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Characterizations of L-convex spaces via domain theory

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Abstract

We provide some new characterizations of L-convex spaces by using the way-below relation in domain theory. For this purpose, we first study the notion of domain finiteness in the lattice-valued case, and then introduce three kinds of spaces: algebraic L-closure spaces, restricted L-hull spaces, and L-entailment spaces. These three spaces are shown to be categorically isomorphic to L-convex spaces. Additionally, we introduce the notion of L-polytopes and prove that a subcollection of a dense L-convex structure is a base if and only if it contains all L-polytopes. These results indicate that in the study of the theory of (fuzzy) convex spaces, (fuzzy) domain theory has important applications.

Keywords: L-convex structure; Continuous lattice; Algebraic L-closure operator; Restricted L-hull operator; L-entailment relation; L-polytope

1. Introduction

Convexity theory has received wide attention in recent years in the study of extremum problems in many areas of applied mathematics. As the axiomatization of convex sets in Euclidean space, a (abstract) convex space on a set is a collection of subsets that contains the empty set and is closed under arbitrary intersections and directed unions. Convexity exists in so many mathematical research areas, such as lattices [12,34], algebras [17,22], metric spaces [20], graphs [10,11,14], and topological spaces [15,35]. Some more details can be found in [36].

Matroid theory plays an important role in combinatorial optimization problems [37]. Many real-world problems can be defined and solved by use of matroid theory. In the classical problem, it is assumed that all weights are precisely known. However, this assumption may be a serious restriction, since in many practical applications the exact values of the weights are unknown in advance. To solve this problem, Shi [29,30] proposed the concept of L-fuzzifying matroids, which is a generalization of matroids. A matroid is a convex space that satisfies the exchange law (see Section 2 in Chapter I in [36] for details). In this case, extending the notion of convex spaces to the fuzzy setting is a particularly meaningful topic in areas of both theoretical research and practical applications. In 1994, the notions of fuzzy convex spaces and hull operators were first proposed by Rosa [26]. Later, Maruyama [18] extended Rosa's definition to a completely distributive lattice-valued setting, resulting in the notion of L-convex spaces. Also, some

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combinatorial properties of L-convex sets in Euclidean spaces were investigated. Pang and Shi [23] proposed several types of L-convex spaces and established their categorical relations. Pang and Zhao [24] recently presented the notions of L-concave structures, concave L-interior operators, and concave L-neighborhood systems, and proved they are all isomorphic to L-convex spaces. In the sense of [18,23,24,26], each convex set is fuzzy, but the convex structure comprising those convex sets is crisp. In a completely different way, Shi and Xiu [32] provided a new approach to the fuzzification of convex spaces and presented the notions of M-fuzzifying convex structures and M-fuzzifying hull operators. Also, their one-to-one correspondence was constructed.

Convexity theory also has a close link with universal algebra. Precisely, a structure on a given set is convex if and only if it is isomorphic to the subalgebra lattice of some universal algebra (which excludes nullary operation). In 1987, Di Nola and Gerla [9] provided a general approach to the notion of fuzzy subalgebras, called *L-subalgebra*, studied also by Biacino and Gerla [5]. Precisely, for a complete lattice L, every L-subalgebra is defined as L-sets $A \in L^U$ in a general universal algebra U that satisfy $A(u_1) \wedge A(u_2) \wedge \cdots \wedge A(u_n) \leq A(f(u_1, u_2, \cdots, u_n))$ for every n-ary operation of the universal algebra. They examined the category whose objects are the triples $\langle U, A, L\rangle$ and studied the notions of morphism, congruence, and factor algebra in the spirit of Di Nola and Gerla [8]. In 1991, Murali [21], independently of Di Nola and Gerla, studied fuzzy subalgebras in the particular case that $L = [0, 1]$. Murali introduced the notion of algebraic fuzzy closure systems and proved that a fuzzy closure system is algebraic if and only if it is closed under directed unions; all fuzzy subalgebras of a given algebra form an algebraic fuzzy closure system. A direct consequence deduced from Murali's result is that all fuzzy subalgebras of a given algebra without nullary operation form a fuzzy convex structure in the sense of Rosa [26]. Fuzzy closure operators and related structures have also been studied by Bělohlávek [2,3], Biacino and Gerla [5], Chakraborty [6], Lowen [16], Mashhour and Ghanim [19], Pavelka [25], and Yao and Lu [38] (all of which are included in [4]).

In the classical setting, it is well known that the properties of directed sup-preservation and domain finiteness (see [7]) are equivalent for a closure operator, and that the hull operator of a convex space is actually a closure operator satisfying domain finiteness. It has been proved by Rosa [26] that fuzzy convex spaces are equivalent to fuzzy closure operators satisfying directed sup-preservation (called *fuzzy hull operators* in [26]). Then a natural question arises: Can L-convex spaces (or equivalently L-hull operators) be characterized by the closure operators satisfying domain finiteness? The problem seems simple, but it turns out to be an intricate one because of the invalidity of substituting only finite sets for finite L-sets. In this article, we give a positive answer to this question by using the way-below relation in domain theory. Moreover, it turns out that the notions of fuzzy hull operators [26] and algebraic fuzzy closure operators [21] are both equivalent to the closure operators satisfying domain finiteness in the case that $L = [0, 1]$. Three main results of the present work are as follows:

- Let L be a complete lattice. Every L-hull operator is domain-finite if and only if L is a continuous lattice.
- The categorical isomorphisms among L-convex spaces, algebraic L-closure spaces, restricted L-hull spaces, and L-entailment spaces are constructed.
- The coherent base and subbase axioms with L-polytopes are established. Precisely, a collection of L-convex sets is a base if and only if it contains all L-polytopes; a collection of L-convex sets is a subbase if and only if it can generate all L-polytopes by a directed supremum operation.

This article is arranged as follows. In Section 2, we review some preliminaries that are needed in the subsequent sections. In Section 3, we summarize some results of the way-below relation between L-sets. In Sections 4–6 we introduce three spaces: algebraic L-closure spaces, restricted L-hull spaces, and L-entailment spaces, which are categorically isomorphic to L-convex spaces. In Section 7, we give the notion of L-polytopes and study its relationship with bases and subbases in L-convex spaces.

2. Preliminaries

In this section, we recall some basic concepts and results on fuzzy sets, lattices, and convex spaces. For undefined notions in this article, refer to [1,13].

Suppose L is a complete lattice. The greatest element and the least element in L are denoted by $\top$ and $\bot$, respectively. For $S \subseteq L$, we write $\bigvee S$ and $\bigwedge S$ for the least upper bound and the greatest lower bound of S, respectively. A subset D of L is called *directed* if it is nonempty and every finite subset of D has an upper bound in D. In particular, we use the convenient notation $x = \bigvee^{\uparrow} D$ to express that the set D is directed and x is its least upper bound.

Definition 2.1 *([7,13]).* Let L be a poset. For $x, y \in L$, x is *way below* y (in symbols $x \ll y$) if for any directed subset $D \subseteq L$ such that $\bigvee^{\uparrow} D$ exists, the relation $y \leq \bigvee^{\uparrow} D$ always implies the existence of some $d \in D$ with $x \leq d$.

Proposition 2.2 *([13]). In a poset L, the following statements hold for all $u, x, y, z \in L$:*

(P1) *$x \ll y$ implies $x \leq y$.*
(P2) *$u \leq x \ll y \leq z$ implies $u \ll z$.*
(P3) *$x \ll z$ and $y \ll z$ imply $x \vee y \ll z$ whenever $x \vee y$ exists in L.*
(P4) *$\perp \ll x$ whenever L has a smallest element $\perp$.*

Definition 2.3 *([13]).* A complete lattice L is called a *continuous lattice* if it satisfies the following *approximation*: for any $x \in L$,

$$x = \bigvee^{\uparrow} \twoheaddownarrow x,$$

where $\twoheaddownarrow x = \{y \in L \mid y \ll x\}$.

Definition 2.4 *([13]).* A binary relation $\prec$ on a poset L is called an *auxiliary relation*, or an *auxiliary order*, if it satisfies the following conditions for all u, x, y, z:

(1) $\perp \prec x$ whenever L has a smallest element $\perp$.
(2) $x \prec y$ implies $x \leq y$.
(3) $u \leq x \prec y \leq z$ implies $u \prec z$.

An auxiliary relation $\prec$ on a dcpo L is called *approximating* if for any $x \in L$ the set $\{u \in L \mid u \prec x\}$ is directed and

$$x = \bigvee^{\uparrow} \{u \in L \mid u \prec x\}.$$

In particular, every way-below relation in a continuous lattice is an approximating auxiliary relation.

Proposition 2.5 *([13]). Let L be a continuous lattice. For all $x, z \in L$, if $x \ll \bigvee^{\uparrow} D$ for a directed set D, then $x \ll d$ for some element $d \in D$.*

Throughout this article, L is always assumed to be a continuous lattice.

We write $\mathbf{2}^X$ and $\mathbf{2}^X_{fin}$ for the power set and the collection of all finite subsets of X, respectively. Each mapping $A : X \longrightarrow L$ is called an L*-set of* X, and the collection of all L-sets of X is usually denoted by L^X. It is easy to verify that L^X is a complete lattice under the pointwise order. We call an L-set A *finite* if its support set $\{x \in X \mid A(x) \neq \perp\}$ is finite. Let L^X_{fin} denote the collection of all finite L-sets of X. For any $A \subseteq X$, its characteristic function $\chi_A \in L^X$ is defined as follows:

$$\chi_A(x) = \begin{cases} \top, & x \in A, \\ \perp, & x \notin A. \end{cases}$$

In this case, the greatest element and the smallest element in L^X are χ_X and $\chi_\emptyset$, respectively.

Given a mapping $f : X \longrightarrow Y$, as usual, we define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by

$$\forall A \in L^X, B \in L^Y, \ f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x), \quad f_L^{\leftarrow}(B) = B \circ f.$$

The notions and some basic properties of L-closure systems and L-closure operators can be found in many publications (e.g., [4,5,19,21,31]).

