Spectral Differences method for Astrophysics

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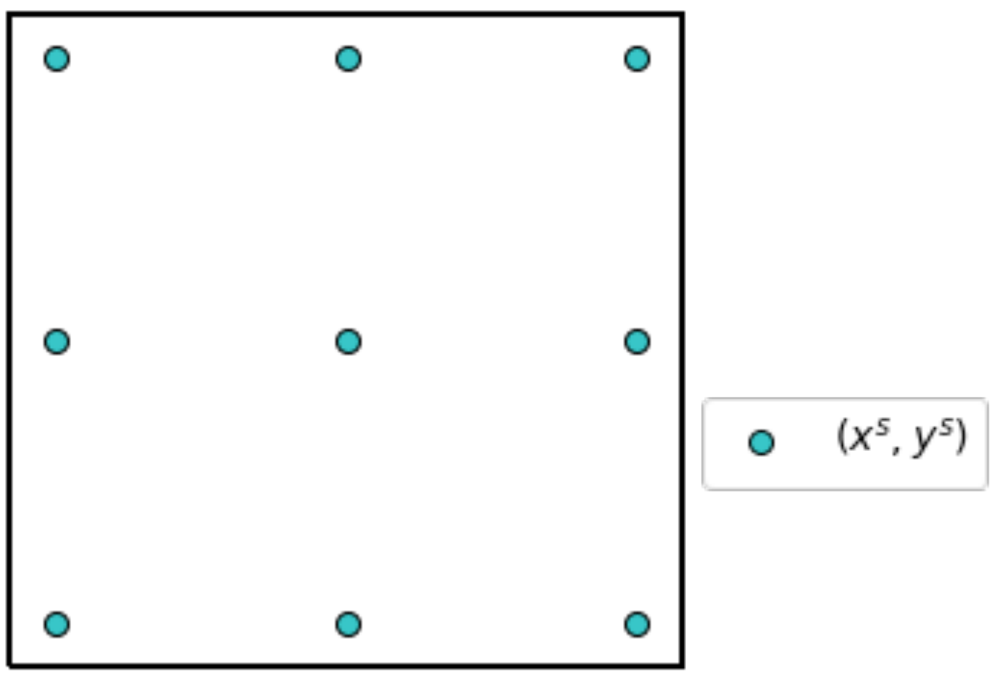
The Spectral Difference Method

- Divide the domain in Cartesian cells or *elements*.
- We define a set of **p+1** solution points $S = \left(x_0^s, x_1^s \dots x_p^s\right)$.
- The solution inside the element is given by Lagrange polynomials of degree **p**:

$$u(x,t) = \sum_{i=0}^{p} U_i(t)\ell_i^s(x).$$

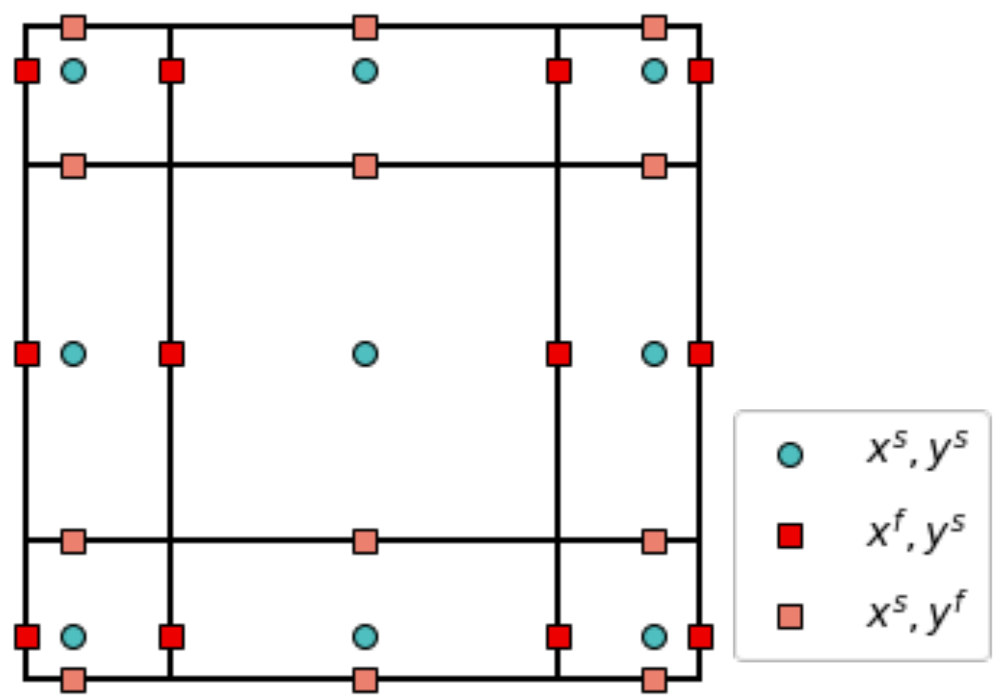


Kopriva 1998
Liu *et al.* 2006

The Spectral Difference Method

- We define a set of **p+2** flux points $F = \left(x_0^f, x_1^f \ldots x_{p+1}^f\right)$
- We compute the flux at the inner flux points as $F_i(t) = F(u(x_i^f, t))$.
- Fluxes at the two end flux points are computed using a Riemann solver.
- The flux inside the element is given by Lagrange polynomials of degree **p+1**:

$$F(x,t) = \sum_{i=0}^{p+1} F_i(t)\ell_i^f(x).$$



- We compute the divergence of the flux at the solution points and update *u* as:

$$\frac{du_i}{dt} = -\sum_{j=0}^{p+1} F_j(t)\ell_j^{\prime f}(x_i^s)$$

Kopriva 1998
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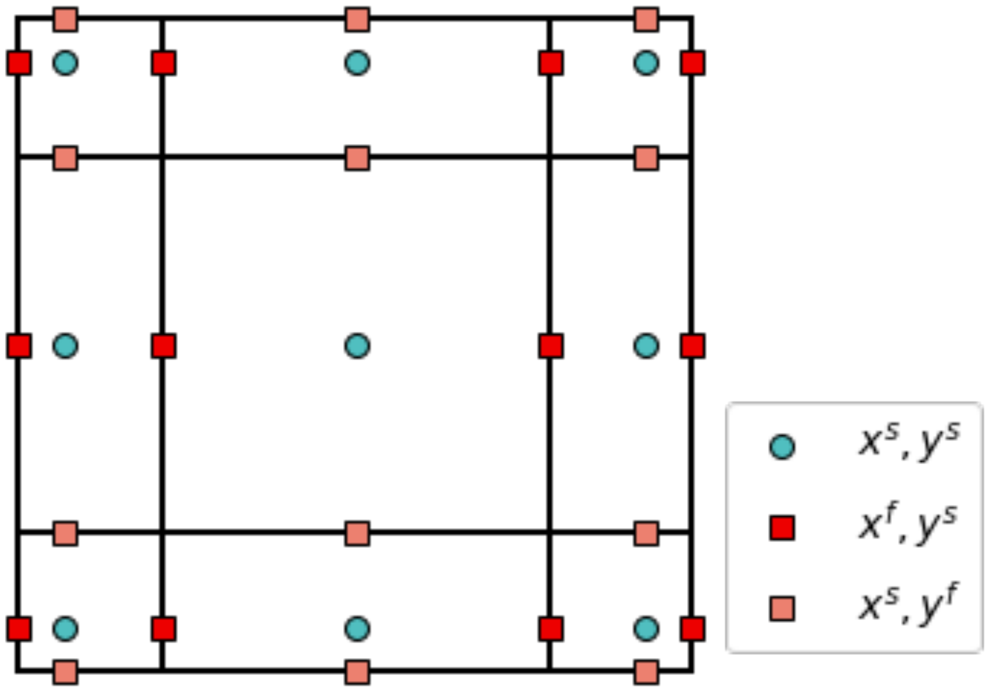
The Spectral Difference Method

- Lagrange basis for polynomials of degree p.
- Polynomials of degree p defined on p+1 nodal interpolation points .

$$\sum_{i=0}^{p} \ell_i(x) = 1, \;\; \ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{p} \frac{x - x_i}{x_j - x_i},$$

$$\ell_i(x_j) = \delta_{ij}, \;\; p(x) = \sum_{i=0}^{p} p(x_i)\ell_i(x).$$

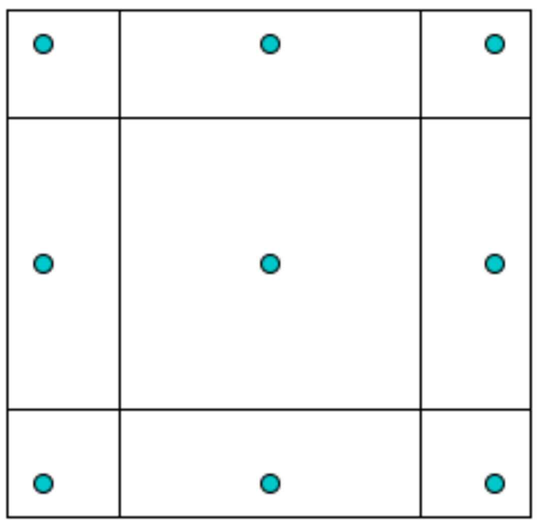


- To ensure stability, the flux points are the Gauss-Legendre quadrature points + the two end points of the element.
- The solution points are the zeros of the Chebyshev polynomials.

