Yichao Zhang, Shi Zhou, Zhongzhi Zhang,
Jihong Guan, Shuigeng Zhou, and Guanrong Chen

Traffic Fluctuations on Weighted Networks

Abstract

Traffic fluctuation has so far been studied on unweighted networks. However, many real traffic systems are better represented and understood as weighted networks, where nodes and links are assigned some weight values representing their physical properties such as capacity and delay. Here, we introduce a general random diffusion (GRD) model to investigate the traffic fluctuations on weighted networks, where a random walk's choice of route is affected not only by the number of links a node has, but also by the weights of individual links. We obtain analytical solutions that characterize the relation between the average traffic and the fluctuations through nodes and links. Our analysis is supported by results of extensive numerical simulations. We observe that the value ranges of the average traffic and the fluctuations, through nodes or links, increase dramatically with the level of heterogeneity in link weights. This highlights the key role that link weight plays in traffic fluctuation and the necessity to study traffic fluctuations on weighted networks.

I. Introduction

In nature and society, many complex systems can be represented as graphs or networks, where nodes represent the elementary units of a system and links stand for the interactions between the nodes. Complex networks have been a research focus in both science and engineering in the last decade [1]–[10].

Recently, attention has been given to the traffic fluctuation problem in communication networks. It is associated with an additive quantity representing the volume of traffic traversing through a node (or a link) in a time interval, and the dependence between its mean and standard deviation [11]. Knowledge on traffic fluctuation is relevant to the design and engineering of real systems such as air transport networks, highway networks, power grids and the Internet, for example how to deploy network resources, how to route traffic efficiently and how to mitigate congestion.

In recent years, there has been a strong research interest in the traffic fluctuation problem, which is key to a wide range of applications in various networked systems [12]–[17]. In particular, researchers are interested in the relation between the mean of traffic and the standard deviation at a given node. This is because various problems of immediate social and economical interests are ultimately constrained by the extent to which the assignment of resources matches supply and demand under realistic conditions, and the resource assignment is essentially governed

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by the 'normal' traffic behavior characterized with large fluctuations.

In many real systems, traffic fluctuations are often affected by specific physical properties of network elements, such as the bandwidth of a cable or the computational power of an Internet router. Such systems are much better described as a more sophisticated form of weighted networks, where the physical properties of network elements are represented by links' weights and nodes' strengths.

In this article, we investigate the traffic fluctuation problem for weighted networks. In Section II, we review the existing works about traffic fluctuations on unweighted networks. In Section III, we introduce some network properties related to our work and define a number of variables that are used in the study of traffic fluctuations. We introduce a general random diffusion (GRD) model, where a general random walker's choice of path is affected by links' weights. In Sections IV and V, we analyze the fluctuations of traffic on weighted networks. We provide analytical solutions to the relation between the fluctuations and the average traffic at nodes in Section IV and on links in Section V, respectively. We also run extensive numerical simulations, which confirm our analysis and illustrate the physical meaning of our results. We summarize our work in Section VI.

Our contributions are four-fold. Firstly, we introduce a more general analytical law which characterizes the traffic fluctuations on weighted networks. Previously existing works belong to a special case of our law. Secondly, our results show that traffic fluctuations on a weighted network can be dramatically different from that on an equivalent unweighted network. This highlights the necessity of studying real systems as weighted networks when network elements have a non-trivial impact on traffic dynamics. Thirdly, in addition to traffic fluctuations through nodes, we also analytically study traffic fluctuations through links. We show that on weighted networks the traffic fluctuations on links are significant therefore should be considered when designing real systems. Finally, we reveal the dependence between a link's traffic properties and the connectivity of the link's two end-nodes.

II. Previous Studies on Traffic Fluctuations

An early discovery was that the average volume of traffic arriving at a node, $\langle f \rangle$, and the fluctuation (standard deviation) of the traffic, σ, follow a power-law relation, i.e. $\sigma \sim \langle f \rangle^{\alpha}$, where the exponent α has two universal values, 1/2 and 1 [12], [13]. This result has attracted a lot of interest from the network research community and it also generated debates. Subsequently, it has been shown numerically that there is a wide spectrum of possible values within the range of $[1/2, 1]$ for α [14].

Lately, some scaling properties of traffic fluctuations were revealed by simulations on unweighted scale-free networks [18], where a navigation algorithm is used to give a preference for less-used links in the traffic history. However, the scenario considered in [18] is unrealistic, because if a link is less used in a network's traffic history, then it indicates that the link is actually not preferred for reasons like having smaller capacity or longer delay, or being more expensive.

Recently, an analytical law [15] shows the dependence of fluctuations on the mean traffic over unweighted networks, revealing that the dependence is governed by the delicate interplay of three factors: the size of the observation window; the noise associated to the fluctuations in the number of packets from time window to time window; and the degrees of nodes. Notice that unweighted networks are relatively simple therefore widely used to represent the connectivity structure of a complex networked system.

On an unweighted network, physical properties of links (and nodes) are ignored so that all links are equal, i.e. each link only represents the existence of a topological connection between two nodes. As is known, however, real systems display different interaction strengths between nodes, which reveal unweighted networks' drawback in link definition. In fact, a traffic path is rarely randomly chosen. This is because links have different physical properties (e.g., bandwidth, delay or cost) and naturally traffic tends to choose a path to achieve better performance, higher efficiency or lower cost. Therefore, weighted networks are clearly more realistic albeit more difficult to deal with.

III. Traffic Fluctuations on Weighted Networks

A. Weighted Networks

As mentioned, a more realistic form of networks is the weighted networks [19]–[21], where each link is assigned a *weight* value to denote a physical property of interest, e.g. the bandwidth of a cable or the length of a road; and similarly, each node is assigned a *strength* to represent, for example, the computational capacity of

Yichao Zhang and Jihong Guan are with Department of Computer Science and Technology, Tongji University, Shanghai 201804, China.
Shi Zhou is with Department of Computer Science, University College London (UCL), Gower Street, London, WC1E 6BT, United Kingdom.
Zhongzhi Zhang and Shuigeng Zhou are with Shanghai Key Lab of Intelligent Information Processing and School of Computer Science, Fudan University, Shanghai, 200433, China. Guanrong Chen is with Department of Electronic Engineering, City University of Hong Kong, Hong Kong SAR, China.

