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ASP 5203, ESTRELLAS VARIABLES 2019-1 HOMEWORK 2.

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ABSTRACT

This homework is based on Marcio Catelan's Second homework (2019). First, mean density and timescales are estimated for white dwarf. Second, several values for "pulsation constant Q" are computed; and relating Period to different astrophysics characteristics, finding that these relationships are the same for every pulsating subclass of star. Third, thermal time scale are computed for stars with different spectral type; finding that adiabatic process is dominant in A, F and G stars; while non-adiabatic process is dominant in O and B stars. Fourth, Energy Conservation Equation (ECS) is found considering the energy as E = E(P, T). Finally, a linearized version of ECS is found considering E = E(ρ, T).

Keywords: White Drawf, "Pulsation Constant Q", Kelvin-Helmholtz time scale, ECS.

Object	Mass $M_\odot$	Radius (Km)
White Dwarf	≤ 1.4	$\sim 5 \cdot 10^3$

Table 1
Parameters for a typical white dwarf, values from Table 1 from Balberg Shapiro 2000. $M_\odot = 1.9899 \cdot 10^{33}$ g.

1. PROBLEM: WHITE DWARF AND NEUTRON STARS.

1.1.

We want to estimate the typical mean density for white dwarf stars. Table 1 from Balberg Shapiro 2000, shows parameters for the Sun, a typical white dwarf and neutron star. Using the value for mass (M) and radius (R) of a typical white dwarf, was possible to estimate the mean density through the next formula:

$$\bar{\rho} \simeq \frac{3M}{4\pi R^3} \tag{1}$$

Using the values from Table 1 we obtained a *mean density* of $\bar{\rho} = \sim 5.3 \cdot 10^7$ g/cm^3. Seeing *Table 2: The equation of state a of cold, catalyzed high density matter* from Balberg Shapiro 2000, the speed of sound that correspons to this density regime ($\bar{\rho}$) is $c_s = 4.37 \cdot 10^8$ cm/s.

1.2.

We estimate the corresponding pulsation timescale, assuming pulsation to be due to pressure waves propagating in the stellar interior. Using the period-mean density relation:

$$\Pi = \frac{2R}{c_s} \tag{2}$$

As we do not have the radius corresponding to the value of $\bar{\rho}$, we use an analogous to the preview equation:

$$\Pi = \sqrt{\frac{3\pi}{2\Gamma G \rho}} \tag{3}$$

Where G is the Gravitational constant, $\rho = \bar{\rho}$ ang Γ is the adiabatic index. According to Figure 1 it is possible to see that we can get Γ.

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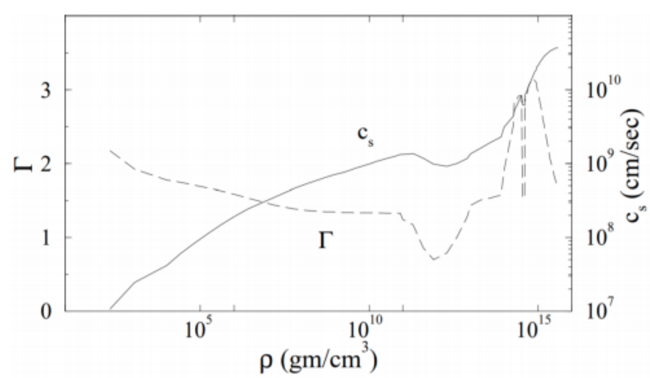

Figure 1. Adapted from Balberg Shapiro 2000, it is possible to see how the speed of sound (denoted c_s) changes with density, in the high-density regime that characterizes white dwarfs and neutron stars.

The value obtained was Π = 16.7 min. According to the article: *The Pulsating White Dwarf Stars, Author(s): G. Fontaine and P. Brassard*, Table 1. this value is slightly higher than the measured. This can be explained because Π does not have information about the pulsation modes, some pulsation modes are theoretically predicted to be more sensitive to the extent of crystallization of the atmosphere material in white dwarf.

2. PROBLEM: PHYSICAL PROPERTIES OF PULSATING VARIABLE STARS.

2.1.

