example1

December 6, 2023

1 Example 1

1.1 Problem

$$(OCP)_1 \begin{cases} Min \int_0^{t_f} x\sqrt{1+u^2}dt \\ \dot{x} = u \\ u \in \end{cases}$$

[11]:
```
# Packages

using Pkg
Pkg.activate(".")
#
using OptimalControl
using LinearAlgebra
using ForwardDiff
using DifferentialEquations
using Roots      # solve an equation f(x)=0 where f is from R to R
# using MINPACK # NLE solver
# using NLsolve
using LaTeXStrings

using Test
```

```
  Activating project at `~/control-toolbox/indirect`
```

1.2 Control-toolbox definition of the problem

[12]:
```
t0 = 0
tf = 2
x0 = 1
xf = 1
@def ocp begin
    t âĹĹ [ t0, tf ], time
    x âĹĹ R, state
```

```
    u âĹĹ R, control
    x(t0) == x0
    x(tf) == xf
    xĪĞ(t) ==  u(t)
    âĹń(x(t)*(1 + u(t)^2)^(1/2)) âĘŠ min
end
```

```
t âĹĹ [t0, tf], time
x âĹĹ R, state
u âĹĹ R, control
x(t0) == x0
x(tf) == xf
ážŃ(t) == u(t)
âĹń(x(t) * (1 + u(t) ^ 2) ^ (1 / 2)) âĘŠ min
```

```
âŤŇaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤňaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤňaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤňaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂ
âŤĆ times âŤĆ state âŤĆ control âŤĆ dynamics âŤĆ

objective âŤĆ constraints âŤĆ
âŤĲaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤijaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤijaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤijaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂ
âŤĆ     âĲĚ âŤĆ     âĲĚ âŤĆ       âĲĚ âŤĆ        âĲĚ âŤĆ          âĲĚ
 ↪âŤĆ
âŤŤaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤťaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤťaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂaŤťaŤĂaŤĂaŤĂaŤĂaŤĂaŤĂ
```

1.3 Hamiltonian flow

The Hamiltonian is

$$H(x,u,p) = -x\sqrt{1+u^2} + up,$$

and for $|p| < |x|$ the maximization of the Hamiltonian have the solution

$$u(x,p) = sign(x)p\sqrt{1./(x^2-p^2)}.$$

Then the true Hamiltonian is $H_r(x,p) = H(x,u(x,p),p)$.

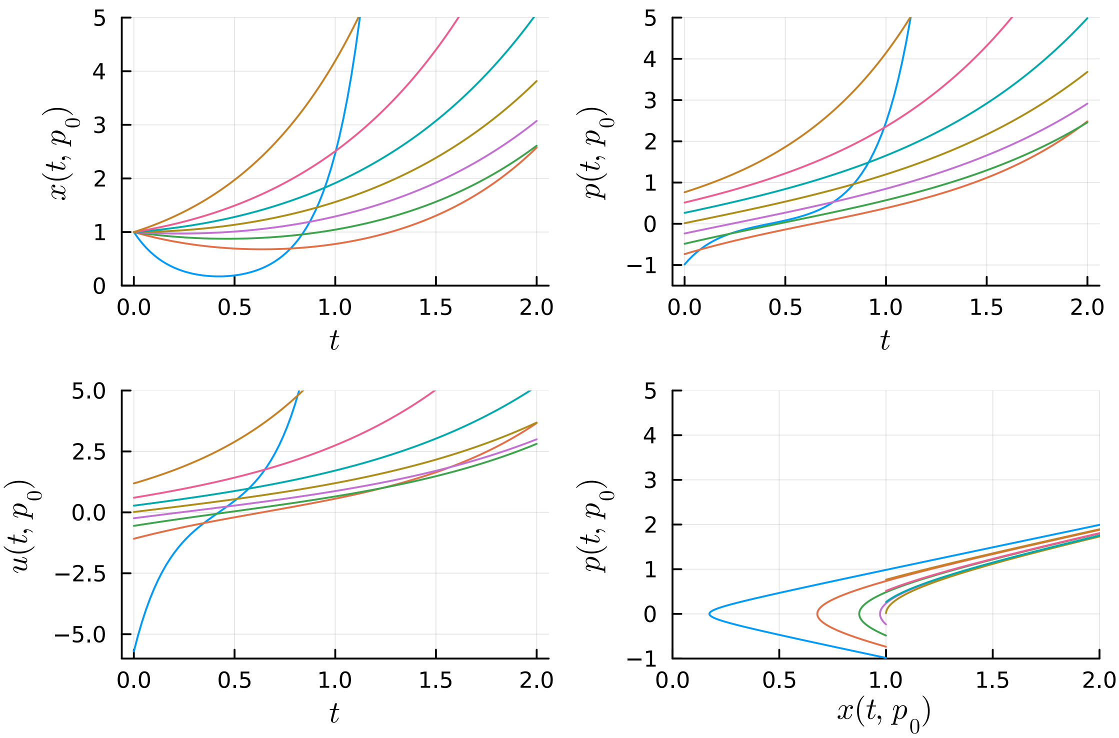
We note $z(t) = (x(t),p(t))$, then the hamiltonian flow is the function $\phi(.,z_0) = \phi(.,x_0,p_0)$ solution of the initial value problem

