Explaining some properties of the Mandelbrot set

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1 Introduction

Given a complex number c, define $f_c(z) := z^2 + c$; the Mandelbrot set is defined as the set of all c such that the sequence $0,\ f_c(0),\ f_c(f_c(0)),\ f_c(f_c(f_c(0))), \ldots$ is bounded (meaning that the absolute value of the terms in the sequence are bounded). This sequence is also notated as $(f_c^n(0))_{n\geq 0}$, where $f_c^n(z) := \underbrace{f_c(\ldots(f_c}_{n}(z)))$ represents function iteration (not to be confused with raising to a power, which would be $f_c(z)^n$).

This simple definition gives rise to an infinitely detailed shape: zooming in along the border, no matter how far, reveals new structure. Patterns in the Mandelbrot set include the Fibonacci numbers, quasi self-similarity, pi, and many others that can be found online. However, most sources I have found fail to give insight into these patterns beyond visual observation, and the ones that don't are highly technical and require an extensive background on iterated function systems and related fields.

My goal here is to be in between these extremes, explaining some patterns somewhat rigorously while remaining accessible, in order to convince the general public that the Mandelbrot set is not completely mysterious. I will only rely on single-variable calculus, polynomials, and a few simplifying assumptions that will be clearly stated along the way (and a bit of linear algebra to prove one specific statement). Analysis of other properties of the Mandelbrot set can be found in [1] and [2].

1.1 Why start at 0?

It is natural to ask why we use the sequence $(f_c^n(0))_{n\geq 0}$ and not $(f_c^n(z))_{n\geq 0}$ for some other starting number z. The reason is that since $z = 0$ is a critical point of $f_c(z) = z^2 + c$ (as $\frac{d}{dz} f_c(z) = 0$ at $z = 0$), the following are true [1]:

- if f_c^n has an attracting cycle (defined later) as $n \to \infty$, then $(f_c^n(0))_{n\geq 0}$ is guaranteed to reach it;
- f_c^n has at most one attracting cycle, because f_c has exactly one critical point.

Since these statements are difficult to prove, we will treat them as our first simplifying assumptions.

2 Periodicity Bulbs

For a given value c, there are three limiting behaviors of $f_c^n(0)$ as n approaches infinity:

1. it diverges to infinity
   - e.g. $c = 1$ gives $0,\ 0^2 + 1 = 1,\ 1^2 + 1 = 2,\ 2^2 + 1 = 5,\ 5^2 + 1 = 26, \ldots$
2. it remains bounded and converges to a cycle
   - e.g. $c = -0.76$ makes the sequence flip between roughly -0.6 and -0.4, with the terms getting closer and closer to these values, converging to a 2-cycle
   - e.g. $c = 0$ gives $0,\ 0,\ 0, \ldots$, converging to a fixed point (1-cycle)
3. it remains bounded but does not converge to a cycle (chaos)

- e.g. $c = -1.9$ (empirically observed)

