Rao Nozzles

Method for increasing performance of converging-diverging nozzles for DukeAERO.

Background

The information on this page comes from two journal articles written by G.V.R. Rao for the American Rocket Society. He developed an approach using Variational Calculus to optimize the diverging section of converging-diverging nozzles. Using computational methods, he then approximated the solutions to the Euler-Lagrange equation given various initial conditions. From this data he determined that the optimal curve could be approximated by a piecewise function consisting of a circular arc close to the throat and a parabolic curve towards the exit. This document describes the method for using his data to create your own Rao nozzle.

Required Parameters

In order to define the curve, a set of parameters must be chosen. The throat radius and expansion ratio can be found by following the process detailed in **Nozzle Theory**, the length fraction is chosen based on size requirements, and the junction and exit angles can be found by using the expansion ratio and length fraction in Fig. 1. Arc fraction does not have that noticeable of an impact on performance, so a typical range of values is given.

Parameter	Symbol	Source
Throat Radius	r_t	Choked flow condition
Expansion Ratio	ε	Area-Mach relation
Length Fraction	L_f	Chosen value (0.6-1.0)
Junction Angle	θ_m	Fig. 1
Exit Angle	θ_e	Fig. 1
Arc Fraction	α	Chosen value (0.35-0.45)

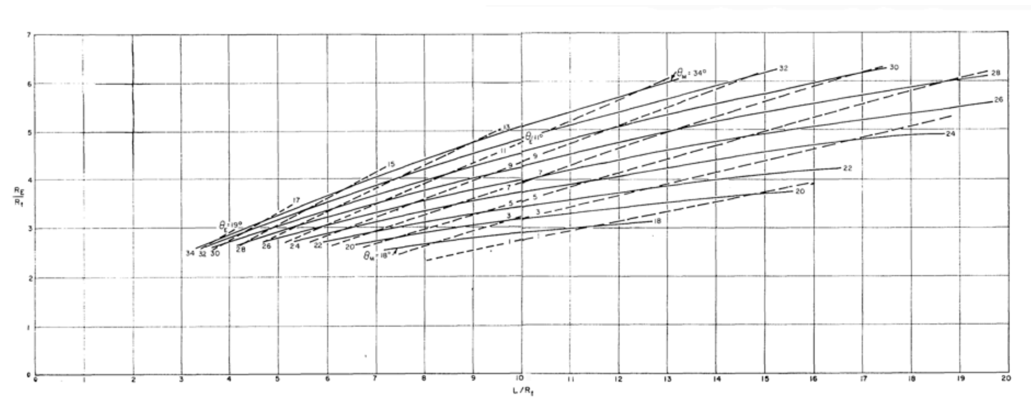

Fig. 1: Junction and exit angle from nozzle length and exit radius normalized by throat radius
(G.V.R. Rao, *Approximation of Optimum Thrust Nozzle Contour*, 1960)

Mathematical Description of Rao Nozzles

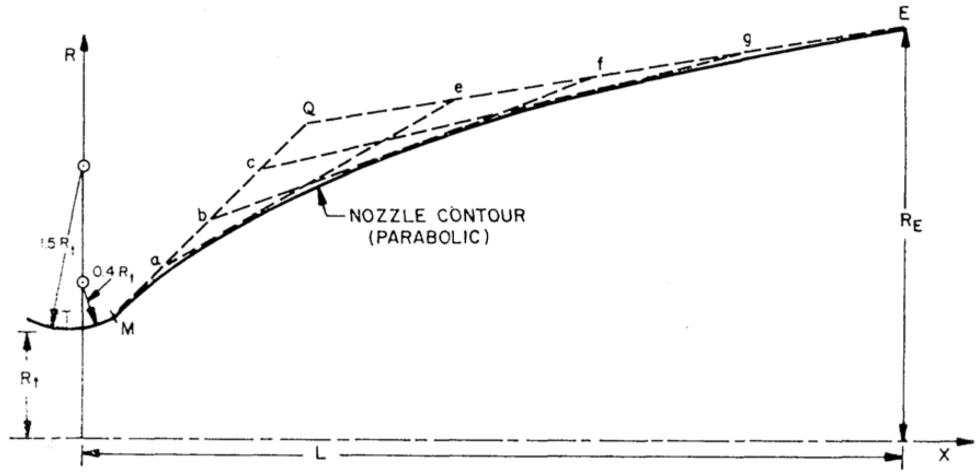

Fig. 2: Example Rao Nozzle
(G.V.R. Rao, *Approximation of Optimum Thrust Nozzle Contour*, 1960)