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Weighted networks have the advantage to encode information of various physical properties of links and nodes. For example, in a weighted social network, a link can indicate that two people know each other while the weight of the link can denote how often they meet each other [21]; in a weighted Internet router network, link weight can represent the bandwidth of a cable and node strength can represent the processing power of a router [22]; in a weighted aviation network, link weight can denote the annual volume of passengers traveling between two airports [21]; in the weighed metabolic network of *E. coli*, link weight can encode the optimal metabolic flux between two metabolites [23]. On the other hand, recently there are some works on random walks based on weighted networks, but they only considered a single random walker [24], [25].

In this article, we study the traffic fluctuation problem on weighted networks and investigate critical questions such as 'what are the impact of different capacity of nodes and links on the fluctuations of traffic passing through them?' 'can we predict the fluctuations?' and 'what are their implications for network resource assignment?'

1) Link Weight

In network research, the node degree, k, is defined as the number of links a node has. When representing real systems by weighted networks, the weight of a link is often related to the degrees of the two end-nodes of the link. For example, the number of scheduled flights between two airports increases with the number of flights each of the two airports has. In such cases, we define the weight of a link between nodes i and j as

$$w_{ij} = w_{ji} = (k_i k_j)^{\theta}, \tag{1}$$

where k_i and k_j are the degrees of the two nodes, and θ is the network's weightiness parameter which characterizes the dependence between the link weight and the node degrees [21], [23], [26]. This definition is well supported by empirical studies [21], [23], [26] and is widely used in research on weighted networks. The introduced weightiness parameter conveniently determines the level of link heterogeneity in a weighted network. When $\theta = 0$, there is no dependence between link weight and node degree, thus all links are equal with $w = 1$, and the network becomes an unweighted network. When $\theta > 0$, it is a weighted network where links have different weights. The larger the θ, the wider the difference between links.

2) Node Strength

The strength of node i is defined as

$$s_i = \sum_{j\in\Gamma(i)} w_{ij} = \sum_{j\in\Gamma(i)} (k_i k_j)^{\theta}, \tag{2}$$

where $\Gamma(i)$ is the set of neighbors of node i, and θ is the same as above. In an unweighted network with $\theta = 0$, node strength $s_i = k_i$ is the same as the node degree. In a weighted network with $\theta > 0$, a node's strength is the sum of the weights of the links connecting to the node. Two nodes with the same degree may have different strength values depending on the weights of their links. For example, consider two airports A and B, both having 4 flight connections, $k_A = k_B = 4$. Airport A will have larger 'strength' than airport B if the former is connected with four well-connected hub airports and the later is connected with four less-connected local airports.

B. General Random Diffusion (GRD) Model

Random walk is a mathematical formalization of a trajectory taking successive random steps. A familiar example is the random walk phenomenon in a liquid or gas, known as Brownian motion [27], [28]. Random walk is also a fundamental dynamic process on complex networks [29]. Random walk on networks has many practical applications, such as navigation and search of information on the World Wide Web and routing on the Internet [30]–[34]. Existing research on traffic fluctuation either studied random walkers traveling on unweighted networks where the choice of a route is random as all links are regarded as equal [12]–[15], or examined a single random walker traveling on a weighted network [24], [25].

Before introducing our model, we firstly introduce the general random walk on weighted networks. Consider a general random walker starting from node i at time $t = 0$ and denote $P_{im}(t)$ as the probability of finding the walker at node m at time t. The probability of finding the walker at node j at the next time step is

We propose a general random diffusion (GRD) model, which describes the traffic fluctuation problem as a large number of independent random walkers traveling simultaneously on a weighted network.

$P_{ij}(t+1) = \sum_m a_{mj} \cdot \Pi_{m\to j} \cdot P_{im}(t)$, where a_{mj} is an element of the network's adjacent matrix. Here, $\Pi_{m\to j}$ is defined to be (w_{mj}/s_m). Thus, the probability $P_{ij}(t)$ for the walker to travel from node i to node j in t time steps is

$$P_{ij}(t) = \sum_{m_1,\ldots,m_{t-1}} \frac{a_{im_1} w_{im_1}}{s_i} \times \frac{a_{m_1m_2} w_{m_1m_2}}{s_{m_1}} \times \cdots \times \frac{a_{m_{t-1}j} w_{m_{t-1}j}}{s_{m_{t-1}}}, \tag{3}$$

i.e., $P_{ij}(t) = \sum_{m_1,\ldots,m_{t-1}} P_{im_1} P_{m_1m_2} \ldots P_{m_{t-1}j}$. Comparing the expressions for P_{ij} and P_{ji}, one can see that $s_i P_{ij}(t) = s_j P_{ji}(t)$. This is a direct consequence of the undirectedness of the network. For the stationary solution, one obtains $P_i^{\infty} = s_i/Z$ with $Z = \sum_i s_i$. Note that the stationary distribution is, up to normalization, equal to s_i, the strength of the node i. This means that the higher strength a node has, the more often it will be visited by a walker.

Next, we propose a general random diffusion (GRD) model, which describes the traffic fluctuation problem as a large number of independent random walkers traveling simultaneously on a weighted network, where a walker's choice of paths is based on the rule mentioned above.

1) Size of the Time Window

We observe traffic arriving at a node (or passing through a link) in time windows of equal size. Each time window consists of M time units, which is defined as the step for a random walker to hop from one node to another.

2) Preferential Choice of a Path

A walker at node i chooses link i–j as the next leg of travel according to the following preferential probability:

$$\Pi_{i\to j} = \frac{w_{ij}}{\sum_{j\in\Gamma(i)} w_{ij}} = \frac{w_{ij}}{s_i}, \tag{4}$$

which is proportional to the weight of the link.

3) Average Traffic

The traffic arriving at node i during a time window is $f_i = \sum_{m=1}^{M} \Delta_i(m)$, where $\Delta_i(m)$ is a random variable representing the number of walkers arriving at node i at the mth time step. The average traffic, $\langle f_i \rangle$, is the mean traffic volume at node i over all time windows. Similarly, f_{ij} is the traffic passing through a link between nodes i and j during a time window, and $\langle f_{ij} \rangle$ is the average link traffic.

4) Traffic Fluctuation

The standard deviation σ_i indicates the fluctuation of traffic volume around the average traffic $\langle f_i \rangle$ at node i over all time windows. Similarly σ_{ij} is the fluctuation of link traffic f_{ij} on the link between nodes i and j.

The present interest on the traffic fluctuation problem is the relation between the average traffic $\langle f \rangle$ (of a node or a link) and the fluctuation σ, and the impact of relevant quantities (time window size M, weightiness parameter θ, and node degree k) on such relation. In the following two sections, we investigate the traffic fluctuations of node and link, respectively.