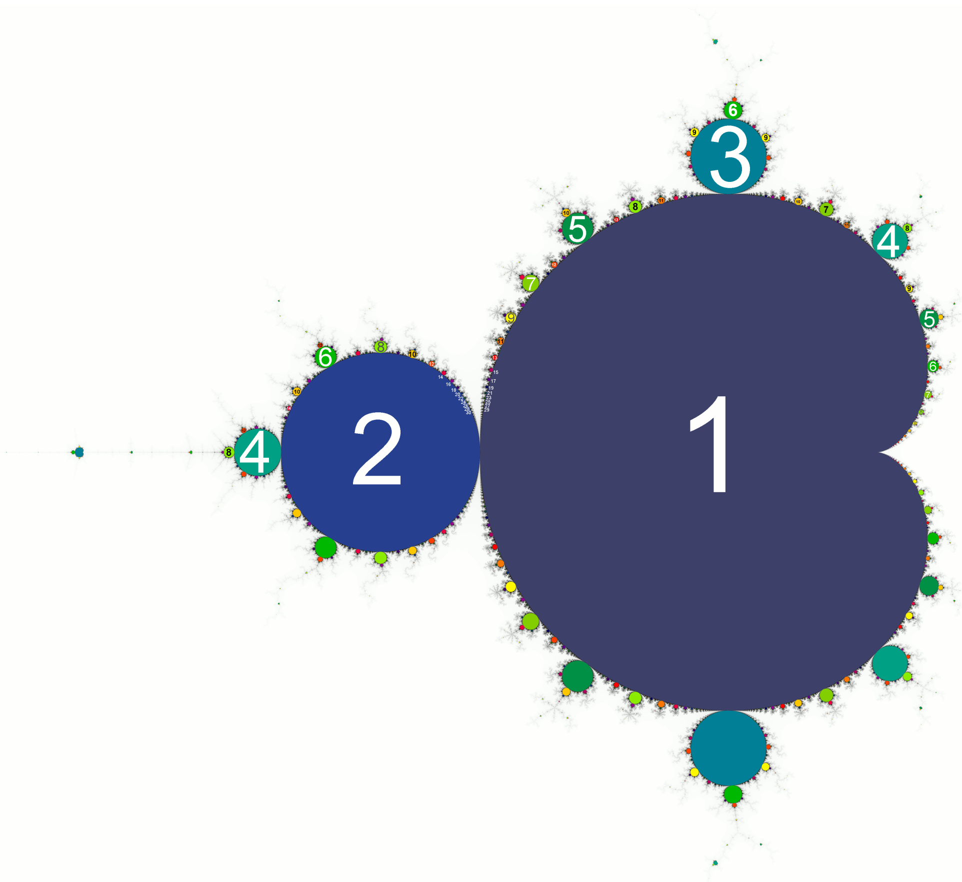
For our purposes, we will only analyze c that make the sequence converge to a cycle.

A fixed point of a function f is an input z such that $f(z) = z$. A fixed point is *attracting* if all nearby inputs w converge to z under repeated iteration. This is equivalent to having $\left|\frac{d}{dz}f(z)\right| < 1$; the intuition is that if $w = z + \delta$ for a small δ, then $f(w) \approx z + \left|\frac{d}{dz}f(z)\right|\delta$, so the distance between w and z is approximately multiplied by $\left|\frac{d}{dz}f(z)\right|$, and we want this distance to get smaller [1].

A k-cycle of f is a sequence $z_0, \ldots, z_{k-1}$ such that $f(z_0) = z_1,\ f(z_1) = z_2, \ldots, f(z_{k-1}) = z_0$ and all z_ℓ are distinct (so that we are not going over a smaller cycle multiple times). This is equivalent to having $f^k(z_0) = z_0$ and $f^\ell(z_0) \neq z_0$ for $\ell < k$, i.e. z_0 is a fixed point of f^k and not of any lower iterate of f. Using our previous analysis, the k-cycle is attracting if and only if $\left|\frac{d}{dz}f^k(z)\right| < 1$ at $z = z_0$.

To simplify analysis, we can drop the $f^\ell(z_0) \neq z_0$ requirement; then a fixed point of f^k is a d-cycle of f, where d is a divisor of k. We can then remove all the non-k-cycles in a different way that we show later.

In the Mandelbrot set, if a given c has some k such that f_c^k has an attracting fixed point, the smallest such k is the *periodicity number* of c If we label each point c with its periodicity number, we get this image (taken from [3]):



We observe that each periodicity number corresponds to a union of finitely many regions we call "bulbs". Additionally, (the boundaries of) two bulbs intersect in at most one point, i.e. they never share a curve of positive length. These intersection points [2] will be our main focus.

[1] If $\left|\frac{d}{dz}f(z)\right| = 1$, things get tricky as we then have to analyze the 2nd-order behavior of f around z, and the fixed point may or may not be attracting, or even one-sided attracting.

[2] The points where two different bulbs touch also happen to be *bifurcation points*, where if we move c through such a point, the points of an attracting cycle will each split ("bifurcate") into several points and form a larger cycle. We do not use this fact, so we will keep calling these points "intersection points".

Formally, the set of all c in the Mandelbrot set with periodicity number **a divisor of** k is

$$B_k := \left\{ c \in \mathbb{C} : \ \exists z \in \mathbb{C} \text{ s.t. } f_c^k(z) = z \ \& \ \left| \frac{d}{dz} f_c^k(z) \right| < 1 \right\}$$

and the boundary of this set is

$$\partial B_k = \left\{ c \in \mathbb{C} : \ \exists z \in \mathbb{C} \text{ s.t. } f_c^k(z) = z \ \& \ \left| \frac{d}{dz} f_c^k(z) \right| = 1 \right\};$$

Due to our assumptions in Section 1.1, describing these sets only requires finding *any* attracting fixed point z of f_c^k, without having to differentiate between fixed points.