Kopriva 1998
Liu *et al.* 2006

Interpolation

Solution points

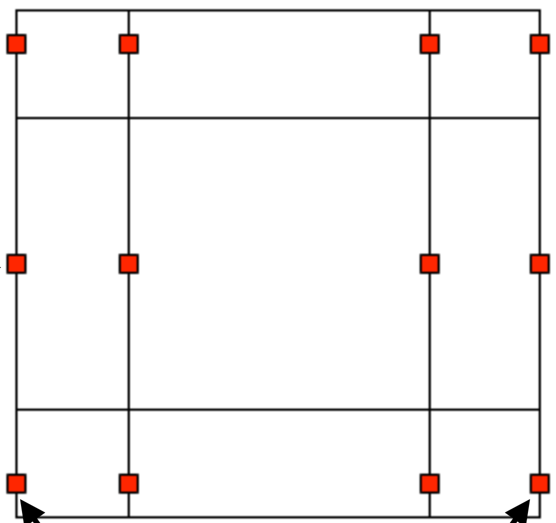


$$L_{ij}^{s,f} = \ell_j^s(x_i^f)$$

$$U(x_i^f, t) = L_{ij}^{s,f} U(x_j^s, t)$$

Flux points

$$F(x_i^f, t) = F\left(U\left(x_i^f, t\right)\right)$$

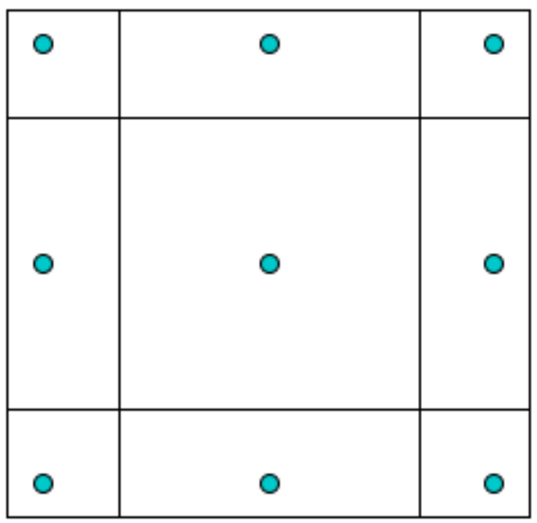


$$\mathbf{F} = \mathbf{F}\left(\mathrm{RP}\right)$$

$$L_{ij}^{\prime f,s} = \ell_j^{\prime f}(x_i^s)$$

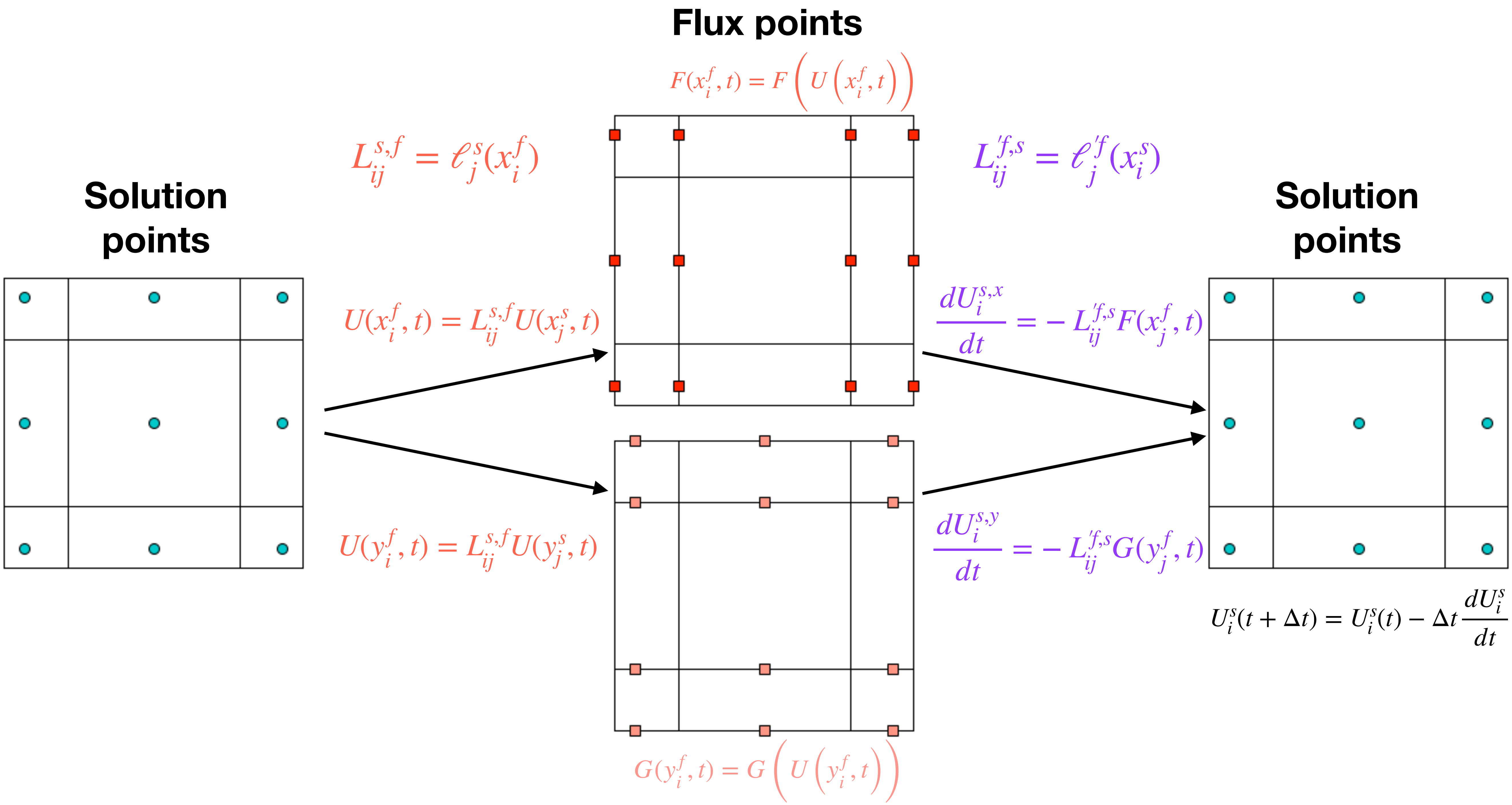
$$\frac{dU_i^s}{dt} = -L_{ij}^{\prime f,s} F(x_j^f, t)$$

Solution points



$$U_i^s(t + \Delta t) = U_i^s(t) - \Delta t \frac{dU_i^s}{dt}$$

Interpolation



Equivalence to FV method

$$u(x,t) = \sum_{m=0}^{p} u_m(t)\ell^s_m(x)$$

$$F(x,t) = \sum_{m=0}^{p+1} F_m(t)\ell^f_m(x)$$

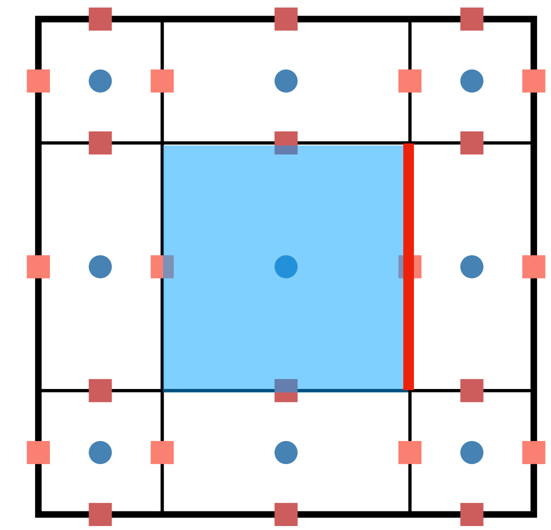
Volume Integral (1D):

$$\bar{u}_{a,m} = \frac{1}{|h_m|}\int_{I_{a,m}} u_a dx$$

Face Integral (1D):

$$\hat{F}_{a,m+1/2} = F_a(x_{m+1/2})$$

$$I_{a,m} = x_a + [x^f_m, x^f_{m+1}]$$



2D case

SD Update:

$$u(t+\Delta t) = u(t) - \Delta t \sum_{k=0}^{p}\sum_{m=0}^{p+1} w_k F_m(t)\ell'^f_m(x^s_m)$$

$$\frac{1}{h_m}\int_{I_{a,m}} \frac{\mathrm{d}}{\mathrm{d}t} u^s_{a,m}(t)dx = -\frac{1}{h_m}\int_{I_{a,m}} \hat{F}'(x)dx = -\frac{\hat{F}^f_{a,m+1} - \hat{F}^f_{a,m}}{h_m},$$

FV Update:

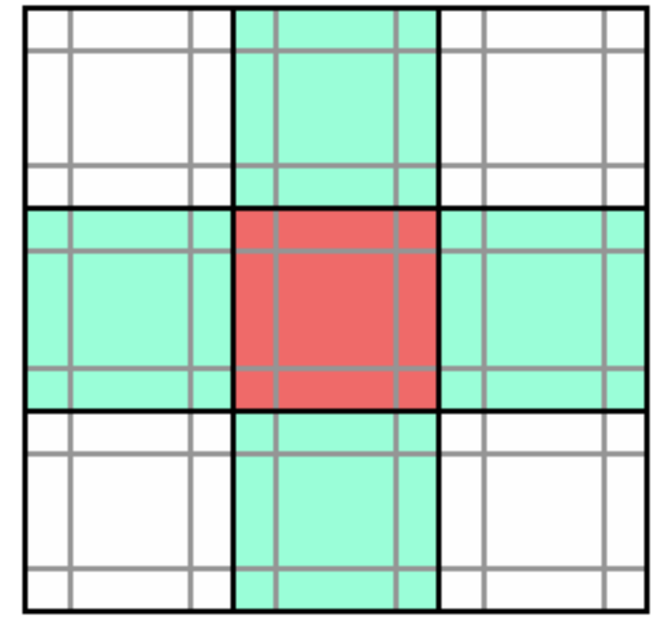
$$\bar{u}_{a,m}(t+\Delta t) = \bar{u}_{a,m}(t) - \Delta t \sum_{k=0}^{p} w_k \frac{\left(\hat{F}^{f,k}_{a,m+1} - \hat{F}^{f,k}_{a,m}\right)}{h_m}.$$

Limiting Criteria

Numerical Admissibility Criteria (NAD):

$$\min(u_{i-1}^n, u_i^n, u_{i+1}^n) \leq u_i^{n+1} \leq \max(u_{i-1}^n, u_i^n, u_{i+1}^n)$$

Cell



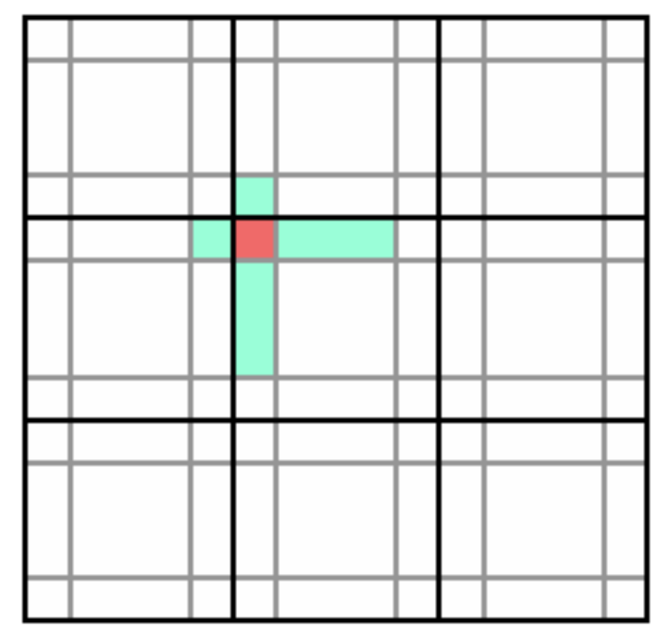
i-1 ***i*** ***i+1***

Detection of smooth extrema:

- If at least the linearized version of the numerical solution spatial derivative presents a monotonous profile.

$$v_h(x) = \partial_x u_i^{n+1} + (x - x_i)\partial_{xx} u_i^{n+1}$$

Subcell



Physical Admissibility Criteria (PAD):

$$\rho_i^{n+1} >= \rho_{\min}$$

$$p_i^{n+1} >= p_{\min}$$

SubCell Trouble correction

Candidate solution:

$$\tilde{u}_m^{n+1} = \hat{u}_m^n - \frac{\left(\hat{F}_{m+1/2} - \hat{F}_{m-1/2}\right)}{\Delta x_m}\Delta t$$

If the subcell is troubled:

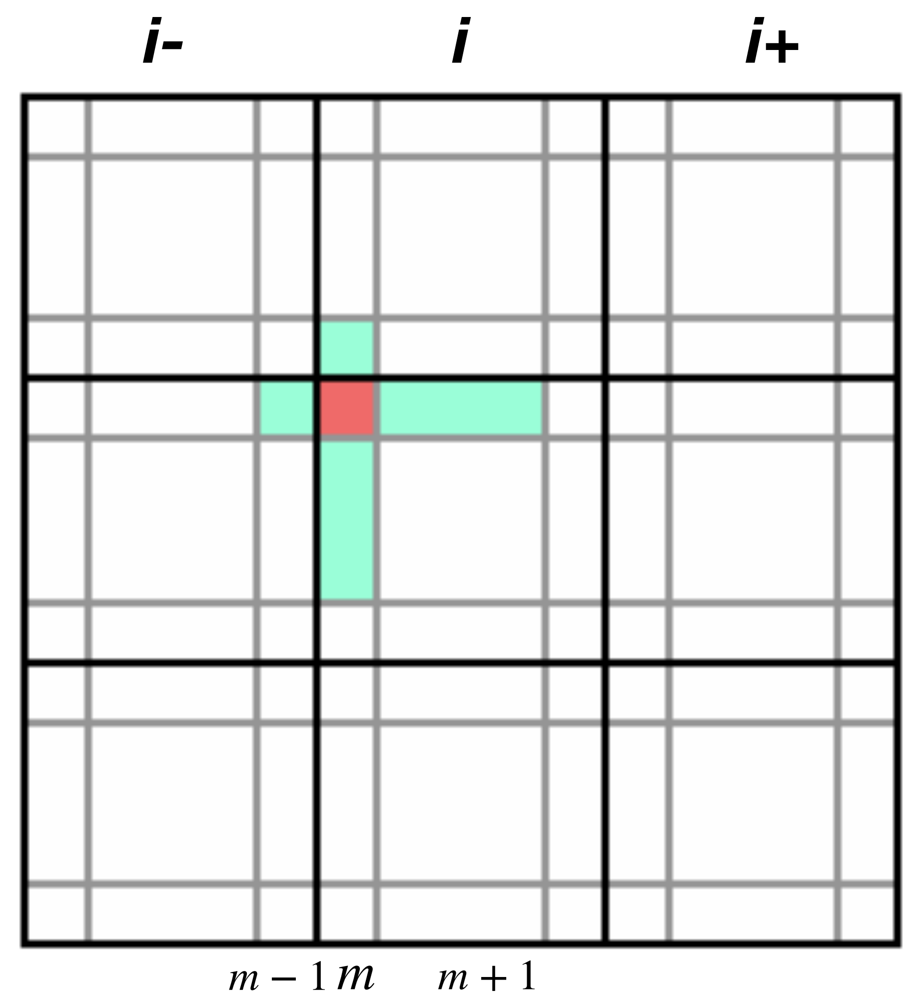
Fall-Back scheme (2nd order Godunov):

$$\hat{F}_{m+1/2} = F\left(RP\{\bar{u}_{m+1}^n, \bar{u}_m^n\}\right)$$

$$\hat{F}_{m-1/2} = F\left(RP\{\bar{u}_m^n, \bar{u}_{m-1}^n\}\right)$$

Corrected solution for troubled and affected cells:

$$\hat{u}_m^{n+1} = \hat{u}_m^n - \frac{\left(\boxed{\hat{F}_{m+1/2}} - \boxed{\hat{F}_{m-1/2}}\right)}{\Delta x_m}\Delta t$$



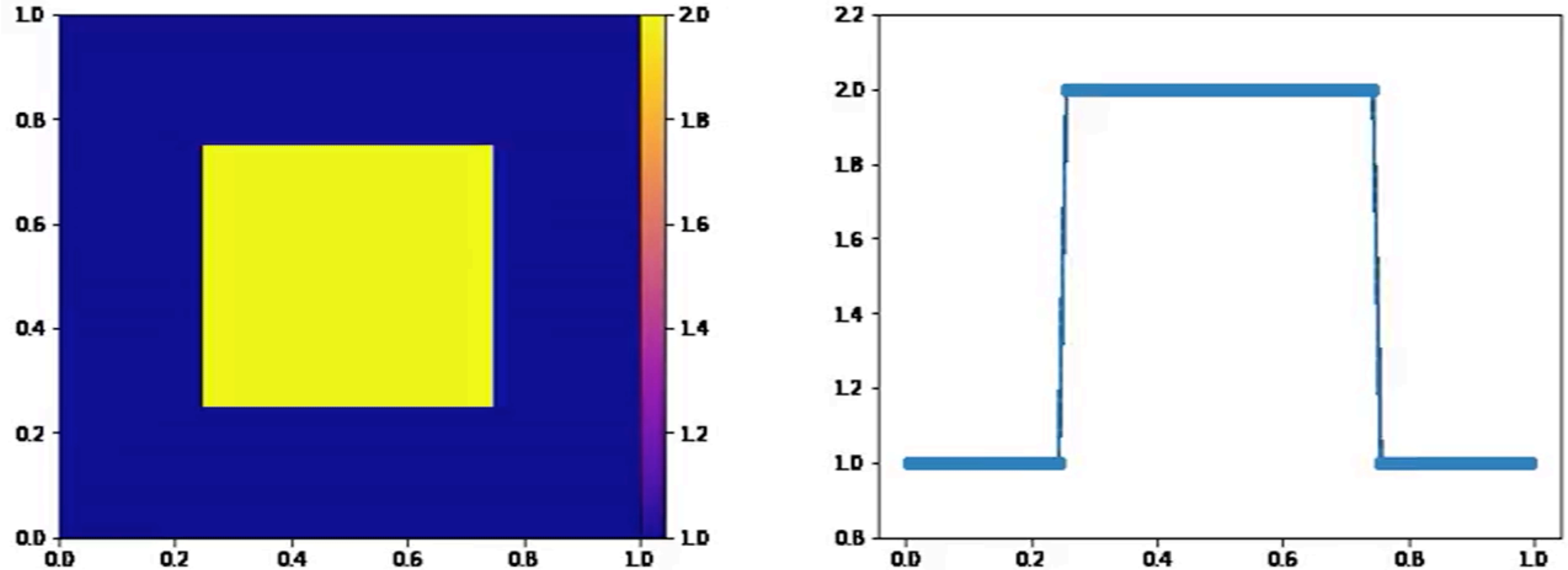
$$\boxed{\bar{u}_{m+1}^{n+1}} = \bar{u}_{m+1}^n - \frac{\left(\hat{F}_{m+3/2} - \boxed{\hat{F}_{m+1/2}}\right)}{\Delta x_{m+1}}\Delta t$$

$$\boxed{\bar{u}_{m-1}^{n+1}} = \bar{u}_{m-1}^n - \frac{\left(\boxed{\hat{F}_{m-1/2}} - \hat{F}_{m-3/2}\right)}{\Delta x_{m-1}}\Delta t$$