IV. Node Traffic Fluctuation on Weighted Networks

A. Analytical Solution

According to the GRD model's preferential choice of path (see (4)), in the stationary regime the number of walkers visiting node i at a single time step can be estimated by

$$\Phi_i(r) = r\frac{s_i}{\sum_{i=1}^{N} s_i}, \tag{5}$$

where r is the number of random walkers traveling on the weighted network and N is the number of nodes. In the GRD model, random walkers are independent and the arrival of walkers at a node is a Poisson process. Thus, the mean number of walkers visiting node i within a window of M time steps is

$$\langle f_i \rangle = \Phi_i(r)M, \tag{6}$$

and the probability that exactly n walkers visit node i within a time window is

$$P_i(n) = e^{-\Phi_i(r)M}\frac{[\Phi_i(r)M]^n}{n!}. \tag{7}$$

In a more general case, the number of walkers, r, observed from time window to time window, is uniformly distributed in $[R-\delta, R+\delta]$, $0 < \delta \le R$, where R is the average number of walkers and the noise constant δ describes the fluctuation. The probability of having r walkers in a time window is

Social networks show assortative mixing where high-degree nodes tend to connect with other high-degree nodes and low-degree nodes with low-degree ones. By contrast, biological and technological networks show disassortative mixing.

$$F(r) = \frac{1}{2\delta + 1}. \tag{8}$$

Thus (7) becomes

$$\psi_i(n) = \sum_{j=0}^{2\delta}\left(\frac{e^{-(s_i/\sum_{i=1}^{N} s_i)(R-\delta+j)M}}{2\delta+1} \times \frac{[(s_i/\sum_{i=1}^{N} s_i)(R-\delta+j)M]^n}{n!}\right). \tag{9}$$

Calculating the first and second moments of f_i yields

$$\langle f_i \rangle = \sum_{n=0}^{\infty} n\psi_i(n) = \frac{s_i}{\sum_{i=1}^{N} s_i} RM \tag{10}$$

and

$$\langle f_i^2 \rangle = \sum_{n=0}^{\infty} n^2 \psi_i(n) = \langle f_i \rangle^2 \left(1 + \frac{\delta^2+\delta}{3R^2}\right) + \langle f_i \rangle. \tag{11}$$

Then, the standard deviation as a function of $\langle f_i \rangle$ is

$$\sigma_i^2 = \langle f_i \rangle \left(1 + \langle f_i \rangle \frac{\delta^2+\delta}{3R^2}\right). \tag{12}$$

This indicates that the relation between the traffic at nodes and its scale does not depend on the weight parameter θ. The traffic fluctuation at node i is given by $\sigma_i^2 = (\sigma_i^{\text{int}})^2 + (\sigma_i^{\text{ext}})^2$. This suggests that the driving force of traffic fluctuation at node i can be ascribed to two factors: one is the internal randomness of the diffusion process, $\sigma_i^{\text{int}} = \sqrt{\langle f_i \rangle}$, and the other is the change in the external environment, $\sigma_i^{\text{ext}} = \langle f_i \rangle \sqrt{(\delta^2+\delta)/(3R^2)}$, i.e. the fluctuation of the number of walkers on the network in different time windows. To be more precise, we will show a class of specific networks in what follows.

1) Neutral Weighted Networks

Networks often exhibit different mixing patterns or degree-degree correlations [35], [36]. For example, social networks show assortative mixing where high-degree nodes tend to connect with other high-degree nodes and low-degree nodes with low-degree ones. By contrast, biological and technological networks show disassortative mixing where high-degree nodes tend to connect with low-degree nodes and vice versa.

Neutral networks show neither assortative nor disassortative mixing. Three popular examples are: (1) the Erdös-Rényi (ER) random graphs [37], generated by random link attachment between nodes and characterized by a Poisson degree distribution; (2) the Barabási-Albert (BA) scale-free graphs [38], generated by the so-called preferential attachment and characterized by a power-law degree distribution; (3) the Watts-Strogatz (WS) small-world graphs [39], generated by rewiring links on regular lattices and characterized by both high clustering [9] and small average distance [40]. These three generic models have been widely studied in network research in the past as well as recently.

2) Node Strength in Terms of Node Degree

Let $P(k_q | k_i)$ be the conditional probability distribution that a k_q-degree node connects with a k_i-degree node [35]. For neutral networks,

$$P(k_q | k_i) = P(k_q | k) = k_q P(k_q)/\langle k \rangle, \tag{13}$$

where $P(k_q)$ is the probability of a node having degree k_q and $\langle k \rangle$ is the average degree [35].

The nearest-neighbors average degree of node i can be estimated as $k_{nn}(k_i) = \sum_{k_q=k_{\min}}^{k_{\max}} k_q P(k_q | k_i)$, where the suffix q stands for a neighbor of node i, and $k_{\min}$ and $k_{\max}$ are the minimum and maximal node degrees in the network, respectively. By mean-field approximation,

$$k_{nn}^{\theta}(k_i) = \sum_{k_q=k_{\min}}^{k_{\max}} k_q^{\theta+1} P(k_q)/\langle k \rangle = \langle k^{\theta+1} \rangle / \langle k \rangle. \tag{14}$$

One can see that $k_{nn}^{\theta}(k_i)$ does not depend on the degree k_i, hence $\sum_{q \in \Gamma(i)} k_q^{\theta} = k_i \cdot k_{nn}^{\theta}(k_i)$. It follows from (2) and (14) that

$$s_i = k_i^{\theta+1} \langle k^{\theta+1} \rangle / \langle k \rangle. \tag{15}$$

For neutral weighted networks, one can rewrite (10) as

$$\langle f_i \rangle = \frac{k_i^{\theta+1}}{N \langle k^{\theta+1} \rangle} RM. \tag{16}$$

When the four quantities, M, R, δ and θ, satisfy the condition that $((\delta^2+\delta)/(3R)) \cdot ((k_i^{\theta+1})/(N\langle k^{\theta+1}\rangle))M \ll 1$, the relation between traffic fluctuation and average traffic as given in (12) is reduced to a power-law scaling, $\sigma \sim \langle f \rangle^{\alpha}$, with $\alpha = 1/2$. When $((\delta^2+\delta)/(3R)) \cdot ((k_i^{\theta+1})/(N\langle k^{\theta+1}\rangle))M$ is not negligible, the exponent α is in the range of $[1/2, 1]$.

