1 Diffusion equations in tokamak

On the diffusion time scale, the (perpendicular) transport in an axisymmetric tokamak can be described by flux-surface averaged transport (diffusion-convection) equations.

$$\frac{\partial \langle y \rangle}{\partial t} - \frac{\partial}{\partial x}\left(D \frac{\partial \langle y \rangle}{\partial x} + v_y \langle y \rangle \right) = S_y$$

Here, $\langle y \rangle$ corresponds to flux-averaged quantities,

$$\langle y \rangle = \frac{\partial}{\partial V} \int y \mathrm{d}V = \frac{\oint \frac{y}{B_{\mathrm{pol}}} \mathrm{d}l_\theta}{\oint \frac{1}{B_{\mathrm{pol}}} \mathrm{d}l_\theta}$$

ψ is the poloidal magnetic flux and x is a normalized flux coordinate, defined as

$$x = \frac{\rho}{\rho_1}, \quad \rho = \sqrt{\frac{\Phi}{\pi B_0}}$$

Φ is the toroidal magnetic flux, ρ_1 is ρ at the last closed flux surface.

In particular, for the magnetic flux (or current) diffusion, we have

$$\frac{\partial \psi}{\partial t} - \frac{\left\langle |\nabla \rho|^2 \big/ R^2 \right\rangle}{\mu_0 \sigma_{\parallel} \rho_1^2 \left\langle 1/R^2 \right\rangle} \frac{\partial^2 \psi}{\partial x^2} +$$
$$\left\{ \frac{\left\langle |\nabla \rho|^2 \big/ R^2 \right\rangle}{\mu_0 \sigma_{\parallel} \rho_1^2 \left\langle 1/R^2 \right\rangle} \frac{\partial}{\partial x} \left[\ln \left(\frac{V' \left\langle |\nabla \rho|^2 \big/ R^2 \right\rangle}{F} \right) \right] + \frac{x}{\rho_1} \frac{\mathrm{d}\rho_1}{\mathrm{d}t} \right\} \frac{\partial \psi}{\partial x} = \frac{B_0}{\sigma_{\parallel} F \left\langle 1/R^2 \right\rangle} j_{\mathrm{ni}}$$

where

$$V' = \partial V / \partial \rho$$

After using

$$g_1 = \left\langle 1/R^2 \right\rangle$$
$$g_2 = \left\langle |\nabla \rho|^2 / R^2 \right\rangle$$

$$\frac{\partial \psi}{\partial t} - \frac{g_2}{\mu_0 \sigma_{\parallel} \rho_1^2 g_1} \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial}{\partial x} \ln \left(\frac{V' g_2}{F} \right) \frac{\partial \psi}{\partial x} \right] + \frac{x}{\rho_1} \frac{\mathrm{d}\rho_1}{\mathrm{d}t} \frac{\partial \psi}{\partial x} = \frac{B_0}{\sigma_{\parallel} F g_1} j_{\mathrm{ni}}$$

The most common boundary condition is a prescribed total plasma current

$$I_{\mathrm{p}} = -\frac{1}{2\pi \mu_0} V' g_2 \left. \frac{\partial \psi}{\partial \rho} \right|_{x=1}$$

On the magnetic axis we have

$$\left. \frac{\partial \psi}{\partial x} \right|_{x=0} = 0$$

2 Equilibrium

MHD equilibrium is described by the Grad-Shafranov equation

$$-\Delta^{*}\psi = \mu_0 R j_\phi = \mu_0 R^2 \frac{\mathrm{d}p}{\mathrm{d}\psi} + F\frac{\mathrm{d}F}{\mathrm{d}\psi}$$

On the diffusion time scale, this equation is valid for every time instant. On the right-hand side appears $p'(\psi)$—the pressure gradient—and $FF'(\psi)$. p' can be calculated from $p(x)$ and $\partial\psi/\partial x$. The calculation of FF' is more difficult. One possibility, which is currently used in CRONOS and ETS-C, is averaging the G-S equation. One obtains

$$FF' = \frac{\mu_0}{g_1}\left(\langle j_\phi/R\rangle - p'\right)$$

The average current term can be calculated as

$$\langle j_\phi/R\rangle = -\frac{\frac{\partial}{\partial x}\left(V' g_2 \frac{\partial\psi}{\partial x}\right)}{\rho_1^2 \mu_0 V'}$$

3 (Numerical) challenges

- Boundary conditions
- Noise from the G-S equation
- Calculation of $\frac{\partial^2\psi}{\partial x^2}$
- Geometric coefficients regularity
- Possible interplay in FF' and G-S calculation