Definition 2.6. A subcollection $\mathcal{C}$ of L^X is called an *L-closure structure on X* if it satisfies the following conditions:

(CS1) $\chi_\emptyset, \chi_X \in \mathcal{C}$.
(CS2) For any $\{A_i \mid i \in I\} \subseteq \mathcal{C}$, $\bigwedge_{i\in I} A_i \in \mathcal{C}$.

A pair $(X, \mathcal{C})$ is called an *L-closure space* if $\mathcal{C}$ is an L-closure structure on X and every $A \in \mathcal{C}$ is an *L-closed set*.

Definition 2.7. An *L-closure operator cl* on X is a mapping on L^X satisfying the following conditions:

(LC1) Normalization: $cl(\chi_\emptyset) = \chi_\emptyset$.
(LC2) Monotonicity: $A \le B$ implies $cl(A) \le cl(B)$.
(LC3) Extension: $A \le cl(A)$.
(LC4) Idempotency: $cl(cl(A)) = cl(A)$.

Proposition 2.8 *([2,3,19,21]). Every L-closure structure $\mathcal{C}$ on X defines an L-closure operator cl by*

$$\forall A \in L^X,\ cl(A) = \bigwedge\{C \in \mathcal{C} \mid A \le C\}.$$

Conversely, every L-closure operator cl on X defines an L-closure structure by

$$\mathcal{C} = \left\{A \in L^X \mid cl(A) = A\right\},$$

and the correspondence between L-closure operators and L-closure structures defined is bijective.

Definition 2.9 *([18,26,27]).* An L-closure structure $\mathcal{C}$ on X is called an *L-convex structure on X* if it satisfies the following condition:

(CS3) For any directed collection $\{D_i \mid i \in I\} \subseteq \mathcal{C}$, $\bigvee^{\uparrow}_{i\in I} D_i \in \mathcal{C}$.

A pair $(X, \mathcal{C})$ is called an *L-convex space* if $\mathcal{C}$ is an L-convex structure on X. Every $A \in \mathcal{C}$ is called an *L-convex set*.

Definition 2.10 *([18,26,27]).* Let $(X, \mathcal{C})$ be an L-convex space. For any $A \in L^X$, we define

$$co(A) = \bigwedge\{B \in \mathcal{C} \mid A \le B\};$$

that is, $co(A)$ is the smallest element of $\mathcal{C}$ that contains A, called the *L-convex hull*, or simply *L-hull*, of A. The operator co is called the *L-convex hull operator*, or simply the *L-hull operator*, on $(X, \mathcal{C})$.

Definition 2.11 *([18,26,27]).* Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a mapping between L-convex spaces. Then f is called

(1) *convexity-preserving* (*CP* for short) if for any $B \in \mathcal{C}_Y$, $f_L^{\leftarrow}(B) \in \mathcal{C}_X$;
(2) *convex-to-convex* (*CC* for short) if for any $A \in \mathcal{C}_X$, $f_L^{\rightarrow}(A) \in \mathcal{C}_Y$.

The category whose objects are L-convex spaces and whose morphisms are CP mappings is denoted by L-**CS**.

3. Some results on way-below relations between L-sets

In this section, we mainly summarize the properties of the way-below relation between L-sets.
Firstly, let us recall a very useful result in domain theory.

Proposition 3.1 *([13]). If $\{L_i \mid i \in I\}$ is a collection of continuous lattices with the smallest element $\perp$, then the product $\prod_{i\in I} L_i$ is also a continuous lattice. Furthermore, for elements $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ in $\prod_{i\in I} L_i$, the way-below relation is given by*

$$x \ll y \text{ if and only if } x_i \ll y_i \text{ for all } i \in I \text{ and } x_i = \perp \text{ for all but finite } i \in I.$$

The following proposition is a special case of Proposition 3.1 by our letting $I = X$ and $L_i = L$ for all $i \in I$.

Proposition 3.2. (1) *L^X is a continuous lattice under the pointwise order.*
(2) *For any $A, B \in L^X$, $A \ll B$ if and only if A is finite and $A(x) \ll B(x)$ for all $x \in X$.*

For any $A \in L^X$, we write $\Downarrow A$ for the collection of all L-sets that are way-below A; that is,

$$\Downarrow A = \left\{F \in L^X \mid F \ll A\right\}.$$

Remark 3.3. Let $A \in L^X$. Clearly, $\Downarrow A = \mathbf{2}^A_{fin}$ whenever $L = \{\bot, \top\}$. Here the way-below relation provides a more accurate way to characterize the relationship between a set A and its finite subsets; that is, F is a finite subset of A if and only if $F \ll A$ in $\left(\mathbf{2}^X, \subseteq\right)$.

By Propositions 2.5 and 3.2, the following two propositions are trivial.

Proposition 3.4. *For any $A \in L^X$, we have the following statements:*

(1) $\chi_\emptyset \ll A$.
(2) *$F_1, F_2 \in \Downarrow A$ implies $F_1 \bigvee F_2 \ll A$.*
(3) *$\Downarrow A$ is directed and $A = \bigvee^{\uparrow} \Downarrow A$.*

Proposition 3.5. *The following statements hold for any $A, B \in L^X$ and any directed collection $\{D_i \mid i \in I\} \subseteq L^X$:*

(1) *If $A \le B$, then $\Downarrow A \subseteq \Downarrow B$.*
(2) $\Downarrow \bigvee^{\uparrow}_{i\in I} D_i = \bigcup_{i\in I} \Downarrow D_i$.

Next we give a lemma that will be used later. The proof is trivial and is omitted here.

Lemma 3.6. *Let $f : X \longrightarrow Y$ be a mapping. Then $F \in L^X_{fin}$ implies $f^{\rightarrow}_L(F) \in L^Y_{fin}$.*

Proposition 3.7. *Let $f : X \longrightarrow Y$ be a mapping and let $A \in L^X$.*

(L1) *$F \ll A$ implies $f^{\rightarrow}_L(F) \ll f^{\rightarrow}_L(A)$.*
(L2) *$F \ll f^{\leftarrow}_L(H)$ if and only if $f^{\rightarrow}_L(F) \ll H$.*

Proof. (L1) Since $F \ll A$, we have $F \in L^X_{fin}$. By Lemma 3.6, we know $f^{\rightarrow}_L(F) \in L^Y_{fin}$. Suppose $\{D_i \mid i \in I\} \subseteq L^Y$ is directed and $f^{\rightarrow}_L(A) \le \bigvee^{\uparrow}_{i\in I} D_i$. We have

$$F \ll A \le f^{\leftarrow}_L\left(\bigvee^{\uparrow}_{i\in I} D_i\right) \le \bigvee^{\uparrow}_{i\in I} f^{\leftarrow}_L(D_i).$$

This shows $F \le f^{\leftarrow}_L(D_j)$ for some $j \in I$, which means $f^{\rightarrow}_L(F) \le D_j$. Hence $f^{\rightarrow}_L(F) \ll f^{\rightarrow}_L(A)$.
(L2) Suppose $F \ll f^{\leftarrow}_L(H)$. By (L1), we obtain $f^{\rightarrow}_L(F) \ll f^{\rightarrow}_L(f^{\leftarrow}_L(H)) \le H$. Conversely, if $f^{\rightarrow}_L(F) \ll H$, then $f^{\rightarrow}_L(F) \le H$. For any $x \in X$, we have

$$F(x) \le \bigvee_{f(z)=f(x)} F(z) = f^{\rightarrow}_L(F)(f(x)) \ll H(f(x)) = f^{\leftarrow}_L(H)(x).$$

Therefore $F \ll f^{\leftarrow}_L(H)$. $\square$

4. Algebraic L-closure operators

As shown in [36], the relation between convex structures and algebraic (also called *domain-finite*) closure operators is bijective. In this section, by applying the way-below relation in domain theory to the definition of domain finiteness, we construct a one-to-one relationship between L-convex structures and algebraic L-closure operators.

In the crisp situation, a closure operator cl on a set X is called *domain-finite* if for any subset $A \subseteq X$

$$cl(A) = \bigcup \{cl(F) \mid F \in 2^{A}_{fin}\}.$$

As shown in Proposition 2.8, the approach to the fuzzification of closure operators is straightforward. However, this approach to the lattice-valued counterpart of domain finiteness fails to be valid if we only generalize the inclusion order "$\subseteq$" to the pointwise order "$\leq$." By Remark 3.3, a more accurate formulation of domain finiteness can be shown as follows:

$$\forall A \in L^X,\ cl(A) = \bigcup \{cl(F) \mid F \in \mathord{\downarrow\!\!\downarrow} A\}.$$

In the following, we extend this formulation of domain finiteness to the lattice-valued case and show that this approach has a decisive role in characterizing L-convex spaces.

Definition 4.1. We call an L-closure operator co on a given set X *algebraic* if it satisfies *the domain finiteness*:

$$\forall A \in L^X,\ co(A) = \bigvee\nolimits^{\uparrow} \{co(F) \mid F \ll A\}.$$

We call a pair (X, co) an *algebraic L-closure space* if co is an algebraic L-closure operator on X.

Remark 4.2. Murali [21, Definition 5.4] defined *the algebraic fuzzy closure operator* c on X in the case that $L = [0, 1]$. Precisely, a fuzzy closure operator $c : [0, 1]^X \longrightarrow [0, 1]^X$ is called *algebraic* if

$$\forall A \in [0, 1]^X,\ c(A) = \bigvee \{c(B) \mid B \text{ is finite and } \forall x \in X, B(x) < A(x)\}.$$

The condition here coincides with the domain finiteness in Definition 4.1 in the case that $L = [0, 1]$.

Theorem 4.3. *Let $(X, \mathcal{C})$ be an L-closure space. Then the following statements are equivalent.*

(1) *$\mathcal{C}$ is an L-convex structure on X.*
(2) *The L-closure operator cl of $(X, \mathcal{C})$ is algebraic.*
(3) *For any directed collection $\{D_i\}_{i\in I} \subseteq L^X$, $cl\left(\bigvee^{\uparrow}_{i\in I} D_i\right) = \bigvee^{\uparrow}_{i\in I} cl(D_i)$.*

Proof. (3)⇒(2) It is clear by Proposition 3.4.