B. Numerical Simulations

We run numerical simulations for the following purpose: (1) to verify the analytical solution; (2) to

The traffic fluctuation function does not follow a simple power-law. Rather, the power-law scaling is within the range of [1/2, 1], and is affected by several parameters including window size and the degree of nodes under study.

examine the impact of parameters, such as window size M and node degree k, on the power-law scaling of the traffic fluctuation function; (3) to compare unweighted networks with $\theta = 0$ with weighted networks with $\theta > 0$.

1) Simulation Settings

Our simulation is based on network graphs generated by the BA model [38], which is neutrally mixing and features a power-law degree distribution $P(k) \sim k^{-3}$. We generate ten BA graphs, each has 5,000 nodes and 25,000 links. We assign link weights and node strengths as defined in (1) and (2), respectively. Initially, we disperse $r = R \pm \delta = 10{,}000 \pm 1{,}000$ random walkers uniformly on nodes. At each time step, all walkers travel one hop according to (4). For a given time window size of M, we observe traffic fluctuation at each node over a large number of time windows.

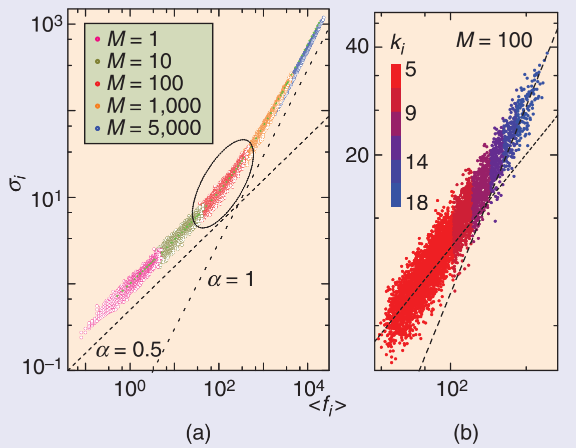

Figure 1. Traffic fluctuation σ_i as a function of average traffic $\langle f_i \rangle$ at node i. (a) Observed in different time window sizes of $M = 1$, 10, 100, 1,000 and 5,000; (b) for nodes of different degrees (with $M = 100$). The green dotted-line in (a) is the analytical solution given in (12). The two black dotted-lines in both (a) and (b) correspond to $\sigma_i \sim \langle f_i \rangle^{\alpha}$ with $\alpha = 0.5$ and 1.0, respectively. Simulation results are obtained on weighted BA networks having 5,000 nodes and 25,000 links with the weightiness parameter $\theta = 0.5$, with average number of random walkers $R = 10^4$ and noise constant of $\delta = 10^3$. For clarity, we only show nodes with degrees smaller than 18. Note that the minimal degree in the networks is 5, with initial number of links $m_0 = 5$ and added number of links $m = 5$ in generating the weighted BA networks.

For each given value of time window size M or weightiness parameter θ, we repeat the simulation for 50 times (with different random seeds) on each of the ten BA networks. Each result shown below is the average over the $10 \times 50 = 500$ simulations.

2) Power-Law Relation

Figure 1(a) shows the relation between traffic fluctuation σ and average traffic $\langle f \rangle$ for different time window sizes M, where the weightiness parameter is set as $\theta = 0.5$. The simulation results overlap with the analytical solution. Both the average traffic and the traffic fluctuation increase with the size of window M. For any given value of M, the two quantities follow a power-law relation $\sigma \sim \langle f \rangle^{\alpha}$. When M is small, the power-law exponent α is close to 1/2; when M increases, the exponent grows towards 1.

Figure 1(b) shows the enlargement of the traffic fluctuation function for the window size $M = 100$, as circled out in Figure 1(a), where data dots are colored by node degrees. For nodes with higher degrees, larger values of σ and $\langle f \rangle$ are observed. For low-degree nodes (e.g. $k = 5$), the power-law exponent is close to $\alpha = 1/2$, whereas for higher-degree nodes (e.g. $k = 18$), the exponent approaches 1.

As predicted by (16), our simulation results confirm that the traffic fluctuation function $\sigma \sim \langle f \rangle^{\alpha}$ does not follow a simple power-law. Rather, the power-law scaling is within the range of [1/2, 1]. It is affected by several parameters including the window size M and the degree of nodes under study. This is consistent with previous studies as their random diffusion model based on unweighted networks is a special case of our general random diffusion model on weighted networks.

3) Impact of the Weightiness Parameter

Figure 2(a) illustrates the solutions of (12) for the weightiness parameter $\theta = 0$, 0.5 and 1 with the time window size set as $M = 100$. Figure 2(b), (c), (d), (e) and (f) show the simulation results for $\theta = 0$, 0.25, 0.5, 0.75, and 1, respectively. For different θ values, the traffic fluctuation curves overlap with each other, and in all cases the high-degree nodes are concentrated at the upper-right end of the curves whereas the low-degree nodes are dispersed alone the lower-left part of the curve. A remarkable difference, however, is that with the increase of θ, the value ranges of $\langle f_i \rangle$ and σ_i

expand significantly towards both directions. This means that comparing with an unweighted network, traffic fluctuation in a weighted network is more acute at high-degree nodes and more stable at low-degree nodes. This is because in a weighted network the node strength $s \sim k^{\theta+1}$ (see (15)) and therefore high-degree nodes deprive more traffic from low-degree ones than in an unweighted network.

4) Neutral Weighted Networks

Figure 3 shows the simulation results on weighted BA, WS small-world networks and random graphs with

