Correcting Ampere's Law

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Abstract

One of the four fundamental equations describing electric and magnetic fields, Ampere's Law describes the magnetic field generated by a current carrying wire. However, there are some scenarios that cause Ampere's Law to deliver inconsistent results. In the attempt to fix these inconsistencies, we find a new term in the equation that provides a beautiful symmetry with Faraday's Law.

1 Examining an RC Circuit

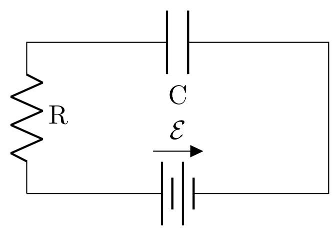

Figure 1.1: A circuit with a battery, a resistor, and a capacitor, all in series

We begin by examining the circuit shown in Figure 1.1. Let $\mathcal{E}$ be the voltage of the battery, R be the resistance of the resistor, and C be the capacitance of the capacitor. We can solve for the charge on the capacitor, $q(t)$, and the current flowing through the circuit, $I(t)$, with Kirchhoff's Voltage Law.

$$\begin{aligned}\mathcal{E} &= \Delta V_R + \Delta V_c \\ &= IR + \frac{q}{C} \\ &= \frac{dq}{dt}R + \frac{q}{C}\end{aligned}$$

The expression above is a first-order differential equation that we can solve by separation of variables.

$$\begin{aligned}\mathcal{E} &= \frac{dq}{dt}R + \frac{q}{C} \\ \frac{1}{\mathcal{E} - \frac{q}{C}}\frac{dq}{dt} &= \frac{1}{R} \\ \int \frac{1}{\mathcal{E} - \frac{q}{C}} dq &= \int \frac{1}{R} dt\end{aligned}$$

$$\begin{aligned}-Cln|\mathcal{E} - \frac{q}{C}| &= \frac{t}{R} + c_1 \\ ln|\mathcal{E} - \frac{q}{C}| &= -\frac{t}{CR} + c_2 \\ q(t) &= C(\mathcal{E} - c_3 e^{-\frac{t}{CR}})\end{aligned}$$

Letting $q(0) = 0$, we have the the charge and current as functions of time as

$$q(t) = C\mathcal{E}(1 - e^{-\frac{t}{CR}}) \tag{1}$$

$$I(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R} e^{-\frac{t}{CR}} \tag{2}$$

2 Finding the Magnetic Field

Let the curve $\mathcal{C}$ be a circle of radius r centered on the wire. Ampere's Law tells us that the path integral of the magnetic field along $\mathcal{C}$ is proportional to the current passing through a surface bounded by $\mathcal{C}$.

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 I \tag{3}$$

But what if we choose a surface that runs between the parallel plates of the capacitor? There is no current between the plates, so Ampere's Law tells us that the path integral of the magnetic field along $\mathcal{C}$ must be zero. However, we could just have easily picked a surface that does run through the wire, giving a non-zero path integral along $\mathcal{C}$. So there must be some other quantity that picks up inside the capacitor. The only other quantity in that region is the electric field, so let's calculate it.

Let d be the distance between the parallel plates, and A be the area of each plate. We also know that the capacitance C is equal to $\frac{1}{\gamma}$, where $\gamma = \frac{d}{\epsilon_0 A}$.

$$\begin{aligned} E &= -\frac{dV}{dr} \\ &= -\frac{\Delta V_C}{d} \\ &= -\frac{q}{Cd} \\ &= -\frac{q}{\epsilon_0 A} \\ q &= -\epsilon_0 A E \end{aligned}$$

If we have a left plate with charge $+q$ an a right plate with charge $-q$, then there exists an electric field pointing to the right. If current now flows away from the left plate to the left, the electric field will decrease, and because it is pointing to the right, this is equivalent to saying it will increase to the left. In this way, the current and the time derivative of the electric field should be in the same direction, so there shouldn't be a sign difference. Flipping the sign and differentiating both sides, we find

$$I = \epsilon_0 A \frac{d}{dt} E \tag{4}$$

Now combining our expression for current (4) with Ampere's Law (3)

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 \epsilon_0 A \frac{d}{dt} E$$

3 Extending the Discovery

The A here is the area of the parallel plates, but it makes sense to extend it to actually be the flux through the surface $\mathcal{S}$ bounded by $\mathcal{C}$. If we make this change, we will get the same result, as the electric field anywhere outside of the two plates is 0. Reformulating our equation, we have

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 \epsilon_0 \iint_{\mathcal{S}} \frac{d}{dt} E \cdot dA$$

I would like to pause for a moment and just notice the resemblance to Faraday's Law. They're almost identical, except for the sign.

$$\oint_{\mathcal{C}} E \cdot d\ell = -\iint_{\mathcal{S}} \frac{d}{dt} B \cdot dA$$

Now in the region where we still have current, the electric field is not necessarily changing, so we still need to account for the current. Adding the original statement of Ampere's Law back in, we arrive at

$$\oint_{\mathcal{C}} B \cdot d\ell = \mu_0 \left(I + \epsilon_0 \iint_{\mathcal{S}} \frac{d}{dt} E \cdot dA \right)$$

This equation is known as the Ampere-Maxwell Law, and the correction we found is known as Maxwell's Addition to Ampere's Law.

