CONTROLLED ALGEBRAIC G-THEORY

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Abstract. This paper extends the notion of geometric control in algebraic K-theory from additive categories with split exact sequences to other exact structures. In particular, we construct exact categories of modules over a noetherian ring filtered by subsets of a metric space and sensitive to the large scale properties of the space. The algebraic K-theory of these categories is related to the controlled K-theory of geometric modules the way G-theory is classically related to K-theory. We recover familiar results in the new setting, including nonconnective controlled excision, equivariant properties, and prove the G-theoretic Novikov conjecture which is shown to be stronger than the usual K-theoretic conjecture.

Contents

1. Introduction

Since the invention of algebraic K-groups of a ring defined using the finitely generated projective R-modules, there existed a companion K-theory defined using arbitrary finitely generated R-modules, called G-theory. Its usefulness comes from the computational tool available in G-theory, the localization exact sequence, and the close relation to K-theory via the Cartan map which becomes an isomorphism when R is a regular ring. The recent success of controlled K-theory in algebra and topology, where the ring involved is usually the regular ring of integers $\mathbb{Z}$, makes it natural to look for a similar controlled analogue of G-theory. This paper constructs and exploits such an analogue.

The bounded control is introduced by fixing a basis B in a free module M and defining a locally finite set function $s\colon B \to X$ into a metric space X. The control comes from restrictions on the maps one allows between the based modules. Since each element x in M is written uniquely as a sum $x = \sum_{b \in B} r_b b$, there is the notion of support $\operatorname{supp}(x) = \{s(b) \mid b \in B \text{ such that } r_b \neq 0\}$ in X. For two sets of choices

Date: April 28, 2009.
1991 *Mathematics Subject Classification.* 18E10, 18E30, 18E35, 18F25, 19D35, 19J99.
The authors acknowledge support from the National Science Foundation.

(M_i, B_i, s_i), $i = 1, 2$, an R-homomorphism $\phi \colon M_1 \to M_2$ is *bounded* if there is a number $D > 0$ such that for every $b \in B_1$ the support $\operatorname{supp} s_2(\phi(b))$ is contained in the metric ball of radius D centered at $s_1(b)$. It is clear that the triples as above with the bounded homomorphisms form a small *bounded category* $\mathcal{B}(X, R)$. It is in fact an additive category since the direct products can be defined in the evident way. To each small additive category $\mathcal{A}$, one associates a sequence of groups $K_i(\mathcal{A})$, $i \in \mathbb{Z}$, or rather a nonconnective spectrum $\mathrm{Spt}(\mathcal{A})$ whose stable homotopy groups are $K_i(\mathcal{A})$, as in [17]. This construction applied to the bounded category gives the *bounded algebraic K-theory* $K_i(X, R)$.

The general goal of this paper is to construct larger categories associated to a proper metric space X and a noetherian ring R and to recover in this context the basic results from controlled K-theory. We are mostly concerned with controlled excision established in section 3. In many ways these categories are more flexible than the bounded categories and allow application of recent powerful results in algebraic K-theory. Their properties are essential for our study of the Borel isomorphism conjecture continued elsewhere but indicated in section 4. In the same section, we prove integral Novikov conjecture in this context which is stronger than the usual statement in algebraic K-theory.

First notice that, given a triple (M, B, s) in $\mathcal{B}(X, R)$, to every subset $S \subset X$ there is associated a free submodule $M(S)$ generated by those $b \in B$ with the property $s(b) \in S$. The restriction to bounded homomorphisms can be described entirely in terms of these submodules. We generalize this as follows. The objects of the new category $\mathbf{U}^{\mathrm{b}}(X, R)$ are left R-modules M filtered by the subsets of X in the sense that they are functors from subsets of X to submodules of M ordered by inclusion so that X maps to M. By abuse of notation we usually denote the functor by the same letter M. We also make several additional assumptions spelled out in Definition 2.13, in particular, that the bounded subsets map to finitely generated submodules. The morphisms are the R-homomorphisms $\phi \colon M_1 \to M_2$ such that the image $\phi(M_1(S))$ is contained in the submodule $M_2(S[D])$ for some $D \geq 0$ specific to ϕ. Here $S[D]$ stands for the metric D-enlargement of S in X. In this context we say an element $x \in M$ is supported on a subset S if $x \in M(S)$. The *boundedly controlled* category $\mathbf{B}(X, R)$ is the full subcategory of $\mathbf{U}^{\mathrm{b}}(X, R)$ on filtered modules M generated by elements supported on subsets of diameter less than d for some number $d > 0$ specific to M.

The additive structure on $\mathbf{B}(X, R)$ gives it the *split exact* structure where the admissible monomorphisms are all split monics and admissible epimorphisms are all split epis. In order to describe a different Quillen exact structure on $\mathbf{B}(X, R)$, we define an additional property a boundedly controlled homomorphism $\phi \colon M_1 \to M_2$ in $\mathbf{U}^{\mathrm{b}}(X, R)$ may or may not have: ϕ is *boundedly bicontrolled* if in addition to containments

$$\phi(M_1(S)) \subset M_2(S[D])$$

as above, there are containments

$$\phi(M_1) \cap M_2(S) \subset \phi M_1(S[D])$$

for all subsets S of X. The admissible monomorphisms in either $\mathbf{B}(X, R)$ or $\mathbf{U}^{\mathrm{b}}(X, R)$ consist of boundedly bicontrolled injections of modules. The admissible epimorphisms in $\mathbf{U}^{\mathrm{b}}(X, R)$ are the boundedly bicontrolled surjections. The admissible epimorphisms in $\mathbf{B}(X, R)$ are the boundedly bicontrolled surjections with kernels in $\mathbf{B}(X,R)$. In both cases the exact sequences are simply the short exact sequences when viewed as sequences in $\mathbf{U}^{\mathrm{b}}(X,R)$ so that all kernels and cokernels are well-defined filtered submodules in the respective category. Notice that split injections and surjections are boundedly bicontrolled, so the split exact structure is an exact subcategory of the new one. This theory is functorial in the space variable X with respect to certain maps called quasi-Lipschitz equivalences, as should be expected from [17].

To a small exact category $\mathbf{E}$, one associates a sequence of groups $K_i(\mathbf{E})$, $i \geq 0$, as in Quillen [19] or a connective spectrum $K(\mathbf{E})$ whose stable homotopy groups are $K_i(\mathbf{E})$. If the exact structure is split, these groups are the same as the K-groups of $\mathbf{E}$ as an additive category.

Suppose Z is a metric subspace of X. There is a construction of an exact category $\mathbf{B}/\mathbf{Z}$ associated to Z and an exact map $\mathbf{B}(X,R) \to \mathbf{B}/\mathbf{Z}$ such that the following is true.

Theorem (Localization, Theorem 4.13)**.** *The sequence*

$$G(Z,R) \longrightarrow G(X,R) \longrightarrow K(\mathbf{B}/\mathbf{Z})$$

is a homotopy fibration.

Localization can be used to construct nonconnective deloopings of $G(X,R)$ which are compatible with the K-theory deloopings as seen in Pedersen–Weibel [16]. We will indicate the corresponding nonconnective spectra with superscripts "$-\infty$".

The following is the analogue of a major tool in many proofs of the Novikov conjecture. If a proper metric space X is the union of subspaces X_1 and X_2, let $\mathbf{B}(X_1,X_2;R)$ stand for the full subcategory of $\mathbf{B}(X,R)$ on the modules supported on the intersection of bounded enlargements of X_1 and X_2 and let $G(X_1,X_2;R)$ denote its K-theory.

Theorem (Nonconnective controlled excision, Theorem 4.24)**.** *There is a homotopy pushout*

$$\begin{array}{ccc} G^{-\infty}(X_1,X_2;R) & \longrightarrow & G^{-\infty}(X_1,R) \\ \big\downarrow & & \big\downarrow \\ G^{-\infty}(X_2,R) & \longrightarrow & G^{-\infty}(X,R) \end{array}$$

Finally, we describe the application to splitting G-theoretic assembly maps and, further, establishing when they are weak equivalences. There is a close relation to the same problem in K-theory.

The left translation action of a group Γ on itself viewed as a metric space with a word metric is by quasi-Lipschitz equivalences, and the fixed object category $\mathbf{B}_{\Gamma,0}(\Gamma,R)^{\Gamma}$ can be defined as a lax limit. If Γ is a torsion-free group with a word metric, the K-theory of $\mathbf{B}_{\Gamma,0}(\Gamma,R)^{\Gamma}$ can be identified with the K-theory of a certain exact subcategory of finitely generated $R[\Gamma]$-modules referred to as $\mathbf{B}(R[\Gamma])$, cf. section 5. This category contains all free finitely generated $R[\Gamma]$-modules but also such important objects as images of idempotents of free modules, which are not necessarily projective. Its exact structure is intermediate between the split exact structure and the usual structure inherited from the abelian category $\mathbf{Mod}(R[\Gamma])$ of all left $R[\Gamma]$-modules.

Recall that the integral assembly map in algebraic K-theory

$$A_K \colon B\Gamma_+ \wedge K^{-\infty}(R) \longrightarrow K^{-\infty}(R[\Gamma])$$

is defined for any group Γ and any ring R and relates the homology of Γ with coefficients in the K-theory of R to the K-theory of the group ring. The integral *Novikov conjecture* for Γ is the statement that A_K is a split injection of spectra. It is expected to be true whenever Γ is a discrete torsion-free group. Now for a noetherian ring R and $G^{-\infty}(R[\Gamma]) = K^{-\infty}\,\mathbf{B}(R[\Gamma])$ there is an analogue

$$A_G \colon B\Gamma_+ \wedge G^{-\infty}(R) \longrightarrow G^{-\infty}(R[\Gamma])$$

which we call the *assembly map* in algebraic G-theory. In this paper we show that it is a split injection for many geometric groups.

Theorem. *Let Γ be a discrete group of finite asymptotic dimension and a finite classifying space. Let R be a noetherian ring. Then the assembly map A_G is a split injection.*

The result is, in fact, stronger than the one in algebraic K-theory whenever the ring R is regular noetherian, for example the integers $\mathbb{Z}$. Notice that from the commutative square

$$\begin{array}{ccc} B\Gamma_+ \wedge K^{-\infty}(R) & \xrightarrow{A_K} & K^{-\infty}(R[\Gamma]) \\ \simeq\downarrow & & \downarrow \\ B\Gamma_+ \wedge G^{-\infty}(R) & \xrightarrow{A_G} & G^{-\infty}(R[\Gamma]) \end{array}$$

the assembly map A_G is, up to homotopy, the composition of A_K followed by the Cartan map $K^{-\infty}(R[\Gamma]) \to G^{-\infty}(R[\Gamma])$. If A_G is a split injection, it follows that A_K is a split injection.

The application to the Borel isomorphism conjecture starts with the observation that the Cartan map in the square above is a weak equivalence when Γ has finite asymptotic dimension [5]. This identifies the assembly maps A_K and A_G. In further installments we show that the controlled G-theory described in this paper possesses deformations, absent in K-theory, which allow us to prove that A_G is an equivalence.

The authors gratefully acknowledge support from the National Science Foundation.

2. Controlled categories of filtered objects

This work is motivated by the delooping of algebraic K-theory of a small additive category in [16] and, in particular, the introduction of bounded control in a cocomplete additive category $\mathbf{A}$ which we briefly recall.

A category is *cocomplete* if it contains colimits of arbitrary small diagrams, cf. Mac Lane [15], chapter V.

Definition 2.1 (Pedersen–Weibel)**.** Let X be a proper metric space, in the sense that all closed metric balls in X are compact. Consider an assignment of an object $F(x)$ in $\mathbf{A}$ to each point x satisfying the local finiteness condition: the subset $\{x \in S \mid F(x) \neq 0\}$ should be finite for every bounded $S \subset X$. Such assignments define *X-graded* objects $\bigoplus_{x \in X} F(x)$ in $\mathbf{A}$ and form objects of a new category $\mathcal{B}(X, \mathbf{A})$. Morphisms are collections of $\mathbf{A}$-morphisms $f(x,y) \colon F(x) \to G(y)$ with the property that there is a number $D > 0$ such that $f(x,y) = 0$ if $\operatorname{dist}(x,y) > D$. If $\mathbf{B}$ is a subcategory of $\mathbf{A}$ closed under the direct sum, one obtains the additive bounded category $\mathcal{B}(X, \mathbf{B})$ as the full subcategory of $\mathcal{B}(X, \mathbf{A})$ on objects F with $F(x) \in \mathbf{B}$ for all $x \in X$. Notice that $\mathbf{B}$ does not need to be cocomplete. The bounded algebraic K-theory $K_i(X, \mathbf{B})$ is the K-theory spectrum associated to $\mathcal{B}(X, \mathbf{B})$.

To generalize this construction from additive to more general exact categories $\mathbf{E}$, first notice that, given an object F in $\mathcal{B}(X, \mathbf{B})$, to every subset $S \subset X$ there is associated a direct sum $F(S)$ generated by those $F(x)$ with $x \in S$. The restriction to bounded homomorphisms can be described entirely in terms of these subobjects.

