Nozzle Theory

Information regarding the design of optimal conical converging-diverging nozzles for DukeAERO.

Definition of Terms

To simplify variable naming, we will use the subscript convention where c denotes chamber conditions, t denotes throat conditions, and e denotes exit conditions.

Parameter	Symbol	Unit
Pressure	P	Pa
Area	A	m^2
Temperature	T	K
Specific Gas Constant	R	$J\ kg^{-1}\ K^{-1}$
Speed of Sound	a	$m\ s^{-1}$
Flow Velocity Magnitude	v	$m\ s^{-1}$
Mach Number	M	dimensionless
Force	F	N
Mass Flow Rate	$\dot{m}$	$kg\ s^{-1}$
Density	ρ	$kg\ m^{-3}$
Ratio of Specific Heats	γ	dimensionless

It is common practice to talk about pressures in terms of PSI, however in all equations pressures must be given in Pa.

Calculating Design Parameters

Chamber Pressure

Chamber pressure is the pressure of the gas in the combustion chamber. Higher chamber pressures lead to increased thrust and efficiency and shorter burn times. However, they also require stronger pressure vessels. As such, choosing a chamber pressure is a complicated task, but the entire nozzle's design and performance depends directly upon it.

A simple method for choosing a chamber pressure is to design a pressure vessel and choose a required factor of safety. Factor of safety (FOS) is the ratio between a component's failure condition and its nominal operating condition. In this case, the failure condition is the pressure at which the motor breaks, and the nominal operating condition is the chamber pressure. FOS is typically between 1.5 and 2.5, so if we design a motor that fails at 1400 PSI and require an FOS of 2, we find our chamber pressure as

$P_c = 1400 / 2 = 700$ PSI

More complicated methods might include calculating required thrust and burn duration based on estimated rocket mass, and then guessing a chamber pressure and comparing the resulting thrust and duration to the required values in an iterative guess-and-check approach.

Mass Flow Rate

Mass flow rate is the amount of mass flowing through the nozzle per second. It is directly determined by your chamber pressure, grain geometry, and propellant density. Mass flow through each cross-sectional slice of the nozzle is the same, and as such $\dot{m}$ is required in the flow state calculations in the following sections.

To find $\dot{m}$, we first calculate the burn rate r_b by Vieille's Law, where

$r_b = k \cdot (P_c)^n$

Where k is the burn rate coefficient and n is the burn rate exponent, both determined through propellant characterization. Note that r_b has units m s^{-1}.

We also need the burn area A_b which is determined by grain geometry. For a simple BATES grain, the burn area is given b

$A_b = 2 \cdot \boldsymbol{\pi} \cdot r \cdot h$

Where r is the inner radius and h is the height of the grain.

Once we have the burn rate and burn area, we can find the mass flow as

$\dot{m} = \boldsymbol{\rho} \cdot A_b \cdot r_{b,}$

Where $\boldsymbol{\rho}$ the propellant density.

Throat Area

The throat area is the cross sectional area of the nozzle at its throat, or the narrowest part, in between the converging and diverging sections. The throat area is chosen by requiring the fluid at the throat to be sonic, with M = 1. Given this condition, we can calculate throat area as

$$A_t = (\dot{m} \cdot \sqrt{(T_c)} / P_c) \cdot \sqrt{(R / \gamma)} \cdot ((\gamma + 1) / 2)^{(\gamma + 1) / (2 (\gamma - 1))}$$

Exit Area

The exit area is the cross sectional area of the nozzle at its exit. Here, we want the gas to be ideally expanded, such that the pressure of the exhaust is equal to the ambient pressure. If the exhaust pressure is higher than the ambient pressure, then the nozzle is underexpanded. If the exhaust pressure is lower than the ambient pressure, then the nozzle is overexpanded. In both scenarios, the exit velocity decreases and performance is lost. As the rocket climbs to higher altitudes, ambient pressure will decrease. As such, it may be beneficial to consider the ambient pressure at altitude when finding exit area.

To solve for the exit area, we need to first find the exit Mach number, given by

$$M_e = \sqrt{((2 / \gamma - 1) \cdot ((P_c / P_a)^{(\gamma - 1) / \gamma} - 1))}$$

Where P_a is the ambient pressure.

Next, we can use the area-Mach relation to find the expansion ratio, the ratio between the exit area and throat area.

$$\boldsymbol{\varepsilon} = A_e / A_t = (1 / M_e) \cdot ((2 / (\gamma + 1)) \cdot (1 + ((\gamma - 1) / 2) M_e^2))^{(\gamma + 1) / (2 \cdot (\gamma - 1))}$$

Now we can find the exit area as

$$A_e = \boldsymbol{\varepsilon} \cdot A_t$$

Converging Section

The converging section of a CD nozzle is the geometry prior to the nozzle throat. The geometry is generally defined by an initial area, a final area, and a half angle. The initial area will be the cross sectional area of your combustion chamber, and the final area will be the throat area. The half angle is the acute angle between the nozzle wall and the axial direction. Typical half angles in the converging section range from 40° to 60° depending on length requirements.

Diverging Section

The diverging section of a CD nozzle is the geometry after the nozzle throat. The geometry is also generally defined by an initial area, a final area, and a half angle. The initial area will be the throat area, and the final area will be the exit area. The half angle in the diverging section is generally 15°.

Though a conical nozzle with a 15° half angle is pretty close to optimal, you can gain a small amount of performance by creating a Rao (also called bell) nozzle, which replaces the straight wall of the diverging section with a circular arc close to the throat and a parabolic curve close to the exit.

Quantifying Force

You may wish to determine how much thrust your nozzle generates. The thrust generated can be quantified as

$$F = \dot{m} \cdot v_e + (P_e - P_a) \cdot A_e$$

Note that if the nozzle is ideally expanded, then $P_e = P_a$ and the thrust reduces to

$$F = \dot{m} \cdot v_e$$

To find the exit velocity from the exit Mach number, we can use the relation

$$v_e = M_e \cdot a = M_e \cdot \sqrt{\gamma \cdot R \cdot T_e}$$

The exit temperature can be calculated using the relation

$$T_e = T_c \cdot \left(1 + \left((\gamma - 1) / 2 \right) \cdot M_e^2 \right)^{-1}$$

If the nozzle is not ideally expanded and the exit pressure is required, then it can be calculated as

$$P_e = P_c \cdot \left(1 + \left((\gamma - 1) / 2 \right) \cdot M_e^2 \right)^{-\gamma / (\gamma - 1)}$$

Notes on Constants

There are a few constants that must be found by characterizing the propellant. The first and most ubiquitous is γ, which is defined as

$$\gamma = C_p / C_v$$

Where C_p is the specific heat under constant pressure and C_v is the specific heat under constant volume. Thus γ is a fundamental constant of the gas. For air, γ is roughly 1.4, and for combustion gasses γ is generally between 1.20 and 1.25.

The second is R, the specific gas constant, which is found as

$R = R_u / mm$

Where R_u is the universal gas constant, roughly 8.314, and mm is the molar mass of the propellant.

The third is $\boldsymbol{\rho}$, the density of the propellant. Note that this is the density of the propellant mixture in its solid form in the grain, not as a gas.

The other constants that appear are k, the burn rate coefficient, and n, the burn rate exponent. Both of these must be found experimentally.