Problem Chosen	2024 MCM/ICM Summary Sheet	Team Control Number
B		2420511

20000 Leagues Under The Sea

Searching for shipwrecks in the Ionian Sea is a difficult task. It requires consideration of complex environmental variables, unstable dynamical systems, the use of multiple equipment, and an efficient and robust localization-search-cooperation approach. We create a model with a stable and coherent mechanism of action and strong generalization to cope with this complex search and rescue problem.

In response to the localization model, we creatively used **triangulation** to lock on to the submarine's position. In addition, we created a dynamics system to predict the position in case the submarine is lost. We also used a **Gaussian process** of sequential **Hamiltonian sampling** of the mechanical elements to check the model uncertainty. Finally, trajectory prediction was completed by **3D scene modeling**.

For equipment preparation, we classify the equipment into main ship equipment and assisting equipment on the rescue vehicle, on the basis of which, this paper solves the **collaborative competition coefficients** among different equipment. In order to determine the search equipment configuration scheme that can produce the best results, we build a multi-objective integer planning model based on **NSGA-II** solution to minimize the cost and maximize the search effect.

For the search model, we selected the temperature, density, flow velocity, and current direction of seawater as **heuristic information** to guide the search. We modeled the heuristic information separately and iteratively performed **covariance-based updating** to determine the optimal search range and initial deployment point through the **CMA-ES** method. In addition, for a single-rescue vehicle, we establish a **Markov process** based on the heuristic information to guide its search and predict the search success probability. For the cooperation of multiple rescue vehicles, we set up two **knowledge migration mechanisms**: experience speed sharing and repetitive search region suppression to increase the search effectiveness.

Considering the exploring of the model, in order to adapt the dynamical system to different sea areas, we modeled the seawater flow velocity at different latitudes and longitudes, by the **geostrophic equation** and six representative sea areas were selected for testing, and the errors were all within 5%. Considering the task of multiple rescue vehicles searching for multiple submarines, we introduce a **queuing mechanism** to minimize the search and rescue time.

In the model testing section, we performed a **sensitivity analysis** of the weighting parameters of the heuristic information. Parallel to this, we tested the robustness and generalizability of our model by making **correlation adjustments** for gravity, temperature and seawater density. Finally, we did **ablation experiments** on the three resistances of the dynamical model to determine how much different mechanical factors affect the localization model.

Keywords: Gaussian process; Collaborative-competitive relationship; Heuristic Search; Markov process; Queuing theory; Ablation Experiment.

Contents

1 Introduction

1.1 Background

Located in the eastern Mediterranean Sea, the Ionian Sea is a mysterious and beautiful sea with numerous magnificent fairy tales and mythological epics. Over the past few centuries, the Ionian Sea has witnessed a rise of ancient Greek civilization, the expansion of the Roman Empire and countless significant historical moments. Since Ionian Sea was once a vital hub for maritime trade and navigation, there are many shipwreck sites at the bottom of the sea. As the lastnon-subducted sector of the Neo-Tethys ocean, the Ionian Seaturns out to be the oldest in situ ocean fragment of the world,which attracts hundreds of tourists, explorers and expeditors to visit each year. Thanks to the development of modern science and technology, submersibles can now be used to explore the remains of shipwrecks beneath the azure blue sea.

It is therefore necessary to develop a reliable rescue system that takes a number of factors into account in order to better ensure the safety of passengers. In this regard, we not only need to consider the potential threat, such as communication defects of the submersible, loss of propulsion and natural hazard, but also need to cultivate specialized rescue personnel to copy with emergency situations.

1.2 Restatement of the problem

Our mission is to assist the MCMS Submarine Company in implementing a safety and rescue system for the submersible. We need to consider the dynamic modeling of a submarine in complex undersea environments for communication localization and position prediction, as well as equipment selection for rescue vehicle and host ship to improve search efficiency. After that, we need to model the collaborative search and rescue strategy and predict the search and rescue probability, and finally extend the model to different geographical areas and more search and rescue targets. Our mission can be divided into five parts:

- Modeling the communication between the host ship and the submarine and predicting the trajectory after the loss of connection based on the dynamics model, and finally doing an uncertainty analysis on the dynamics variables.
- Find the optimal equipment configuration for host ship and rescue vehicle, taking into account both the choice of equipment and the number of equipment selected.
- Find the optimal search scope, establish a search strategy for rescue vehicles and a mechanism for cooperation between them.
- Extension of the model to different environments and missions with more search targets. At this point it is important to do a correction of the dynamics model and a supplement to the search strategy
- The model was tested from three perspectives: parameters, variables, and components. And we need to give the results of the analysis

2 Foundation of the model

2.1 Assumptions and Justifications

- **The position of a submarine can only be transmitted by acoustic signals.** Since electrical signals are too fast so the measurement position error is large. After the failure of the sonar data such as temperature and density can be transmitted by electrical signals.
- **A submarine stops moving only when it reaches a point of neutral buoyancy and sinks to the bottom.** We don't take into account the chance of being affected by typhoons, marine life impacts, tsunamis, and so on.
- **Assuming that sudden changes in temperature and density do not affect the model** Neglecting the effects of abrupt changes in seawater density and temperature due to phenomena such as river-sea interfaces or deep-sea hot springs.
- **Uninterrupted communication between rescue vehicles in real time** Position information between rescue vehicles and seawater information measured by various devices can be highly shared in real time without interruption.
- **The probability of sinking a submarine can be fitted to some distribution** We model the distribution of crash probability over time with an exponential distribution.
- **Equipments are all available and undamaged** All equipment costs are within acceptable limits and work together in real time during the search and rescue process.

2.2 Nomenclature

Table 1: Major Notations

Symbol	Definition
v_w	The sea water velocity
v_s	Submarine speed
ρ_p	Local seawater density
f_{fri}	Frictional resistance to submarine travel
f_{oce}	Current resistance to submarine travel
f_{flo}	The buoyant force on the traveling submarine
M_s	The mass of submarine
F_{aut}	Submarine motive power
C_d	Coefficient of friction of a submarine
C_p	Pressure coefficient of a submarine
ω_p	Angular velocity of Earth's rotation at current position
g_p	Local acceleration of gravity
P_k	Probability of success of search and rescue submarine by $k-th$ rescue vehicle

3 Model Design and Promotion

3.1 Locating Model

3.1.1 Signal Locating Design

We know that it only takes three satellites that can communicate with each other to cover every location on Earth with exploration space. Inspired by this, we also use triangulation to determine the location of submarines. As shown in the **Figure 1**, the host ship sends acoustic signals to the submarine through the interrogator-receiver, and the submarine returns a signal immediately after receiving the signal through the FSS (Free Swimming Submersible) equipment, which measures the interval time Δt_1. In this scenario, the host ship's range line sinks the TSS (Tetherd Submersible) equipment into the sea at a predetermined angle. A similar acoustic signal reception and propagation is performed to obtain the time interval Δt_2 between the TSS equipment and the submarine and the time interval Δt_3 between the host ship and the TSS equipment.

Considering that we know the speed of propagation of acoustic signals in the water, then we can get the distance between the host ship, the TSS, and the submarine carrying the FSS, and according to the principle of plane geometry, their relative positions are uniquely determined in the plane consisting of these three. However, when our consideration space becomes three-dimensional, the trajectory of the possible positions of the submarine becomes an arc as shown in the figure. So we lay out another TSS and use the same method to determine another curve of the trajectory of the possible position of the submarine as shown in **Figure 2**, and the intersection of the two is the finalized position of the submarine. The above can be modeled as follow:

$$\begin{cases} (p_h - p_s)^2 = (v_w \Delta t_{1i})^2 = (x_h - x_s)^2 + (y_h - y_s)^2 + (z_h - z_s)^2 \\ (p_{TSSi} - p_s)^2 = (v_w \Delta t_{2i})^2 \\ p_h - p_{TSSi} = v_w \Delta t_{3i} \vec{e_i} \\ i = 1, 2 \end{cases} \tag{1}$$

p_h, p_s, p_{TSSi} in the above equation refers to the position of the host ship, the submarine and the $i-th$ TSS in the three-dimensional space respectively. And $\Delta t_{1i}, \Delta t_{2i}, \Delta t_{3i}$ denote the specific values of the above time intervals in the plane where the TSSi is located. Ultimately, $\vec{e_i}$ refers to the direction

