The main idea is to assume that the process for finding the optimal piecewise fitting with heterogeneous primitives is done in two steps, as shown in Fig. 1. The top decision layer is to decide the types of primitives and their respective intervals. Assuming that these decisions have been made, the lower layer optimizes the piecewise fitting.

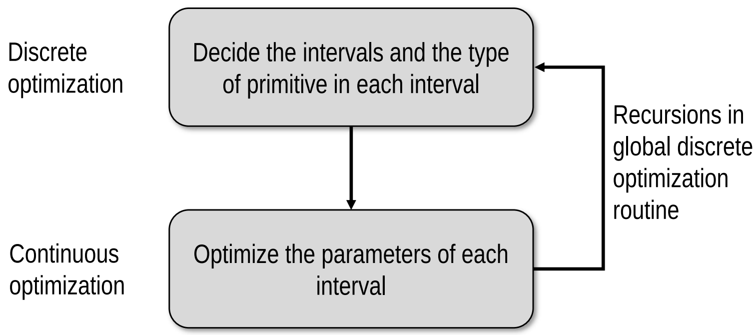

Figure 1: Two-layer approach in solving the optimal piecewise fitting problem.

This document focuses on the lower layer of optimization, i.e., how to find the best parameters for the MoveL and MoveJ primitives **after** we have decided the intervals and the type of primitive to be used in each interval. See the illustration in Fig. 2. Here, we assume that there are four intervals started by a MoveJ primitive, followed by two MoveL primitives, and ended by another MoveJ primitive. The points where two primitives of **different types** meet are called *connection points*, as shown in the figure. The points where two primitives of the **same type** meet are called *boundary points*.

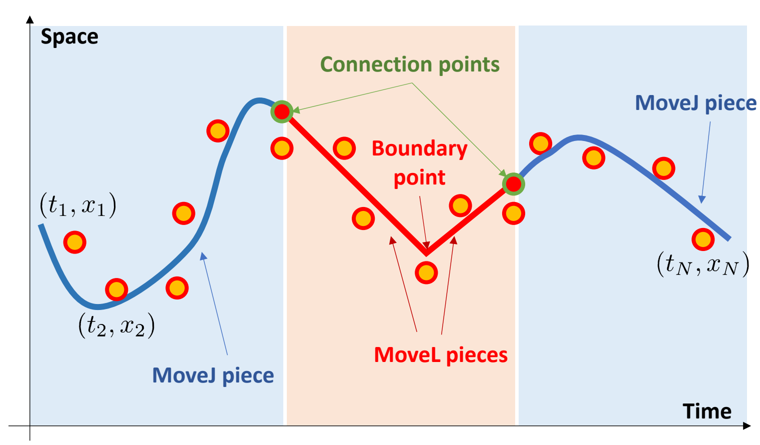

Figure 2: An illustration for the problem setup.

Problem Definition

The problem that we solve in this document can be formally stated as follows.

We are given a time-series of desired end-effector positions (t_i, x_i) for $i = 1, \cdots, N$, where t_i indicates the index of the data and $x_i \in \mathbb{R}^k$ is the desired end-effector position at the time index t_i. Through inverse kinematic calculation (and additional pose assumptions), we assume that we also have the desired joint angles data (t_i, q_i) for $i = 1, \cdots, N$, where $q_i \in \mathbb{R}^\ell$ is the desired joint angles (can be expressed as angles or quaternions) at the time index t_i. Generally, we denote the forward kinematic relation $T(\cdot)$ such that for any joint angles $q \in \mathbb{R}^\ell$ the corresponding end-effector position is given by

$$x = T(q),\ x \in \mathbb{R}^k. \tag{1}$$

The time-series is then divided into M intervals. We denote the length of the j-th interval as L_j, and the data in the j-th interval as $(t_{j,i}, x_{j,i})$ and $(t_{j,i}, q_{j,i})$ for $i = 1, \cdots, L_j$. We assume that the end points of intervals **overlap**. That is,

$$(t_{j,L_j}, x_{j,L_j}) = (t_{j+1,1}, x_{j+1,1}),\ (t_{j,L_j}, q_{j,L_j}) = (t_{j+1,1}, q_{j+1,1}). \tag{2}$$

We also assume that each of the M intervals have been designated to be fitted by a MoveL or MoveJ primitive.