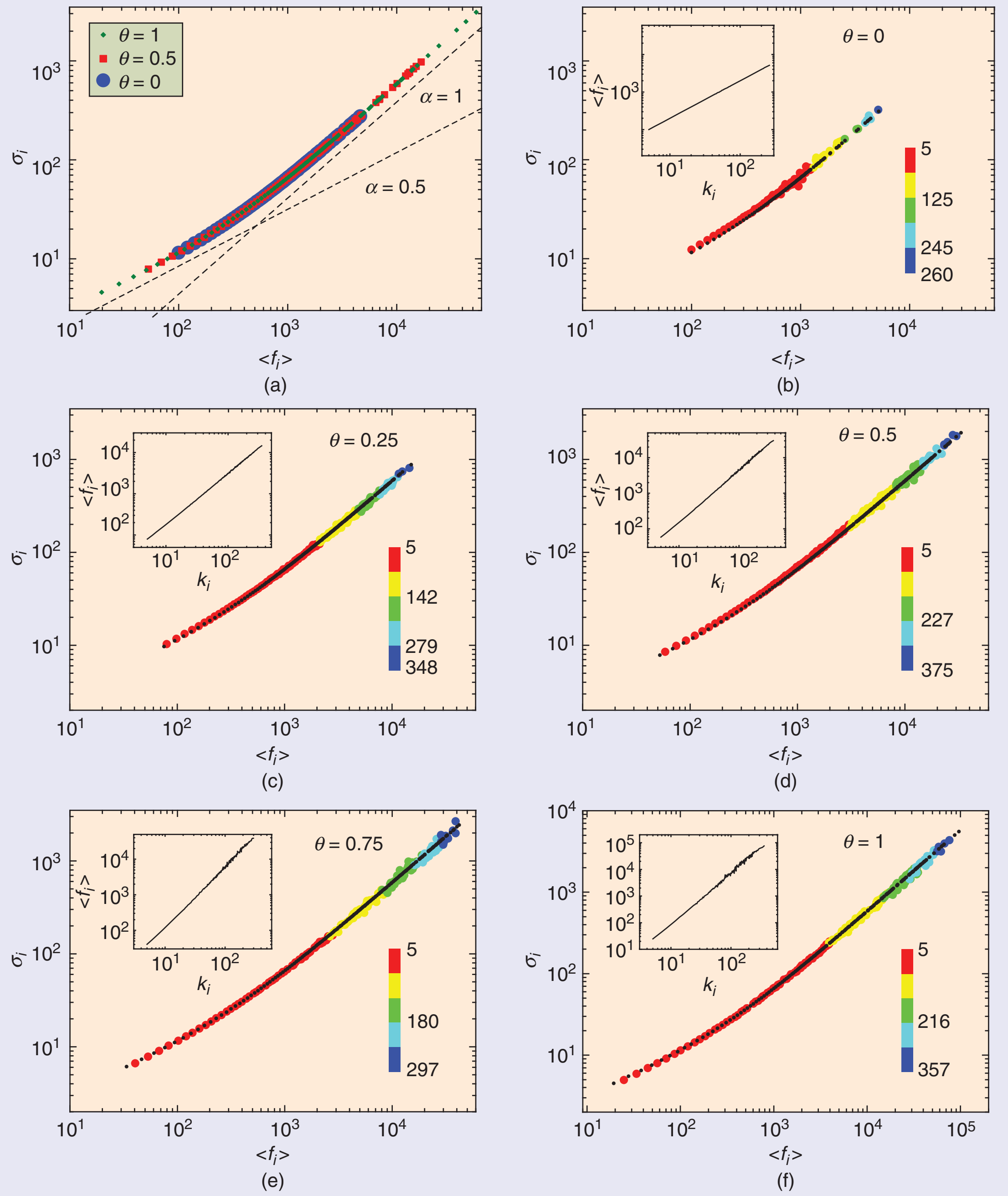

Figure 2. Traffic fluctuation σ_i as a function of the average traffic volume $\langle f_i \rangle$ at node i for different values of weightiness parameter $\boldsymbol{\theta}$. (a) shows analytical solutions of (12) for $\theta = 0$, 0.5, and 1; (b), (c), (d), (e) and (f) show simulation results for $\boldsymbol{\theta} = 0$, 0.25, 0.5, 0.75, and 1, respectively, where results are shown for all nodes and colored by node degree k_i, and the black dotted lines are the analytical solutions. Time window size M is set as 100 and other parameters are as before. The insets of (b), (c) and (d) show $\langle f_i \rangle$ as a function of k_i on a log-log scale, which is predicted by (16).

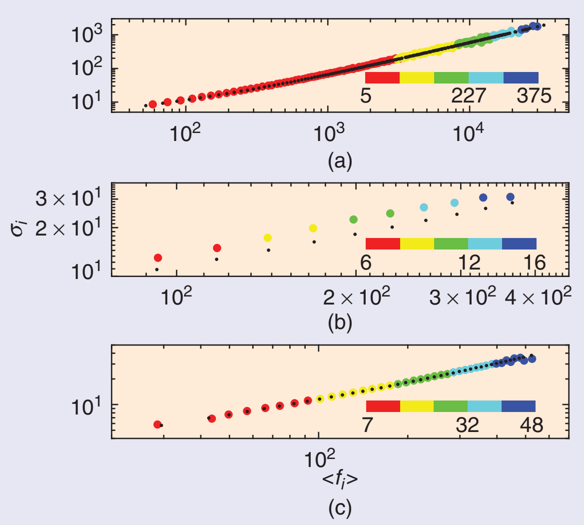

Figure 3. Traffic fluctuation σ_i as a function of the average traffic volume $\langle f_i \rangle$ at node i for neutral networks with $\theta = 0.5$, $M = 100$, $R = 10^4$ and $\delta = 10^3$. The black dotted lines are the analytical solutions given by (12) and (16). (a) shows the simulation obtained on weighted BA networks with 5,000 nodes and 25,000 links; (b) shows the simulation obtained on weighted WS small-world networks with rewiring probability 0.1, 5,000 nodes and 25,000 links; (c) shows the simulation obtained on weighted ER random graphs with connected probability 0.002, 5,000 nodes and 25,000 links.

$\theta = 0.5$. As shown in the panels (a) and (c), the simulation results on the scale of f_i are remarkably consistent with the solution given by (12). However, one can find that the numerical results in the panel (b) deviate from (12) apparently. This is because most of the links (at least 80% in our simulations) in the WS networks are still regularly connected, where 80% is predicted by the small rewiring probability 0.1. Obviously, the conditional probability distribution $P(k_q | k_i)$ does not completely match (13).

Our results suggest that if a real system should be described as a weighted network with $\theta = 1$ but instead an unweighted network with $\theta = 0$ is used, then one actually underestimates the $\langle f_i \rangle$ and σ_i values for higher-degree nodes but overestimates the values for lower-degree nodes by as large as one order in magnitude. This highlights the importance of choosing a proper (weighted) network model for traffic fluctuation research.