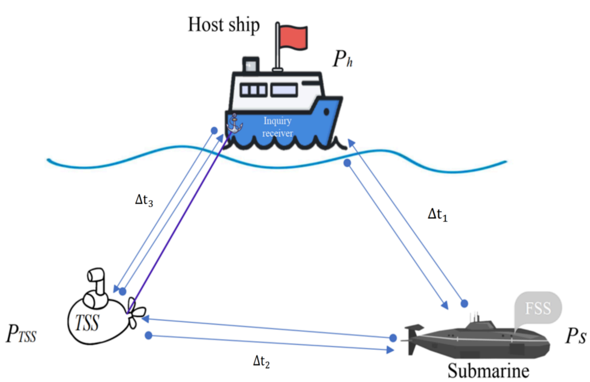


Figure 1: Plane communication model

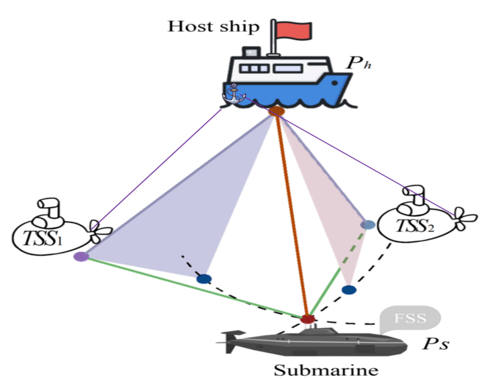


Figure 2: Space communications model

vector of the host ship when it puts TSSi. Since the temperature, depth and salinity of seawater affect the propagation of acoustic signals, which undoubtedly affects our position determination, we used the *W. D. Wilson formula* to correct for the speed of sound in water:

$$v_w = 1449.2 + 4.6tp - 5.5 \times 10^{-2}tp^2 + 2.9 \times 10^{-4}tp^3 + (1.34 - 0.01tp)(S - 35) + 1.7 \times 10^{-2}Z$$

where tp denotes sea water temperature, S denotes sea water salinity, and Z denotes sea water depth. These indicators are based on the seawater at the current position of the submersible. And all these variables can be obtained by **CTD** (Conductivity Temperature Depth).

3.1.2 Dynamical System Modeling

We first consider the motion of the submarine in an inertial system. We can make the following statement about the force of gravity, resistance and eventual acceleration on it:

$$g = g_p - w_p^2 r_p \tag{2}$$

$$f = f_{oce} + f_{flo} + f_{fri} \tag{3}$$

The gravitational acceleration on the submarine is the local gravitational acceleration g_p minus the centripetal acceleration of the Earth's rotation $w_p^2 r_p$. And the resistance to motion consists of ocean current disturbance f_{oce}, motion friction f_{fri} and buoyancy f_{flo}. Also, we introduce the submarine autonomous power F_{aut} into the speed update:

$$v_n' = v_n + \frac{d(F_{aut} + f + g)}{dM_s} dt \tag{4}$$

where M_s is mass of the submarine. Finally we introduce the inertial system into the oceanic reference system, a transformation that requires consideration of the parameters as below:

Heading angle ψ : the angle between the projection of the submarine's longitudinal axis on the local horizontal plane and the local geographic north direction.

Pitch angle θ : the angle between the positive direction of the vertical axis of the submarine and its horizontal projection line.

Roll angle ϕ : the angle between the positive direction of the vertical axis of the submarine and the plumb plane where the longitudinal axis of the submarine is located.

We can migrate the mass point on the inertial system[Liu et al.] to the reference system by using the angle transformation[Williams et al.], and the migration coefficient C_{NED}^{FRD} can be expressed as follows:

$$C_{NED}^{FRD}(\psi, \theta, \Phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By means of migration coefficients, we update the velocity in the reference system:

$$v' = C_{NED}^{FRD}(\psi, \theta, \Phi) v_n' \tag{5}$$

However, we remain unclear about f_{oce}, f_{flo} and f_{fri}. So we started by modeling friction and buoyancy (the minus between pressures) using the Reynolds stress model(RSM). Here we explain just the x-axis in three-dimensional space as an example, where v_s and v_w denote the speed of the submarine and the speed of the water, respectively, δ_{sw} is a coefficient related only to the angle of the velocity pinch.

$$f_{frix} = C_d\rho\overline{v_s v_w} = \mu(\frac{\partial v_s}{\partial x} + \frac{\partial v_w}{\partial x}) - \frac{2}{3}\rho\frac{\partial v_s}{\partial x}\delta_{sw} \tag{6}$$

$$p_x = C_p\Delta x = \rho\frac{v_s v_w\frac{\partial v_w}{\partial x} + 2\Delta x\frac{\partial v_s}{\partial x}\frac{\partial v_w}{\partial x}}{\frac{\partial v_s}{\partial x} + \frac{\partial v_w}{\partial x}} \tag{7}$$

$$f_{flox} = \sum_{\Theta\in(x,y,z)} \Delta p_x\frac{d\Theta(x,y,z)}{dx} \tag{8}$$

We can simplify the complex equation by expressing the coefficient of friction and the coefficient of pressure[Lapenna et al., Qu et al.] in **Eq 6** and **Eq 7**. The determination of these coefficients requires knowledge of the flow velocity and density of the nearby waters, the density can be measured with the previously mentioned **CTD**; and we need to use the new equipment **ADCP**(Acoustic Doppler Current Profiler) to measure the flow velocity of the waters. We modeled the submarine as an ellipsoid and simulated the forces during the submarine's travel using the Ionian Sea's watershed data, thus fitting the distribution of friction coefficients and pressure coefficients of each part of the submarine as shown in **Figures 3** and **Figure 4**. At the same time, we sampled the coefficients of the uniaxials[Serge et al.], fitted and estimated them as shown in **Figure 5** and **Figure 6** and found that they have a strong fit and confidence that they can be used in the dynamic simulation of the submarine. Finally, we establish the relationship between the effect of the oceann current on the submarine and its flow rate Q and the angle β between its flow direction and v_s, with k being the corresponding coefficients:

$$f_{oce} = kQ^2 cos\beta \tag{9}$$

3.1.3 Uncertainty Analysis

Now, let's review our dynamics model, where the forces affecting the motion of the model can be represented as the following quintet $(g, F_{aut}, f_{fri}, f_{oce}, f_{flo})$. The first three of these are known.

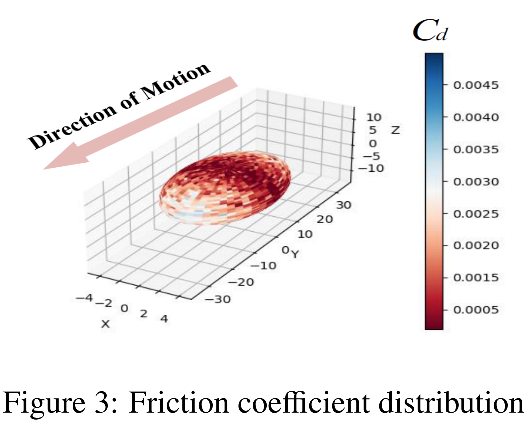


Figure 3: Friction coefficient distribution

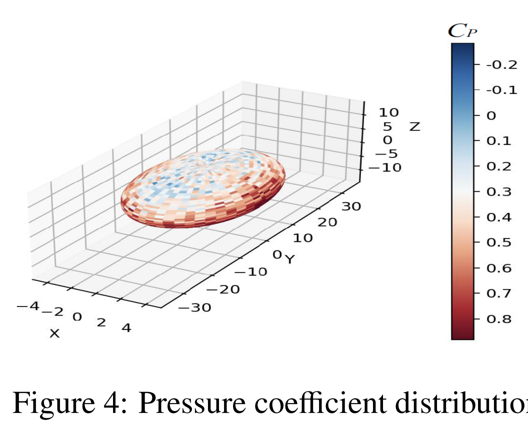


Figure 4: Pressure coefficient distribution

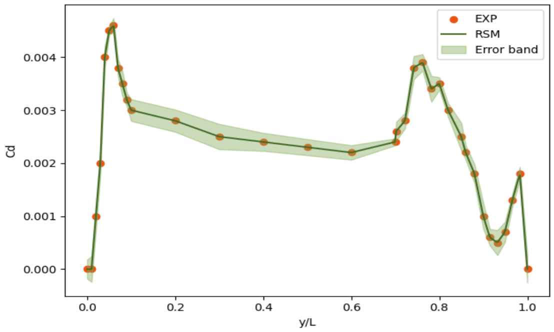


Figure 5: Friction coefficient curve

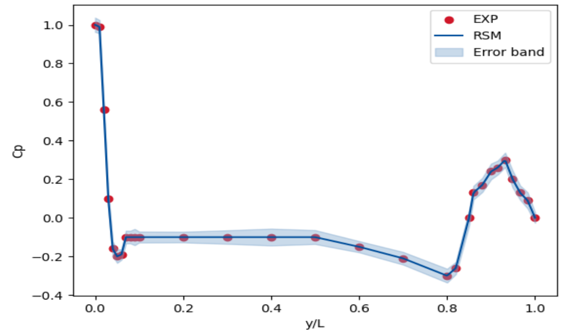


Figure 6: Pressure coefficient curve

The latter two positions are because the flow density and current information is unknown in the parameters of f_{oce} and f_{flo}. We can only obtain the information once it is available from the additional equipment. For this reason we design four mechanistic groups for uncertainty testing by means of control variables as shown in **Table 2**.

