Complex Economics Dynamics 1

Homework assignment 1

Important

a) Submit the assignment as one .pdf file prepared with LaTeX or one of its variants, with a font size of at least 10pt.

b) **The first line of the .pdf document should contain the name or names of the submitters.**

c) You are allowed to code in `julia`, `matlab`, `python`, or `R`. If you want to use another programming language, please consult first with the tutorials teacher.

d) Submit any code as a `filename.txt` file: for instance, if you submit the file `foo.py`, rename it to `foo.py.txt`.

e) Clear, concise, well-documented code gets more points.

Question 1. Write a function `traj = rk4(f,t0,x0,tau,n)` that takes the following input arguments:

- `f`: a function $f(t, x)$ of two arguments $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$;
- `t0`: an initial time t_0;
- `x0`: an initial state x_0;
- `tau`: a final time τ;
- `n`: a number of gridpoints n.

The output `traj` should be an $(n + 1) \times (m + 1)$ matrix $\mathit{traj}_{t,i}$, such that $\mathit{traj}_{t,1} = t_0 + th$, where $h = (\tau - t_0)/n$, and $\mathit{traj}_{t,j} = s^{\mathrm{RK}}_{n,j}(t_0 + th)$, where $s^{\mathrm{RK}}_{n,j}(t)$ is the numerical approximation of j'th component of the solution $s(t)$ of the differential equation $\dot{s} = f(t, s)$, $s(t_0) = x_0$ that is obtained by using the fourth order Runge–Kutta method described in Chapter 3 of the lecture notes.

Question 2. Test the function written in Question 1 by integrating the equation $\dot{s} = s$, $s(0) = 1$, whose solution is $s(t) = \mathrm{e}^t$.

Choose successively the number of gridpoints equal to $n = 1, 2, 4, 8, 16, \cdots$.

Make a table with the following columns:

- n: the number of grid points
- h: the step size;
- $s^{\mathrm{RK}}_n(1)$: the numerical approximation of $s(1)$ given the number of grid points;
- $|s^{\mathrm{RK}}_n(1) - \mathrm{e}|$: the absolute approximation error;
- $|s^{\mathrm{RK}}_n(1) - s^{\mathrm{RK}}_{n/2}(1)|$: the difference between the current and the previous approximation
- $|s^{\mathrm{RK}}_n(1) - s^{\mathrm{RK}}_{n/2}(1)|/|s^{\mathrm{RK}}_{n/2}(1) - s^{\mathrm{RK}}_{n/4}(1)|$: the ratio of the two last approximation differences

Explain any regularities you see.

Question 3. Consider the dynamical system generated by $\dot{s} = f(s)$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given as

$$f(x) = \begin{pmatrix} -x_2 \\ \lambda - \mu x_2 - x_1^2 - x_1 x_2, \end{pmatrix},$$

and where $\lambda, \mu \in \mathbb{R}$ are parameters.

a. Find all steady states of the vector field.

b. Mark in the (λ, μ) parameter space the parameter region for which the system has no steady states and the region where the system has two steady states.

c. Determine the stability type of the two steady states for all values of the parameters.

d. One of the steady states changes its stability type from locally stable to unstable. Determine the parameter curve separating the corresponding parameter regions.

Question 4. Consider the dynamical system generated by $\dot{s} = f(s)$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given as

$$f(x) = \begin{pmatrix} -x_2 \\ \lambda - \mu x_2 - x_1^2 - x_1 x_2, \end{pmatrix},$$

and where $\lambda, \mu \in \mathbb{R}$ are parameters.

a. Compute and plot the function $s_1(t)$ for $t \in [0, 160]$ and $s(0) = (0.499999, 0)$, $\lambda = 0.25$, and μ taking successively the values 0.55, 0.45, and 0.35.

b. To understand why the resulting graphs are very different, do the following. Find the saddle steady state $\bar{x}$ for the given parameter values, and compute its eigenvectors v_1 and v_2, where v_1 is the eigenvector associated to the postive eigenvalue, and v_2 that associated to the negative eigenvalue.

   Set $h = 10^{-6}$.

   Compute and plot the forward orbits starting at $\bar{x} + hv_1$ and $\bar{x} - hv_1$. You'll have to take the length of the time interval between 10 and 200 — experiment to get it right.

   Also compute and plot the backward orbits starting at $\bar{x} - hv_2$ and $\bar{x} + hv_2$.

   Do this for all three parameter combinations.

c. Use the plots obtained under b. to explain the different forms of the plots obtained under a.

d. Between $(\lambda, \mu) = (0.25, 0.45)$ and $(\lambda, \mu) = (0.25, 0.35)$ there has been a qualitative change of the dynamics. Describe the change as well as the intermediate 'bifurcating' situation.