V. Link Traffic Fluctuation on Weighted Networks

A. Analytical Solution

In the GRD model, random walkers on a weighted network travel independently and, therefore, the number of walkers passing through a link is a Poisson process. As given in (4), the probability that a walker at node i chooses link $i-j$ as the next leg of travel is w_{ij}/s_i. Thus, for r random walkers in a weighted network, the average number of walkers passing through link $i-j$ (from node i to node j as well as from node j to node i) during a time window M is

$$\langle f_{ij} \rangle = \Omega_{ij}(r)M,$$

where

$$\Omega_{ij}(r) = r\left(\frac{s_i}{\sum_{i=1}^N s_i} \cdot \frac{w_{ij}}{s_i} + \frac{s_j}{\sum_{i=1}^N s_i} \cdot \frac{w_{ij}}{s_j}\right), \tag{17}$$

and the probability of $f_{ij} = n$ in a time window is

$$Q_{ij}(n) = e^{-\Omega_{ij}(r)M} \frac{[\Omega_{ij}(r)M]^n}{n!}. \tag{18}$$

Similarly, as the above analysis on node traffic fluctuation, for a more general case where the number of random walkers r from time window to time window is distributed in $[R-\delta, R+\delta]$, the probability of $f_{ij} = n$ in a time window is

$$\Gamma_{ij}(n) = \sum_{j=0}^{2\delta}\left(\frac{e^{\frac{2w_{ij}}{\langle k^{\theta+1}\rangle^2 N}(R-\delta+j)M}}{2\delta+1} \times \frac{e^{\frac{2w_{ij}}{\langle k^{\theta+1}\rangle^2 N}(R-\delta+j)M}}{n!}\right). \tag{19}$$

Calculating the first and second moments of f_{ij} gives

$$\langle f_{ij} \rangle = \sum_{n=0}^{\infty} n \Gamma_{ij}(n) = \frac{2w_{ij}}{\sum_{i=1}^N s_i} RM, \tag{20}$$

and

$$\langle f_{ij}^2 \rangle = \sum_{n=0}^{\infty} n^2 \Gamma_{ij}(n) = \langle f_{ij} \rangle^2 \left(1 + \frac{\delta^2 + \delta}{3R^2}\right) + \langle f_{ij} \rangle. \tag{21}$$

Thus, the standard deviation as a function of the average traffic $\langle f_{ij} \rangle$ is

$$\sigma_{ij}^2 = \langle f_{ij} \rangle \left(1 + \langle f_{ij} \rangle \frac{\delta^2 + \delta}{3R^2}\right). \tag{22}$$

This also indicates the relation between the traffic on links and its scale is irrelevant to θ. For neutral weighted networks, using (1) and (15), one can rewrite (20) as

$$\langle f_{ij} \rangle = \frac{2(k_i k_j)^\theta \langle k \rangle MR}{\langle k^{\theta+1} \rangle^2 N}. \tag{23}$$

If $((\delta^2+\delta)/(3R)) \cdot ((2(k_i kj)^\theta \langle k \rangle M)/(\langle k^{\theta+1} \rangle^2 N)) \ll 1$, (22) is reduced to a power-law scaling, $\sigma_{ij} \sim \langle f_{ij} \rangle^\alpha$, with $\alpha = 1/2$. Conversely, as $((\delta^2+\delta)/(3R)) \cdot (2(k_i k_j)^\theta \langle k \rangle M)/(\langle k^{\theta+1} \rangle^2 N))$, increases to 1, the exponent α will leave 1/2 for 1.

B. Numerical Simulations

Here, we use the same simulation settings as Section IV(B).

1) Power-Law Relation

In Figure 4, we plot the relation between the traffic fluctuation σ_{ij} and the average traffic $\langle f_{ij} \rangle$ on link $i-j$ for three different values of time window size M. The simulation results are in good agreement with the analytical solution. As predicted by (22) and (23), the average traffic and the fluctuation increase with M. The two quantities follow a power-law scaling, $\sigma_{ij} \sim \langle f_{ij} \rangle^{\alpha}$, where the exponent α is 1/2 for small values of M and approaches 1 with larger M. Such behavior is similar to the traffic fluctuation on nodes.

2) Impact of Weightiness Parameter

In Figure 5(a), the range of scale for $\theta = 1$ is $[0.5031, 0.9602]$ while for $\theta = 0$ it is nearly 0. For $\theta = 0$, the links in the simulation form a dense group on the plot, representing almost equal fluctuation properties [18]. This unaccounted fact can be explained by the solutions of (22) and (23). The dashed lines are for reference and correspond to $\sigma_i \sim \langle f_i \rangle^{\alpha}$, with $\alpha = 1/2$ and $\alpha = 1$, respectively. Clear comparison among unweighted networks ($\theta = 0$) and weighted ones

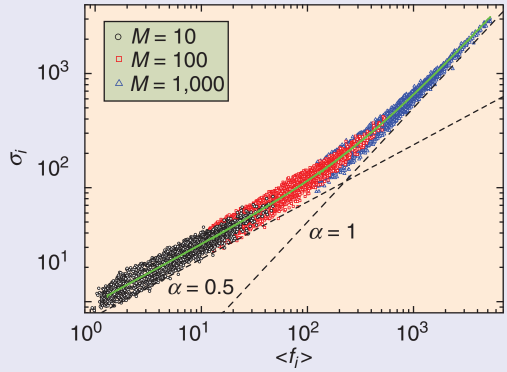

Figure 4. Traffic fluctuation σ_{ij} as a function of average traffic $\langle f_{ij} \rangle$ on link $i-j$ between nodes i and j with time window size of $M = 10$, 100, and 1,000, respectively. The green line is the analytical solution given by (22). The black dotted lines correspond to $\sigma_{ij} \sim \langle f_{ij} \rangle^{\alpha}$ with $\alpha = 0.5$ and 1, respectively. The simulation results are obtained on weighted BA networks having 5,000 nodes and 25,000 links, with $\theta = 0.5$, $R = 10^4$ and $\delta = 10^3$.