Table 2: Uncertainty Analysis

	Test Statistic	Reference Value
1	(g, F_{aut}, f_{fri})	$(f_{oce}, f_{flo}) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$
2	$(g, F_{aut}, f_{fri}, f_{oce})$	$f_{flo} \sim N(\mu_2, \sigma_2^2)$
3	$(g, F_{aut}, f_{fri}, f_{flo})$	$f_{oce} \sim N(\mu_1, \sigma_1^2)$
4	$(g, F_{aut}, f_{fri}, f_{oce}, f_{flo})$	None

Our test recognizes the reference quantities as all Gaussian distributions and is making discrete decisions in continuous time, so it is basically recognized as a Gaussian process. In a Gaussian process[Wilson et al.], we take values for normally distributed quantities using Hamiltonian Monte Carlo sampling[Lotfi et al.]. Taking the first set of tests as an example, the bi-normal overall distribution is modeled by the sampler as the distribution of electrons outside the nucleus of the atom according to the probability distribution in space as shown in **Figure 7**, and the sampling process is modeled as the movement of electron.

We symbolize the uncertainty of the process by the degree of dispersion of the sample, which can specifically be fitted with the kernel function of a Gaussian process:

$$K = \sum_{i,j} K(x_i, x_j) = \sum_{i,j} \sigma^2 \exp\left(-\frac{\|x_i - x_j\|^2}{2l^2}\right) \tag{10}$$

where σ and l are hyperparameters of the kernel function. Our results have been represented in **Figure 8**, with the light blue bars representing the 95% confidence intervals.

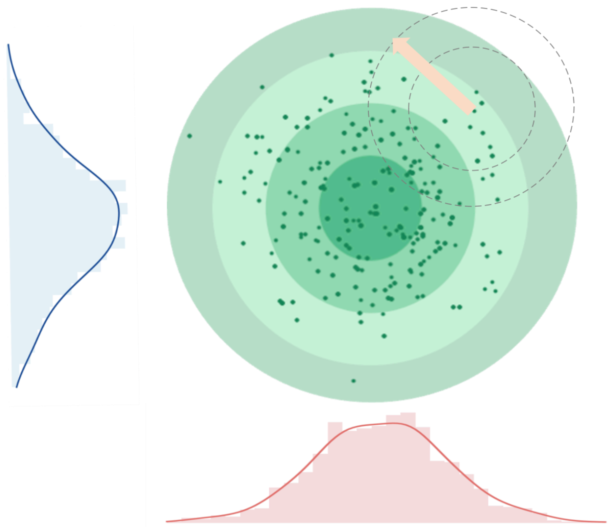


Figure 7: Friction coefficient curve

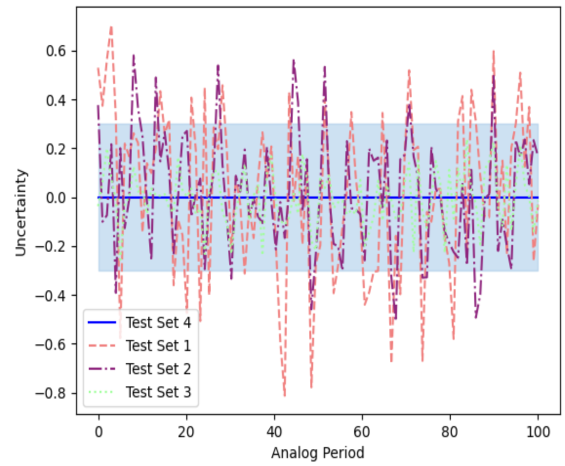


Figure 8: Pressure coefficient curve

3.1.4 Modeling and Locating

Now there is one last point we need to consider. Before that we considered the communication model, on the basis of which we can find the position of the submarine, as long as the communication equipment does not fail. Then we considered the dynamics model, which allows us to predict the submarine's course over time in the event of a failure of the communication equipment. Finally we consider the case when the submarine loses power. In this case, we need to make a small correction to the dynamical system, i.e., deleting the role of F_{aut}:

$$v_n' = v_n + \frac{d(f+g)}{dM_s}dt \tag{11}$$

In the event of a loss of power, the localization model will terminate under two conditions: encountering a point of neutral buoyancy and sinking to the bottom, so we need to patch the model with constraints:

$$\text{s.t.} \begin{cases} f_{flo} \neq M_s g_p, \\ p_s \neq p_{bottom}, \end{cases}$$

Finally we simulated the navigation process as shown in **Figure 9**.

3.2 Equipment Preparation

3.2.1 Establishment of multi-objective optimization model

In order to improve the performance of a search and location system, it is necessary to equip the main ship with appropriate searching equipment and the rescue submarine with the appropriate equipment to accomplish the assistance. Through extensive research we have identified 20 types of searching equipment that can be fitted to either the main ship or the rescue submarine.

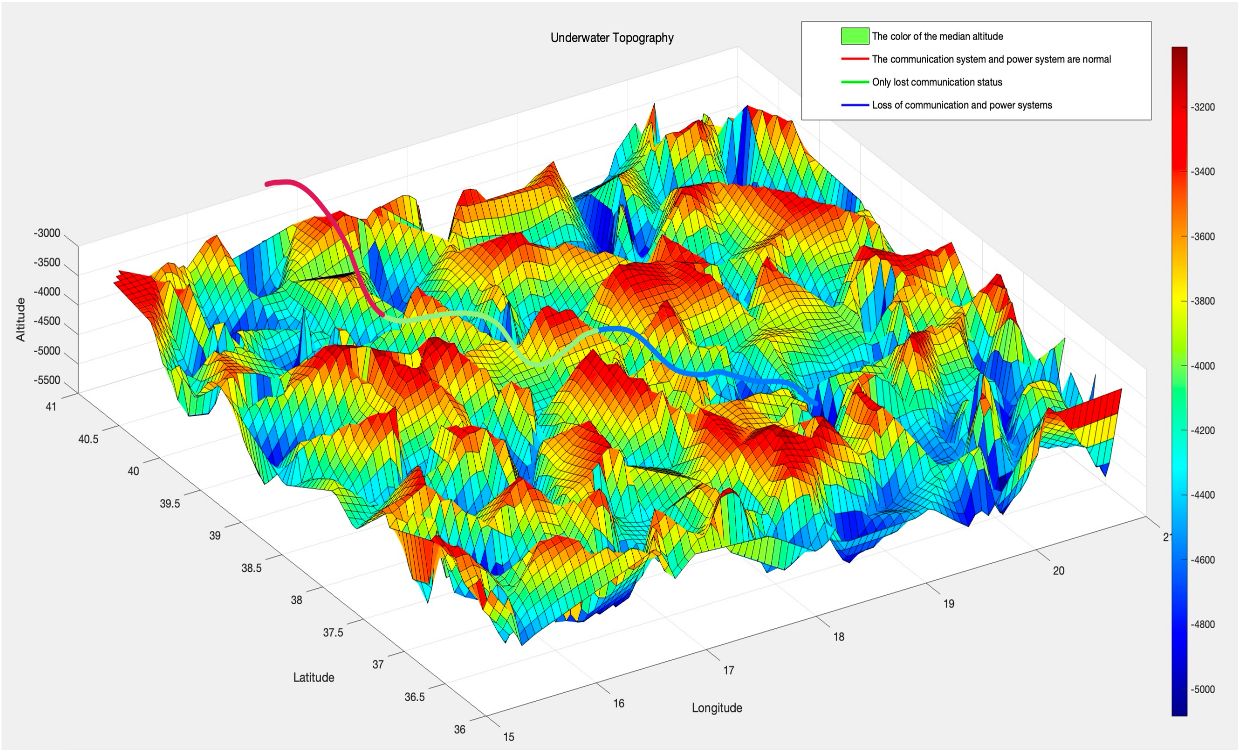


Figure 9: Navigation process

Table 3: Rescue equipment list

loaded position	Equipment	Basic Usage
On the rescue submarine	Autonomous Underwater Vehicle(AUV)	Undertake specific missions in the deep sea independently
	Ultra-short baseline sonar (USBLS)	Generate ultra-baseline acoustic signals
	long baseline sonar (LBLS)	Generate long-baseline acoustic signals
	short baseline sonar (SBLS)	Generate short-baseline acoustic signals
	Side-scan sonar (SSS-M)	Seabed topographic mapping and lateral-target detection
	Multibeam sonar (MBS)	Generate high-resolution underwater image
	Mechanical arm (MA)	Perform delicate maneuvers in complex marine environment
	Collision sonar (CS)	Detect obstacles or collisions with the environment
	Imaging sanor (IS)	Generate detailed real-time images of the underwater terrain and avoid obstacles
	Radiocommunication equipment (RE-M)	Use very highor ultra high frequency radio waves to transmit and receive data or voice communications over a short distance
	Non-Pyrotechnic marine distress signals (NPMDS)	Signal for help or express distress in marine emergencies