Discontinuous Solution

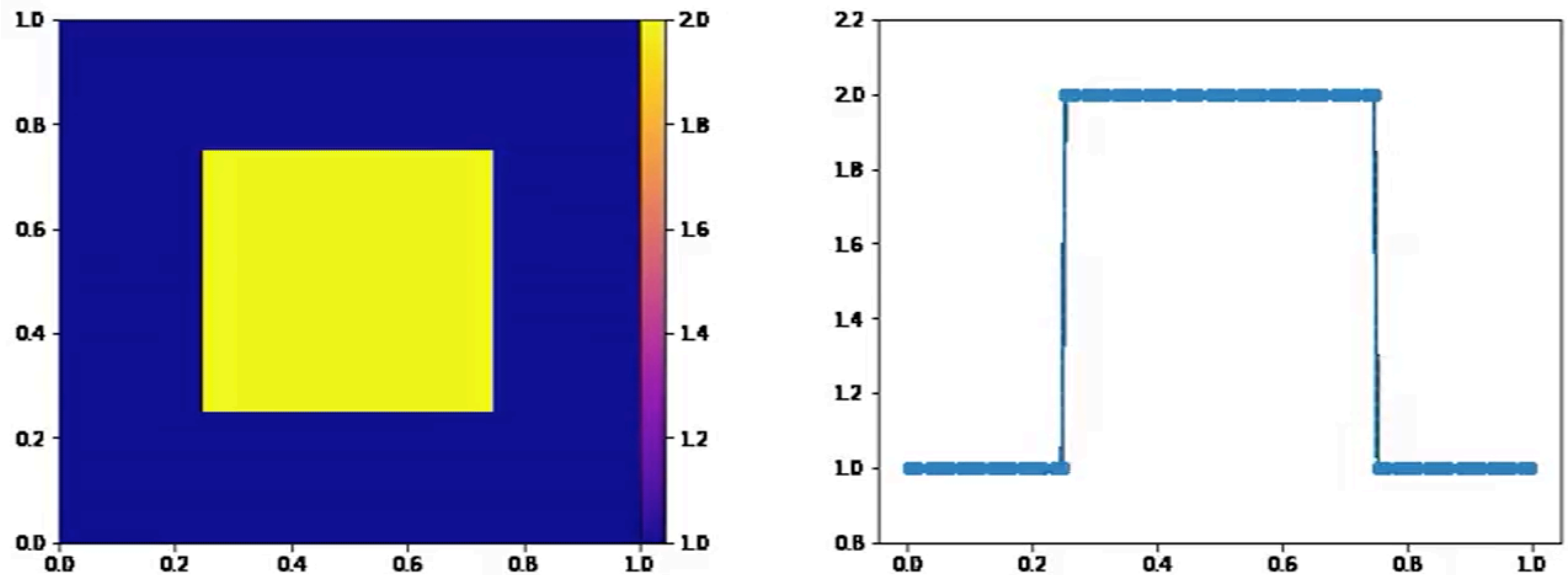
Finite Volume 2nd order Godunov

D.o.f = 40 x 2



SD-ADER + Fall-Back scheme 4th order

D.o.f = 20 x 4



Discontinuous Solution

Finite Volume 2nd order Godunov

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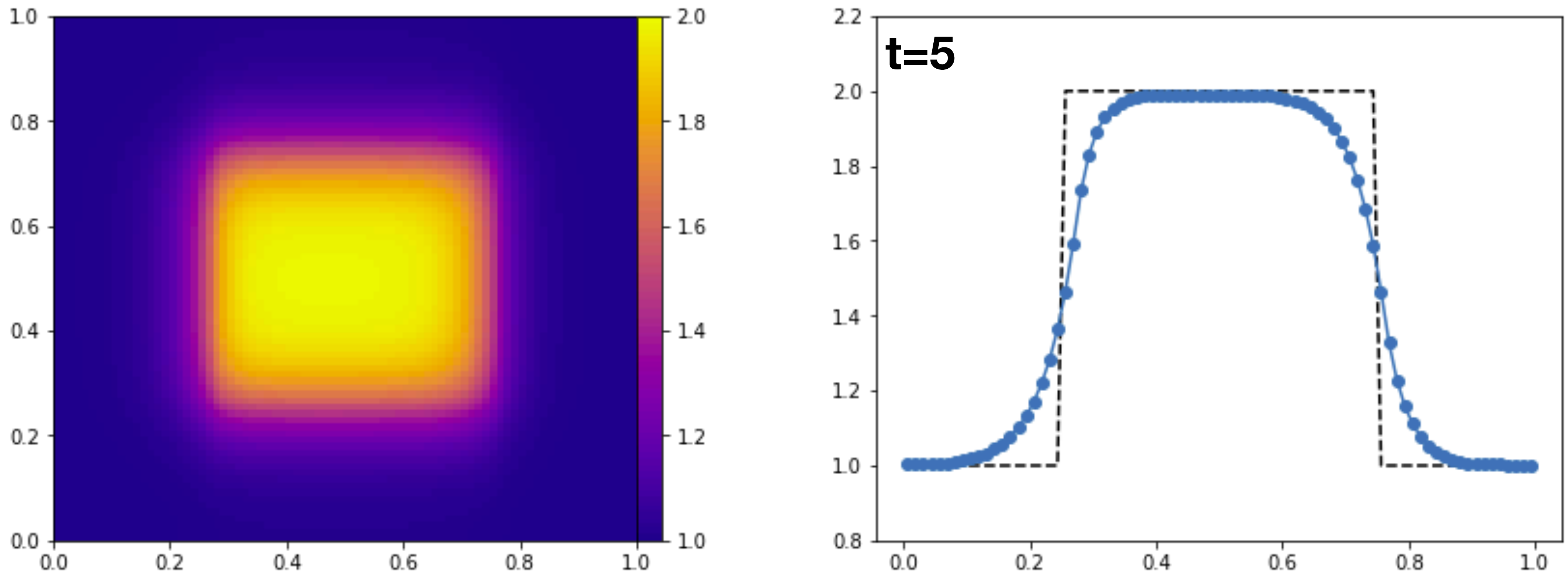
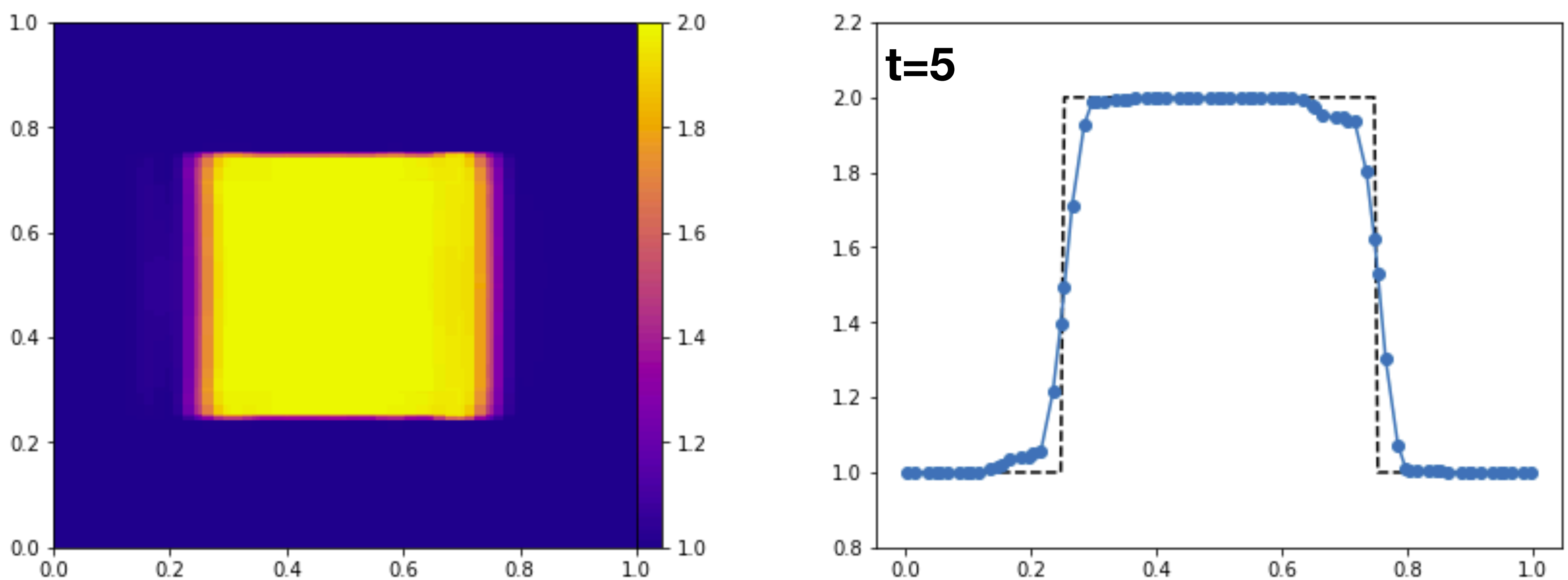
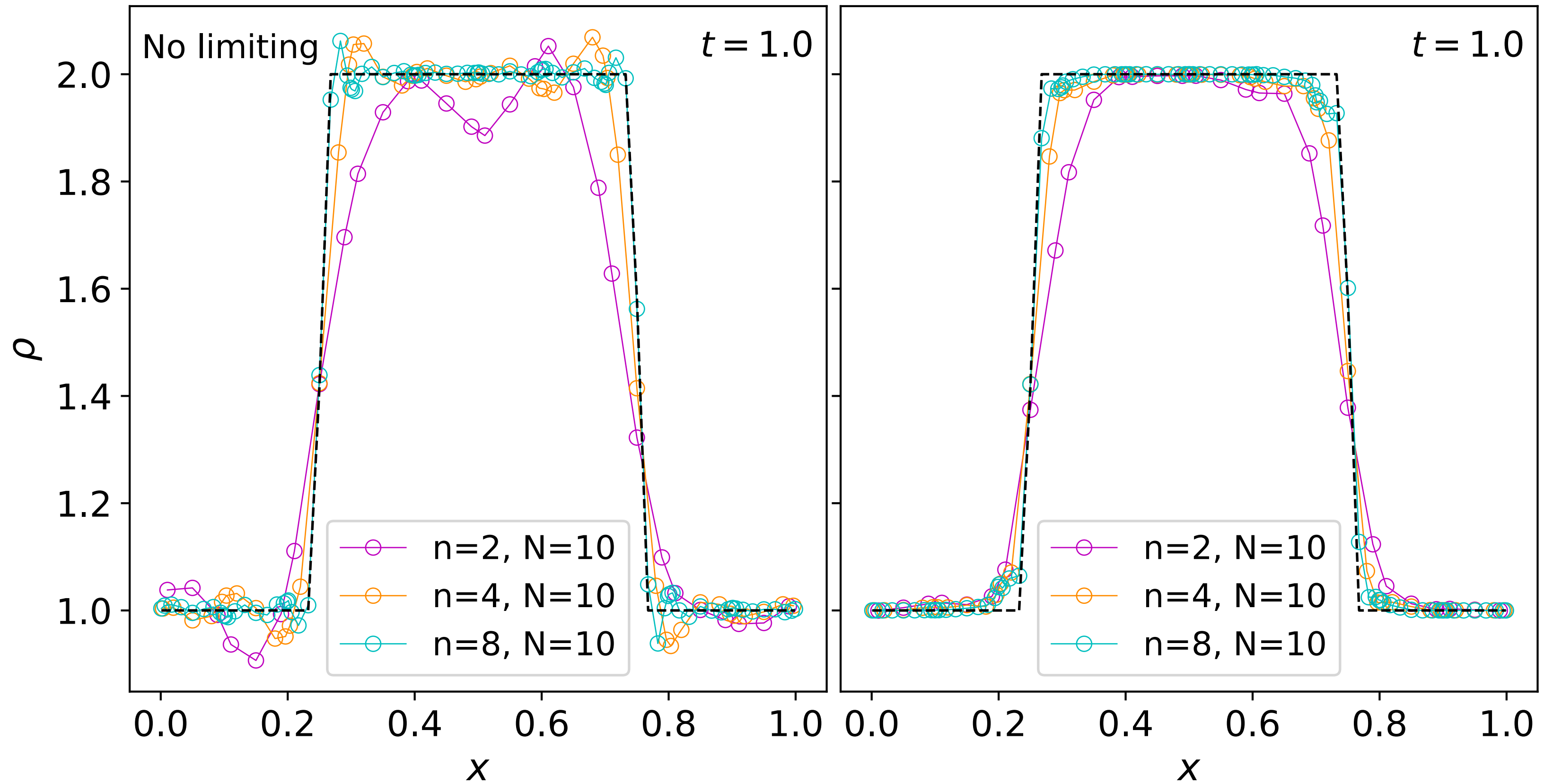


