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ASP 5203, ESTRELLAS VARIABLES 2019-1
HOMEWORK 3.

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ABSTRACT

This homework is based on Marcio Catelan's third homework (2019). First, based on up-to-date values for δ Cep (A) we estimate the relative range of the effective temperature in this star. Second, we model a Zero Age Main-Sequence (ZAMS) Star (M = $6.5M_{\odot}$); calculating its pulsations using the Linear Adiabatic Wave Equation (LAWE). With this we find that periods for pulsations in the fundamental mode, first, second and third overtone. We also comment some LAWE numerical results obtained. Third, using the one-zone model we estimate the periods and mean radius of a star of $6.5M_{\odot}$. Fourth, we classify image of spherical harmonics giving them a quantum number based only on their appearance. Fifth, using two stars models (for $1M_{\odot}$ and $15M_{\odot}$) we analyzed some properties about p and g-modes.

Keywords: stars: oscillations and pulsations, variables: Cepheids

1. PROBLEM: EFFECTIVE TEMPERATURE VARIATION OF THE STAR DURING ITS PULSATION CYCLE.

1.1.

The objective of this problem is to evaluate the relative range of variation in the effective temperature of the star during its pulsation cycle, using estimates of the radius and mass of the star δ Cep. We will assume that the radius of the star changes by 12% during a pulsation cycle.

From General Astrophysics it is known that the luminosity of a star is basically:

$$Luminosity = Area \cdot Flux \quad (1)$$

If it is assume that the star is sphere and that radiates as a perfect blackbody, equation (1) can be written as:

$$Luminosity = L = 4\pi r^2 \cdot \sigma T_{eff}^4 \quad (2)$$

where:

1. L is luminosity.
2. R is the radius of the star.
3. T_{eff} is the effective temperature of the star.
4. $\sigma = 5.67 \cdot 10^{-8} \frac{W}{m^2 \cdot K^4}$ is the Stefan-Boltzmann constant.

We can obtain T_{eff} from equation (2):

$$T_{eff} = (\frac{L}{4\pi r^2 \cdot \sigma})^{\frac{1}{4}} \quad (3)$$

In this way T_{eff} depends on the luminosity and the radius. As δ Cep varies L and r during its pulsation, we need to get an approximate value for these two variables on their maximum and minimum.

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δ Cep (A)
Parameters

Parameter	Value
$m_{V,max}$	4.37
$m_{V,min}$	3.48
$<m_V>$	4.07
Distance (pc)	273.0 ± 8.0
Mass ($M_{\odot}$)	4.5 ± 0.3
Radius ($R_{\odot}$)	44.5
Luminosity ($L_{\odot}$)	2000
T_{eff} (K)	5500 - 6800
$<T_{eff}>$ (K)	6150

Figure 1. δ Cep (A) parameters, taken from Benedict et al. (2002); van Leeuwen (2007); Majaess et al. (2012).

We can correlate the mass with the luminosity and diameter of a star through this equation:

$$m_{star} - m_{\odot} = -2.5log_{10}[\frac{L_{star}}{L_{\odot}} \cdot (\frac{d_{\odot}}{d_{star}})^2] \quad (4)$$

where:

- m is the apparent magnitude.
- d the distance.

We are going to use the values for L and d summarized in Figure 1. One important thing to point out is that these values correspond to δ Cep A, the brightest star in the δ Cep system, that is formed by four stars.

The estimation for L an T_{eff} that we are going to obtained will be compared with the values given in Figure 1. Using equation (4) it is possible to estimate the luminosity of the star when its magnitude reaches its maximum and minimum. Thus, from equation (4) we can obtain:

$$\frac{L}{L_{\odot}} = 10^{(m_{\odot} - m_{star})/2.5} \cdot (\frac{d_{star}}{d_{\odot}})^2 \quad (5)$$

Using $m_{V,max}$ and $m_{V,min}$ values from Figure 1 in equation (5) we obtain the values for L presented in Figure 2.

Sun (☉) Parameters	
Parameter	Value
m_V	-26.74
Distance (pc)	4.84×10^{-6}
Mass (kg)	1.988×10^{30}
Radius (m)	3.828×10^{26}
Luminosity (W)	6.963×10^{8}
T_{eff} (K)	5772

Figure 2. Sun parameters, taken from Williams (2013) (NASA).