Definition 2.2 (Quillen exact categories)**.** Let $\mathbf{C}$ be an additive category. Suppose $\mathbf{C}$ has two classes of morphisms $\mathbf{m}(\mathbf{C})$, called *admissible monomorphisms*, and $\mathbf{e}(\mathbf{C})$, called *admissible epimorphisms*, and a class $\mathcal{E}$ of *exact* sequences, or extensions, of the form

$$C^{\cdot}\colon \quad C' \xrightarrow{i} C \xrightarrow{j} C''$$

with $i \in \mathbf{m}(\mathbf{C})$ and $j \in \mathbf{e}(\mathbf{C})$ which satisfy the three axioms:

a) any sequence in $\mathbf{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$; the canonical sequence

$$C' \xrightarrow{\mathrm{incl}_1} C' \oplus C'' \xrightarrow{\mathrm{proj}_2} C''$$

is in $\mathcal{E}$; for any sequence $C^{\cdot}$, i is a kernel of j and j is a cokernel of i in $\mathbf{C}$,

b) both classes $\mathbf{m}(\mathbf{C})$ and $\mathbf{e}(\mathbf{C})$ are subcategories of $\mathbf{C}$; $\mathbf{e}(\mathbf{C})$ is closed under base-changes along arbitrary morphisms in $\mathbf{C}$ in the sense that for every exact sequence $C' \to C \to C''$ and any morphism $f\colon D'' \to C''$ in $\mathbf{C}$, there is a pullback commutative diagram

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "="'] & D \arrow[r, "j'"] \arrow[d, "f'"] & D'' \arrow[d, "f"] \\ C' \arrow[r] & C \arrow[r, "j"] & C'' \end{tikzcd}$$

where $j'\colon D \to D''$ is an admissible epimorphism; $\mathbf{m}(\mathbf{C})$ is closed under cobase-changes along arbitrary morphisms in $\mathbf{C}$ in the (dual) sense that for every exact sequence $C' \to C \to C''$ and any morphism $g\colon C' \to D'$ in $\mathbf{C}$, there is a pushout diagram

$$\begin{tikzcd} C' \arrow[r, "i"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "g'"'] & C'' \arrow[d, "="] \\ D' \arrow[r, "i'"] & D \arrow[r] & C'' \end{tikzcd}$$

where $i'\colon D' \to D$ is an admissible monomorphism,

c) if $f\colon C \to C''$ is a morphism with a kernel in $\mathbf{C}$, and there is a morphism $D \to C$ so that the composition $D \to C \to C''$ is an admissible epimorphism, then f is an admissible epimorphism; dually for admissible monomorphisms.

According to Keller [12], axiom (c) follows from the other two. We will use the standard notation $\rightarrowtail$ for admissible monomorphisms and $\twoheadrightarrow$ for admissible epimorphisms.

Recall that an *abelian category* is an additive category with kernels and cokernels such that every morphism f is balanced, that is, the canonical map from the coimage $\mathrm{coim}(f) = \mathrm{coker}(\ker f)$ to the image $\mathrm{im}(f) = \ker(\mathrm{coker}\, f)$ is an isomorphism. An abelian category has the canonical exact structure where all kernels and cokernels are respectively admissible monomorphisms and admissible epimorphisms.

A *subobject* of a fixed object F is a monic $m\colon F' \to F$. The collection of all subobjects of F forms a category where morphisms are morphisms $j\colon F' \to F''$ between two subobjects of F such that $m''j = m'$. Notice that such j are also monic. If the category is an exact category, there is the subcategory of *admissible subobjects* of F represented by admissible monomorphisms. If both m' and m'' are admissible, it follows from exactness axiom 3 that j is also an admissible monomorphism.

Given two subobjects $m'\colon F' \to F$, $m''\colon F'' \to F$, the *intersection* $F' \cap F''$, which is the pullback of m' along m'', is a subobject of F and can be written as the kernel of a morphism. If F' and F'' are admissible subobjects then the intersection $F' \cap F''$ is an admissible subobject.

Now let $\mathbf{A}$ be a cocomplete abelian category. The power set $\mathcal{P}(X)$ of a proper metric space X is partially ordered by inclusion which makes it into the category with subsets of X as objects and unique morphisms (S,T) when $S \subset T$. A *presheaf* of $\mathbf{A}$-objects on X is a functor $F\colon \mathcal{P}(X) \to \mathbf{A}$. This corresponds to the usual notion of presheaf of $\mathbf{A}$-objects on the discrete topological space X^δ if the chosen Grothendieck topology on $\mathcal{P}(X)$ is the partial order given by inclusion, cf. section II.1 of [11]. We will use terms which are standard in sheaf theory such as *structure maps*, when referring to the morphisms $F(S,T)$.

Definition 2.3. A presheaf F is an *X-filtered object* if all structure maps of F are monomorphisms. For each presheaf F there is an associated X-filtered object given by $F_X(S) = \operatorname{im} F(S,X)$.

Suppose F is a filtered object. Given a subobject $F' \subset F(X)$ in $\mathbf{A}$, define the *standard filtration* of F' induced from F by the formula $F'(S) = F(S) \cap F'$. In other words, $F'(S)$ is the image of the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & F(S) \arrow[d] \\ F' \arrow[r] & F(X) \end{tikzcd}$$

If $F' = F(T)$ for some subset $T \subset X$, a different *strict filtration* is defined by $F(T)(S) = F(T \cap S)$.

Definition 2.4. The objects of the *uncontrolled* category $\mathbf{U}(X, \mathbf{A})$ are the X-filtrations of objects F in $\mathbf{A}$. The morphisms $F \to G$ are the morphisms $F(X) \to G(X)$ in $\mathbf{A}$.

Let $S[D]$ denote the subset $\{x \in X \mid \operatorname{dist}(x,S) \le D\}$. A morphism $f\colon F \to G$ in $\mathbf{U}(X, \mathbf{A})$ is *boundedly controlled* if there is a number $D \ge 0$ such that the image of ϕ restricted to $F(S)$ is a subobject of $G(S[D])$ for every subset $S \subset X$. The *bounded category* $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ is the full subcategory of $\mathbf{U}(X, \mathbf{A})$ on the objects with the property $F(\emptyset) = 0$ and the boundedly controlled morphisms.

If f in addition has the property that for all subsets $S \subset X$ the pullback $\operatorname{im}(f) \cap G(S)$ factors through $fF(S[D])$ as a subobject of the image of f, then it is called *boundedly bicontrolled*. In this case we say that f has filtration degree D and write $\operatorname{fil}(f) \le D$.

Lemma 2.5. *Let $f_1\colon F \to G$, $f_2\colon G \to H$ be in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ and $f_3 = f_2 f_1$.*

(1) *If f_1, f_2 are boundedly bicontrolled morphisms and either $f_1\colon F(X) \to G(X)$ is an epi or $f_2\colon G(X) \to H(X)$ is a monic, then f_3 is also boundedly bicontrolled.*

(2) *If f_1, f_3 are boundedly bicontrolled and f_1 is epic then f_2 is also boundedly bicontrolled; if f_3 is only boundedly controlled then f_2 is also boundedly controlled.*

(3) *If f_2, f_3 are boundedly bicontrolled and f_2 is monic then f_1 is also boundedly bicontrolled; if f_3 is only boundedly controlled then f_1 is also boundedly controlled.*

Proof. Suppose $\mathrm{fil}(f_i) \leq D$ and $\mathrm{fil}(f_j) \leq D'$ for $\{i,j\} \subset \{1,2,3\}$, then in fact $\mathrm{fil}(f_{6-i-j}) \leq D+D'$ in each of the three cases. For example, there are factorizations

$$f_2 G(S) \subset f_2 f_1 F(S[D]) = f_3 F(S[D]) \subset H(S[D+D'])$$
$$f_2 G(X) \cap H(S) \subset f_3 F(S[D']) = f_2 f_1 F(S[D']) \subset f_2 G(S[D+D'])$$

which verify part 2 with $i = 1$, $j = 3$. □

Lemma 2.6. *In any additive category, given a morphism h,* $\ker(h) = 0$ *if and only if h is monic. Similarly,* $\mathrm{coker}(h) = 0$ *if and only if h is epic.*

Proof. Left to the reader. □

Proposition 2.7. $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ *is an additive category with kernels and cokernels.*

Proof. Additive properties are inherited from $\mathbf{A}$. In particular, the biproduct is given by the filtration-wise operation $(F \oplus G)(S) = F(S) \oplus G(S)$ in $\mathbf{A}$. For any boundedly controlled morphism $f\colon F \to G$, the kernel of f in $\mathbf{A}$ has the standard X-filtration K where $K(S) = \ker(f) \cap F(S)$ which gives the kernel of f in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$. The canonical monic $\kappa\colon K \to F$ has filtration degree 0. It follows from part 3 of Lemma 2.5 that K has the universal properties of the kernel in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$.

Similarly, let I be the standard X-filtration of the image of f in $\mathbf{A}$ by $I(S) = \mathrm{im}(f) \cap G(S)$. Then there is a presheaf C over X with $C(S) = G(S)/I(S)$ for $S \subset X$. Of course $C(X)$ is the cokernel of f in $\mathbf{A}$. Recall that there is an X-filtered object C_X associated to C given by $C_X(S) = \mathrm{im}\, C(S, X)$. The canonical morphism $\sigma\colon G(X) \to C(X)$ gives a filtration 0 morphism $\sigma\colon G \to C_X$ since

$$\mathrm{im}(\sigma G(S, X)) = \mathrm{im}\, C(S, X) = C_X(S).$$

This in conjunction with part 2 of Lemma 2.5 also verifies the universal cokernel properties of C_X and σ in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$. □

Remark 2.8. For an explicit description of a boundedly controlled morphism in $\mathbf{U}(\mathbb{Z}, \mathbf{Mod}(R))$ which is an isomorphism of left R-modules but whose inverse is not boundedly controlled, we refer to Example 1.5 of [16]. This indicates that in general $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ is not an abelian category and that under any embedding of such $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ in an abelian category the kernels and/or cokernels of some morphisms will be different from those in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$.

Remark 2.9. One consequence of Remark 2.8 is that $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ is not a balanced category. It is easy to see that a morphism in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ is balanced if and only if it is boundedly bicontrolled. So an isomorphism in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ is boundedly bicontrolled.

Definition 2.10. The *admissible monomorphisms* $\mathbf{mU}^{\mathrm{b}}(X,\mathbf{A})$ in $\mathbf{U}^{\mathrm{b}}(X,\mathbf{A})$ consist of boundedly bicontrolled morphisms $m\colon F_1 \to F_2$ such that $m\colon F_1(X) \to F_2(X)$ is a monic in $\mathbf{A}$. The *admissible epimorphisms* $\mathbf{eU}^{\mathrm{b}}(X,\mathbf{A})$ are the boundedly bicontrolled morphisms $e\colon F_1 \to F_2$ such that $e\colon F_1(X) \to F_2(X)$ is an epi in $\mathbf{A}$. The class $\mathcal{E}$ of exact sequences consists of the sequences

$$E^{\cdot}\colon \quad E' \xrightarrow{i} E \xrightarrow{j} E''$$

with $i \in \mathbf{mU}^{\mathrm{b}}(X,\mathbf{A})$ and $j \in \mathbf{eU}^{\mathrm{b}}(X,\mathbf{A})$ which are exact at E in the sense that $\mathrm{im}(i)$ and $\ker(j)$ represent the same subobject of E.

Theorem 2.11. $\mathbf{U}^{\mathrm{b}}(X,\mathbf{A})$ *is an exact category.*

Proof. (a) It follows from Lemma 2.5 that any short exact sequence $F^{\cdot}$ isomorphic to some $E^{\cdot} \in \mathcal{E}$ is also in $\mathcal{E}$, that

$$F' \xrightarrow{[\mathrm{id},0]} F' \oplus F'' \xrightarrow{[0,\mathrm{id}]^T} F''$$

is in $\mathcal{E}$, and that $i = \ker(j)$, $j = \mathrm{coker}(i)$ in any $E^{\cdot} \in \mathcal{E}$.

(b) The collections of morphisms $\mathbf{mU}^{\mathrm{b}}(X,\mathbf{A})$ and $\mathbf{eU}^{\mathrm{b}}(X,\mathbf{A})$ are closed under composition by part 1 of Lemma 2.5. Given $E^{\cdot} \in \mathcal{E}$ and any $f\colon A \to E'' \in \mathbf{U}^{\mathrm{b}}(X,\mathbf{A})$, there is a base change diagram

$$\begin{array}{ccccc} E' & \longrightarrow & E \times_f A & \xrightarrow{j'} & A \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ E' & \longrightarrow & E & \xrightarrow{j} & E'' \end{array}$$

where $m\colon E \times_f A \to E \oplus A$ is the kernel of the epi $j\,\mathrm{pr}_1 - f\,\mathrm{pr}_2\colon E \oplus A \to E''$ and $f' = \mathrm{pr}_1 m$, $j' = \mathrm{pr}_2 m$. The X-filtration is given by

$$(E \times_f A)\,(S) = E \times_f A \cap (E(S) \times A(S))\,,$$

so that j' is boundedly controlled and has the same kernel as j. In fact,

$$\mathrm{im}(j') \cap A(S) \subset (E \times_f A)\,(S[D(f) + D(j)])$$

since $fA(S) \subset E''(S[D(f)])$, so j' is boundedly bicontrolled of filtration degree $D(f) + D(j)$. Now given an admissible subobject $F \subset E \times A$, the restriction $j'|F$ is the pullback of the admissible epimorphism $f(F) \to fj'(F)$. This shows that the class of admissible epimorphisms is closed under base change by arbitrary morphisms in $\mathbf{U}^{\mathrm{b}}(X,\mathbf{A})$. Cobase changes by admissible monomorphisms are similar. $\square$

Definition 2.12. A full subcategory $\mathbf{H}$ of a small exact category $\mathbf{C}$ is said to be *closed under extensions* in $\mathbf{C}$ if $\mathbf{H}$ contains a zero object and for any exact sequence $C' \to C \to C''$ in $\mathbf{C}$, if C' and C'' are isomorphic to objects from $\mathbf{H}$ then so is C. A *Grothendieck subcategory* of an exact category is a subcategory which is closed under isomorphisms, exact extensions, admissible subobjects, and admissible quotients.

