcal{U}}}{\operatorname{hocolim}}\, G_X^{-\infty}(Y,Y')_{<A}.$$

Then there is a weak equivalence

$$\mathcal{G}_X(Y,Y';\mathcal{U}) \simeq G_X^{-\infty}(Y,Y').$$

3.6. **Functoriality, Equivariant Fibred G-theory.** Recall from Definition 2.3.1 that a map $f\colon X \to Y$ between proper metric spaces is a *coarse map* if it is proper and there is a real positive function l such that

$$T \in \mathcal{B}_r(X) \implies f(T) \in \mathcal{B}_{l(r)}(Y),$$

where $\mathcal{B}_d(X)$ denotes the collection of subsets of X with diameter bounded by d.

The map f is a *coarse equivalence* if there is a coarse map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded maps.

3.6.1. **Proposition.** *A coarse map $f\colon X \to Y$ induces a functor*

$$f_*\colon\ \mathbf{U}(X,R) \longrightarrow \mathbf{U}(Y,R).$$

If f is a coarse equivalence then f_* is an equivalence of categories. In this case, one also has a functor

$$f_*\colon\ \mathbf{BL}(X,R) \longrightarrow \mathbf{BL}(Y,R).$$

Proof. Define $f_*(F)(S) = F(f^{-1}(S))$. □

3.6.2. **Corollary.** *Consider the category of proper metric spaces X and coarse equivalences. Then $\mathbf{BL}(X,R)$ is a covariant functor in the space variable to small exact categories and exact functors. Composing with the covariant functor $G^{-\infty}$ gives the spectrum-valued functor $G^{-\infty}(X,R)$.*

A subset W of a metric space X is *boundedly dense* or *commensurable* if $W[d] = X$ for some $d \geq 0$.

3.6.3. **Proposition.** *For a commensurable metric subspace W of X, there is a natural exact equivalence of categories $\mathbf{BL}(W,R) \to \mathbf{BL}(X,R)$ and the induced weak homotopy equivalence $G^{-\infty}(W,R) \simeq G^{-\infty}(X,R)$.*

Proof. Any surjective coarse equivalence $f\colon X \to Y$ induces two functors on filtered modules. One is the contravariant $f^*\colon\ \mathbf{BL}(Y,R) \to \mathbf{BL}(X,R)$ given by $f^*F(S) = F(f(S))$. The other is the covariant functor f_* as in Proposition 3.6.1, so that $f^*f_* = \mathrm{id}$. Even when f is not surjective, there is the endofunctor $\omega = f^{-1}f$ of $\mathcal{P}(X)$ which induces an endofunctor ω_* of $\mathbf{BL}(X,R)$. If $f\colon X \to X$ is a bounded function, there is always an isomorphism $\omega_*(F) \cong F$ induced by the identity on $F(X)$. This shows that $f_*F \cong F$ for all $F \in \mathbf{BL}(X,R)$.

Now if $W \subset X$ is commensurable, there is a bounded surjection $f\colon X \to W$, so f induces a natural transformation $\eta\colon\ \mathrm{id} \to f_*$ where all $\eta(F)$ are isomorphisms. □

3.6.4. **Corollary.** *If X is a bounded metric space then we have the natural equivalence $\mathbf{BL}(X,R) \cong \mathbf{BL}(\text{point},R) = \mathbf{Modf}(R)$, which induces a weak equivalence $G^{-\infty}(X,R) \simeq G^{-\infty}(R)$.*

Given a geometric action of Γ on a metric space X, there are natural actions of Γ on $\mathbf{U}(X,R)$ and $\mathbf{B}(X,R)$ induced from the action on the power set $\mathcal{P}(X)$.

The equivariant nonconnective G-theory and the corresponding fixed point spectra are modeled on Definition 2.1.10 for equivariant K-theory. This is a sequence of constructions performed in the following specific order. Let $\mathbf{BL}^{\Gamma,0}(X,R)$ be the full subcategory of the functor category $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathbf{BL}(X,R))$ on the functors that send the morphisms of $\boldsymbol{E\Gamma}$ to filtration 0 maps. One similarly has $\mathbf{BL}^{\Gamma,0}(X \times \mathbb{R}^k, R)$, assuming the trivial action of Γ on the Euclidean factor. Now the category $\mathbf{BL}^{\Gamma,0}(X \times \mathbb{R}^k, R)^\Gamma$ can be defined as the category of pairs (F,ψ), where F is an object of $\mathbf{BL}^{\Gamma,0}(X \times \mathbb{R}^k, R)$ and ψ is a function on Γ with $\psi(\gamma)$ isomorphism in $\mathrm{Hom}(F, \gamma F)$ subject to the usual lax limit conditions in Definition 2.1.8.

3.6.5. **Proposition.** *The fixed point category* $\mathbf{BL}^{\Gamma,0}(X,R)^\Gamma$ *is exact.*

Proof. The exact structure is inherited from $\mathbf{BL}(X,R)$ in the sense that a morphism $\phi\colon (F,\psi) \to (F',\psi')$ is an admissible monomorphism or epimorphism if the map $\phi\colon F \to F'$ is an admissible monomorphism or epimorphism in $\mathbf{BL}(X,R)$, respectively. The fact that this is an exact structure follows from the proofs of Theorems 3.1.9, 3.1.22, and 3.1.26 by observing that all constructions in those proofs produce equivariant objects and morphisms. □

We now define the nonconnective spectra

$$G^{-\infty}(X,R) = \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k K(\mathbf{BL}^{\Gamma,0}(X \times \mathbb{R}^k, R)),$$

$$G^{-\infty}(X,R)^\Gamma = \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k K(\mathbf{BL}^{\Gamma,0}(X \times \mathbb{R}^k, R)^\Gamma).$$

It can be shown as in [3] that the fixed points of the spectrum $G^{-\infty}(X,R)$ are equivalent to $G^{-\infty}(X,R)^\Gamma$. Since all maps we use are induced from geometric maps between metric spaces and so commute with all relevant colimits, we may ignore the distinction between these two spectra.