2.1 Periods 1

To find the period 1 bulb (equivalent to B_1), we want z such that $f_c(z) = z$ and $\left| \frac{d}{dz} f_c(z) \right| < 1$: then we have $z^2 + c = z$ and $|2z| < 1$. Here, it is best to write c in terms of z instead of solving everything in terms of c, so $B_1 = \left\{ z - z^2 : |z| < \frac{1}{2} \right\}$. This shape forms what is known as a *cardioid*, and because it is the largest bulb in the Mandelbrot set it is often called the "main cardioid".

2.2 Period 2

To find the period 2 bulb, we want $f_c^2(z) = z$ but $f_c(z) \neq z$. By a fundamental property of polynomials, we notice that $f_c^2(z) - z$ must be a polynomial multiple of $f_c(z) - z$, since having $f_c(z) - z = 0$ forces [3] $f_c^2(z) - z = 0$.

To factor $f_c^2(z) - z$, we note that it equals $f_c\Big(\underbrace{(f_c(z) - z)}_{=:w} + z \Big) - z = (w + z)^2 + c - z = w^2 + 2zw + \underbrace{z^2 + c - z}_{=w} = w(w + 2z + 1) = w(z^2 + z + 1 + c)$. Since we want $w \neq 0$, we must have $z^2 + z + 1 + c = 0$.

We also need $\left| \frac{d}{dz} f_c^2(z) \right| < 1$. Using the chain rule, $\frac{d}{dz} f_c^2(z) = 2 f_c(z) \left(\frac{d}{dz} f_c(z) \right) = 2(z^2 + c) \cdot 2z$. We can simplify this expression, using the fact that $z^2 + z + 1 + c = 0$:

$$2(z^2 + c) \cdot 2z = -4(z + 1)z = -4(z^2 + z) = 4(1 + c),$$

which amazingly cancels out z. Thus, the period-2 bulb is $\left\{ c : |1 + c| < \frac{1}{4} \right\}$, a circular disk centered at -1 with radius $\frac{1}{4}$. By the fundamental theorem of algebra, any such c will yield at least one solution in z.

Higher-period bulbs appear much more difficult to describe (I couldn't find a way to parameterize their points), and consist of multiple pieces, whereas the period-1 and period-2 bulbs each are one piece.

3 Intersections between periodicity bulbs

Now we get to our main focus, which is finding the intersection points between some pairs of periodicity bulbs (that are feasible by directly manipulating polynomials).

3.1 Periods 1 and k

We will first loosen the period-k requirement to allow all divisors of k, then restrict to period k at the end.

Our first instinct is to solve $f_c^k(z) - z = 0$ and $f_c(z) - z = 0$, but doing so would result in all points in the period-1 bulb being included. However, because both $f_c^k(z) - z$ and $f_c(z) - z$ are polynomials, and the former equals 0 whenever the latter equals 0, there must exist some polynomial g_k such that $(f_c^k(z) - z) = (f_c(z) - z) g_k(z)$. Furthermore, this means B_k is the union of B_1 and the set $\{c : \exists z \text{ s.t. } g_k(z) = 0\}$, if we ignore the stability constraint $\left| \frac{d}{dz} f_c^k(z) \right| < 1$. This suggests that what we really want to solve is $g_k(z) = 0$ and $f_c(z) - z = 0$.

[3] If $f_c(z) - z = 0$, then $f_c(z) = z$, so $f_c(f_c(z)) = f_c(z) = z$.

Notice that we have two unknowns (z, c) and two complex polynomial equations, so we should expect there to be a finite number of solutions. To simplify things, we will continue ignoring the stability constraint, since it is not a complex polynomial (analyzing it would require splitting complex numbers into their real and imaginary parts); remarkably, the solutions we find will satisfy this constraint anyway.

Also notice that $g_k(z)$ is defined even when $f_c(z) - z = 0$, since g_k is obtained by dividing $f_c^k(z) - z$ by $f_c(z) - z$ as *polynomial expressions*, not as numbers; earlier we saw from finding the period-2 bulb that $g_2(z) = z^2 + z + 1 + c$. A simpler analogy is how $\frac{z^2-1}{z-1} = z + 1$ makes sense even when $z - 1 = 0$, because the right-hand side $z + 1$ is the unique polynomial such that multiplying it by $z - 1$ yields $z^2 - 1$.

