

Actividad No. (3)
2024 Cálculo Vectorial.
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1 : De la página 963 del libro que seguimos de Cálculo de Varias Variables de Stewart, entre el ejercicio 3 y el ejercicio 12, escoja y resuelva 5 ejercicios.

a : $f(x, y) = 3x + y; x^2 + y^2 = 10$

$$\begin{aligned} F(x, y) &= 3x + y \\ G(x, y) &= x^2 + y^2 - 10 \end{aligned}$$

$$\begin{aligned} \nabla F &< F_x, F_y > = < 3, 1 >; \nabla G < G_x, G_y > = < 2x, 2y > \\ \nabla F &= \lambda \nabla G \end{aligned}$$

$$\begin{aligned} 2\lambda x &= 3 \\ 2\lambda y &= 1 \\ x^2 + y^2 &= 10 \end{aligned}$$

$$(i)x = -\frac{3}{2\lambda} \quad (ii)y = -\frac{1}{2\lambda} \quad (iii) \left(-\frac{3}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 = 10$$

$$\left(-\frac{3}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 = 10$$

$$\frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10$$

$$\frac{9}{4\lambda^2} \cdot 4\lambda^2 + \frac{1}{4\lambda^2} \cdot 4\lambda^2 = 10 \cdot 4\lambda^2 \quad \text{simplificando : } 10 = 40\lambda^2$$

$$\begin{aligned} \lambda^2 &= \frac{1}{4} \\ \lambda &= \sqrt{\frac{1}{4}}, \lambda = -\sqrt{\frac{1}{4}} \\ \lambda &= \frac{1}{2}, \lambda = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{Para : } \lambda &= \frac{1}{2} \\ x &= -\frac{3}{2\lambda} = -\frac{3}{2 \cdot \left(\frac{1}{2}\right)} = -3 \quad y = -\frac{1}{2\lambda} = -\frac{1}{2 \cdot \left(\frac{1}{2}\right)} = -1 \end{aligned}$$

$$\begin{aligned} \text{Para : } \lambda &= -\frac{1}{2} \\ x &= -\frac{3}{2\lambda} = -\frac{3}{2 \cdot \left(-\frac{1}{2}\right)} = 3 \quad y = -\frac{1}{2\lambda} = -\frac{1}{2 \cdot \left(-\frac{1}{2}\right)} = 1 \end{aligned}$$

R// Puntos críticos: $(-3, -1)$ y $(3, 1)$ $\lambda = -\frac{1}{2}; \frac{1}{2}$

b : $f(x, y) = x^2 + y^2; xy = 1$

$$\begin{aligned} F(x, y) &= x^2 + y^2 \\ G(x, y) &= xy - 1 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x}(x^2 + y^2) &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) = 2x & \frac{\partial}{\partial y}(x^2 + y^2) &= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) = 2y \\ \frac{\partial}{\partial x}(xy - 1) &= \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial x}(1) = y & \frac{\partial}{\partial y}(xy - 1) &= \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial y}(1) = x \end{aligned}$$

$$\begin{aligned} \nabla F &< F_x, F_y > = < 2x, 2y >; \nabla G < G_x, G_y > = < y, x > \\ \nabla F &= \lambda \nabla G \end{aligned}$$

$$\begin{aligned}2x + \lambda y &= 0 \\2y + \lambda x &= 0 \\xy - 1 &= 0\end{aligned}$$

$$\begin{aligned}(i) \quad \lambda &= -\frac{2x}{y} \\(ii) \quad 2y &= -\frac{\left(-\frac{2x}{y}\right)^2 y}{2} - \frac{2x^2}{y} \\2y^2 &= -2x^2 \\y &= \pm x\end{aligned}$$

$$\begin{aligned}(iii) \quad xy &= 1 \\x \frac{1}{x} &= 1 \text{ se cumple sí } x \neq 0 \\\lambda &= -\frac{2(1)}{(1)} = -2 \\\lambda &= -\frac{2(-1)}{(-1)} = -2\end{aligned}$$

$$\mathbf{R// Puntos críticos:} \quad (1, 1) \text{ y } (-1, -1) \quad \lambda = -2$$

$$c : f(x, y) = e^{xy}; x^3 + y^3 = 16$$

$$\begin{aligned}F(x, y) &= e^{xy} \\G(x, y) &= x^3 + y^3 - 16\end{aligned}$$

$$\frac{\partial}{\partial x}(e^{xy}) = e^{xy} \frac{\partial}{\partial x}(xy) = e^{xy} y; \quad \frac{\partial}{\partial y}(e^{xy}) = e^{xy} \frac{\partial}{\partial y}(xy) = e^{xy} x;$$