Parameter	Computed Value
r_{mean} ($R_\odot$)	44.5
r_{min} ($R_\odot$)	41.83
r_{max} ($R_\odot$)	47.17
Δr ($R_\odot$)	5.34
L_{min} ($L_\odot$)	~1136
L_{max} ($L_\odot$)	~2579
L_{mean} ($L_\odot$)[1]	~1858
$<L>$ ($L_\odot$)[2]	~1498
ΔL ($L_\odot$)	~1450
ε_L (%)[3]	~7.1
$T_{eff,min}$ (K)	~5468
$T_{eff,max}$ (K)	~6688
$<T_{eff}>$ (K)	~5678
ΔT_{eff} (K)	~1220
ε_{Teff} (%)	~7.8

Figure 3. Main minimum and maximum values obtained for r, L and T_{eff} for δ Cep (A). $_1$ indicates the mean luminosity from the average of L_{min} y L_{max}, ^{2} indicates mean luminosity from $< L >$ y ^{3} indicates the error between L_{mean} and the luminosity presented in Figure 1.

Basically, what we did was:

1. Compute L_{min} y L_{max} using equation (5).
2. Consider r_{min} = 0.94r and r_{max} = 1.06r.
3. Using equation (3) compute the limits for T_{eff}.

Main results are presented in Figure 3.

2. PROBLEM: ZERO AGE MAIN-SEQUENCE (ZAMS) AND LINEAR ADIABATIC WAVE EQUATION (LAWE).

2.1.

For the ZAMS structure of a star we will use the values shown in Table 1. We will model and focus on a starwith mass M = 6.5 $M_\odot$. Values for this mass were interpolated/ extrapolated using initial points the values given by 1, 3 and 15 $M_\odot$.

2.2.

We have to remember that spherical harmonics have associated "quantum numbers":

- Principal quantum number, n, where 1 $\leq$ n.
- Azimuthal quantum number (angular momentum), l, where 0 $\leq$ l $\leq$ n-1.
- Magnetic quantum number (Projection of angular momentum), m, where -l $\leq$ m$\leq$l.

$M/M_\odot$	P_c (dyn cm^{-2})	T_c (K)	R (cm)	$L/L_\odot$
1	1.482 10^{17}	1.442 10^{7}	6.932 10^{10}	0.9083
3	1.141 10^{17}	2.347 10^{7}	1.779 10^{11}	98.35
6.5	8.889 10^{16}	2.618 10^{7}	1.863 10^{11}	2440
15	2.769 10^{16}	3.275 10^{7}	3.289 10^{11}	1.96 10^{4}

Table 1
Table given in homework task, in file `readme.pdf`. Values for M = 6.5 $M_\odot$ were computed interpolating linearly, except for L value; which was interpolated using an quadratic function. From left to right, columns are: Mass ($M_\odot$), central pressure (in cgs), central temperature (K), radius and Luminosity ($L_\odot$) of our model star; assuming X = 0.74 and Y = 0.24 for abundances.

n	Period (s)
0	6178
1	4700
2	3658
3	2987

Table 2
Estimated pulsating periods, using star modeled with mass equal to 6.5$M_\odot$ for fundamental mode and first three overtones.

If we choose l = 0 this means that the star pulsates radially. Using the star modeled in part i) we execute `puls.exe`, which integrates the LAWE numerically. We give to this program, as an input, the output given by `ZAMS.exe`. We hand to this program estimated periods from 200 to 6500 s, varying intervals in ΔP_{guess} = 200 s, until there is no more radial nodes in Y_1(Y_1 = $\delta r/r$). With these input we find the values presented in Table 2.

2.3.

In this part we simply plot output given by `puls.exe` for the model of our star of 6.5$M_\odot$. These result are in Figure 4 and Figure 5.

It is possible to see that:

- δP/P decreases in outer regions of the star.
- δP/P decreases in higher orders for higher overtones.
- δr/r increases in outer regions of the star.
- δr/r increases in higher order for lower overtones. So it reaches its maximum value in the surface of the star in the fundamental mode.
- From the δr/r vs r/R_{star} plot, number n says how many times our curve cross the value δr/r = 0.

2.4.