(1)⇒(3) Suppose $\{D_i \mid i \in I\} \subseteq L^X$ is directed. Since cl is monotone, the collection $\{cl(D_i) \mid i \in I\} \subseteq \mathcal{C}$ is directed, and hence $\bigvee^{\uparrow}_{i\in I} cl(D_i) \in \mathcal{C}$. It follows that

$$cl\left(\bigvee_{i\in I}^{\uparrow} D_i\right) \leq cl\left(\bigvee_{i\in I}^{\uparrow} cl(D_i)\right) = \bigvee_{i\in I}^{\uparrow} cl(D_i),$$

the reverse inequality is trivial by (CL2). Therefore $\bigvee^{\uparrow}_{i\in I} cl(D_i) = cl\left(\bigvee^{\uparrow}_{i\in I} D_i\right)$.

(2)⇒(1) It suffices to verify (CS3). Suppose $\{D_i \mid i \in I\} \subseteq \mathcal{C}$ is directed. Then $cl(D_i) = D_i$ for any $i \in I$. By Proposition 3.5, we have

$$
\begin{aligned}
cl\left(\bigvee\nolimits^{\uparrow}_{i\in I} D_i\right) &= \bigvee\nolimits^{\uparrow}\{cl(F) \mid F \ll \bigvee\nolimits^{\uparrow}_{i\in I} D_i\} \\
&= \bigvee\nolimits^{\uparrow}\{cl(F) \mid \exists i \in I, F \ll D_i\} \\
&= \bigvee\nolimits^{\uparrow}\bigcup\nolimits_{i\in I}\{cl(F) \mid F \ll D_i\} \\
&= \bigvee\nolimits^{\uparrow}_{i\in I}\bigvee\nolimits^{\uparrow}\{cl(F) \mid F \ll D_i\} \\
&= \bigvee\nolimits^{\uparrow}_{i\in I} cl(D_i) = \bigvee\nolimits^{\uparrow}_{i\in I} D_i.
\end{aligned}
$$

This shows $\bigvee^{\uparrow}_{i\in I} D_i \in \mathcal{C}$, as desired. □

Theorem 4.3 shows that every L-hull operator is an algebraic L-closure operator. Also, the algebraic L-closure operators here coincide with the fuzzy hull operators in [26] in the case that $L = [0, 1]$.

Corollary 4.4. *The relationship between L-convex structures and algebraic L-closure operators is bijective.*

For a complete lattice L, the requirement that L is continuous is also necessary for Theorem 4.3. The proof is straightforward by Theorem 4.3 and the fact that L^X is an L-convex structure (the finest one) on X.

Theorem 4.5. *Let L be a complete lattice. For any L-convex space $(X, \mathcal{C})$, its L-hull operator is algebraic if and only if L is a continuous lattice.*

Next we show that L-hull operators can also be used to characterize CC and CP mappings.

Theorem 4.6. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a mapping between L-convex spaces. Then the following statements are equivalent.*

(1) *f is CP.*
(2) *For any $F \in L^X_{fin}$, $f^{\rightarrow}_L(co_X(F)) \leq co_Y(f^{\rightarrow}_L(F))$.*
(3) *For any $A \in L^X$, $f^{\rightarrow}_L(co_X(A)) \leq co_Y(f^{\rightarrow}_L(A))$.*

Proof. (1)⇒(2) Since $co_Y(f^{\rightarrow}_L(F)) \in \mathcal{C}_Y$, we have $F \leq f^{\leftarrow}_L(co_Y(f^{\rightarrow}_L(F))) \in \mathcal{C}_X$. This means $co_X(F) \leq f^{\leftarrow}_L(co_Y(f^{\rightarrow}_L(F)))$. Hence $f^{\rightarrow}_L(co_X(F)) \leq f^{\leftarrow}_L(co_Y(f^{\rightarrow}_L(F))$.

(2)⇒(3) By Lemma 3.7, we have

$$
\begin{aligned}
f^{\rightarrow}_L(co_X(A)) &= f^{\rightarrow}_L\left(\bigvee\nolimits^{\uparrow}_{F\ll A} co_X(F)\right) \\
&= \bigvee\nolimits^{\uparrow}_{F\ll A} f^{\rightarrow}_L(co_X(F)) \\
&\leq \bigvee\nolimits^{\uparrow}_{F\ll A} co_Y(f^{\rightarrow}_L(F)) \\
&= co_Y\left(f^{\rightarrow}_L\left(\bigvee\nolimits^{\uparrow}_{F\ll A} F\right)\right) \\
&= co_Y\left(f^{\rightarrow}_L(A)\right).
\end{aligned}
$$

(3)⇒(1) Suppose $B \in \mathcal{C}_Y$. We have

$$f^{\rightarrow}_L(co_X(f^{\leftarrow}_L(B))) \leq co_Y(f^{\rightarrow}_L(f^{\leftarrow}_L(B))) \leq co_Y(B) = B.$$

This shows $co_X(f^{\leftarrow}_L(B)) \leq f^{\leftarrow}_L(B)$. Hence $f^{\leftarrow}_L(B) \in \mathcal{C}_X$. □

The proof of the following theorem is dually analogous to Theorem 4.6 and is omitted here.

Theorem 4.7. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a mapping between L-convex spaces. Then the following statements are equivalent.*

(1) *f is CC.*
(2) *For any $F \in L^X_{fin}$, $co_Y(f_L^{\rightarrow}(F)) \leq f_L^{\rightarrow}(co_X(F))$.*
(3) *For any $A \in L^X$, $co_Y(f_L^{\rightarrow}(A)) \leq f_L^{\rightarrow}(co_X(A))$.*

Definition 4.8. A mapping $f : (X, co_X) \longrightarrow (Y, co_Y)$ between algebraic L-closure spaces is called *L-closure-preserving* (*LCP* for short) if for any $A \in L^X$, $f_L^{\rightarrow}(co_X(A)) \leq co_Y\left(f_L^{\rightarrow}(A)\right)$.

Theorem 4.9. *If $f : (X, co_X) \longrightarrow (Y, co_Y)$ and $g : (Y, co_Y) \longrightarrow (Z, co_Z)$ are two LCP mappings, then the composite mapping $g \circ f : (X, co_X) \longrightarrow (Z, co_X)$ is an LCP mapping.*

The category whose objects are algebraic L-closure spaces and whose morphisms are L-LCP mappings is denoted by L-**AC**.

The following is immediate by Theorems 4.4 and 4.6.

Theorem 4.10. *L-**AC** is isomorphic to L-**CS**.*

5. Restricted L-hull operators

Domain finiteness shows that convex structures can be completely determined by their polytopes (hulls of finite sets). Motivated by this, a more concise axiom system, called a *restricted hull operator*, defined by restriction of the domain of hull operators to the collection of finite sets, can also characterize convex structures [36].

A restricted hull operator on X is a mapping $h : \mathbf{2}^X_{fin} \longrightarrow \mathbf{2}^X$ satisfying the following properties:

(H1) $h(\emptyset) = \emptyset$-
(H2) For any $F \in \mathbf{2}^X_{fin}$, $F \subseteq h(F)$.
(H3) For any $F, G \in \mathbf{2}^X_{fin}$, $G \subseteq h(F)$ implies $h(G) \subseteq h(F)$.

A pair (X, h) is called a *restricted hull space* if h is a restricted hull operator on X.

In the following, we extend this concept to the lattice-valued case by restricting the domain of L-hull operators to finite L-sets.

Definition 5.1. A mapping $\mathfrak{h} : L^X_{fin} \longrightarrow L^X$ is called a *restricted L-hull operator on X* if it satisfies the following conditions:

(LH1) $\mathfrak{h}(\chi_\emptyset) = \chi_\emptyset$.
(LH2) For any $F \in L^X_{fin}$, $F \leq \mathfrak{h}(F)$.
(LH3) For any $F \in L^X_{fin}$, $G \ll \mathfrak{h}(F)$ implies $\mathfrak{h}(G) \leq \mathfrak{h}(F)$.
(LH4) For any $F \in L^X_{fin}$, $\mathfrak{h}(F) = \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}(G)$.

We call a pair $(X, \mathfrak{h})$ a *restricted L-hull space* if $\mathfrak{h}$ is a restricted L-hull operator on X.

Remark 5.2. In the crisp situation (i.e., $L = \{\bot, \top\}$), (LH4) holds naturally. (LH1)–(LH3) can reduce to (H1)–(H3), respectively. In this sense, the notion of restricted L-hull operators is a reasonable generalization of restricted hull operators.

Proposition 5.3. *If $\mathfrak{h} : L^X_{fin} \longrightarrow L^X$ is a mapping satisfying* (LH4)*, then* (LH3) *is equivalent to the following condition:*

(LH3*) *For any $F \in L^X_{fin}$, $G \leq \mathfrak{h}(F)$ implies $\mathfrak{h}(G) \leq \mathfrak{h}(F)$.*

Proof. It suffices to verify (LH3)⇒(LH3*). Suppose $F \in L^X_{fin}$ and $G \leq \mathfrak{h}(F)$. Then by (LH3), $\mathfrak{h}(E) \leq \mathfrak{h}(F)$ for all $E \ll G$. It follows from (LH4) that

$$\mathfrak{h}(G) = \bigvee_{E \ll G}^{\uparrow} \mathfrak{h}(E) \leq \mathfrak{h}(F).$$

The proof is complete. □

We introduce some properties on restricted L-hull operators in the following proposition; these can be proved straightforwardly.

Proposition 5.4. *Let $(X, \mathfrak{h})$ be a restricted L-hull space.*

(1) *For any $F \in L^X_{fin}$, $\mathfrak{h}(F) \in L^X_{fin}$ implies $\mathfrak{h}(\mathfrak{h}(F)) = \mathfrak{h}(F)$.*
(2) *For any $A, B \in L^X_{fin}$, $A \leq B$ implies $\mathfrak{h}(A) \leq \mathfrak{h}(B)$.*

Definition 5.5. A mapping $f : (X, \mathfrak{h}_X) \longrightarrow (Y, \mathfrak{h}_Y)$ between restricted L-hull spaces is called *restricted L-hull-preserving* (*RHP* for short) if for any $F \in L^X_{fin}$, $f^{\rightarrow}_L(\mathfrak{h}(F)) \leq \mathfrak{h}(f^{\rightarrow}_L(F))$.