$$\frac{\partial}{\partial x}(x^3 + y^3 - 16) = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(16) = 3x^2; \quad \frac{\partial}{\partial y}(x^3 + y^3 - 16) = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(16) = 3y^2;$$

$$\begin{aligned}\nabla F &< Fx, Fy > = < e^{xy}y, e^{xy}x >; \nabla G < Gx, Gy > = < 3x^2, 3y^2 > \\\nabla F &= \lambda \nabla G\end{aligned}$$

$$\begin{aligned}e^{xy}y &= \lambda 3x^2 \\e^{xy}x &= \lambda 3y^2 \\x^3 + y^3 - 16 &= 0\end{aligned}$$

$$(i) \quad \lambda = \frac{e^{xy}y}{3x^2}; \quad x \neq 0 \quad (ii) \quad \lambda = \frac{e^{xy}x}{3y^2}; \quad y \neq 0 \rightarrow \quad \frac{e^{xy}y}{3x^2} = \frac{e^{xy}x}{3y^2} = \lambda \rightarrow \quad ; \frac{y}{x^2} = \frac{x}{y^2}; \rightarrow \quad y^3 = x^3;$$

$$(iii) \quad x^3 + x^3 - 16 = 0 \rightarrow \quad 2x^3 = 16 \rightarrow \quad x^3 = 8 \rightarrow \quad x = 2;$$

$$(ii) \quad y^3 = (2)^3 \rightarrow \quad y^3 = 8 \rightarrow \quad y = \sqrt[3]{8} = 2$$

$$(i) \quad \lambda = \frac{e^{xy}y}{3x^2} \rightarrow \quad \lambda = \frac{e^{2(2)} \cdot 2}{3(2)^2} \rightarrow \quad \lambda = \frac{e^4}{6} = 9,0996$$

$$\mathbf{R// Punto crítico:} \quad (2, 2) \quad \lambda = \frac{e^4}{6} = 9,0996$$

$$d : f(x, y, z) = x^2 y^2 z^2; x^2 + y^2 + z^2 = 1$$

$$\begin{aligned}F(x, y, z) &= x^2 y^2 z^2 \\G(x, y, z) &= x^2 + y^2 + z^2 - 1\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}(x^2 y^2 z^2) &= y^2 z^2 \frac{\partial}{\partial x}(x^2) = 2xy^2 z^2; & \frac{\partial}{\partial y}(x^2 y^2 z^2) &= x^2 z^2 \cdot 2y^{2-1} = 2x^2 y z^2; \\ \frac{\partial}{\partial z}(x^2 y^2 z^2) &= x^2 y^2 \frac{\partial}{\partial z}(z^2) = 2x^2 y^2 z;\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 1) &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial x}(1) = 2x; \\ \frac{\partial}{\partial y}(x^2 + y^2 + z^2 - 1) &= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial y}(1) = 2y; \\ \frac{\partial}{\partial z}(x^2 + y^2 + z^2 - 1) &= \frac{\partial}{\partial z}(x^2) + \frac{\partial}{\partial z}(y^2) + \frac{\partial}{\partial z}(z^2) - \frac{\partial}{\partial z}(1) = 2z;\end{aligned}$$

$$\begin{aligned}\nabla F < Fx, Fy, Fz > &= < 2xy^2 z^2, 2x^2 y z^2, 2x^2 y^2 z >; \nabla G < Gx, Gy, Gz > = < 2x, 2y, 2z > \\ \nabla F &= \lambda \nabla G\end{aligned}$$

$$\begin{aligned}2xy^2 z^2 &= \lambda 2x; \\ 2x^2 y z^2 &= \lambda 2y; \\ 2x^2 y^2 z &= \lambda 2z; \\ x^2 + y^2 + z^2 &= 1\end{aligned}$$

$$\begin{aligned}(i) \quad \lambda &= y^2 z^2; \quad (ii) \lambda = x^2 z^2; \quad (iii) \lambda = x^2 y^2 \rightarrow x^2 = y^2 = z^2 \\ (iiii) \quad 3x^2 &= 1; \quad x^2 = \frac{1}{3}; \quad x = \pm \frac{1}{\sqrt{3}}; \rightarrow y = \pm \frac{1}{\sqrt{3}}; \quad z = \pm \frac{1}{\sqrt{3}};\end{aligned}$$

$$\lambda = y^2 z^2 = \left(\frac{1}{\sqrt{3}}\right)^2 * \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