The pulsation constant Q can be written as:

$$P\sqrt{\frac{<\rho>}{<\rho_\odot>}} \simeq Q \tag{4}$$

where P corresponds to pulsation period (not pressure) of the star, $<\rho>$ is the mean star density, $<\rho_\odot>$ is the means density of the Sun ($\sim 1408\frac{kg}{m^3}$) and Q is the pulsation "constant".
In this problem we make some assumptions, the stars:

$\log P$	M_V	$B-V$	$\log L/L_\odot$	$\log R/R_\odot$	$\log T_{\rm eff}$	$\log \mathcal{M}/\mathcal{M}_\odot$	$\log g$
				Classical Cepheids			
0.4	−2.4	0.49	2.81	1.41	3.76	0.54	2.2
0.6	−2.9	0.57	3.05	1.55	3.75	0.61	1.9
0.8	−3.5	0.66	3.30	1.70	3.74	0.68	1.7
1.0	−4.1	0.75	3.55	1.85	3.72	0.75	1.5
1.2	−4.7	0.84	3.80	2.00	3.71	0.82	1.3
1.4	−5.3	0.93	4.06	2.15	3.70	0.89	1.0
1.6	−5.8	1.01	4.32	2.29	3.70	0.97	0.8
1.8	−6.4	1.10	4.58	2.44	3.69	1.04	0.6
				δ Sct/Dwarf Cepheids[b]			
−1.4	+2.7	0.20	0.90	0.07	3.91	0.21	4.5
−1.0	+1.7	0.25	1.22	0.37	3.88	0.25	3.9
−0.7	+0.8	0.29	1.58	0.59	3.86	0.31	3.6
				RR Lyrae stars[c]			
−0.5	0.7	0.20	1.47	0.54	3.86	−0.26	3.1
−0.3	0.7	0.25	1.57	0.67	3.82	−0.26	2.8
−0.1	0.7	0.38	1.59	0.76	3.78	−0.26	2.7

Figure 2. Adapted table from "*Astrophysical Quantities, 4th edition*", this provides average physical properties of different classes of pulsating variables stars along the "classical" instability strip, the first column shows the logarithm of the pulsation period (in days), referring in all classes to the fundamental mode.

- are homogeneous, so we can use $<\rho>$.
- are perfect spheres.
- the means density is equal to the density, this means $<\rho>=\rho$.

With this simplifications we know the volume and the density of a star, so it is possible to write the density in function of the mass and radius:

$$V = \frac{4}{3}\pi r^3 \tag{5}$$

and

$$\rho = \frac{M}{V} \tag{6}$$

$$\Rightarrow \rho = \frac{3M}{4\pi r^3} \tag{7}$$

Using equation (7), it is possible to write equation (4). in terms of the mass, period, radius and sun density, obtaining:

$$Q \simeq P\sqrt{\frac{3M}{4\pi r^3 < \rho_\odot >}} \tag{8}$$

In this way, we compute Q in function of the parameters given by the table from "*Astrophysical Quantities, 4th edition*", see Figure 2.

Using $logP$, $log\frac{R}{R_\odot}$ and $log\frac{M}{M_\odot}$ was possible to obtain Q for each subclass, results are presented in Figure 3, Figure 4 and Figure 5:

Seeing the result for the values of Q in the different subclasses, we find that Q $\sim$ 0.04 days. We can see that for higher values period and radius, Q increases, for this reason the Classical Cepheids and RR Lyrae stars have higher Q values. In a general picture it is possible to say that Q does not have big changes in these 3 subclasses.

Classical Cepheids			
log P	log R/R⊙	log M/M⊙	Q
0.4	1.41	0.54	0.0355
0.6	1.55	0.61	0.0376
0.8	1.70	0.68	0.0385
1.0	1.85	0.75	0.0394
1.2	2.00	0.82	0.0394
1.4	2.15	0.89	0.0394
1.6	2.29	0.97	0.0394
1.8	2.44	1.04	0.0394
		Mean:	0.0403

Figure 3. Values used for computing Q for Classical Cepheids, P and Q are in days. The mean value for the "pulsation constant Q" is $\sim$0.0403 days.

δ Sct/Dwarf Cepheids			
log P	log R/R⊙	log M/M⊙	Q
-1.4	0.07	0.21	0.0394
-1,0	0.37	0.25	0.0368
-0.7	0.59	0.31	0.0368
		Mean:	0.0377

Figure 4. Values used for computing Q for δ Sct/Cepheids, P and Q are in days. The mean value for the "pulsation constant Q" is $\sim$0.0377 days.