$$(IVP)\left\{\begin{array}{l} \dot{z} = \vec{H}(z) = \begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix} = \begin{pmatrix}\dfrac{\partial H(z)}{\partial p}\\ -\dfrac{\partial H(z)}{\partial x}\end{pmatrix} \\ z(0) = z_0 = (x_0,p_0). \end{array}\right.$$

[13]:

```
# Compute the  Flow
```

```
u(x, p) = sign(x[1])*p[1]*sqrt(1. /(x[1]^2-p[1]^2))  # contol function

# the Flow function of the control-toolbox package computes the hamiltonian flow
ocp_flow = Flow(ocp, u)  # ocp_flow.ode_sol is the result of the solve function␣
 ↪from the DifferentailEquations package

Int_p0 = -0.985:0.25:0.98   # intervalle of p_0
ÎŤt = (tf - t0)/100          #
plt_x = plot()          # plot of the state x(t)
plt_p = plot()          # plot of the costate p(t)
plt_u = plot()          # plot of the control u(t)
plt_phase = plot()      # plot (x,p)

for p0 in Int_p0        # plot for each p_0 in Int_p0
    flow_p0 = ocp_flow((t0, tf), x0, p0, reltol = 1e-8, abstol = 1e-8, saveat =␣
 ↪ÎŤt)
    T = flow_p0.ode_sol.t
    X = flow_p0.ode_sol[1,:]
    P = flow_p0.ode_sol[2,:]
    par = atanh(p0./x0)
    plot!(plt_x,T,X)
    plot!(plt_p,T,P)
    plot!(plt_u,T,u.(X,P))
    plot!(plt_phase,X,P)
    #plot!(plt,flow_p0, control_style=(label="u for p_0",)) # pb car pas les␣
 ↪mÃłme ordonnÃľes
end

#flow_sol = f((t0, tf), x0, p_sol, saveat = ÎŤt)

#plot!(plt, flow_sol, control_style=(label="u_sol",))

#plot!(plt[5], ylims=(-6, 6))
plot!(plt_x,xlabel = L"t",ylabel=L"x(t,p_0)",legend=false, ylims=(0.,5.))
plot!(plt_p,xlabel = L"t",ylabel=L"p(t,p_0)",legend=false, ylims=(-1.5,5.))
plot!(plt_u,xlabel = L"t",ylabel=L"u(t,p_0)",legend=false, ylims=(-6.,5.))
plot!(plt_phase,xlabel = L"x(t,p_0)",ylabel=L"p(t,p_0)",legend=false, xlims=(0.
 ↪,2.), ylims=(-1.,5.))
plot(plt_x,plt_p,plt_u,plt_phase,layout = (2,2))
```



1.4 Conjugate points

The time τ is said to be conjugate to the the time $t_0 = 0$ if the solution of the Jacobi equation

$$\dot{\delta z}(t) = \frac{\partial \vec{H}}{\partial z}(z(t, z_0))\delta z(t)$$

with the initial condition $\delta z(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is vertical at time τ, that is $\delta x(\tau) = 0$.

