

1 Diffusion equations in tokamak

On the diffusion time scale, the (perpendicular) transport in an axisymmetric tokamak can be described by flux-surface averaged transport (diffusion-convection) equations.

$$\frac{\partial \langle y \rangle}{\partial t} - \frac{\partial}{\partial x} \left(D \frac{\partial \langle y \rangle}{\partial x} + v_y \langle y \rangle \right) = S_y$$

Here, $\langle y \rangle$ corresponds to flux-averaged quantities,

$$\langle y \rangle = \frac{\partial}{\partial V} \int y dV = \frac{\oint \frac{y}{B_{\text{pol}}} dl_\theta}{\oint \frac{1}{B_{\text{pol}}} dl_\theta}$$

ψ is the poloidal magnetic flux and x is a normalized flux coordinate, defined as

$$x = \frac{\rho}{\rho_1}, \quad \rho = \sqrt{\frac{\Phi}{\pi B_0}}$$

Φ is the toroidal magnetic flux, ρ_1 is ρ at the last closed flux surface.

In particular, for the magnetic flux (or current) diffusion, we have

$$\begin{aligned} & \frac{\partial \psi}{\partial t} - \frac{\langle |\nabla \rho|^2 / R^2 \rangle}{\mu_0 \sigma_{\parallel} \rho_1^2 \langle 1/R^2 \rangle} \frac{\partial^2 \psi}{\partial x^2} + \\ & \left\{ \frac{\langle |\nabla \rho|^2 / R^2 \rangle}{\mu_0 \sigma_{\parallel} \rho_1^2 \langle 1/R^2 \rangle} \frac{\partial}{\partial x} \left[\ln \left(\frac{V' \langle |\nabla \rho|^2 / R^2 \rangle}{F} \right) \right] + \frac{x}{\rho_1} \frac{d\rho_1}{dt} \right\} \frac{\partial \psi}{\partial x} = \frac{B_0}{\sigma_{\parallel} F \langle 1/R^2 \rangle} j_{\text{ni}} \end{aligned}$$

where

$$V' = \partial V / \partial \rho$$

After using

$$\begin{aligned} g_1 &= \langle 1/R^2 \rangle \\ g_2 &= \langle |\nabla \rho|^2 / R^2 \rangle \end{aligned}$$

$$\frac{\partial \psi}{\partial t} - \frac{g_2}{\mu_0 \sigma_{\parallel} \rho_1^2 g_1} \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial}{\partial x} \ln \left(\frac{V' g_2}{F} \right) \frac{\partial \psi}{\partial x} \right] + \frac{x}{\rho_1} \frac{d\rho_1}{dt} \frac{\partial \psi}{\partial x} = \frac{B_0}{\sigma_{\parallel} F g_1} j_{\text{ni}}$$

The most common boundary condition is a prescribed total plasma current

$$I_p = -\frac{1}{2\pi\mu_0} V' g_2 \frac{\partial \psi}{\partial \rho} \Big|_{x=1}$$

On the magnetic axis we have

$$\frac{\partial \psi}{\partial x} \Big|_{x=0} = 0$$

2 Equilibrium

MHD equilibrium is described by the Grad-Shafranov equation

$$-\Delta^* \psi = \mu_0 R j_\phi = \mu_0 R^2 \frac{dp}{d\psi} + F \frac{dF}{d\psi}$$

On the diffusion time scale, this equation is valid for every time instant. On the right-hand side appears $p'(\psi)$ —the pressure gradient—and $FF'(\psi)$. p' can be calculated from $p(x)$ and $\partial\psi/\partial x$. The calculation of FF' is more difficult. One possibility, which is currently used in CRONOS and ETS-C, is averaging the G-S equation. One obtains

$$FF' = \frac{\mu_0}{g_1} (\langle j_\phi / R \rangle - p')$$

The average current term can be calculated as

$$\langle j_\phi / R \rangle = - \frac{\frac{\partial}{\partial x} \left(V' g_2 \frac{\partial \psi}{\partial x} \right)}{\rho_1^2 \mu_0 V'}$$

3 (Numerical) challenges

* Boundary conditions * Noise from the G-S equation * Calculation of $\frac{\partial^2 \psi}{\partial x^2}$ * Geometric coefficients regularity * Possible interplay in FF' calculation