Optimal Vehicle Controls on a Surface

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Abstract

The advent of self-driving vehicles has revolutionized the automotive industry. As automated vehicles become more prevalent, the benefits of optimal control become more apparent. Our goal is to explore techniques for optimizing fuel usage and arrival time given an arbitrary surface.

1 Background

Individuals have been working on self-driving vehicles since the 1920s. At the 1939 New York World's Fair, Norman Bel Geddes created the Futurama exhibit that put forth a vision of "cars of 1960 and the highways on which they drive will have in them devices which will correct the faults of human beings as drivers."[1] Although his timeline was not entirely accurate, many other bright individuals contributed a tremendous amount to this field until Japan's Tsukuba Mechanical Engineering Laboratory put forth the first semi-automated car in 1977. Progress in developing self-driving vehicles continued as the technology improved, and has captivated the minds of millions today. However, despite the many advancements, engineers still face many challenges as they continue to design these cars. Two such challenges include determining how to minimize the time it takes for a car to travel from one point to another and how to optimize among a mixture of other controls (such as gas consumption). Our team has decided to answer both of these questions in one approach using a combination of numerical optimization methods.

We model this with a 2-dimensional surface that represents the surface the car is traversing, where the x-axis represents the car's position and the y-axis represents the height of both the surface and car (which follows the curve of the surface itself). Additionally, we have a starting point and ending point from which we optimize time and vehicle controls. We also give the condition that the vehicle start and end at rest.

2 Mathematical Representation

The surface the car travels is a curve in 2-dimensional space, or in other words, the path travelled by the car is given by the function $f(x)$, where x is position. The car starts at position $(A, f(A))$ and ends at $(B, f(B))$. The starting and ending speeds are zero, meaning the car starts from rest and comes to a complete stop. For our formulation, we track speed, denoted by v, rather than velocity. This is to simplify our dynamics. For simplicity, we

[1] Geddes, N. B. *Magic Motorways.* Random House 1940

assume that the endpoint B is zero. We choose this formulation in order to shift our axis origin and thus simplify the solution described later.

We must first analyze the physical dynamics by looking at the tangent vector to the graph $f(x)$. The unit gradient (tangent) vector to the surface is given by $(1, f'(x))/||(1, f'(x))|| = (1, f'(x))/\sqrt{1+f'(x)^2}$. At any given time our speed will be split into our x and y coordinates by this vector, thus we have

$$\dot{x} = \frac{v}{\sqrt{1+f'(x)^2}}. \tag{1}$$

We choose not to track the y state since it can be found using $y = f(x)$. To find acceleration, consider all the forces acting on the car. These include gravity $g = 9.8\frac{m}{s^2}$ pointing in the (0,-1) direction, a normal force on the surface, and our acceleration u. Note that the angle of the tangent vector off horizontal is the same as the normal vector off vertical. We call this angle θ. The sum of the normal force and the gravitational force give a total non-input force

$$F = m\Big((0,-g) + gcos(\theta)\frac{(-f'(x),1)}{\sqrt{1+f'(x)^2}}\Big) = m\Big((0,-g) + g\frac{(-f'(x),1)}{1+f'(x)^2}\Big) \tag{2}$$

$$= m\Big(\frac{(-gf'(x), -g+g+gf'(x)^2)}{1+f'(x)^2}\Big) = m\Big(-gf'(x)\frac{(1,f'(x))}{1+f'(x)^2}\Big) \tag{3}$$

Since $\cos(\theta)$ is just the x component of the unit tangent vector. Dividing by mass and taking the norm we obtain the change in speed:

$$\dot{v} = u - g\frac{f'(x)}{\sqrt{1+f'(x)^2}}, \tag{4}$$

where u is an acceleration input control and pushes in the direction of the tangent vector. We want to find a balance of gas used (we say gas used is proportional to u by a factor of one-fifth) and time to destination. The importance of time versus gas expenditure is given by a user-defined constant λ. We constrain our input acceleration to the interval [-5,5]. The resulting functional to minimize is

$$J(u) = \int_0^T \lambda + (1-\lambda)\Big(\frac{u}{5}\Big)^2 dt \tag{5}$$

subject to

$$\dot{x} = \frac{v}{\sqrt{1+f'(x)^2}}, \quad x(0) = x_0, \quad x(T) = 0 \tag{6}$$

$$\dot{v} = u - g\frac{f'(x)}{\sqrt{1+f'(x)^2}}, \quad v(0) = 0, \quad v(T) = 0 \tag{7}$$

$$u \in [-5, 5] \tag{8}$$

3 Solution

We solve this using Pontryagin's Maximum Principle. Our Hamiltonian is

$$H = p_1 \frac{v}{\sqrt{1+f'(x)^2}} + p_2\Big(u - g\frac{f'(x)}{\sqrt{1+f'(x)^2}}\Big) - \lambda - (1-\lambda)\Big(\frac{u}{5}\Big)^2 \tag{9}$$

