1 The current diffusion equation

On the diffusion time scale, the (perpendicular) transport in an axisymmetric tokamak can be described by flux-surface averaged transport (diffusion-convection) equations.

$$\frac{\partial \langle y \rangle}{\partial t} - \frac{\partial}{\partial x}\left(D \frac{\partial \langle y \rangle}{\partial x} + v_y \langle y \rangle \right) = S_y \tag{1}$$

Here, $\langle y \rangle$ corresponds to flux-averaged quantities,

$$\langle y \rangle = \frac{\partial}{\partial V} \int y \mathrm{d}V = \frac{\oint \frac{y}{B_{\mathrm{pol}}} \mathrm{d}l_\theta}{\oint \frac{1}{B_{\mathrm{pol}}} \mathrm{d}l_\theta} \tag{2}$$

ψ is the poloidal magnetic flux and x is a normalized flux coordinate, defined as

$$x = \frac{\rho}{\rho_1}, \quad \rho = \sqrt{\frac{\Phi}{\pi B_0}} \tag{3}$$

Φ is the toroidal magnetic flux, ρ_1 is ρ at the last closed flux surface. In particular, for the magnetic flux (or current) diffusion, we have

$$\begin{gathered} \frac{\partial \psi}{\partial t} - \frac{\left\langle |\nabla \rho|^2 / R^2 \right\rangle}{\mu_0 \sigma_\parallel \rho_1^2 \langle 1/R^2 \rangle} \frac{\partial^2 \psi}{\partial x^2} + \\ \left\{ \frac{\left\langle |\nabla \rho|^2 / R^2 \right\rangle}{\mu_0 \sigma_\parallel \rho_1^2 \langle 1/R^2 \rangle} \frac{\partial}{\partial x} \left[\ln \left(\frac{V' \left\langle |\nabla \rho|^2 / R^2 \right\rangle}{F} \right) \right] + \frac{x}{\rho_1} \frac{\mathrm{d}\rho_1}{\mathrm{d}t} \right\} \frac{\partial \psi}{\partial x} = \frac{B_0}{\sigma_\parallel F \langle 1/R^2 \rangle} j_{\mathrm{ni}} \end{gathered} \tag{4}$$

where

$$V' = \partial V / \partial \rho \tag{5}$$

After using

$$\begin{aligned} g_1 &= \left\langle 1/R^2 \right\rangle \\ g_2 &= \left\langle |\nabla \rho|^2 / R^2 \right\rangle \end{aligned} \tag{6}$$

the equation can be written as

$$\frac{\partial \psi}{\partial t} - \frac{g_2}{\mu_0 \sigma_\parallel \rho_1^2 g_1} \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial}{\partial x} \ln \left(\frac{V' g_2}{F} \right) \frac{\partial \psi}{\partial x} \right] + \frac{x}{\rho_1} \frac{\mathrm{d}\rho_1}{\mathrm{d}t} \frac{\partial \psi}{\partial x} = \frac{B_0}{\sigma_\parallel F g_1} j_{\mathrm{ni}} \tag{7}$$

1.1 Boundary conditions

On the magnetic axis we have (from the geometry)

$$\left. \frac{\partial \psi}{\partial x} \right|_{x=0} = 0 \tag{8}$$

On the plasma boundary ($x = 1$), the most common boundary condition is a prescribed total plasma current

$$I_{\mathrm{p}} = -\frac{1}{2\pi \mu_0} V' g_2 \left. \frac{\partial \psi}{\partial \rho} \right|_{x=1} \tag{9}$$

In case of free-boundary equilibrium simulations, the plasma current is no longer presribed. Hence different boundary conditions must be used. As the magnetic flux must be consistent in the transport and equilibrium equations, the natural boundary condition (at $x = 1$) would be

$$\psi^{\mathrm{diff}} = \psi^{\mathrm{equi}} \tag{10}$$

This is similar to prescribing the loop voltage in fixed boundary simulation, which is known to be prone to numerical errors. For this reason, we have derived two different boundary conditions for FBE simulations. The first one follows from (10) and using $L_{\mathrm{i}} I_{\mathrm{p}} = \psi_0 - \psi_1$. This allows us to calculate an I_{p} predictor, which enforces the ψ consistency:

$$I_{\mathrm{p}}^{*} = I_{\mathrm{p}} \left(1 + \frac{\psi_1^{\mathrm{diff}} - \psi_1^{\mathrm{eq}}}{\psi_0^{\mathrm{eq}} - \psi_1^{\mathrm{eq}}} \right) \tag{11}$$

The second possibility is similar to the approach in DINA [1]:

$$\frac{\psi_1}{\tilde{L}_{\mathrm{ext}}} - \gamma C \left.\frac{\partial \psi}{\partial \rho}\right|_{\rho=1} = \frac{1}{\tilde{L}_{\mathrm{ext}}} \left(\tilde{\psi}_1 + \psi_0^{\mathrm{ext}} - \tilde{\psi}_0^{\mathrm{ext}} \right) + \tilde{I}_{\mathrm{p}}. \tag{12}$$

2 Equilibrium

MHD equilibrium is described by the Grad-Shafranov equation

$$-\Delta^{*}\psi = \mu_0 R j_{\phi} = \mu_0 R^2 \frac{\mathrm{d}p}{\mathrm{d}\psi} + F \frac{\mathrm{d}F}{\mathrm{d}\psi} \tag{13}$$

On the diffusion time scale, this equation is valid for every time instant. On the right-hand side appears $p'(\psi)$—the pressure gradient—and $FF'(\psi)$. p' can be calculated from $p(x)$ and $\partial\psi/\partial x$. The calculation of FF' is more difficult. One possibility, which is currently used in CRONOS and ETS-C, is averaging the G-S equation. One obtains

$$FF' = \frac{\mu_0}{g_1} \left(\langle j_{\phi}/R \rangle - p' \right) \tag{14}$$

The average current term can be calculated as

$$\langle j_{\phi}/R \rangle = -\frac{\frac{\partial}{\partial x}\left(V' g_2 \frac{\partial \psi}{\partial x} \right)}{\rho_1^2 \mu_0 V'} \tag{15}$$

Since the current density is generally continuous, it follows that

$$\psi \in C^2,\ g_2 \in C^1,\ V' \in C^1 \tag{16}$$

3 (Numerical) challenges

- Boundary conditions
- Noise from the G-S equation
- Calculation of $\frac{\partial^2 \psi}{\partial x^2}$
- Geometric coefficients regularity
- Possible interplay in FF' and G-S calculation

References

[1] R. R. Khayrutdinov and V. E. Lukash. Studies of plasma equilibrium and transport in a tokamak fusion device with the inverse-variable technique. *Journal of Computational Physics*, 109(2):193–201, 1993.