It is known [19] that a subcategory closed under extensions inherits an exact structure from $\mathbf{C}$.

Now let $\mathbf{E}$ be a Grothendieck subcategory of $\mathbf{A}$ and let F be an object of $\mathbf{U}^{\mathrm{b}}(X,\mathbf{A})$.

Definition 2.13. (1) F is $\mathbf{E}$-*local* if $F(V)$ is an object of $\mathbf{E}$ for every bounded subset $V \subset X$.

(2) F is *lean* or D-*lean* if there is a number $D \geq 0$ such that for every subset S of X

$$F(S) \subset \sum_{x \in S} F(B_D(x)),$$

where $B_D(x)$ is the metric ball of radius D centered at x.

(3) F is *insular* or d-*insular* if there is a number $d \geq 0$ such that

$$F(T) \cap F(U) \subset F(T[d] \cap U[d])$$

for every pair of subsets T, U of X.

Notice that a d-insular object has the property that for any subset $T \subset X$, $F(T) \cap F(U) = 0$ whenever $T \cap U[2d] = \emptyset$.

Remark 2.14. It is clear that properties (1), (2), and (3) are preserved under isomorphisms in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$.

Proposition 2.15. (1) *Lean objects are closed under exact extensions in* $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$, *that is, if*

$$E' \longrightarrow E \longrightarrow E''$$

is an exact sequence in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$, *and* E', E'' *are lean, then* E *is lean.*

(2) *Insular objects are closed under exact extensions in* $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$.

(3) *If in the exact sequence above the object* E *is lean and insular then*

(a) E' *is insular,*

(b) E'' *is lean,*

(c) E'' *is insular if and only if* E' *is lean.*

Proof. Let

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ and let $b \geq 0$ be a common filtration degree for both f and g as boundedly bicontrolled maps.

(1) Assume that both E' and E'' are D-lean as objects of $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$. Consider $E(S)$, then $gE(S) \subset E''(S[b])$ and so

$$gE(S) \subset \sum_{x \in S[b]} E''(B_D(x)).$$

For each $x \in S[b]$,

$$E''(B_D(x)) \subset gE(B_{D+b}(x)),$$

so

$$E(S) \subset \sum_{x \in S[b]} E(B_{D+2b}(x)) + \sum_{x \in S[b]} fE'(B_{D+2b}(x)).$$

Therefore

$$E(S) \subset \sum_{x \in S[b]} E(B_{D+3b}(x)) \subset \sum_{x \in S} E(B_{D+4b}(x)),$$

so E is $(D + 4b)$-lean.

(2) Assuming that both E' and E'' are d-insular, for any pair of subsets T and U of X,

$$\begin{aligned} &g(E(T) \cap E(U)) \\ &\subset E''(T[b]) \cap E''(U[b]) \\ &\subset E''(T[b+d] \cap U[b+d]). \end{aligned}$$

Now we have

$$\begin{aligned} &E(T) \cap E(U) \\ &\subset E(T[2b+d] \cap U[2b+d]) + fE' \cap E(T[2b+d]) \cap E(U[2b+d]) \\ &\subset E(T[2b+d] \cap U[2b+d]) + f(E'(T[3b+d]) \cap E'(U[3b+d]) \\ &\subset E(T[2b+d] \cap U[2b+d]) + fE'(T[3b+2d] \cap U[3b+2d]) \\ &\subset E(T[4b+2d] \cap U[4b+2d]). \end{aligned}$$

So E is $(4b+2d)$-insular.

(3a) Suppose E is d-insular. Given subsets T and U of X,

$$\begin{aligned} &f(E'(T) \cap E'(U)) \\ &\subset fE'(T) \cap fE'(U) \\ &\subset E(T[b]) \cap E(U[b]) \\ &\subset E(T[b+d]) \cap E(U[b+d]), \end{aligned}$$

so

$$E'(T) \cap E'(U) \subset E'(T[2b+d] \cap U[2b+d]).$$

Thus E' is $(2b+d)$-insular.

(3b) If E is D-lean then for any $S \subset X$, $E''(S) \subset gE(S[b])$. Since

$$E(S[b]) \subset \sum_{x \in S[b]} E(B_D(x)),$$

$$E''(S) \subset \sum_{x \in S[b]} E''(B_{D+b}(x)).$$

So

$$E''(S) \subset \sum_{x \in S} E''(B_{D+2b}(x)),$$

and E'' is $(D+2b)$-lean.

(3c) Suppose E' is D-lean. For any pair of subsets T, $U \subset X$,

$$E''(T) \cap E''(U) \subset gE(T[b]) \cap gE(U[b]).$$

Given $z \in E''(T) \cap E''(U)$, let $y' \in E(T[b])$ and $y'' \in E(U[b])$ so that $g(y') = g(y'') = z$. Now

$$k = y' - y'' \in \big(E(T[b]) + E(U[b])\big) \cap \ker(g),$$

so there is $\overline{k} \in E'(T[2b]) + E'(U[2b]) \subset E'(T[2b] \cup U[2b])$ with $f(\overline{k}) = k$. Since E' is D-lean,

$$\overline{k} \in \sum_{x \in T[2b] \cup U[2b]} E'(B_D(x)) = \sum_{x \in T[2b]} E'(B_D(x)) + \sum_{y \in U[2b]} E'(B_D(y)).$$

Hence,

$$\overline{k} \in E'(T[2b+D]) + E'(U[2b+D]).$$

This allows us to write $\overline{k} = \overline{k}_1 + \overline{k}_2$, where $\overline{k}_1 \in E'(T[2b+D])$ and $\overline{k}_2 \in E'(U[2b+D])$. Now $k = f\overline{k}_1 + f\overline{k}_2$. Notice that

$$y' = y'' + k = y'' + f\overline{k}_1 + f\overline{k}_2.$$

So

$$y = y' - f\overline{k}_1 = y'' + f\overline{k}_2$$

has the property

$$y \in E(T[3b + D]) \cap E(U[3b + D]) \subset E(T[3b + 2D] \cap U[3b + 2D]),$$

and $g(y) = z$. Hence

$$z \in E''(T[4b + 2D] \cap U[4b + 2D]).$$

We conclude that E'' is $(4b + 2D)$-insular. The converse is proved similarly; it is not used in this paper. □

Definition 2.16. An object F of $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ is called *strict* if there exists an order preserving function $\ell\colon \mathcal{P}(X) \to \mathbb{R}^{\geq 0}$ such that for every $S \subset X$ the subobject $F(S)$ is $\mathbf{E}$-local, ℓ_S-lean and ℓ_S-insular with respect to the strict filtration $F(S)(T) = F(S \cap T)$. In this case, each $F(S)$ is also strict.

Proposition 2.17. *If F is strict then $F(S)$ is strict with respect to the strict filtration, for all subsets $S \subset X$.*

Proof. Immediate from definitions. □

Unlike the subcategory of lean and insular objects, the subcategory of strict objects is not necessarily closed under isomorphisms.

Definition 2.18. The *boundedly controlled* category $\mathbf{B}(X, \mathbf{E})$ is the full subcategory of $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ on objects which are isomorphic to strict objects.

The terminology adopted here is convenient and should not suggest relations to boundedly controlled spaces and maps introduced earlier by Anderson and Munkholm [1].

Remark 2.19. The exact subcategory $\mathbf{E}$ is not assumed to be cocomplete. In fact, the construction is most interesting when it is not. Notice also that the notation $\mathbf{B}(X, \mathbf{E})$ does not suggest that the objects F have the terminal piece $F(X)$ in $\mathbf{E}$, unlike the notation for $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ where $F(X)$ are in $\mathbf{A}$. The object $F(X)$ is contained in the cocompletion of $\mathbf{E}$ in $\mathbf{A}$.

Theorem 2.20. *$\mathbf{B}(X, \mathbf{E})$ is closed under extensions in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$.*

Proof. Let

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

be an exact sequence in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{E})$ and let $b \geq 0$ be a common filtration degree for both f and g as boundedly bicontrolled maps. We will also assume, without loss of generality, that both E' and E'' are ℓ-strict for some function $\ell \geq 0$. We need to check that E is isomorphic to a strict object.

Since $\mathbf{E}$ is a Grothendieck subcategory of $\mathbf{A}$, for every bounded subset $V \subset X$ the restriction $g|E(V)\colon E(V) \to gE(V)$ is an admissible epimorphism onto an admissible subobject of $E''(V[D])$, which is in $\mathbf{E}$. The kernel of $g|E(V)$ is the admissible subobject $\ker(g) \cap E(V)$ of $E(V)$, which is also in $\mathbf{E}$. So $E(V)$ is in $\mathbf{E}$ by closure under extensions in $\mathbf{A}$.

E is lean and insular by parts (1) and (2) of Proposition 2.15. To see that E is isomorphic to a strict object, consider $S \subset X$ so that $E''(S[b])$ is $\ell_{S[b]}$-lean and $\ell_{S[b]}$-insular. The induced epic

$$g\colon E(S[2b]) \cap g^{-1}E''(S[b]) \longrightarrow E''(S[b])$$

extends to another epic

$$g'\colon fE'(S[3b]) + E(S[2b]) \cap g^{-1}E''(S[b]) \longrightarrow E''(S[b])$$

with $\ker(g') = E'(S[3b])$. Since $\ker(g')$ and $E''(S[b])$ are $\mathbf{E}$-local, lean, and insular, the filtration

$$\widehat{E}(S) = fE'(S[3b]) + E(S[2b]) \cap g^{-1}E''(S[b])$$

for each $S \subset X$ makes $\widehat{E}$ ϕ-strict for the function $\phi_S = \ell_{S[3b]}$. Clearly, the identity map $\mathrm{id}\colon E(X) \to E(X)$ gives an isomorphism $\mathrm{id}\colon \widehat{E} \to E$ with $\mathrm{fil}(\mathrm{id}) \leq 4b$. □

Corollary 2.21. $\mathbf{B}(X, \mathbf{E})$ *is an exact category in the sense of Quillen. The additive category* $\mathcal{B}(X, \mathbf{E})$ *of geometric objects with the split exact structure is an exact subcategory of* $\mathbf{B}(X, \mathbf{E})$.

Proof. The X-filtration of the geometric objects in $\mathcal{B}(X, \mathbf{E})$ is the obvious one with $F(S) = \bigoplus_{x \in S} F(x)$, and the structure maps are the boundedly controlled inclusions and projections onto direct summands. □

Recall that a morphism $e\colon F \to F$ is an idempotent if $e^2 = e$. Categories in which every idempotent is the projection onto a direct summand of F are called *idempotent complete*. Abelian categories are clearly idempotent complete. Thus $\mathbf{A}$ and its Grothendieck subcategories, which are abelian, are idempotent complete.

Corollary 2.22. *The subcategory* $\mathbf{B}(X, \mathbf{E})$ *is idempotent complete.*

Proof. Since the restriction of an idempotent e to the image of e is the identity, every idempotent is boundedly bicontrolled of filtration 0. It follows easily that the splitting of e in $\mathbf{A}$ is in fact a splitting in $\mathbf{B}(X, \mathbf{E})$. □

Proposition 2.23. *The subcategory* $\mathbf{B}(X, \mathbf{E})$ *is closed under admissible quotients of strict objects. Precisely, given a boundedly bicontrolled epic* $f\colon F \to G$ *in* $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ *where both* F *and the kernel* $k\colon K \to F$ *filtered by* $K(S) = K \cap F(S)$ *are strict, the cokernel* G *is isomorphic to a strict object.*

Proof. Suppose $\mathrm{fil}(f) \leq b$, then from the assumptions

$$K(S[b]) \longrightarrow F(S[b]) \longrightarrow fF(S[b])$$

is an exact sequence in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$ for any subset $S \subset X$. Since $F(S[b])$ is lean and insular and $K(S[b])$ is lean, the quotient $fF(S[b])$ is lean and insular by Proposition 2.15. It is clear that $fF(S[b])$ is also $\mathbf{E}$-local. Thus the object $\widehat{G}$ with filtration $\widehat{G}(S) = fF(S[b])$ is strict. There is an isomorphism $G \cong \widehat{G}$ induced by the identity map because for all $S \subset X$ we have $G(S) \subset \widehat{G}(S)$ and $\widehat{G}(S) \subset G(S[2b])$. □

For additional flexibility, one may want to impose weaker requirements on objects in $\mathbf{U}^{\mathrm{b}}(X, \mathbf{A})$. Restricting as in Definition 2.18 to objects F with a fixed locally finite covering $\mathcal{U} \subset \mathcal{P}(X)$ by bounded subsets $U \in \mathcal{B}(X)$ such that $F(X) = \sum_{U \in \mathcal{U}} F(U)$ gives another exact category. In this case, one may also relax the bounded control conditions on the maps to those of Lipschitz type. Similar modifications have become useful in recent work of Hambleton–Pedersen [10] and Pedersen–Weibel [18] in controlled K-theory.

3. Localization in controlled categories

Definition 3.1. Let F be an object of $\mathbf{B}(X, \mathbf{E})$ and Z be a subset of X. We say F is *supported near* Z if there is a number $d \geq 0$ such that $F(X) \subset F(Z[d])$.