With this correction, we now have the four governing equations of classical electromagnetism, known collectively as Maxwell's Equations. Written out, they are

$$\begin{aligned}
\text{Gauss's Law} \quad & \oint\!\!\oint_{\mathcal{S}} E \cdot dA = \frac{q_{encl}}{\epsilon_0} \\
\text{Gauss's Law for Magnetism} \quad & \oint\!\!\oint_{\mathcal{S}} B \cdot dA = 0 \\
\text{Faraday's Law} \quad & \oint_C E \cdot d\ell = -\iint_{\mathcal{S}} \frac{d}{dt} B \cdot dA \\
\text{Ampere-Maxwell Law} \quad & \oint_C B \cdot d\ell = \mu_0 \left(I + \epsilon_0 \iint_{\mathcal{S}} \frac{d}{dt} E \cdot dA \right)
\end{aligned}$$

Let's consider empty space, without any charges or currents. Reducing the four equations with these conditions yields

$$\oint\!\!\oint_{\mathcal{S}} E \cdot dA = 0 \qquad \oint_C E \cdot d\ell = -\iint_{\mathcal{S}} \frac{d}{dt} B \cdot dA$$

$$\oint\!\!\oint_{\mathcal{S}} B \cdot dA = 0 \qquad \oint_C B \cdot d\ell = \mu_0 \epsilon_0 \iint_{\mathcal{S}} \frac{d}{dt} E \cdot dA$$

Now, imagine if we have a circular loop L_1 with normal vector $\hat{\imath}$ and center P_1, pictured in Figure 3.1. Suppose now that there is also a time-varying electric field that increases in the $\hat{\imath}$ direction. Then there is a time-varying electric flux passing through L_1. By the Ampere-Maxwell Law, there must also be a magnetic field around L_1.

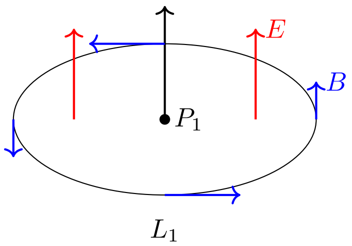

Figure 3.1: A loop L_1 with normal vector $\hat{\imath}$ and center P_1

Now, take a loop L_2 identical to L_1, but place its center somewhere on L_1 with normal vector $\hat{\jmath}$, pictured in Figure 3.2. Because there is a magnetic field B on L_1, there is a magnetic flux through L_2. If B now varies over time, Faraday's Law states that there is also an electric field around L_2, but in the opposite direction of the rate of change of the magnetic flux. In this case the magnetic flux is increasing into the page, so the electric field must rotate out of the page.

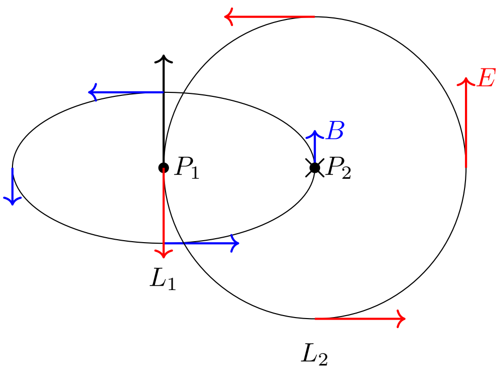

Figure 3.2: A second loop L_2 with normal vector $\hat{\jmath}$ and center P_2 on L_1

Once again, take a loop L_3 identical to the previous two and place its center somewhere on L_2 with normal vector $\hat{\imath}$, pictured in Figure 3.3. Because there is a electric field on L_2, there is an electric flux through L_3. If E now varies over time, the Ampere-Maxwell's Law states that there is also a magnetic field around L_3.

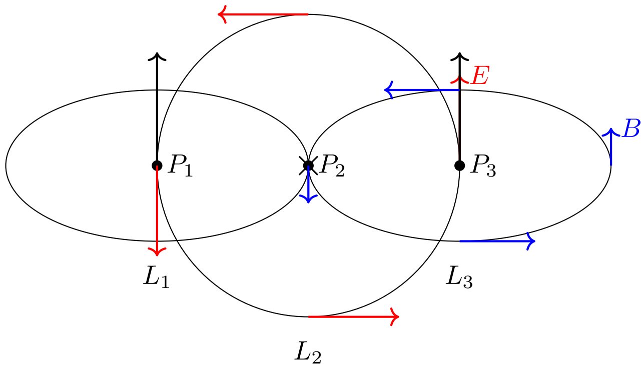

Figure 3.3: A third loop L_3 with normal vector $\hat{\imath}$ and center P_3 on L_2

At this point you can probably guess what happens next. We begin the whole sequence again, acting as though L_3 were L_1, and a similar thing will occur. What we have just described is the mechanism through which electric and magnetic fields propagate through space. What we are looking at here is a conceptual understanding of the mathematical description behind electromagnetic waves–light! If you still need to be convinced that what we are looking at here is light, let's consider one last result.

Faraday's Law and the Ampere-Maxwell Law are very similar, except for the sign and $\mu_0\epsilon_0$. It turns out that $\mu_0 \approx 1.3 \cdot 10^{-6}\frac{H}{m}$ and $\epsilon_0 \approx 9 \cdot 10^{-12}\frac{F}{m}$. So what is their product?

$$\mu_0\epsilon_0 \approx 1.2 \cdot 10^{-17}\frac{HF}{m^2}$$

That doesn't really seem like anything. What if we take the reciprocal?

$$\frac{1}{\mu_0\epsilon_0} \approx 8.6 \cdot 10^{16}\frac{m^2}{HF}$$

That seems nonsensical too. Also, what actually is a Henry times a Farad?

$$HF = \left(\frac{kg \cdot m^2}{s^2 \cdot A^2}\right)\left(\frac{s^4 \cdot A^2}{kg \cdot m^2}\right) = s^2$$

Oh, interesting! So $\frac{1}{\mu_0\epsilon_0}$ has units $\frac{m^2}{s^2}$? If we take the square root, then we are left with an object that has the units $\frac{m}{s}$.

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} \approx 2.9 \cdot 10^8\frac{m}{s}$$

It's the speed of light!