RR Lyrae stars			
log P	log R/R⊙	log M/M⊙	Q
-0.5	0.54	-0.26	0.0360
-0.3	0.67	-0.26	0.0364
-0.1	0.76	-0.26	0.0422
		Mean: 0.0382	

Figure 5. Values used for computing Q for RR Lyrae stars, P and Q are in days. The mean value for the "pulsation constant Q " is $\sim$0.0382 days.

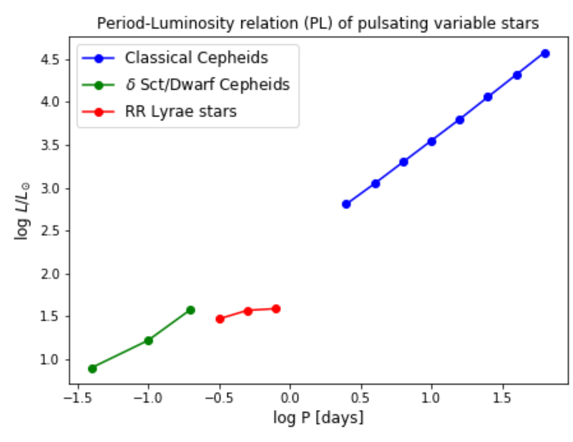

Figure 6. Period-luminosity (PL) relation for each subclass, from the data given by Figure 2.

2.2.

In this part we evaluate the existence of period-luminosity (PL), period-radius and period-surface gravity relation for the three subclasses of Figure 2.

It is possible to see that the PL (Figure 6) relation is directly proportional for each subclass, specially for Classical Cepheids where the increase is almost linear. We can see a sequence, when the period and the

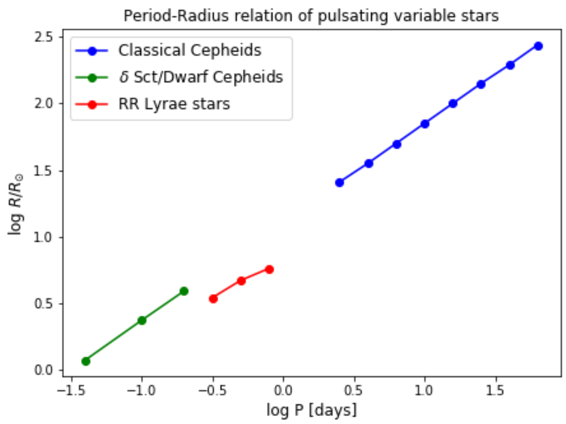

Figure 7. Period-radius relation for each subclass, from the data given by Figure 2.

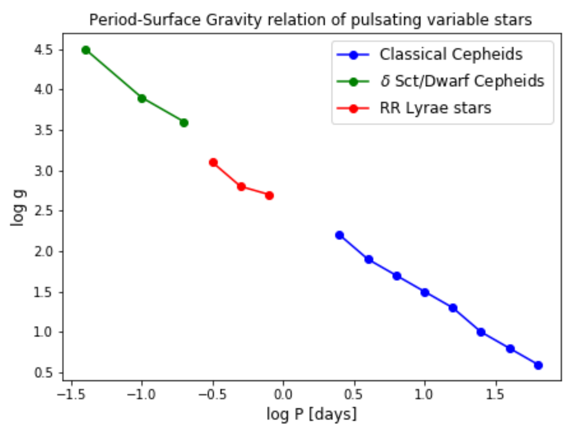

Figure 8. Period-surface gravity relation for each subclass, from the data given by Figure 2.

luminosity increase we pass from δ Sct/Cepheids to RR Lyrae stars to Classical Cepheids.

In the case of the period-ratio relation (Figure 7) we can see a similar behaviour as the PL relation for the 3 subclasses.

For the period-surface gravity (Figure 8) the relation is inversely proportional for each subclass, for longer periods, smaller surface gravity, this can be due to density of the stars, between δ Sct/Cepheids and Classical Cepheid are araound 3 magnitude order of difference.

2.3.