Let $\mathbf{B}(X, \mathbf{E})_{<Z}$ be the full subcategory of $\mathbf{B}(X, \mathbf{E})$ on objects supported near Z. If $\mathbf{B}_d(X, \mathbf{E})_{<Z}$ denotes the full subcategory of $\mathbf{B}(X, \mathbf{E})$ with objects F as above then

$$\mathbf{B}(X, \mathbf{E})_{<Z} = \underset{d}{\underrightarrow{\operatorname{colim}}}\, \mathbf{B}_d(X, \mathbf{E})_{<Z}.$$

Proposition 3.2. $\mathbf{B}(X, \mathbf{E})_{<Z}$ *is a Grothendieck subcategory of* $\mathbf{B}(X, \mathbf{E})$.

Proof. First we show closure under exact extensions. Let

$$F' \xrightarrow{f} F \xrightarrow{g} F''$$

be an exact sequence in $\mathbf{B}(X, \mathbf{E})$. Let b be a common filtration degree of f and g and let d', $d'' \geq 0$ be numbers with $F' = F'(Z[d'])$ and $F'' = F''(Z[d''])$. Since $F = I + M$, where $I = \operatorname{im}(f)$ and M is any subobject $M \subset F$ with $g(M) = F''$, it suffices to show that for some $d \geq 0$

$$I(X) = I(Z[d]) \subset F(Z[d]),$$

and that M can be chosen to be a subobject of $F(Z[d])$. Indeed,

$$I(X) = fF'(X) = fF'(Z[d']) \subset F(Z[d' + b]),$$
$$F''(X) = gF(X) \cap F''(Z[d'']) \subset gF(Z[d'' + b]).$$

Let $M = F(Z[d'' + b])$. If we choose $d = \max\{d' + b, d'' + b\}$ then $F = F(Z[d])$ is in $\mathbf{B}(X, R)_{<Z}$.

Now suppose $f \colon F' \to F$ is an admissible subobject in $\mathbf{B}(X, \mathbf{E})$, which is a boundedly bicontrolled monic with $\operatorname{fil}(f) \leq b$, $F = F(Z[d])$, and F is c-insular. Also suppose F and F' are respectively D- and D'-lean, then notice that

$$fF'(B_{D'}(x)) \subset F(B_{D'+b}(x)),$$

while $F(X) \subset F(Z[d])$. Therefore,

$$F'(B_{D'}(x)) = 0$$

for

$$x \in X - Z[d + D + D' + b + 2c].$$

This means that

$$F' = F'(Z[d + D + 2D' + b + 2c]).$$

On the other hand, if $g \colon F \to F''$ is an admissible quotient of filtration b then it is easy to see that $F'' = F''(Z[d + D + b])$ is also in $\mathbf{B}(X, \mathbf{E})_{<Z}$. Since $\mathbf{B}(X, \mathbf{E})_{<Z}$ is clearly closed under isomorphisms, this proves the assertion. □

Given an object $F \in \mathbf{B}(X, \mathbf{E})$ and a subset $T \subset X$, we will need a construction of an admissible subobject $\widetilde{F}$ of F in $\mathbf{B}(X, \mathbf{E})$ such that $F(T) \subset \widetilde{F} \subset F(T[D])$ for some $D \geq 0$.

Choose a strict F' isomorphic to F in $\mathbf{B}(X, \mathbf{E})$ and assume the chosen isomorphism and its inverse are of filtration b.

Lemma 3.3. *The subobject $\widetilde{F} = F'(T[b])$ is an admissible subobject of F in $\mathbf{B}(X,\mathbf{E})$ and satisfies $F(T) \subset \widetilde{F} \subset F(T[2b])$.*

Proof. The cokernel G' of the inclusion $k \colon F'(T[b]) \to F$ is in $\mathbf{B}(X,\mathbf{E})$ by Proposition 2.23. We can view $F'(T[b])$ as an admissible subobject of F with the cokernel G isomorphic to G'. $\square$

Definition 3.4. A class of morphisms Σ in an additive category $\mathbf{C}$ *admits a calculus of right fractions* if

(1) the identity of each object is in Σ,
(2) Σ is closed under composition,
(3) each diagram $F \xrightarrow{f} G \xleftarrow{s} G'$ with $s \in \Sigma$ can be completed to a commutative square

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle s} \\ F & \xrightarrow{f} & G \end{array}$$

with $t \in \Sigma$, and
(4) if f is a morphism in $\mathbf{C}$ and $s \in \Sigma$ such that $sf = 0$ then there exists $t \in \Sigma$ such that $ft = 0$.

In this case there is a construction of the *localization* $\mathbf{C}[\Sigma^{-1}]$ which has the same objects as $\mathbf{C}$. The morphism sets $\mathrm{Hom}(F,G)$ in $\mathbf{C}[\Sigma^{-1}]$ consist of equivalence classes of diagrams

$$(s,f)\colon \quad F \xleftarrow{s} F' \xrightarrow{f} G$$

with the equivalence relation generated by $(s_1, f_1) \sim (s_2, f_2)$ if there is a map $h \colon F'_1 \to F'_2$ so that $f_1 = f_2 h$ and $s_1 = s_2 h$. Let $(s|f)$ denote the equivalence class of (s,f). The composition of morphisms in $\mathbf{C}[\Sigma^{-1}]$ is defined by

$$(s|f) \circ (t|g) = (st'|gf')$$

where g' and s' fit in the commutative square

$$\begin{array}{ccc} F'' & \xrightarrow{f'} & G' \\ \Big\downarrow{\scriptstyle t'} & & \Big\downarrow{\scriptstyle t} \\ F & \xrightarrow{f} & G \end{array}$$

from axiom 3.

Proposition 3.5. *The localization $\mathbf{C}[\Sigma^{-1}]$ is a category. The morphisms of the form $(\mathrm{id}\,|s)$ where $s \in \Sigma$ are isomorphisms in $\mathbf{C}[\Sigma^{-1}]$. The rule $P_\Sigma(f) = (\mathrm{id}\,|f)$ gives a functor $P_\Sigma \colon \mathbf{C} \to \mathbf{C}[\Sigma^{-1}]$ which is universal among the functors making the morphisms Σ invertible.*

Proof. The proofs of these facts can be found in Chapter I of [9]. The inverse of $(\mathrm{id}\,|s)$ is $(s|\,\mathrm{id})$. $\square$

Suppose $\mathbf{E}$ is a Grothendieck subcategory of a cocomplete abelian category $\mathbf{A}$, and let $\mathbf{Z}$ be the subcategory $\mathbf{B}(X,\mathbf{E})_{<Z}$ of $\mathbf{B} = \mathbf{B}(X,\mathbf{E})$ for a fixed choice of $Z \subset X$. Let the class of *weak equivalences* Σ consist of all finite compositions of admissible monomorphisms with cokernels in $\mathbf{Z}$ and admissible epimorphisms with kernels in $\mathbf{Z}$. We will show that the class Σ admits a calculus of right fractions.

Definition 3.6. A Grothendieck subcategory $\mathbf{Z} \subset \mathbf{B}$ is *right filtering* if each morphism $f \colon F \to G$ in $\mathbf{B}$, where G is an object of $\mathbf{Z}$, factors through an admissible epimorphism $e \colon F \to \overline{G}$, where $\overline{G}$ is in $\mathbf{Z}$.

Lemma 3.7. *The subcategory $\mathbf{Z} = \mathbf{B}(X, \mathbf{E})_{<Z}$ of $\mathbf{B} = \mathbf{B}(X, \mathbf{E})$ is right filtering.*

Proof. Suppose first that F and G are strict with the characteristic functions ℓ_F and ℓ_G respectively. Let $L_F = \ell_F(X)$ and $L_G = \ell_G(X)$. If $G = G(Z[d_G])$ and the given morphism $f \colon F \to G$ is bounded by d then we have

$$fF(B_{L_G}(x)) \subset G(B_{L_G+b}(x)) = 0$$

for all

$$x \in X - Z[d_G + L_G + 2L_G + d + L_F].$$

Let

$$E = F(X - Z[d_G + L_G + 2L_G + d + L_F]),$$

then $fE = 0$. Now E is an admissible subobject of F by Lemma 3.3; let $\overline{G}$ be the cokernel of the inclusion. Since

$$\overline{G}(B_{L_F}(x)) = 0$$

for all

$$x \in X - Z[d_G + L_G + 2L_G + d + L_F],$$

we have

$$\overline{G} = \overline{G}(Z[d_G + 2L_G + 2L_G + d + L_F])$$

as an object of $\mathbf{Z}$. The required factorization is the right square in the map between the two exact sequences

$$\begin{tikzcd} E \arrow[r] \arrow[d, "i"'] & F \arrow[r, "j'"] \arrow[d, "="] & \overline{G} \arrow[d] \\ K \arrow[r, "k"] & F \arrow[r, "f"] & G \end{tikzcd}$$

If F and G are not strict, one considers a map $f' \colon F' \to G'$ between strict objects isomorphic to F and G and chooses the subobject

$$E = F'(X - Z[d_G + L_G + 2L_G + d + L_F + 4b])$$

of F' for an appropriate value of b, as in the proof of Lemma 3.3. $\square$

Corollary 3.8. *The class Σ admits a calculus of right fractions.*

Proof. This follows from Lemma 3.7, see Lemma 1.13 of [21]. $\square$

Definition 3.9. The *quotient category* $\mathbf{B}/\mathbf{Z}$ is the localization $\mathbf{B}[\Sigma^{-1}]$.

It is clear that the quotient $\mathbf{B}/\mathbf{Z}$ is an additive category, and P_Σ is an additive functor. In fact, we have the following.

Theorem 3.10. *The short sequences in $\mathbf{B}/\mathbf{Z}$ which are isomorphic to images of exact sequences from $\mathbf{B}$ form a Quillen exact structure.*

Proof. This follows from Proposition 1.16 of Schlichting [21]. Since $\mathbf{Z}$ is right filtering by Lemma 3.7, it remains to check that the subcategory $\mathbf{Z}$ is *right s-filtering* in $\mathbf{B}$, that is to show that if $f\colon F \to G$ is an admissible monomorphism with F in $\mathbf{Z}$ then there exist E in $\mathbf{Z}$ and an admissible epimorphism $e\colon G \to E$ such that the composition ef is an admissible monomorphism.

Again, suppose that F and G are strict with the characteristic functions ℓ_F and ℓ_G and let $L_F = \ell_F(X)$ and $L_G = \ell_G(X)$. Assume that $F = F(Z[d_F])$, $\mathrm{fil}(f) \leq d$, and let $G' = G(X - Z[d_F + 2L_F + 2L_G + d + L_G])$. Let $e\colon G \to E$ be the cokernel of the inclusion, then $f(F) \cap G' = 0$, so that ef is an admissible monomorphism with $\mathrm{fil}(ef) = \mathrm{fil}(f) \leq d$. If F and G are not strict, one makes the obvious adjustments. □

The main tool in proving controlled excision in the boundedly controlled K-theory will be the following localization theorem.

Theorem 3.11 (Theorem 2.1 of Schlichting [21])**.** *Let* $\mathbf{Z}$ *be an idempotent complete right s-filtering subcategory of an exact category* $\mathbf{B}$*. Then the sequence of exact categories* $\mathbf{Z} \to \mathbf{B} \to \mathbf{B}/\mathbf{Z}$ *induces a homotopy fibration of K-theory spectra*

$$K(\mathbf{Z}) \to K(\mathbf{B}) \to K(\mathbf{B}/\mathbf{Z}).$$

4. Bounded excision theorem

The proof of controlled excision in the boundedly controlled G-theory requires the context of Waldhausen K-theory of derived categories.

Definition 4.1 (Waldhausen categories)**.** A *Waldhausen category* is a category $\mathbf{D}$ with a zero object 0 together with two chosen subcategories of *cofibrations* $\mathrm{co}(\mathbf{D})$ and *weak equivalences* $\boldsymbol{w}(\mathbf{D})$ satisfying the four axioms:

(1) every isomorphism in $\mathbf{D}$ is in both $\mathrm{co}(\mathbf{D})$ and $\boldsymbol{w}(\mathbf{D})$,
(2) every map $0 \to D$ in $\mathbf{D}$ is in $\mathrm{co}(\mathbf{D})$,
(3) if $A \to B \in \mathrm{co}(\mathbf{D})$ and $A \to C \in \mathbf{D}$ then the pushout $B \cup_A C$ exists in $\mathbf{D}$, and the canonical map $C \to B \cup_A C$ is in $\mathrm{co}(\mathbf{D})$,
(4) ("gluing lemma") given a commutative diagram

$$\begin{tikzcd} B \arrow[d] & A \arrow[l, "a"'] \arrow[r] \arrow[d] & C \arrow[d] \\ B' & A' \arrow[l, "a'"'] \arrow[r] & C' \end{tikzcd}$$

in $\mathbf{D}$, where the morphisms a and a' are in $\mathrm{co}(\mathbf{D})$ and the vertical maps are in $\boldsymbol{w}(\mathbf{D})$, the induced map $B \cup_A C \to B' \cup_{A'} C'$ is also in $\boldsymbol{w}(\mathbf{D})$.

A Waldhausen category $\mathbf{D}$ with weak equivalences $\boldsymbol{w}(\mathbf{D})$ is often denoted by $\boldsymbol{w}\mathbf{D}$ as a reminder of the choice. A functor between Waldhausen categories is exact if it preserves cofibrations and weak equivalences.

A Waldhausen category may or may not satisfy the following additional axioms.

Saturation axiom 4.2. Given two morphisms $\phi\colon F \to G$ and $\psi\colon G \to H$ in $\mathbf{D}$, if any two of ϕ, ψ, or $\psi\phi$, are in $\boldsymbol{w}(\mathbf{D})$ then so is the third.

Extension axiom 4.3. Given a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & G & \longrightarrow & H \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\mu} \\ F' & \longrightarrow & G' & \longrightarrow & H' \end{array}$$

with exact rows, if both ϕ and μ are in $\boldsymbol{w}(\mathbf{D})$ then so is ψ.