Given a proper metric space Y with a left Γ-action by either isometries or coarse equivalences, there are G-theory analogues of equivariant theories K_i^Γ, K_c^Γ, K_p^Γ from Definition 2.4.1. We are specifically interested in the analogue of K_p^Γ.

3.6.6. **Definition** (Coarse Equivariant Theories, continued)**.** Suppose Γ acts on Y by coarse equivalences.

(4) $g_p^\Gamma(Y)$ is defined as the K-theory spectrum of the exact category $\mathbf{G}_p^\Gamma(Y)$ of functors

$$\theta\colon\ \boldsymbol{E\Gamma} \longrightarrow \mathbf{G}_\Gamma(Y)$$

such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of Γ-filtered modules.

The nonconnective equivariant G-theory spectrum $G_p^\Gamma(Y)$ is a nonconnective delooping of $g_p^\Gamma(Y)$. Thus there are maps

$$g_p^\Gamma(Y \times \mathbb{R}^{k-1}) \longrightarrow \Omega g_p^\Gamma(Y \times \mathbb{R}^k),$$

where the action of Γ on $\mathbb{R}^k$ is trivial, and the nonconnective equivariant theory is defined as the colimit

$$G_p^\Gamma(Y) \overset{\mathrm{def}}{=} \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k g_p^\Gamma(Y \times \mathbb{R}^k).$$

Similarly for the fixed points

$$G^{\Gamma}_p(Y)^{\Gamma} = \underset{\overrightarrow{k>0}}{\mathrm{hocolim}}\ \Omega^k g^{\Gamma}_p(Y \times \mathbb{R}^k)^{\Gamma}.$$

3.6.7. **Definition** (Cartan maps)**.** The natural inclusions $\mathcal{C}^{\Gamma}_i(Y) \to \mathbf{G}^{\Gamma}_p(Y)$ induce maps between diagrams involved in delooping $k^{\Gamma}_i(Y)$ and $g^{\Gamma}_p(Y)$, so we obtain the induced equivariant maps between the colimits $\kappa \colon K^{\Gamma}_i(Y) \to G^{\Gamma}_p(Y)$ and therefore the induced Cartan map of fixed points

$$\kappa^{\Gamma} \colon K^{\Gamma}_i(Y)^{\Gamma} \longrightarrow G^{\Gamma}_p(Y)^{\Gamma}.$$

Let Y' be a subset of Y. We will relativize the equivariant constructions.

3.6.8. **Definition.** $\mathbf{G}_{\Gamma}(Y)_{<Y'}$ denotes the full subcategory of $\mathbf{G}_{\Gamma}(Y)$ on objects F such that there is a number $k \geq 0$ and an order preserving function $\lambda \colon \mathcal{B}(\Gamma) \to [0, +\infty)$ such that $F(S) \subset F(S[k])(C[\lambda(S)])$ for each bounded subset $S \subset \Gamma$. This is a right filtering Grothendieck subcategory. In particular, there is an exact quotient category $\mathbf{G}_{\Gamma}(Y)/\,\mathbf{G}_{\Gamma}(Y)_{<Y'}$ which we denote by $\mathbf{G}_{\Gamma}(Y, Y')$.

3.6.9. **Definition.** Given a proper metric space Y with a left Γ-action, a subset Y' is called *coarsely* Γ*-invariant* or simply *coarsely invariant* if for each element γ of Γ there is a number $t(\gamma)$ with $\gamma \cdot Y' \subset Y'[t(\gamma)]$. The subset Y' is further Γ*-invariant* if the function $t(\gamma) = 0$.

The following is now clear.

3.6.10. **Proposition.** *If Y' is a coarsely invariant subset of Y, the subcategory $\mathbf{G}_{\Gamma}(Y)_{<Y'}$ is invariant under the action of Γ on $\mathbf{G}_{\Gamma}(Y)$, so there is a left Γ-action on the quotient $\mathbf{G}_{\Gamma}(Y)/\,\mathbf{G}_{\Gamma}(Y)_{<Y'}$, and one obtains the equivariant relative theory $G^{\Gamma}_p(Y, Y')$ and the corresponding fixed point spectrum $G^{\Gamma}_p(Y, Y')^{\Gamma}$.*

4. Fibrewise Excision Theorems

We finally develop useful excision results with respect to specific coverings of the variable Y in $G^{\Gamma}_p(Y)^{\Gamma}$ and some other related spectra. After that, we include a sketch of the application of these excision theorems to the computation of the K-theoretic assembly maps.

4.1. **Equivariant Localization and Excision.** Recall from that whenever a proper metric space Y possesses a left Γ-action, we call a subset Y' *coarsely invariant* if there is a function $t \colon \Gamma \to \mathbb{R}$ such that $\gamma \cdot Y' \subset Y'[t(\gamma)]$. In this case, the Grothendieck subcategory $\mathbf{G}^{\Gamma}_p(Y)_{<Y'}$ is invariant under the action of Γ. The most important situation of interest to us is an action of Γ on Y by bounded coarse equivalences. Then $t(\gamma)$ can be taken to be the bound of the coarse equivalence $\alpha(\gamma)$, which shows that all subsets of Y are coarsely invariant. For a general coarsely invariant subset Y', there is a well-defined left Γ-action on the quotient $\mathbf{G}^{\Gamma}_p(Y, Y')$, and we obtain the equivariant relative theory $G^{\Gamma}_p(Y, Y')$.