Define the fitting objective of the j-th interval as a function of the fitting parameters α_j and β_j as

$$V_j(\alpha_j, \beta_j) \triangleq \begin{cases} \sum_{i=1}^{L_j} \|\alpha_j t_{j,i} + \beta_j - x_{j,i}\|^2, & \text{if fitted with MoveL,} \\ \sum_{i=1}^{L_j} \|\alpha_j t_{j,i} + \beta_j - q_{j,i}\|^2, & \text{if fitted with MoveJ.} \end{cases} \tag{3}$$

Note that here we slightly abuse the notation as α_j and β_j would have different dimensions, depending on whether the j-th interval is fitted with a MoveL or MoveJ primitive. Then, the problem that we are solving can be expressed as:

$$\min_{\alpha_j, \beta_j} \sum_{j=1}^{M} V_j(\alpha_j, \beta_j), \tag{4}$$

subject to the **continuity constraints**

$$\hat{x}_{j,L_j} = \hat{x}_{j+1,1},\ \forall j \in \{1, \cdots, M-1\}, \tag{5}$$

where we define for any $j \in 1, \cdots, M$ and $i \in 1, \cdots, L_j$,

$$\hat{x}_{j,i} \triangleq \begin{cases} \alpha_j t_{j,i} + \beta_j, & \text{if the } j\text{-th interval is fitted with MoveL,} \\ T\left(\alpha_j t_{j,i} + \beta_j\right), & \text{if the } j\text{-th interval is fitted with MoveJ.} \end{cases} \tag{6}$$

Proposed Solution

If all the pieces are defined as MoveL, the constraints in Eq. (5) are linear and the problem is a simple Quadratic Program. The same is true if all pieces are MoveJ because the constraints in Eq. (5) can be expressed as linear constraints in q.

To deal with the general case, we divide the computation procedure into a part for the MoveL pieces and a part for the MoveJ pieces. Multiple adjacent pieces of primitives of the same type are calculated together. For example, in Fig. 2, the two MoveL pieces in the red shaded intervals are calculated together because the continuity constraint between them is a linear constraint. Connection between MoveJ and MoveL pieces are established using the connection points as equality constraints.

Now, let us describe a procedure that will be used as a building block for our overall solution. In the procedure, we assume that we are given $(n+1)$ connected intervals of MoveL pieces, corresponding to $j = m, \cdots, m+n$ for some m. We want to find the best MoveL paremeters that can fit the data in these intervals. Further, we assume that the connection points that bookend these intervals are **determined**, and denoted as x_{begin} and x_{end}. Thus, the optimization problem that we seek to solve is

$$\min \sum_{j=m}^{n} \sum_{i=1}^{L_j} \|\alpha_j t_{j,i} + \beta_j - x_{j,i}\|^2, \tag{7}$$

with α_j and β_j, $j = m, \cdots, m+n$ as the optimization variables, subject to the boundary point continuity constraints

$$\alpha_j t_{j,L_j} + \beta_j = \alpha_{j+1} t_{j+1,1} + \beta_{j+1}, \ \forall j \in \{m, \cdots, m+n-1\}, \tag{8}$$

and the connection point constraints

$$\alpha_m t_{m,1} + \beta_m = x_{\text{begin}}, \tag{9}$$

$$\alpha_{m+n} t_{m+n,L_{m+n}} + \beta_{m+n} = x_{\text{end}}. \tag{10}$$

The optimization problem described in Eqs. (7) - (10) is a Quadratic Program with linear equality constraints. It can be solved using the standard Lagrange multiplier technique by formulating the Lagrangian

$$\begin{aligned} L(\alpha, \beta, \gamma, \lambda_{m,\text{begin}}, \lambda_{m+n,\text{end}}) \triangleq \sum_{j=m}^{n} \sum_{i=1}^{L_j} \|\alpha_j t_{j,i} + \beta_j - x_{j,i}\|^2 + \gamma_j^T \left(\alpha_j t_{j,L_j} + \beta_j - \alpha_{j+1} t_{j+1,1} - \beta_{j+1}\right) + \\ \lambda_{m,\text{begin}}^T \left(\alpha_m t_{m,1} + \beta_m - x_{\text{begin}}\right) + \lambda_{m+n,\text{end}}^T \left(\alpha_{m+n} t_{m+n,L_{m+n}} + \beta_{m+n} - x_{\text{end}}\right). \end{aligned} \tag{11}$$