Here we evaluate a period-luminosity-color relation or PLC relation for Classical Cepheids. From Bono et al. (1999b) it was found that " *PLC relations based on different photometric bands present pros and cons, and indeed the PLC (V,B-V) relation, in comparison with the PLC (V,V-I) relation, is less affected by errors on both reddening corrections and colors but is more affected by a spread in metallicity*". In that work they obtained a (V, B-V) relation:

$$< M_V > = -2.83 - 3.57 logP$$

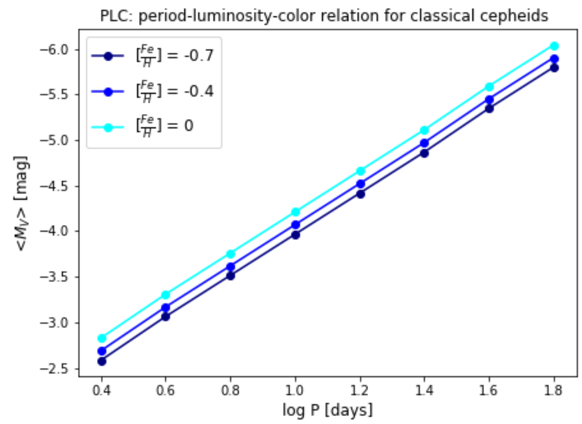

Figure 9. Period-luminosity-color (PLC) relation for Classical Cepheids, equation (9) for 3 values of metallicity was computed. With process is based on G.Boro et al., 1999, "On the Period-Luminosity-Color Relation of Classical Cepheids".

$$+ 2.92[< B > - < V >] \text{ -0.35 [Fe/H]} \quad (9)$$

For this reason the PLC relation was compute for 3 different values of metallicity.
We also use the color because *the value of the color coefficient in the optical PLC (V, B-V) relation is quite similar (see Bono et al. 1999b) to the extinction parameter, and thus in these bands the Wesenheit function and the PLC relation present the same behavior.*

3. PROBLEM: PULSATIONAL AND THERMAL TIMES SCALES.

Here we evaluate the ratio between the pulsational and Kelvin-Helmholtz (or thermal) time scales for stars of different spectral types and luminosity classes. Table 15.8 from "*Astrophysical Quantities, 4th edition*" give us certain parameter for Main Sequence (V), Giant (III) and Supergiants (I) types of stars, these parameters are:

1. Spectral Type.
2. Mass of the star, $M/M_{\odot}$.
3. Radius of the star, $R/R_{\odot}$.
4. Surface gravity log $g/g_{\odot}$.
5. Mean density, $\rho/\rho_{\odot}$.
6. Rotational velocity.

It is known that the Kevin-Helmholtz time scale (or thermal time scale) can be calculated as:

$$\tau_{KH} \simeq \frac{GM^2}{2RL} \quad (10)$$

Where G is the Gravitational Constant, M is mass, R the radius and L the luminosity of the star.
It is possible to see that we need the luminosity of a star to compute the thermal time scale, for this reason we only choose Main Sequence (MS) stars, because

for these objects we can estimate a luminosity from the mass-luminosity relation:

$$\frac{L}{L_{\odot}} = (\frac{M}{M_{\odot}})^{\alpha} \tag{11}$$

where $1 < \alpha < 6$, for the Main Sequence we adopted a values of $\alpha = 3.5$. Unfortunately, this mass-luminosity relation for MS stars is a good estimation only for stars with mass in a range of $2 < M/M_{\odot} < 20$, but we used it anyways. In this way, it was possible to obtain an estimated luminosity for MS stars and then calculate τ_{KH}, see Table 2.

To compute the pulsational time scale we used equation (4) obtaining:

$$P \simeq \frac{Q}{\sqrt{<\rho> / <\rho_{\odot}>}} \tag{12}$$

In Problem 2 we obtained an estimated value for Q $\simeq$ 0.04 days, also we used that $<\rho> / <\rho_{\odot}> = \rho/\rho_{\odot}$.

From Table 2, we can appreciate that P$/\tau_{KH} \ll 1$ for every spectral type, also P$/\tau_{KH}$ decreases with the mass. Therefore, it is expected that Pulsating Stars which belong to O and B type should have non-adiabatic pulsation mechanism, in the case of A, F and G spectral type, it is expected adiabatic mechanism for pulsations.