Nevertheless, it is difficult to directly evaluate $g_k(v)$ at a point v where $f_c(v) - v = 0$. Instead, we can calculate it as $\lim_{z \to v} \frac{f_c^k(z)-z}{f_c(z)-z}$, where here we treat the numerator and denominator as numbers; this works because we know a priori that g_k is a polynomial, so it is continuous everywhere.

Since both the numerator and denominator in the limit approach 0, we can invoke L'Hopital's rule: $\lim_{z \to v} \frac{f_c^k(z)-z}{f_c(z)-z} = \left. \frac{\frac{d}{dz}\left(f_c^k(z)-z\right)}{\frac{d}{dz}\left(f_c(z)-z\right)} \right|_{z=v}$. By the chain rule, $\frac{d}{dz} f_c^k(z) = 2 f_c^{k-1}(z) \cdot \frac{d}{dz} f_c^{k-1}(z) = \cdots = \prod_{0 \le \ell < k} 2 f_c^\ell(z)$. Finally, since $f_c(v) - v = 0$, $f_c(v) = v$, so when $z = v$ the expression simplifies to $(2v)^k$. Thus, $g_k(v) = \frac{(2v)^k - 1}{2v-1} = 1 + 2v + \cdots + (2v)^{k-1}$; here we again perform division between *polynomials* instead of numbers, which can be justified by noting that g_k is a priori a polynomial (or by applying L'Hopital's rule again if necessary).

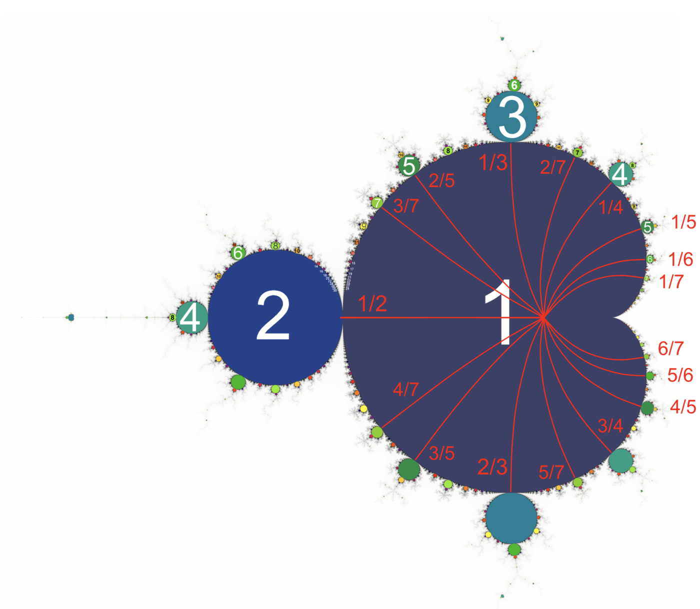
Thus, we need to solve $1 + 2v + \cdots + (2v)^{k-1} = 0$ and $f_c(v) - v = 0$; the first equation is equivalent to having $2v$ be a k-th root of unity that is not 1, and the second is equivalent to having $c = v - v^2$. Since $|2v| = 1 \Rightarrow |v| = \frac{1}{2}$, such c is on ∂B_1, so we have solved the intersection points of the main cardioid with bulbs of period a divisor of k.

To restrict ourselves to only period k, we repeat this analysis for all strict divisors of k and remove the resulting points from our set: then the set of intersections of period 1 and period k is precisely

$$C_{1,k} := \left\{ z - z^2 : \ z = \frac{1}{2} e^{\frac{\ell}{k} 2\pi i}, \ \ell \in \mathbb{Z}, \ \gcd(\ell, k) = 1 \right\}.$$

Due to the GCD (greatest common divisor) constraint, each bulb that touches the main cardioid naturally corresponds to a rational number $\frac{\ell}{k}$ strictly between 0 and 1, written in lowest terms, and that bulb must have period k.