Remark 5.6. By Lemma 3.6, it holds that $F \in L^X_{fin}$ implies $f^{\rightarrow}_L(F) \in L^Y_{fin}$. This shows that RHP mappings in Definition 5.5 are well defined.

We can immediately show the following result.

Proposition 5.7. *If $f : (X, \mathfrak{h}_X) \longrightarrow (Y, \mathfrak{h}_Y)$ and $g : (Y, \mathfrak{h}_Y) \longrightarrow (Z, \mathfrak{h}_Z)$ are two RHP mappings between restricted L-hull spaces, then the composite mapping $g \circ f : (X, \mathfrak{h}_X) \longrightarrow (Z, \mathfrak{h}_Z)$ is also RHP.*

The category whose objects are restricted L-hull spaces and whose morphisms are RHP mappings is denoted by L-**RHS**. Next we consider the relationship between L-**CS** and L-**RHS**.

Proposition 5.8. *Let $(X, \mathfrak{h})$ be a restricted L-hull space. Define a collection $\mathcal{C}^{\mathfrak{h}} \subseteq L^X$ as follows:*

$$\mathcal{C}^{\mathfrak{h}} = \left\{ A \in L^X \mid \forall F \ll A,\ \mathfrak{h}(F) \leq A \right\}.$$

Then $\mathcal{C}^{\mathfrak{h}}$ is an L-convex structure on X.

Proof. It suffices to prove (CS1)–(CS3).

(CS1) Note that $\Downarrow \chi_\emptyset = \{\chi_\emptyset\}$ and $\mathfrak{h}(\chi_\emptyset) = \chi_\emptyset$. Then $\chi_\emptyset \in \mathcal{C}^{\mathfrak{h}}$. The proof that $\chi_X \in \mathcal{C}^{\mathfrak{h}}$ is trivial.

(CS2) Suppose $\{A_i \mid i \in I\} \subseteq \mathcal{C}^{\mathfrak{h}}$. If $F \ll \bigwedge_{i \in I} A_i$, then by Proposition 3.5, $F \ll A_i$ for all $i \in I$. Note that $\{A_i \mid i \in I\} \subseteq \mathcal{C}^{\mathfrak{h}}$. We have $\mathfrak{h}(F) \leq A_i$ for all $i \in I$. That is, $\mathfrak{h}(F) \leq \bigwedge_{i \in I} A_i$. Hence $\bigwedge_{i \in I} A_i \in \mathcal{C}^{\mathfrak{h}}$.

(CS3) Suppose $\{D_i \mid i \in I\} \subseteq \mathcal{C}^{\mathfrak{h}}$ is directed. If $F \ll \bigvee^{\uparrow}_{i \in I} D_i$, then by Proposition 3.5 there exists $j \in I$ such that $F \ll D_j$. Since $D_j \in \mathcal{C}^{\mathfrak{h}}$, we have $\mathfrak{h}(F) \leq D_j \leq \bigvee^{\uparrow}_{i \in I} D_i$. Hence $\bigvee^{\uparrow}_{i \in I} D_i \in \mathcal{C}^{\mathfrak{h}}$. □

Proposition 5.9. *Let $\left(X, \mathcal{C}^{\mathfrak{h}}\right)$ be the L-convex space induced by a restricted L-hull space $(X, \mathfrak{h})$ and let $co^{\mathfrak{h}}$ be the L-hull operator on $\left(X, \mathcal{C}^{\mathfrak{h}}\right)$.*

(1) *For any $F \in L^X_{fin}$, $\mathfrak{h}(F) \in \mathcal{C}^{\mathfrak{h}}$.*
(2) *For any $A \in L^X$, $\bigvee^{\uparrow}_{F \ll A} \mathfrak{h}(F) \in \mathcal{C}^{\mathfrak{h}}$.*
(3) *For any $A \in L^X$, $co^{\mathfrak{h}}(A) = \bigvee^{\uparrow}_{F \ll A} \mathfrak{h}(F)$.*
(4) *For any $F \in L^X_{fin}$, $co^{\mathfrak{h}}(F) = \mathfrak{h}(F)$.*

Proof. By (LH3), the proof of (1) and (2) is trivial.

(3) Let $B = \bigvee^{\uparrow}_{F \ll A} \mathfrak{h}(F)$. We need only to show $B \leq co^{\mathfrak{h}}(A)$. By Proposition 3.4, $A \leq B \in \mathcal{C}^{\mathfrak{h}}$. For any $C \in \mathcal{C}^{\mathfrak{h}}$, if $A \leq C$, then for any $F \ll A$, we have $F \ll C$, meaning that $\mathfrak{h}(F) \leq C$. It follows that $B = \bigvee^{\uparrow}_{F \ll A} \mathfrak{h}(F) \leq C$. Thus

$$\bigvee^{\uparrow}_{F \ll A} \mathfrak{h}(F) \leq \bigwedge\{C \in \mathcal{C}^{\mathfrak{h}} \mid A \leq C\} = co^{\mathfrak{h}}(A).$$

(4) By (3) and (LH4), we have $co^{\mathfrak{h}}(F) = \bigvee^{\uparrow}\{\mathfrak{h}(E) \mid E \ll F\} = \mathfrak{h}(F)$. $\square$

Proposition 5.10. *Let $f : (X, \mathfrak{h}_X) \longrightarrow (Y, \mathfrak{h}_Y)$ be an RHP mapping between restricted L-hull spaces. Then $f : \left(X, \mathcal{C}^{\mathfrak{h}_X}\right) \longrightarrow \left(Y, \mathcal{C}^{\mathfrak{h}_Y}\right)$ is CP.*

Proof. Suppose $A \in L^X$. We have

$$\begin{aligned} f_L^{\rightarrow}\left(co^{\mathfrak{h}_X}(A)\right) &= f_L^{\rightarrow}\left(\bigvee{}^{\uparrow}\{co^{\mathfrak{h}_X}(F) \mid F \ll A\}\right) \\ &= \bigvee{}^{\uparrow}\{f_L^{\rightarrow}\left(co^{\mathfrak{h}_X}(F)\right) \mid F \ll A\} \\ &\leq \bigvee{}^{\uparrow}\{co^{\mathfrak{h}_Y}\left(f_L^{\rightarrow}(F)\right) \mid F \ll A\} \\ &\leq co^{\mathfrak{h}_Y}\left(\bigvee{}^{\uparrow}\{f_L^{\rightarrow}(F) \mid F \ll A\}\right) \\ &= co^{\mathfrak{h}_Y}\left(f_L^{\rightarrow}(A)\right). \end{aligned}$$

Hence f is CP. $\square$

By Propositions 5.8 and 5.10, we obtain a functor $\mathbb{G} : L$-**RHS**$\longrightarrow L$-**CS** defined by

$$\mathbb{G}(X, \mathfrak{h}) = \left(X, \mathcal{C}^{\mathfrak{h}}\right) \text{ and } \mathbb{G}(f) = f.$$

Proposition 5.11. *Let $(X, \mathcal{C})$ be an L-convex space. Define a mapping $\mathfrak{h}^{\mathcal{C}} : L^X_{fin} \longrightarrow L^X$ as follows:*

$$\forall F \in L^X_{fin},\ \mathfrak{h}^{\mathcal{C}}(F) = \bigwedge\{C \in \mathcal{C} \mid F \leq C\}.$$

Then $\mathfrak{h}^{\mathcal{C}}$ is a restricted L-hull operator.

Proof. The proof of (LH1) and (LH2) is trivial.

(LH3) Suppose $F \in L^X_{fin}$ and $G \leq \mathfrak{h}^{\mathcal{C}}(F)$. Then by (SC2), we have $\mathfrak{h}^{\mathcal{C}}(F) = co(F) \in \mathcal{C}$. It follows that $\mathfrak{h}^{\mathcal{C}}(G) = co(G) \leq \mathfrak{h}^{\mathcal{C}}(F)$.

(LH4) Suppose $F \in L^X_{fin}$. We need only to prove $\mathfrak{h}(F) = \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}^{\mathcal{C}}(G)$. Since $\downarrow F$ is directed and $\mathfrak{h}^{\mathcal{C}}$ is monotone, $\left\{\mathfrak{h}^{\mathcal{C}}(G) \mid G \ll F\right\} \subseteq \mathcal{C}$ is directed. It follows that $\bigvee^{\uparrow}_{G \ll F} \mathfrak{h}^{\mathcal{C}}(G) \in \mathcal{C}$. Furthermore,

$$F = \bigvee^{\uparrow}_{G \ll F} G \leq \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}^{\mathcal{C}}(G) \in \mathcal{C}.$$

Hence $\mathfrak{h}^{\mathcal{C}}(F) \leq \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}^{\mathcal{C}}(G)$. The reverse inequality is trivial. Therefore $\mathfrak{h}(F) = \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}^{\mathcal{C}}(G)$. $\square$

For an L-convex space $(X, \mathcal{C})$, the operator $\mathfrak{h}^{\mathcal{C}}$ coincides with the L-hull operator of $(X, \mathcal{C})$ when its domain is restricted to finite L-sets; that is, $\mathfrak{h}^{\mathcal{C}} = co_X\mid_{L^X_{fin}}$. Then the following result is clear by Proposition 4.6.

Proposition 5.12. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a CP mapping between L-convex spaces. Then $f : \left(X, \mathfrak{h}^{\mathcal{C}_X}\right) \longrightarrow \left(Y, \mathfrak{h}^{\mathcal{C}_Y}\right)$ is RHP.*

By Propositions 5.11 and 5.12, we obtain a functor $\mathbb{H}: L$-**CS**$\longrightarrow L$-**RHS** defined by

$$\mathbb{H}(X, \mathcal{C}) = \left(X, \mathfrak{h}^{\mathcal{C}}\right) \text{ and } \mathbb{H}(f) = f.$$

Theorem 5.13. L-**RHS** *is isomorphic to* L-**CS**.

Proof. It suffices to verify that (1) $\mathcal{C}^{\mathfrak{h}^{\mathcal{C}}} = \mathcal{C}$ and (2) $\mathfrak{h}^{\mathcal{C}^{\mathfrak{h}}} = \mathfrak{h}$ for any L-convex space $(X, \mathcal{C})$ and any restricted L-hull space $(X, \mathfrak{h})$.