4. PROBLEM: ENERGY CONSERVATION EQUATION.

In this problem it will be obtained a form for the energy conservation equation based on the process seen in classes, starting from a functional dependence of the form:

$$E = E(P, T). \tag{13}$$

P is the pressure and T the temperature.
It is known that the heat rate change per unit of mass is given by:

$$\epsilon_g = \frac{dQ}{dt} = \frac{\partial E}{\partial t} + \frac{\partial}{\partial t}(\frac{1}{\rho}) \tag{14}$$

Since E = E(P, T), it is possible to write:

$$\epsilon_g = \frac{dQ}{dt} = (\frac{\partial E}{\partial T})_P \frac{\partial T}{\partial t} + (\frac{\partial E}{\partial P})_T \frac{\partial P}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{dt} \tag{15}$$

Using:

$$\frac{\partial T}{\partial t} = T\frac{lnT}{\partial t} \tag{16}$$

$$\frac{\partial P}{\partial t} = P\frac{lnP}{\partial t} \tag{17}$$

$$\frac{\partial \rho}{\partial t} = \rho\frac{ln\rho}{\partial t} \tag{18}$$

So, the rate of heat gained or lost per unit of mass is:

$$\frac{\partial Q}{\partial t} = (\frac{\partial E}{\partial T})_P T\frac{\partial lnT}{\partial t} + (\frac{\partial E}{\partial P})_T P\frac{\partial lnP}{\partial t} - \frac{P}{\rho^2}\rho\frac{ln\rho}{\partial t} \tag{19}$$

That is equal to:

$$\frac{\partial Q}{\partial t} = (\frac{\partial E}{\partial T})_P T\frac{\partial lnT}{\partial t} + (\frac{\partial E}{\partial P})_T P\frac{\partial lnP}{\partial t} - \frac{P}{\rho^2}\frac{ln\rho}{\partial t} \tag{20}$$

The expression $\frac{dQ}{dT} = 0$ because we are considering an adiabatic process, so we can write:

$$\frac{\partial lnT}{\partial t} = \frac{\frac{P}{\rho}\frac{\partial lnP}{\partial t} - (\frac{\partial E}{\partial t})_T P\frac{\partial lnP}{\partial t}}{T(\frac{\partial E}{\partial T})_P} \tag{21}$$

Dividing for $\frac{\partial lnP}{\partial t}$ in both sides:

$$\frac{\partial lnT}{\partial t} = \frac{\frac{P}{\rho}\frac{\partial lnP}{\partial t} - (\frac{\partial E}{\partial t})_T P}{T(\frac{\partial E}{\partial T})_P} \tag{22}$$

The right side of the last equation is by definition $\frac{\Gamma_2 - 1}{\Gamma_2}$:

$$\frac{\Gamma_2 - 1}{\Gamma_2} = \frac{\frac{P}{\rho}\frac{\partial lnP}{\partial t} - (\frac{\partial E}{\partial t})_T P}{T(\frac{\partial E}{\partial T})_P} = \frac{\partial lnT}{\partial t} \tag{23}$$

If we multiply by $\frac{lnP}{\partial t}$, we can obtain:

$$\frac{\Gamma_2 - 1}{\Gamma_2}\frac{lnP}{\partial t} = \frac{\frac{P}{\rho}\frac{\partial lnP}{\partial t} - (\frac{\partial E}{\partial t})_T P}{T(\frac{\partial E}{\partial T})_P}\frac{lnP}{\partial t} \tag{24}$$

If we divide equation (20) by $T\frac{\partial E}{\partial t}$, the second member of the right side of this equation corresponds to the numerator of the right side of equation (24) So we obtain:

$$\frac{\frac{dQ}{dt}}{T(\frac{\partial E}{\partial T})} = \frac{\partial lnT}{\partial t} - (\frac{\Gamma_2 - 1}{\Gamma_2})\frac{\partial lnP}{\partial t} \tag{25}$$

$\Leftrightarrow$

$$\frac{\partial lnT}{\partial t} = (\frac{\Gamma_2 - 1}{\Gamma_2})\frac{\partial lnP}{\partial t} - (T\frac{\partial E}{\partial t})^{-1}\frac{dQ}{dt} \tag{26}$$

If we consider P constant, we have $C_p = \frac{\partial E}{\partial T}$, so:

$$\frac{\partial lnT}{\partial t} = (\frac{\Gamma_2 - 1}{\Gamma_2})\frac{\partial lnP}{\partial t} - (TC_p)^{-1}\frac{dQ}{dt} \tag{27}$$