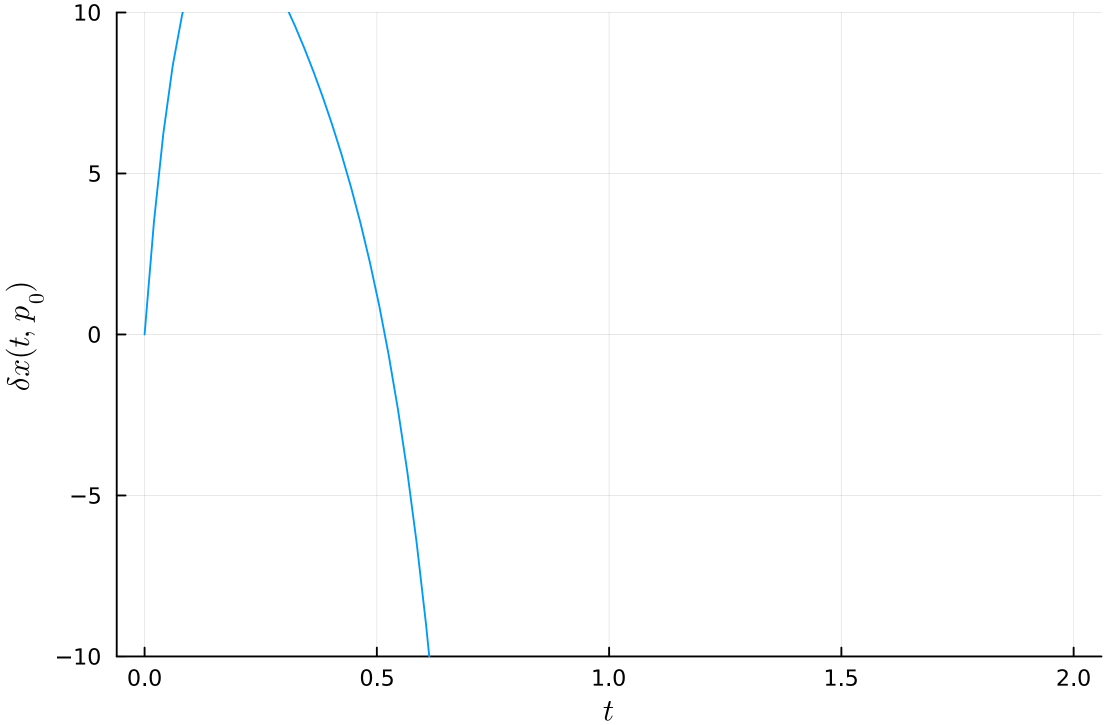
We first compute by automatic differentiation the flow of the Jacobi equation with the initial condition

$$\delta z(t, x_0, p_0) = \frac{\partial z}{\partial p_0} z(t, x_0, p_0)$$

[41]:
```
#
# Conjugate points
#
include("./example1-analytique.jl")
"""
    Compute the flow of the Jacobi equation for the initial condition Îťz(0) =␣
 ↪(0,1)
    sol = flow_jacobi(T,x0,p0)
    input :
```

```
    T : time where we want Îťz(t)
        list or array (t0,t1,...,tN)
    x0 : initial state
         Real(n), here a real
    p0 : initial costate
         Real(n), here a real
    output :
    sol : (Îťz(t0),Îťz(t1),...,Îťz(tN))
          Vector of vector of dimension 2n, the type of the element are the same␣
↪as the type of p0 because we use after the automatic differentiation.
          To modifie if n > 1

"""
function flow_jacobi(T,x0,p0)
    n = length(x0)
    t0 = T[1]
    nb_t = length(T)
    # type because we use ForwardDiff
    sol = [Vector{typeof(p0[1])}(undef,2n) for _ in 1:nb_t] # Vector -> Matrix␣
↪if n >  1
                                                             # typeof(p0) ->␣
↪typeof(p0[1]), type of the element of p0
    for i in 1:nb_t
        t = T[i]
        # flow_p0 compute de flow at t oh the hamiltonian system
        flow_p0(p_0) = ocp_flow((t0, t), x0, p_0, reltol = 1e-10, abstol =␣
↪1e-10).ode_sol.u[end]
        # temp is the solution at t of the derivative with respect to p0 of the␣
↪flow
        temp = ForwardDiff.derivative(flow_p0, p0)  # p0 is a real
        #sol[i] = det(temp[1:n,1:n])
        sol[i] = temp
    end
    return sol
end

p0 = -0.985
t0tf = (0., 2.)
#println(flow_jacobi(t0tf,x0,p0)[end] - flow_jacobi_ana(t0tf,x0,p0).u[end])
#@test isapprox(flow_jacobi(t0tf,x0,p0)[end] ,flow_jacobi_ana(t0tf,x0,p0).u[end];
↪ rtol = 1.e-3)

Int_t0tf = range(t0,stop=tf,length=100)
sol = flow_jacobi(Int_t0tf,x0,p0)
plt_conj1 = plot()
```

```
plot!(plt_conj1,Int_t0tf,[sol[i][1] for i in 1:length(sol)])  # as n=1 the det␣
↪is the number
plot!(plt_conj1,xlabel = L"t",ylabel=L"\delta x(t,p_0)",legend=false, ylims=(-10.
↪,10.))
plot(plt_conj1)
```