Using Pontryagin's Maximum Principle (PMP) to maximize H with respect to u, we must have that

$$0 = H_u = p_2 - \frac{2}{25}(1-\lambda)u \tag{10}$$

$$\implies u = \frac{25p_2}{1-\lambda} \tag{11}$$

And applying our constraints on u gives

$$u = max(-5, min(\frac{25p_2}{1-\lambda}, 5))$$

Substituting this into our system dynamics and applying the rest of PMP, we get the ODE system:

$$\dot{p_1} = -H_x = -p_1 v \frac{d}{dx}\frac{1}{\sqrt{1+f'(x)^2}} + p_2 g \frac{d}{dx}\frac{f'(x)}{\sqrt{1+f'(x)^2}} \tag{12}$$

$$\dot{p_2} = -H_v = -\frac{p_1}{\sqrt{1+f'(x)^2}} \tag{13}$$

$$\dot{x} = \frac{v}{\sqrt{1+f'(x)^2}}, \quad x(0) = x_0, \quad x(T) = 0 \tag{14}$$

$$\dot{v} = u - g\frac{f'(x)}{\sqrt{1+f'(x)^2}}, \quad v(0) = v_0, \quad v(T) = 0 \tag{15}$$

To solve this, we use the shooting method to "guess" the initial conditions of p_1 and p_2 that will get us the correct end conditions of x and v. We apply Broyden's Method[2], a 2D variant of the secant method, and use a centered finite difference to estimate the initial derivative guess. Unfortunately, this method, like all Newton-based methods, is extremely sensitive to initial guesses. In our experiments, we found that being off by even a tenth in either direction was enough to cause divergence in more complex surfaces.

We also noted that we have made no progress in finding what our optimal time will be. We instead rely on fixing T, solving the problem and evaluating the cost, and repeating for various values of T. We then choose our optimal time to be the one that minimizes the cost.

To do so, we begin with the largest value of T that we find reasonable. Note that our controls will be rather small with such a large amount of time to get to our end state. This helps narrow down finding our initial guess for $p_1(0)$ and $p_2(0)$ as they will also likely be small. To find a good guess, we plot a heatmap of various values of $p_1(0)$ and $p_2(0)$ with $||[x(T), v(T)]^T||$ as the output. An example of this can be seen in Figure 1.

Once we found the correct $p_1(0)$ and $p_2(0)$ for the largest value of T, simply iterate to the next largest value of T, using the resulting $p_1(0)$ and $p_2(0)$ of the previous iteration as

[2] `https://neos-guide.org/content/broydens-method`

our initial guesses. Note $p_1(0)$ and $p_2(0)$ shouldn't have changed significantly as T hasn't changed significantly. After doing this for many values of T, we'll eventually have that no control will get us to the end state since T will be too small. At this point, simply choose the T along with it's corresponding $p_1(0)$ and $p_2(0)$ that minimize the cost. This can be seen in Figure 2 for various values of λ. This will be the solution. It looks something like Figure 3.

