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4 authors:

Chong Shen
Nanjing Normal University

Hadrian Andradi
Universitas Gadjah Mada

Dongsheng Zhao
National Institute of Education (NIE), Singapore

Fu-Gui SHI
Beijing Institute of Technology

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SI_2-TOPOLOGY ON T_0 SPACES

CHONG SHEN, HADRIAN ANDRADI*, DONGSHENG ZHAO, AND FUGUI SHI

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ABSTRACT. This paper provides a new approach to defining a topology by using irreducible sets from a given T_0 space. This derived topology, called SI_2-topology, leads to a weak sobriety and continuity, called δ-sobriety and SI_2-continuity, respectively. It is proved that the δ-sober C-spaces are exactly the s_2-topological spaces on s_2-continuous posets. Moreover, it turns out that the SI_2-topology on a T_0 space can be described completely in terms of convergence, and this convergence structure is topological whenever the given space is SI_2-continuous.

1. INTRODUCTION

Domain theory, initiated by Scott in the late 60's, has some close relations to topology. The main focus of the study of domain theory is originally on complete lattices, and later transferred to the general domains (continuous dcpos). Afterwards, many authors studied topology and continuity in general posets setting (see [3, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16]).

Comparing to posets, T_0 spaces provide a more general platform for domain theory, as one can always view a poset as T_0 space (e.g., the Scott space of the poset). Recently, Zhao and Ho [18] introduced a method of deriving a new topology (called SI-topology) out of a given T_0 space. The motivating ingredients are the irreducible sets and two well-known topologies defined on posets: the Alexandroff topology and the Scott topology. Notably, the irreducible subsets of

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*corresponding author.

a poset with respect to the Alexandroff topology are exactly the directed sets, and the Scott topology on a poset are exactly the Alexandroff open which are inaccessible by Alexandroff-irreducible suprema. In the case when the Alexandroff space is replaced with some T_0 space, the derived Scott topology is exactly the Zhao and Ho's SI-topology.

Given a poset, one cannot guarantee that every directed subset has a supremum, let alone any subset, while directed sets have an important role in the study of domain theory. Two treatments that are acted upon this imperfection are: (1) considering only those directed sets whose supremum exists and (2) looking at the cuts of directed sets instead of the principal ideals generated by directed sets (see Erné's work in [5] for more detail). Using the latter treatment, one can define a new topology, called s_2-topology, a new convergence structure, called s_2-convergence, and two new notions of continuity, namely s_2-continuity and s_2-quasicontinuity (see [14, 15, 16]). The two continuities are deeply related to the s_2-way-below relation which is defined based on treatment (2) above, i.e, for any two elements x and y of a poset P, x is s_2-way-below y if and only if for every directed set D such that $y \in D^\delta$, it holds that $x \leq d$ for some $d \in D$. The set D^δ here refers to the set of all lower bounds of the set of all upper bounds of D. Note that in the definition of the usual way-below relation, the part '$y \in D^\delta$' is replaced by '$y \in \downarrow \bigvee D$, whenever $\bigvee D$ exists', which is the treatment (1).

This paper mainly studies the so-called "Erné's s_2 approach" on the context of T_0 spaces. First, by using the irreducible sets and the specialization order of a given T_0 space, we define a new topology, called SI_2-topology. This topology induces a weak notion of sobriety: δ-sober, in the same sense as SI-topology induces k-bounded sobriety in [18]. We prove that a T_0 space is δ-sober if and only if its SI_2-topology coincides with the original topology. We then introduce a new way-below relation on T_0 spaces in the manner of Erné, and deploy it to define SI_2-continuity, following closely a recent work in [1] (as we consider irreducibility instead of directedness). This SI_2-continuity can characterize the spaces whose SI_2-topology forms a completely distributive lattice. We then obtain that the δ-sober C-spaces are exactly the s_2-topological spaces on s_2-continuous posets. Furthermore, we prove that the SI_2-topology on a T_0 space X can be induced by a certain convergence structure, and the SI_2-continuity entails the convergence to be topological.

2. Preliminaries

In this section, we recall some basic definitions and results that will be used later. For undefined notions in this paper, the reader can refer to [2, 7, 8].

Let P be a poset. For any $a, b \in P$, a is *way below* b, denoted by $a \ll b$, if for any directed subset $D \subseteq P$, $b \leq \bigvee D$ implies $a \in \downarrow D$ whenever $\bigvee D$ exists. P is a *continuous poset* if for any $a \in P$, the set $\twoheaddownarrow a := \{b \in P : b \ll a\}$ is directed and $a = \bigvee \twoheaddownarrow a$. A subset U of P is called *Scott open* if it is upper and for any directed set D, $\bigvee D \in U$ implies $D \cap U \neq \varnothing$ whenever $\bigvee D$ exists. Denote the set of all Scott open subsets of P by $\sigma(P)$.

For a subset A of a poset P, $A^{\uparrow}$ denotes the set of all upper bounds of A, and $A^{\downarrow}$ the set of all lower bounds of A. We shall use A^{δ} instead of $(A^{\uparrow})^{\downarrow}$.

Definition 2.1 ([5])**.** Let P be a poset.

(i) For any $x, y \in P$, we say x *approximates* y, written $x \ll_2 y$, if for any directed subset D of P, $y \in D^{\delta}$ implies $x \in \downarrow D$. Let $\twoheaddownarrow_2 x := \{y \in P : y \ll_2 x\}$ and $\twoheaduparrow_2 x := \{y \in P : x \ll_2 y\}$.

(ii) P is called s_2*-continuous* if for any $x \in P$, $\twoheaddownarrow_2 x$ is directed and $\bigvee \twoheaddownarrow_2 x = x$.

An upper subset U of a poset P is called s_2*-open* [5] if for any directed subset D of P , $D^{\delta} \cap U \neq \varnothing$ implies $D \cap U \neq \varnothing$. The collection of all s_2-open subsets of P forms a topology, called s_2*-topology*, and is denoted by $s_2(P)$. A subset C of P is called s_2*-closed* if the complement of C is s_2-open. Obviously, C is s_2-closed if and only if it is a lower set and for any directed subset D of P, $D \subseteq C$ implies $D^{\delta} \subseteq C$. In fact, if P is a dcpo, the s_2-topology on P coincides with the Scott-topology.