A *cylinder functor* on $\mathbf{D}$ is a functor C from the category of morphisms $f\colon F \to G$ in $\mathbf{D}$ to $\mathbf{D}$ together with three natural transformations $j_1\colon F \to C(f)$, $j_2\colon G \to C(f)$, and $p\colon C(f) \to G$ such that $pj_2 = \mathrm{id}_G$ and $pj_1 = f$ for all f, and which has a number of properties listed in point 1.3.1 of [24] which will be rather automatic for the functors we construct later.

Cylinder axiom 4.4. A cylinder functor C satisfies this axiom if for all morphisms $f\colon F \to G$ the required map p is in $\boldsymbol{w}(\mathbf{D})$.

Let $\mathbf{D}$ be a small Waldhausen category with respect to two categories of weak equivalences $\boldsymbol{v}(\mathbf{D}) \subset \boldsymbol{w}(\mathbf{D})$ with a cylinder functor T both for $\boldsymbol{v}\mathbf{D}$ and for $\boldsymbol{w}\mathbf{D}$ satisfying the cylinder axiom for $\boldsymbol{w}\mathbf{D}$. Suppose also that $\boldsymbol{w}(\mathbf{D})$ satisfies the extension and saturation axioms. Define $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$ to be the full subcategory of $\boldsymbol{v}\mathbf{D}$ whose objects are F such that $0 \to F \in \boldsymbol{w}(\mathbf{D})$. Then $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$ is a small Waldhausen category with cofibrations $\mathrm{co}(\mathbf{D}^{\boldsymbol{w}}) = \mathrm{co}(\mathbf{D}) \cap \mathbf{D}^{\boldsymbol{w}}$ and weak equivalences $\boldsymbol{v}(\mathbf{D}^{\boldsymbol{w}}) = \boldsymbol{v}(\mathbf{D}) \cap \mathbf{D}^{\boldsymbol{w}}$. The cylinder functor T for $\boldsymbol{v}\mathbf{D}$ induces a cylinder functor for $\boldsymbol{v}\mathbf{D}^{\boldsymbol{w}}$. If T satisfies the cylinder axiom then the induced functor does so too.

Theorem 4.5 (Approximation theorem)**.** *Let $E\colon \mathbf{D}_1 \to \mathbf{D}_2$ be an exact functor between two small saturated Waldhausen categories. It induces a map of K-theory spectra*

$$K(E)\colon K(\mathbf{D}_1) \longrightarrow K(\mathbf{D}_2).$$

Assume that $\mathbf{D}_1$ has a cylinder functor satisfying the cylinder axiom. If E satisfies two conditions:

1. *a morphism $f \in \mathbf{D}_1$ is in $\boldsymbol{w}(\mathbf{D}_1)$ if and only if $E(f) \in \mathbf{D}_2$ is in $\boldsymbol{w}(\mathbf{D}_2)$,*
2. *for any object $D_1 \in \mathbf{D}_1$ and any morphism $g\colon E(D_1) \to D_2$ in $\mathbf{D}_2$, there is an object $D_1' \in \mathbf{D}_1$, a morphism $f\colon D_1 \to D_1'$ in $\mathbf{D}_1$, and a weak equivalence $g'\colon E(D_1') \to D_2 \in \boldsymbol{w}(\mathbf{D}_2)$ such that $g = g'E(f)$,*

then $K(E)$ is a homotopy equivalence.

Proof. This is Theorem 1.6.7 of [25]. The presence of the cylinder functor with the cylinder axiom allows to make condition 2 weaker than that of Waldhausen, see point 1.9.1 in [24]. □

Definition 4.6. In any additive category, a sequence of morphisms

$$E^{\cdot}\colon \quad 0 \longrightarrow E^1 \xrightarrow{d_1} E^2 \xrightarrow{d_2} \ \ldots \ \xrightarrow{d_{n-1}} E^n \longrightarrow 0$$

is called a *(bounded) chain complex* if the compositions $d_{i+1}d_i$ are the zero maps for all $i = 1, \ldots, n-1$. A *chain map* $f\colon F^{\cdot} \to E^{\cdot}$ is a collection of morphisms $f^i\colon F^i \to E^i$ such that $f^i d_i = d_i f^i$. A chain map f is *null-homotopic* if there are morphisms $s_i\colon F^{i+1} \to E^i$ such that $f = ds+sd$. Two chain maps $f, g\colon F^{\cdot} \to E^{\cdot}$ are *chain homotopic* if $f-g$ is null-homotopic. Now f is a *chain homotopy equivalence* if there is a chain map $h\colon E^i \to F^i$ such that the compositions fh and hf are chain homotopic to the respective identity maps.

The Waldhausen structures on categories of bounded chain complexes are based on homotopy equivalence as a weakening of the notion of isomorphism of chain complexes.

Definition 4.7. A sequence of maps in an exact category is called *acyclic* if it is assembled out of short exact sequences in the sense that each map factors as the composition of the cokernel of the preceding map and the kernel of the succeeding map.

It is known that the class of acyclic complexes in an exact category is closed under isomorphisms in the homotopy category if and only if the category is idempotent complete, which is also equivalent to the property that each contractible chain complex is acyclic, cf. [13, sec. 11].

Definition 4.8. Given an exact category $\mathbf{E}$, there is a standard choice for the Waldhausen structure on the derived category $\mathbf{E}'$ of bounded chain complexes in $\mathbf{E}$ where the degree-wise admissible monomorphisms are the cofibrations and the chain maps whose mapping cones are homotopy equivalent to acyclic complexes are the weak equivalences $\boldsymbol{v}(\mathbf{E}')$.

Proposition 4.9. *The category $\boldsymbol{v}\mathbf{E}'$ is a Waldhausen category satisfying the extension and saturation axioms and has cylinder functor satisfying the cylinder axiom.*

Proof. The pushouts along cofibrations in $\mathbf{E}'$ are the complexes of pushouts in each degree. All standard Waldhausen axioms including the gluing lemma are clearly satisfied. The saturation and the extension axioms are also clear. The cylinder functor C for $\boldsymbol{v}\mathbf{E}'$ is defined using the canonical homotopy pushout as in point 1.1.2 in Thomason–Trobaugh [24]. Given a chain map $f\colon F \to G$, $C(f)$ is the canonical homotopy pushout of f and the identity $\mathrm{id}\colon F \to F$. With this construction, the map $p\colon C(f) \to G$ is a chain homotopy equivalence, so the cylinder axiom is also satisfied. □

Definition 4.10. There are two choices for the Waldhausen structure on the bounded derived category $\mathbf{B}' = \mathbf{B}'(X, \mathbf{E})$. One is $\boldsymbol{v}\mathbf{B}'$ as in Definition 4.8. Given a metric subspace Z in X, the other choice for the weak equivalences $\boldsymbol{w}(\mathbf{B}')$ is the chain maps whose mapping cones are homotopy equivalent to acyclic complexes in the quotient $\mathbf{B}/\mathbf{Z}$.

Corollary 4.11. *The categories $\boldsymbol{v}\mathbf{B}'$ and $\boldsymbol{w}\mathbf{B}'$ are Waldhausen categories satisfying the extension and saturation axioms and have cylinder functors satisfying the cylinder axiom.*

Proof. All axioms and constructions, including the cylinder functor, for $\boldsymbol{w}\mathbf{B}'$ are inherited from $\boldsymbol{v}\mathbf{B}'$. □

The K-theory functor from the category of small Waldhausen categories $\mathbf{D}$ and exact functors to connective spectra is defined in terms of $S.$-construction as in Waldhausen [25]. It extends to simplicial categories $\mathbf{D}$ with cofibrations and weak equivalences and inductively delivers the connective spectrum $n \mapsto |\boldsymbol{w}S.^{(n)}\,\mathbf{D}\,|$. We obtain the functor assigning to $\mathbf{D}$ the connective Ω-spectrum

$$K(\mathbf{D}) = \Omega^\infty |\boldsymbol{w}S.^{(\infty)}\,\mathbf{D}\,| = \underset{\overrightarrow{n \geq 1}}{\operatorname{colim}}\; \Omega^n |\boldsymbol{w}S.^{(n)}\,\mathbf{D}\,|$$

representing the Waldhausen algebraic K-theory of $\mathbf{D}$. For example, if $\mathbf{D}$ is the additive category of free finitely generated R-modules with the canonical Waldhausen structure, then the stable homotopy groups of $K(\mathbf{D})$ are the usual K-groups of the ring R. In fact, there is a general identification of the two theories. Recall that for any exact category $\mathbf{E}$, the derived category $\mathbf{E}'$ has the Waldhausen structure $\boldsymbol{v}\mathbf{E}'$ as in Definition 4.8.

Theorem 4.12. *The Quillen K-theory of an exact category $\mathbf{E}$ is equivalent to the Waldhausen K-theory of $\boldsymbol{v}\mathbf{E}'$.*

Proof. The proof is based on repeated applications of the additivity theorem, cf. Thomason's Theorem 1.11.7 [24]. Thomason's proof of his Theorem 1.11.7 can be repeated verbatim here. It is in fact simpler in this case since condition 1.11.3.1 is not required. □

Let $\mathbf{E}$ be a Grothendieck subcategory of a cocomplete abelian category $\mathbf{A}$ and let X be a proper metric space with subspace Z. We will use the notation $\mathbf{B} = \mathbf{B}(X, \mathbf{E})$ and $\mathbf{Z} = \mathbf{B}(X, \mathbf{E})_{<Z}$.

Theorem 4.13 (Localization)**.** *If $\mathbf{E}$ is idempotent complete, there is a homotopy fibration*

$$K(Z, \mathbf{E}) \longrightarrow K(X, \mathbf{E}) \longrightarrow K(\mathbf{B}/\mathbf{Z}).$$

This is a direct consequence of Theorem 3.11 as soon as we identify $K(Z, \mathbf{E})$ with $K(\mathbf{Z}) = K(X, \mathbf{E})_{<Z}$.

Recall that the *essential full image* of a functor $F\colon \mathbf{C} \to \mathbf{D}$ is the full subcategory of $\mathbf{D}$ whose objects are those D such that $D \cong F(C)$ for some C from $\mathbf{C}$.

There is a fully faithful embedding $\epsilon\colon \mathbf{B}(Z, \mathbf{E}) \to \mathbf{B}(X, \mathbf{E})$ defined by associating to each filtered object $F \in \mathbf{B}(Z, \mathbf{E})$ the extension $\epsilon(F) \in \mathbf{B}(X, \mathbf{E})$ given by $\epsilon(F)(S) = F(S \cap Z)$. It is clear that $\epsilon(F)$ is strict for strict F.

Lemma 4.14. *The essential full image of $\mathbf{B}(Z, \mathbf{E})$ in $\mathbf{B}(X, \mathbf{E})$ is the Grothendieck subcategory $\mathbf{B}(X, \mathbf{E})_{<Z}$.*

Proof. Of course for each F in $\mathbf{B}(Z, \mathbf{E})$, the image $\epsilon(F)$ is in $\mathbf{B}(X, \mathbf{E})_{<Z}$. Now if $G(X) = G(Z[d])$ is an object of $\mathbf{B}(X, \mathbf{E})_{<Z}$ then there is a bounded function $r\colon Z[d] \to Z$, bounded by d, which gives an object $R = R(G)$ of $\mathbf{B}(Z, \mathbf{E})$ by the assignment $R(S) = G(r^{-1}(S))$. If G is strict then the new object R is $\mathbf{E}$-local and strict with $\ell_R = \ell_G + d$. Since the identity map $\mathrm{id}\colon R \to G$ is boundedly bicontrolled with $\mathrm{fil}(\mathrm{id}) \leq 2d$, it is an isomorphism in $\mathbf{B}(X, \mathbf{E})$. □

Corollary 4.15. *For any pair of proper metric spaces $Z \subset X$, there is a weak equivalence $K(Z, \mathbf{E}) \simeq K(X, \mathbf{E})_{<Z}$.*

Now Theorem 4.13 follows from the localization fibration in Theorem 3.11.

The computational tools from nonconnective bounded K-theory, the controlled excision theorems [3, 16, 17], can now be adapted to $\mathbf{B}(X, \mathbf{E})$. We will obtain a direct analogue, which is one of the main results of this paper.

Let $\mathbf{E}$ be a Grothendieck subcategory of a cocomplete abelian category $\mathbf{A}$ and let X be a proper metric space. Suppose X_1 and X_2 are subspaces in a proper metric space X, and $X = X_1 \cup X_2$. We use the notation $\mathbf{B} = \mathbf{B}(X, \mathbf{E})$, $\mathbf{B}_i = \mathbf{B}(X, \mathbf{E})_{<X_i}$ for $i = 1$ or 2, and $\mathbf{B}_{12}$ for the intersection $\mathbf{B}_1 \cap \mathbf{B}_2$.

Now there is a commutative diagram

$$(\dagger)\qquad \begin{array}{ccccc} K(\mathbf{B}_{12}) & \longrightarrow & K(\mathbf{B}_1) & \longrightarrow & K(\mathbf{B}_1/\mathbf{B}_{12}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle K(I)} \\ K(\mathbf{B}_2) & \longrightarrow & K(\mathbf{B}) & \longrightarrow & K(\mathbf{B}/\mathbf{B}_2) \end{array}$$

where the rows are homotopy fibrations. We should not expect the map induced by the rightmost exact inclusion $I\colon \boldsymbol{w}\mathbf{B}'_1 \to \boldsymbol{w}\mathbf{B}'$ to be an equivalence of categories as in similar applications in [3] and [22], but we claim that $K(I)$ is almost a weak equivalence.