(1) We have

$$\begin{aligned}
\mathcal{C}^{\mathfrak{h}^{\mathcal{C}}} &= \left\{A \in L^X \mid \forall F \ll A, \mathfrak{h}^{\mathcal{C}}(F) \le A\right\} \\
&= \left\{A \in L^X \mid \textstyle\bigvee^{\uparrow}_{F \ll A} \mathfrak{h}^{\mathcal{C}}(F) \le A\right\} \\
&= \left\{A \in L^X \mid \textstyle\bigvee^{\uparrow}_{F \ll A} co(F) \le A\right\} \\
&= \left\{A \in L^X \mid co(A) \le A\right\} \\
&= \mathcal{C}.
\end{aligned}$$

(2) Suppose $F \in \mathscr{A}$. We have $\mathfrak{h}^{\mathcal{C}^{\mathfrak{h}}}(F) = \bigwedge\{B \in \mathcal{C}^{\mathfrak{h}} \mid F \le B\}$. On one hand, if $E \ll \mathfrak{h}(F)$, then $E \le \mathfrak{h}(F)$. By (LH3), we obtain $\mathfrak{h}(E) \le \mathfrak{h}(F)$, meaning that $\mathfrak{h}(F) \in \mathcal{C}^{\mathfrak{h}}$. On the other hand, if $B \in \mathcal{C}^{\mathfrak{h}}$ and $F \le B$, then $G \le B$ for all $G \ll F$. It follows that $\mathfrak{h}(G) \le B$. By (LH4), we obtain $\mathfrak{h}(F) = \bigvee^{\uparrow}_{G \ll F} \mathfrak{h}(G) \le B$. Therefore $\mathfrak{h}^{\mathcal{C}^{\mathfrak{h}}}(F) = \bigwedge\{B \in \mathcal{C}^{\mathfrak{h}} \mid F \le B\} = \mathfrak{h}(F)$. □

6. *L*-entailment spaces

The relationship between L-sets of an L-convex space $(X, \mathcal{C})$ and their L-hulls can be abstracted to a binary relation on L^X, called an L*-entailment relation*, and we will show that algebraic L-closure operators are equivalent to L-entailment relations. The axiom system of L-entailment relations is given as follows.

Definition 6.1. We call a binary relation $\preccurlyeq$ on L^X an L*-entailment relation* if it satisfies the following conditions: for any $A, B, C \in L^X$,

(ER1) $\chi_\emptyset \preccurlyeq \chi_\emptyset$;
(ER2) $A \preccurlyeq B$ implies $A \le B$;
(ER3) $A \preccurlyeq \bigwedge\left\{B_i \in L^X \mid i \in I\right\}$ if and only if $A \preccurlyeq B_i$ for all $i \in I$;
(ER4) $A \preccurlyeq B$ if and only if $F \preccurlyeq B$ for all $F \ll A$;
(ER5) if $A \preccurlyeq B$, then there exists $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq B$.

A pair $(X, \preccurlyeq)$ is called an L*-entailment space* if $\preccurlyeq$ is an L-entailment relation on L^X.

Lemma 6.2. *Let $\preccurlyeq$ be an L-entailment relation on L^X.*

(1) *$\chi_\emptyset \preccurlyeq A$ for all $A \in L^X$.*
(2) *If $A \le B \preccurlyeq C \le D$, then $A \preccurlyeq D$.*
(3) *If $A \preccurlyeq B$, then there exists $C \in L^X$ such that $A \le C \preccurlyeq C \le B$.*

Proof. (1) The proof is straightforward by (ER1) and (ER3).

(2) By (ER3) we have $B \preccurlyeq D$. For any $F \ll A$, since $A \le B$, we have $F \ll B$. Thus $F \preccurlyeq D$ by (ER4), implying that $A \preccurlyeq D$.

(3) Let $\mathcal{D} = \{D \in L^X \mid A \preccurlyeq D\}$ and let $C = \bigwedge\{D \in L^X \mid D \in \mathcal{D}\}$. Obviously, we have $C \le B \in \mathcal{D}$. Moreover from (ER3), we can obtain $A \preccurlyeq C$, which implies that C is the minimum element in $\mathcal{D}$. By (ER5), we know that there exists

$E \in L^X$ such that $A \preccurlyeq E \preccurlyeq C$. By the definition of C we know $E \geq C$. From (ER2) and $E \preccurlyeq C$, we have $E \leq C$. It follows that $E = C$. Therefore $A \leq C \preccurlyeq C \leq B$. □

Proposition 6.3. *A binary relation $\preccurlyeq$ on L^X satisfies* (ER4) *if and only if it satisfies the following condition:*

(ER4*) *For any directed collection $\{B_i \mid i \in I\} \subseteq L^X$, $\bigvee^{\uparrow}_{i\in I} B_i \preccurlyeq A$ if and only if $B_i \preccurlyeq A$ for all $i \in I$.*

Proof. It suffices to show (ER4*) $\Rightarrow$ (ER4). Suppose $\bigvee^{\uparrow}_{i\in I} B_i \preccurlyeq A$. For any $j \in I$, to prove $B_j \preccurlyeq A$, by (ER4) it suffices to check $F \preccurlyeq A$ for all $F \ll B_j$. Since

$$F \ll \bigvee\nolimits^{\uparrow}_{i\in I} B_i \preccurlyeq A,$$

by (ER4), we have $F \preccurlyeq A$, as desired. Conversely, let $B_i \preccurlyeq A$ for any $i \in I$. Then for any $F \ll \bigvee^{\uparrow}_{i\in I} B_i$, by Proposition 3.5 there exists $j \in I$ such that $F \ll B_j$. Since $B_j \preccurlyeq A$, we have $F \preccurlyeq A$. Hence by (ER4) $\bigvee^{\uparrow}_{i\in I} B_i \preccurlyeq A$. □

By Definition 6.1 and Lemma 6.2, every L-entailment relation is an auxiliary order with the interpolation property on L^X. However in general, it is not approximating. This can be shown by the following example.

Example 6.4. Let $X = \{x, y\}$. For any $B \in \mathbf{2}^X$, define a binary relation $\preccurlyeq$ on $\mathbf{2}^X$ as follows:

$$\{A \mid A \preccurlyeq B\} = \begin{cases} \{\emptyset\}, & x \notin B, \\ \{C \mid C \subseteq B\}, & x \in B. \end{cases}$$

It is easy to check that $\preccurlyeq$ is an L-entailment relation ($L = \mathbf{2}$), but it is not approximating.

The following theorem shows that every L-convex structure can induce an L-entailment relation. Its proof is straightforward.

Theorem 6.5. *Let $(X, \mathcal{C})$ be an L-convex space. Define a binary $\preccurlyeq_{\mathcal{C}}$ on L^X as follows:*

$$\forall A, B \in L^X,\ A \preccurlyeq B \text{ iff } co(A) \leq B.$$

Then $\preccurlyeq_{\mathcal{C}}$ is an L-entailment relation on L^X.

From an L-entailment relation, we can also obtain an L-convex structure.

Theorem 6.6. *Let $(X, \preccurlyeq)$ be an L-entailment space. Define an operator on L^X as follows:*

$$\forall A \in L^X,\ co_{\preccurlyeq}(A) = \bigwedge\{G \in L^X \mid A \preccurlyeq G\}.$$

(1) *For any $G \in L^X$, $co_{\preccurlyeq}(A) \leq G$ if and only if $A \preccurlyeq G$.*
(2) *The operator $co_{\preccurlyeq}$ is an algebraic L-closure operator, and hence it can induce an L-convex structure on X, denoted by $\mathcal{C}_{\preccurlyeq}$.*

Proof. (1) It is straightforward to prove that $A \preccurlyeq G$ implies $co_{\preccurlyeq}(A) \leq G$. Conversely, suppose $co_{\preccurlyeq}(A) \leq G$. Then by (ER3) in Definition 6.1, we know that $A \preccurlyeq co_{\preccurlyeq}(A) \leq G$. Hence $A \preccurlyeq G$.

(2) We only prove (LC4) and domain finiteness.

To prove $co_{\preccurlyeq}(co_{\preccurlyeq}(A)) = co_{\preccurlyeq}(A)$, we need only to prove $co_{\preccurlyeq}(co_{\preccurlyeq}(A)) \leq co_{\preccurlyeq}(A)$. Let $co_{\preccurlyeq}(A) \leq G$. Then $A \preccurlyeq G$. By (ER5) we know that there exists $C \in L^X$ such that $A \preccurlyeq C \preccurlyeq G$. This implies $co_{\preccurlyeq}(C) \leq G$ and $co_{\preccurlyeq}(A) \leq C$. Thus we have $co_{\preccurlyeq}(co_{\preccurlyeq}(A)) \leq co_{\preccurlyeq}(C) \leq G$. Therefore $co_{\preccurlyeq}(co_{\preccurlyeq}(A)) \leq co_{\preccurlyeq}(A)$.

By (ER4) we can obtain that

$$\bigvee^{\uparrow}_{F \ll A} co_{\preccurlyeq}(F) \leq co_{\preccurlyeq}(A).$$

It remains to prove $co_{\preccurlyeq}(A) \leq \bigvee^{\uparrow}_{F \in \ll A} co_{\preccurlyeq}(F)$. For convenience, set $G = \bigvee^{\uparrow}_{F \ll A} co_{\preccurlyeq}(F)$. Then $co_{\preccurlyeq}(F) \leq G$ for any $F \ll A$. This implies $co_{\preccurlyeq}(A) \leq G$. Hence

$$co_{\preccurlyeq}(A) \leq \bigvee^{\uparrow}_{F \ll A} co_{\preccurlyeq}(F).$$

Therefore $co_{\preccurlyeq}(A) = \bigvee^{\uparrow}_{F \ll A} co_{\preccurlyeq}(F)$. $\square$

The following is straightforward by Theorem 6.6.

Corollary 6.7. *For an L-entailment relation $\preccurlyeq$ on L^X, it follows that*

$$\mathcal{C}_{\preccurlyeq} = \{A \in L^X \mid A \preccurlyeq A\}.$$

By Theorems 6.5 and 6.6, the following two theorems are straightforward.