The quotient map of categories induces the equivariant map

$$G^{\Gamma}_p(Y) \longrightarrow G^{\Gamma}_p(Y, Y')$$

and the map of fixed point spectra

$$G^{\Gamma}_p(Y)^{\Gamma} \longrightarrow G^{\Gamma}_p(Y, Y')^{\Gamma}.$$

More generally, if Y'' is another coarsely invariant subset of Y that is coarsely antithetic to Y', then the intersection $Y'' \cap Y'$ is coarsely invariant in both Y and Y'. Therefore, there is an equivariant map

$$G^{\Gamma}_p(Y'', Y'' \cap Y') \longrightarrow G^{\Gamma}_p(Y, Y')$$

and the map of fixed points

$$G^{\Gamma}_p(Y'', Y'' \cap Y')^{\Gamma} \longrightarrow G^{\Gamma}_p(Y, Y')^{\Gamma}.$$

Suppose C is a coarsely invariant subset of Y.

4.1.1. **Definition.** Define $\mathbf{G}^{\Gamma}_p(Y)^{\Gamma}_{<C}$ to be the full subcategory of $\mathbf{G}^{\Gamma}_p(Y)^{\Gamma}$ on the objects F with $F(S)$ contained in $\mathbf{G}(Y,R)_{<C}$ for all bounded subsets $S \subset \Gamma$. Similarly, $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{<C}$ is the full subcategory of $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$ on objects in $\mathbf{G}^{\Gamma}_p(Y)^{\Gamma}_{<C}$.

4.1.2. **Proposition.** *$\mathbf{G}^{\Gamma}_p(Y)^{\Gamma}_{<C}$ and $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{<C}$ are Grothendieck subcategories of respectively $\mathbf{G}^{\Gamma}_p(Y)^{\Gamma}$ and $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$ and are idempotent complete.*

Proof. Since $\psi(\gamma) \colon F \to \gamma F$ are isomorphisms, all γF are objects of $\mathbf{G}_p(Y)_{<C}$ if F is in $\mathbf{G}_p(Y)_{<C}$. In fact, if $\mathrm{fil}(\psi(\gamma)) \le (b_\gamma, \theta_\gamma)$ and

$$F(X, Y) \subset F((X, C)[K, k_{x_0}])$$

then

$$(\gamma F)(X, Y) \subset F((X, C)[K + b_\gamma, k_{x_0} + (\theta_\gamma)_{x_0}]).$$

The rest of the proof follows that of Proposition 3.2.20. $\square$

It is straightforward to verify that these Grothendieck subcategories are right filtering and s-right filtering using the corresponding properties of the Γ-translates.

4.1.3. *Notation.* Let $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{>C}$ denote the quotient of the embedding

$$\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{<C} \longrightarrow \mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$$

and $G^{\Gamma}_p(Y, Y')^{\Gamma}_{>C}$ be the corresponding nonconnective K-theory spectrum.

4.1.4. **Theorem.** *For a pair of coarsely invariant mutually antithetic subsets Y' and C of Y, there is a homotopy fibration*

$$G^{\Gamma}_p(Y, Y')^{\Gamma}_{<C} \longrightarrow G^{\Gamma}_p(Y, Y')^{\Gamma} \longrightarrow G^{\Gamma}_p(Y, Y')^{\Gamma}_{>C}.$$

Proof. The proof is an application of the localization theorem. $\square$

Suppose C_1 and C_2 are two coarsely invariant mutually antithetic subsets of Y such that $Y = C_1 \cup C_2$ and let

$$\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{<C_1, C_2} = \mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{<C_1} \cap \mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{<C_2}$$

be the intersection of two full subcategories in $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}$. Since the action is by bounded coarse equivalences, if $F \in \mathbf{G}^{\Gamma}_p(Y, Y')_{<C_i}$ then $\gamma F \in \mathbf{G}^{\Gamma}_p(Y, Y')_{<C_i}$ for all $\gamma \in \Gamma$ and $i = 1$, 2. Also $\mathbf{G}^{\Gamma}_p(Y, Y')^{\Gamma}_{<C_1, C_2}$ is clearly closed under extensions, so it is an exact category.

4.1.5. **Theorem.** *The commutative square*

$$\begin{array}{ccc} G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_1,C_2} & \longrightarrow & G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_1} \\ \big\downarrow & & \big\downarrow \\ G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_2} & \longrightarrow & G^{\Gamma}_p(Y,Y')^{\Gamma} \end{array}$$

induced by inclusions is a homotopy pushout of spectra.

Proof. The square is the left square in the map of two fibration sequences

$$\begin{array}{ccccc} G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_1,C_2} & \longrightarrow & G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_1} & \longrightarrow & G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_1,>C_2} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_2} & \longrightarrow & G^{\Gamma}_p(Y,Y')^{\Gamma} & \longrightarrow & G^{\Gamma}_p(Y,Y')^{\Gamma}_{>C_2} \end{array}$$

The rightmost vertical map is an equivalence by an application of the Approximation Theorem as in the proof of Lemma 3.5.3. □

We will say that three subsets S, T, and V form a *coarsely antithetic triple* if they are pairwise coarsely antithetic.

The following are examples of antithetic triples:

(1) half-spaces $\mathbb{R}^{n-1} \times [0, +\infty)$, $\mathbb{R}^{n-1} \times (-\infty, 0]$, and the line $\{0\} \times (-\infty, +\infty)$ in $\mathbb{R}^n = \mathbb{R}^{n-1} \times (-\infty, +\infty)$,

(2) $T(D^n_+)$, $T(D^n_-)$, and $T(\partial D^n)$ where T is the construction from section 2.6, D^n is the unit disk in $\mathbb{R}^n$, and $D^n_{\pm} = D^n \cap \mathbb{R}^{n-1} \times [0, \pm\infty)$.