Let Z be a subset of X, so $\mathbf{Z} = \mathbf{B}(X,\mathbf{E})_{<Z}$ is a Grothendieck subcategory of $\mathbf{B}$, and recall that $\mathbf{C}^\frown$ is the idempotent completion of an exact category $\mathbf{C}$.

Lemma 4.16. *If $f^\cdot \colon F^\cdot \to G^\cdot$ is either a degreewise admissible monomorphism with cokernels in $\mathbf{Z}$ or a degreewise admissible epimorphism with kernels in $\mathbf{Z}$ then $f^\cdot$ is a weak equivalence in $\boldsymbol{v}((\mathbf{B}/\mathbf{Z})^\frown)'$.*

Proof. We need to see that the mapping cone $Cf^\cdot$ is the zero complex in the bounded homotopy category of $\mathbf{B}/\mathbf{Z}$. In the first case, $Cf^\cdot$ is weakly equivalent to the cokernel of $f^\cdot$, by Lemma 11.6 of [13], which is zero in $\mathbf{B}/\mathbf{Z}$. In the second case, $F^\cdot$ is weakly equivalent to the mapping cone of $\ker(f^\cdot)$ in $\mathbf{B}/\mathbf{Z}$, which is again weakly equivalent to $G^\cdot$. □

An exact subcategory $\mathbf{C}$ of an exact category $\mathbf{E}$ is *cofinal* if it is closed under extensions and for every E in $\mathbf{E}$ there is E' so that $E \oplus E'$ is isomorphic to an object from the subcategory $\mathbf{C}$.

Theorem 4.17 (Cofinality theorem, Staffeldt [22])**.** *If $\mathbf{C}$ is cofinal in $\mathbf{E}$ then the Waldhausen K-theory sequence $K(\boldsymbol{v}\mathbf{C}') \to K(\boldsymbol{v}\mathbf{E}') \to BG$, where $G = K_0(\mathbf{E})/K_0(\mathbf{C})$, is a fibration.*

Lemma 4.18. *$K(I)\colon K(\boldsymbol{w}\mathbf{B}'_1) \to K(\boldsymbol{w}\mathbf{B}')$ is a weak equivalence of spectra in positive dimensions.*

Proof. Applying the Cofinality theorem to the inclusion $I\colon \mathbf{B}_1/\mathbf{B}_{12} \to (\mathbf{B}/\mathbf{B}_2)^\frown$, for any E in $(\mathbf{B}/\mathbf{B}_2)^\frown$ choose E' so that $E \oplus E'$ is isomorphic to an object

$$F^\cdot\colon \quad 0 \longrightarrow F^1 \xrightarrow{\phi_1} F^2 \xrightarrow{\phi_2} \ \ldots \ \xrightarrow{\phi_{n-1}} F^n \longrightarrow 0$$

in $(\mathbf{B}/\mathbf{B}_2)'$. Fixing a nonnegative number b that serves as a control bound for all ϕ_i, we define $E'^i = F^i(X_1[ib])$ and define ϕ'_i to be the restrictions of ϕ_i to E'^i. This gives a chain subcomplex $E'^\cdot$ of $F^\cdot$ in $\mathbf{B}'_1$ so that the mapping cone of the inclusion is homotopy equivalent to the cokernel and so is contractible. Therefore, it is acyclic in the idempotent complete category $(\mathbf{B}/\mathbf{B}_2)^\frown$. □

Let $\mathbb{Z}$, $\mathbb{Z}^{\geq 0}$, and $\mathbb{Z}^{\leq 0}$ denote the metric spaces of integers, nonnegative integers, and nonpositive integers with the restriction of the usual metric on the real line $\mathbb{R}$. Let $\mathbf{E}$ be an idempotent complete Grothendieck category of an abelian category $\mathbf{A}$. Then for any proper metric space X, we have the following instance of commutative diagram $(\dagger)$

$$\begin{array}{ccccc} K(X,\mathbf{E}) & \longrightarrow & K(X\times\mathbb{Z}^{\geq 0},\mathbf{E}) & \longrightarrow & K(\mathbf{B}_1/\mathbf{B}_{12}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle K(I)} \\ K(X\times\mathbb{Z}^{\leq 0},\mathbf{E}) & \longrightarrow & K(X\times\mathbb{Z},\mathbf{E}) & \longrightarrow & K(\mathbf{B}/\mathbf{B}_2) \end{array}$$

Lemma 4.19. *The spectra $K(X\times\mathbb{Z}^{\geq 0},\mathbf{E})$ and $K(X\times\mathbb{Z}^{\leq 0},\mathbf{E})$ are contractible.*

Proof. This follows from the fact that these controlled categories are flasque, that is, the usual shift functor T in the positive (respectively negative) direction along $\mathbb{Z}^{\geq 0}$ (respectively $\mathbb{Z}^{\leq 0}$) interpreted in the obvious way is an exact endofunctor, and there is a natural equivalence $1\oplus\pm T\cong\pm T$. Contractibility follows from the additivity theorem, cf. Pedersen–Weibel [16]. □

In view of Lemma 4.18, we obtain a map $K(X,\mathbf{E})\to\Omega K(X\times\mathbb{Z},\mathbf{E})$ which induces isomorphisms of K-groups in positive dimensions. Iterations of this construction give weak equivalences

$$\Omega^k K(X\times\mathbb{Z}^k,\mathbf{E})\longrightarrow\Omega^{k+1}K(X\times\mathbb{Z}^{k+1},\mathbf{E})$$

for $k\geq 2$.

Definition 4.20. The *nonconnective controlled K-theory* of $\mathbf{E}$, relative to the embedding $\epsilon\colon\ \mathbf{E}\to\mathbf{A}$, over a proper metric space X is the spectrum

$$K^{-\infty}_{\epsilon}(X,\mathbf{E})\overset{\text{def}}{=}\underset{\overrightarrow{k}}{\operatorname{hocolim}}\ \Omega^k K(X\times\mathbb{Z}^k,\mathbf{E}).$$

Since $\mathbf{B}(X,\mathbf{E})$ can be identified with $\mathbf{E}$ for a bounded metric space X, this definition gives the *nonconnective K-theory* of $\mathbf{E}$

$$K^{-\infty}_{\epsilon}(\mathbf{E})\overset{\text{def}}{=}\underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k K(\mathbb{Z}^k,\mathbf{E}).$$

As $K^{-\infty}_{\epsilon}(\mathbf{E})$ is an Ω-spectrum in positive dimensions, the positive homotopy groups of $K^{-\infty}(\mathbf{E})$ coincide with those of $K(\mathbf{E})$, as desired. The class group $K_{\epsilon,0}(\mathbf{E})$ is the class group of the idempotent completion $K_0(\mathbf{E}^{\frown})$.

The first known delooping of the K-theory of a general exact category with these properties is due to M. Schlichting [20], however the construction here is different and is required in the excision theorem ahead.

Example 4.21. If $\mathbf{E}$ is an arbitrary small exact category, there is the full Gabriel–Quillen embedding of $\mathbf{E}$ in the cocomplete abelian category $\mathbf{A}$ of left exact functors $\mathbf{E}^{\text{op}}\to\mathbf{Mod}(\mathbb{Z})$ with the standard exact structure. The embedding is always closed under extensions in $\mathbf{A}$. It is not necessarily a Grothendieck subcategory, as when $\mathbf{E}$ is not balanced. But if $\mathbf{E}$ is abelian, for example, this gives a canonical delooping of $K(\mathbf{E})$.

Example 4.22. One may start with the cocomplete abelian category $\mathbf{Mod}(R)$ of modules over a ring R with the standard abelian exact structure where the admissible monomorphisms and epimorphisms are respectively all monics and epis. If R is a noetherian ring, the subcategory $\mathbf{E}$ may be taken to be the noncocomplete abelian category of finitely generated R-modules $\mathbf{Modf}(R)$. Now $K^{-\infty}(\mathbf{E})$ gives the algebraic G-theory of R which we denote by $G^{-\infty}(R)$. We also use notation $\mathbf{B}(X,R)$ for $\mathbf{B}(X,\mathbf{E})$. Now the *nonconnective controlled G-theory of X-filtered modules over R* is $G^{-\infty}(X,R)=K^{-\infty}\,\mathbf{B}(X,\mathbf{E})$.

Example 4.23. The negative K-theory of a regular ring R is trivial in the sense that $K_i(\mathbf{Modf}(R)) = 0$ for all $i < 0$. This is well-known in Bass' theory [2]. A proof that the negative K-theory is trivial for general abelian categories can be given using the same strategy as in chapter 9 of [20].

Of course, when the exact category $\mathbf{E}$ is itself cocomplete, its K-theory is contractible because of the Eilenberg swindle type argument.

We finally prove the main result of this section.

Theorem 4.24 (Nonconnective excision)**.** *There is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc} K^{-\infty}(\mathbf{B}_{12}) & \longrightarrow & K^{-\infty}(\mathbf{B}_1) \\ \big\downarrow & & \big\downarrow \\ K^{-\infty}(\mathbf{B}_2) & \longrightarrow & K^{-\infty}(\mathbf{B}) \end{array}$$

where the maps of spectra are induced from the exact inclusions.

Proof. Let us write $S^k\,\mathbf{B}$ for $\mathbf{B}(X \times \mathbb{Z}^k, \mathbf{E})$ whenever $\mathbf{B}$ is the boundedly controlled category for a general metric space X. If $\mathbf{Z}$ is a subset of X, consider the fibration

$$K(Z, \mathbf{E}) \longrightarrow K(X, \mathbf{E}) \longrightarrow K(\mathbf{B}/\mathbf{Z})$$

from Theorem 4.13. Notice that there is a map

$$K(\mathbf{B}/\mathbf{Z}) \to \Omega K(S\,\mathbf{B}/S\,\mathbf{Z})$$

which is a weak equivalence in positive dimensions by the Five Lemma. If one defines

$$K^{-\infty}(\mathbf{B}/\mathbf{Z}) = \underset{\overrightarrow{k}}{\operatorname{hocolim}}\ \Omega^k K(S^k\,\mathbf{B}/S^k\,\mathbf{Z}),$$

there is an induced fibration

$$K^{-\infty}(Z, \mathbf{E}) \longrightarrow K^{-\infty}(X, \mathbf{E}) \longrightarrow K^{-\infty}(\mathbf{B}/\mathbf{Z})$$

The theorem follows from the commutative diagram

$$\begin{array}{ccccc} K^{-\infty}(\mathbf{B}_{12}) & \longrightarrow & K^{-\infty}(\mathbf{B}_1) & \longrightarrow & K^{-\infty}(\mathbf{B}_1/\mathbf{B}_{12}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle K^{-\infty}(I)} \\ K^{-\infty}(\mathbf{B}_2) & \longrightarrow & K^{-\infty}(\mathbf{B}) & \longrightarrow & K^{-\infty}(\mathbf{B}/\mathbf{B}_2) \end{array}$$

and the fact that now $K^{-\infty}(I)\colon\ K^{-\infty}(\mathbf{B}_1/\mathbf{B}_{12}) \to K^{-\infty}(\mathbf{B}/\mathbf{B}_2)$ is a weak equivalence. □

Remark 4.25. As in other versions of controlled K-theory, there is no excision theorem similar to Theorem 4.24 which employs the connective K-theory. This time the reason is that the map $K(I)$ is not necessarily a weak equivalence. The difference is detected at the level of K_0 which makes the use of cofinality theorem essential to the proof. To give an idea why condition 2 in the approximation theorem fails when applied to the inclusion of Waldhausen categories $I\colon\ \boldsymbol{w}\mathbf{B}'_1 \to \boldsymbol{w}\mathbf{B}'$, suppose $E^{\cdot}$ is a chain complex in $\mathbf{B}_1$, $F^{\cdot}$ is a chain complex in $\mathbf{B}$, and $g\colon E^{\cdot} \to F^{\cdot}$. One needs to construct a subcomplex $E'^{\cdot}$ of $F^{\cdot}$ such that the inclusion h is a weak equivalence in $\boldsymbol{w}\mathbf{B}'$ and $hf = g$ for some $f\colon E^{\cdot} \to E'^{\cdot}$. Of course, we can assume that all $F^i = F^i(X_1[d])$ for some fixed number d and that some b serves as a control bound for all ϕ_i and can indeed easily define E'^i as in the proof of Lemma 4.18. This gives a chain subcomplex of $F^\cdot$ in $\mathbf{B}'_1$ with cokernel in $\mathbf{B}'_2$. The mapping cone $C(h)$ of the inclusion h is contractible in $\mathbf{B}/\mathbf{Z}$. The point is that unless $\mathbf{B}/\mathbf{Z}$ is idempotent complete, this does not necessarily imply that $C(h)$ is acyclic.

5. Equivariant theory and the Novikov conjecture

First we establish functoriality of $G^{-\infty}(X,R)$ from Example 4.22 of the kind one should expect in a boundedly controlled theory. Recall that a map $f\colon X \to Y$ of metric spaces is a *Lipschitz equivalence* if there is a number $k \geq 1$ such that

$$k^{-1}\operatorname{dist}(x_1,x_2) \leq \operatorname{dist}(f(x_1),f(x_2)) \leq k\operatorname{dist}(x_1,x_2)$$

for all x_1, $x_2 \in X$. We say f is *quasi-Lipschitz equivalence* if there is a real positive function l such that

$$\begin{aligned} \operatorname{dist}(x_1,x_2) \leq r &\implies \operatorname{dist}(f(x_1),f(x_2)) \leq l(r), \\ \operatorname{dist}(f(x_1),f(x_2)) \leq r &\implies \operatorname{dist}(x_1,x_2) \leq l(r). \end{aligned}$$

For example, any bounded function $f\colon X \to X$ with $\operatorname{dist}(x,\phi(x)) \leq D$, for all $x \in X$ and a fixed D, is quasi-Lipschitz with $l(r) = r + 2D$. An isometry $g\colon X \to Y$ is quasi-Lipschitz with $l = \mathrm{id}$. If only the first of the two conditions is satisfied, the map f is called *bornological*.