Two functions, f, g are orthogonal in an interval I = [a,b] if:

$$< f(x), g(x) >= \int_a^b fg dx = 0 \quad (6)$$

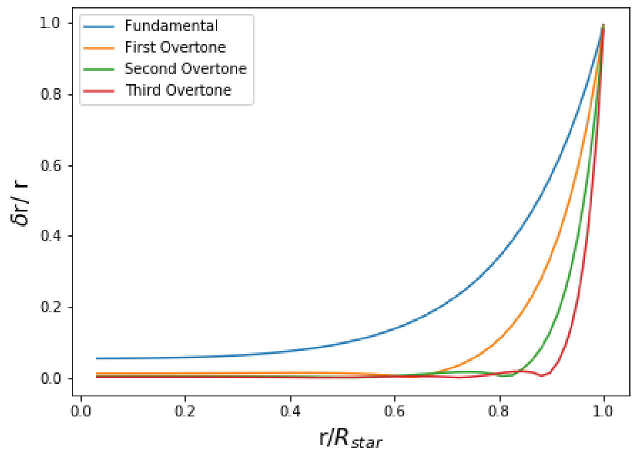

Figure 4. Output values for $\delta r/r$ given by `puls.exe` in function of the position inside the star.

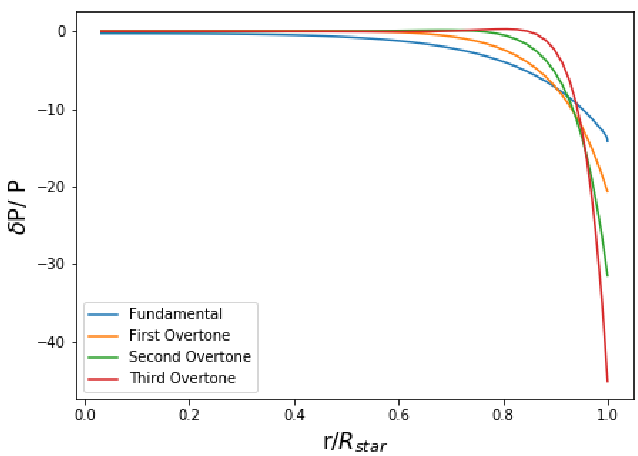

Figure 5. Output values for $\delta P/P$ given by `puls.exe` in function of the position inside the star.

In this case is useful to know (and prove) that:

$$\int_0^R \frac{\delta r_n}{r} \frac{\delta r_{n+1}}{r} 4\pi\rho(r)dr = 0 \tag{7}$$

Since $\rho(r)$ is dependent of the radius we can do:

$$\rho(r) = \rho_0(r) + \delta\rho(r) \tag{8}$$

And as we know from the LAWE it is:

$$\rho(r) = \rho_0(r)(1 + \frac{1}{\Gamma_1}\frac{\delta P}{P}) \tag{9}$$

Thus, equation (6) is now:

$$\int_0^1 \frac{\delta r_n}{r} \frac{\delta r_{n+1}}{r} 4\pi x^2 R\rho(x)(1 + \frac{1}{\Gamma_1}\frac{\delta P}{P})dx = 0 \tag{10}$$

where x = r/R and the Jacobian $R^2\rho(r)dr = \rho(x)dx$.

We can approximate the integral as a Riemann sum because we are working with certain values not at continuous function:

$$\int_0^1 h(x)dx = \sum_{i=1}^{n} h(x_i)(x_{i+1} - x_i) \tag{11}$$

Doing this we can compute the following result:

$$4\pi\int_0^1 \frac{\delta r_0}{r}\frac{\delta r_1}{r}\rho(x)x^2(1 + \frac{\delta\rho}{\rho})dx \sim -10^{-4} \tag{12}$$

$$4\pi\int_0^1 \frac{\delta r_0}{r}\frac{\delta r_2}{r}\rho(x)x^2(1 + \frac{\delta\rho}{\rho})dx \sim -10^{-4} \tag{13}$$

$$4\pi\int_0^1 \frac{\delta r_0}{r}\frac{\delta r_3}{r}\rho(x)x^2(1 + \frac{\delta\rho}{\rho})dx \sim -10^{-4} \tag{14}$$

$$4\pi\int_0^1 \frac{\delta r_1}{r}\frac{\delta r_3}{r}\rho(x)x^2(1 + \frac{\delta\rho}{\rho})dx \sim -10^{-4} \tag{15}$$

$$4\pi\int_0^1 \frac{\delta r_1}{r}\frac{\delta r_2}{r}\rho(x)x^2(1 + \frac{\delta\rho}{\rho})dx \sim -10^{-5} \tag{16}$$

$$4\pi\int_0^1 \frac{\delta r_2}{r}\frac{\delta r_3}{r}\rho(x)x^2(1 + \frac{\delta\rho}{\rho})dx \sim -10^{-5} \tag{17}$$

We can see that the result of these integral are practically 0, therefore, the eigenfunction are orthogonal.