4.1.6. **Lemma.** *Suppose Y' is a subset of Y and $\{C_1, C_2\}$ is a covering such that all three subsets are coarsely invariant and form a coarsely antithetic triple. If the action of Γ on (Y, Y') is trivial then there is a weak equivalence*

$$G^{\Gamma}_p(C_1 \cap C_2, C_1 \cap C_2 \cap Y')^{\Gamma} \longrightarrow G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_1,C_2}.$$

Proof. Apply Lemma 3.3.24 to each isomorphic copy γF of F in (F, ψ) expressed as an object of the lax limit. □

4.1.7. **Theorem.** *Suppose Y' is a subset of Y and $\{C_1, C_2\}$ is a covering such that all three subsets are coarsely invariant and form a coarsely antithetic triple. Suppose the action of Γ on Y is trivial. Then the commutative square*

$$\begin{array}{ccc} G^{\Gamma}_p(C_1 \cap C_2, Y' \cap C_1 \cap C_2)^{\Gamma} & \longrightarrow & G^{\Gamma}_p(C_1, Y' \cap C_1)^{\Gamma} \\ \big\downarrow & & \big\downarrow \\ G^{\Gamma}_p(C_2, Y' \cap C_2)^{\Gamma} & \longrightarrow & G^{\Gamma}_p(Y,Y')^{\Gamma} \end{array}$$

is a homotopy pushout.

Proof. This follows from Theorem 4.1.5, using Lemma 3.3.24 to verify that the maps

$$G^{\Gamma}_p(C_1 \cap C_2, Y' \cap C_1 \cap C_2)^{\Gamma} \longrightarrow G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_1,C_2}$$

and

$$G^{\Gamma}_p(C_i, Y' \cap C_i)^{\Gamma} \longrightarrow G^{\Gamma}_p(Y,Y')^{\Gamma}_{<C_i}$$

induced by inclusions are weak equivalences. □

4.1.8. **Definition.** Given a finite coarse covering $\mathcal{U}$ of Y, there are two kinds of homotopy colimit constructions. For any action of Γ by bounded coarse equivalences, one can define

$$\mathcal{G}_p^\Gamma(Y, Y'; \mathcal{U})^\Gamma = \underset{\overrightarrow{U_i \in \mathcal{U}}}{\text{hocolim}}\; G_p^\Gamma(Y, Y')^\Gamma_{<U_i}.$$

For the trivial action, one also has

$$\mathcal{G}_p^\Gamma(\mathcal{U}, Y' \cap \mathcal{U})^\Gamma = \underset{\overrightarrow{U_i \in \mathcal{U}}}{\text{hocolim}}\; G_p^\Gamma(U_i, Y' \cap U_i)^\Gamma.$$

Inductive applications of Theorems 4.1.5 and 4.1.7 give

4.1.9. **Theorem.** *Suppose the action of Γ on (Y, Y') is by bounded coarse equivalences and $\mathcal{U}$ is a finite coarse covering of Y such that all subsets $U \in \mathcal{U}$ and Y' are pairwise coarsely antithetic. Then there is a weak equivalence*

$$G_p^\Gamma(Y, Y')^\Gamma_{<\mathcal{U}} \longrightarrow G_p^\Gamma(Y, Y')^\Gamma.$$

If the action of Γ is in fact trivial, then there is a weak equivalence

$$\mathcal{G}_p^\Gamma(Y, Y'; \mathcal{U})^\Gamma \longrightarrow G_p^\Gamma(Y, Y')^\Gamma.$$

Here is a couple of useful observations. The first is a restatement of the excision theorem.

4.1.10. **Theorem.** *Suppose the action of Γ on (Y, Y') is by bounded coarse equivalences, and a proper subset U of Y is coarsely invariant under the action of Γ. If U, $Y - U$, and Y' form a coarse covering by pairwise antithetic subsets, there is an equivalence*

$$G_p^\Gamma(Y, Y')^\Gamma \simeq G_p^\Gamma(Y - U, Y' - U)^\Gamma.$$

If the action of Γ is trivial, then there is a similar equivalence

$$K_i^\Gamma(Y, Y')^\Gamma \simeq K_i^\Gamma(Y - U, Y' - U)^\Gamma.$$

Proof. The fact in G-theory follows from Proposition 3.5.13. The K-theory statement requires similar Karoubi filtration arguments. The details are left to the reader. □

4.1.11. **Theorem.** *Suppose Γ acts on a metric space Y by bounded coarse equivalences, and Y' is an invariant metric subspace. Let Y_0 and Y_0' be the same metric spaces but with Γ acting trivially. Then there is a weak equivalence*

$$\varepsilon \colon G_p^\Gamma(Y, Y')^\Gamma \xrightarrow{\simeq} G_p^\Gamma(Y_0, Y_0')^\Gamma.$$

Proof. The category $\mathbf{G}_p^\Gamma(Y, Y')$ has the left action by Γ induced from the diagonal action on $\Gamma \times Y$. Recall from section 2.4 that an object of $\mathbf{G}_p^\Gamma(Y, Y')^\Gamma$ is determined by an object F of $\mathbf{G}_\Gamma(Y, Y')$ and isomorphisms $\psi(\gamma) \colon F \to \gamma F$ which are of filtration 0 when projected to Γ.

Given two objects $(F, \{\psi(\gamma)\})$ and $(G, \{\phi(\gamma)\})$, a morphism $\lambda \colon (F, \{\psi(\gamma)\}) \to (G, \{\phi(\gamma)\})$ is given by a morphism $\lambda \colon F \to G$ in $\mathbf{G}_\Gamma(Y, Y')$ such that the collection of morphisms $\gamma\lambda \colon \gamma F \to \gamma G$ satisfies

$$\psi(\gamma) \circ \lambda = \gamma\lambda \circ \phi(\gamma)$$

for all γ in Γ.