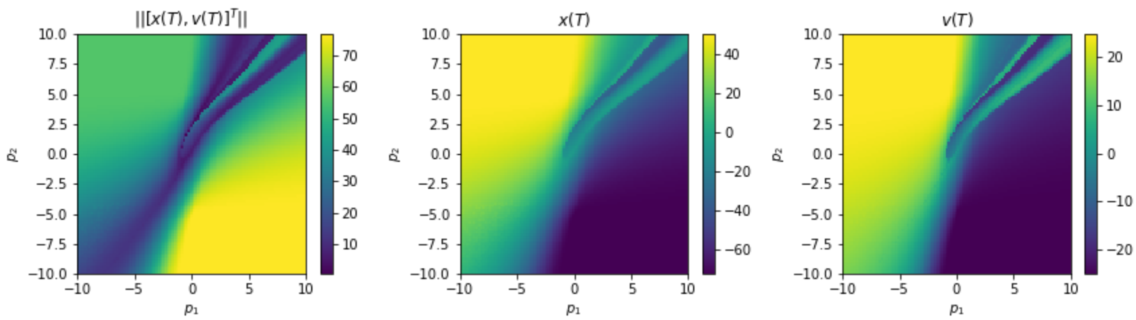

Figure 1: Search for Initial Conditions

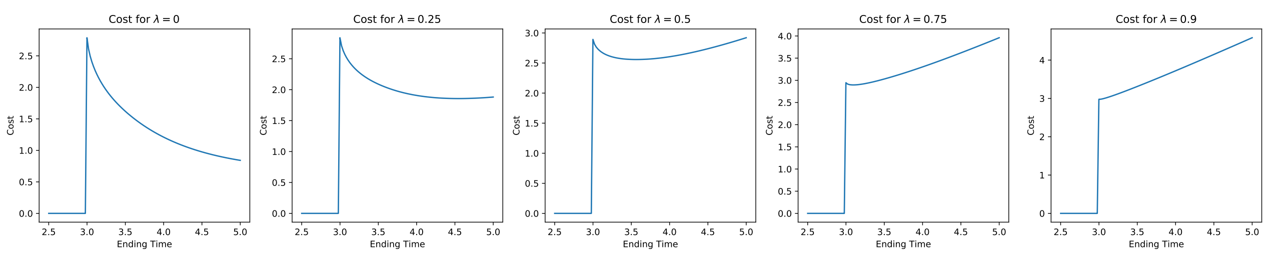

Figure 2: Search for Optimal Time

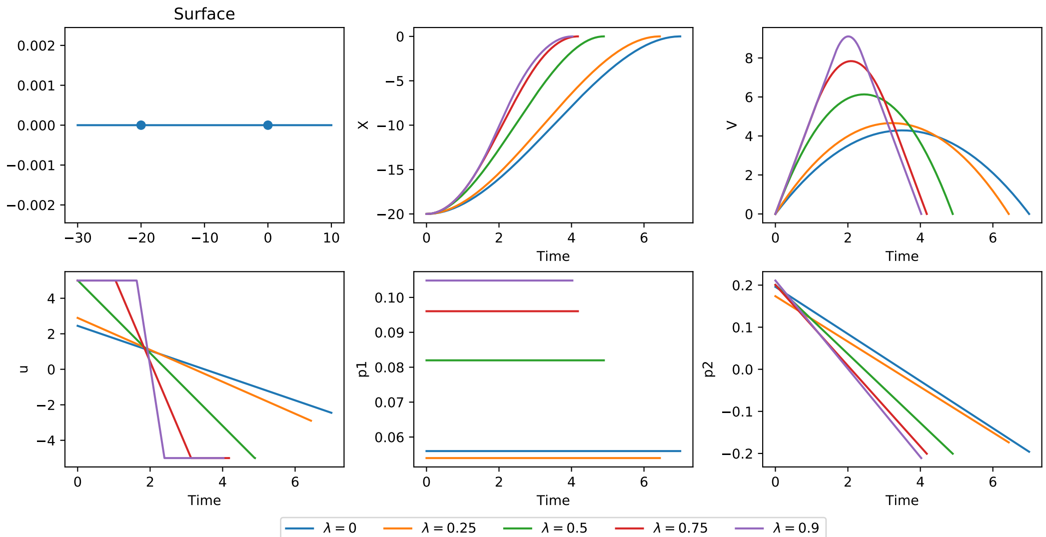

Figure 3: Flat Surface Solution

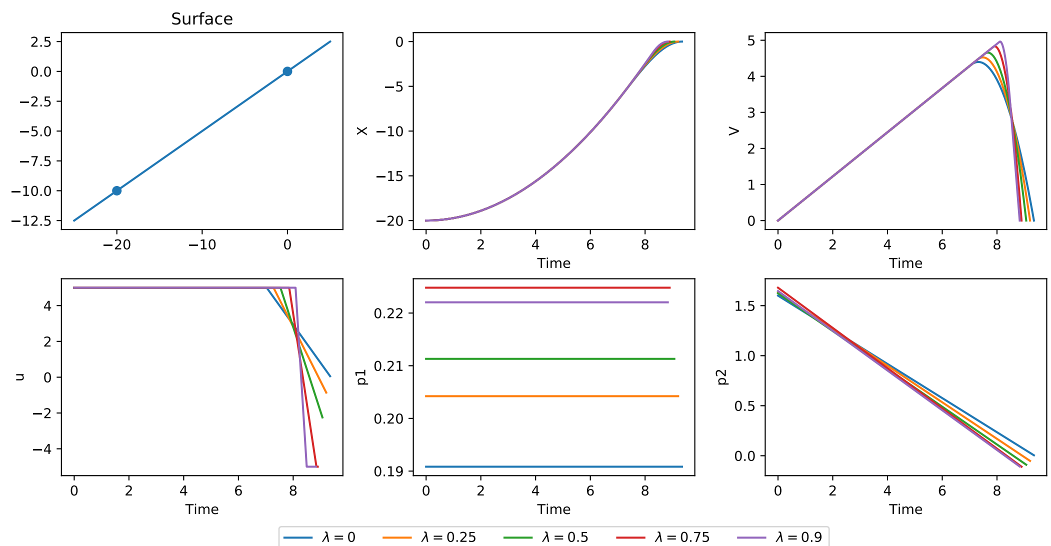

Figure 4: Angled Surface Solution

4 Interpretation

Our model of minimizing both the controls and time necessary to guide a vehicle from point A to point B in $\mathbb{R}^2$ gives us dynamics and functional as given by equations (4), (5), (6), and (7).

The value λ is a multiplicative constant that allows us to find the optimal balance between prioritizing minimal time and minimal control. x and v are the vehicle's horizontal position and velocity, respectively, with the origin set as our final position. u is an acceleration control between -5 and 5, a model choice that increases the vehicle's ability to drive up steep hills, as defined by $f(x)$, against the downward acceleration due to gravity g. Minimizing this control would optimize gas consumption without the expense of time.