Circular Arc

The first section of the diverging portion of the nozzle consists of a circular arc. We find the radius of this arc as

$$r_c = \boldsymbol{\alpha} \cdot r_t$$

The arc continues until the slope of the curve matches the junction angle at point M. This curve can be described mathematically by the equations

$$x(\boldsymbol{\theta}) = r_c \cdot \sin(\boldsymbol{\theta})$$

$$r(\boldsymbol{\theta}) = r_t + r_c \cdot (1 - \cos(\boldsymbol{\theta}))$$

To find the junction point M, we can evaluate both of these expressions at $\boldsymbol{\theta}_m$. This yields the junction point as

$$x_m = r_c \cdot \sin(\boldsymbol{\theta}_m)$$

$r_m = r_t + r_c \cdot (1 - \cos(\theta_m))$

Parabolic Curve

The second section of the diverging portion of the nozzle consists of a parabolic curve. In Fig. 2, the parabola starts at point M at angle θ_m, and ends at point E with θ_e. In the previous section we calculated the coordinates of point M, so now we will find the coordinates of point E, and then the equation of the curve.

The radius at point E will be the exit radius. The exit radius can be found from the throat radius and the expansion ratio by

$r_e = r_t \cdot \sqrt{\varepsilon}$

The axial position of point E is determined by the expansion ratio and the length fraction. Length fraction is the ratio of the nozzle length to a conical nozzle with a 15° half angle. The length of a conical nozzle can be found as

$L_c = r_t \cdot (\sqrt{\varepsilon} - 1) / \tan(15°)$

Thus the axial position of point E is

$x_e = L_f \cdot L_c$

Although the curve is a parabola, it may be canted, and as such cannot be simply represented by the standard vertex form. It is therefore simpler to fit a quadratic Bezier curve, which is mathematically equivalent to a parabola, to our conditions. A quadratic Bezier curve can be defined by three points, P_0, P_1, and P_2, where P_0 and P_2 are the endpoints. As a function of a parameter t from 0 to 1, the points P on the curve are given by

$P(t) = (1 - t)^2 \cdot P_0 + 2 \cdot t \cdot (1 - t) \cdot P_1 + t^2 \cdot P_2$

Note that $P(0) = P_0$ and $P(1) = P_2$ as we would expect. If we split this into components, we have the axial component x(t) and radial component r(t) given by

$x(t) = (1 - t)^2 \cdot x_m + 2 \cdot t \cdot (1 - t) \cdot x_1 + t^2 \cdot x_e$

$r(t) = (1 - t)^2 \cdot r_m + 2 \cdot t \cdot (1 - t) \cdot r_1 + t^2 \cdot r_e$

The challenge now is to solve for $P_1 = (x_1, r_1)$, which we will find is constrained by the junction and exit angles.

Finding the slope of the curve using chain rule and setting the slope at t = 0 to $\tan(\theta_m)$ and the slope at t = 1 to $\tan(\theta_e)$, we obtain a system of two equations and two unknowns. Solving the system for x_1 and r_1 we find

$x_1 = (r_m - r_e + x_e \cdot \tan(\theta_e) - x_m \cdot \tan (\theta_m)) / (\tan(\theta_e) - \tan(\theta_m))$

$r_1 = r_e + (x_1 - x_e) \cdot \tan(\theta_e)$

Plugging this into our two equations we find the equation for the parabolic arc.

Evaluation and Testing

It should be noted that while this method does increase performance, it does so only slightly. In CFD simulation testing DukeAERO received only 1-2% performance gains in both thrust and specific impulse over an otherwise identical conical nozzle with 15° half angle. These results align with Rao's finding in his own journal entries, which list performance gains of around 2.5%.

The real strength of this method is for nozzles with high expansion ratios. A high expansion ratio conical nozzle can grow exceedingly long, increasing space and mass requirements. To decrease size and mass over conical nozzles, small length fraction Rao nozzles are used, as they provide only minimal drops in thrust output despite being far shorter and lighter.