Every T_0 space X can be viewed as a poset under the specialization order $\leq$, defined as follows:

$$\forall x, y \in X,\ x \leq y \text{ if and only if } x \in \mathrm{cl}_X(\{y\}).$$

An open subset U of T_0 space X is called SI-*open* [18] if for any irreducible set F with $\bigvee F$ existing, the follows statement holds:

$$\bigvee F \in U \Rightarrow F \cap U \neq \varnothing.$$

The collection of all SI-open sets, denoted by $\mathcal{O}_{\mathrm{SI}}(X)$, is a topology on X, called the SI-*topology*. Denote the derived space $(X, \mathcal{O}_{\mathrm{SI}}(X))$ by $\mathrm{SI}(X)$.

Definition 2.2 ([18])**.** Let X be a T_0 space.

(1) For $x, y \in X$, define $x \ll_{\mathrm{SI}} y$ if for any irreducible set F in X, $y \leq \bigvee F$ implies $x \in \downarrow F$ whenever $\bigvee F$ exists. Denote the set $\{x \in X : x \ll_{\mathrm{SI}} y\}$ by $\twoheaddownarrow_{\mathrm{SI}} y$, and the set $\{x \in X : y \ll_{\mathrm{SI}} x\}$ by $\twoheaduparrow_{\mathrm{SI}} y$.

(2) X is called SI-*continuous* if for any $x \in X$, the following conditions hold:
   (i) $\Uparrow_{\mathrm{SI}} x$ is open in X, and
   (ii) $\Downarrow_{\mathrm{SI}} x$ is directed and $\bigvee \Downarrow_{\mathrm{SI}} x = x$.

Throughout this paper, when we say X is a space, it always means X is a T_0 topological space and $\mathcal{O}(X)$ is the underlying topology of the space.

3. SI_2-topology induced by irreducible sets

In this section we define a new topology, called the SI_2-topology, from a given space by using specialization order and irreducible sets. Some basic properties of this new topology are investigated.

Recall that a nonempty subset F of a space X is *irreducible* if for any closed subsets A, B of X, $F \subseteq A \cup B$ implies $F \subseteq A$ or $F \subseteq B$ [7, 8]. One can easily check that a nonempty set F is irreducible if and only for any open sets U and V such that $F \cap U \neq \varnothing$ and $F \cap V \neq \varnothing$, one has $F \cap U \cap V \neq \varnothing$.

Definition 3.1. Let X be a space. A subset U of X is called SI_2-*open* if the following conditions are satisfied:

(i) U is an open set in X, and
(ii) for any irreducible set F in X, $F^{\delta} \cap U \neq \varnothing$ implies $F \cap U \neq \varnothing$.

The set of all SI_2-open sets in X is denoted by $\mathcal{O}_{\mathrm{SI}_2}(X)$. Complements of SI_2-open sets are called SI_2-*closed sets*.

Using the definition of irreducible sets, we can easily verify the following result.

Proposition 3.2. *For any space X, $\mathcal{O}_{\mathrm{SI}_2}(X)$ is a topology on X, called the* SI_2-*topology*.

Remark 3.3. If P is a poset and $X = (P, \gamma(P))$, where $\gamma(P)$ is the Alexandroff topology, i.e., $U \in \gamma(P)$ if and only if U is an upper set, then $\mathcal{O}_{\mathrm{SI}_2}(X)$ is the s_2-topology on P systematically studied by Erné [5]. Furthermore, if P is a dcpo, then $\mathcal{O}_{\mathrm{SI}_2}(X)$ is the Scott topology on P.

For a space X, denote the space $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$ by $\mathrm{SI}_2(X)$. It is clear that $\mathcal{O}_{\mathrm{SI}_2}(X) \subseteq \mathcal{O}_{\mathrm{SI}}(X)$, but the reverse containment is generally not true. An example is shown as follows.

Example 3.4. Let $X = (B, \gamma(B))$, where $B = \{a, b, c\} \cup \mathbb{N}$ is as shown in Figure 1, with $\mathbb{N}$ is equipped with its usual order. It is easy to check that $U := \{a, b, c\}$ is SI-open but not SI_2-open by considering the irreducible set $\mathbb{N}$.

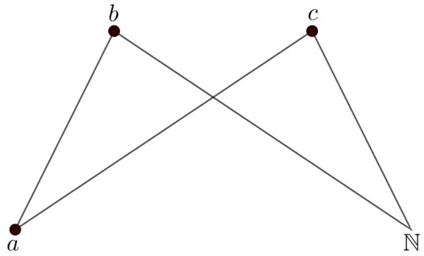

FIGURE 1. The poset B

Proposition 3.5. (1) *For any space X, it holds that* $\mathrm{cl}_X(\{x\}) = \mathrm{cl}_{\mathrm{SI}_2(X)}(\{x\})$.
(2) *The specialization order of spaces X and $\mathrm{SI}_2(X)$ coincide.*

PROOF. (1) Clearly, $\mathrm{cl}_X(\{x\}) \subseteq \mathrm{cl}_{\mathrm{SI}_2(X)}(\{x\})$ because every SI_2-closed set is closed in X. To establish the reverse containment, it suffices to show that $X \setminus \mathrm{cl}_X(\{x\}) \subseteq X \setminus \mathrm{cl}_{\mathrm{SI}_2(X)}(\{x\})$. Since $\mathrm{int}_{\mathrm{SI}_2(X)}(X \setminus \{x\}) = X \setminus \mathrm{cl}_{\mathrm{SI}_2(X)}(\{x\})$, to achieve the desired containment it amounts to showing that $X \setminus \mathrm{cl}_X(\{x\})$ is an SI_2-open set (contained in $X \setminus \{x\}$). For this purpose, suppose F is an irreducible set in X such that $F \cap (X \setminus \mathrm{cl}_X(\{x\})) = \varnothing$. Then, with respect to the specialization order of X, $F \subseteq {\downarrow}x$, which implies $F^\delta \subseteq ({\downarrow}x)^\delta = {\downarrow}x = \mathrm{cl}_X(\{x\})$. Thus $F^\delta \cap (X \setminus \mathrm{cl}_X(\{x\})) = \varnothing$. Therefore $X \setminus \mathrm{cl}_X(\{x\})$ is SI_2-open. Hence $\mathrm{cl}_X(\{x\}) \supseteq \mathrm{cl}_{\mathrm{SI}_2(X)}(\{x\})$.