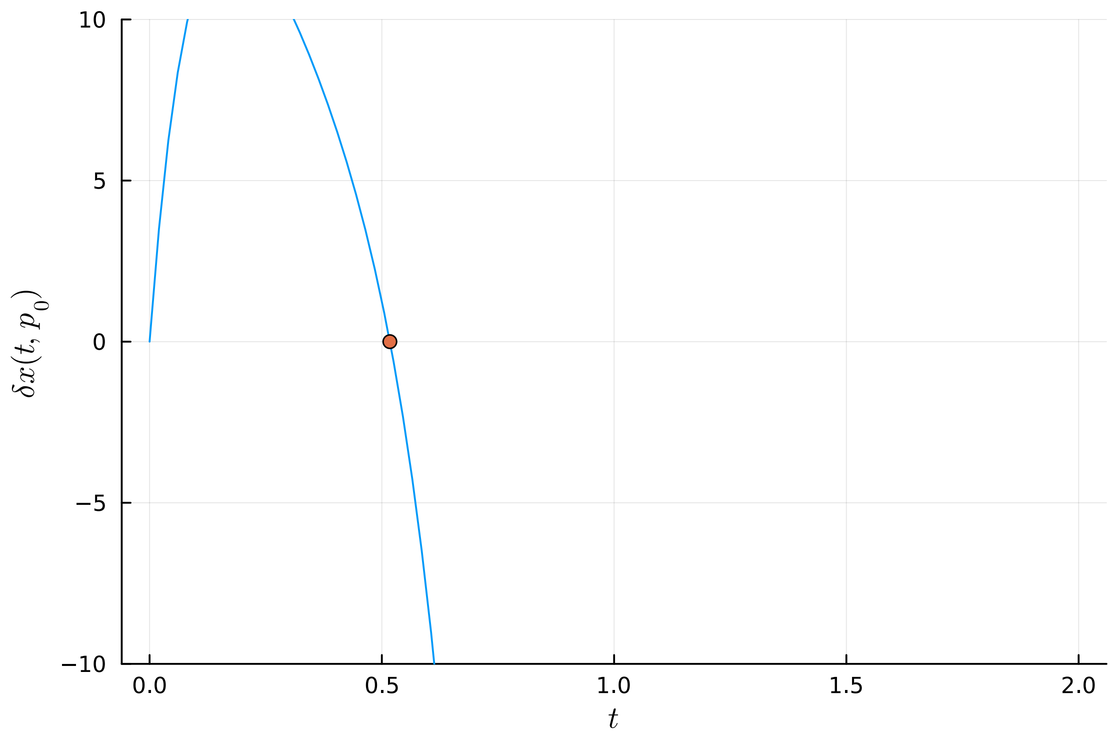


Then we numerically compute the conjugate point by solving $\delta x(t) = \delta z(t, x_0, p_0)_1 = 0$, for $x_0 = 1.$ and $p_0 = 0.985$.