3. PROBLEM: MODELING A STAR AND COMPUTING ITS OSCILLATION PERIOD.

The following is Problem 14.3 in Carrol Ostlies book An Introduction to Modern Astrophysics (2nd Ed.). In this problem, were going to carry out a nonlinear calculation of the radial pulsation of the named one-zone model. The equations that describe the oscillation of this model obey the Newtons Seconds law for the forces on a shell:

$$m\frac{dv}{dt} = -G\frac{Mm}{R^2} + 4\pi R^2 P \tag{18}$$

The definition of the velocity, v, of the mass shell is:

$$v = \frac{dR}{dt} \tag{19}$$

Also, if we assume that the expansion and contraction of the gas are adiabatic, we can say that:

$$P_i V_i^\gamma = P_f V_f^\gamma \tag{20}$$

Where P is the pressure, V the volume, and γ is the adiabatic index. Initial (i) and final (f) subscripts refer to any two instants during the pulsation cycle.

3.1.

Here, equation (18) is an equation between forces. The left side is basically $m\frac{dv}{dt} = m\ddot{R}$. We can divide the right side in two sub-sides: a gravitational one (with constant G), which is, Newton Gravitational Attraction between two bodies (in this case, the mass shell of the star and its environment); the other side is based on the pressure on a shell that fights against the gravity. So, basically, its a kind of hydrostatic equilibrium. But, in hydrostatic equilibrium we assume that dv/dt = 0, there is no movement. In this case dv/dt $\neq$ 0.

3.2.

We have to prove

$$P_i R_i^{3\gamma} = P_f R_f^{3\gamma} \tag{21}$$

using equation (18).
Assuming a model of a star spherical, we can calculate its volume V:

$$V = \frac{4}{3}\pi R^3 \tag{22}$$

Replacing equation (22) in equation (20) we can obtain:

$$P_i(\frac{4}{3}\pi R_i^3)^\gamma = P_f(\frac{4}{3}\pi R_f^3)^\gamma \tag{23}$$

where R_i and R_f are the "initial" and "final" radius. We can simplify equation (23):

$$P_i(R_i^3)^\gamma = P_f(R_f^3)^\gamma \tag{24}$$

that is equivalent to:

$$P_i R_i^{3\gamma} = P_f R_f^{3\gamma} \tag{25}$$

That is exactly what we wanted to prove.

3.3.

In this part, instead of taking derivatives we will use discrete differences:

$$\frac{dv}{dt} = \frac{(v_f - v_i)}{\Delta t} \tag{26}$$

$$\frac{dR}{dt} = \frac{(R_f - R_i)}{\Delta t} \tag{27}$$

Replacing equation (26) in equation (18) we obtain:

$$m\frac{(v_f - v_i)}{\Delta t} = -G\frac{Mm}{R_i^2} + 4\pi R_i^2 P_i \tag{28}$$

If we multiply both sides by $\Delta t/m$ and then add v_i, we obtained:

$$v_f = -G\frac{M}{R_i^2}\Delta T + 4\pi R_i^2 P_i \frac{\Delta T}{m} + v_i \tag{29}$$

Factoring by Δt:

$$v_f = v_i + (\frac{4\pi R_i^2 P_i}{m} - G\frac{M}{R_i^2})\Delta t \tag{30}$$

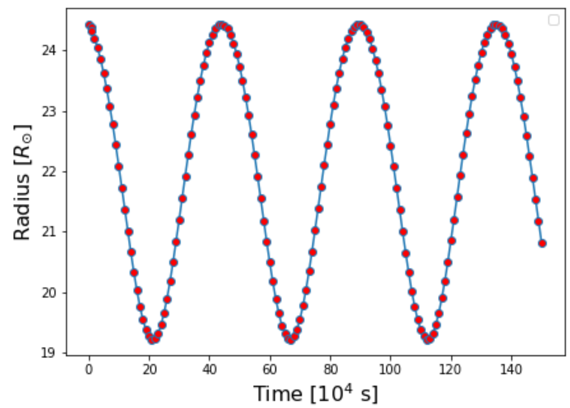

Figure 6. Results of radius for our one-zone model.
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Replacing equation (27) in equation (19) it is obtained:

$$v = v_f = \frac{R_f - R_i}{\Delta t} \tag{31}$$

Multiplying both side by Δt and then adding R_i results:

$$R_f = R_i + v_f \Delta t \tag{32}$$

3.4.