(2) It is a straightforward consequence of (1). □

It is known that a lower subset C of a poset P is s_2-closed if and only if for any directed set D, $D \subseteq C$ implies $D^\delta \subseteq C$ (see [16]). We have a similar characterization for SI_2-topology.

Proposition 3.6. *A closed subset C of a space X is SI_2-closed if and only if for any irreducible set F in X, $F \subseteq C$ implies $F^\delta \subseteq C$.*

The following result shows that SI_2-open sets in a T_0 space can be defined by only using irreducible closed sets and the specialization order.

Proposition 3.7. *An open set U of a space X is SI_2-open if and only if for any irreducible closed set F in X, $F^\delta \cap U \neq \varnothing$ implies $F \cap U \neq \varnothing$.*

PROOF. We only need to prove the sufficiency. Suppose U is an open subset of X satisfying the stated condition and G is an irreducible set (need not be closed) such that $G^\delta \cap U \neq \varnothing$. Then $\mathrm{cl}_X(G)$ is an irreducible closed set and $(\mathrm{cl}_X(G))^\delta \cap U \neq \varnothing$. Hence by the assumption, $\mathrm{cl}_X(G) \cap U \neq \varnothing$, implying $G \cap U \neq \varnothing$. Therefore U is SI_2-open. $\square$

Proposition 3.8. *Let X be a space and $U \subseteq X$. Then U is clopen in X if and only if U is clopen in $\mathrm{SI}_2(X)$. In particular, if X is zero-dimensional (i.e., the clopen sets form a base for X), then so is $\mathrm{SI}_2(X)$.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. It suffices to prove (1) $\Rightarrow$ (2). Suppose without loss of generality that U is a nontrivial clopen subset of X. Let F be an irreducible set in X such that $F^\delta \cap U \neq \varnothing$. Note that U is both upper and lower with respect to the specialization order on X, and then one can easily deduce that $F \cap U \neq \varnothing$.
The last statement is immediate from the definition of zero-dimensional spaces. $\square$

Recall that a space X is connected if whenever $X = U \cup V$ for two disjoint clopen sets U and V, then either $U = \varnothing$ or $U = X$. The following result can be deduced from Proposition 3.8.

Theorem 3.9. *A space X is connected if and only if $\mathrm{SI}_2(X)$ is connected.*

Clearly, if a space X is compact, then $\mathrm{SI}_2(X)$ is compact.

4. δ-sobriety

Recall that a T_0 space X is *sober* if every irreducible closed subset of X is the closure of a unique singleton [7, 8].

Definition 4.1. A space X is called δ-sober, if for any irreducible closed set F, $F^\delta = F$ holds.

Remark 4.2.
(1) Clearly, every sober space is a δ-sober space.
(2) A space X is said to be *k-bounded sober* [18] if for any irreducible closed set F of X with $\bigvee F$ existing, there is a unique point $x \in X$ such that $F = \mathrm{cl}_X(\{x\})$. Note that in a δ-sober space Y, if E is an irreducible closed set with $\bigvee E$ existing, then $E = E^\delta = \downarrow \bigvee E = \mathrm{cl}_X(\{\bigvee E\})$. This shows that every δ-sober space is k-bounded sober. Thus we have the following chain of implications:
$$\text{sobriety} \Rightarrow \delta\text{-sobriety} \Rightarrow k\text{-bounded sobriety}.$$
(3) If X is a δ-sober space whose specialization order of X is a cut-complete order, i.e., every set in $\delta(X) := \{A^\delta : A \subseteq X\}$ has a supremum, then X is sober.

The following examples show that δ-sobriety is generally different from sobriety and k-bounded sobriety.

Example 4.3. (1) Let $X = (B, \gamma(B))$, where B is the poset given in Example 3.4. Then X is clearly k-bounded sober. It is not δ-sober since $\mathbb{N}$ is irreducible closed in X but $\mathbb{N}^\delta = \mathbb{N} \cup \{a\} \neq \mathbb{N}$.

(2) The space $X = (\mathbb{Q}, \sigma(\mathbb{Q}))$, where $\mathbb{Q}$ is the set of all rational numbers with the usual order, is δ-sober but not sober.

Proposition 4.4. *A space X is δ-sober if and only if $\mathcal{O}_{\mathrm{SI}_2}(X) = \mathcal{O}(X)$.*

Proof. Let X be a δ-sober space. It suffices to show $\mathcal{O}(X) \subseteq \mathcal{O}_{\mathrm{SI}_2}(X)$. Suppose $U \in \mathcal{O}(X)$. If F is an irreducible closed set in X satisfying $F^\delta \cap U \neq \varnothing$, then $F \cap U \neq \varnothing$ as $F = F^\delta$. Therefore, $U \in \mathcal{O}_{\mathrm{SI}_2}(X)$.

Now suppose that $\mathcal{O}_{\mathrm{SI}_2}(X) = \mathcal{O}(X)$, that is, every closed set in X is SI_2-closed. Let F be any irreducible closed set in X. Then by Proposition 3.6 and the fact that $F \subseteq F$, we obtain $F^\delta \subseteq F$, implying $F^\delta = F$. Thus X is δ-sober. $\square$

Theorem 4.5. *Let X be a space. If $\tau \subseteq \mathcal{O}(X)$ is a δ-sober topology on X that induces the same specialization order as $\mathcal{O}(X)$, then τ is contained in $\mathcal{O}_{\mathrm{SI}_2}(X)$.*

Proof. Let $U \in \tau$ and F be an irreducible set in X such that $F^\delta \cap U \neq \varnothing$. Since $\tau \subseteq \mathcal{O}(X)$, it follows that F is also an irreducible set in (X, τ). Then we have $F^\delta = F$, and hence $F \cap U \neq \varnothing$. Therefore $U \in \mathcal{O}_{\mathrm{SI}_2}(X)$. $\square$