SD-ADER 4th order + Fall-Back scheme

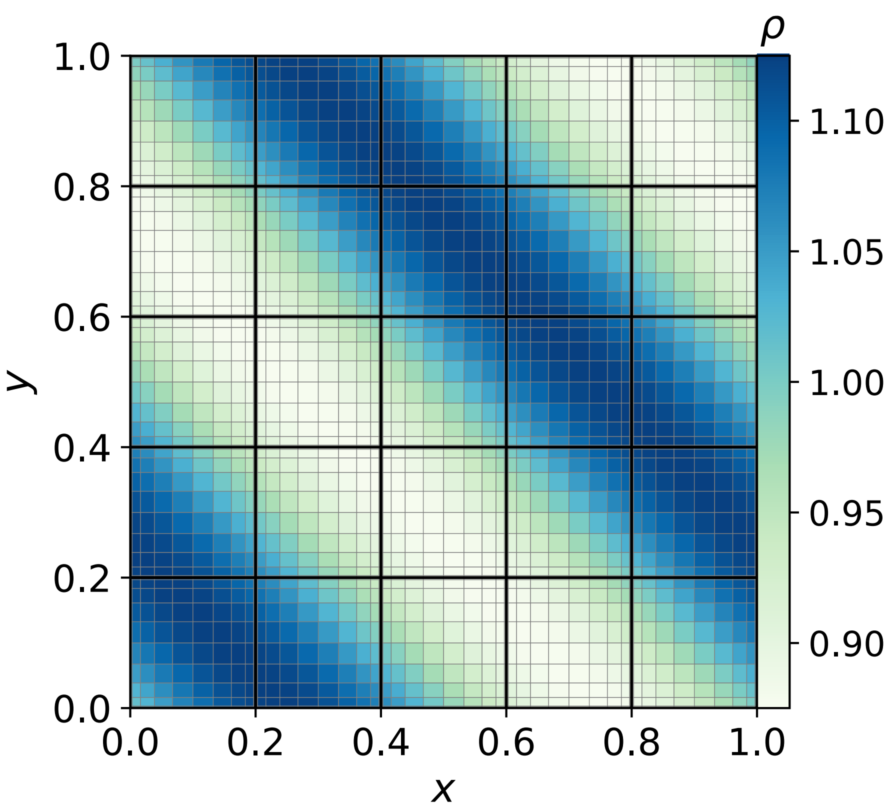
D.o.f = 20 x 4

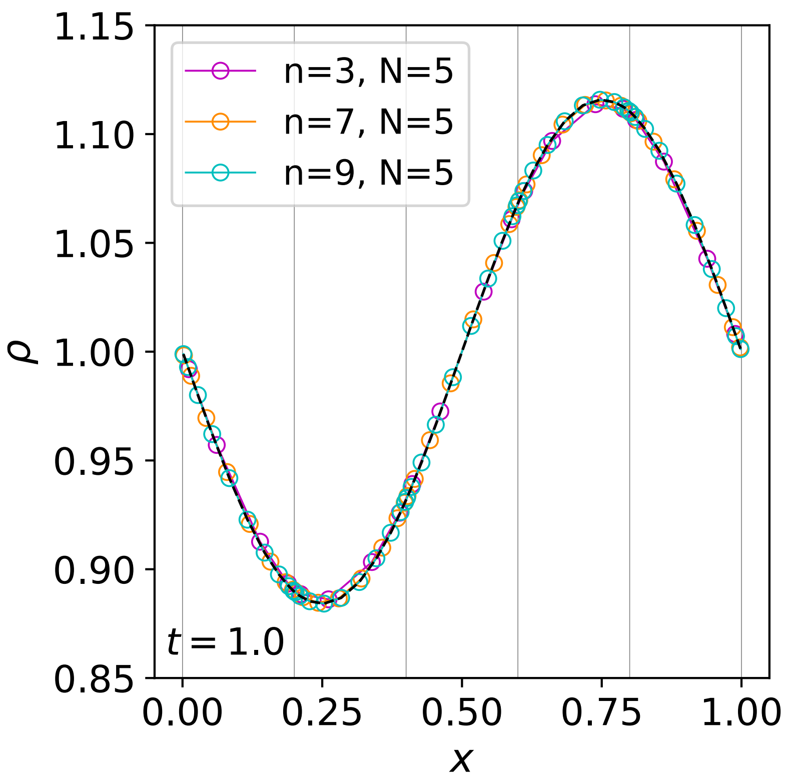


Limited SD-ADER



Suppresses oscillations for discontinuous solutions





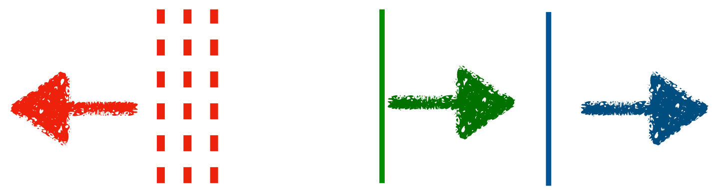
Conserves the accuracy for smooth non-linear solutions