If we plot the curves $z - z^2$ for $z = re^{\frac{\ell}{k} 2\pi i}$, $0 < r < \frac{1}{2}$, and reduced fractions $\frac{\ell}{k}$, we get the *internal rays* of the main cardioid, which start at the origin and meet a bulb:



We furthermore notice that as $\frac{\ell}{k}$ increases, the intersection point $z - z^2$ moves more counterclockwise around the origin, implying that the relative angular order of the bulbs is the same as the order of their corresponding numbers. To prove this, we show that $\det \begin{bmatrix} \mathrm{Re}(c) & \mathrm{Re}(c') \\ \mathrm{Im}(c) & \mathrm{Im}(c') \end{bmatrix} > 0$ for $c := z - z^2$, $z := \frac{1}{2}e^{i\theta}$, and $c' := \frac{d}{d\theta}c$:

- geometrically, c' is the velocity vector of c as the angle θ changes, and the determinant inequality requires c' to point counterclockwise relative to c;
- some inspection shows that the determinant is equal to $\mathrm{Re}(ic \cdot \overline{c'})$;
- defining $z' := \frac{d}{d\theta}z$, we have $c' = z' - 2zz' = z'(1-2z)$ and $z' = iz$;
- so $\mathrm{Re}(ic \cdot \overline{c'}) = \mathrm{Re}(iz(1-z) \cdot \overline{iz(1-2z)}) = \mathrm{Re}(z(1-z) \cdot \overline{z}(1-2\overline{z}))$
$= \mathrm{Re}(|z|^2(1-z)(1-2\overline{z})) = |z|^2\mathrm{Re}((1-z)(1-2\overline{z})) = |z|^2\mathrm{Re}(1-z-2\overline{z}+2|z|^2)$
$= |z|^2\left(1+2|z|^2 - 3\mathrm{Re}(z)\right)$;
- since $|z| = \frac{1}{2}$, this expression equals $\frac{1}{4}\left(\frac{3}{2} - 3\mathrm{Re}(z)\right) = \frac{1}{4}\cdot\frac{3}{2}(1-2\mathrm{Re}(z))$, which must be nonnegative because $\mathrm{Re}(z) \le \frac{1}{2}$.

3.2 Periods 2 and $2k$

The previous analysis generalizes to intersections between period 2 and $2k$. This time we solve $h_k(z) := \frac{f_c^{2k}(z)-z}{f_c^2(z)-z} = 0$ and $g_2(z) := \frac{f_c^2(z)-z}{f_c(z)-z} = 0$, where again we use polynomial division (so the expressions are defined even when the denominators are 0). Notice that having $f_c^2(z) - z = 0$ instead of $g_2(z) = 0$ is incorrect, as it would include intersections between periods 1 and $2k$.

Using L'Hopital's, for any v such that $g_2(v) = 0$ we have $h_k(v) = \left.\frac{\frac{d}{dz}\left(f_c^{2k}(z)-z\right)}{\frac{d}{dz}(f_c^2(z)-z)}\right|_{z=v}$. Since $\frac{d}{dz}f_c^{2k}(z) = \prod_{0\le\ell<2k} 2f_c^{\ell}(z)$ and $f_c^2(v) = v$, $\left.\frac{d}{dz}f_c^{2k}(z)\right|_{z=v} = \left(\prod_{0\le\ell<2} 2f_c^{\ell}(v)\right)^k = \left(\left.\frac{d}{dz}f_c^2(z)\right|_{z=v}\right)^k$.

Letting $w_k(z) := \frac{d}{dz}f_c^k(z)$, we solve $1 + w_2(z) + \cdots + w_2(z)^{k-1} = 0$ and $g_2(z) = z^2 + z + 1 + c = 0$. Since $w_2(z) = 2(z^2+c)\cdot 2z$, which simplifies to $4(1+c)$ on the surface where $g_2(z) = 0$, we have that $4(1+c)$ is a k-th root of unity not equal to 1. We check that such c is on ∂B_2.

After removing all solutions for c that correspond to strict divisors of k, we have that periods 2 and $2k$ intersect at

$$C_{2,k} := \left\{-1 + \frac{1}{4}e^{\frac{\ell}{k}2\pi i} : \ell \in \mathbb{Z},\ \gcd(\ell,k) = 1\right\}.$$

Generalizing our analysis further to intersections between periods m and mk seems very difficult for $m \ge 3$; already for $m = 3$ and $k = 2$ SymPy fails to find any algebraic solutions for (z, c).

3.3 Fibonacci

If we look at the bulbs touching the main cardioid, there is an interesting pattern: the largest bulb between the period 2 and the (top) period 3 bulb has period $2 + 3 = 5$; the largest between the period 3 and the period 5 bulbs has period $3 + 5 = 8$; and so on, producing the Fibonacci numbers.