Proposition 4.6. *A continuous mapping $f : X \longrightarrow Y$ is continuous between $\mathrm{SI}_2(X)$ and $\mathrm{SI}_2(Y)$ if and only if $f(F^\delta) \subseteq [f(F)]^\delta$ holds for any irreducible set F in X.*

Proof. Let $f : \mathrm{SI}_2(X) \longrightarrow \mathrm{SI}_2(Y)$ be a continuous mapping and F be an irreducible set in X. Suppose there exists $x \in F^\delta$ such that $f(x) \notin [f(F)]^\delta$. Then we can find $y \in [f(F)]^\uparrow$ such that $f(x) \nleq y$, implying $f(x) \in Y \setminus \mathrm{cl}_Y(\{y\})$. Note that $\mathrm{cl}_Y(\{y\}) = \mathrm{cl}_{\mathrm{SI}_2(Y)}(\{y\})$ by Proposition 3.5 (1), which implies that $Y \setminus \mathrm{cl}_Y(\{y\})$ is an SI_2-open set in Y containing x. Since f is a continuous mapping between $\mathrm{SI}_2(X)$ and $\mathrm{SI}_2(Y)$, there exists an SI_2-open set U in X such that $x \in U$ and $f(U) \subseteq Y \setminus \mathrm{cl}_Y(\{y\})$. Thus $U \cap F \neq \varnothing$ as $x \in U \cap F^\delta \neq \varnothing$. Let $z \in U \cap F$. It follows that $f(z) \in Y \setminus \mathrm{cl}_Y(\{y\})$, meaning that $f(z) \nleq y$, a contradiction to $y \in [f(F)]^\uparrow$. Hence $f(F^\delta) \subseteq [f(F)]^\delta$.

Now assume $f(F^\delta) \subseteq [f(F)]^\delta$ holds for every irreducible set F in X. Let V be an arbitrary SI_2-open subset of Y. It suffices to prove $f^{-1}(V)$ is an SI_2-open set in X. If G be an irreducible set in X such that $G^\delta \cap f^{-1}(V) \neq \varnothing$, then $f(G^\delta) \cap V \neq \varnothing$, implying $[f(G)]^\delta \cap V \neq \varnothing$ because $f(G^\delta) \subseteq [f(G)]^\delta$. Note that the continuous image of any irreducible set is also irreducible. Thus $f(G)$ is an irreducible set in Y, and $f(G) \cap V \neq \varnothing$. It follows that $G \cap f^{-1}(V) \neq \varnothing$, showing that $f^{-1}(V)$ is an SI_2-open set. Hence f is a continuous mapping between $\mathrm{SI}_2(X)$ and $\mathrm{SI}_2(Y)$. □

Proposition 4.7. *Let P be a chain.*

(1) *$(P, s_2(P))$ is a δ-sober space.*
(2) *$(P, \sigma(P))$ is a δ-sober space.*

Proof. First note that every subset of a chain is directed, and therefore irreducible with respect to both $(P, s_2(P))$ and $(P, \sigma(P))$. Then it is trivial to check statement (1), and statement (2) follows from the fact that $s_2(P) \subseteq \sigma(P)$. □

Proposition 4.8. *Let P be a poset.*

(1) *If P is s_2-continuous, then $(P, s_2(P))$ is a δ-sober space.*
(2) *If P is continuous, then $(P, \sigma(P))$ is a δ-sober space.*

Proof. (1) Let P be an s_2-continuous poset. Suppose there is an irreducible closed set F in $(P, s_2(P))$ such that $F^\delta \neq F$. Then there exists $x \in F^\delta$ such that $x \notin F$. Note that $\mathord{\downarrow}_2 x$ is directed and $\bigvee \mathord{\downarrow}_2 x = x$, which means $\bigvee \mathord{\downarrow}_2 x \in P \setminus F$. Thus $(\mathord{\downarrow}_2 x)^\delta \cap (P \setminus F) \neq \varnothing$, implying $\mathord{\downarrow}_2 x \cap (P \setminus F) \neq \varnothing$ as $P \setminus F$ is s_2-open. Then there exists $y \notin F$ such that $y \ll_2 x$. Note that the space $(P, s_2(P))$ is a C-space by [5, Theorem 2.13]. As $x \in F^\delta$, in virtue of [18, Lemma 7.4], $x \in D^\delta$ for some $D \subseteq F = \downarrow F$. Since $y \ll_2 x$, it holds that $y \in \downarrow D \subseteq \downarrow F = F$, which is a contradiction. Hence $(P, s_2(P)$ is a δ-sober space.

(2) It is trivial by the fact that $s_2(P) \subseteq \sigma(P)$ and conclusion (1). □

5. SI_2-continuous spaces

In this section, we introduce the notion of SI_2-continuous spaces, which generalizes the notion of s_2-continuous posets and is closely connected to δ-sobriety.

First, we introduce a new way-below-like relation on a space X. For any $x, y \in X$, define $x \ll_{\mathrm{SI}_2} y$, if for every irreducible set F in X, $y \in F^\delta$ implies $x \in \downarrow F$. Denote the set $\{y \in X : y \ll_{\mathrm{SI}_2} x\}$ by $\mathord{\downarrow}_{\mathrm{SI}_2} x$ and the set $\{y \in X : x \ll_{\mathrm{SI}_2} y\}$ by $\mathord{\uparrow}_{\mathrm{SI}_2} x$.

Remark 5.1. (1) $x \ll_{\mathrm{SI}_2} y$ implies $x \leq y$.

(2) $u \leq x \ll_{\mathrm{SI}_2} y \leq v$ implies $u \ll_{\mathrm{SI}_2} v$.

(3) For any $x \in X$, $\mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow x) \subseteq \mathord{\uparrow}_{\mathrm{SI}_2} x$. In fact, if $y \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow x)$ and F is an irreducible set in X such that $y \in F^\delta$, then $F^\delta \cap \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow x) \neq \varnothing$.

Thus $F \cap \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow x) \neq \varnothing$, implying $F \cap \uparrow x \neq \varnothing$. This shows $x \in \downarrow F$. Hence $x \ll_{\mathrm{SI}_2} y$.