Proposition 5.1. *Consider the category of proper metric spaces X and quasi-Lipschitz equivalences and the category of noetherian rings R, then $\mathbf{B}(X,R)$ is a bifunctor covariant in the first variable and contravariant in the second variable to small exact categories and exact functors. Composing with the covariant functor $K^{-\infty}$ from Example 4.22 gives the spectrum-valued bifunctor $G^{-\infty}(X,R)$.*

Proof. If $f\colon X \to Y$ is a quasi-Lipschitz equivalence, the functor $f_*\,\mathbf{B}(X,R) \to \mathbf{B}(Y,R)$ is given on objects by $f_*F(S) = F(f^{-1}(S))$. Using the containment $f^{-1}(S)[D] \subset f^{-1}(S[l(D)])$, one sees that if $\phi \in \mathbf{B}(X,R)$ is a boundedly bicontrolled morphism with $\operatorname{fil}(\phi) \leq D$ then $f_*\phi$ is boundedly bicontrolled with $\operatorname{fil}(f_*\phi) \leq l(D)$. □

A subset W of a metric space X is *boundedly dense* or *commensurable* if $W[D] = X$ for some $D \geq 0$.

Proposition 5.2. *For a commensurable metric subspace W of X, there is a natural exact equivalence of categories $\mathbf{B}(W,R) \to \mathbf{B}(X,R)$ and the induced weak homotopy equivalence $G^{-\infty}(W,R) \simeq G^{-\infty}(X,R)$.*

Proof. Any surjective quasi-Lipschitz equivalence $f\colon X \to Y$ induces two functors on filtered modules. One is contravariant $f^*\colon \mathbf{B}(Y,R) \to \mathbf{B}(X,R)$ given by $f^*F(S) = F(f(S))$; the other is covariant $f_*\colon \mathbf{B}(X,R) \to \mathbf{B}(Y,R)$ given by $f_*F(S) = F(f^{-1}(S))$, so that $f^*f_* = \mathrm{id}$. Even when f is not surjective, there is the endofunctor $\omega = f^{-1}f$ of $\mathcal{P}(X)$ which induces an endofunctor ω_* of $\mathbf{B}(X,R)$. If $f\colon X \to X$ is bounded, that is $d(x,f(x)) \leq D$ for some $D \geq 0$ and all $x \in X$, there is always an isomorphism $\omega_*(F) \cong F$ induced by the identity on $F(X)$. This shows that $f_*F \cong F$ for all $F \in \mathbf{B}(X,R)$. Now if $W \subset X$ is commensurable, there is a bounded surjection $f\colon X \to W$, so f induces a natural transformation $\eta\colon \mathrm{id} \to f_*$ where all $\eta(F)$ are isomorphisms. □

Corollary 5.3. *If X is a bounded metric space then the natural equivalence*

$$\mathbf{B}(X,R) \cong \mathbf{B}(\text{point},R) = \mathbf{Modf}(R)$$

induces a weak equivalence $G^{-\infty}(X,R) \simeq G^{-\infty}(R)$ on the level of K-theory.

Given a group Γ with a left action on X by quasi-Lipschitz equivalences, there is a natural action of Γ on $\mathbf{B}(X,R)$ induced from the action on the power set $\mathcal{P}(X)$. However, this is not the correct choice for a useful equivariant controlled theory for essentially the same reasons as in the discussion of geometric modules in [4, ch. VI].

Definition 5.4. Let $\boldsymbol{E\Gamma}$ be the category with the object set Γ and the unique morphism $\mu\colon \gamma_1 \to \gamma_2$ for any pair γ_1, $\gamma_2 \in \Gamma$. There is a left Γ-action on $\boldsymbol{E\Gamma}$ induced by the left multiplication in Γ.

If $\mathcal{C}$ is a small category with left Γ-action, then the category of functors $\mathcal{C}_\Gamma = \mathrm{Fun}(\boldsymbol{E\Gamma},\mathcal{C})$ is another category with the Γ-action given on objects by $\gamma(F)(\gamma') = \gamma F(\gamma^{-1}\gamma')$ and $\gamma(F)(\mu) = \gamma F(\gamma^{-1}\mu)$. It is always nonequivariantly equivalent to $\mathcal{C}$. The fixed subcategory $\mathrm{Fun}(\boldsymbol{E\Gamma},\mathcal{C})^\Gamma \subset \mathcal{C}_\Gamma$ consists of equivariant functors and equivariant natural transformations.

Explicitly, when $C = \mathbf{B}(X,R)$ with the Γ-action described above, the objects of $\mathbf{B}_\Gamma(X,R)^\Gamma$ are the pairs (F,ψ) where $F \in \mathbf{B}(X,R)$ and ψ is a function on Γ with $\psi(\gamma) \in \mathrm{Hom}(F,\gamma F)$ such that

$$\psi(1) = 1 \quad \text{and} \quad \psi(\gamma_1\gamma_2) = \gamma_1\psi(\gamma_2)\psi(\gamma_1).$$

These conditions imply that $\psi(\gamma)$ is always an isomorphism as in [23]. The set of morphisms $(F,\psi) \to (F',\psi')$ consists of the morphisms $\phi\colon F \to F'$ in $\mathbf{B}(X,R)$ such that the squares

$$\begin{array}{ccc} F & \xrightarrow{\psi(\gamma)} & \gamma F \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\gamma\phi} \\ F' & \xrightarrow{\psi'(\gamma)} & \gamma F' \end{array}$$

commute for all $\gamma \in \Gamma$. A slightly more refined theory is obtained by replacing $\mathbf{B}_\Gamma(X,R)$ with the full subcategory $\mathbf{B}_{\Gamma,0}(X,R)$ of functors sending all morphisms of $\boldsymbol{E\Gamma}$ to filtration 0 maps. So $\mathbf{B}_{\Gamma,0}(X,R)^\Gamma$ consists of (F,ψ) with $\mathrm{fil}\,\psi(\gamma) = 0$ for all $\gamma \in \Gamma$.

Proposition 5.5. *The fixed point category $\mathbf{B}_{\Gamma,0}(X,R)^\Gamma$ is exact.*

Proof. The exact structure is inherited from $\mathbf{B}(X,R)$ in the sense that a morphism $\phi\colon (F,\psi) \to (F',\psi')$ is an admissible monomorphism or epimorphism if the map $\phi\colon F \to F'$ is in $\mathbf{mB}(X,R)$ or $\mathbf{eB}(X,R)$ respectively. The fact that this is an exact structure follows from the proof of Theorem 2.11 by observing that all constructions in that proof produce equivariant objects and morphisms. □

The usual exact structure in the category of finitely generated $R[\Gamma]$-modules $\mathbf{Modf}(R[\Gamma])$ for a noetherian ring R consists of respectively injective module homomorphisms and surjective homomorphisms with finitely generated kernels. Notice that when $R[\Gamma]$ is itself noetherian, so that $\mathbf{Modf}(R[\Gamma])$ is an abelian category, this coincides with the conventional choice of all injections for admissible monomorphisms and all surjections for admissible epimorphisms. However, there is a reasonable conjecture of P. Hall that only polycyclic-by-finite groups have noetherian group rings, cf. Question 32 in Farkas [8].

We are going to define a new exact structure on a subcategory $\mathbf{B}(R[\Gamma])$ of $\mathbf{Modf}(R[\Gamma])$ and relate it to the exact category $\mathbf{B}_{\Gamma,0}(X,R)^{\Gamma}$.

Recall that the *word metric* on a finitely generated group Γ with a fixed generating set Ω is the path metric induced from the condition that $\mathrm{dist}(\gamma, \omega\gamma) = 1$ whenever $\gamma \in \Gamma$ and $\omega \in \Omega$. This clearly makes Γ a proper metric space. We will use the notation $B_d(\gamma)$ for the metric ball of radius d centered at γ.

Definition 5.6. Given a finitely generated $R[\Gamma]$-module F, fix a finite generating set Σ and define a Γ-filtration of the R-module F by $F(S) = \langle S\Sigma\rangle_R$, the R-submodule of F generated by $S\Sigma$. Let $s(F,\Sigma)$ stand for the resulting Γ-filtered R-module.

Lemma 5.7. *Every $R[\Gamma]$-homomorphism between finitely generated modules $\phi\colon F \to G$ is boundedly controlled as an R-homomorphism between the filtered R-modules $s(F,\Sigma_F) \to s(G,\Sigma_G)$ with respect to any choice of the finite generating sets Σ_F and Σ_G.*

Proof. Consider $x \in F(S) = \langle S\Sigma_F\rangle_R$, then

$$x = \sum_{s,\sigma} r_{s,\sigma} s\sigma$$

for a finite collection of pairs $s \in S$, $\sigma \in \Sigma_F$. Since $F(\{e\}) = \langle \Sigma_F\rangle_R$ for the identity element e in Γ, there is a number $d \geq 0$ such that $\phi F(\{e\}) \subset G(B_d(e))$. Therefore,

$$\phi(x) = \sum_{s,\sigma} r_{s,\sigma}\phi(s\sigma) = \sum_{s,\sigma} r_{s,\sigma} s\phi(\sigma) \subset \sum_{s\in S} sG(B_d(e)) \subset G(S[d])$$

because the left translation action by any element $s \in S$ on $B_d(e)$ in Γ is an isometry onto $B_d(s)$. $\square$

Corollary 5.8. *Given a finitely generated $R[\Gamma]$-module F and two choices of finite generating sets Σ_1 and Σ_2, the filtered R-modules $s(F,\Sigma_1)$ and $s(F,\Sigma_2)$ are isomorphic as Γ-filtered R-modules.*

Proof. The identity map and its inverse are boundedly controlled as maps between $s(F,\Sigma_1)$ and $s(F,\Sigma_2)$ by Lemma 5.7. $\square$

Corollary 5.9. *Finitely generated $R[\Gamma]$-modules F with filtrations $s(F,\Sigma)$, with respect to arbitrary finite generating sets Σ, are locally finitely generated and lean. If $s(F,\Sigma)$ is insular and Σ' is another finite generating set then $s(F,\Sigma')$ is also insular.*

Proof. For a finite subset S, the submodule $F(S)$ is generated by the finite set $S\Sigma$. Since $F(x) = \langle x\Sigma\rangle_R$,

$$F(S) = \sum_{x\in\Sigma} \langle x\Sigma\rangle_R = \langle S\Sigma\rangle_R,$$

so $s(F,\Sigma)$ is 0-lean. The second claim follows from Corollary 5.8. $\square$

Definition 5.10. Let $\mathbf{B}(R[\Gamma])$ be the full subcategory of $\mathbf{Modf}(R[\Gamma])$ on R-modules F which are *strict* as filtered modules $s(F,\Sigma)$ with respect to some choice of the finite generating set Σ.

Let $\mathbf{B}_{\times}(R[\Gamma])$ be the category of objects which are pairs (F,Σ) with F in $\mathbf{B}(R[\Gamma])$ and Σ a finite generating set for F. The morphisms are the $R[\Gamma]$-homomorphisms between the modules.

Lemma 5.7 shows that the map

$$s\colon\ \mathbf{B}_\times(R[\Gamma]) \longrightarrow \mathbf{B}(\Gamma, R)$$

described in Definition 5.6 is a functor. In fact, it is a functor

$$s_\Gamma\colon\ \mathbf{B}_\times(R[\Gamma]) \longrightarrow \mathbf{B}_{\Gamma,0}(\Gamma, R)^\Gamma$$

by interpreting $s_\Gamma(F,\Sigma) = (F,\psi)$ with $F = s(F,\Sigma)$ and $\psi(\gamma)\colon F \to \gamma F$ induced from $s\sigma \mapsto \gamma^{-1}s\sigma$. Since $(\gamma F)(S) = \langle \gamma^{-1}(S)\Sigma\rangle_R$, it follows that the object $s_\Gamma(F,\Sigma)$ lands in $\mathbf{B}_{\Gamma,0}(\Gamma,R)^\Gamma$, and s sends all $R[\Gamma]$-homomorphisms to Γ-equivariant homomorphisms.

Lemma 5.11. *Let $F \in \mathbf{B}_{\Gamma,0}(\Gamma,R)^\Gamma$ and let Σ be a finite generating set for the $R[\Gamma]$-module F. Then the identity homomorphism* $\mathrm{id}\colon s_\Gamma(F,\Sigma) \to F$ *is boundedly controlled with respect to the induced and the original filtrations of F.*

Proof. If Σ is contained in $F(B_d(e))$, where e is the identity element in Γ, then $\gamma\Sigma \subset F(B_d(\gamma))$ for all $\gamma \in \Gamma$, and $s(F,\Sigma)(S) = \langle S\Sigma\rangle_R \subset F(S[d])$ for all subsets $S \subset \Gamma$. □

Both functors s and s_Γ are additive with respect to the obvious additive structure in $\mathbf{B}_\times(R[\Gamma])$ where $(F,\Sigma_F) \oplus (G,\Sigma_G) = (F\oplus G, \Sigma_F \times \Sigma_G)$. Let the *admissible monomorphisms* $\phi\colon (F,\Sigma_F) \to (G,\Sigma_G)$ in $\mathbf{B}_\times(R[\Gamma])$ be the injections $\phi\colon F \to G$ of $R[\Gamma]$-modules ϕ such that $s(\phi)\colon s(F,\Sigma_F) \to s(G,\Sigma_G)$ is a boundedly bicontrolled homomorphism of Γ-filtered R-modules. This is equivalent to requiring that $s(\phi)$ be an admissible monomorphism in $\mathbf{B}(\Gamma,R)$. Let the *admissible epimorphisms* be the morphisms ϕ such that $s(\phi)$ are admissible epimorphisms in $\mathbf{B}(\Gamma,R)$.

