Theorem: lcm(a, b) $\times$ gcd(a, b) = ab for any positive integers a, b.

Proof: First a

Lemma: If m > 0, lcm (ma, mb) = m $\times$ lcm (a, b).

Since lcm(ma, mb) is a multiple of ma, which is a multiple of m, we have m | lcm (ma, mb).

Let mh_1 = lcm(ma, mb), and set h_2 = lcm(a, b).

Then $ma \mid mh_1 \Rightarrow a \mid h_1$ and $mb \mid mh_1 \Rightarrow b \mid h_1$.

That says h_1 is a common multiple of a and b; but h_2 is the least common multiple, so

$$h_1 \geq h_2. \qquad (1)$$

Next, $a \mid h_2 \Rightarrow am \mid mh_2$ and $b \mid h_2 \Rightarrow bm \mid mh_2$.

Since mh_2 is a common multiple of ma and mb, and mh_1 = lcm(ma, mb), we have $mh_2 \geq mh_1$, i.e.

$$h_2 \geq h_1. \qquad (2)$$

From (1) and (2), $h_1 = h_2$.

Therefore, lcm(ma, mb) = $mh_1 = mh_2 = m \times$ lcm(a, b); proving the Lemma.

Conclusion of Proof of Theorem:

Let g = gcd(a, b). Since g | a, g | b, let a = gc and b = gd.

From a result in the text, gcd(c, d) = gcd(a/g, b/g) = 1.

Now we will prove that lcm(c, d) = cd. (3)

Since c | lcm(c, d), let lcm(c, d) = kc.

Since d | kc and gcd(c, d) = 1, d | k and so dc $\leq$ kc.

However, kc is the least common multiple and dc is a common multiple, so kc $\leq$ dc.

Hence kc = dc, i.e. lcm(c, d) = cd.

Finally, using the Lemma and (3), we have:

lcm(a, b) x gcd(a, b) = lcm(gc, gd) x g = g x lcm(c, d) $\times$ g = gcdg = (gc)(gd) = ab.

QED