Definition 5.2. A space X is called SI_2-*continuous* if for any $x \in X$, the following conditions hold:

(i) $\Uparrow_{\mathrm{SI}_2} x$ is open in X, and
(ii) $\Downarrow_{\mathrm{SI}_2} x$ is an irreducible set in X with $\bigvee \Downarrow_{\mathrm{SI}_2} x = x$.

Remark 5.3. (1) For a poset P, one can easily verify that the space $(P, \gamma(P))$ is SI_2-continuous if and only if the poset P is s_2-continuous.
(2) The condition (ii) in Definition 5.2 is equivalent to either one of the following statements.
(ii*) $\Downarrow_{\mathrm{SI}_2} x$ is an irreducible set in X and $x \in [\Downarrow_{\mathrm{SI}_2} x]^{\delta}$.
(ii**) There exists an irreducible subset F of $\Downarrow_{\mathrm{SI}_2} x$ such that $\bigvee F = x$.

Lemma 5.4. *Let X be an SI_2-continuous space. If F is an irreducible set in X, then so is $\Downarrow_{\mathrm{SI}_2} F := \{\Downarrow_{\mathrm{SI}_2} d : d \in F\}$. In addition, if $\bigvee F$ exists, then $\bigvee \Downarrow_{\mathrm{SI}_2} F$ exists and $\bigvee \Downarrow_{\mathrm{SI}_2} F = \bigvee F$.*

Proof. Let U and V be two open subsets of X such that $\Downarrow_{\mathrm{SI}_2} F \cap U \neq \varnothing$ and $\Downarrow_{\mathrm{SI}_2} F \cap V \neq \varnothing$. Then there exist $a \in \Downarrow_{\mathrm{SI}_2} F \cap U$ and $b \in \Downarrow_{\mathrm{SI}_2} F \cap V$. So, we can find elements $d, e \in F$ such that $a \ll_{\mathrm{SI}_2} d$ and $b \ll_{\mathrm{SI}_2} e$, implying $\Uparrow_{\mathrm{SI}_2} a \cap F \neq \varnothing$ and $\Uparrow_{\mathrm{SI}_2} b \cap F \neq \varnothing$. As F is irreducible and $\Uparrow_{\mathrm{SI}_2} a$ and $\Uparrow_{\mathrm{SI}_2} b$ are open in X, we have $\Uparrow_{\mathrm{SI}_2} a \cap \Uparrow_{\mathrm{SI}_2} b \cap F \neq \varnothing$. Thus there exists $z \in \Uparrow_{\mathrm{SI}_2} a \cap \Uparrow_{\mathrm{SI}_2} b \cap F$, and it follows that $a \in \Downarrow_{\mathrm{SI}_2} z \cap U \neq \varnothing$ and $b \in \Downarrow_{\mathrm{SI}_2} z \cap V \neq \varnothing$. As $\Downarrow_{\mathrm{SI}_2} z$ is irreducible, it holds that $\Downarrow_{\mathrm{SI}_2} z \cap U \cap V \neq \varnothing$. Let $k \in \Downarrow_{\mathrm{SI}_2} z \cap U \cap V$. Note that $k \ll_{\mathrm{SI}_2} z$ and $z \in F$, thus $k \in \Downarrow_{\mathrm{SI}_2} F$ and it follows $k \in \Downarrow_{\mathrm{SI}_2} F \cap U \cap V \neq \varnothing$. Hence $\Downarrow_{\mathrm{SI}_2} F$ is irreducible. $\square$

Theorem 5.5. *If X is an SI_2-continuous space, then the relation $\ll_{\mathrm{SI}_2}$ is interpolative.*

Proof. Suppose $x, y \in X$ satisfying $x \ll_{\mathrm{SI}_2} y$. First, by Lemma 5.4, we know that $D_y := \bigcup \{\Downarrow_{\mathrm{SI}_2} d : d \ll_{\mathrm{SI}_2} y\}$ is irreducible and $y = \bigvee D_y$, implying $x \in \downarrow D_y = D_y$. Thus there exists $d \ll_{\mathrm{SI}_2} y$ such that $x \ll_{\mathrm{SI}_2} d$. This completes the proof. $\square$

A little surprising fact regarding the SI_2-continuity is that the irreducibility given in condition (ii) in Definition 5.2 can be strengthened to directedness.

Proposition 5.6. *A space X is SI_2-continuous if and only if for any $x \in X$,*

(i*) *$\Uparrow_{\mathrm{SI}_2} x$ is SI_2-open in X, and*
(ii*) *$\Downarrow_{\mathrm{SI}_2} x$ is directed with $\bigvee \Downarrow_{\mathrm{SI}_2} x = x$.*

PROOF. The sufficiency is immediate as every directed subset of X is irreducible. Now assume that X is SI_2-continuous. We shall prove that (i*) and (ii*) hold.

(i*) Let F be an irreducible set in X such that $F^{\delta} \cap \Uparrow_{\mathrm{SI}_2} x \neq \varnothing$. Then there exists $y \in F^{\delta}$ satisfying $x \ll_{\mathrm{SI}_2} y$. By the interpolation property of $\ll_{\mathrm{SI}_2}$, there exists $z \in X$ such that $x \ll_{\mathrm{SI}_2} z \ll_{\mathrm{SI}_2} y$. It follows from $y \in F^{\delta}$ that $z \in \downarrow F$. Hence $F \cap \Uparrow_{\mathrm{SI}_2} x \neq \varnothing$.

(ii*) Let $y, z \in \Downarrow_{\mathrm{SI}_2} x$. By Theorem 5.5, $\Uparrow_{\mathrm{SI}_2} y \cap \Downarrow_{\mathrm{SI}_2} x \neq \varnothing$ and $\Uparrow_{\mathrm{SI}_2} z \cap \Downarrow_{\mathrm{SI}_2} x \neq \varnothing$. Note that $\Downarrow_{\mathrm{SI}_2} x$ is irreducible and both $\Uparrow_{\mathrm{SI}_2} y$ and $\Uparrow_{\mathrm{SI}_2} z$ are open. It follows that $\Uparrow_{\mathrm{SI}_2} y \cap \Uparrow_{\mathrm{SI}_2} z \cap \Downarrow_{\mathrm{SI}_2} x \neq \varnothing$. Then there exists $u \in \Downarrow_{\mathrm{SI}_2} x$ such that $y \leq u$ and $z \leq u$. Hence $\Downarrow_{\mathrm{SI}_2} x$ is directed. The conclusion $x = \bigvee \Downarrow_{\mathrm{SI}_2} x$ is trivial by the SI_2-continuity of X. □

One can see from Proposition 5.6 that it is reasonable and meaningful in some sense to transfer the role of directedness in poset to irreducibility in T_0 space.