Theorem 6.8. *For an L-entailment space $(X, \preccurlyeq)$, $\preccurlyeq_{\mathcal{C}_{\preccurlyeq}} = \preccurlyeq$.*

Theorem 6.9. *For an L-convex space $(X, \mathcal{C})$, $\mathcal{C}_{\preccurlyeq_{\mathcal{C}}} = \mathcal{C}$.*

Now we consider the mappings between two L-entailment spaces.

Definition 6.10. Let $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ be two L-entailment spaces and let $f : X \longrightarrow Y$ be a mapping. We call the mapping f

(1) *L-entailment relation dual-preserving* (*ERDP* for short) if for any $U, V \in L^Y$, $U \preccurlyeq_Y V$ implies $f_L^{\leftarrow}(U) \preccurlyeq_X f_L^{\leftarrow}(V)$;
(2) *L-entailment relation preserving* (*ERP* for short) if for any $A, B \in L^X$, $A \preccurlyeq_X B$ implies $f_L^{\rightarrow}(A) \preccurlyeq_Y f_L^{\rightarrow}(B)$;
(3) *isomorphic* if f is a bijective, ERD, and PERP mapping.

The following theorem can be proved easily.

Theorem 6.11. (1) *If $f : (X, \preccurlyeq_X) \longrightarrow (Y, \preccurlyeq_Y)$ and $g : (Y, \preccurlyeq_Y) \longrightarrow (Z, \preccurlyeq_Z)$ are two ERDP mappings, then the composite mapping $g \circ f : (X, \preccurlyeq_X) \longrightarrow (Z, \preccurlyeq_Z)$ is also ERDP.*

(2) *If $f : (X, \preccurlyeq_X) \longrightarrow (Y, \preccurlyeq_Y)$ and $g : (Y, \preccurlyeq_Y) \longrightarrow (Z, \preccurlyeq_Z)$ are two ERP mappings, then the composite mapping $g \circ f : (X, \preccurlyeq_X) \longrightarrow (Z, \preccurlyeq_Z)$ is also ERP.*

The category whose objects are L-entailment spaces and whose morphisms are ERDP mappings is denoted by L-**ERS**.

Theorem 6.12. (1) *If $f : (X, \preccurlyeq_X) \longrightarrow (Y, \preccurlyeq_Y)$ is an ERDP mapping, then $f : (X, \mathcal{C}_{\preccurlyeq_X}) \longrightarrow (Y, \mathcal{C}_{\preccurlyeq_Y})$ is a CP mapping.*

(2) *If $f : (X, \preccurlyeq_X) \longrightarrow (Y, \preccurlyeq_Y)$ is an ERP mapping, then $f : (X, \mathcal{C}_{\preccurlyeq_X}) \longrightarrow (Y, \mathcal{C}_{\preccurlyeq_Y})$ is a CC mapping.*

Proof. (1) Let A be an L-convex set in $(Y, \mathcal{C}_{\preccurlyeq_Y})$. Then $A = \bigwedge\{G \in L^Y \mid A \preccurlyeq_Y G\}$. Hence $f_L^{\leftarrow}(A) = \bigwedge\{f_L^{\leftarrow}(G) \mid G \in L^Y, A \preccurlyeq_Y G\}$. It is trivial that $f_L^{\leftarrow}(A) \preccurlyeq_X f_L^{\leftarrow}(G)$ whenever $A \preccurlyeq_Y G$. Then we have

$$\begin{aligned} f_L^{\leftarrow}(A) &= \bigwedge\{f_L^{\leftarrow}(G) \mid G \in L^Y, A \preccurlyeq_Y G\} \\ &\geq \bigwedge\{E \in L^X \mid f_L^{\leftarrow}(A) \preccurlyeq_X E\} \\ &\geq f_L^{\leftarrow}(A). \end{aligned}$$

This implies $f_L^{\leftarrow}(A) = \bigwedge\{E \in L^X \mid f_L^{\leftarrow}(A) \preccurlyeq_X E\} = co_{\preccurlyeq_X}\big(f_L^{\leftarrow}(A)\big)$; that is, $f_L^{\leftarrow}(A)$ is an L-convex set in $(X, \mathcal{C}_{\preccurlyeq_X})$. Therefore $f : (X, \mathcal{C}_{\preccurlyeq_X}) \longrightarrow (Y, \mathcal{C}_{\preccurlyeq_Y})$ is a CP mapping.

(2) The proof is similar to (1). □

Theorem 6.13. (1) *If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a CP mapping, then $f : (X, \preccurlyeq_{\mathcal{C}_X}) \longrightarrow (Y, \preccurlyeq_{\mathcal{C}_Y})$ is an ERDP mapping.*

(2) *If $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a CC mapping, then $f : (X, \preccurlyeq_{\mathcal{C}_X}) \longrightarrow (Y, \preccurlyeq_{\mathcal{C}_Y})$ is an ERP mapping.*

Proof. (1) The L-hull operators of X and Y are denoted by co_X and co_Y, respectively. Let $A, B \in L^Y$ with $A \preccurlyeq_{\mathcal{C}_Y} B$. Then $co_Y(A) \le B$. It follows that $f_L^{\leftarrow}(co_Y(A)) \le f_L^{\leftarrow}(B)$. Since $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ is a CP mapping, we have $co_X(f_L^{\leftarrow})(A) \le f_L^{\leftarrow}(co_Y(A))$. Thus we obtain $co_X(f_L^{\leftarrow}(A)) \le f_L^{\leftarrow}(co_Y(A)) \le f_L^{\leftarrow}(B)$. This shows $f_L^{\leftarrow}(A) \preccurlyeq_{\mathcal{C}_X} f_L^{\leftarrow}(B)$. Therefore $f : (X, \preccurlyeq_{\mathcal{C}_X}) \longrightarrow (Y, \preccurlyeq_{\mathcal{C}_Y})$ is an ERDP mapping.

(2) The proof is dually analogous to (1). □

Now we define a functor $\mathbb{S}$: L-**ERS** $\to$ L-**CS** by

$$\mathbb{S}(X, \preccurlyeq) = (X, \mathcal{C}_{\preccurlyeq}), \quad \mathbb{S}(f) = f.$$

By Theorems 6.8, 6.9, 6.12, and 6.13, we can obtain the following theorem.

Theorem 6.14. L-**ERS** *is isomorphic to* L-**CS**.

A summary of the relationships among L-convex spaces, algebraic L-closure spaces, restricted L-hull spaces, and L-entailment spaces is shown as follows.

$$\begin{aligned} &\mathcal{C}^{\preccurlyeq} = \{A \mid A \preccurlyeq A\} \\ &\mathcal{C}^{co} = \{C \mid co(A) = A\} \\ &\mathcal{C}^{\mathfrak{h}} = \{A \mid \forall F \ll A, \mathfrak{h}(F) \le A\} \end{aligned}$$

$$\begin{tikzcd} & (X, \mathcal{C}) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ (X, \mathfrak{h}) \arrow[dr, leftrightarrow] & & (X, \mathfrak{h}) \arrow[dl, leftrightarrow] \\ & (X, \preccurlyeq) & \end{tikzcd}$$

$$\begin{aligned} &\forall A \in L^X, \\ &co^{\mathfrak{h}}(A) = \textstyle\bigvee^{\uparrow}_{F \ll A} \mathfrak{h}(F) \\ &co^{\mathcal{C}}(A) = \textstyle\bigwedge\{B \in \mathcal{C} \mid A \le B\} \\ &co^{\preccurlyeq}(A) = \textstyle\bigwedge\{B \in L^X \mid A \preccurlyeq B\} \end{aligned}$$

$$\begin{aligned} &\forall F \in L^X_{fin}, \\ &\mathfrak{h}^{co}(F) = co(F) \\ &\mathfrak{h}^{\mathcal{C}}(F) = \textstyle\bigwedge\{A \in \mathcal{C} \mid F \le A\} \\ &\mathfrak{h}^{\preccurlyeq}(F) = \textstyle\bigwedge\{A \in L^X \mid F \preccurlyeq A\} \end{aligned}$$

$$\begin{aligned} &\forall A, B \in L^X, \\ &A \preccurlyeq^{co} B \Leftrightarrow co(A) \le B \\ &A \preccurlyeq^{\mathfrak{h}} B \Leftrightarrow \forall F \le A, \mathfrak{h}(F) \le B \\ &A \preccurlyeq^{\mathcal{C}} B \Leftrightarrow \textstyle\bigwedge\{C \in \mathcal{C} \mid A \le C\} \le B \end{aligned}$$

7. Polytopes and bases of L-convex spaces

In convexity theory, polytopes and bases are closely connected. Precisely, the collection of all polytopes (convex hulls of finite sets) of a convex space $(X, \mathcal{C})$ forms the smallest base of $\mathcal{C}$. In other words, every base of a convex space must contain the collection of all polytopes. However, this classic result fails to hold in the lattice-valued case if the notion of polytopes is just regarded as L-hulls of finite sets (consider the example in Remark 7.2). In this section, an approach to the definition of L-polytopes is presented, and it turns out that every base of an L-convex structure includes the collection of all L-polytopes.

7.1. L-polytopes

In the classical case, the polytopes are of the type $co(F)$, where F is a finite set of X, and obviously it satisfies $F \ll co(F)$ in the lattice of power set $\mathbf{2}^X$. On this basis, the notion of L-polytopes is presented as follows.

Definition 7.1. Let $(X, \mathcal{C})$ be an L-convex space. We call an L-convex set C an *L-polytope* if there exists an L-set $F \ll C$ such that $C = co(F)$.

Remark 7.2. Every L-polytope C of an L-convex space $(X, \mathcal{C})$ is of the form $co(F)$ for a finite L-set F of X, but the converse may not be true. We give an example. Let $X = \{x\}$, $L = [0, 1]$, and $\mathcal{C} = \{x_\lambda \mid \lambda \in [0.5, 1]\}$. It is easy to check that $\mathcal{C}$ is an L-convex structure on X, and that $co(x_1) = x_1 \in \mathcal{C}$, which is not an L-polytope.