On the host ship	Side-scan sonar (SSS-H)	Seabed topographic mapping and lateral-target detection
	Remote operated vehicle (ROV)	Unmanned underwater vehicle operated by remote controller to undertake specific missions
	Ultra-short baseline array sonar (USBLA)	Determine the accurate position of a submersible according to the acoustic signals(offers a range of a few hundred meters and accuracy within a few centimeters to a few decimeters)
	long baseline array sonar (LBLA)	A position system provides a range of a few kilometers and accuracy within a few meters
	short baseline array sonar (SBLA)	A position system covers a range of a few hundred meters and accuracy within a few centimeters to a few meters
	Radiocommunication equipment (RE-M)	Use very highor ultra high frequency radio waves to transmit and receive data or voice communications over a short distance
	Towed array sonar (TAS)	Use to detect and locat underwater object, and provide detailed acoustic images of the underwater environment
	Underwater Locator Beacon (ULB)	Emit signals to help locate the submarine
	Wreck buoy (WB)	Deployed near a sunken ship or underwater wreck to mark its location and provide a reference point for divers or maritime navigation

Initially, we define $N = \{n_1, n_2, ..., n_{20}\}$ as the decision variables to represent the actual amount of the equipment. We apply integer programming ideas into our model so that each element inside set N should be an integer.

According to the Equipment bove, we build a two-dimension matrix $X^{4\times 20}$ to represent the searching ablity of each equipmet.(Including rating scores for Searching Frequency, Searching Scale, Searching accuracy, Response time). In the function below, we use $X_{\bullet i}, i \in \{1, 2, 3, ..., 20\}$ to represent the searching ablity column vector for each sigle equipment.

Considering that there is collaboration or competition between different searching devices, this paper uses the matrix $R_{ij}, i, j \in \{1, 2, 3, ..., 20\}$ to represent this relationship.

When the two pieces of equipment are loaded on the main ship and the rescue submarine, respectively.

$$R_{ij} = \frac{X_{\bullet i} \cdot X_{\bullet j}}{|X_{\bullet i}| \cdot \left|X_{\bullet j}\right|}$$

When both types of equipment are loaded on the main vessel or rescue submarine at the same time

$$R_{ij} = -\frac{X_{\bullet i} \cdot X_{\bullet j}}{|X_{\bullet i}| \cdot \left|X_{\bullet j}\right|}$$

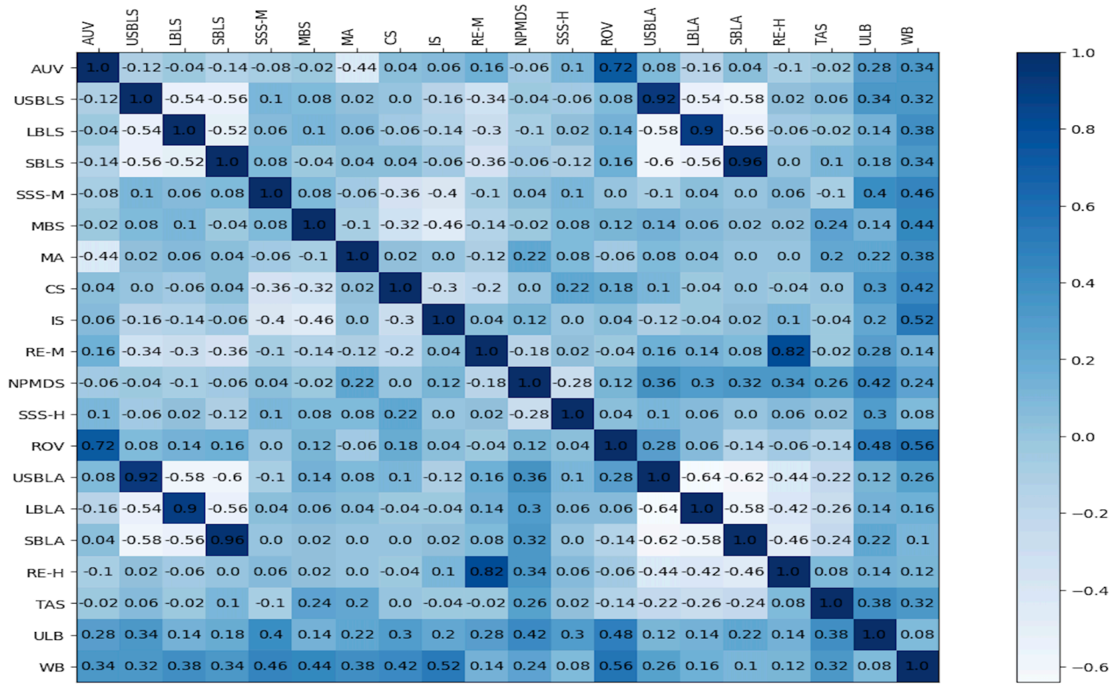


Figure 10: correlation coefficient for different equipments

First of all, we should maximize our searching localization system performance:

$$Min \quad F_1 = -\sum_{i=1}^{20}\sum_{j=1}^{20} R_{ij}\sqrt{(1+\alpha_i \ln(n_i))(\mu \cdot x_i) \times (1+\alpha_j \ln(n_j))(\mu \cdot x_j)}$$

We assume $\mu^{1\times 4}$ as a row vector that measures the relative importance weights of frequency, range, accuracy and speed for our searching localization system, and $\alpha_i, i \in \{1,2,3,...,20\}$ as the impact factor of the device contribution to the searching localization system in the case of incremental increase in number. So as to link this impact factor to the equipment amount, we use $1+\alpha_i \ln(n_i)$ curve to simulate relationship between device searching capability and quantity.

Secondly,to achieve cost minimization we construct the following objective function:

$$Min \quad F_2 = \sum_{i=1}^{20} n_i \gamma \cdot Y_i$$

We should consider the foundamental costs for the searching localization equipment, such as maintenance costs, procurement costs, personnel training costs and loading costs, for which we use the $Y^{4\times 20}$ matrix to represent the costs of 20 kinds of equipment. Use $\gamma^{1\times 4}$Matrix to represent the cost measurement coefficient.

To list our constraint conditions, we should keep our quantitative influence factor α positive, and the sum of relative importance weight factor μ and γ is equal to 1. Moreover, the coefficient of collaborative competition inside R matric should be range from -1 to 1. Associated with the reality, it is necessary to constrain the maximal expense M_d and minimal search effect M_e.

$$Min \quad F_1 = -\sum_{i=1}^{20}\sum_{j=1}^{20} R_{ij}\sqrt{(1+\alpha_i \ln(n_i))(\mu \cdot x_i) \times (1+\alpha_j \ln(n_j))(\mu \cdot x_j)}$$

$$Min \quad F_2 = \sum_{i=1}^{20} n_i \gamma \cdot Y_i$$

$$s.t. = \begin{cases} \alpha_i \geq 0 \\ \sum_{k=1}^{4} \mu_k = 1 \\ \sum_{k=1}^{4} \gamma_k = 1 \\ -1 \leq R_{ij} \leq 1 \\ F_1 \leq -M_e \\ F_2 \leq M_d \end{cases}$$

3.2.2 Model Solution

Algorithm 1: Multi-objective integer planning algorithm based on NSGA-II

Input: $\mu^{1\times4}, \gamma^{1\times4}, R^{20\times20}, \alpha^{1\times20}, X^{4\times20}, Y^{4\times20}$