Because our problem is two-fold (finding the balance between both optimal time and optimal control), we rely on a few ideas to find our solution. First, using the Hamiltonian (8) and applying the Pontryagin Maximum Principle (10) as indicated previously, we solve the ODE system with the Shooting Method. This allows us to guess the initial values for p_1 and p_2 until our boundary conditions are satisfied.

Moreover, rather than finding a *global* optimum for our final value of T, we optimize over a chosen range of time values to find a *local* optimum. With these guesses set, we solve the system over a grid of values (see Figure 1) and take the minimizers.

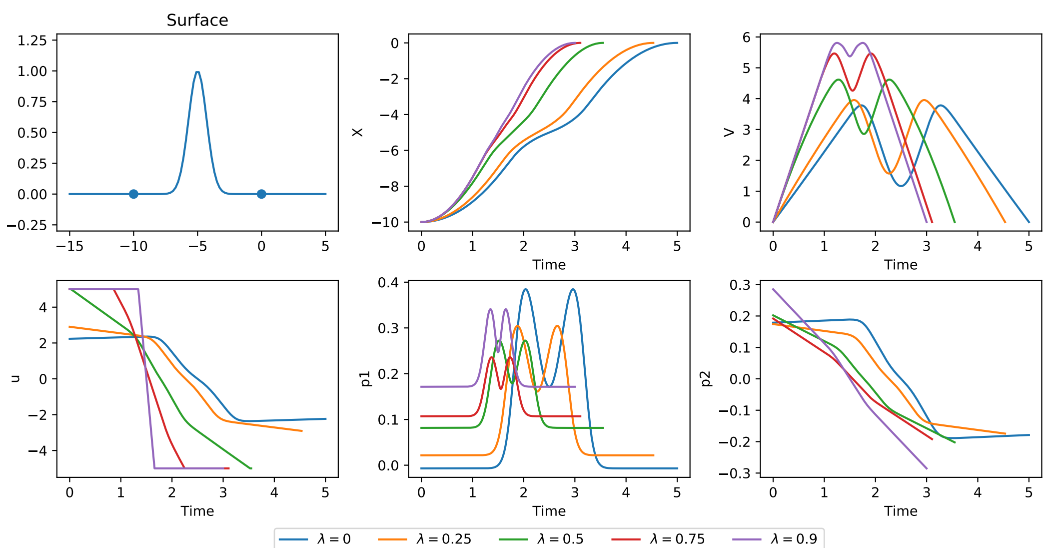

Figure 5: Gaussian Solution

For various values of λ, we can compute the optimal value for T. Lower values for λ correspond to less gas consumption (i.e. gas usage is more heavily penalized), whereas higher values will penalize gas consumption less, end time more, and allow the vehicle to attain a higher velocity and an earlier ending time.