Using the definition of ϵ_g:

$$\frac{\partial lnT}{\partial t} = (\frac{\Gamma_2 - 1}{\Gamma_2})\frac{\partial lnP}{\partial t} - (TC_p)^{-1}\epsilon_g \tag{28}$$

Since,

$$\frac{\partial L}{\partial m} = \epsilon_n - \epsilon_\nu - \epsilon_g = \epsilon_n - \epsilon_\nu - \frac{dQ}{dT} \tag{29}$$

Using this expression, we find:

$$\frac{\partial lnT}{\partial t} = (\frac{\Gamma_2 - 1}{\Gamma_2})\frac{\partial lnP}{\partial t} - (TC_p)^{-1}(\epsilon - \frac{\partial L_r}{\partial m}) \tag{30}$$

that is the Energy Conservation Equation.

Main Sequence. V.

Sp	$M/M_{\odot}$	$R/R_{\odot}$	$\log g/g_{\odot}$	$\log\rho/\rho_{\odot}$	Estimated Luminosity. $L/L_{\odot}$	τ_{KH} (days)	P (days)	P/τ_{KH}
O3	$1.20{\cdot}10^{2}$	$1.50 \cdot 10$	$-3.00 \cdot 10^{-1}$	-1.50	$1.89 \cdot 10^{7}$	$2.89 \cdot 10^{5}$	$7.11 \cdot 10^{-3}$	$2.46 \cdot 10^{-8}$
O5	$6.00 \cdot 10$	$1.20 \cdot 10$	$-4.00 \cdot 10^{-1}$	-1.50	$1.67 \cdot 10^{6}$	$1.02 \cdot 10^{6}$	$7.11 \cdot 10^{-3}$	$6.95 \cdot 10^{-9}$
O6	$3.70 \cdot 10$	$1.00 \cdot 10$	$-4.50 \cdot 10^{-1}$	-1.45	$3.08 \cdot 10^{5}$	$2.53 \cdot 10^{6}$	$7.53 \cdot 10^{-3}$	$2.97 \cdot 10^{-9}$
O8	$2.30 \cdot 10$	8.50	$-5.00 \cdot 10^{-1}$	-1.40	$5.84 \cdot 10^{4}$	$6.08 \cdot 10^{6}$	$7.98 \cdot 10^{-3}$	$1.31 \cdot 10^{-9}$
B0	$1.75 \cdot 10$	7.40	$-5.00 \cdot 10^{-1}$	-1.40	$2.24 \cdot 10^{4}$	$1.05 \cdot 10^{7}$	$7.98 \cdot 10^{-3}$	$7.58 \cdot 10^{-10}$
B3	7.60	4.80	$-5.00 \cdot 10^{-1}$	-1.15	$1.21 \cdot 10^{3}$	$5.67 \cdot 10^{7}$	$1.06 \cdot 10^{-2}$	$1.88 \cdot 10^{-10}$
B5	5.90	3.90	$-4.00 \cdot 10^{-1}$	-1.00	$4.99{\cdot}10^{2}$	$1.02 \cdot 10^{8}$	$1.26 \cdot 10^{-2}$	$1.24 \cdot 10^{-10}$
B8	3.80	3.00	$-4.00 \cdot 10^{-1}$	$8.50 \cdot 10^{-1}$	$1.07{\cdot}10^{2}$	$2.57 \cdot 10^{8}$	$1.06 \cdot 10^{-1}$	$4.15 \cdot 10^{-10}$
