x_j	f_j	$g''(x_j)$	f'_j
-1.0000	0.0385	$+\ 0.0000$	?
-0.9511	0.0424	$-\ 0.1342$	?
-0.8090	0.0576	$+\ 1.5096$	?
-0.5878	0.1038	$-\ 2.1089$	?
-0.3090	0.2952	$+16.6458$	?
$+0.0000$	1.0000	-30.4676	?
$+0.3090$	0.2952	$+16.6458$	?
$+0.5878$	0.1038	$-\ 2.1089$	?
$+0.8090$	0.0576	$+\ 1.5096$	?
$+0.9511$	0.0424	$-\ 0.1342$	?
$+1.0000$	0.0385	$+\ 0.0000$	?

Table 1: Part A.

Part A. (35%)

We use cubic spline interpolation with free run-out conditions to interpolate ($N + 1 = 11$) data points (x_j, f_j), where $j = 0, 1, 2, ..., 10$, as listed in Table 1. The independent variable x is in the range of $[-1, 1]$.

Let $g(x)$ denote the collection of all the cubics for the entire range of x and $g''(x_j)$ be the second derivative of the cubics at x_j, where $j = 0, 1, 2, ..., 10$, as listed in Table 1.

Please find f'_j, where $j = 0, 1, 2, ..., 10$, based on cubic spline interpolation. In other words, please find $g'(x_j)$, where $j = 0, 1, 2, ..., 10$, as an approximation of f'_j, where $j = 0, 1, 2, ..., 10$.

Part B. (35%)

Trapezoidal rule with end-correction for one interval $x_i \le x \le x_{i+1}$ is

$$I_i = h_i \frac{f_i + f_{i+1}}{2} - \frac{1}{12} h_i^3 \frac{f'_{i+1} - f'_i}{h_i} + O(h_i^5),$$

where $h_i = x_{i+1} - x_i$, $f_i = f(x_i)$ and $f_{i+1} = f(x_{i+1})$, $f'_i = f'(x_i)$ and $f'_{i+1} = f'(x_{i+1})$. The order of the leading error term is $O(h_i^5)$.

Please derive the complete form of the leading error term for the trapezoidal rule with end-correction for one interval.

Part C. (30%)

C.1 Let $f''(x_j)$ be related to $f''(x_{j-1})$, $f''(x_{j+1})$, $f(x_{j-1})$, $f(x_j)$, $f(x_{j+1})$. Please derive the Padé scheme for the derivatives $f''(x_{j-1})$, $f''(x_j)$, $f''(x_{j+1})$, i.e.

$$c_1 f''(x_{j-1}) + c_2 f''(x_j) + c_3 f''(x_{j+1}) = c_4 f(x_{j-1}) + c_5 f(x_j) + c_6 f(x_{j+1}) + O(h^?),$$

with the leading error term as small as possible, where $h = x_{j+1} - x_j = x_j - x_{j-1}$ and $c'_i s$ are unknown coefficients. Please also derive the complete form of the leading error term.

C.2 Please derive $(k'h)^2$ and express $(k'h)^2$ in terms of (kh) for the Padé scheme in C.1, where k' is the modified wavenumber. (Note that $k = 2\pi n/L$, $n = 0, 1, 2, ..., N/2$, where L is the period and $h = L/N$ is the grid spacing. The grid points are $x_j = hj$, $j = 0, 1, 2, ..., N - 1$.)

You may NOT discuss with others about the exam. You may use notes, books and calculators. No late exam will be accepted.

You must certify in writing that you take the exam honestly. Please sign and date after your written statement. Your exam will NOT be graded without your written statement.