By domain finiteness, every L-convex set is the supremum of a directed subcollection of L-hulls of finite L-sets. However, it generally does not hold that every L-convex set is the supremum of a directed subcollection of L-polytopes. This leads to the following definition.

Definition 7.3. We call an L-convex space $(X, \mathcal{C})$ *dense* if the L-polytopes are directed-join dense in $\mathcal{C}$; that is, every L-convex set is the supremum of a directed subcollection of L-polytopes.

Proposition 7.4. *The compact elements of the lattice of an L-convex structure are precisely the L-polytopes.*

Proof. Let $(X, \mathcal{C})$ be an L-convex space. Suppose C is an L-polytope of $(X, \mathcal{C})$. Then there exists a finite L-set F of X such that $F \ll C$ and $C = co(F)$. If $\{D_i \mid i \in I\} \subseteq \mathcal{C}$ is directed such that $C \leq \bigvee^{\uparrow}_{i \in I} D_i$, then $F \leq D_i$ for some $i \in I$. Hence $C = co(F) \leq co(D_i) = D_i$. This means that $C \ll C$, so C is a compact. Conversely, if $C \in \mathcal{C}$ is compact, then we have

$$C \ll C = \bigvee{}^{\uparrow}\{co(F) \mid F \ll C\}.$$

Then $C \ll co(E)$ for some $E \ll C$, implying that $C = co(E)$. So C is an L-polytope. □

A straightforward consequence of Proposition 7.4 is as follows.

Corollary 7.5. *Let $(X, \mathcal{C})$ be an L-convex space. Then the following statements are equivalent.*

(1) *$(X, \mathcal{C})$ is a dense L-convex space.*
(2) *$\mathcal{C}$ is an algebraic lattice under the pointwise order.*
(3) *For any $A \in L^X$, $co(A) = \bigvee^{\uparrow}\{co(F) \mid F \ll co(F) \leq A\}$.*

Theorem 7.6. *Let $(X, \mathcal{C})$ be an L-convex space and let C be a nonempty L-convex set. Then the following statements are equivalent.*

(1) *C is an L-polytope.*
(2) *C is not the supremum of any directed collection of proper L-convex subsets.*
(3) *C is not the supremum of any chain of proper L-convex subsets.*

Proof. It is trivial that (2)⇒(3).

(1)⇒(2) Let $\{D_i \mid i \in I\} \subseteq \mathcal{C}$ be directed such that $D_i < C$ (i.e., $D_i \leq C$ and $D_i \neq C$) for all $i \in I$ and $C = \bigvee^{\uparrow}_{i \in I} D_i$. Since C is an L-polytope, there exists an L-set $F \ll C$ such that $C = co(F)$. It follows that

$$F \ll co(F) = C = \bigvee_{i \in I}{}^{\uparrow} D_i,$$

which means $F \leq D_j$ for some $j \in I$. Therefore $co(F) \leq co(D_j) = D_j$, contradicting $D_j < co(F)$.

(3)⇒(1) Assume C is an L-convex set that is not an L-polytope. Let $\mathcal{D}$ be a collection of all L-convex sets defined as follows:

$$\mathcal{D} = \{D \in \mathcal{C} \mid D < C \text{ such that for all} F \ll C, co(D \vee F) \neq C\}.$$

As C is not an L-polytope, we have $\chi_\emptyset \in \mathcal{D}$, which shows that $\mathcal{D}$ is nonempty. Let $\mathcal{M} \subseteq \mathcal{D}$ be a maximal chain (so $\mathcal{M} \neq \emptyset$), and consider the L-convex set $M^* = \bigvee^{\uparrow} \mathcal{M}$. If $M^* = C$, then the proof is complete. So assume $M^* \neq C$, i.e., $M^* < C$. Then there exists a point $x \in X$ satisfying $M^*(x) < C(x)$. Thus there exists $\lambda \in L$ such that $\lambda \ll C(x)$ and $\lambda \nleq M^*(x)$, and hence $x_\lambda \ll C$ with $x_\lambda \nleq M^*(x)$.

Now suppose $M^* \in \mathcal{D}$. For any $F \ll C$, since $x_\lambda \ll C$, we have $F \vee x_\lambda \ll C$. It follows that

$$co(co(M^* \vee x_\lambda) \vee F) = co(M^* \vee x_\lambda \vee F) \neq C$$

for any $F \ll C$. This shows that $co(M^* \vee x_\lambda)$ is also in $\mathcal{D}$. Note that $M^* < co(M^* \vee x_\lambda)$ for any $M^* \in \mathcal{M}$, and this allows us to enlarge the chain $\mathcal{M}$ in $\mathcal{D}$. So M^* is not in $\mathcal{D}$, meaning that there exists $G \ll C$ such that $co(M^* \vee G) = C$. Consider the following chain in $\mathcal{D}$, consisting of proper L-convex subsets of C:

$$\mathcal{N} = \{co(M \vee G) \mid M \in \mathcal{M}\}.$$

Now note that

$$\bigvee\nolimits^{\uparrow} \mathcal{N} = \bigvee_{M\in\mathcal{M}}^{\uparrow} co(M \vee G) = co\left(\bigvee_{M\in\mathcal{M}}^{\uparrow} M \vee G\right) = co(M^* \vee G) = C.$$

The proof is complete. $\square$

The result in Theorem 7.6 shows that if an L-polytope C is the directed supremum of a collection of L-convex sets, then C must be a member of this collection.

7.2. Bases and subbases of L-convex spaces

Definition 7.7. Let $(X, \mathcal{C})$ be an L-convex space and $\mathcal{S}, \mathcal{B} \subseteq \mathcal{C}$. Then

(1) the collection $\mathcal{S}$ is called a *subbase of* $\mathcal{C}$ if $\mathcal{C}$ is the coarsest L-convex structure including $\mathcal{S}$;
(2) the collection $\mathcal{B}$ is called a *base of* $\mathcal{C}$ if every member of $\mathcal{C}$ is the supremum of a directed subcollection of $\mathcal{B}$.

Note that every base of $\mathcal{C}$ is a subbase of $\mathcal{C}$.

Lemma 7.8. *Let $(X, \mathcal{B})$ be an L-closure space with the L-closure operator cl. Then for any $B \in \mathcal{B}$, we have $B = \bigvee^{\uparrow}_{F\ll B} cl(F)$.*

Proof. Suppose $B \in \mathcal{B}$. It follows that

$$B = \bigvee\nolimits^{\uparrow}_{F\ll B} F \leq \bigvee\nolimits^{\uparrow}_{F\ll B} cl(F) \leq cl(B) = B.$$

Therefore $B = \bigvee^{\uparrow}_{F\ll B} cl(F)$. $\square$

Proposition 7.9. *Let $\mathcal{S} \subseteq L^X$ containing $\chi_\emptyset$.*

(1) $\mathcal{B} = \{\bigwedge \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{S}\}$ is an L-closure structure.
(2) $\mathcal{C} = \{\bigvee^{\uparrow} \mathcal{D} \mid \mathcal{D} \subseteq \mathcal{B}$ is directed$\}$ is an L-convex structure.
Hence we have
(3) $\mathcal{S} \subseteq \mathcal{B} \subseteq \mathcal{C}$;
(4) $\mathcal{S}$ is a subbase of $\mathcal{C}$, and $\mathcal{B}$ is a base of $\mathcal{C}$.

Proof. (1) The proof is straightforward.

(2) It suffices to prove (CS1), (CS2), and (CS3).

(CS1) The proof is trivial since $\chi_\emptyset, \chi_X \in \mathcal{B}$.

(CS2) Suppose $\{A_i \mid i \in I\} \subseteq \mathcal{C}$. For any $i \in I$, there exists a collection $\mathcal{D}_i = \{D_{i,j} \mid j \in J_i\} \subseteq \mathcal{B}$ such that $A_i = \bigvee^{\uparrow} \mathcal{D}_i$. For any $x \in X$, we have

$$\bigwedge\nolimits_{i\in I} A_i(x) = \bigwedge\nolimits_{i\in I} \bigvee\nolimits^{\uparrow}_{j\in J_i} D_{i,j}(x) = \bigvee\nolimits^{\uparrow}_{f\in\prod_{i\in I} J_i} \bigwedge\nolimits_{i\in I} D_{i,f(i)}(x).$$

This means $\bigwedge_{i\in I} A_i = \bigvee^{\uparrow}_{f\in\prod_{i\in I} J_i} \bigwedge_{i\in I} D_{i,f(i)}$. Since for any $f \in \prod_{i\in I} J_i$, $\{D_{i,f(i)} \mid i \in I\} \subseteq \mathcal{B}$, it follows that $\bigwedge_{i\in I}\{D_{i,f(i)} \mid i \in I\} \subseteq \mathcal{B}$, and hence $\bigwedge_{i\in I} A_i = \bigvee^{\uparrow}_{f\in\prod_{i\in I} J_i} \bigwedge_{i\in I} D_{i,f(i)} \in \mathcal{C}$.

(CS3) Suppose $\{D_i \mid i \in I\} \subseteq \mathcal{C}$ is directed. For any $i \in I$, there exists a collection $\mathcal{D}_i = \{D_{i,j} \mid j \in J_i\} \subseteq \mathcal{B}$ such that $D_i = \bigvee^{\uparrow} \mathcal{D}_i$. For any $i \in I$ and $j \in J_i$, $D_{i,j} \in \mathcal{B}$, by Lemmas 7.8 and 3.5, it follows that $D_{i,j} = \bigvee_{F\ll D_{i,j}} cl(F)$. In addition, we have

$$\begin{aligned}
\bigvee^{\uparrow}_{i\in I} D_i &= \bigvee^{\uparrow}_{i\in I} \bigvee^{\uparrow}_{j\in J_i} D_{i,j} \\
&= \bigvee^{\uparrow}_{i\in I} \bigvee^{\uparrow}_{j\in J_i} \bigvee^{\uparrow}\{cl(F) \mid F \ll D_{i,j}\} \\
&= \bigvee^{\uparrow}_{i\in I} \bigvee^{\uparrow} \bigcup_{j\in J_i}\{cl(F) \mid F \ll D_{i,j}\} \\
&= \bigvee^{\uparrow}_{i\in I} \bigvee^{\uparrow}\{cl(F) \mid F \in \bigcup_{j\in J_i} \Downarrow D_{i,j}\} \\
&= \bigvee^{\uparrow}_{i\in I} \bigvee^{\uparrow}\{cl(F) \mid F \ll \bigvee_{j\in J_i} D_{i,j}\} \\
&= \bigvee^{\uparrow}_{i\in I} \bigvee^{\uparrow}\{cl(F) \mid F \ll D_i\} \\
&= \bigvee^{\uparrow}\{cl(F) \mid F \ll \bigvee_{i\in I} D_i\},
\end{aligned}$$

implying that $\bigvee^{\uparrow}_{i\in I} D_i \in \mathcal{C}$. Therefore $\mathcal{C}$ is an L-convex structure on X.