A0	2.90	2.40	$-3.00 \cdot 10^{-1}$	$-7.00 \cdot 10^{-1}$	$4.15 \cdot 10$	$4.81 \cdot 10^{8}$	$1.79 \cdot 10^{-2}$	$3.71 \cdot 10^{-11}$
A5	2.00	1.70	$-1.50 \cdot 10^{-1}$	$-4.00 \cdot 10^{-1}$	$1.13 \cdot 10$	$1.19 \cdot 10^{9}$	$2.52 \cdot 10^{-2}$	$2.13 \cdot 10^{-11}$
F0	1.60	1.50	$-1.00 \cdot 10^{-1}$	$-3.00 \cdot 10^{-1}$	5.18	$1.88 \cdot 10^{9}$	$2.83 \cdot 10^{-2}$	$1.51 \cdot 10^{-11}$
F5	1.40	1.30	$-1.00 \cdot 10^{-1}$	$-2.00 \cdot 10^{-1}$	3.25	$2.65 \cdot 10^{9}$	$3.18 \cdot 10^{-2}$	$1.20 \cdot 10^{-11}$
G0	1.05	1.10	$-5.00 \cdot 10^{-2}$	$-1.00 \cdot 10^{-1}$	1.19	$4.82 \cdot 10^{9}$	$3.57 \cdot 10^{-2}$	$7.40 \cdot 10^{-12}$
G5	$9.20 \cdot 10^{-1}$	$9.10 \cdot 10^{-1}$	$5.00 \cdot 10^{-2}$	$-1.00 \cdot 10^{-1}$	$7.47 \cdot 10^{-1}$	$7.10 \cdot 10^{9}$	$3.57 \cdot 10^{-2}$	$5.02 \cdot 10^{-12}$
K0	$7.90 \cdot 10^{-1}$	$8.50 \cdot 10^{-1}$	$5.00 \cdot 10^{-2}$	$1.00 \cdot 10^{-1}$	$4.38 \cdot 10^{-1}$	$9.56 \cdot 10^{9}$	$4.49 \cdot 10^{-2}$	$4.70 \cdot 10^{-12}$
K5	$6.70 \cdot 10^{-1}$	$7.20 \cdot 10^{-1}$	$1.00 \cdot 10^{-1}$	$2.50 \cdot 10^{-1}$	$2.46 \cdot 10^{-1}$	$1.44 \cdot 10^{10}$	$5.33 \cdot 10^{-2}$	$3.69 \cdot 10^{-12}$
M0	$5.10 \cdot 10^{-1}$	$6.00 \cdot 10^{-1}$	$1.50 \cdot 10^{-1}$	$3.50 \cdot 10^{-1}$	$9.47 \cdot 10^{-2}$	$2.61 \cdot 10^{10}$	$5.98 \cdot 10^{-2}$	$2.29 \cdot 10^{-12}$
M2	$4.00 \cdot 10^{-1}$	$5.00 \cdot 10^{-1}$	$3.00 \cdot 10^{-1}$	$8.00 \cdot 10^{-1}$	$4.05 \cdot 10^{-2}$	$4.51 \cdot 10^{10}$	$1.00 \cdot 10^{-1}$	$2.23 \cdot 10^{-12}$
M5	$2.10 \cdot 10^{-1}$	$2.70 \cdot 10^{-1}$	$5.00 \cdot 10^{-1}$	1.00	$4.24 \cdot 10^{-3}$	$2.20 \cdot 10^{11}$	$1.26 \cdot 10^{-1}$	$5.76 \cdot 10^{-13}$
M8	$6.00 \cdot 10^{-2}$	$1.00 \cdot 10^{-1}$	$5.00 \cdot 10^{-1}$	1.20	$5.29 \cdot 10^{-5}$	$3.88 \cdot 10^{12}$	$1.59 \cdot 10^{-1}$	$4.10 \cdot 10^{-14}$