Lemma 5.7. *If X is SI_2-continuous, then for any $x \in X$, $\Uparrow_{\mathrm{SI}_2} x = \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow x)$.*

PROOF. Since $\Uparrow_{\mathrm{SI}_2} x \subseteq \uparrow x$ and $\Uparrow_{\mathrm{SI}_2} x$ is SI_2-open, we have that $\Uparrow_{\mathrm{SI}_2} x \subseteq \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow x)$. Conversely, it follows from Remark 5.1(3) that $\Uparrow_{\mathrm{SI}_2} x \supseteq \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow x)$. □

A space X is called a *C-space*, if $\mathcal{O}(X)$ is a completely distributive lattice, or equivalently, for every $U \in \mathcal{O}(X)$ and $x \in U$, there exists $u \in U$ such that $x \in \mathrm{int}_X(\uparrow u)$ [4].

In the following theorem, we obtain a necessary and sufficient condition for a T_0 space to be SI_2-continuous.

Theorem 5.8. *A space X is SI_2-continuous if and only if $\mathrm{SI}_2(X)$ is a C-space.*

PROOF. Suppose $\mathrm{SI}_2(X)$ is a C-space. Let $D := \{u : x \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u)\}$. Next we prove $\bigvee D = x$. First, for any $u \in D$, we have $x \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u) \subseteq \uparrow u$, showing that x is an upper bound of D. Further, let v be any upper bound of D. Then we assert $x \leq v$. Otherwise by Proposition 3.5 (1), $x \in X \setminus \mathrm{cl}_X(\{v\}) = X \setminus \mathrm{cl}_{\mathrm{SI}_2(X)}(\{v\})$. Thus there exists $w \in X \setminus \mathrm{cl}_{\mathrm{SI}_2(X)}(\{v\})$ such that $x \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow w)$. It follows that $w \in D$, and hence $w \leq v$ as $v \in D^{\uparrow}$, which contradicts $w \in X \setminus \mathrm{cl}_{\mathrm{SI}_2(X)}(\{v\})$. Furthermore, if $x \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u_1)$ and $x \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u_2)$, then $x \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u_1) \cap \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u_2)$. So there exists $u_3 \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u_1) \cap \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u_2)$ such that $x \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u_3)$. This means the set D is directed. By Remark 5.1, $D \subseteq \Downarrow_{\mathrm{SI}_2} x$, and hence $\Downarrow_{\mathrm{SI}_2} x$ is directed and $\bigvee \Downarrow_{\mathrm{SI}_2} x = x$.

Now we prove $\Uparrow_{\mathrm{SI}_2} y$ is SI_2-open for all $y \in X$. By the above argument we can see that $z \in \Uparrow_{\mathrm{SI}_2} y \Rightarrow \exists u \in \uparrow y, z \in \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u)$. Thus $\Uparrow_{\mathrm{SI}_2} y = \bigcup_{u \in \uparrow y} \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow u)$, which is SI_2-open in X. Therefore X is SI_2-continuous.

Conversely, assume X is SI_2-continuous. Let $U \in \mathcal{O}_{\mathrm{SI}_2}(X)$ and $x \in U$. Note that $\Downarrow_{\mathrm{SI}_2} x$ is irreducible in X and $\bigvee \Downarrow_{\mathrm{SI}_2} x = x$, which means $x \in [\Downarrow_{\mathrm{SI}_2} x]^{\delta} \cap U \neq \varnothing$, and thus $\Downarrow_{\mathrm{SI}_2} x \cap U \neq \varnothing$. Then there exists $y \ll_{\mathrm{SI}_2} x$ such that $y \in U$. By Lemma 5.7, $x \in \Uparrow_{\mathrm{SI}_2} y = \mathrm{int}_{\mathrm{SI}_2(X)}(\uparrow y)$, and therefore $\mathrm{SI}_2(X)$ is a C-space. $\square$

Proposition 5.9. *For any* SI_2*-continuous space* X*,* $\mathrm{SI}_2(X)$ *is a* δ*-sober space.*

Proof. Let F be an irreducible closed set in $\mathrm{SI}_2(X)$. We need to prove $F^{\delta} = F$. Suppose there exists $x \in F^{\delta}$ such that $x \notin F$. Note that $x = \bigvee \Downarrow_{\mathrm{SI}_2} x$ if and only if $x \in (\Downarrow_{\mathrm{SI}_2} x)^{\delta}$, and then $x \in (\Downarrow_{\mathrm{SI}_2} x)^{\delta} \cap (X \setminus F)$. As $X \setminus F$ is SI_2-open and $\Downarrow_{\mathrm{SI}_2} x$ is irreducible, it holds that $\Downarrow_{\mathrm{SI}_2} x \cap (X \setminus F) \neq \varnothing$. Thus there exists $y \in (X \setminus F)$ such that $y \ll_{\mathrm{SI}_2} x$. Since $x \in F^{\delta}$, we have $y \in \downarrow F = F$, a contradiction. Hence $F^{\delta} = F$. $\square$

In [5, Theorem 2.13], Erné proved that a poset P is s_2-continuous if and only if $(P, s_2(P))$ is a C-space, and by Proposition 4.8, it is also a δ-sober space.

Theorem 5.10. *The* δ*-sober* C*-spaces are exactly the* s_2*-topological spaces induced by* s_2*-continuous posets.*

Proof. By virtue of Proposition 4.8 and [5, Theorem 2.13], it suffices to prove that every δ-sober C-space is of the form $(P, s_2(P))$ for some s_2-continuous poset P. Let X be a δ-sober C-space. Define $x \prec y$ if and only if $y \in \mathrm{int}_X(\uparrow x)$. We proceed by establishing three claims as follows:

Claim 1. $x \prec y$ implies $x \ll_2 y$. Let D be a directed subset of X such that $y \in D^{\delta}$. Then $y \in [\mathrm{cl}_X(D)]^{\delta} = \mathrm{cl}_X(D)$. Note that $y \in \mathrm{int}_X(\uparrow x)$, implying $\mathrm{int}_X(\uparrow x) \cap D \neq \varnothing$. Thus there exists $z \in \mathrm{int}_X(\uparrow x) \cap D$, which shows $z \in D$ and $x \leq z$.