Given $(F, \{\psi(\gamma)\})$, define $(F_0, \{\psi_0(\gamma)\})$ by $F_0 = F$ and $\psi_0(\gamma) = \mathrm{id}_F$ for all $\gamma \in \Gamma$. Then $\psi(\gamma)^{-1}$ give a natural isomorphism Z_F from F to F_0 and induce an equivalence

$$\zeta \colon G^{\Gamma}_{p}(Y, Y')^{\Gamma} \simeq G^{\Gamma}_{p}(Y_0, Y'_0)^{\Gamma}.$$

Of course, the bound for the isomorphism $\psi(\gamma)^{-1}$ can vary with γ. □

4.2. **Fibrewise excision for lax limits.** Theorem 4.1.10 is certainly an example of excision results one can prove with the lax limit categories that model the fixed points in algebraic K-theory. Our goal in this section is a theorem of a different kind that became crucial for our own work on the Borel rigidity conjecture and should have other applications. The distinguishing feature is the nature of the exact category that we study. Formal properties of the category $\mathbf{G}^{\bullet}_{p}(Y, Y')^{\bullet}$ resemble those of the category of fixed objects $\mathbf{G}^{\Gamma}_{p}(Y, Y')^{\Gamma}$ while the action is not uniquely specified. This is precisely the type of excision result that is not available in (fibred) bounded K-theory.

First, we recapitulate the main points of the construction of $\mathbf{G}^{\Gamma}_{p}(Y, Y')^{\Gamma}$ and the spectrum $G^{\Gamma}_{p}(Y, Y')^{\Gamma}$ in a revisionist way that can be generalized. Suppose Γ acts on the proper metric space Y by coarse equivalences and Y' is a coarsely invariant subspace. The objects of $\mathbf{G}^{\Gamma}_{p}(Y, Y')$ are the functors $\theta \colon \boldsymbol{E\Gamma} \to \mathbf{G}_{\Gamma}(Y, Y')$ such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of Γ-filtered modules. Recall that this category has the left action by Γ induced from the diagonal action on $\Gamma \times Y$. This is the category used in the description of $\mathbf{G}^{\Gamma}_{p}(Y, Y')^{\Gamma}$. So, again, a fixed object in $\mathbf{G}^{\Gamma}_{p}(Y, Y')^{\Gamma}$ is determined by an object F of $\mathbf{G}_{\Gamma}(Y, Y')$ and isomorphisms $\psi(\gamma) \colon F \to \gamma F$ which are of filtration 0 when projected to Γ. We will exploit the fact that this category and its exact structure can be described independently from the construction of the equivariant functor category $\mathbf{G}^{\Gamma}_{p}(Y, Y')$. The spectrum $g^{\Gamma}_{p}(Y, Y')^{\Gamma}$ can be defined as the K-theory of the exact category $\mathbf{G}^{\Gamma}_{p}(Y, Y')^{\Gamma}$.

Let $\mathbb{R}^k$ be the Euclidean space with the trivial action of Γ. Then the product $\Gamma \times \mathbb{R}^k$ is given the Γ-action defined by $\gamma(\gamma', x) = (\gamma\gamma', x)$. By using the diagonal actions on $\Gamma \times \mathbb{R}^k \times Y$, one obtains the equivariant categories $\mathbf{G}_{\Gamma}(\mathbb{R}^k \times Y, \mathbb{R}^k \times Y')$ and also $\mathbf{G}^{\Gamma,k}_{p}(Y, Y')$ where the objects are the functors $\theta \colon \boldsymbol{E\Gamma} \to \mathbf{G}_{\Gamma}(\mathbb{R}^k \times Y, \mathbb{R}^k \times Y')$ such that the morphisms $\theta(f)$ are of filtration 0 when viewed as homomorphisms of Γ-filtered modules. The K-theory of $\mathbf{G}^{\Gamma,k}_{p}(Y, Y')^{\Gamma}$ is denoted by $g^{\Gamma,k}_{p}(Y, Y')^{\Gamma}$. Now the nonconnective delooping of $g^{\Gamma}_{p}(Y, Y')^{\Gamma}$ is constructed as

$$G^{\Gamma}_{p}(Y, Y')^{\Gamma} = \underset{k>0}{\underrightarrow{\mathrm{hocolim}}}\, \Omega^k g^{\Gamma,k}_{p}(Y, Y')^{\Gamma}.$$

The details of the defininition of $\mathbf{G}^{\bullet}_{p}(Y, Y')^{\bullet}$ and the corresponding nonconnective spectrum $G^{\bullet}_{p}(Y, Y')^{\bullet}$ are mostly straightforward extensions of the summary above. It should be helpful to point out that the category $\mathbf{G}^{\bullet}_{p}(Y, Y')^{\bullet}$ by itself is not a lax limit with respect to a specific action of Γ. However lax limits such as $\mathbf{G}^{\Gamma}_{p}(Y, Y')^{\Gamma}$ are going to be exact subcategories of $\mathbf{G}^{\bullet}_{p}(Y, Y')^{\bullet}$. In order to stress the lax limit origin of the construction without committing to a specific action, we use the $^{\bullet}$ superscript.

4.2.1. **Definition.** The category $\mathbf{G}^{\bullet}_{p}(Y, Y')^{\bullet}$ has objects which are sets of data $(\{F_\gamma\}, \alpha, \{\psi_\gamma\})$ where

- F_γ is an object of $\mathbf{G}_{\Gamma}(Y, Y')$ for each γ in Γ,

- α is an action of Γ on Y by bounded coarse equivalences,
- ψ_γ is an isomorphism $F_e \to F_\gamma$ induced from α_γ,
- ψ_γ has filtration 0 when viewed as a morphism in $\mathbf{U}(\Gamma, \mathbf{U}(Y))$,
- $\psi_e = \mathrm{id}$,
- $\psi_{\gamma_1\gamma_2} = \gamma_1 \psi_{\gamma_2} \psi_{\gamma_1}$.

The morphisms $(\{F_\gamma\}, \alpha, \{\psi_\gamma\}) \to (\{F'_\gamma\}, \alpha', \{\psi'_\gamma\})$ are collections $\{\phi_\gamma\}$, where each ϕ_γ is a morphism $F_\gamma \to F'_\gamma$ in $\mathbf{G}_\Gamma(Y, Y')$, such that the squares

$$\begin{tikzcd} F_e \arrow[r, "\psi_\gamma"] \arrow[d, "\phi_e"'] & F_\gamma \arrow[d, "\phi_\gamma"] \\ F'_e \arrow[r, "\psi'_\gamma"] & F'_\gamma \end{tikzcd}$$

commute for all $\gamma \in \Gamma$.

