Assignment 3: MA211 Discrete Mathematics
(Functions)

1. Let
$$g(x) = \begin{cases} 2|x| + 3, & \text{if } x \leq 0 \\ 5 & \text{if } 0 < x \leq 3 \\ -x^2 & \text{otherwise} \end{cases}$$
Compute the following:
   (a) $g(-3.4)$
   (b) $g(0)$
   (c) $g(3)$

2. Let $n \in \mathbb{N}$. A positive integer d is a proper factor of n if d is a factor of n and $d < n$. For example, the proper factors of 12 are 1, 2, 3, 4, and 6. Let $\sigma : \mathbb{N} \to \mathbb{N}$ defined by $\sigma(n) =$ sum of the proper factors of n. (σ is the lowercase Greek letter, sigma.) Compute $\sigma(n)$ for each value of n, where p and q are distinct primes. [A positive integer n such that $\sigma(n) = n$ is a perfect number.]
   (a) 6
   (b) 38
   (c) 39
   (d) pq
   (e) p^2

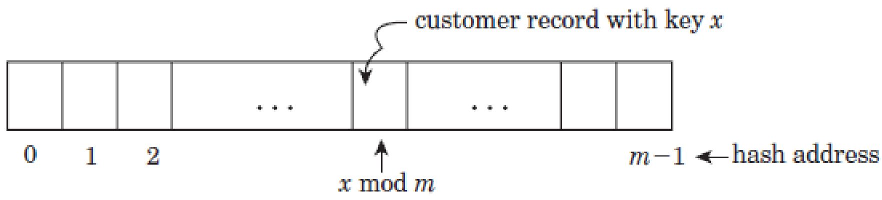
3. If $B \subseteq A \subseteq X$, then $f(A) - f(B) \subseteq f(A - B)$.

4. A **characteristic function** over a set S is defined as
$$f_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0, \text{ otherwise.} \end{cases}$$
For such a function prove that $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_{A\cap B}(x)$.

5. Let $x = 3.456$ and $y = 2.789$. Compute each.
   (a) $\lfloor x + y \rfloor$
   (b) $\lfloor x \rfloor + \lceil y \rceil$
   (c) -$\lfloor x \rfloor + \lfloor -y \rfloor$
   (d) $\lceil xy \rceil$
   (e) $\lfloor x \rfloor \lfloor y \rfloor$

6. The **mod function** $f(x, y) = x \mod y$ denotes the *remainder* when an integer x is divided by a positive integer y. For example, 23 mod 5=3. Today is Thursday. Use mod function to find the day of the week after 100 days from today?

7. Banks use nine-digit account numbers to create and maintain customer accounts. Customer records are stored in an array in a computer and can be accessed fairly easily and quickly using their unique keys, which in this case are the account numbers. Access is often accomplished using the hashing function h(x) = x mod m, where x denotes the key (account number) and m the number of cells in the array; h(x) denotes the hash address of the customer record with key x.



Let $m = 1009$, then the key $x = 207630764$ (account number) will be stored at location 762 as $h(207630764) = 762$.
Student records are maintained in a table using the hashing function h(x) = x mod 9767, where x denotes the student's social security number. Compute the location in the table corresponding to the given key, where the record is stored.

   (a) 012-34-5678
   (b) 876-54-3210

8. The **div function** $g(x, y) = x \text{ div } y$ denotes the *quotient* when x is divided by y. For example, 23 div 5=4.
Consider a standard deck of 52 playing cards. They are originally assigned the numbers 0 through 51 in order. Use the suit labels 0 = clubs, 1 = diamonds, 2 = hearts, and 3 = spades to identify each suit, and the card labels 0 = ace, 1 = deuce, 2 = three, . . . , and 12 = king to identify the cards in each suit. Suppose card x is drawn at random from a well-shuffled deck, where $0 \leq x \leq 51$. How do we identify the cards?
First, we need to determine the suit to which the card belongs. It is given by x div 13. Next, we need to determine the card within the suit; this is given by x mod 13. Thus card x is card (x mod 13) in suit (x div 13). What is the identity of card 50.

9. Two finite sets have the same cardinality if and only if there exists a bijection between them.

10. Prove the following:

   (a) If fg is surjective then f is surjective.
   (b) If fg is injective then g is injective.
   (c) If fg is bijective then f is surjective and g is injective.

11. Prove that any set S of three integers contains at least two integers whose sum is even.

12. If 10 points are selected inside an equilateral triangle of unit side, then at least two of them are no more than 1/3 of a unit apart.

13. Mark each sentence as true or false. Assume the composites and inverses are defined:

(a) Every function is invertible.

(b) Every invertible function is bijective.

(c) The composition of two injections is injective.

(d) The composition of two invertible functions is invertible.

14. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Prove that $(fog)^{-1} = g^{-1} o f^{-1}$. This property is called the **Shoe-sock property** as we first wear socks then shoes but we take them off in opposite order.

15. Using the big-oh notation, estimate the growth of each function.

(a) $f(n) = \lg(5n!)$

(b) $f(n) = \sum_{i=1}^{n} \lfloor i/2 \rfloor$

(c) $f(n) = \sum_{i=1}^{n} \lceil i/2 \rceil$

16. Verify:

(a) $2^n = O(n!)$

(b) $\sum_{i=1}^{n} i^k = O(n^{k+1})$

(c) $\sum_{i=1}^{n} \frac{1}{n(n+1)} = O(1)$.

17. Let $f(n) = O(h(n))$ and $g(n) = O(h(n))$. Prove that $(f.g)(n) = O((h(n))^2)$.

18. Let $f, g : \mathbb{N} \to \mathbb{R}$. Prove that $f(n) = O(g(n))$ if and only if $A|g(n)| \leq |f(n)| \leq B|g(n)|$ for some constants A and B.