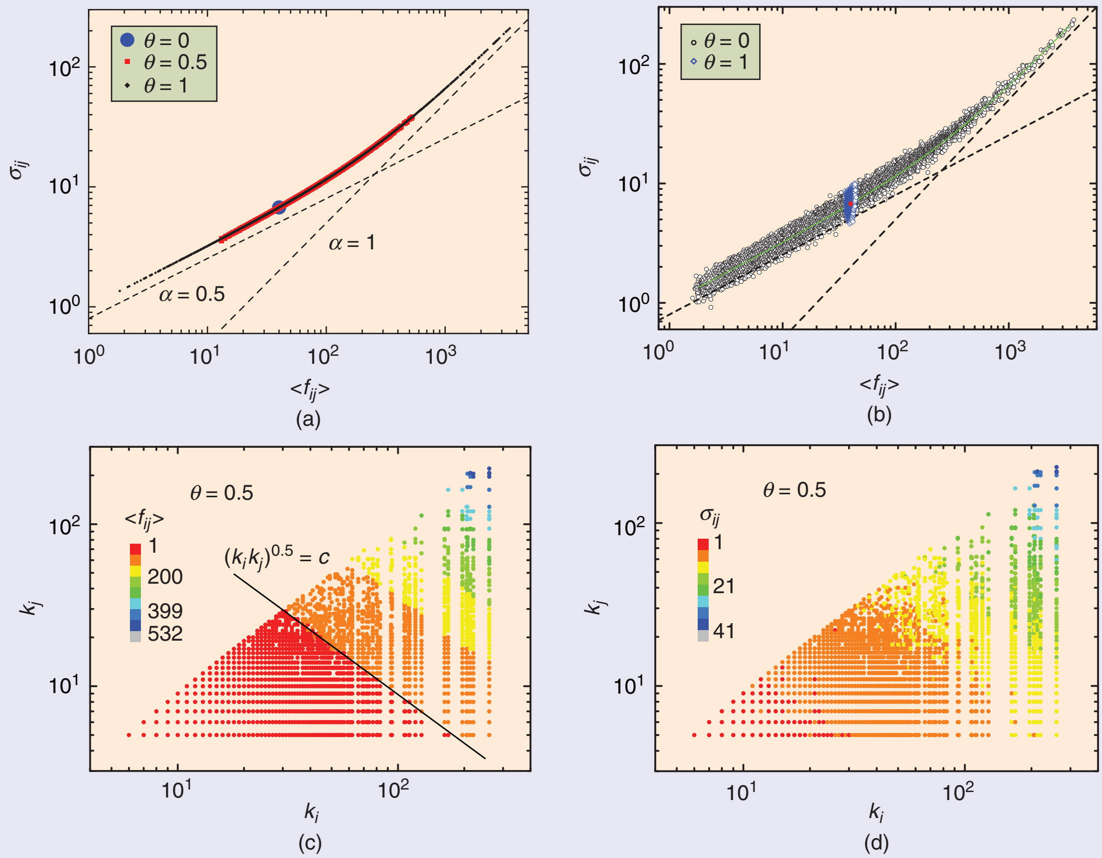

Figure 5. Traffic fluctuation σ_{ij} and average traffic $\langle f_{ij} \rangle$ on link $i-j$ with parameters $M = 100$, $R = 10^4$ and $\delta = 10^3$. (a) Shows analytical solutions of (22) for $\theta = 0$, 0.5 and 1, respectively. (b), (c) and (d) are simulation results. (b) shows σ_{ij} as a function of $\langle f_{ij} \rangle$ for $\theta = 0$ and 1, respectively. (c) shows the average traffic $\langle f_{ij} \rangle$ on link $i - j$ as a function of degrees of the two end-nodes of the link, k_i and k_j, where $\theta = 0.5$, the value of $\langle f_{ij} \rangle$ is given by a color bar, and a reference is given by (23). Similarly, (d) shows the traffic fluctuation σ_{ij} as a function of k_i and k_j.

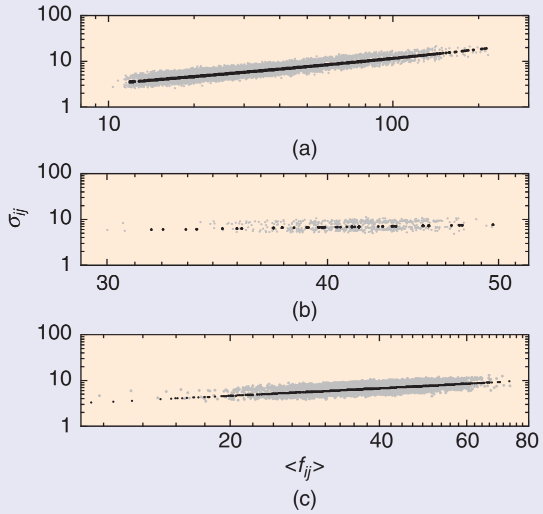

Figure 6. Traffic fluctuation σ_{ij} and average traffic $\langle f_{ij} \rangle$ on link $i-j$ with parameters $M = 100$, $R = 10^4$ and $\delta = 10^3$. The black dotted lines are the analytical solutions given by (22) and (23). (a) shows the simulation obtained on weighted BA networks having 5,000 nodes and 25,000 links with $\theta = 0.5$, $R = 10^4$ and $\delta = 10^3$. (b) shows the simulation obtained on weighted WS small-world networks with a rewiring probability 0.1, 5,000 nodes and 25,000 links. (c) shows the simulation obtained on weighted ER random graphs with connected probability 0.002, 5,000 nodes and 25,000 links.

($\theta > 0$) can be observed in this figure in panel (b). The green and red dots reflect the solution of (22). As shown in the figure, the differences of f_{ij} and σ_{ij} for different node pairs crop up when $\theta = 1$. However, this process actually is not as sudden as it looks.

3) Node Degree

In Figures 5(c) and (d), we show the middle case of $\theta = 0.5$ numerically for different node pairs. In panel (c), the plots are colored by f_{ij}. For all the links, we focus on the results obtained for pairs of k_i and k_j (restricting $k_j > 10$ to enhance the speed of loading figures), where $k_i > k_j$. One can easily find that f_{ij} is directly proportional to the product of two end-nodes' degrees, k_i and k_j, when $\theta > 0$, while they are almost a constant when $\theta = 0$. Likewise, σ_{ij} is directly proportional to $k_i k_j$ when $\theta = 0.5$, but they are rather stable when $\theta = 0$ (see panel (b)).

4) Neutral Weighted Networks

Figure 6 shows the simulation results on weighted BA, WS small-world networks and ER random graphs with $\theta = 0.5$. As can be seen from the panels (a) and (c), the simulation results on the scale of f_{ij} are consistent with the solution given by (22). In panel (b), one can find that the numerical results deviate from (22) again. The behavior confirms our observations from Fig. 6 and the discussion in Section IV-B4, from another angle of view.

C. Discussion

One simple observation from the simulation results is that for a traffic network, the traffic on different roads differs, the wider of which can have a larger traffic. At the same time, the traffic on roads with heavy loads fluctuates more dramatically, depending on the road conditions such as rush hours. Indeed, the interactions among walkers should not be ignored in realistic scenarios, e.g., transport of information packets, signals, molecules, rumors, diseases, to name just a few, whereas these interactions vary widely from case to case.