The exact structure on $\mathbf{G}^\bullet_p(Y, Y')^\bullet$ is induced from that on $\mathbf{G}_\Gamma(Y, Y')$. First we observe that for any action α on Y by bounded coarse equivalences, a subset Y' is coarsely invariant, so there is the induced action on the pair (Y, Y'). Now the lax limit category $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_\alpha$ is a subcategory of $\mathbf{G}^\bullet_p(Y, Y')^\bullet$. The embedding E_α is realized by sending the object (F, ψ) of $\mathbf{G}_\Gamma(Y, Y')^\Gamma_\alpha$ to $(\{\alpha_\gamma F\}, \alpha, \{\psi(\gamma)\})$. On the morphisms, $E_\alpha(\phi) = \{\alpha_\gamma \phi\}$.

A morphism ϕ in $\mathbf{G}^\bullet_p(Y, Y')^\bullet$ is an admissible monomorphism if $\phi_e \colon F \to F'$ is an admissible monomorphism in $\mathbf{G}_\Gamma(Y, Y')$. This of course implies that all structure maps ϕ_γ are admissible monomorphisms. Similarly, a morphism ϕ is an admissible epimorphism if $\phi_e \colon F \to F'$ is an admissible epimorphism. It is clear that for each action α by bounded coarse equivalences the inclusion functor E_α is exact.

4.2.2. **Definition.** The spectrum $g^\bullet_p(Y, Y')^\bullet$ is defined as the K-theory spectrum of $\mathbf{G}^\bullet_p(Y, Y')^\bullet$. If $g^\Gamma_p(Y, Y')^\Gamma_\alpha$ stands for the K-theory of $\mathbf{G}^\Gamma_p(Y, Y')^\Gamma_\alpha$ then we get the induced map of spectra

$$\varepsilon_\alpha \colon g^\Gamma_p(Y, Y')^\Gamma_\alpha \longrightarrow g^\bullet_p(Y, Y')^\bullet.$$

We have the equivariant categories $\mathbf{G}_\Gamma(\mathbb{R}^k \times Y, \mathbb{R}^k \times Y')_\alpha$ and thus $\mathbf{G}^{\Gamma,k}_p(Y, Y')_\alpha$. So there is the lax limit $\mathbf{G}^{\Gamma,k}_p(Y, Y')^\Gamma_\alpha$. Similarly, there are categories $\mathbf{G}^{\bullet,k}_p(Y, Y')^\bullet$ and the evident exact inclusions

$$E^k \colon \ \mathbf{G}^{\Gamma,k}_p(Y, Y')^\Gamma_\alpha \longrightarrow \mathbf{G}^{\bullet,k}_p(Y, Y')^\bullet.$$

4.2.3. *Notation.* The K-theory of $\mathbf{G}^{\Gamma,k}_p(Y, Y')^\Gamma_\alpha$ will be denoted by $g^{\Gamma,k}_p(Y, Y')^\Gamma_\alpha$. The K-theory of $\mathbf{G}^{\bullet,k}_p(Y, Y')^\bullet$ will be denoted by $g^{\bullet,k}_p(Y, Y')^\bullet$.

Now the nonconnective delooping of $g^\Gamma_p(Y, Y')^\Gamma_\alpha$ can be constructed as

$$G^\Gamma_p(Y, Y')^\Gamma_\alpha \ = \ \underset{k>0}{\underrightarrow{\operatorname{hocolim}}} \ \Omega^k g^{\Gamma,k}_p(Y, Y')^\Gamma_\alpha.$$

4.2.4. **Definition.** We define the nonconnective delooping of $g^\bullet_p(Y, Y')^\bullet$ as

$$G^\bullet_p(Y, Y')^\bullet \ = \ \underset{k>0}{\underrightarrow{\operatorname{hocolim}}} \ \Omega^k g^{\bullet,k}_p(Y, Y')^\bullet.$$

The exact inclusions E^k induce a map of nonconnective spectra

$$\varepsilon_\alpha \colon G^\Gamma_p(Y, Y')^\Gamma_\alpha \longrightarrow G^\bullet_p(Y, Y')^\bullet.$$

The following is a corollary to the proof of Theorem 4.1.11.

4.2.5. **Corollary.** *Consider actions of Γ on a metric space Y by bounded coarse equivalences and let Y' be a metric subspace. Let Y_0 and Y'_0 be the same metric spaces with the trivial action of Γ. Then there are a natural transformation from the identity functor on $\mathbf{G}^\bullet_p(Y,Y')^\bullet$ to the functor*

$$Z\colon\ \mathbf{G}^\bullet_p(Y,Y')^\bullet \longrightarrow \mathbf{G}^\Gamma_p(Y_0,Y'_0)^\Gamma,$$

where each $Z(F)$ is an isomorphism, and the induced weak equivalence

$$\zeta\colon G^\bullet_p(Y,Y')^\bullet \xrightarrow{\ \simeq\ } G^\Gamma_p(Y_0,Y'_0)^\Gamma.$$

In particular, the maps ε_α are weak equivalences.

The localization and excision results for $G^\bullet_p(Y,Y')^\bullet$ are straightforward extrapolations of those for $G^\Gamma_p(Y,Y')^\Gamma$. There is the familiar construction of the full subcategory $\mathbf{G}^\bullet_p(Y,Y')^\bullet_{<C}$ associated to any subset C of Y, which is a Grothendieck subcategory of $\mathbf{G}^\bullet_p(Y,Y')^\bullet$, and there is an exact quotient category $\mathbf{G}^\bullet_p(Y,Y')^\bullet_{>C}$ obtained by localizing away from C.