Table 2

This table is a re-adapted version of Table 15.8 from "*Astrophysical Quantities, 4th edition*". The data shows properties for `V` type star, i.e., stars in the Main Sequence (MS). From left to right, the columns shows: Spectral type, mass of the star ($M/M_{\odot}$), radius of the star ($R/R_{\odot}$), surface gravity $\log g/g_{\odot}$, $\log\rho/\rho_{\odot}$, estimated luminosity with mass-luminosity relation, thermal time scale (τ_{KH}, in days) computed, period (P in days) and ratio between P and τ_{KH}.

5. PROBLEM: LINEARIZED VERSION OF ENERGY CONSERVATION EQUATION.

If we express the internal energy as E = E(ρ, T) and using a similar process as the Problem 4., it is possible to obtain the Energy Conservation Equation as follows:

$$\frac{\partial lnT}{\partial t} = (\Gamma_3 - 1)\frac{\partial lnP}{\partial t} - (TC_p)^{-1}(\epsilon - \frac{\partial L_r}{\partial m}) \qquad (31)$$

Using, small perturbations in T $\rightarrow T_0 + \delta T$ and $\rho \rightarrow \rho_0 + \delta\rho$:

$$(T_0 + \delta T)^{-1}\frac{\partial T_0 + \partial\delta T}{\partial t} = (\Gamma_3 - 1)(\rho_0 + \delta\rho)^{-1}\frac{\partial\rho_0 + \partial\delta\rho}{\partial t}$$

$$+ (\mathrm{C}_v[T_0 + \delta T])^{-1}[(\epsilon - \tfrac{\partial L_r}{\partial m})_0 + \delta(\epsilon - \tfrac{\partial L_r}{\partial m})](32)$$

Expanding:

$$(T_0{+}\delta T)^{-1}(\frac{\partial T_0}{\partial t}{+}\frac{\partial\delta T}{\partial t}) = (\Gamma_3{-}1)(\rho_0{+}\delta\rho)^{-1}(\frac{\partial\rho_0}{\partial t}{+}\frac{\partial\delta\rho}{\partial t})$$

$$+ (\mathrm{C}_v[T_0 + \delta T])^{-1}[(\epsilon - \tfrac{\partial L_r}{\partial m})_0 + \delta(\epsilon - \tfrac{\partial L_r}{\partial m})](33)$$

Here, T_0 and ρ_0 are the temperature and density where the system is in equilibrium, we consider these values constant so the temporal derivative is 0. This is also consider for $(\epsilon - \frac{\partial L_r}{\partial m})_0$. We have:

$$(T_0 + \delta T)^{-1}\frac{\partial\delta T}{\partial t} = (\Gamma_3 - 1)(\rho_0 + \delta\rho)^{-1}\frac{\partial\delta\rho}{\partial t}$$

$$+ (\mathrm{C}_v[T_0 + \delta T])^{-1}\delta(\epsilon - \tfrac{\partial L_r}{\partial m})(34)$$

Expressing $(T_0 + \delta T)^{-1} = \frac{1}{T_0(1+\frac{\delta T}{T_0})}$ and $(\rho_0 + \delta\rho)^{-1} = \frac{1}{\rho_0(1+\frac{\delta\rho}{\rho_0})}$, the last equation can be written as:

$$\frac{1}{T_0(1+\frac{\delta T}{T_0})}\frac{\partial\delta T}{\partial t} = (\Gamma_3 - 1)\frac{1}{\rho_0(1+\frac{\delta\rho}{\rho_0})}\frac{\partial\delta\rho}{\partial t}$$

$$+ \mathrm{C}_v^{-1}[\tfrac{1}{T_0(1+\frac{\delta T}{T_0})}]\delta(\epsilon - \tfrac{\partial L_r}{\partial m})(35)$$

Using Taylor:

$$\frac{1}{T_0}(1+\frac{\delta T}{T_0})\frac{\partial\delta T}{\partial t} = (\Gamma_3 - 1)\frac{1}{\rho_0}(1+\frac{\delta\rho}{\rho_0})\frac{\partial\delta\rho}{\partial t}$$

$$+ \mathrm{C}_v^{-1}[\tfrac{1}{T_0}(1+\tfrac{\delta T}{T_0})]\delta(\epsilon - \tfrac{\partial L_r}{\partial m})(36)$$

Expanding:

$$\frac{1}{T_0}\frac{\partial\delta T}{\partial t} - \frac{\delta T}{T_0^2}\frac{\partial\delta T}{\partial t} = (\Gamma_3 - 1)[\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial t} - \frac{\delta\rho}{\rho^2}\frac{\partial\delta\rho}{\partial t}]$$

$$+ \tfrac{1}{C_v T_0}\ \delta(\epsilon - \tfrac{\partial L_r}{\partial m}) - \tfrac{1}{C_v}\tfrac{\delta T}{T_0^2}\delta(\epsilon - \tfrac{\partial L_r}{\partial m})(37)$$

As we are considering small perturbations, we can ignore terms of 2nd order or higher, obtaining the linearized version of the energy conservation equation when we express the internal energy as E = E(ρ, T):

$$\frac{1}{T_0}\frac{\partial\delta T}{\partial t} = (\Gamma_3 - 1)\frac{1}{\rho_0}\frac{\partial\delta\rho}{\partial t} + (T_0 C_v)^{-1}\delta(\epsilon - \frac{\partial L_r}{\partial m}) \quad (38)$$

Note : all the programs used and codes are available in Public GitHub repository.