Claim 2. The set $D := \{x : x \prec y\}$ is directed with supremum y. Take any $x_1, x_2 \in D$. Then $y \in \mathrm{int}_X(\uparrow x_1) \cap \mathrm{int}_X(\uparrow x_2)$. Since X is a C-space, there exists $x \in \mathrm{int}_X(\uparrow x_1) \cap \mathrm{int}_X(\uparrow x_2)$ such that $y \in \mathrm{int}_X(\uparrow x)$. This implies $x \in D$ and $x_1, x_2 \leq x$. Hence D is directed. Let $z \in D^{\uparrow}$. Note that for every open set U in X containing y, there exists $u \in U$ such that $y \in \mathrm{int}_X(\uparrow u)$ (as X is a C-space). It follows from $u \in D$ that $u \leq z$, implying $z \in U$. Thus $y \leq z$. Clearly, $y \in D^{\uparrow}$. Therefore, $y = \bigvee D$.

Claim 3. $\mathcal{O}(X) = s_2(X, \leq)$. First, by using Propositions 3.7 and 4.4, it is trivial that every open subset of space X is s_2-open, and hence $\mathcal{O}(X) \subseteq s_2(X, \leq)$.

To prove the reverse containment, we first prove that $x \prec y$ is equivalent to $x \ll_2 y$. Now suppose $x \ll_2 y$. By Claim 2, the set $D_y := \{x : x \prec y\}$ is directed and $y = \bigvee D_y$. We then obtain $x \in \downarrow D_y$, and hence $y \in \mathrm{int}_X(\uparrow x)$, which implies $x \prec y$. Since we have $x \prec y$ if and only if $x \ll_2 y$, by Claim 2, $(X, \leq)$ is s_2-continuous. Moreover, $\Uparrow_2 x = \mathrm{int}_X(\uparrow x) \in \mathcal{O}(X)$. By virtue of [5, Theorem 2.13], any s_2-open set in $(X, \leq)$ is in $\mathcal{O}(X)$. This completes the proof. □

In the following, we will provide a characterization of SI_2-open sets in terms of convergence.

Definition 5.11. A net $(x_i)_{i\in I}$ in a space X is said to SI_2-*converge* to a point x of X, denoted by $x_i \xrightarrow{\mathrm{SI}_2} x$, if there exists an irreducible set F in X such that

(i) $x \in F^\delta$, and
(ii) for any $U \in \mathcal{O}(X)$, $F \cap U \neq \varnothing$ implies $x_i \in U$ eventually.

The topology induced by the SI_2-convergence is denoted by τ_{SI_2}, i.e., $V \in \tau_{\mathrm{SI}_2}$ if and only if for any net $(x_i)_{i\in I}$, $x_i \xrightarrow{\mathrm{SI}_2} x$ and $x \in V$ imply $x_i \in V$ eventually.

Theorem 5.12. *The two topologies τ_{SI_2} and $\mathcal{O}_{\mathrm{SI}_2}(X)$ coincide on any given space X.*

Proof. Let $V \in \tau_{\mathrm{SI}_2}$. Next we prove $V \in \mathcal{O}_{\mathrm{SI}_2}(X)$ in two steps.

(i) Suppose that V is not open in X. Then there exists $x \in V$ such that for every $W \in I_x := \{W \in \mathcal{O}(X) : x \in W\}$, $W \nsubseteq V$. We equip I_x with reverse inclusion order. Then I_x is a directed ordered set. For each $W \in I_x$, choose $x_W \in W \setminus V$ to form a net $(x_W)_{W\in I_x}$. It is clear that $\{x\}$ is irreducible and $x \in \{x\}^\delta$. Let $U \in \mathcal{O}(X)$ such that $x \in U$. Then for every $W \in I_x$ such that $W \subseteq U$ we have that $x_W \in U$. Hence $x_W \xrightarrow{\mathrm{SI}_2} x$. Since $V \in \tau_{\mathrm{SI}_2}$, $x_W \in V$ for some $W \in I_x$, which is a contradiction. Therefore V is open in X.

(ii) Let F be an irreducible set in X such that $F^\delta \cap V \neq \varnothing$. Then there exists $x \in F^\delta \cap V$. Define $I_F = \{(e, O) \in F \times \mathcal{O}(X) : e \in O\}$ and equip I_F with $\leq$ defined as follows:

$$(e_1, O_1) \leq (e_2, O_2) \text{ if and only if } O_2 \subseteq O_1.$$

Irreducibility of F gives that I_F is directed. For each $(e, O) \in I_F$, we let $x_{(e,O)} = e$. If $U \in \mathcal{O}(X)$ such that $F \cap U \neq \varnothing$, then there exists $d \in X$ such that $(d, U) \in I_F$. For every $(e, O) \geq (d, U)$ we have that $x_{(e,O)} = e \in U$. Hence $x_{(e,O)} \xrightarrow{\mathrm{SI}_2} x$. Since $x \in V$ and $V \in \tau_{\mathrm{SI}_2}$, we have that $x_{(e,O)} = e \in V$ for some $e \in F$. Hence $F \cap V \neq \varnothing$.

