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Fidelity of single qubit maps

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Abstract

We describe a simple way of characterizing the average fidelity between a unitary (or anti-unitary) operator and a general operation on a single qubit, which only involves calculating the fidelities for a few pure input states, and discuss possible applications to experimental techniques including nuclear magnetic resonance (NMR). © 2002 Elsevier Science B.V. All rights reserved.

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In quantum information theory [1] it is often useful to compare the effects of two processes applied to a quantum system. The basic building blocks of quantum information processing are transformations (maps) on two level quantum systems known as quantum bits or qubits. Ideally, we would like to be able to compare any two single qubit maps, but unfortunately this is not always straightforward. The comparison is, however, much simpler if one map is unitary or anti-unitary. A natural approach to compare two maps is to calculate the state fidelity of their output states given identical inputs. The Uhlmann state fidelity of two density operators (ρ_1, ρ_2) is given by [2]

$$F(\rho_1, \rho_2) = \left(\mathrm{Tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right)\right)^2. \tag{1}$$

This may be interpreted as the maximal overlap of all purifications of ρ_1 and ρ_2. Under a unitary or anti-unitary transformation, a pure input state maps to a pure output state and in this case we can simplify the state fidelity (1) to [3]

$$F\big(|\psi\rangle\langle\psi|, \rho\big) = \mathrm{Tr}\big(|\psi\rangle\langle\psi|\rho\big). \tag{2}$$

The state fidelity of a unitary (or anti-unitary) map U and a general linear, trace-preserving, transformation $\mathcal{M}$ acting on an initially pure state $|\psi\rangle\langle\psi|$ is given by

$$F_{|\psi\rangle\langle\psi|} = \mathrm{Tr}\big(U|\psi\rangle\langle\psi|U^\dagger \mathcal{M}\big[|\psi\rangle\langle\psi|\big]\big). \tag{3}$$

The average map fidelity can then be defined by integrating over all pure input states,

$$\bar{F} = \frac{1}{4\pi}\int F_{|\psi\rangle\langle\psi|}\, d\Omega \tag{4}$$

(where the integral is over the surface of the Bloch sphere), and this definition is widely used [4–7]. There

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is, however, a simplification: using the fact that $|\psi\rangle\langle\psi|$ can be written in terms of the Pauli spin matrices and the identity matrix [8],

$$\begin{aligned}|\psi_{\theta,\phi}\rangle\langle\psi_{\theta,\phi}| &= \frac{1}{2}\left(\mathbf{1}+\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}\right)\\ &= \frac{1}{2}(\sigma_0+\sin\theta\cos\phi\sigma_x+\sin\theta\sin\phi\sigma_y+\cos\theta\sigma_z)\\ &= \sum_{j=0,x,y,z} c_j(\theta,\phi)\frac{\sigma_j}{2},\end{aligned} \tag{5}$$

we can now express Eq. (4) as

$$\begin{aligned}\bar{F} &= \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\mathrm{Tr}\left(U\left[\sum_j c_j(\theta,\phi)\frac{\sigma_j}{2}\right]U^\dagger \times\mathcal{M}\left[\sum_k c_k(\theta,\phi)\frac{\sigma_k}{2}\right]\right)\sin\theta\,d\phi\,d\theta\\ &= \sum_{jk}\left(\frac{1}{4\pi}\int_\theta\int_\phi c_jc_k\sin\theta\,d\phi\,d\theta\right)\times\mathrm{Tr}\left(U\frac{\sigma_j}{2}U^\dagger\mathcal{M}\left[\frac{\sigma_k}{2}\right]\right),\end{aligned} \tag{6}$$

where we have used the linearity of U and $\mathcal{M}$. When integrated over the Bloch sphere the coefficients of the off-diagonal terms go to zero, while the diagonal terms survive [9], leaving

$$\begin{aligned}\bar{F} &= \sum_{jk}\left(\frac{2\delta_{j0}\delta_{k0}+\delta_{jk}}{3}\right)\mathrm{Tr}\left(U\frac{\sigma_j}{2}U^\dagger\mathcal{M}\left[\frac{\sigma_k}{2}\right]\right)\\ &= \mathrm{Tr}\left(U\frac{\sigma_0}{2}U^\dagger\mathcal{M}\left[\frac{\sigma_0}{2}\right]\right) + \frac{1}{3}\sum_{j=x,y,z}\mathrm{Tr}\left(U\frac{\sigma_j}{2}U^\dagger\mathcal{M}\left[\frac{\sigma_j}{2}\right]\right)\\ &= \frac{1}{2}+\frac{1}{3}\sum_{j=x,y,z}\mathrm{Tr}\left(U\frac{\sigma_j}{2}U^\dagger\mathcal{M}\left[\frac{\sigma_j}{2}\right]\right),\end{aligned} \tag{7}$$

where we have used the unit trace of σ_0 and the fact that $\mathcal{M}$ is trace-preserving.