Output: A set of solutions that satisfy Pareto optimality

Initialize the parent population matrix $P_t^{50\times20}$;
while *terminal criteria is not satisfied* **do**
  Create an empty population matrix $Q_t^{50\times20}$;
  while $|Q_t| < |P_t|$ **do**
    Use Roulette-based selection to select 2 individuals from P_t
    Perform crossover and mutation to create offspring
    Add offspring to Q_t;
  end
  Merge parents and offspring population,$P_t = P_t \cup Q_t$
  Perform non-dominated sorting on P_t
  Perform crowding calculation on P_t
  P_t = Truncate(P_t)
end

3.2.3 Final Equipments

On the host ship									
Equipment	ROV	USBLA	TAS	WB	ULB	RE-H			
Quantity	2	1	1	1	1	1			
On the rescue submarine									
Equipment	AUV	USBLS	MA	IS	SSS	CS	RE-M	NPMDS	MBS
Quantity	1	1	1	1	2	4	1	1	2

3.3 Searching Model

3.3.1 Heuristic Information Modeling

We decided to rescue the submarine using a heuristic search guide rescue vehicle. Before that, we need to model the heuristic information. We selected temperature, density, seawater current speed and current flow direction as our heuristic information, which can be measured by carrying CTD and ADCP on the rescue vehicle. Sea water temperature T can be considered to be basically related to the depth d of the sea water, and we collected sea water and temperature data for the Ionian Sea and curve-fitted them based on the sample points:

$$\begin{bmatrix} \sum d_i^6 & \sum d_i^5 & \sum d_i^4 & \sum d_i^3 \\ \sum d_i^5 & \sum d_i^4 & \sum d_i^3 & \sum d_i^2 \\ \sum d_i^4 & \sum d_i^3 & \sum d_i^2 & \sum d_i \\ \sum d_i^3 & \sum d_i^2 & \sum d_i & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ T_0 \end{bmatrix} = \begin{bmatrix} \sum T_i d_i^3 \\ \sum T_i d_i^2 \\ \sum T_i d_i \\ \sum T_i \end{bmatrix}$$

then we obtain the following equation:

$$T(d) = -1.72 \times 10^{-9} \cdot d^3 + 1.06 \times 10^{-5} \cdot d^2 - 2.18 \times 10^{-2} \cdot d + 17.4 \tag{12}$$

Similarly, the density of seawater can be essentially thought of as depth-dependent modeling We performed curve fitting based on the obtained data and obtained the following equation:

$$\begin{bmatrix} \sum d_i^6 & \sum d_i^5 & \sum d_i^4 & \sum d_i^3 \\ \sum d_i^5 & \sum d_i^4 & \sum d_i^3 & \sum d_i^2 \\ \sum d_i^4 & \sum d_i^3 & \sum d_i^2 & \sum d_i \\ \sum d_i^3 & \sum d_i^2 & \sum d_i & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ \rho_0 \end{bmatrix} = \begin{bmatrix} \sum \rho_i d_i^3 \\ \sum \rho_i d_i^2 \\ \sum \rho_i d_i \\ \sum \rho_i \end{bmatrix}$$

And the final result is:

$$\rho(d) = 2.19 \times 10^{-13} \cdot d^3 - 1.64 \times 10^{-9} \cdot d^2 + 3.82 \times 10^{-6} \cdot d + 1.025 \tag{13}$$

The relationship between these two heuristic variables and depth is given in **Figure 11**. Next, consider the modeling of seawater flow velocity in space, which is undoubtedly relevant to every dimension in space. We set the location of the rescue vehicle as the origin and consider its upward and downward current velocity modeling. In the x-y direction, the seawater flow rate is mainly affected by the Earth's deflection force, we can model this with the laws of conservation of momentum and conservation of mass for fluids:

$$\frac{\partial \rho_w}{\partial t} + \nabla \cdot (\rho \mathbf{v_w}) = 0 \tag{14}$$

$$\frac{\partial \mathbf{v_w}}{\partial t} + (\mathbf{v_w} \cdot \nabla)\mathbf{v_w} = -\frac{1}{\rho}\nabla \cdot \sigma + \mathbf{w_p^2 r_p} \tag{15}$$

In the vertical direction, the force on the fluid is simpler, just consider the effect of gravity:

$$\frac{\partial \mathbf{v_w}}{\partial t} = \mathbf{g_p} \tag{16}$$

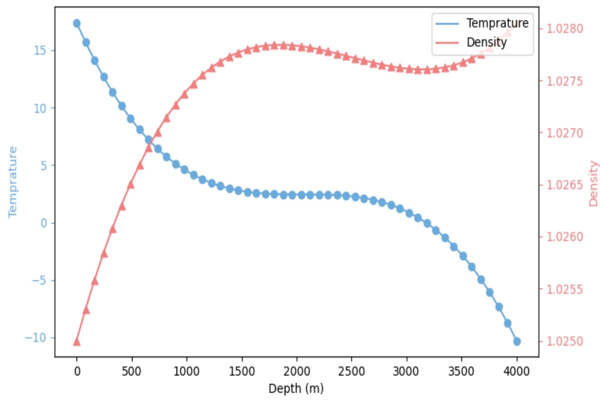


Figure 11: Temperature and density fitting

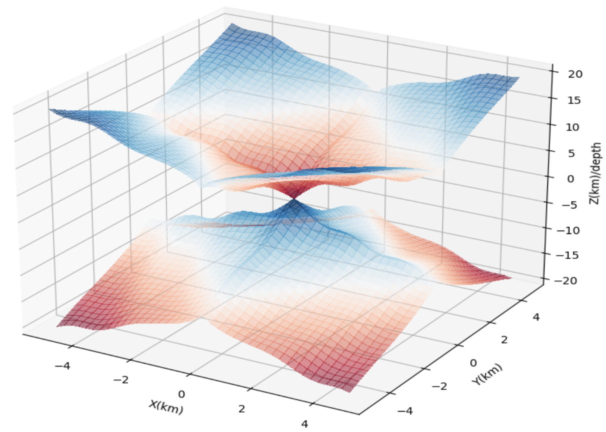


Figure 12: Flow rate modeling

Dynamics simulation modeling is performed above and below the location of the rescue vehicle, then we can get the results as shown in **Figure 12**. The flow velocity in the water above the submarine decays gradually with extents, and the flow velocity in the water below gradually increases with extents. We now model the last heuristic information - the perspective of the ocean currents passing over the rescue vehicle. Since the change in angle is extremely fragile and very poorly robust over the search range, we decide to convert it to curvature before modeling it:

$$\kappa = \frac{\|d\beta\|}{\|dS\|} \tag{17}$$

Also, we know that if we let

$$\begin{cases} x(\theta) = r(\theta)\cos\theta \\ y(\theta) = r(\theta)\sin\theta \end{cases} \tag{18}$$

and then:

$$\kappa = \frac{\left|r^2 + 2r'^2 - rr''\right|}{(r^2 + r'^2)^{\frac{3}{2}}} \tag{19}$$

In this way, we can smooth the curvature by the existing position and thus predict the curvature within the neighborhood of the existing position.

3.3.2 Initialization Deployment

We use the CMA-ES[Nikolaus et al.] method for the initial search space range determination. First we initialize a search position covariance matrix using the submarine position predicted by the localization model as the sample center:

$$X_k^1 \sim N(p_s, C^{(0)}) \tag{20}$$

where $C^{(0)}$ is the initial covariance matrix. Then we select the top μ samples with higher fitness scores as elite individuals. The fitness score here refers to how close the four heuristics for these locations are to the heuristics for the submarine's location:

$$fitness = \omega_1(T_p - T_s) + \omega_2(\rho_p - \rho_s) + \omega_3(v_{wp} - v_{ws}) + \omega_4(\beta_p - \beta_s) \tag{21}$$

With the first μ samples selected, we can update the mean of the next generation of populations by weighted average:

$$m(g+1) = \sum_{i=1}^{\mu} w_i x_{i:(g+1)} \tag{22}$$

$$\sum_{i=1}^{\mu} w_i = 1, \quad w_1 \geq w_2 \geq \ldots \geq w_\mu > 0 \tag{23}$$

Here g denotes the number of iterations the population is in, and $m(g+1)$ denotes the mean value that needs to be updated for the next generation of the population Next, we update the covariance of the next generation population distribution as well:

$$C_\mu^{(g+1)} = \sum_{i=1}^{\mu} w_i \left(x_i^{(g+1)} - m^{(g)}\right)\left(x_i^{(g+1)} - m^{(g)}\right)^T \tag{24}$$

This generates new solutions until the maximum number of iterations λ is reached:

$$x_k^{(g+1)} \sim m^{(g)} + \sigma^{(g)} N(0, C^{(g)}) \quad \text{for } k = 1, ..., \lambda \tag{25}$$

where σ^g is the update step we set for ourselves, the whole process is shown in the **Figure 13**.