Now, using equation (21) and equation (32) we modeled a star, using the values:

- $P_i = 5.6\ 10^4\ \mathrm{N}m^{-2}$.
- $R_i = 1.7\ 10^{10}$ m.
- $v_i = 0$ m/s.
- $\Delta t = 10^4$ s.
- $t_i = 0$ s.
- $t_f = 1.5\ 10^6$ s.

We iterate 150 times the procedure to obtain v_f, R_f and P_f, keeping a Δt constant and using the previous values obtained to calcule to next.
Results are plot in Figure 6, Figure 7 and Figure 8.

3.5.

Finally, for our graphs we measure the following parameters:

- A period, P of ~46 (10^4 s) = ~5.32 days.
- An equilibrium radius, $R_0 = \sim 2.61\ R_\odot$.

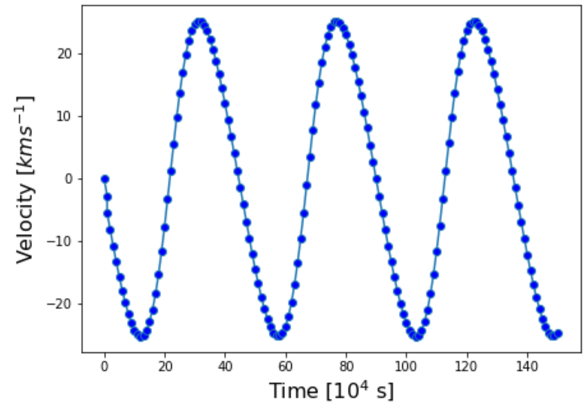

Figure 7. Results of velocity for our one-zone model.
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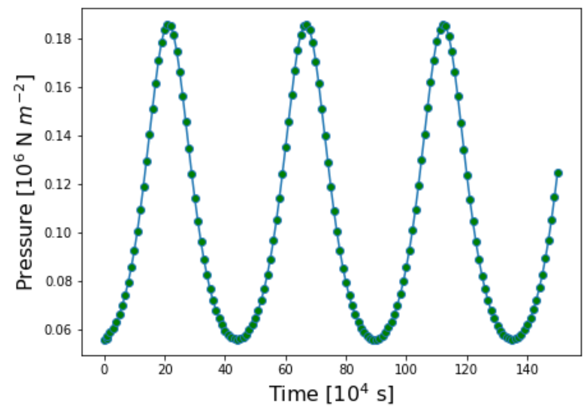

Figure 8. Results of pressure for our one-zone model.
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4. PROBLEM: SURFACE BEHAVIOUR OF NON-RADIAL PULSATION MODES IN THE SURFACE OF A STAR.

In this problem we estimate the quantum numbers that describes a spherical harmonic only based on its appearance.

Total number of lines says the quantity of the number l (angular degree), the lines that cross perpendicularly the equator of the star are the quantity of |m| (m: azimuthal order). The result for *(m, l)* of Figure 9 are in Table 3. It is possible to see that i., iii. and iv. corresponds to *zonal modes* ($| m | = 0$). ii., vii. and ix. are *sectoral modes* ($| m | =$ l). Other combination of *(l, m)* corresponds to *tesseral modes*.

In every case it is impossible to say anything about the radial order because the images shows only the behaviour in the surface of the star.

5. PROBLEM: P-MODES AND G-MODES.

5.1.

The program `puls.exe` gives as output a period (therefore, a frequency), order of n, r/R_{star}, $\delta r/r_0 = \xi_r$ and $\delta P/P$. We run `puls.exe` giving as inputs the values for the rows corresponding to $1M_\odot$ and $15M_\odot$ from tableelel, the chemical compositions are X = 0.74 and Y = 0.24. The frequency comes from $\nu = \frac{1}{P}$. The results are shown in Table 4.