Conclusions (3) and (4) are immediate consequences of (1) and (2). □

7.3. Relationships between L-polytopes and bases

Theorem 7.10. *Let $(X, \mathcal{C})$ be a dense L-convex space and $\mathcal{B} \subseteq \mathcal{C}$. Then $\mathcal{B}$ is a base of $\mathcal{C}$ if and only if it contains all L-polytopes.*

Proof. The necessity holds from Propositions 7.6 and 7.9. The sufficiency is trivial by the domain finiteness. □

Proposition 7.11. *Let $(X, \mathcal{B})$ be an L-closure space with the L-closure operator cl and let $\mathcal{C}$ be the L-convex structure generated by $\mathcal{B}$ with the L-hull operator co. Then for any finite L-set F,*

(1) *$co(F) \le cl(F)$;*
(2) *if $co(F)$ is an L-polytope, then $co(F) = cl(F)$.*

Proof. Let $\mathcal{C} = \left\{\bigvee^{\uparrow} \mathcal{D} \mid \mathcal{D} \subseteq \mathcal{B} \text{ is directed}\right\}$. Then by Proposition 7.9, $\mathcal{C}$ is the L-convex structure generated by $\mathcal{B}$.

(1) Since $\mathcal{B} \subseteq \mathcal{C}$, we have

$$co(F) = \bigwedge\{C \in \mathcal{C} \mid F \le C\} \le \bigwedge\{B \in \mathcal{B} \mid F \le B\} = cl(F).$$

(2) It suffices to show $cl(F) \le co(F)$. Since $\mathcal{B}$ is a base of $\mathcal{C}$, by Proposition 7.10 we have $F \le co(F) \in \mathcal{B}$. Note that

$$cl(F) = \bigwedge\{B \in \mathcal{B} \mid F \le B\} \le co(F) \in \mathcal{B}.$$

Then $cl(F) \le co(F)$. By (1), we have $co(F) = cl(F)$. □

Remark 7.12. In Proposition 7.11 (1), that $cl(F) \le co(F)$ is generally incorrect. We give an example. Let $X = \{x\}$, $L = [0, 1]$, and $\mathcal{B} = \{x_\lambda \mid \lambda \in [0, 0.5) \bigcup \{1\}\}$. It is easy to check that $\mathcal{B}$ is an L-closure structure, and it generates the L-convex structure $\mathcal{C} = \{x_\lambda \mid \lambda \in [0, 0.5] \bigcup \{1\}\}$. For $x_{0.5}$, we have

$$cl(x_{0.5}) = \bigwedge\{B \in \mathcal{B} \mid x_{0.5} \le B\} = x_1.$$

However,

$$co(x_{0.5}) = \bigwedge\{B \in \mathcal{C} \mid x_{0.5} \leq B\} = x_{0.5}.$$

Hence $co(x_{0.5}) \neq cl(x_{0.5})$.

After closing a subcollection $\mathcal{S}$ of L^X for arbitrary infima, we can obtain an L-closure structure that operates as a base for an L-convex structure. This leads to the following characterization of a subbase.

Theorem 7.13. *Let $(X, \mathcal{C})$ be a dense L-convex space and let $\mathcal{S}$ be a subcollection of $\mathcal{C}$. Then $\mathcal{S}$ is a subbase of $\mathcal{C}$ if and only if every L-polytope is the infimum of a subcollection of $\mathcal{S}$.*

As a consequence of Theorems 7.10 and 7.13, we have the following result.

Corollary 7.14. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a mapping between dense L-convex spaces and let $\mathcal{B}$ (or $\mathcal{S}$) be a base (or a subbase) of $(Y, \mathcal{C}_Y)$. Then the following statements are equivalent.*

(1) *f is CP.*
(2) *$f_L^{\leftarrow}(C) \in \mathcal{C}_X$ for all $C \in \mathcal{B}$.*
(3) *$f_L^{\leftarrow}(C) \in \mathcal{C}_X$ for all $C \in \mathcal{S}$.*

Lemma 7.15. *Let $f : X \longrightarrow Y$ be a surjective mapping. Then for any $G \in L^Y_{fin}$, there exists $F \in L^X_{fin}$ such that $f_L^{\rightarrow}(F) = G$.*

Proof. For any $y \in Y$, fix a point $x_y \in f^{-1}(y)$, and let $X^* = \{x_y \mid y \in Y\}$. Now we define $F \in L^X$ by

$$F(x) = \begin{cases} G(y), & x = x_y \in X^*, \\ \perp, & x \notin X^*. \end{cases}$$

It is straightforward to check F is finite and $f_L^{\rightarrow}(F) = G$. □

Lemma 7.16. *Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a mapping between L-convex spaces and let $F \in L^X$ such that $F \ll co_X(F)$.*

(1) *$f_L^{\rightarrow}(F) \ll f_L^{\rightarrow}(co_X(F))$.*
(2) *If f is CP, then $f_L^{\rightarrow}(F) \ll co_Y(f_L^{\rightarrow}(F))$.*

Proof. (1) Suppose $\{D_i \mid i \in I\} \subseteq L^Y$ is a directed collection such that $f_L^{\rightarrow}(co_X(F)) \leq \bigvee^{\uparrow}_{i\in I} D_i$. It follows that $co_X(F) \leq f_L^{\leftarrow}\left(\bigvee^{\uparrow}_{i\in I} D_i\right) = \bigvee^{\uparrow}_{i\in I} f_L^{\leftarrow}(D_i)$. Since $F \ll co_X(F)$, there exists $j \in I$ such that $F \leq f_L^{\leftarrow}(D_j)$, implying that $f_L^{\rightarrow}(F) \leq D_j$. Therefore $f_L^{\rightarrow}(F) \ll f_L^{\rightarrow}(co_X(F))$.

(2) The proof is straightforward by use of (1). □

Theorem 7.17. *A CP and CC image of a dense L-convex space is dense.*

Proof. Let $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ be a CC and CP surjective mapping from a dense L-convexity $\mathcal{C}_X$ to an L-convexity $\mathcal{C}_Y$. By Corollary 7.5, it suffices to show that for any $B \in L^Y$,

$$co_Y(B) = \bigvee{}^{\uparrow}\{co_Y(E) \mid E \ll co_Y(E) \leq B\}.$$

Let $B \in L^Y$. Since f is surjective, there exists $A \in L^X$ such that $f_L^{\rightarrow}(A) = B$. We have

$$\begin{aligned}
co_Y(B) &= co_Y(f_L^{\rightarrow}(A)) = f_L^{\rightarrow}(co_X(A)) \\
&= f_L^{\rightarrow}\left(\bigvee\{co_X(F) \mid F \ll co_X(F) \le A\}\right) \\
&= \bigvee\{f_L^{\rightarrow}(co_X(F)) \mid F \ll co_X(F) \le A\} \\
&= \bigvee\{co_Y(f_L^{\rightarrow}(F)) \mid F \ll co_X(F) \le A\} \\
&\le \bigvee\{co_Y(E) \mid E \ll co_Y(E) \le B\}.
\end{aligned}$$

The last inequality holds because of Lemma 7.15. Since the reverse inequality holds obviously, we have that $co_Y(B) = \bigvee^{\uparrow}\{co_Y(E) \mid E \ll co_Y(E) \le B\}$. So $\mathcal{C}_Y$ is dense. □

8. Conclusions and remarks for further work

In this article, we have presented some new characterizations of L-convex spaces via domain theory. The main results are (i) that we provided a natural extension of the domain finiteness to the fuzzy setting, (ii) L-convex structures are categorically isomorphic to algebraic L-closure operators, and (iii) the interrelationship between L-polytopes and the base (subbase) of an L-convex spaces is established. These major results fully demonstrate the validity of this new approach to dealing with the finiteness of L-sets.

We close the article with some problems and tasks for further exploration on this topic.

(1) In Section 7, we introduced the notion of dense L-convex spaces (see Definition 7.3 in detail), which is a natural generalization of convex spaces and is more compatible with the L-polytopes. However, the relationship between L-convex spaces and dense L-convex spaces is still unknown. Precisely, we wonder whether the category of dense L-convex spaces is a reflective (or coreflective) subcategory of L-convex spaces.

(2) Shi and Xiu [33] introduced a more generalized fuzzy convex structure, called an (L, M)-*fuzzy convex structure*. It will be interesting to generalize the descriptions of L-convex structures to the (L, M)-fuzzy case.

(3) In convexity theory, besides convex structures, there is another essential structure called a *convex system* (see [36, 1.17]). The difference with convex structures is simply that the universal set need not be convex. Many concepts and results in convex structures can actually be formulated in terms of convex systems. We conjecture that the idea of domain finiteness in this article can also be used to characterize L-convex systems in [28].

(4) In the classical setting, the combined structure of a topology (the collection all closed sets) and a convex structure on the same set is called a *topological convex structure* if all polytopes are closed (see [36]). The compatible connection between L-polytopes and bases in the article provides a possibility to extend topological convex structures and the related concepts to the fuzzy setting.

(5) Murali [21] introduced the notions of fuzzy subalgebras and algebraic fuzzy closure systems, and one of the main results is that the lattice of fuzzy subalgebras of a universal algebra is an algebraic fuzzy closure system. Remark 4.2 shows that algebraic L-closure operators coincide with algebraic fuzzy closure operators defined by Murali [21] in the case that $L = [0, 1]$. In the spirit of domain finiteness introduced in this article, it is possible to generalize the work on algebraic fuzzy closure systems from the unit interval to any continuous lattice.

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