Given a finite coarse covering $\mathcal{U}$ of Y, we define the homotopy colimit

$$\mathcal{G}^\bullet(Y,Y')^\bullet_{<\mathcal{U}} \;=\; \underset{\overrightarrow{U_i\in\mathcal{U}}}{\mathrm{hocolim}}\; G^\bullet_p(Y,Y')^\bullet_{<U_i}.$$

4.2.6. **Theorem.** *Suppose the action of Γ on Y is by bounded coarse equivalences. If $\mathcal{U}$ is a finite coarse covering of Y such that the family of all subsets U in $\mathcal{U}$ together with Y' are pairwise coarsely antithetic, then the natural map induced by inclusions*

$$\delta\colon \mathcal{G}^\bullet(Y,Y')^\bullet_{<\mathcal{U}} \longrightarrow G^\bullet_p(Y,Y')^\bullet$$

is a weak equivalence.

Proof. First, observe that for any coarsely antithetic triple Y', C_1, C_2 there is a homotopy pushout of spectra

$$\begin{array}{ccc}
G^\bullet_p(Y,Y')^\bullet_{<C_1,C_2} & \longrightarrow & G^\bullet_p(Y,Y')^\bullet_{<C_1} \\
\big\downarrow & & \big\downarrow \\
G^\bullet_p(Y,Y')^\bullet_{<C_2} & \longrightarrow & G^\bullet_p(Y,Y')^\bullet
\end{array}$$

obtained from the map of the fibration sequences

$$\begin{array}{ccccc}
G^\bullet_p(Y,Y')^\bullet_{<C_1,C_2} & \longrightarrow & G^\bullet_p(Y,Y')^\bullet_{<C_1} & \longrightarrow & G^\bullet_p(Y,Y')^\bullet_{<C_1,>C_2} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
G^\bullet_p(Y,Y')^\bullet_{<C_2} & \longrightarrow & G^\bullet_p(Y,Y')^\bullet & \longrightarrow & G^\bullet_p(Y,Y')^\bullet_{>C_2}
\end{array}$$

The map $G^\bullet_p(Y,Y')^\bullet_{<C_1,>C_2} \to G^\bullet_p(Y,Y')^\bullet_{>C_2}$ is an equivalence by a familiar application of the Approximation Theorem, which is a generalizagion of Lemma 3.5.3. Using this homotopy pushout inductively with $\mathcal{U}$ gives the theorem.

It might be instructive to spell out exactly what happens in the crucial application of the Approximation Theorem. To check the second condition of the Approximation Theorem, consider a chain complex $F^\cdot$ in $\mathbf{G}^\bullet_p(Y,Y')^\bullet_{<C_1,>C_2}$. By the nature of the objects, all maps in $F^\cdot$ and their control features are determined by the values on the objects F_e^i of $\mathbf{G}_\Gamma(Y,Y')$. So we can specify $F^\cdot$ by the chain complex

$$F_e^\cdot\colon \quad 0 \longrightarrow F_e^1 \xrightarrow{\phi_1} F_e^2 \xrightarrow{\phi_2} \ \dots\ \xrightarrow{\phi_{n-1}} F_e^n \longrightarrow 0$$

in $\mathbf{G}_p(Y,Y')_{<C_1,>C_2}$. Given a chain complex $G^\cdot$ in $\mathbf{G}_p^\bullet(Y,Y')^\bullet_{>C_2}$, we can apply the same discussion to $G^\cdot$. Now a chain map $g\colon F^\cdot \to G^\cdot$ is given by a chain map from $F_e^\cdot$ to

$$G_e^\cdot\colon \quad 0 \longrightarrow G_e^1 \xrightarrow{\psi_1} G_e^2 \xrightarrow{\psi_2} \ \dots\ \xrightarrow{\psi_{n-1}} G_e^n \longrightarrow 0$$

in $\mathbf{G}_p(Y,Y')_{>C_2}$. Also observe that if F_e^i is supported on a member $\mathcal{C}$ of the coarse covering $\mathcal{U}$ then all F_γ^i are too. This allows to apply the rest of the argument for Lemma 3.5.3 verbatim. □

We will now specialize to the situation that is studied in [7] and prove a very specific excision result.

Let X be the universal cover of an aspherical manifold M of dimension n with fundamental group Γ. There is a proper metric on X which is commensurable with the given word metric on Γ when it is viewed as a free orbit in X. Suppose the manifold M is embedded in a sphere S^{n+k} for sufficiently large k. Let Y be the universal cover of the small tubular neighborhood N of the embedding. There is a metric on Y commensurable with the metric on M. Let ∂Y denote the topological boundary of Y.

Recall two geometric constructions: the left-bounded metric from 2.6.3 and the cone construction from 2.6.10. In [7], the authors construct the exact category $\mathbf{G}_p^\bullet((TY)^{bdd},(T\partial Y)^{bdd})^\bullet$ and its nonconnective K-theory $G_p^\bullet((TY)^{bdd},(T\partial Y)^{bdd})^\bullet$.

Choose a closed Euclidean ball B in N and let $\widehat{B}$ be an arbitrary lift of B in Y expressed by a homeomorphism $\sigma\colon B \to \widehat{B}$. We define the category

$$\mathbf{G}_p^\bullet((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{>\complement T\widehat{B}}$$

as the exact quotient determined by the choice of the subset $\complement T\widehat{B}$ of $(TY)^{bdd}$ which is the complement of $T\widehat{B}$. We have the resulting nonconnective spectrum

$$G_p^\bullet((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{>\complement T\widehat{B}}.$$

This is the spectrum we would like to compute.

4.2.7. *Notation.* We will need to localize these constructions further to certain subsets of TY. In order to clear the slate for further subscripts, we will use the following notation for the quotient category

$$\mathbf{G}^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet,$$

and for the spectrum

$$G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet.$$

We will apply the excision theorem and show that there is a weak equivalence

$$e\colon G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet \longrightarrow \Sigma^{n+k+1}G^{\Gamma,0}(X,R)^\Gamma.$$

The construction of e will require several families of subsets of TY.