[42]:
```
# compute the fisrt conjugate point
function F(t0,ÏĎ,x0,p0)
    tspan = (t0,ÏĎ)
#    return [flow_jacobi(tspan,xâĆĂ,p0)(ÏĎ)[1]]
    return flow_jacobi(tspan,x0,p0)[end]
end

# compute ÏĎ0
Îťx(ÏĎ) = F(t0,ÏĎ,x0,p0)[1]

#sol = nlsolve(Îťx,[0.5])
using Roots
#ÏĎ0 = sol.zero
ÏĎ0 = find_zero(Îťx, (0.4, 0.6))
```

```
println("For p0 = ", p0, " tau_0 = ", ÏĎ0)

plot!(plt_conj1,[ÏĎ0],[flow_jacobi((t0,ÏĎ0),x0,p0)[end]],seriestype=:scatter)
```

```
For p0 = -0.985 tau_0 = 0.51728931341542
```



1.5 Compute the conjugate loci

We compute cojugate loci by path following algorithm

We define $F(\tau, p_0) = \delta x(\tau, p_0)$ and we suppose that $\dfrac{\partial F}{\partial \tau}(\tau, p_0)$ is inversible, then by the implicit function theorem the conjugate time is a function of p_0. So, as here $p_0 \in$, we can compute them by solving the initial value problem

$$(IVP_{conj.points}) \begin{cases} \dot{\tau} = -\dfrac{\partial F}{\partial \tau}(\tau, p_0)^{-1} \dfrac{\partial F}{\partial p_0}(\tau, p_0) \\ \tau(p_0) = \tau_0. \end{cases}$$

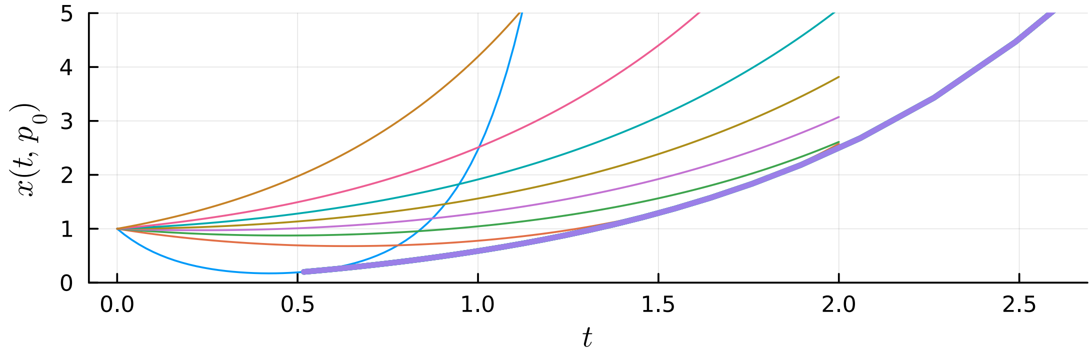
1.5.1 Remark

The derivative $\dfrac{\partial F}{\partial \tau}(\tau, p_0) = \dfrac{\partial \delta x}{\partial \tau}(\tau, p_0)$ is equal to the first component of the second member of the Jacobi equation $\dfrac{\partial \vec{H}}{\partial z}(z(t, z_0))\delta z(t)$.