Here we use a shortened notations α, β, and γ to denote $(\alpha_j)_{j=m}^n$, $(\beta_j)_{j=m}^n$, and $(\gamma_j)_{j=m}^{n-1}$ respectively. The variables γ, $\lambda_{m,\text{begin}}$, and $\lambda_{m+n,\text{end}}$ are the Lagrange multipliers (or dual variables). Hereafter, we **assume that we have a solver that can obtain the (unique) solution to this optimization problem**, providing us with the optimal primal and dual variables $(\alpha^*, \beta^*, \gamma^*, \lambda^*_{m,\text{begin}}, \lambda^*_{m+n,\text{end}})$. For example, the software tool CVX provides this functionality.

Remark 1 *If we want to fit the pieces with MoveJ instead, the procedure is largely the same, except that the variables x are replaced by q. The right-hand side of the connection point constraints in Eq. (9) - (10) are then replaced with the corresponding points in the joint angle space q_{begin} and q_{end}, where*

$$x_{\text{begin}} = T(q_{\text{begin}}), \ x_{\text{end}} = T(q_{\text{end}}). \tag{12}$$

The overall procedure to solve the optimization problem outlined in the previous section is shown below. Assume that the set C consists of the indices of the intervals that end with a connection point. Thus, an integer $j \in C$ if and only if the j-th interval and the $j+1$-st interval are fitted with primitives of different types.

STEP 1. Assume an arbitrary value for the connection point constraints. That is for each $j \in C$, pick an arbitrary $q_{j,\text{end}}$ and set the connection point continuity constraints

$$\left.\begin{aligned} \alpha_j t_{j,L_j} + \beta_j = T(q_{j,\text{end}}), \\ \alpha_{j+1} t_{j,1} + \beta_{j+1} = q_{j,\text{end}}, \end{aligned}\right\} \text{if the } j\text{-th interval is fitted with MoveL,} \tag{13}$$

or

$$\left.\begin{aligned} \alpha_j t_{j,L_j} + \beta_j = q_{j,\text{end}}, \\ \alpha_{j+1} t_{j,1} + \beta_{j+1} = T\left(q_{j,\text{end}}\right), \end{aligned}\right\} \text{if the } j\text{-th interval is fitted with MoveJ.} \tag{14}$$

In practice, we can pick $q_{j,\text{end}}$ to be equal to the desired joint angles,

$$q_{j,\text{end}} = q_{j,L_j}, \tag{15}$$

which was provided to us.

STEP 2. For each of the connected intervals between the connection points, solve the optimization problem described in Eqs. (7) - (10). Obtain the optimal primal and dual variable values.

STEP 3. We adjust the connection point continuity constraints as follows. For each $j \in C$,

If the j-th interval is fitted with MoveL,

$$q_{j,\text{end}}^{\text{new}} = q_{j,\text{end}} + \delta \cdot \left(\left(\frac{\partial T}{\partial q}(q_{j,\text{end}}) \right)^T \lambda_{j,\text{end}}^* + \lambda_{j+1,\text{begin}}^* \right). \tag{16}$$

If the j-th interval is fitted with MoveJ,

$$q_{j,\text{end}}^{\text{new}} = q_{j,\text{end}} + \delta \cdot \left(\lambda_{j,\text{end}}^* + \left(\frac{\partial T}{\partial q}(q_{j,\text{end}}) \right)^T \lambda_{j+1,\text{begin}}^* \right). \tag{17}$$

Here, δ is the step size of the constraint update. In practice, δ can be chosen using a line search algorithm. The symbol $\frac{\partial T}{\partial q}$ refers to the forward kinematic Jacobian matrix.

STEP 4. Repeat from STEP 1, by replacing the old connection point $q_{j,\text{end}}$ with the new one $q_{j,\text{end}}^{\text{new}}$ for each $j \in C$. Iterate until convergence.