R// Puntos críticos:

$$\begin{aligned}&\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ &\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \\ &\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \\ &\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \\ &\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ &\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \\ &\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ &\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\ &\lambda = \frac{1}{9}\end{aligned}$$

$$e : f(x, y, z) = xyz; x^2 + 2y^2 + 3z^2 = 6$$

$$\begin{aligned}F(x, y, z) &= xyz \\ G(x, y, z) &= x^2 + 2y^2 + 3z^2 - 6\end{aligned}$$

$$\frac{\partial}{\partial x}(xyz) = yz \frac{\partial}{\partial x}(x) = yz; \quad \frac{\partial}{\partial y}(xyz) = xz \frac{\partial}{\partial y}(y) = xz; \quad \frac{\partial}{\partial z}(xyz) = xy \frac{\partial}{\partial z}(z) = xy;$$

$$\begin{aligned}\frac{\partial}{\partial x}(x^2 + 2y^2 + 3z^2 - 6) &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(2y^2) + \frac{\partial}{\partial x}(3z^2) - \frac{\partial}{\partial x}(6) = 2x; \\ \frac{\partial}{\partial y}(x^2 + 2y^2 + 3z^2 - 6) &= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial y}(3z^2) - \frac{\partial}{\partial y}(6) = 4y; \\ \frac{\partial}{\partial z}(x^2 + 2y^2 + 3z^2 - 6) &= \frac{\partial}{\partial z}(x^2) + \frac{\partial}{\partial z}(2y^2) + \frac{\partial}{\partial z}(3z^2) - \frac{\partial}{\partial z}(6) = 6z;\end{aligned}$$

$$\begin{aligned}\nabla F < Fx, Fy, Fz > &= < yz, xz, xy >; \nabla G < Gx, Gy, Gz > = < 2x, 4y, 6z > \\ \nabla F &= \lambda \nabla G\end{aligned}$$

$$\begin{aligned}
yz &= \lambda 2x \\
xz &= \lambda 4y \\
xy &= \lambda 6z \\
x^2 + 2y^2 + 3z^2 &= 6
\end{aligned}$$

$$(i) \quad \lambda = \frac{yz}{2x} \quad (ii) \quad \lambda = \frac{xz}{4y} \quad (iii) \quad \lambda = \frac{xy}{6z}$$

$$(i) \& (ii) \quad \frac{yz}{2x} = \frac{xz}{4y} \rightarrow 4y^2 = 2x^2 \rightarrow x^2 = 2y^2$$

$$(i) \& (iii) \quad \frac{yz}{2x} = \frac{xy}{6z} \rightarrow 3z^2 = y^2$$

$$2y^2 + 6z^2 + 3z^2 = 6$$

$$2(3z^2) + 9z^2 = 6$$

$$6z^2 + 9z^2 = 6$$

$$15z^2 = 6$$

$$z^2 = \frac{6}{15}$$

$$z^2 = \frac{2}{5}$$

$$z = \pm \sqrt{\frac{2}{5}};$$

$$y^2 = 3z^2 = 3 \left(\frac{2}{5} \right) = \frac{6}{5} \quad y = \pm \sqrt{\frac{6}{5}};$$

$$x^2 = 2y^2 = 2 \left(\frac{6}{5} \right) = \frac{12}{5}$$

$$x = \pm \sqrt{\frac{12}{5}};$$

$$\lambda = \frac{\sqrt{\frac{6}{5}} \cdot \sqrt{\frac{2}{5}}}{2 \cdot \sqrt{\frac{12}{5}}} = \frac{\sqrt{\frac{12}{25}}}{2 \cdot \sqrt{\frac{12}{5}}} = \frac{\sqrt{12} \cdot 5}{\sqrt{25} \cdot \sqrt{24}} = \frac{5\sqrt{12}}{5\sqrt{24}} = \frac{\sqrt{12}}{\sqrt{24}} = \frac{\sqrt{12}}{\sqrt{2 \cdot 12}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

R// Puntos críticos:

$$(x, y, z) = \left(\sqrt{\frac{12}{5}}, \sqrt{\frac{6}{5}}, \sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2},$$

$$(x, y, z) = \left(-\sqrt{\frac{12}{5}}, -\sqrt{\frac{6}{5}}, -\sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2},$$

$$(x, y, z) = \left(\sqrt{\frac{12}{5}}, -\sqrt{\frac{6}{5}}, -\sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2},$$

$$(x, y, z) = \left(-\sqrt{\frac{12}{5}}, \sqrt{\frac{6}{5}}, \sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2},$$

$$(x, y, z) = \left(-\sqrt{\frac{12}{5}}, -\sqrt{\frac{6}{5}}, \sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2},$$

$$(x, y, z) = \left(\sqrt{\frac{12}{5}}, \sqrt{\frac{6}{5}}, -\sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2},$$

$$(x, y, z) = \left(-\sqrt{\frac{12}{5}}, \sqrt{\frac{6}{5}}, -\sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2},$$

$$(x, y, z) = \left(\sqrt{\frac{12}{5}}, -\sqrt{\frac{6}{5}}, \sqrt{\frac{2}{5}} \right), \quad \lambda = \frac{\sqrt{2}}{2}.$$