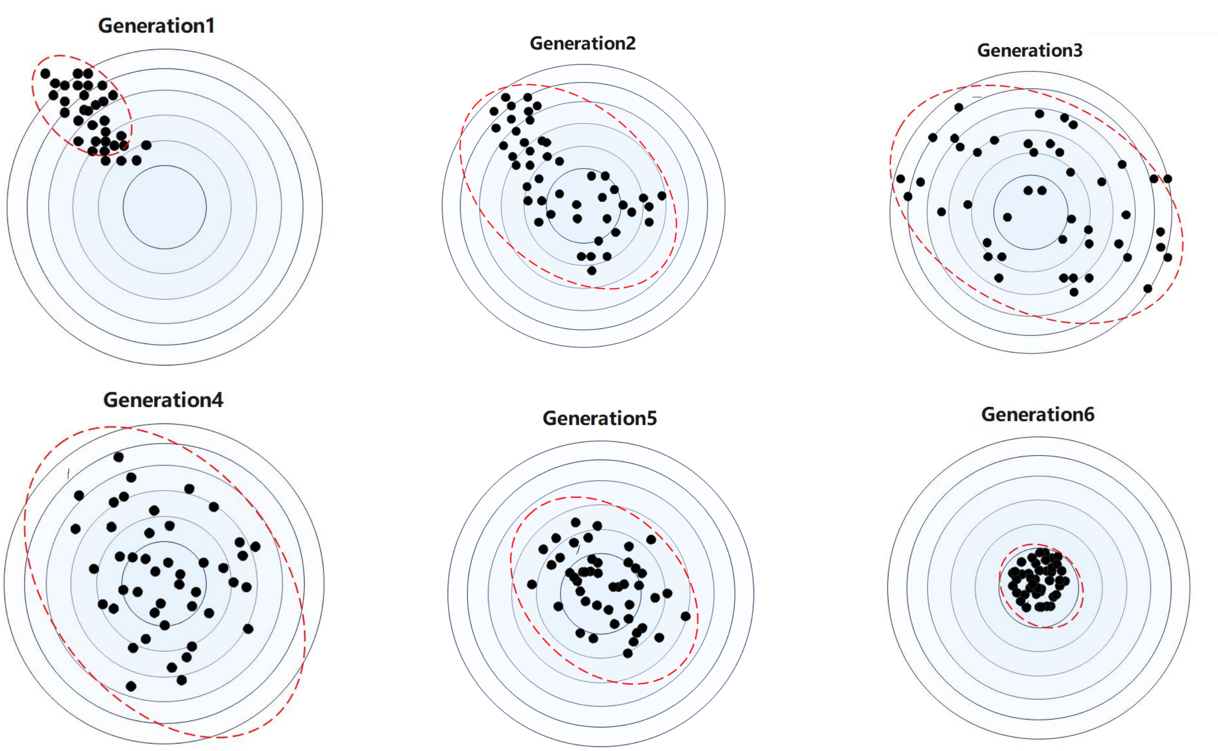


Figure 13: The searching process of CMA-ES

We arrange the n-rescue vehicles used for search and rescue all clockwise on the search boundary, and in order to allow us to do as much global search as possible, set their initial positions to

$p_1, p_2.....p_n$, we require that their Euclidean distance is maximized:

$$\max_{p_1,p_2,...,p_n} \sum_{i=1}^{n-1} \sqrt{p_i^2 - p_{i+1}^2} = \max_{p_1,p_2,...,p_n} \sum_{i=1}^{n-1} \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 + (z_i - z_{i+1})^2}$$

3.3.3 Heuristic Search

We first consider the heuristic search for a single submarine, with the basic idea being to measure the gap between the available heuristic information and the submarine heuristic information one at a time, thus guiding itself to update in the direction of higher probability searches[Han et al.].

In the process of updating, we need to take into account the degree of variation of different heuristic information, for example, the variation of temperature T is much less drastic than the variation of flow rate v_w, and if we set the updating of these two pieces of information on the velocity to be of equal step size, then our global search effectiveness will be greatly reduced. For this reason we need to limit the update step size by the differential of the information at the current location, which is formulated as follows:

$$v_{pz} = \omega v_{pz} + \omega_1 \frac{T_p - T_s}{\frac{\partial T(z)}{\partial z}} + \omega_2 \frac{\rho_p - \rho_s}{\frac{\partial \rho(z)}{\partial z}} \tag{26}$$

$$v_p = \omega v_p + \omega_3 \frac{v_{wp} - v_s}{\frac{\partial v_w}{\partial r_p}} + \omega_4 \frac{\kappa_p - \kappa_s}{\frac{\partial \kappa}{\partial r_p}} \tag{27}$$

Where temperature and density only affect the update of the velocityrescue vehicle v_p in the depth z dimension, while the seawater flow velocity v_w and the direction of the current β (curvature κ) affects the update of the velocity v_p in all the dimensions.

Note that here our quintuple $(\omega, \omega_1, \omega_2, \omega_3, \omega_4)$ is going to be updated with the probability of possibly finding the submarine, and we can think of the submarine search and rescue process as a Markov process, which is a process of constantly making decisions based on the environment and then being rewarded for them before changing the current decisions. At each decision moment, we define that our rescue vehicle has gone through the following speeds $v_1, v_2.....v_n$. Where we can obtain v_{max} and v_{min}. Note that both max and min here are relative to the current sequence at the time of computation. We set the reward to R and the discount rate to γ. Then we can model R as:

$$R = \frac{v_n - v_{n-1}}{v_{max} - v_{min}} + \gamma \frac{v_n - v_{n-1}}{v_{max} - v_{min}} + ... + \gamma^{n-1} \frac{v_2 - v_1}{v_{max} - v_{min}} \tag{28}$$

Through the Markov process, we use the previous information to update the current speed, the larger the magnitude of the previous speed update, it means that the current heuristic information is more conducive to correcting the speed, we will adjust the corresponding step weight; the smaller the magnitude of the previous speed update, it means that in the current dimension is close to the optimal solution, if the step is too large, it may deviate from the optimal solution, so we should adjust the corresponding step weight down. And at the same time, the introduction of the discount factor allows us to focus more on the impact of speed updates at nearby points in time on decision making, to the effect of targeting. Our update formula can be described as:

$$(\omega, \omega_3, \omega_4) = (\omega, \omega_3, \omega_4) \cdot \lambda_1 \frac{R}{n} \tag{29}$$

$$(\omega_1, \omega_2) = (\omega_1, \omega_2) \cdot \lambda_2 \frac{R_z}{n} \tag{30}$$

We still use the Gaussian kernel function from the previous section to model the uncertainty and thus determine the probability$P$ of finding the submarine:

$$P = 1 - K(R) = 1 - \sum_{i,j} K(R_i, R_j) = \sum_{i,j} \sigma^2 \exp\left(-\frac{\|R_i - R_j\|^2}{2l^2}\right) \tag{31}$$

This means that the more our R stabilizes, i.e., the more our Markov process stabilizes, the higher our probability of search and rescue success. Finally, when we consider the work of multiple rescue vehicles, we want to introduce knowledge migration formulas to increase the collaboration between them, making their search more efficient. The μ rescue vehicles with the highest P values in the current state are selected each time. Their knowledge is migrated to the other rescue vehicles:

$$TE_{i-t} = \omega_i(v_i - v_t) \tag{32}$$

$$v_t = v_t + \sum_{i=1}^{\mu} \sum_{t=\mu+1}^{n} TE_{i-t} \tag{33}$$

$$\sum_{i=1}^{\mu} \omega_i = 1, \quad \omega_1 \geq \omega_2 \geq \ldots \geq \omega_\mu > 0 \tag{34}$$

The next problem we have to consider is that we want to prevent multiple rescue vehicles from repeatedly taking the searched path while searching, although it may be a shortcut to the submarine in terms of greedy strategy, there is no possibility of successful search and rescue on this path, and secondly it may cause our search space to converge at the wrong location to the detriment of our fast global search, we simulate the ACO algorithm[Yu et al.,Yang et al.] to design an inhibitory pheromone, which controls rescue vehicles to try not to pass through paths that they have already traveled to each other. Consider the rescue vehicle passing through an arbitrary search path $path(i, j)$ leaving the pheromone $\tau_{i,j}$, consider our volatilization rate as ϱ, and the pheromone volatilizes according to the volatilization rate at a certain periodicity, then the update formula is:

$$\tau_{i,j}^{k} = \begin{cases} \frac{1}{P_k} & \textit{if the } k_{th} \textit{ rescue vehicle passed the path}(i, j) \\ 0 & \textit{otherwise} \end{cases} \tag{35}$$