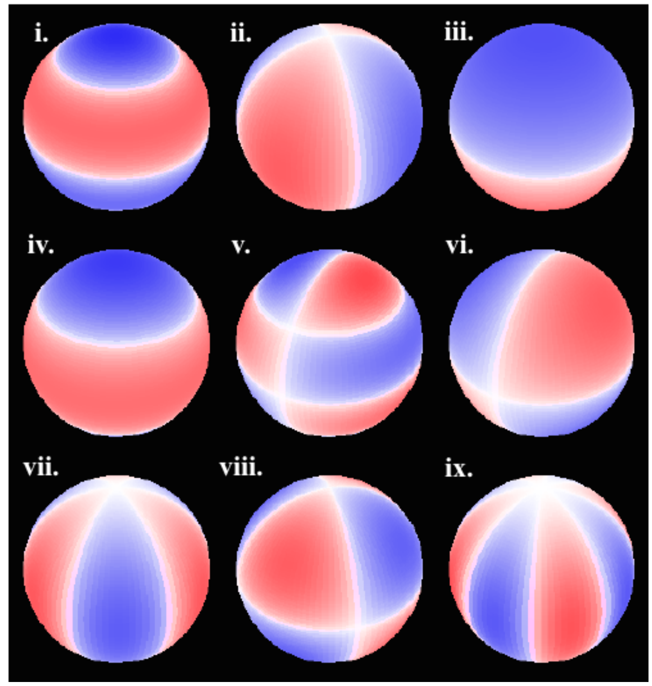

Figure 9. These figures show different behaviours of non-radial modes of pulsation in the surface of a star. It is assumed that the images are not edge on, but with an inclination angle $0 < i < 90$. Every case is labeled by a Roman numerals.
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Roman numeral (m: azimuthal order.)	l: angular degree	$\mid m \mid$
i.	2	0
ii.	2	2
iii.	1	0
iv.	1	0
v.	3	1
vi.	2	1
vii.	3	3
viii.	3	2
ix.	4	4

Table 3
Values of (m,l) for every case of Figure 9

Mass ($M_\odot$)	n	Period (s)	$\nu(\mu Hz)$	l	p-order	g-order
1	25	19726	50.6	5	-	24
1	7	4112	242.6	10	-	5
1	11	253	3952	100	10	-
15	2	1960	510	100	1	-
15	13	204	4901	1000	7	-

Table 4
Inputs and outputs obtained from `puls.exe`.

We can see that:

- ν increases when l increases.
- p-modes decreases when l increases.
- g-modes decreases when l increases and are high for small l and big periods.

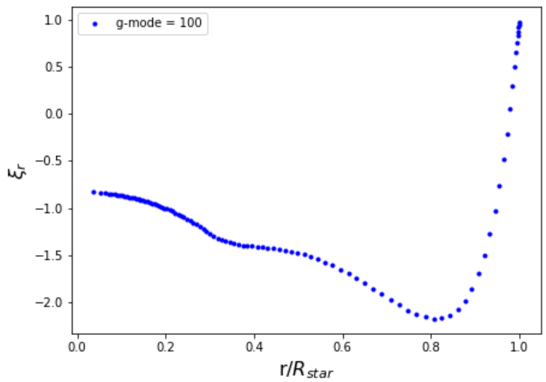

Figure 10. ξ_r in function of $\delta r/r_0$ for a model with $1M_\odot$.
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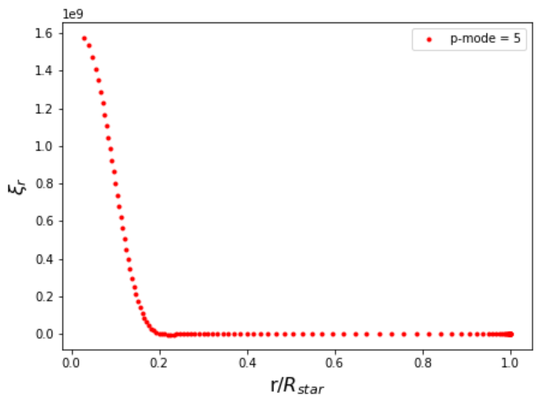

Figure 11. ξ_r in function of $\delta r/r_0$ for a model with $15M_\odot$.
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5.2.

Since they are different types of oscillations (acoustic and gravitational ones) p-modes they increase during its travel through the interior of the star; and for g-modes the maximum value is at the atmosphere of the star (were getting further from the mass center).

Note : all the programs used and codes are available in Public GitHub repository.