We may assume that the ball B is a metric ball in $\mathbb{R}^{n+k}$ centered at c with radius R and contained entirely in the interior of N. Let $h\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ be the linear map $h(x) = c + Rx$, so h restricts to a linear homeomorphism $h\colon D^{n+k} \to B$ from the unit disk $D^{n+k} = 0[1]$ onto the chosen ball $B \subset N$.

4.2.8. *Notation.* We identify the following subsets of $\mathbb{R}^{n+k}$:

$$\begin{aligned}
&E_i^+ = \{(x_1, \ldots, x_{n+k}) \mid x_l = 0 \text{ for all } l > i,\ x_i \geq 0\},\\
&E_i^- = \{(x_1, \ldots, x_{n+k}) \mid x_l = 0 \text{ for all } l > i,\ x_i \leq 0\},\\
&E_i = E_i^- \cup E_i^+, \text{ for } 1 \leq i \leq n+k, \text{ and}\\
&E_0 = E_0^- = E_0^+ = \{(0, \ldots, 0)\}.
\end{aligned}$$

There are also related to $E_i^{\pm}$ subsets

$$\begin{aligned}
&D_0 = D_0^- = D_0^+ = \{(0, \ldots, 0)\},\\
&D_i^{\pm} = E_i^{\pm} \cap D^{n+k}, \text{ for } 1 \leq i \leq n+k, \text{ and}\\
&D_i = D_i^- \cup D_i^+ = D_{i+1}^- \cap D_{i+1}^+.
\end{aligned}$$

The images of D_i^* under the linear homeomorphism h will be called B_i^*.

The subsets E_i^* form a coarsely antithetic covering of $\mathbb{R}^{n+k}$. It is easy to see that TE_i^* form a coarsely antithetic covering of $T\mathbb{R}^{n+k}$. Therefore, we obtain a coarsely antithetic covering $\mathcal{E}$ of $T\mathbb{R}^{n+k}$ by the subsets $Th(E_i^*)$. Notice that this covering is closed under coarse intersections since it includes the subsets $Th(E_i)$ for $0 \leq i < n+k$, where E_i are the intersections $E_{i+1}^- \cap E_{i+1}^+$.

4.2.9. *Notation.* Now we define the following collection of subsets of $V = T\widehat{B}$:

$$\begin{aligned}
&V' = T\partial\widehat{B},\\
&V_i^* = T\sigma(B_i^*), \text{ for } 0 \leq i \leq n+k, \text{ and}\\
&V_i = (T\sigma h(D_i))\,.
\end{aligned}$$

The subsets $\{V_i^*\}$ can be thought of as a coarsely antithetic covering of $T\widehat{B}$. These can be extended to a coarsely antithetic covering of the metric space $\overline{V} = (TY)^{bdd}$:

$$\begin{aligned}
&\overline{V'} = T\partial\widehat{B} \cup \complement T\widehat{B},\\
&\overline{V_i^*} = T\sigma(B_i^*) \cup \complement T\widehat{B} \ \text{ for } 0 \leq i \leq n+k.
\end{aligned}$$

4.2.10. **Definition.** Let $\mathcal{U}$ be the coarse antithetic covering of $T\widehat{B}$ by V_i^*. There is a homotopy pushout

$$\mathcal{G}^{\Gamma}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})_{<\mathcal{U}} = \underset{\overrightarrow{\mathcal{U}}}{\operatorname{hocolim}}\ G^{\bullet}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet}_{<\overline{V_i^*}}.$$

From Theorem 4.2.6 we have the following consequence.

4.2.11. **Corollary.** *There is a weak equivalence*

$$\delta\colon \mathcal{G}^{\Gamma}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet}_{<\mathcal{U}} \longrightarrow G^{\Gamma}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet}.$$

4.2.12. **Theorem.** *There is a weak equivalence*

$$e\colon G^{\bullet}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet} \longrightarrow \Sigma^{n+k+1} G^{-\infty}(R[\Gamma])^{\bullet}.$$

Proof. Let C_1 and C_2 be any two of the subsets $\overline{V_i^*}$. The intersection category defined as $\mathbf{G}^{\bullet}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet}_{<C_1} \cap \mathbf{G}^{\bullet}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet}_{<C_2}$ will be denoted by $\mathbf{G}^{\bullet}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet}_{<C_1, C_2}$. It is clearly a Grothendieck subcategory of $\mathbf{G}^{\bullet}_{T\widehat{B}}((TY)^{bdd}, (T\partial Y)^{bdd})^{\bullet}$.

Applying Theorem 4.2.6 to $\overline{V}$, $\overline{V'}$, C_1, C_2, we obtain a homotopy pushout square

$$\begin{array}{ccc} G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<C_1,C_2} & \longrightarrow & G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<C_1} \\ \downarrow & & \downarrow \\ G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<C_2} & \longrightarrow & G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet \end{array}$$

Here $G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<C_i}$ are contractible by Lemma 3.5.4. If $C_1 = \overline{V_i^*}$ and $C_2 = \overline{V_j^*}$ with $j \le i$, so that $C_2 \subset C_1$, then $G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<C_1,C_2}$ is also contractible. If $C_1 = \overline{V_i^\pm}$ and $C_2 = \overline{V_i^\mp}$ then

$$G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<C_1,C_2} = G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<\overline{V_{i-1}}}.$$

This gives weak equivalences

$$\Sigma\, G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<\overline{V_{i-1}}} \simeq G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<\overline{V_i}},$$

and, therefore, the equivalence

$$\begin{aligned} &G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<\mathcal{U}} \\ &\qquad \simeq \Sigma^{n+k} G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet_{<\overline{V_0}} \simeq \Sigma^{n+k+1} G^{-\infty}(R[\Gamma]). \end{aligned}$$

Combining this with Corollary 4.2.11, we obtain a map

$$e\colon G^\bullet_{T\widehat{B}}((TY)^{bdd},(T\partial Y)^{bdd})^\bullet \longrightarrow \Sigma^{n+k+1} G^{-\infty}(R[\Gamma])$$

which is a weak equivalence. □

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