Since we now know that each bulb touching the main cardioid corresponds to a rational number, we can be more specific about which bulbs we are referring to: the largest bulb between $\frac{1}{2}$ and $\frac{1}{3}$ is $\frac{2}{5}$; the largest between $\frac{2}{5}$ and $\frac{1}{3}$ is $\frac{3}{8}$; between $\frac{2}{5}$ and $\frac{3}{8}$ is $\frac{5}{13}$; and so on. The pattern seems to be that the largest bulb between $\frac{F_{n-2}}{F_n}$ and $\frac{F_{n-1}}{F_{n+1}}$ is $\frac{F_n}{F_{n+2}}$, where F_n is the n-th Fibonacci number, defined as $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$.

To prove this Fibonacci pattern, we make one final simplifying assumption: bulbs with higher period are smaller (specifically, the smallest bulb of period k is larger than the largest bulb of period $k+1$, for all k). We make this assumption due to the difficulty of calculating the exact area of bulbs with period 3 or more.

Since we also know that the relative order of bulbs matches the order of their corresponding numbers, finding the largest bulb between $\frac{a}{b}$ and $\frac{c}{d}$ (written as reduced fractions) is equivalent to finding the rational number $\frac{p}{q}$ with smallest q such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$. This problem is closely related to something called the *mediant* (or *Farey sum*) of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, defined as $\frac{a+c}{b+d}$. In fact, if $bc - ad = 1$, the mediant is the optimal answer for $\frac{p}{q}$. We give a compressed version of the proof in [4]:

- if $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$, then $\frac{p}{q} - \frac{a}{b} > 0$ and $\frac{c}{d} - \frac{p}{q} > 0$;
- letting $x := bp - aq$ and $y := cq - dp$, we have $x > 0$ and $y > 0$ by multiplying out denominators in the previous two inequalities;
- since x and y are integers, we have in fact $x \geq 1$ and $y \geq 1$;
- we want to use x and y to bound q, and we are not interested in p, so we seek to eliminate p; some inspection shows that $dx + by = (bc - ad)q$;
- thus, $q = \frac{dx+by}{bc-ad} \geq \frac{b+d}{bc-ad}$
- when $bc - ad = 1$, $q \geq b + d$;
- finally, to show $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$, we have $\frac{a+c}{b+d} - \frac{a}{b} = \frac{(a+c)b-a(b+d)}{(b+d)b} = \frac{bc-ad}{(b+d)b}$ and $\frac{c}{d} - \frac{a+c}{b+d} = \frac{c(b+d)-(a+c)d}{d(b+d)} = \frac{bc-ad}{d(b+d)}$, both of which are positive because $\frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} > 0 \Rightarrow bc - ad > 0$.

In our situation, we want to find the lowest-denominator fraction between $\frac{F_{n-2}}{F_n}$ and $\frac{F_{n-1}}{F_{n+1}}$. Conveniently, the Fibonacci numbers satisfy $F_{n-2}F_{n+1} - F_{n-1}F_n = (-1)^{n+1}$, which can be proven by induction:

- base case $n = 2$: $F_0F_3 - F_1F_2 = 0 \cdot 2 - 1 \cdot 1 = (-1)^{2+1}$
- $F_{n-2}F_{n+1} - F_{n-1}F_n = (-1)^{n+1}$
  $\Rightarrow F_{n-1}F_{n+2} - F_nF_{n+1}$
  $= F_{n-1}(\cancel{F_{n+1}} + F_n) - (\cancel{F_{n-1}} + F_{n-2})F_{n+1}$
  $= F_{n-1}F_n - F_{n-2}F_{n+1}$
  $= -(-1)^{n+1}$
  $= (-1)^{n+2}$

This property also shows that the sequence of bulbs we visit is zigzagging (i.e. $\frac{F_1}{F_3} < \frac{F_2}{F_4} > \frac{F_3}{F_5} < \frac{F_4}{F_6} > \dots$). Regardless of what the relative order between $\frac{F_{n-2}}{F_n}$ and $\frac{F_{n-1}}{F_{n+1}}$ are, though, we have shown that the lowest-denominator fraction between them is the mediant $\frac{F_{n-2}+F_{n-1}}{F_n+F_{n+1}} = \frac{F_n}{F_{n+2}}$, confirming the pattern.

References

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