2 : Encuentre por el método de los multiplicadores de Lagrange, los puntos críticos de las funciones sujetas a las restricciones indicadas:

a : $f(x, y, z) = 2x^2 + xy + y^2 + z$ sujeto a $x + 2y + 4z = 3$

$$L(x, y, z, \lambda) = 2x^2 + xy + y^2 + z - \lambda(x + 2y + 4z - 3)$$

$$\begin{aligned} \frac{\partial L}{\partial x} (2x^2) + \frac{\partial}{\partial x} (xy) + \frac{\partial}{\partial x} (y^2) + \frac{\partial}{\partial x} (z) - \frac{\partial}{\partial x} (\lambda(x + 2y + 4z - 3)) \\ \frac{\partial L}{\partial x} = 4x + y + 0 + 0 - \lambda \\ \frac{\partial L}{\partial x} = 4x + y - \lambda \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial y} (2x^2) + \frac{\partial}{\partial y} (xy) + \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial y} (z) - \frac{\partial}{\partial y} (\lambda(x + 2y + 4z - 3)) \\ \frac{\partial L}{\partial y} = 0 + x + 2y + 0 - 2\lambda \\ \frac{\partial L}{\partial y} = x + 2y - 2\lambda \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial z} (2x^2) + \frac{\partial}{\partial z} (xy) + \frac{\partial}{\partial z} (y^2) + \frac{\partial}{\partial z} (z) - \frac{\partial}{\partial z} (\lambda(x + 2y + 4z - 3)) \\ \frac{\partial L}{\partial z} = 0 + 0 + 0 + 1 - 4\lambda \\ \frac{\partial L}{\partial z} = 1 - 4\lambda \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \lambda} (2x^2) + \frac{\partial}{\partial \lambda} (xy) + \frac{\partial}{\partial \lambda} (y^2) + \frac{\partial}{\partial \lambda} (z) - \frac{\partial}{\partial \lambda} (\lambda(x + 2y + 4z - 3)) \\ \frac{\partial L}{\partial \lambda} = 0 + 0 + 0 + 0 - (x + 2y + 4z - 3) \\ \frac{\partial L}{\partial \lambda} = x + 2y + 4z - 3 = 0 \end{aligned}$$

$$4x + y - \lambda = 0 \quad (i)$$

$$x + 2y - 2\lambda = 0 \quad (ii)$$

$$1 - 4\lambda = 0 \quad (iii)$$

$$x + 2y + 4z - 3 = 0 \quad (iv)$$

$$(iii), 1 - 4\lambda = 0 \quad \rightarrow \quad \lambda = \frac{1}{4}.$$

(i) y (ii), resolvemos para x y y .

$$x = 0, \quad y = \frac{1}{4}$$

$$(iv) \quad x = 0, \quad y = \frac{1}{4}, \quad \text{y } \lambda = \frac{1}{4} \text{ resolvemos } z:$$

$$z = \frac{5}{8}$$

R// Puntos críticos:

$$x = 0, \quad y = \frac{1}{4}, \quad z = \frac{5}{8}, \quad \text{y } \lambda = \frac{1}{4}$$

b : $f(x, y, z) = xyz^2$ sujeto a $x - y + z = 20$

$$L(x, y, z, \lambda) = xyz^2 - \lambda(x - y + z - 20)$$

$$\frac{\partial L}{\partial x} = yz^2 - \lambda = 0$$

$$\frac{\partial L}{\partial y} = xz^2 + \lambda = 0$$

$$\frac{\partial L}{\partial z} = 2xyz - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x - y + z - 20 = 0$$

$$yz^2 - \lambda = 0 \quad (i)$$

$$xz^2 + \lambda = 0 \quad (ii)$$

$$2xyz - \lambda = 0 \quad (iii)$$

$$x - y + z - 20 = 0 \quad (iv)$$

para: λ

$$(i) \quad \lambda = yz^2$$

$$(ii) \quad \lambda = -xz^2$$

$$(iii) \quad \lambda = 2xyz$$

$$yz^2 = -xz^2 \quad (i)$$

$$yz^2 = 2xyz \quad (ii)$$

(iv):

$$x - y + z - 20 = 0 \quad (iii)$$

R//: Puntos críticos:

$$(x, y, z) = (0, 0, 20)$$

$$(x, y, z) = (5, -5, 10)$$

$$(x, y, z) = (10, -10, 0)$$

$$(x, y, z) = (y + 20, y, 0) \quad \text{para cualquier valor de } y$$