Note how different λ values intuitively affect the position, velocity, and control, while maintaining similar graph structures. Expensive fuel results in generally lower control output and lower speeds, while inexpensive fuel leads to higher speeds and quicker arrival at the origin.

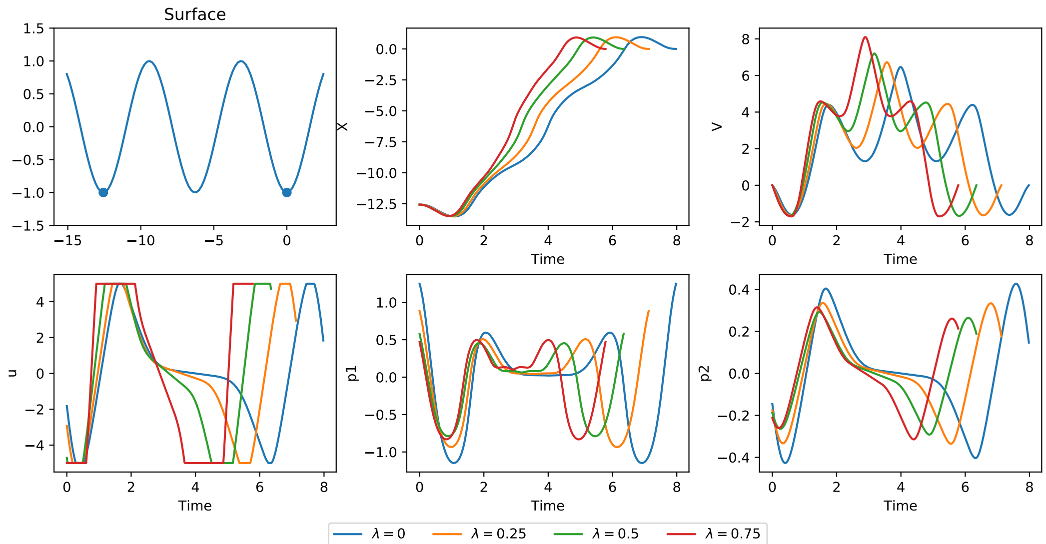

Figure 6: Cosine Solution

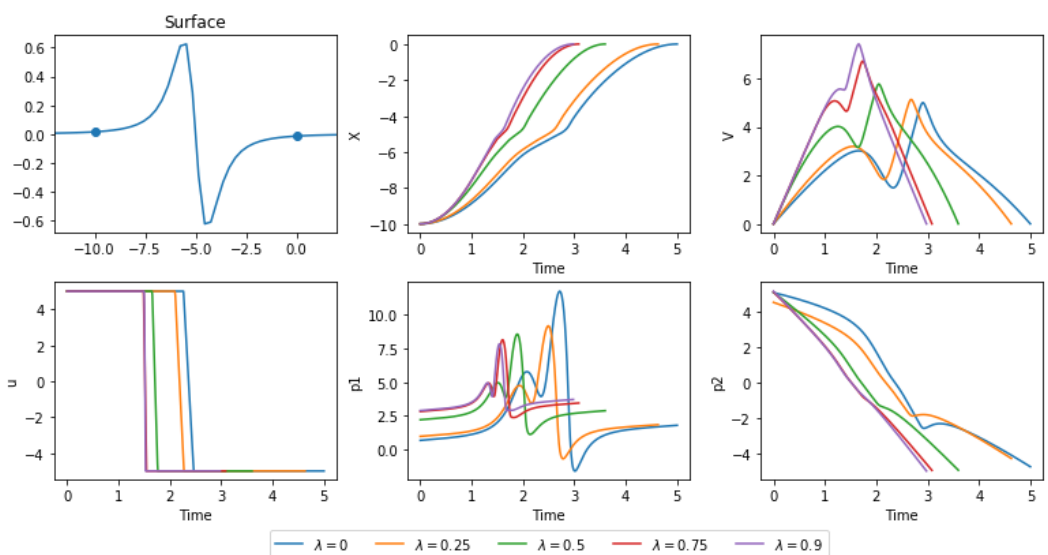

Figure 7: $-2x(1+x^2)^{-2}$ Solution