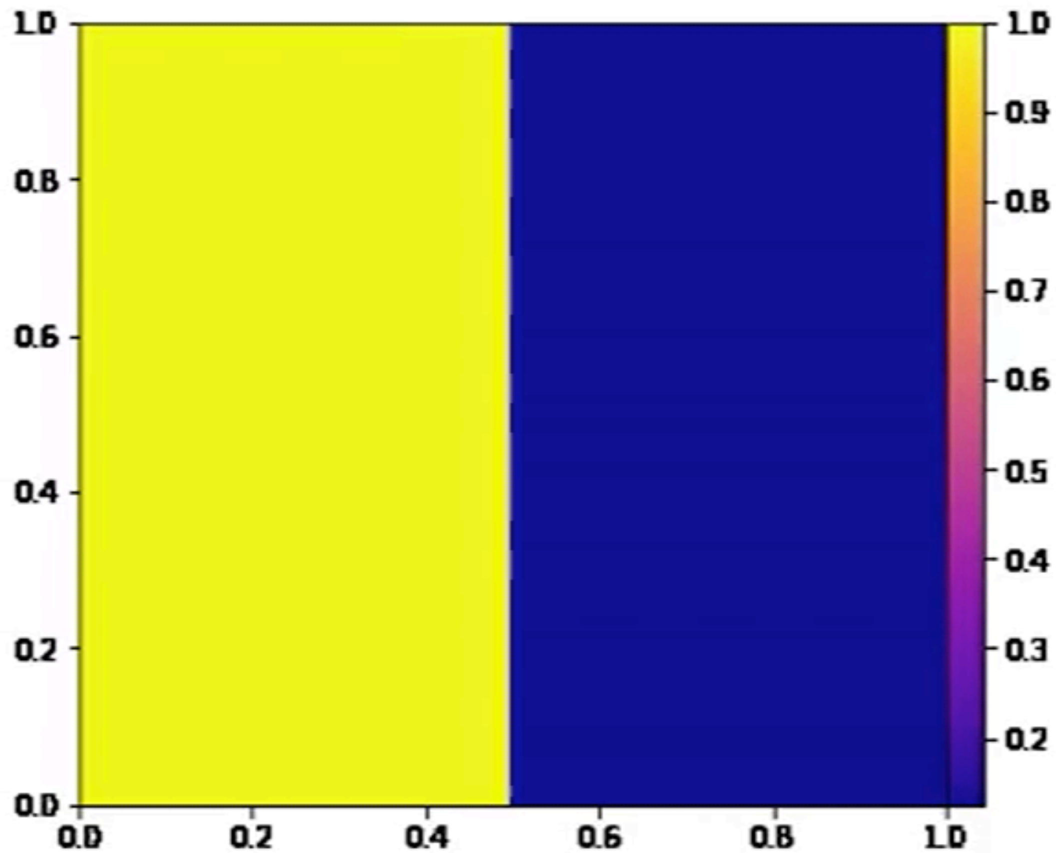
1D Hydro Tests

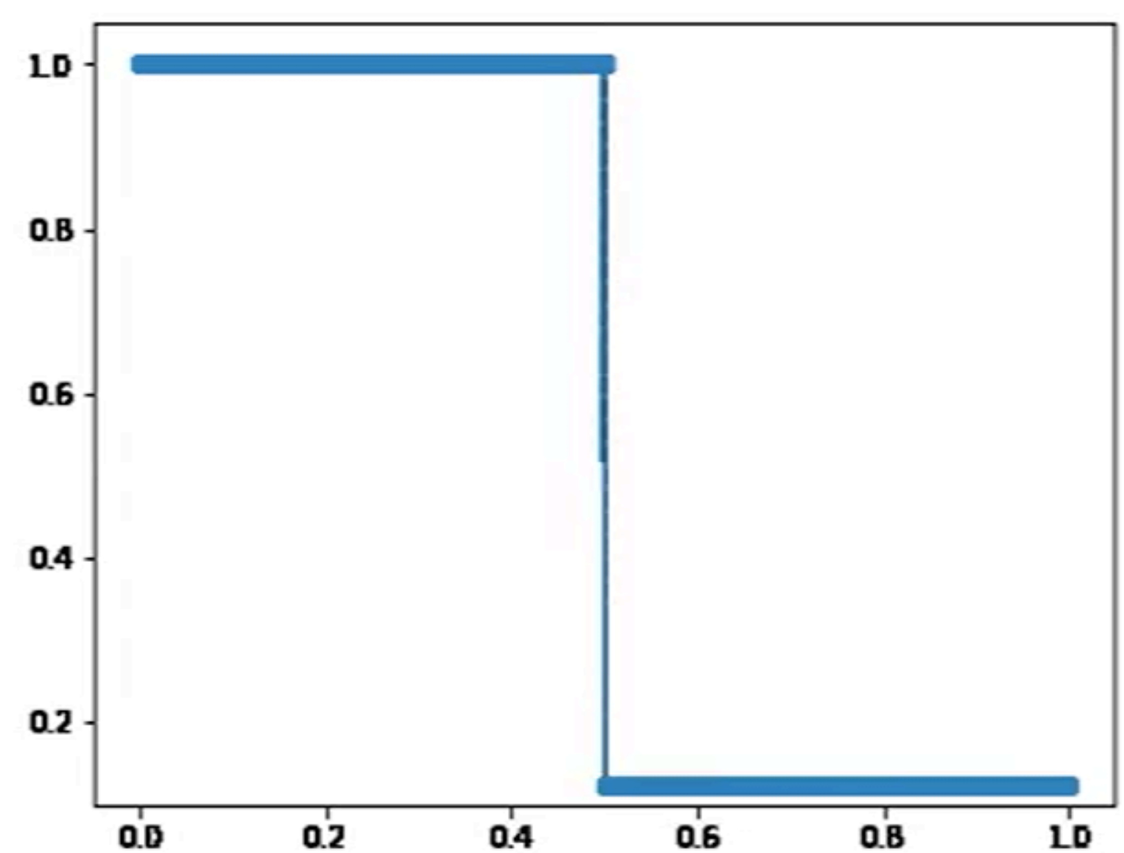
Sod Shock Tube

$$\rho^L = 1, \quad p^L = 1, \quad v_x^L = 0 \qquad\qquad \rho^R = 0.125, \quad p^R = 0.1, \quad v_x^R = 0$$

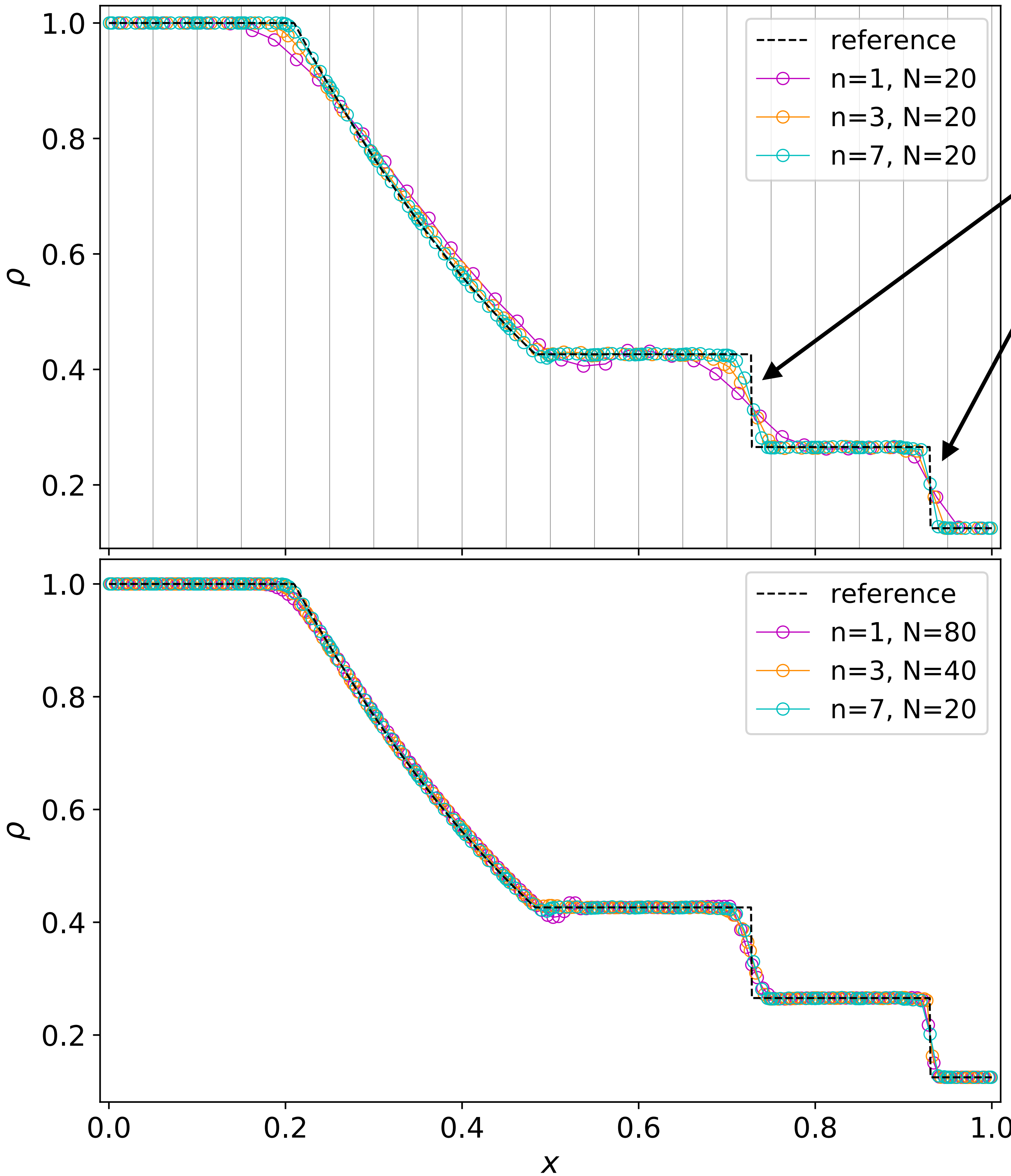
-A rarefaction, a contact discontinuity, and a shock wave.