Proposition 5.12. *The choice of admissible morphisms defines an exact structure on $\mathbf{B}_\times(R[\Gamma])$ such that both s and s_Γ are exact functors.*

Proof. When checking Quillen's axioms in $\mathbf{B}_\times(R[\Gamma])$, all required universal constructions are performed in $\mathbf{B}(R[\Gamma])$ with the canonical choices of finite generating sets. In particular, Σ in the pushout $B \cup_A C$ is the image of the product set $\Sigma_B \times \Sigma_C$ in $B \times C$. The fact that all candidates for admissible morphisms are boundedly bicontrolled in $\mathbf{B}(\Gamma,R)$ or $\mathbf{B}_{\Gamma,0}(\Gamma,R)^\Gamma$ follows from the proof of Theorem 2.11. Exactness of s and s_Γ is immediate. □

Definition 5.13. We give $\mathbf{B}(R[\Gamma])$ the minimal exact structure that makes the forgetful functor $p\colon\ \mathbf{B}_\times(R[\Gamma]) \to \mathbf{B}(R[\Gamma])$ sending (F,Σ) to F an exact functor. In other words, an $R[\Gamma]$-homomorphism $\phi\colon F \to G$ is an *admissible monomorphism* or *epimorphism* if for some choice of finite generating sets, $\phi\colon (F,\Sigma_F) \to (G,\Sigma_G)$ is respectively an admissible monomorphism or epimorphism in $\mathbf{B}_\times(R[\Gamma])$. Corollary 5.8 shows that if $\phi\colon F \to G$ is boundedly bicontrolled as a map of filtered R-modules $s(F,\Sigma_F) \to s(G,\Sigma_G)$ then it is boundedly bicontrolled with respect to any other choice of finite generating sets, so this structure is well-defined.

Notation 5.14. The new exact category will be referred to as $\mathbf{B}(R[\Gamma])$, with the corresponding K-theory spectrum $G^{-\infty}(R[\Gamma])$.

Let (F,ψ) be an object of $\mathbf{B}_{\Gamma,0}(\Gamma,R)^\Gamma$. One may think of $\gamma F \in \mathbf{B}(\Gamma,R)$, $\gamma \in \Gamma$, as the module F with a new Γ-filtration. Now the R-module structure $\eta\colon R \to \operatorname{End} F$ induces an $R[\Gamma]$-module structure $\eta(\psi)\colon R[\Gamma] \to \operatorname{End} F$ given by

$$\sum_\gamma r_\gamma \gamma \mapsto \sum_\gamma \eta(r_\gamma)\psi(\gamma)$$

since the sums are taken over a finite subset of Γ. It is easy to see that this defines a map

$$\pi\colon \mathbf{B}_{\Gamma,0}(\Gamma,R)^\Gamma \longrightarrow \mathbf{B}(R[\Gamma])$$

by sending (F,ψ) to F, so that $p=\pi s_\Gamma$. Notice however that in general π is not exact as the identity homomorphism in Lemma 5.11 is not necessarily an isomorphism.

In the rest of the paper we assume Γ is torsion-free. The exact functors p and s_Γ induce maps in nonconnective K-theory

$$G^{-\infty}(R[\Gamma]) \xleftarrow{\ p\ } K^{-\infty}\,\mathbf{B}_\times(R[\Gamma]) \xrightarrow{\ s_\Gamma\ } G^{-\infty}_{\Gamma,0}(\Gamma,R)^\Gamma.$$

We claim that both of these maps are weak equivalences.

Proposition 5.15. *The functor f induces a weak equivalence*

$$K^{-\infty}\,\mathbf{B}_\times(R[\Gamma]) \simeq G^{-\infty}(R[\Gamma]).$$

Proof. This follows from the Approximation theorem applied to p'. The two categories are saturated, and $\mathbf{B}_\times(R[\Gamma])'$ has a cylinder functor satisfying the cylinder axiom which is constructed as the canonical homotopy pushout with the canonical product basis, see section 1 of [24]. The first condition of the Approximation theorem is clear. For the second condition, let $(F_1^\cdot,\Sigma_1^\cdot)$ be a complex in $\mathbf{B}_\times(R[\Gamma])$ and let $g\colon F_1^\cdot\to F_2^\cdot$ be a chain map in $\mathbf{B}(R[\Gamma])'$. For each $R[\Gamma]$-module F_2^i choose any finite generating set Σ_2^i, then using $f=g$ and $g'=\mathrm{id}$, we have $g=g'p(f)$. $\square$

Proposition 5.16. *The functor s_Γ induces a weak homotopy equivalence*

$$K^{-\infty}\,\mathbf{B}_\times(R[\Gamma]) \simeq G^{-\infty}_{\Gamma,0}(\Gamma,R)^\Gamma.$$

Proof. The target category is again saturated and has a cylinder functor satisfying the cylinder axiom. To check condition 2 of the approximation theorem, let

$$E^\cdot\colon \quad 0\longrightarrow (E^1,\Sigma_1)\longrightarrow (E^2,\Sigma_2)\longrightarrow\ \ldots\ \longrightarrow (E^n,\Sigma_n)\longrightarrow 0$$

be a complex in $\mathbf{B}_\times(R[\Gamma])$,

$$(F^\cdot,\psi.)\colon \quad 0\longrightarrow (F^1,\psi_1)\xrightarrow{f_1}(F^2,\psi_2)\xrightarrow{f_2}\ \ldots\ \xrightarrow{f_{n-1}}(F^n,\psi_n)\longrightarrow 0$$

be a complex in $\mathbf{B}_{\Gamma,0}(\Gamma,R)^\Gamma$, and $g\colon s'_\Gamma(E^\cdot)\to(F^\cdot,\psi.)$ be a chain map. Each F^i can be thought of as an $R[\Gamma]$-module, and there is a chain complex

$$F^\cdot\colon \quad 0\longrightarrow F^1\xrightarrow{f_1}F^2\xrightarrow{f_2}\ \ldots\ \xrightarrow{f_{n-1}}F^n\longrightarrow 0$$

in $\mathbf{Modf}(R[\Gamma])$. Choose arbitrary finite generating sets Ω_i in F^i for all $1\le i\le n$. Now

$$\pi_\Omega F^\cdot\colon \quad 0\longrightarrow (F^1,\Omega_1)\xrightarrow{f_1}(F^2,\Omega_2)\xrightarrow{f_2}\ \ldots\ \xrightarrow{f_{n-1}}(F^n,\Omega_n)\longrightarrow 0$$

is a chain complex in $\mathbf{B}_\times(R[\Gamma])$. The chain map g is degree-wise an $R[\Gamma]$-homomorphism, so there is a corresponding chain map $f\colon E^\cdot\to\pi_\Omega F^\cdot$ which coincides with g on modules. On the other hand, the degree-wise identity gives a chain map $g'\colon s'_\Gamma(\pi_\Omega F^\cdot)\to F^\cdot$ in $\mathbf{B}_{\Gamma,0}(\Gamma,R)^\Gamma$ by Lemma 5.11. This g' is a quasi-isomorphism, as required. $\square$

Corollary 5.17. *Let Γ be a finitely generated torsion-free group and R be a noetherian ring. There is a weak equivalence*

$$G^{-\infty}_{\Gamma,0}(\Gamma,R)^\Gamma \simeq G^{-\infty}(R[\Gamma]).$$

Corollary 5.18. *Let Γ be a finitely generated torsion-free group acting freely, properly discontinuously and cocompactly on a proper metric space X and let R be a noetherian ring. There is a weak homotopy equivalence*

$$G^{-\infty}_{\Gamma,0}(X,R)^{\Gamma} \simeq G^{-\infty}(R[\Gamma]).$$

Proof. Let $p\colon X \to$ point be the geometric collapse. For any $x \in X$ such that the embedding i of the orbit Γx with the word metric is commensurable in X, there is a commutative diagram

$$\begin{array}{ccc} G^{-\infty}_{\Gamma,0}(\Gamma x,R)^{\Gamma} & \xrightarrow{\pi_*} & G^{-\infty}(R[\Gamma]) \\ \big\downarrow{\scriptstyle i_*} & & {\scriptstyle =}\big\downarrow \\ G^{-\infty}_{\Gamma,0}(X,R)^{\Gamma} & \xrightarrow{\pi_*} & G^{-\infty}(R[\Gamma]) \end{array}$$

The top π_* is a weak equivalence by Corollary 5.17. The vertical map i_* is a weak equivalence as in Proposition 5.2, so the lower map π_* is a weak equivalence. □

Definition 5.19. For a discrete group Γ and a ring R there is an assembly map

$$A_K\colon B\Gamma_+ \wedge K^{-\infty}(R) \longrightarrow K^{-\infty}(R[\Gamma]).$$

When the ring R is regular noetherian, so that $G^{-\infty}(R)$ and $K^{-\infty}(R)$ can be naturally identified, the assembly map in G-theory A_G is simply the composition of A_K and the canonical Cartan map $C\colon K^{-\infty}(R[\Gamma]) \to G^{-\infty}(R[\Gamma])$ induced by inclusion of categories. The integral *Novikov conjecture* in algebraic G-theory is the statement that this is a split injection of spectra.

Remark 5.20. Notice that whenever the assembly map A_G is split injective, the map A_K is also split injective, so this conjecture is stronger than the K-theoretic conjecture when the ring R is regular.

Remark 5.21. The standard exact structure on $\mathbf{Modf}(R[\Gamma])$ has all injective and surjective $R[\Gamma]$-homomorphisms with finitely generated cokernels and kernels as admissible morphisms so that the exact sequences are the traditional short exact sequences. Let the corresponding K-theory spectrum be $G^{-\infty}_m(R[\Gamma])$. One might attempt to replace $G^{-\infty}_{\times}(R[\Gamma])$ with $G^{-\infty}_m(R[\Gamma])$ as the target of the assembly A_G. However, W. Lück has pointed out that this map would not be weakly injective even in the simple case when R is a commutative ring and Γ is the free group on two generators, cf. Remark 2.23 in [14].

This underscores the importance of choosing the coarse version $G^{-\infty}(R[\Gamma])$ as our approximation of $K^{-\infty}(R[\Gamma])$.

The equivariant assembly map in G-theory can be defined as in [4]. There is an equivariant natural transformation α_G from the bounded version of the equivariant locally finite homology $h^{lf}(X, G^{-\infty}(R))$ to $G^{-\infty}_{\Gamma,0}(X,R)$, see Definition II.14, loc. cit. When Γ acts cocompactly, the fixed points fit in the commutative diagram

$$\begin{array}{ccccc} B\Gamma_+ \wedge G^{-\infty}(R) & \xrightarrow{\alpha^{\Gamma}_K} & K^{-\infty}_{\Gamma,0}(X,R)^{\Gamma} & \xrightarrow{\simeq} & K^{-\infty}(R[\Gamma]) \\ {\scriptstyle \alpha^{\Gamma}_G}\big\downarrow & & \big\downarrow & & \big\downarrow \\ G^{-\infty}_{\Gamma,0}(X,R)^{\Gamma} & \xleftarrow[s_{\Gamma *}]{\simeq} & K^{-\infty}\,\mathbf{B}_{\times}(R[\Gamma]) & \xrightarrow[s_{\Gamma *}]{\simeq} & G^{-\infty}(R[\Gamma]) \end{array}$$

If we restrict to those equivariant objects in $K^{-\infty}_{\Gamma,0}(X,R)^{\Gamma}$ that have the function ϕ map the generating set B to a single point then the map

$$i\colon\ K^{-\infty}_{\Gamma,0}(X,R)^{\Gamma} \to G^{-\infty}_{\Gamma,0}(X,R)^{\Gamma}$$

is well-defined and is a diagonal in the left square. The composition

$$\alpha_G\colon\ h^{lf}(X, G^{-\infty}(R)) \xrightarrow{\alpha_K} K^{-\infty}_{\Gamma,0}(X,R) \xrightarrow{i} G^{-\infty}_{\Gamma,0}(X,R)$$

is the *assembly map in boundedly controlled G-theory.*

Controlled excision is the main technical tool from controlled K-theory used to prove integral Novikov conjectures. It is used to see that in specific cases the equivariant assembly map in bounded K-theory is a homotopy equivalence. In particular, the argument in [6] applies to groups of finite asymptotic dimension which have a finite classifying space.

Theorem 5.22. *The G-theoretic assembly map*

$$A_G\colon B\Gamma_+ \wedge G^{-\infty}(R) \longrightarrow G^{-\infty}(R[\Gamma])$$

is a split injection for any geometrically finite group Γ of finite asymptotic dimension and a noetherian ring R.

Proof. The main step in the proofs of the Novikov conjecture in algebraic K-theory [4, 6] to which we referred above is the application of homotopy fixed points to reduce the study of the map A_K to the nonequivariant study of the equivariant map α_K. This is shown to be a weak equivalence by using controlled excision to compute the target. With the excision results from section 3 and the equivariant properties established here, the proofs can be repeated verbatim obtaining splittings of the assembly maps α_G for the same collection of groups. □

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