From (i) and (ii), we have that $\tau_{\mathrm{SI}_2} \subseteq \mathcal{O}_{\mathrm{SI}_2}(X)$. Now let $V \in \mathcal{O}_{\mathrm{SI}_2}(X)$ and $(x_i)_{i\in I}$ be a net SI_2-converging to $x \in V$. By definition, there exists an irreducible set F in X such that $x \in F^\delta$ and for every $U \in \mathcal{O}(X)$, $F \cap U \neq \varnothing$ implies $x_i \in U$ eventually. Since $x \in F^\delta \cap V$ and $V \in \mathcal{O}_{\mathrm{SI}_2}(X)$, we have that $F \cap V \neq \varnothing$. Since $V \in \mathcal{O}(X)$, we have that $x_i \in V$ eventually. Therefore $V \in \tau_{\mathrm{SI}_2}$. This completes the proof. □

Proposition 5.13. *If X is an SI_2-continuous space, then the SI_2-convergence is topological.*

Proof. It suffices to prove that if $(x_i)_{i\in I}$ topologically converges to x in $\mathrm{SI}_2(X)$ then $x_i \xrightarrow{\mathrm{SI}_2} x$. By SI_2-continuity of X, we have that $F := \twoheaddownarrow_{\mathrm{SI}_2} x$ is irreducible and $x \in F^\delta$. Now let $U \in \mathcal{O}(X)$ such that $F \cap U \neq \varnothing$. Then there exists $u \in U$ such that $u \ll_{\mathrm{SI}_2} x$. By Proposition 5.6, $\twoheaduparrow_{\mathrm{SI}_2} u$ is open in $\mathrm{SI}_2(X)$. Then by Theorem 5.12, we have that $x_i \in \twoheaduparrow u$ eventually. Since $\twoheaduparrow u \subseteq U$, we have $x_i \in U$ eventually. Hence $x_i \xrightarrow{\mathrm{SI}_2} x$, as desired. □

The converse of Proposition 5.13 above fails to hold in general, as shown by the following example.

Example 5.14. Let $X = \mathbb{N}$ be the set of all natural numbers endowed with the co-finite topology. It can be verified that X is δ-sober. So, $\mathrm{SI}_2(X) = X$ by Proposition 4.4. We have the following statements.

(1) The SI_2-convergence structure in X is topological.
Let $(x_i)_{i\in I}$ topologically converge to x in $\mathrm{SI}_2(X)$. Set $F := \{x\}$. We have $x \in F^\delta$. Let U be in $\mathcal{O}(X) = \mathcal{O}_{\mathrm{SI}_2}(X)$ such that $F \cap U \neq \varnothing$. Then U is an SI_2-open set in X containing x. Hence $x_i \in U$ eventually. We have that $x_i \xrightarrow{\mathrm{SI}_2} x$. Hence SI_2-convergence in X is topological.

(2) X is not SI_2-continuous.
For every $x \in X$, since $x \ll_{\mathrm{SI}_2} x$, we have $\twoheaduparrow_{\mathrm{SI}_2} x = \{x\}$ which is clearly not SI_2-open. Therefore X is not SI_2-continuous.

6. Summary and future work

In this paper we defined a new topology, called SI_2-topology, on any T_0 space which generalizes the s_2-topology on a poset introduced by Erné. The SI_2-topology is generally finer than the SI-topology introduced by Zhao and Ho and enjoy some favorable properties. In [16], Zhang and Xu studied the s_2-quasicontinuous poset. It is possible to further consider the SI_2-quasicontinuous spaces. We leave that as future work.

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References

[1] Andradi, H., Shen, C., Ho, W. K., Dongsheng Z.: A new convergence inducing the SI-topology. Filomat **32**(17): 6017–6029 (2018)

[2] Engelking, R.: General Topology. Polish Scientific Publishers, Warszawa (1997)

[3] Erné, M.: A completion-invariant extension of the concept of continuous lattices. Continuous Lattices, Springer, Berlin, Heidelberg, pp.45–60 (1981)

[4] Erné, M.: The ABC of order and topology. Category Theory at Work **18**: 57–83 (1991)

[5] Erné, M.: Scott convergence and Scott topology on partially ordered sets II. Continuous Lattices, Springer, Berlin, Heidelberg, pp.61–96 (1981)

[6] Erné, M.: Z-continuous posets and their topological manifestation. Applied Categorical Structures **7**(1–2), 31–70 (1999)

[7] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M.W., Scott, D.S.: Continuous Lattices and Domains. Cambridge University Press, New York (2003)

[8] Goubault-Larrecq, J.: Non-Hausdorff Topology and Domain Theory. Cambridge University Press, New York (2013)

[9] Luo, S., Xu, X.: On Monotone Determined Spaces. Electronic Notes in Theoretical Computer Science **333**, 63–72 (2017)

[10] Mao, X., Xu, L.: Quasicontinuity of posets via Scott topology and sobrification. Order **23**(4), 359–369 (2006)

[11] Mao, X., Xu, L.: Meet continuity properties of posets. Theoretical Computer Science **410**(42): 4234–4240 (2009)

[12] Mislove, M.: Algebraic posets, algebraic dcpos and models of concurrency. Topology and Category theory in Computer Science, Oxford University Press, Oxford, pp. 75–111 (1991)

[13] Wright, J.B., Wagner, E.G., Thatcher, J.W.: A uniform approach to inductive posets and inductive closure. Theoretical Computer Science **7**(1), 55–77 (1978)

[14] Zhang, W., Xu, X.: Meet precontinuous posets. Electronic Notes in Theoretical Computer Science **301**, 179–188 (2014)

[15] Zhang, W., Xu, X.: Completely precontinuous posets. Electronic Notes in Theoretical Computer Science **301**, 169–178 (2014)

[16] Zhang, W., Xu, X.: s_2-Quasicontinuous posets. Theoretical Computer Science **574**, 78–85 (2015)

[17] Zhao, D., Fan, T.: Dcpo-completion of posets. Theoretical Computer Science **411**(22–24): 2167–2173 (2010)

[18] Zhao, D., Ho, W. K.: On topologies defined by irreducible sets. Journal of Logical and Algebraic Methods in Programming **84**(1), 185–195 (2015)

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Chong Shen, School of Mathematics and Statistics, Beijing Institute of Technology, Fangshan District Beijing 100081, China
Email address: shenchong0520@163.com

Hadrian Andradi, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616, and Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Indonesia 55281
Email address: hadrian.andradi@gmail.com (Corresponding author)

Dongsheng Zhao, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616
Email address: dongsheng.zhao@nie.edu.sg

Fu-Gui Shi, School of Mathematics and Statistics, Beijing Institute of Technology, Fangshan District Beijing 100081, China
Email address: fuguishi@bit.edu.cn