Sod Shock Tube

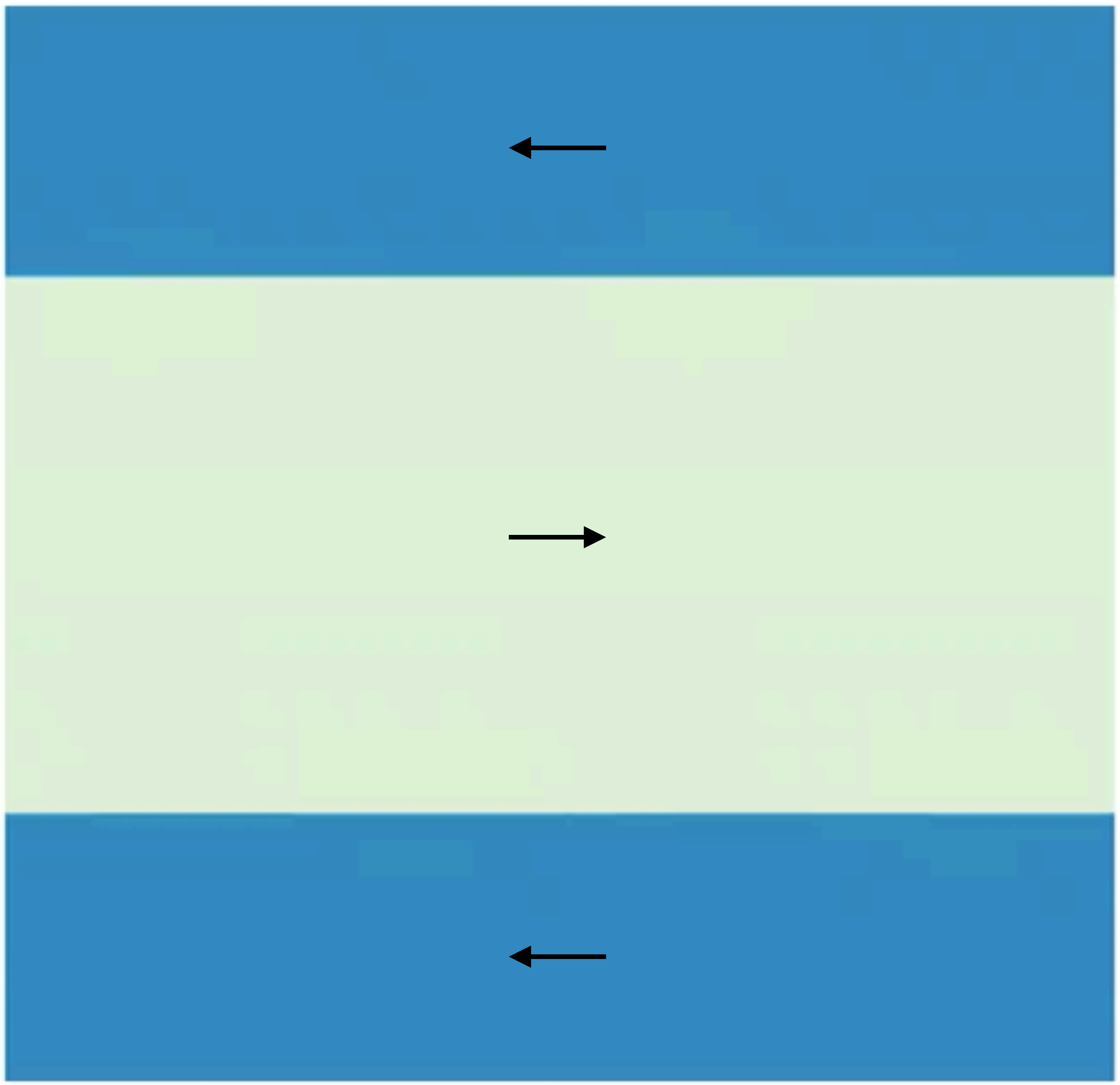


Discontinuities described within elements

-Benefits from increasing order at constant N

-Slight benefit from increasing order at constant #DOF.