Expressing the average fidelity in this form may not seem helpful as the Pauli spin matrices do not represent proper states. However, in NMR experiments where the states are highly mixed, single qubit states can be represented by $\{(1/2)\sigma_x,(1/2)\sigma_y,(1/2)\sigma_z\}$ [10–12] and therefore we can use Eq. (7) directly. One application of this approach is to characterise the behaviour of composite rotation sequences [13,14], which are widely used in NMR to reduce the effects of systematic errors. In conventional NMR experiments [13] composite rotations are used to effect particular motions on the Bloch sphere (such as inversion, which takes a spin from $+z$ to $-z$), and it suffices to determine the point-to-point fidelity, but when used in NMR implementations of quantum computation [15] the initial state is unknown. One approach used to date is Levitt's quaternion fidelity [14,15] but this has the major disadvantage that it can only be used to asses the theoretical behaviour of a rotation sequence and cannot be determined by experiment. The average fidelity approach outlined above provides a simple approach which can be used for *both* theoretical *and* experimental studies.

For experimental and theoretical work with pure state techniques, we require a more appropriate form and so we use the substitutions

$$\begin{aligned}\frac{\sigma_j}{2} &= \frac{\mathbf{1}+\sigma_j}{2}-\frac{\mathbf{1}}{2}=\rho_j-\rho_0\\ &= \frac{\mathbf{1}}{2}-\frac{\mathbf{1}-\sigma_j}{2}=\rho_0-\rho_{-j},\end{aligned} \tag{8}$$

where $\rho_{\pm j}$ represents a pure state in the $(\pm j)$-direction and ρ_0 is the maximally mixed state. This gives the two equivalent expressions

$$\bar{F}=\frac{1}{2}+\frac{1}{3}\sum_{j=x,y,z}\Big(\mathrm{Tr}\big(U\rho_jU^\dagger\mathcal{M}[\rho_j]\big)-\mathrm{Tr}\big(U\rho_jU^\dagger\mathcal{M}[\rho_0]\big)\Big), \tag{9}$$

$$\bar{F}=\frac{1}{2}+\frac{1}{3}\sum_{j=x,y,z}\Big(\mathrm{Tr}\big(U\rho_{-j}U^\dagger\mathcal{M}[\rho_{-j}]\big)-\mathrm{Tr}\big(U\rho_{-j}U^\dagger\mathcal{M}[\rho_0]\big)\Big), \tag{10}$$

and taking the average of (9) and (10) yields

$$\bar{F}=\frac{1}{2}+\frac{1}{6}\sum_{j=x,y,z}\Big(\mathrm{Tr}\big(U\rho_jU^\dagger\mathcal{M}[\rho_j]\big)+\mathrm{Tr}\big(U\rho_{-j}U^\dagger\mathcal{M}[\rho_{-j}]\big)-\mathrm{Tr}\big(U(\rho_j+\rho_{-j})U^\dagger\mathcal{M}[\rho_0]\big)\Big)$$

$$
\begin{aligned}
&= \frac{1}{2} + \frac{1}{6} \sum_{j=x,y,z} \Big(\mathrm{Tr}\big(U\rho_j U^\dagger \mathcal{M}[\rho_j]\big) \\
&\qquad + \mathrm{Tr}\big(U\rho_{-j} U^\dagger \mathcal{M}[\rho_{-j}]\big) \\
&\qquad - 2\,\mathrm{Tr}\big(U\rho_0 U^\dagger \mathcal{M}[\rho_0]\big)\Big) \\
&= \frac{1}{2} + \frac{1}{6} \sum_{j=x,y,z} \Big(\mathrm{Tr}\big(U\rho_j U^\dagger \mathcal{M}[\rho_j]\big) \\
&\qquad + \mathrm{Tr}\big(U\rho_{-j} U^\dagger \mathcal{M}[\rho_{-j}]\big) - 1\Big) \\
&= \frac{1}{6} \sum_{j=\pm x,\pm y,\pm z} \Big(\mathrm{Tr}\big(U\rho_j U^\dagger \mathcal{M}[\rho_j]\big)\Big). \qquad (11)
\end{aligned}
$$

Hence, the fidelity of the map $\mathcal{M}$ with the unitary or anti-unitary map U can be calculated by simply averaging the fidelities of the six axial pure states on the Bloch sphere, $\{\rho_{+x}, \rho_{-x}, \rho_{+y}, \rho_{-y}, \rho_{+z}, \rho_{-z}\}$. We note that the average map fidelity ($\bar{F}$) can in fact be characterized by only four pure states, $\{(1/2)(1+(1/\sqrt{3})(+\sigma_x+\sigma_y+\sigma_z)), (1/2)(1+(1/\sqrt{3})(-\sigma_x-\sigma_y+\sigma_z)), (1/2)(1+(1/\sqrt{3})(-\sigma_x+\sigma_y-\sigma_z)), (1/2)(1+(1/\sqrt{3})(+\sigma_x-\sigma_y-\sigma_z))\}$. Indeed, the fidelity can be characterized using any four pure states forming a regular tetrahedron, or any six forming a regular octahedron; however, the pure states at the six cardinal points provide a particularly natural approach.

An obvious application of this result is to compare a desired unitary operation with its actual implementation that (due to experimental imperfections) may be more closely represented by a superoperator. A practical advantage of characterizing the fidelity by just testing six states is that this approach provides a simple means to verify the map fidelity by experiment. Similarly, we can also use this result to calculate the fidelity of a unitary or superoperator approximation to an anti-unitary map [4] in a convenient and intuitive manner.

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