VI. Conclusions

In summary, we have investigated the traffic fluctuation problem on weighted networks, which is a more general and realistic representation of real traffic systems. Previous studies focusing on nodes are the most simple case of our model. Noticeably, comparatively few investigations have been reported in the literature relative to the fluctuations on links, for both unweighted and weighted networks.

Most importantly, we have introduced a general random diffusion (GRD) model, which describes a swarm of random walkers traveling simultaneously on a weighted network. Based on the model, we are able to provide analytic solutions to characterize the relation between the mean traffic and its fluctuation for both nodes and links. We have discussed the impact of key parameters on the traffic fluctuations. Main parameters include the size of time window, node strength, and link weight. To verify the analytic results, we have taken neutral networks with a specific link weight definition for example. Our analysis indicates that the relation between the link traffic and its scale is irrelevant to the weight parameter θ. Simultaneously, we find the scales of traffic on weighted links with $\theta > 0$ are much wider than the unweighted ones, on which the traffic are rather stable, confirmed by analytical prediction with remarkable accuracy. Thus, both simulations and analysis have suggested that the link weights could have an impact on the way in which networks operate, including information transmission through networks and resources assignment for efficient communications over networks.

One significant observation is that earlier studies based on unweighted networks could lead to unrealistic or even misleading conclusions, with identical link traffic and fluctuation for all links. Therefore, the new GRD model based on weighted networks provides a proper framework for future research. For generality, we began with this simple model to take a step forward in the analytical investigation of the corresponding problems. We believe our rigorous solutions are capable of promoting related studies on interacting walkers in the near future.

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Yichao Zhang received the B.Sc and M.Sc. degrees in Material Physics from Shanghai University, China in 2004 and 2008, respectively. Currently he is a Ph.D. candidate at the Department of Computer Science and Technology of Tongji University, Shanghai, China. His research interests include topological structures of weighted networks, random diffusion on weighted networks, social network analysis, and games on social networks.

Shi Zhou (M'01) received the B.Sc. and M.Sc. degrees in Electronic Engineering from Zhejiang University, Hangzhou, China in 1995 and 1998, respectively. He received the Ph.D. degree in Telecommunications from Queen Mary University of London, UK in 2004. Currently he is an Assistant Professor at the Department of Computer Science, University College London (UCL), UK.

Dr. Zhou has extensive publications since 2004 in the fields of Internet modeling and complex networks, with an emphasis on characterizing and modeling the Internet topology at the autonomous systems level. He was awarded a prestigious Royal Academy of Engineering/EPSRC Research Fellowship in 2007 in support of his research on networks science.

Dr. Zhou is a Committee Member of the Internet Specialist Group of the British Computer Society (BCS). He is a reviewer of a number of IEEE journals (*Transactions on Networking, Transactions on Communications, Communications Letters* and *Communications Magazine*) and physics journals (*Physical Review E, Physica A, New Journal of Physics* and *European Physical Journal B*). He served as a Program Committee member of many IEEE conferences and workshops.

Zhongzhi Zhang received the B.Sc. degree in Applied Mathematics from Anhui University, Hefei, China in 1997 and Ph.D. degree in Management Science and Engineering from Dalian University of Technology, Dalian, China, in 2006. From July 2006 to June 2008, he was a Post-Doctorate Research Fellow in Fudan University. Currently he is an Associate Professor at the School of Computer Science, Fudan University, China.

Dr. Zhang's research interests include structural and dynamical properties of complex networks. He has published more than 60 papers in international journals since 2006 in the field of network modeling and dynamics, receiving over 500 citations with h_index 14 according to the Thomson Reuters ISI Web of Science.

Dr. Zhang is a Committee Member of the Chinese Society of Complexity Science. He received the Excellent Doctoral Dissertation Award of Liaoning Province, China, and the Excellent Post-Doctor Award of Fudan University, in 2007 and 2008 respectively. He is a reviewer for numerous international journals, including *IEEE Transactions on Circuits and Systems, EPL, European Physical Journal B, Journal of Physics A, Physica A*, among others.

Jihong Guan received the B.Sc. degree from Huazhong Normal University, China in 1991, the M. Sc. degree from Wuhan Technical University of Surveying and Mapping in 1991, and the Ph.D. degree from Wuhan University, China in 2002. Currently she is a Professor at the Department of Computer Science and Technology of Tongji University in Shanghai, prior to which she was a Professor at the School of Computer, Wuhan University. Her research interests include databases, data mining, distributed computing, bioinformatics, complex networks and geographic information systems.

Shuigeng Zhou received the B.Sc. degree in Electronic Engineering from Huazhong University of Science and Technology, China in 1988, the M.Sc. in Electronic Engineering from University of Electronic Science and Technology of China in 1991, and the Ph.D. degree in Computer Science from Fudan University, Shanghai in 2000. Currently he is a Professor at the Shanghai Key Lab of Intelligent Information Processing and the School of Computer of Fudan University, Shanghai, China. He was a senior engineer at the Shanghai Academy of Spaceflight Technology from 1991 to 1997 and a post-doctoral research fellow at the State Key Lab of Software Engineering, Wuhan University from 2000 to 2002. His research interests include data management in distributed environments, data mining, bioinformatics and complex networks.

Guanrong Chen (M'87, SM'92, F97) received the M.Sc. degree in Computer Science from Zhongshan University, Guangzhou, China in 1981 and the Ph.D. degree in Applied Mathematics from Texas A&M University, College Station, USA in 1987. Currently he is a Chair Professor and the Founding Director of the Centre for Chaos and Complex Networks at the City University of Hong Kong, prior to which he was a tenured Full Professor in the University of Houston, Texas.

Prof. Chen has numerous publications since 1981 in the fields of nonlinear dynamics, control systems and complex networks, receiving more than 16,000 non-self citations with h_index 62, and is an ISI Highly Cited Researcher in Engineering.

Prof. Chen served and is serving as Editor-in-Chief for the *IEEE Circuits and Systems Magazine* (1998–2011) and for the International Journal of Bifurcation and Chaos (since 2010), and as Deputy Editor-in-Chief, Advisory Editor and Associate Editors for several IEEE Transactions and international journals since 1991. He received four best journal paper awards in the past, along with the 2008 State Natural Science Award of China, the 2010 Ho-Leung-Ho-Lee Science and Technology Progress Award in China, and the 2011 Euler Gold Medal from the Euler Foundation, Russia. He was conferred Honorary Doctorate by the Saint Petersburg State University, Russia in 2011, and is Honorary Professor at different ranks in some twenty universities worldwide.

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