Homework 4

Due: November 23, 2022

Note: No late homework will be accepted. You may discuss with your classmates but you may not plagiarize. You need to turn in your analysis and also your code written in Octave or Matlab.

Part A. (30%)

A.1 Show the discrete orthogonality of cosines

$$\sum_{j=0}^{N} \frac{1}{c_j} \cos k x_j \cos k' x_j = \begin{cases} 0 & \text{if} \quad k \neq k' \\ \frac{1}{2} c_k N & \text{if} \quad k = k' \end{cases}$$

where $x_j = \pi j / N$, $j = 0, 1, 2, ..., N$ and

$$c_k = \begin{cases} 2 & \text{if} \quad k = 0, N \\ 1 & \text{otherwise.} \end{cases}$$

(Hint: by substituting complex exponential representations for cosines.)

A.2 The discrete cosine series is defined by

$$f_j = \sum_{k=0}^{N} a_k \cos k x_j \quad j = 0, 1, 2, ..., N,$$

where $x_j = \pi j / N$.

Prove that the coefficients of the series are

$$a_k = \frac{2}{N} \frac{1}{c_k} \sum_{j=0}^{N} \frac{1}{c_j} f_j \cos k x_j \quad k = 0, 1, 2, ..., N.$$

Part B. (20%)

Differentiate the following functions using two methods: FFT and central difference formula

$$f'_j = \frac{f_{j+1} - f_{j-1}}{2h}.$$

When you use the central difference formula, compute the derivative only at the interior points but not at the boundary points. For each method, use $N = 16$ and $N = 32$. Plot your results based on FFT and central difference formula as symbols (for example, squares or triangles) and the exact derivative as a continuous line.

B.1

$$f(x) = \sin 3x + 3 \cos 6x \quad 0 \leq x < 2\pi$$

B.2

$$f(x) = 6x - x^2 \quad 0 \le x < 2\pi$$

Which method works better in B.1? Which method works better in B.2? Can you explain the reason?

Part C. (30%)

Here are two functions $f(x)$ and $g(x)$ defined in the interval $(0, 2\pi)$, i.e.

$$f(x) = \sin(2x) + 0.1\sin(15x),$$

$$g(x) = \sin(2x) + 0.1\cos(15x).$$

C.1 Use $N = 32$ grid points, i.e. $x_j = 2\pi j/N$, $j = 0, 1, 2, ..., N-1$. Compute $f_j = f(x_j)$, $g_j = g(x_j)$ and $H_j = f_j g_j$. Compute the FFT of H_j, i.e. $\hat{H}_k$, $k = -N/2, -N/2+1, ..., -1, 0, 1, ..., N/2-1$? What is the real function that $\hat{H}_k$ represents?

C.2 Use $N = 32$ grid points and compute the FFT of f_j, i.e. $\hat{f}_k$, and the FFT of g_j, i.e. $\hat{g}_k$. Compute $\hat{h}_m$ using the convolution sum

$$\hat{h}_k = \sum_{m=-N/2}^{N/2-1} \hat{f}_m \hat{g}_{k-m},$$

where $k = -N/2, -N/2+1, ..., -1, 0, 1, ..., N/2-1$. What is the real function that $\hat{h}_k$ represents?

C.3 Use trigonometric identities to show the exact result of $E(x) = f(x)g(x)$. Use $N = 32$ grid points and compute $E_j = E(x_j)$ and the FFT of E_j, i.e. $\hat{E}_k$. Does $\hat{E}_k$ represent $E(x)$ correctly? Do you see any difference among $\hat{E}_k$, $\hat{H}_k$ and $\hat{h}_k$? Which is correct? Why?

Part D. (20%)

We use the Chebyshev derivative matrix operator to differentiate $u(x) = 4(x^2 - x^4)e^{-x/2}$ in the range $-1 \le x \le 1$. Let vector $\mathbf{x}$ represent the collocation points $x_j = \cos(\pi \mathrm{j}/\mathrm{N})$, $j = 0, 1, 2, ..., N$, and vector $\mathbf{u}$ represent the values of $u(x)$ at the collocation points. Construct the $(N+1) \times (N+1)$ Chebyshev collocation derivative matrix $\mathbf{D}$ using (6.46) or (6.47) in the textbook.

D.1 For $N = 7$, write down the vectors $\mathbf{x}$ and $\mathbf{u}$, the derivative matrix $\mathbf{D}$, and the first derivative of $u(x)$ at the collocation points, i.e. $\mathbf{u}'$, via $\mathbf{u}' = \mathbf{D}\mathbf{u}$. Plot the first derivative $\mathbf{u}'$ at the collocation points using symbols and the exact first derivative using a continuous line.

D.2 For $N = 7$, write down the vectors $\mathbf{x}$ and $\mathbf{u}$, the second derivative matrix $\mathbf{D_2}$, and the second derivative of $u(x)$ at the collocation points, i.e. $\mathbf{u}''$, via $\mathbf{u}'' = \mathbf{D_2}\mathbf{u}$. Plot the second derivative $\mathbf{u}''$ at the collocation points using symbols and the exact second derivative using a continuous line.