This means that the greater the probability of a successful search and rescue for a rescue vehicle that passes through the path, the less inhibitory pheromones it will leave behind down that path. The updating formula for the inhibitory pheromone is:

$$\tau_{i,j} = (1 - \varrho)\tau_{i,j} + \sum_{k=1}^{n} \tau_{i,j}^{k} \tag{36}$$

The final pheromone works in such a way that any rescue vehicle at point i and with velocity v_k in the same direction as $path(i, j)$ will be subject to a deflection of the velocity, and considering a deflection angle of β, then:

$$\beta = \arccos(P_k \cdot \tau_{i,j}) \tag{37}$$

Ultimately, our search and rescue process can be represented as **Figure 14**

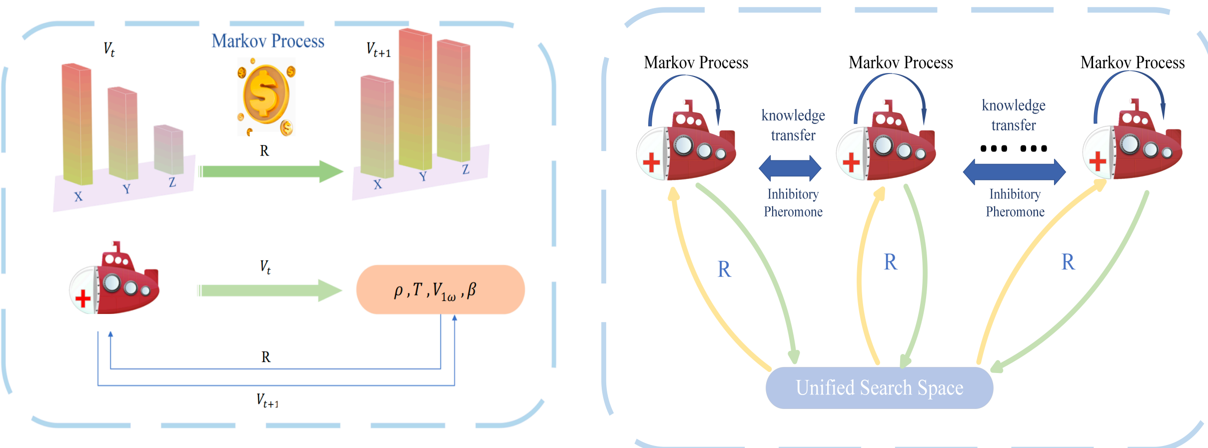


Figure 14: Search Process

3.4 Model Extrapolating

3.4.1 Scenario Promotion

When rescuing in different seas, the density ρ and temperature T involved in our equations can be refitted to the data, and the current information can be reacquired, but the only thing that is not well determined is the current speed v_w of the sea water, which changes greatly due to the rotation of the earth, and which needs to be re-modeled according to the latitude and longitude. Consider a given location with latitude and longitude coordinates p, we make the local gravitational acceleration g_p, the angular velocity of the Earth's rotation ω_p, and the lift to the center of the Earth r_p, and since the horizontal pressure gradient force of the seawater and the geostrophic force will be in equilibrium, we can obtain the following equation (Take the x-direction as an example):

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + w_p^2 r_p \cdot v_w \tag{38}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g_p \tag{39}$$

By solving the above equation, we get:

$$v = \frac{1}{\rho w_p^2 r_p}\frac{\partial P}{\partial x} = \frac{g_p}{w_p^2 r_p}\tan\vartheta \tag{40}$$

Here ϑ is the angle between the local isobaric and equipotential surfaces, which is determined by the local latitude and longitude. Finally, we correct the seawater density ρ and pressure P with the change of water depth in z dimension to get the prediction of seawater flow rate, and we represent the results in **Figure 15**

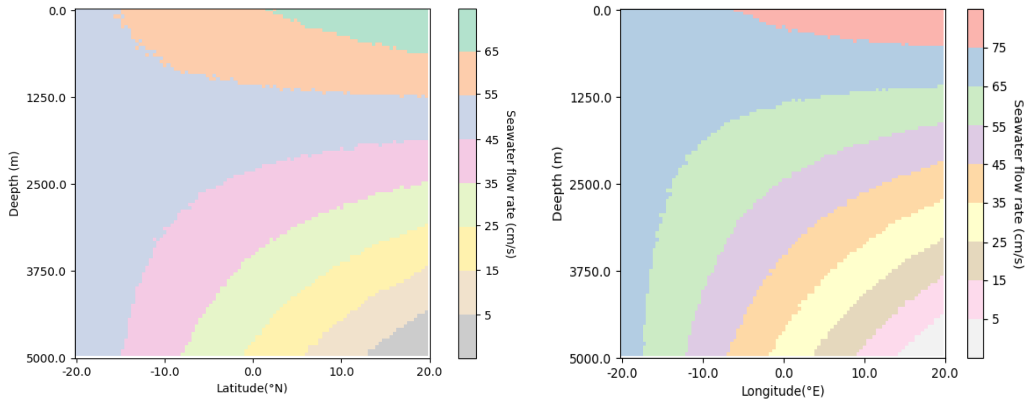


Figure 15: Sea water flow rate analysis

In order to validate our results, we selected the average flow velocity of seawater at water depths of 2000 m in six seas, including the Ionian Sea and the Caribbean Sea, as a reference, and the results are shown in **Table 4** The errors between the predicted and real values are all within 5%.

Table 4

Sea	True flow velocity at 2000m below sea level (cm/s)	Predit flow velocity at 2000m below sea level (cm/s)	Relative Error
Ionian Sea	20.97	21.73	0.036242251
Caribbean Sea	14.78	15.25	0.028915663
South China Sea	16.23	17.03	0.031799729
Hawaii	24.47	19.50	0.040457703
Maldives	16.23	18.40	0.049291436
Bali	34.48	29.48	0.02987396

3.4.2 Multi-Object Rescue

When faced with the task of multiple resuce vehicles rescuing multiple submarines, we can model it as a queueing theory[Joseph W,Muhammad et al.] problem. We set the probability of each submarine being lost to follow an exponential distribution with parameter α, such that $X(t)$ denotes the number of submarines lost at time t, then:

$$P(X(t) = n) = \frac{(at)^n e^{-at}}{n!} \tag{41}$$

Thus $X(t)$ obeys a Poisson distribution with parameter αt, which has expectation:

$$E(X(t)) = \alpha t \tag{42}$$

Thus the average arrival rate λ is:

$$\lambda = \frac{1}{E(X(t))} = \frac{1}{\alpha t} \tag{43}$$

Previously we have modeled the probability P_{ij} of successful search and rescue of each submarine j by each rescue vehicle i based on a Markov process, and if we consider the process of search and rescue of submarines by each rescue vehicle as a process of a server serving a user, then the service rate μ can be expressed as:

$$\mu_{ij} = \frac{1}{P_{ij}} \tag{44}$$

Thus the traffic intensity ρ_j of each submarine can be expressed as:

$$\rho_j = \sum_{i=1}^{n_j} \frac{\lambda_j}{\mu_{ij}} \tag{45}$$

where n_j is considered to be the number of all rescue vehicles supporting submarine j. Completing a SAR task is considered as the process of the server solving the client's demand according to the queuing pattern, we set the average queue length as L_q and the average SAR time as W_q, according to the *Littleequation*, we can get the following equation:

$$W_q = \frac{1}{m} \sum_{j=1}^{m} \frac{L_{qj}}{\lambda_j} = \frac{1}{m} \sum_{j=1}^{m} \frac{\rho_j^2}{\lambda_j (1 - \rho_j)} \tag{46}$$

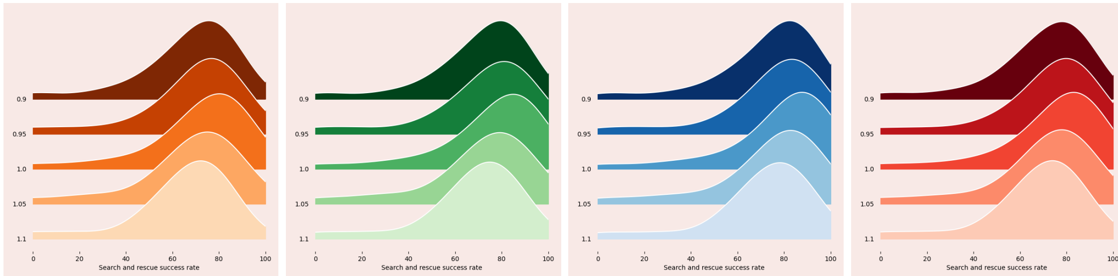
$$Min\ W_q \tag{47}$$

At each moment, we set the target submarine for each rescue vehicle with the goal of minimizing W_q, and then we can just inspire the rescue vehicle to make a target update with the information of that target submarine.

4 Sensitivity Analysis and Model Testing

4.1 Sensitivity Analysis

We consider doing a sensitivity test on the quaternion $(\omega_1, \omega_2, \omega_3, \omega_4)$ of the weights of the heuristic information, updating the model by making small changes to them within a range of 10%, and then doing Monte Carlo sampling on the rescue vehicle to determine the distribution of the search and rescue success rate of the samples after 72h of the Golden Rescue Time, and the distributions are shown as shown in **Figure 16**, which reveals that our model is the most sensitive to the heuristic information of the seawater flow rate, and the least sensitive to the heuristic information of the temperature

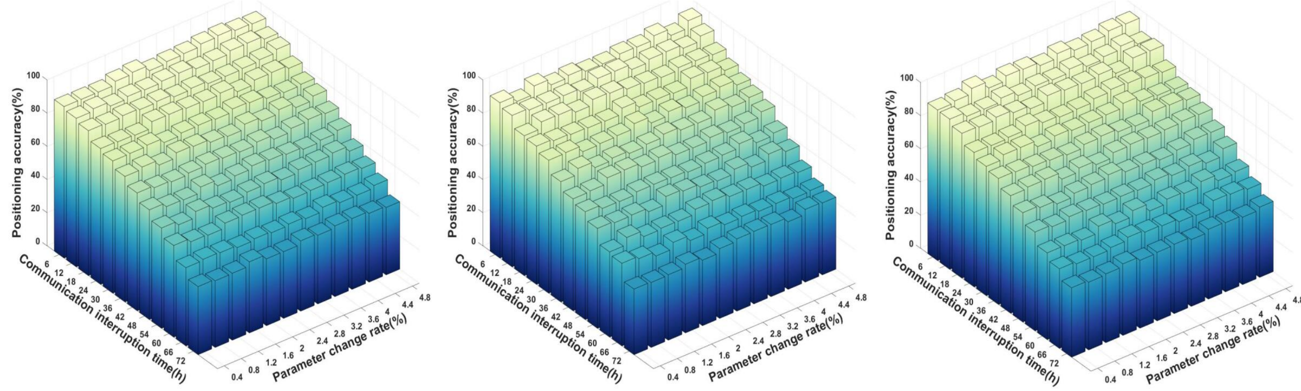


(a) ω_1 (b) ω_2 (c) ω_3 (d) ω_4

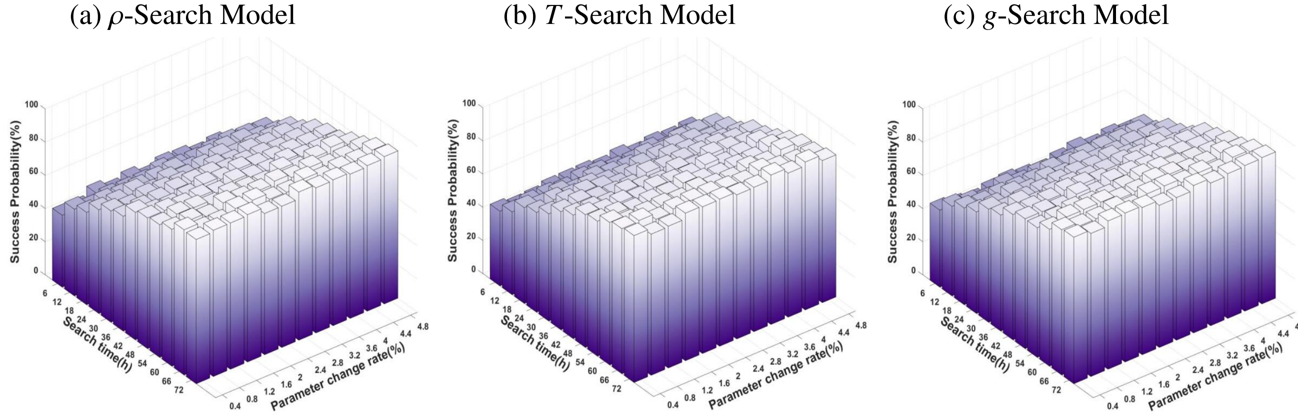
Figure 16: Sensitivity test for heuristic information weights

4.2 Model Testing

We choose density ρ, temperature T and Local gravitational acceleration g as the variables for model testing to test whether our model is robust to changes in the scene, and consider their effects on the localization model and the search and rescue model, respectively. Small correlation corrections were applied to both increases and decreases in variables. The final result is shown in the **Figure 17**. It can be seen that our model is quite robust to changes in the environment.



(a) ρ-Search Model (b) T-Search Model (c) g-Search Model



(d) ρ-Positioning Model (e) T-Positioning Model (f) g-Positioning Model

Figure 17: Model Testing

4.3 Ablation Experiment

Finally, we perform ablation experiments on the model; eliminating each variable of the resistance triad $(f_{fri}, f_{oce}, f_{flo})$ from the model in turn. The lost time is set as $1h$, and the search and rescue period is set as the golden rescue time $72h$, and the obtained localization accuracy and search and rescue success probability are shown in **Table 5**. It can be seen that f_{flo} has the greatest impact on our dynamical system model, f_{oce} is next, and f_{fri} is the smallest.

Table 5: Ablation Experiment

Ablation Element	Localization Accuracy(%)	Search and Rescue Success Probability(%)
f_{fri}	83.1	81.9
f_{oce}	72	76.8
f_{flo}	34.3	51.7

5 Strengths and Weaknesses

Strengths

- **Multidimensional modeling of environmental information**
  Our model takes into account the fact that the forces are all basic forces in physics and are simple and easy to compute. However, this does not affect the comprehensiveness of the dimensions we consider. We have a very comprehensive modeling of the friction and pressure of a submarine, the density, temperature and currents of seawater, and the variation of the flow velocity with latitude and longitude.

- **Full utilization of own experience and inspirational information**
  We make full use of the existing equipments to obtain environmental information heuristic search, at the same time, rescue vehicles can also simulate the time-varying Markov reward process based on the search information to motivate their own searches, which is a very classical and healthy state transfer process that helps us to complete the search task efficiently.

- **Highly effective cooperation mechanisms**
  Our model fully takes into account the cooperation between rescue vehicles in search and rescue missions, with both empirical speed sharing and duplicate search area suppression. We also introduce a queueing theory model to help them minimize the search and rescue time.

- **Prudent equipment selection**
  We consider the choice and number of equipments in a balanced way, and quantify the competition and collaboration of equipment, skillfully optimizing the solution problem with a non-dominated ordering algorithm.

- **Weak data dependency**
  In contrast to many deep learning models, our model is fully explicit in its mechanism of action, and our model is weakly data-dependent and sufficiently robust to face certain rescue scenarios in the sea of missing data, which are extremely prone to occur in the real world.

- **Comprehensive model evaluation**
  We modeled Gaussian process and Hamiltonian sampling simulation model uncertainty, performed sensitivity analysis of parameters, model checking of variables and ablation experiments of components on the model; the checking of the model was very adequate.

Weakness

- **Ignoring the effects of extreme environments**
  As mentioned in the assumptions, we neglected to model seawater density and temperature in extreme environments, which can have an impact on our models.

- **No consideration was given to the possibility of failure of the rescue vehicle**
  We consider communication between rescue vehicles, where heuristic messages are delivered in real-time synchronization, without considering communication failures

- **Real scenarios are not tested enough**
  Due to the lack of ocean data, there is no way to test our model against a wide range of scenarios in the global ocean area.

6 Conclusion

This paper presents a dynamics modeling-based localization search and rescue model to solve the submarine salvage problem. A number of mathematical processes and cooperative mechanisms are creatively proposed. Future work should focus on the following areas: 1. Acquiring more marine datasets to cope with more complex and changing rescue environments.2. Establishing more effective teamwork search mechanisms, which requires the introduction of superior knowledge transfer methods, as well as consideration of rescue vehicle failure problems and response scenarios.3. Many of the dynamical modeling problems have yet to be described by more concise equations to reduce the model complexity and increase the solution efficiency.

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