2D Hydro Tests

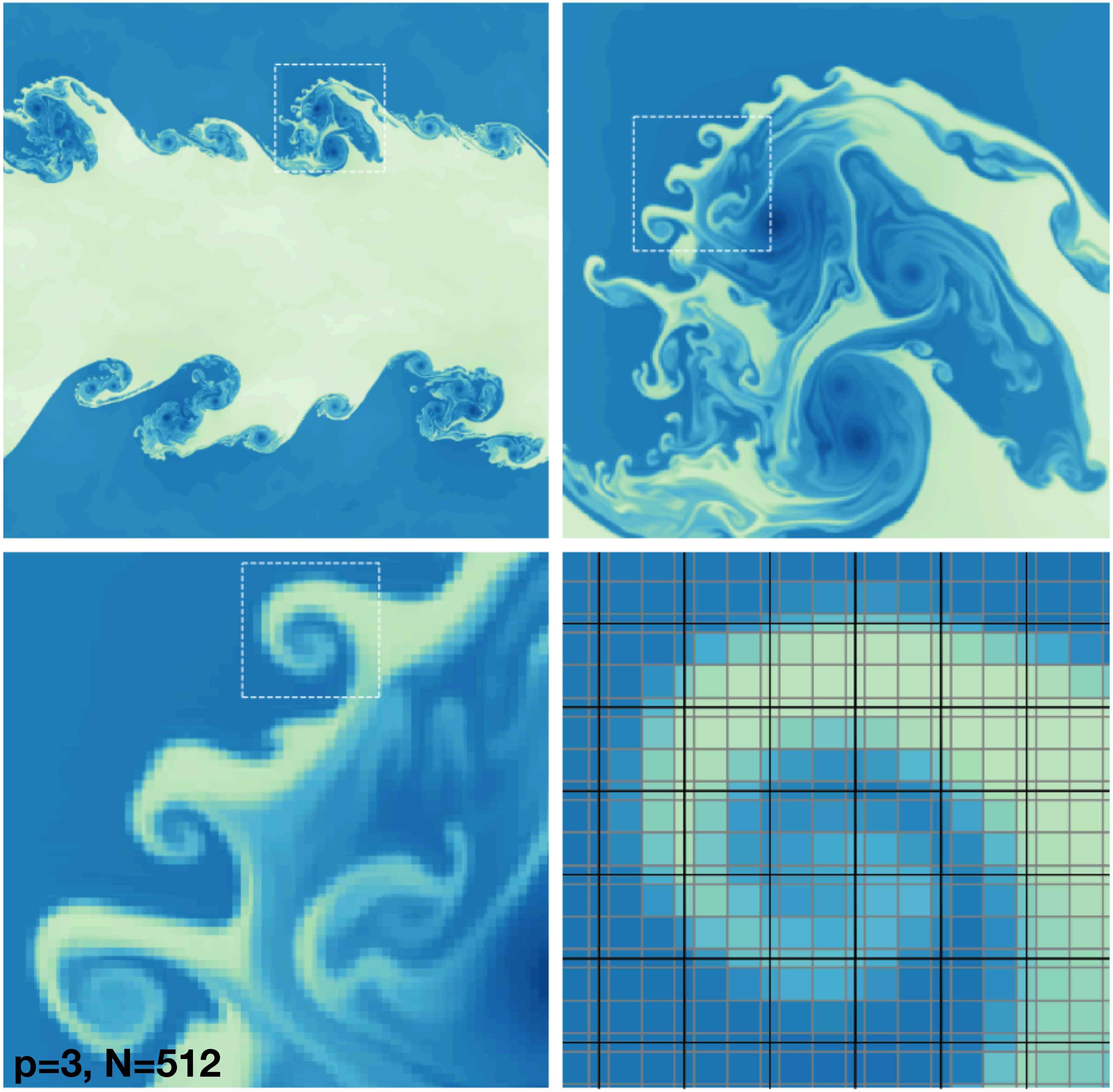


p=3, N=512

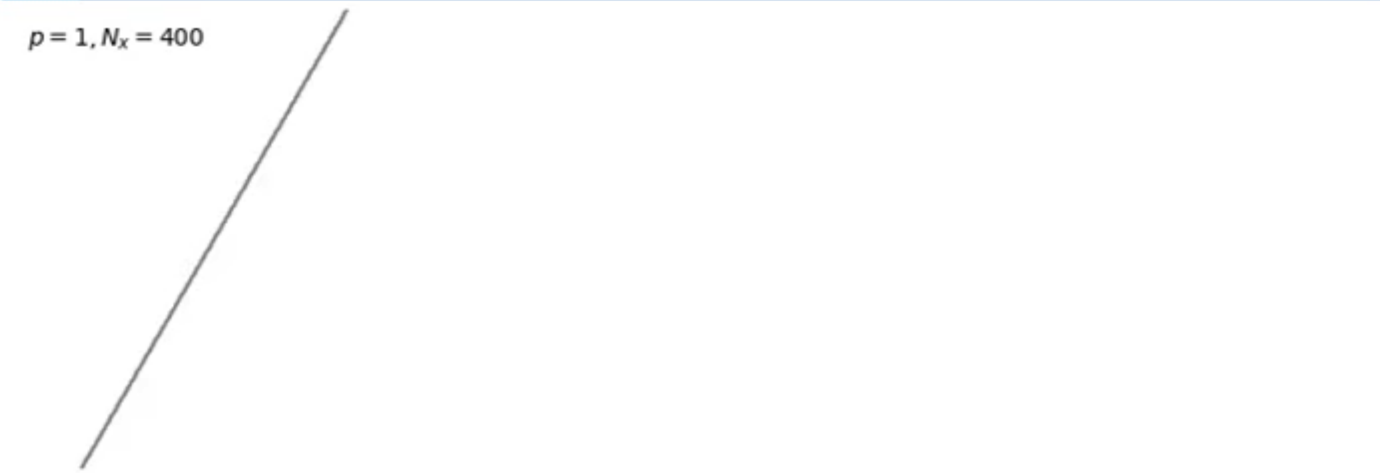
Kevin-Helmhotz Instability

-Instabilities triggered by shearing flows.

-The scale of the eddies depends on the resolution (of both elements and control volumes) and the numerical diffusion.



-Able to describe eddy within subcells

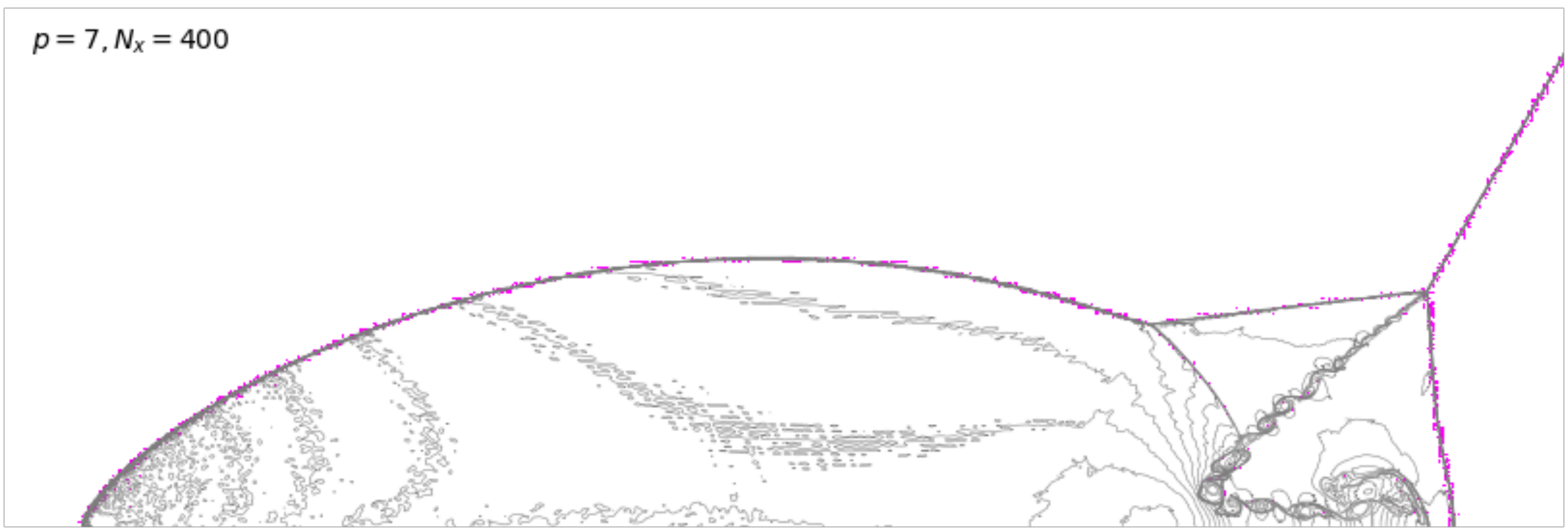


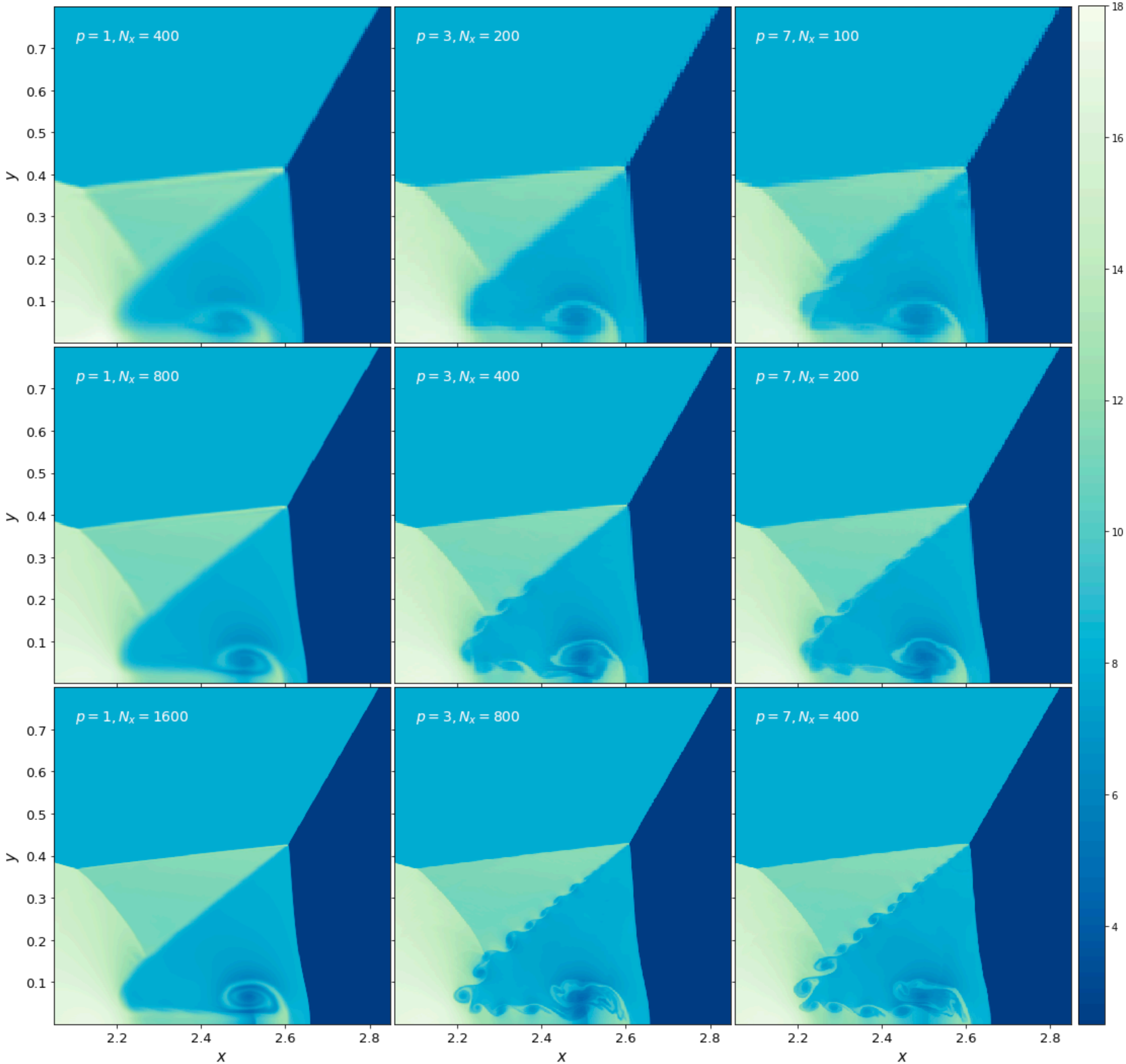
Double Mach Reflection

-Test involving a strong shock hitting a reflective boundary

-Instabilities are generated, with size depending on resolution.


$p = 7, N_x = 400$


$p = 7, N_x = 400$



-Increasing the order has a large impact

-Better results with p=7 and 4x less DOF than with p=1

Thanks!

An arbitrary high-order spectral difference method for the induction equation
MH Veiga, DA Velasco-Romero, Q Wenger, R Teyssier 2019

Spectral Difference method with a posteriori limiting: Application to the Euler equations in one and two space dimensions
D Velasco-Romero, MH Veiga, R Teyssier 2023