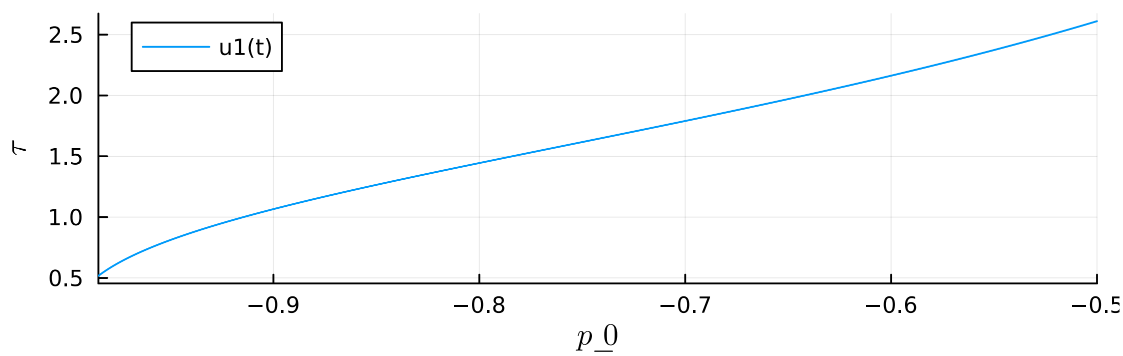
[45]:
```
# Test of Hvec
    Hvec! = ocp_flow.rhs! #
    #z0 = [1.,-0.1]
    par = 0.
    z0 = [x0 , p0]
    zpoint = similar(z0)
    Hvec!(zpoint,z0,par,t0)
    atol = 1.e-12
    @test  zpoint âĽĹ H_vec(z0) atol = atol    # test for the Hvec! function

#



#conjugates points by path following
"""
    Compute the right hanf side of the conj.points IVP equation
    tau_point = rhs_path(tau , par, p0)
    For the structure of the rhs_path see DifferentialEquaions  package
"""
function rhs_path(tau , par, p0)
    n = length(p0)
    ÏĎ = tau[1]      # tau is a vector
    z0 = [x0,p0]
    Hvec!(zpoint,z0) = ocp_flow.rhs!(zpoint,z0,par,ÏĎ)
    Hvec!(zpoint,z0)      # zpoint is the second member of the Hamiltonian flow
    dHvec = Matrix{typeof(p0[1])}(undef,2n,2n) #zeros(2*n,2*n)
    z = ocp_flow((t0, ÏĎ), x0, p0, reltol = 1e-8, abstol = 1e-8).ode_sol(ÏĎ) #␣
 ↪compute z(ÏĎ)
    # Compute matrix \dfrac{\partial \vec{H}}{\partial z}(z(t,z_0)) : the first␣
 ↪part of the rhs of the variational equation
    ForwardDiff.jacobian!(dHvec,Hvec!,zpoint,z)
    Îťz = flow_jacobi((t0,ÏĎ),x0,p0)[end]
    derivee_ÏĎ = (dHvec*Îťz)[1]     # #derivative w.r.t. ÏĎ

    Ftau(p0) = F(t0,ÏĎ,x0,p0)[1] # First componant of the flow of the Jacobi␣
 ↪equation at ÏĎ
    derivee_p0 = ForwardDiff.derivative(Ftau, p0) #derivative w.r.t. p0
    return [-(1/derivee_ÏĎ)*derivee_p0]
end

p0=-0.985
```

```
ÏĎ = 0.5
x0 = 1.
rhs_jacobi = rhs_path(0.5 , 1., p0)
println("x0 = ", x0, ", p0 = ", p0)


@test isapprox(rhs_path(0.5 , 1., p0),rhs_path_ana(0.5 , 1., p0); rtol = 1.e-7)

#Ftau(p0) =␣
 ↪F(t0,ÏĎ,x0,p0)[1]#flow_jacobi((t0,ÏĎ),x0,p0)[end]#F(t0,ÏĎ,x0,p0)[end][1]


function conj_point(p0span, ÏĎ0)
    pb = ODEProblem(rhs_path,[ÏĎ0],p0span,[1.])
    sol = DifferentialEquations.solve(pb, reltol = 1e-8, abstol = 1e-8)
    return sol
end

# conjugate point

println("p0 = ", p0)

p0span = (p0, -0.5)
sol = conj_point(p0span,ÏĎ0)
plt_conj_point = plot(sol,xlabel = L"p\_0", ylabel = L"\tau")
TT = sol.u
nb_t = length(sol.t)
T = zeros(nb_t)
X = zeros(nb_t)
for i in 1:nb_t
    #println(T[i])
    #println(sol.t[i])
    T[i] = TT[i][1]
    X[i] = x(T[i][1],x0,sol.t[i])
end

plot!(plt_x,T,X,linewidth=3)
plot(plt_x,plt_conj_point,layout = (2,1))
```

```
x0 = 1.0, p0 = -0.985
p0 = -0.985
```



$x(t, p_0)$
5
4
3
2
1
0
0.0
0.5
1.0
1.5
2.0
2.5
t



u1(t)
τ
2.5
2.0
1.5
1.0
0.5
−0.9
−0.8
−0.7
−0.6
−0.5
p_0

[44]:

[ ]: