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On T_0 spaces determined by well-filtered spaces ☆

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ABSTRACT

We first introduce and study two new classes of subsets in T_0 spaces — Rudin sets and WD sets lying between the class of all closures of directed subsets and that of irreducible closed subsets. Using such subsets, we define three new types of topological spaces — DC spaces, Rudin spaces and WD spaces. Rudin spaces lie between WD spaces and DC spaces, while DC spaces lie between Rudin spaces and sober spaces. Using Rudin sets and WD sets, we formulate and prove a number of new characterizations of well-filtered spaces and sober spaces. For a T_0 space X, it is proved that X is sober iff X is a well-filtered Rudin space iff X is a well-filtered WD space. We also prove that every locally compact T_0 space is a Rudin space, and every core compact T_0 space is a WD space. One immediate corollary is that every core compact well-filtered space is sober, giving a positive answer to Jia-Jung problem. Using WD sets, we present a more direct construction of the well-filtered reflections of T_0 spaces, and prove that the products of any collection of well-filtered spaces are well-filtered. Our study also leads to a number of problems, whose answers will deepen our understanding of the related spaces and structures.

1. Introduction

In the theory of non-Hausdorff topological spaces, the *d*-spaces, well-filtered spaces and sober spaces form three of the most important classes. Rudin's Lemma is a useful tool in topology and plays a crucial role in domain theory (see [2–6,8–29]). In recent years, it has been used to study the various aspects of well-filtered spaces and sober spaces, initiated by Heckmann and Keimel [11]. In this paper, inspired by the

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topological version of Rudin's Lemma by Heckmann and Keimel, Xi and Lawson's work [26] on well-filtered spaces and our recent work [23,28] on sober spaces and well-filtered reflections of T_0 spaces, we introduce and investigate two new classes of subsets in T_0 spaces — Rudin sets and WD sets lying between the class of all closures of directed subsets and that of irreducible closed subsets. Using such subsets, we introduce and study three new types of topological spaces — directed closure spaces (DC spaces for short), Rudin spaces and well-filtered determined spaces (WD spaces for short). Rudin spaces lie between WD spaces and DC spaces, and DC spaces lie between Rudin spaces and sober spaces. We shall prove that closed subspaces, retracts and products of Rudin spaces (resp. WD spaces) are again Rudin spaces (resp., WD spaces). Using Rudin sets and WD sets, we formulate and prove a number of new characterizations of well-filtered spaces and sober spaces. For a T_0 space X, it is proved that X is sober iff X is a well-filtered Rudin space iff X is a well-filtered WD space. In [4], Erné proved that in a locally hypercompact T_0 space X, every irreducible closed subset of X is the closure of a directed subset of X. So locally hypercompact T_0 spaces are DC spaces. Furthermore, we prove that every locally compact T_0 space is a Rudin space and every core compact T_0 space is a WD space. As a corollary we reduce that every core compact well-filtered space is sober, giving a positive answer to Jia-Jung problem [15], which was first answered by Lawson and Xi [20] using a different method.

It is well-known that the category of all sober spaces (d-spaces) is reflective in the category of all T_0 spaces (see [6,18,23–25]). But for quite a long time, it is not known whether the category of all well-filtered spaces is reflective in the category of all T_0 space. Recently, following Keimel and Lawson's method [18], which originated from Wyler's method [25], Wu, Xi, Xu and Zhao [9] gave a positive answer to the above problem. Following Ershov's method of constructing the d-completion of T_0 spaces, Shen, Xi, Xu and Zhao presented a different construction of the well-filtered reflection of T_0 spaces. In the current paper, using WD sets, we present a more direct construction of the well-filtered reflections of T_0 spaces, and prove that products of well-filtered spaces are well-filtered. Some major properties of well-filtered reflections of T_0 spaces are investigated. Comparatively, the technique presented in this paper is not just more direct, but also more simple. In addition, it can be directly applied to the general K-ifications considered by Keimel and Lawson [18]. In a forthcoming article we will use a similar technique to set up the K-ification theory of T_0 spaces. Our study also leads to a number of open problems, whose answers will deepen our understanding of the related spaces and structures.

2. Preliminary

In this section, we briefly recall some fundamental concepts and notations that will be used in the paper. Some basic properties of irreducible sets and compact saturated sets are presented.

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). Define $A^{\uparrow} = \{x \in P : x \text{ is an upper bound of } A \text{ in } P\}$. Dually, define $A^{\downarrow} = \{x \in P : x \text{ is a lower bound of A in P}\}$. The set $A^{\delta} = (A^{\uparrow})^{\downarrow}$ is called the *cut* generated by A. Let $P^{(<\omega)} = \{F \subseteq P : F \text{ is a nonempty finite set}\}$ and $\mathbf{Fin} P = \{\uparrow F : F \in P^{(<\omega)}\}$. For a nonempty subset A of P, define $max(A) = \{a \in A : a \ \text{ is a maximal element of } A\}$ and $min(A) = \{a \in A : a \ \text{ is a minimal element of } A\}$.

A nonempty subset D of a poset P is *directed* if every two elements in D have an upper bound in D. The set of all directed sets of P is denoted by $\mathcal{D}(P)$. $I \subseteq P$ is called an *ideal* of P if I is a directed lower subset of P. Let $\mathrm{Id}(P)$ be the poset (with the order of set inclusion) of all ideals of P. Dually, we define the notion of *filters* and denote the poset of all filters of P by $\mathrm{Filt}(P)$. P is called a *directed complete poset*, or *dcpo* for short, if for any $D \in \mathcal{D}(P)$, $\bigvee D$ exists in P. P is said to be *Noetherian* if it satisfies the *ascending chain condition* (ACC for short): every ascending chain has a greatest member. Clearly, P is Noetherian iff every directed set of P has a largest element (equivalently, every ideal of P is principal).

As in [6], the *upper topology* on a poset Q, generated by the complements of the principal ideals of Q, is denoted by $\upsilon(Q)$. A subset U of Q is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D with $\bigvee D$ existing, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of Q form a topology, called the *Scott topology* on Q and denoted by $\sigma(Q)$. The space $\Sigma Q = (Q, \sigma(Q))$ is called the *Scott space* of Q. The upper sets of Q form the (*upper*) *Alexandroff topology* $\alpha(Q)$.

The category of all T_0 spaces is denoted by $\mathbf{Top}_0$. For a subcategory $\mathbf{K}$ of the category $\mathbf{Top}_0$, the objects of $\mathbf{K}$ will be called $\mathbf{K}$-spaces. For $X \in \mathbf{Top}_0$, we use $\leq_X$ to denote the *specialization order* of X: $x \leq_X y$ iff $x \in \overline{\{y\}}$). In the following, when a T_0 space X is considered as a poset, the order always refers to the specialization order if no other explanation. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X, and let $\mathcal{S}^u(X) = \{\uparrow x : x \in X\}$. Define $\mathcal{S}_c(X) = \{\overline{\{x\}} : x \in X\}$ and $\mathcal{D}_c(X) = \{\overline{D} : D \in \mathcal{D}(X)\}$.

Remark 2.1. Let X be a T_0 space, $C \subseteq X$ and $x \in X$. Then the followings are equivalent:

(1) $x \in C^{\uparrow}$;
(2) $C \subseteq \downarrow x$;
(3) $\overline{C} \subseteq \downarrow x$;
(4) $x \in \overline{C}^{\uparrow}$.

Therefore,

$$\bigcap_{c \in C} \uparrow c = C^{\uparrow} = \overline{C}^{\uparrow} = \bigcap_{b \in \overline{C}} \uparrow b \text{ and } C^{\delta} = \bigcap\{\downarrow x : C \subseteq \downarrow x\} = \bigcap\{\downarrow x : \overline{C} \subseteq \downarrow x\} = \overline{C}^{\delta}.$$

For a T_0 space X and a nonempty subset A of X, A is *irreducible* if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X. Clearly, every subset of X that is directed under $\leq_X$ is irreducible. X is called *sober*, if for any $F \in \mathsf{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$. The category of all sober spaces and continuous mappings is denoted by $\mathbf{Sob}$.

The following two lemmas on irreducible sets are well-known.

Lemma 2.2. *Let X be a space and Y a subspace of X. Then the following conditions are equivalent for a subset $A \subseteq Y$:*

(1) *A is an irreducible subset of Y.*
(2) *A is an irreducible subset of X.*
(3) *$\mathrm{cl}_X A$ is an irreducible closed subset of X.*

Lemma 2.3. *If $f : X \longrightarrow Y$ is continuous and $A \in \mathsf{Irr}(X)$, then $f(A) \in \mathsf{Irr}(Y)$.*

Remark 2.4. If Y is a subspace of a space X and $A \subseteq Y$, then by Lemma 2.2, $\mathsf{Irr}(Y) = \{B \in \mathsf{Irr}(X) : B \subseteq Y\} \subseteq \mathsf{Irr}(X)$ and $\mathsf{Irr}_c(Y) = \{B \in \mathsf{Irr}(X) : B \in \mathcal{C}(Y)\} \subseteq \mathsf{Irr}(X)$. If $Y \in \mathcal{C}(X)$, then $\mathsf{Irr}_c(Y) = \{C \in \mathsf{Irr}_c(X) : C \subseteq Y\} \subseteq \mathsf{Irr}_c(X)$.

Lemma 2.5. ([23]) *Let $X = \prod_{i \in I} X_i$ be the product space of T_0 spaces $X_i (i \in I)$. If A is an irreducible subset of X, then $\mathrm{cl}_X(A) = \prod_{i \in I} \mathrm{cl}_{X_i}(p_i(A))$, where $p_i : X \longrightarrow X_i$ is the ith projection for each $i \in I$.*

Lemma 2.6. *Let $X = \prod_{i \in I} X_i$ be the product space of T_0 spaces $X_i (i \in I)$ and $A_i \subseteq X_i$ $(i \in I)$. Then the following two conditions are equivalent:*

(1) $\prod_{i\in I} A_i \in \mathsf{Irr}(X)$.
(2) $A_i \in \mathsf{Irr}(X_i)$ *for each* $i \in I$.

Proof. (1) $\Rightarrow$ (2): By Lemma 2.3.

(2) $\Rightarrow$ (1): Let $A = \prod_{i\in I} A_i$. For $U, V \in \mathcal{O}(X)$, if $A \cap U \neq \emptyset \neq A \cap V$, then there exist $I_1, I_2 \in I^{(<\omega)}$ and $(U_i, V_j) \in \mathcal{O}(X_i) \times \mathcal{O}(X_j)$ for all $(i,j) \in I_1 \times I_2$ such that $\bigcap_{i\in I_1} p_i^{-1}(U_i) \subseteq U$, $\bigcap_{j\in I_2} p_j^{-1}(V_j) \subseteq V$ and $A \cap \bigcap_{i\in I_1} p_i^{-1}(U_i) \neq \emptyset \neq A \cap \bigcap_{j\in I_2} p_i^{-1}(V_j)$. Let $I_3 = I_1 \cup I_2$. Then I_3 is finite. For $i \in I_3 \setminus I_1$ and $j \in I_3 \setminus I_2$, let $U_i = X_i$ and $V_j = X_j$. Then for each $i \in I_3$, we have $A_i \cap U_i \neq \emptyset \neq A_i \cap V_i$, and whence $A_i \cap U_i \cap V_i \neq \emptyset$ by $A_i \in \mathsf{Irr}(X_i)$. It follows that $A \cap \bigcap_{i\in I_1} p_i^{-1}(U_i) \cap \bigcap_{j\in I_2} p_i^{-1}(V_j) \neq \emptyset$, and consequently, $A \cap U \cap V \neq \emptyset$. Thus $A \in \mathsf{Irr}(X)$. $\square$

Applying Lemma 2.5 and Lemma 2.6, we obtain the following corollary.

Corollary 2.7. *Let* $X = \prod_{i\in I} X_i$ *be the product space of* T_0 *spaces* $X_i (i \in I)$. *If* $A \in \mathsf{Irr}_c(X)$, *then* $A = \prod_{i\in I} p_i(A)$ *and* $p_i(A) \in \mathsf{Irr}_c(X_i)$ *for each* $i \in I$.

A T_0 space X is called *irreducible complete*, *r-complete* for short, if for any $A \in \mathsf{Irr}(X)$, $\bigvee A$ exists in X. For a subset B of X, $\bigvee B$ exists in X iff $\bigvee \overline{B}$ exists in X, and $\bigvee B = \bigvee \overline{B}$ if they exist in X. So X is irreducible complete iff $\bigvee A$ exists in X for all $A \in \mathsf{Irr}_c(X)$.

Remark 2.8. Every sober space is irreducible complete. In fact, if X is a sober space and $A \in \mathsf{Irr}(X)$, then there is an $x \in X$ such that $\overline{A} = \overline{\{x\}}$, and hence $\bigvee A = \bigvee \overline{A} = \bigvee \overline{\{x\}} = x$.

Let L be the complete lattice constructed by Isbell [14]. Then ΣL is irreducible complete, but is non-sober.

Proposition 2.9. *For any poset* P, *the space* $(P, \upsilon(P))$ *is sober iff it is irreducible complete, where* $\upsilon(P)$ *is the upper topology on* P.

Proof. If the upper topology $\upsilon(P)$ is sober, then $(P, \upsilon(P))$ is irreducible complete by Remark 2.8. Conversely, if $(P, \upsilon(P))$ is irreducible complete, we show that $\upsilon(P)$ is sober. For $A \in \mathsf{Irr}((P, \upsilon(P)))$, if $\mathrm{cl}_{\upsilon(P)} A = P$, then P is irreducible in $(P, \upsilon(P))$ and hence has a largest element $\top$ since $(P, \upsilon(P))$ is irreducible complete. So $P = \downarrow \top = \overline{\{\top\}}$. If $\mathrm{cl}_{\upsilon(P)} A \neq P$, then there is a nonempty family $\{\downarrow F_i : i \in I\} \subseteq \mathbf{Fin}(P)$ such that $\mathrm{cl}_{\upsilon(P)} A = \bigcap_{i\in I} \downarrow F_i$. For each $i \in I$, $\mathrm{cl}_{\upsilon(P)} A \subseteq \downarrow F_i$, and hence $\mathrm{cl}_{\upsilon(P)} A \subseteq \downarrow x_i$ for some $x_i \in F_i$ by the irreducibility of A. Therefore, $\mathrm{cl}_{\upsilon(P)} A = \bigcap_{i\in I} \downarrow x_i \supseteq \bigcap\{\downarrow x : A \subseteq \downarrow x\} = A^\delta \supseteq \mathrm{cl}_{\upsilon(P)} A$. Since $(P, \upsilon(P))$ is irreducible complete, $\bigvee A$ exists in P, and consequently, $\mathrm{cl}_{\upsilon(P)} A = A^\delta = \downarrow \bigvee A = \overline{\{\bigvee A\}}$. Thus $\upsilon(P)$ is sober. $\square$

For any topological space X, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, let $\Diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \bigcap A \neq \emptyset\}$ and $\Box_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. The symbols $\Diamond_{\mathcal{G}} A$ and $\Box_{\mathcal{G}} A$ will be simply written as $\Diamond A$ and $\Box A$ respectively if there is no ambiguous occurrence. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \mathsf{Irr}(X)$, then $\{\Diamond_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$. The space $P_H(\mathcal{C}(X) \setminus \{\emptyset\})$ is called the *Hoare power space* or *lower space* of X and is denoted by $P_H(X)$ for short (cf. [22]). Clearly, $P_H(X) = (\mathcal{C}(X) \setminus \{\emptyset\}, \upsilon(\mathcal{C}(X) \setminus \{\emptyset\}))$. So $P_H(X)$ is always sober by Proposition 2.9 (or [29, Corollary 4.10]). The *upper Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Box_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ as a base, and the resulting space is denoted by $P_S(\mathcal{G})$.

Remark 2.10. Let X be a T_0 space.

(1) If $\mathcal{S}_c(X) \subseteq \mathcal{G}$, then the specialization order on $P_H(\mathcal{G})$ is the set inclusion order, and the *canonical mapping* $\eta_X : X \longrightarrow P_H(\mathcal{G})$, given by $\eta_X(x) = \overline{\{x\}}$, is an order and topological embedding (cf. [6,9,22]).
(2) The space $X^s = P_H(\mathsf{Irr}_c(X))$ with the canonical mapping $\eta_X : X \longrightarrow X^s$ is the *sobrification* of X (cf. [6,9]).

For a space X, a subset A of X is called *saturated* if A equals the intersection of all open sets containing it (equivalently, A is an upper set in the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X and endow it with the *Smyth preorder*, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. X is called *well-filtered* if it is T_0, and for any open set U and filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. The category of all well-filtered spaces with continuous mappings is denoted by $\mathbf{Top}_w$. The space $P_S(\mathsf{K}(X))$, denoted shortly by $P_S(X)$, is called the *Smyth power space* or *upper space* of X (cf. [10,22]). It is easy to verify that the specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \sqsubseteq$). The *canonical mapping* $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$, is an order and topological embedding (cf. [10,11,22]). Clearly, $P_S(\mathcal{S}^u(X))$ is a subspace of $P_S(X)$ and X is homeomorphic to $P_S(\mathcal{S}^u(X))$.

Lemma 2.11. *Let X be a T_0 space and $A \subseteq X$. Then the following three conditions are equivalent:*

(1) $A \in \mathsf{Irr}(X)$.
(2) $\xi_X(A) \in \mathsf{Irr}(P_S(X))$.
(3) $\xi_X(A) \in \mathsf{Irr}(P_S(\mathcal{S}^u(X)))$.

Moreover, the following two conditions are equivalent:

(a) $A \in \mathsf{Irr}_c(X)$.
(b) $\xi_X(A) \in \mathsf{Irr}_c(P_S(\mathcal{S}^u(X)))$.

Proof. (1) ⇒ (2): By Lemma 2.3.
(2) ⇒ (3): By Remark 2.4 and $P_S(\mathcal{S}^u(X)))$ is a subspace of $P_S(X)$.
(3) ⇒ (1) and (a) ⇔ (b): Since $x \mapsto \uparrow x : X \longrightarrow P_S(\mathcal{S}^u(X))$ is a homeomorphism. □

Remark 2.12. Let X be a T_0 space and $\mathcal{A} \subseteq \mathsf{K}(X)$. Then $\bigcap \mathcal{A} = \bigcap \overline{\mathcal{A}}$, here the closure of $\mathcal{A}$ is taken in $P_S(X)$. Clearly, $\bigcap \overline{\mathcal{A}} \subseteq \bigcap \mathcal{A}$. On the other hand, for any $K \in \overline{\mathcal{A}}$ and $U \in \mathcal{O}(X)$ with $K \subseteq U$ (that is, $K \in \Box U$), we have $\mathcal{A} \bigcap \Box U \neq \emptyset$, and hence there is a $K_U \in \mathcal{A} \bigcap \Box U$. Therefore, $K = \bigcap\{U \in \mathcal{O}(X) : K \subseteq U\} \supseteq \bigcap\{K_U : U \in \mathcal{O}(X) \text{ and } K \subseteq U\} \supseteq \bigcap \mathcal{A}$. It follows that $\bigcap \overline{\mathcal{A}} \supseteq \bigcap \mathcal{A}$. Thus $\bigcap \mathcal{A} = \bigcap \overline{\mathcal{A}}$.

Lemma 2.13. ([16,22]) *Let X be a T_0 space. If $\mathcal{K} \in \mathsf{K}(P_S(X))$, then $\bigcup \mathcal{K} \in \mathsf{K}(X)$.*

Corollary 2.14. ([16,22]) *For any T_0 space X, the mapping $\bigcup : P_S(P_S(X)) \longrightarrow P_S(X)$, $\mathcal{K} \mapsto \bigcup \mathcal{K}$, is continuous.*

Proof. For $\mathcal{K} \in \mathsf{K}(P_S(X))$, $\bigcup \mathcal{K} \in \mathsf{K}(X)$ by Lemma 2.13. For $U \in \mathcal{O}(X)$, we have $\bigcup^{-1}(\Box U) = \{\mathcal{K} \in \mathsf{K}(P_S(X)) : \bigcup \mathcal{K} \in \Box U\} = \{\mathcal{K} \in \mathsf{K}(P_S(X)) : \mathcal{K} \subseteq \Box U\} = \eta^{-1}_{P_S(X)}(\Box(\Box U)) \in \mathcal{O}(P_S(P_S(X)))$. Thus $\bigcup : P_S(P_S(X)) \longrightarrow P_S(X)$ is continuous. □

As in [4], a topological space X is *locally hypercompact* if for each $x \in X$ and each open neighborhood U of x, there is $\uparrow F \in \mathbf{Fin}X$ such that $x \in \operatorname{int} \uparrow F \subseteq \uparrow F \subseteq U$. A space X is called a *C-space* if for each $x \in X$ and each open neighborhood U of x, there is $u \in X$ such that $x \in \operatorname{int} \uparrow u \subseteq \uparrow u \subseteq U$. A set $K \subseteq X$ is called *supercompact* if for any family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, $K \subseteq \bigcup_{i \in I} U_i$ implies $K \subseteq U_i$ for some $i \in I$. It is easy to check that the supercompact saturated sets of X are exactly the sets $\uparrow x$ with $x \in X$ (see [11, Fact 2.2]).

It is well-known that X is a C-space iff $\mathcal{O}(X)$ is a *completely distributive* lattice (cf. [2]). A space X is called *core compact* if $(\mathcal{O}(X), \subseteq)$ is a *continuous lattice* (cf. [6]).

Theorem 2.15. ([6]) *Let X be a sober space. Then X is locally compact iff X is core compact.*

For a T_0 space X and a nonempty subset C of X, it is easy to see that C is compact iff $\uparrow C \in \mathsf{K}(X)$. The following result is well-known (see, e.g., [2, pp. 2068]).

Lemma 2.16. *Let X be a T_0 space and $C \in \mathsf{K}(X)$. Then $C = \uparrow min(C)$ and $min(C)$ is compact.*

For a T_0 space $X, \mathcal{U} \subseteq \mathcal{O}(X)$ is called an *open filter* if $\mathcal{U} \in \sigma(\mathcal{O}(X)) \bigcap \mathrm{Filt}(\mathcal{O}(X))$. For $K \in \mathsf{K}(X)$, let $\Phi(K) = \{U \in \mathcal{O}(X) : K \subseteq U\}$. Then $\Phi(K) \in \mathrm{OFilt}(\mathcal{O}(\mathrm{X}))$ and $K = \bigcap \Phi(K)$. Obviously, $\Phi : \mathsf{K}(X) \longrightarrow \mathrm{OFilt}(\mathcal{O}(\mathrm{X})), K \mapsto \Phi(K)$, is an order embedding. It is well-known that Φ is an order isomorphism iff X is sober (see [6,13] or Theorem 5.8 in this paper).

3. *d*-spaces and directed closure spaces

In this section, we give some equational characterizations of d-spaces. Based on directed sets, we introduce the concept of directed closure spaces, and discuss some basic properties of them.

A T_0 space X is called a *d-space* (or *monotone convergence space*) if X (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. [6,25]).

Definition 3.1. A T_0 space X is called *directed bounded*, *d-bounded* for short, if for any $D \in \mathcal{D}(X)$, D has an upper bound in X, that is, there is an $x \in X$ such that $D \subseteq \downarrow x = \overline{\{x\}}$.

Clearly, we have the following implications:

$$d\text{-space} \Rightarrow \text{directed completeness} \Rightarrow d\text{-boundedness}.$$

For a poset P with a largest element $\top$, any *order compatible* topology τ on P (that is, $\leq_\tau$ agrees with the original order on P) is d-bounded.

Proposition 3.2. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is d-bounded.*
(2) *For any $D \in \mathcal{D}(X)$, $D^\uparrow = \bigcap_{d \in D} \uparrow d \neq \emptyset$.*
(3) *For any $D \in \mathcal{D}(X)$ and $A \in \mathcal{C}(X)$, if $D \subseteq A$, then $\bigcap_{d \in D} \uparrow (A \cap \uparrow d) \neq \emptyset$.*
(4) *For any $D \in \mathcal{D}(X)$ and $A \in \mathsf{Irr}_c(X)$, if $D \subseteq A$, then $\bigcap_{d \in D} \uparrow (A \cap \uparrow d) \neq \emptyset$.*

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (4): Trivial.

(2) $\Rightarrow$ (3): $\emptyset \neq \bigcap_{d \in D} \uparrow d \subseteq \bigcap_{d \in D} \uparrow (A \cap \uparrow d)$.

(4) $\Rightarrow$ (2): Since $D \in \mathcal{D}(X)$, $\overline{D} \in \mathsf{Irr}_c(X)$. By condition (4), $\bigcap_{d \in D} \uparrow d = \bigcap_{d \in D} \uparrow (\overline{D} \cap \uparrow d) \neq \emptyset$. $\square$

Proposition 3.3. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is a d-space.*
(2) *$\mathcal{D}_c(X) = \mathcal{S}_c(X)$.*
(3) *For any $D \in \mathcal{D}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{d \in D} \uparrow d \subseteq U$ implies $\uparrow d \subseteq U$ (i.e., $d \in U$) for some $d \in D$.*

(4) *For any filtered family* $\mathcal{K} \subseteq \mathcal{S}^u(X)$ *and* $U \in \mathcal{O}(X)$, $\bigcap \mathcal{K} \subseteq U$ *implies* $K \subseteq U$ *for some* $K \in \mathcal{K}$.
(5) *For any* $D \in \mathcal{D}(X)$ *and* $A \in \mathcal{C}(X)$, *if* $D \subseteq A$, *then* $A \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$.
(6) *For any* $D \in \mathcal{D}(X)$ *and* $A \in \mathsf{Irr}_c(X)$, *if* $D \subseteq A$, *then* $A \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$.
(7) *For any* $D \in \mathcal{D}(X)$, $\overline{D} \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$.

Proof. (1) $\Leftrightarrow$ (2): Clearly, (1) $\Rightarrow$ (2). Conversely, if condition (2) holds, then for each $D \in \mathcal{D}(X)$ and $A \in \mathcal{C}(X)$ with $D \subseteq A$, there is $x \in X$ such that $\overline{D} = \overline{\{x\}}$, and consequently, $\bigvee D = x$ and $\bigvee D \in A$ since $\overline{D} \subseteq A$. Thus X is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$, proving X is a d-space.

(1) $\Rightarrow$ (3): Since X is a d-space, $\uparrow \bigvee D = \bigcap_{d\in D} \uparrow d \subseteq U \in \sigma(X)$. Therefore, $\bigvee D \in U$, and whence $d \in U$ for some $d \in D$.

(3) $\Leftrightarrow$ (4): For $\mathcal{K} = \{\uparrow x_i : i \in I\} \subseteq \mathcal{S}^u(X)$, $\mathcal{K}$ is filtered in $\mathcal{S}^u(X)$ with the Smyth order iff $\{x_i : i \in I\} \in \mathcal{D}(X)$.

(3) $\Rightarrow$ (5): If $A \cap \bigcap_{d\in D} \uparrow d = \emptyset$, then $\bigcap_{d\in D} \uparrow d \subseteq X \setminus A$. By condition (3), $\uparrow d \subseteq X \setminus A$ for some $d \in D$, which is in contradiction with $D \subseteq A$.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7): Trivial.

(7) $\Rightarrow$ (1): For each $D \in \mathcal{D}(X)$ and $A \in \mathcal{C}(X)$ with $D \subseteq A$, by condition (7), $\overline{D} \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$. Select an $x \in \overline{D} \cap \bigcap_{d\in D} \uparrow d$. Then $D \subseteq \downarrow x \subseteq \overline{D}$, and hence $\overline{D} = \downarrow x$ and $\bigvee D = x$. Therefore, $\bigvee D \in A$ because $\overline{\{\bigvee D\}} = \overline{D} \subseteq A$. Thus X is a d-space. $\square$

In the following, we shall give some equational characterizations of d-spaces.

Proposition 3.4. *For a* T_0 *space* X, *the following conditions are equivalent:*

(1) X *is a* d*-space.*
(2) X *is* d*-bounded (especially,* X *is a dcpo), and* $\uparrow\left(A \cap \bigcap_{d\in D} \uparrow d\right) = \bigcap_{d\in D} \uparrow(A \cap \uparrow d)$ *for any* $D \in \mathcal{D}(X)$ *and* $A \in \mathcal{C}(X)$.
(3) X *is* d*-bounded (especially,* X *is a dcpo), and* $\uparrow(A \cap \bigcap \mathcal{K}) = \bigcap_{K\in\mathcal{K}} \uparrow(A \cap K)$ *for any filtered family* $\mathcal{K} \subseteq \mathcal{S}^u(X)$ *and* $A \in \mathcal{C}(X)$.
(4) X *is* d*-bounded (especially,* X *is a dcpo), and* $\uparrow\left(A \cap \bigcap_{d\in D} \uparrow d\right) = \bigcap_{d\in D} \uparrow(A \cap \uparrow d)$ *for any* $D \in \mathcal{D}(X)$ *and* $A \in \mathsf{Irr}_c(X)$.
(5) X *is* d*-bounded (especially,* X *is a dcpo), and* $\uparrow(A \cap \bigcap \mathcal{K}) = \bigcap_{K\in\mathcal{K}} \uparrow(A \cap K)$ *for any filtered family* $\mathcal{K} \subseteq \mathcal{S}^u(X)$ *and* $A \in \mathsf{Irr}_c(X)$.

Proof. (1) $\Rightarrow$ (2): Since X is a d-space, X is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$. For $D \in \mathcal{D}(X)$ and $A \in \mathcal{C}(X)$, clearly, $\uparrow\left(A \cap \bigcap_{d\in D} \uparrow d\right) \subseteq \bigcap_{d\in D} \uparrow(A \cap \uparrow d)$. Conversely, if $x \notin \uparrow\left(A \cap \bigcap_{d\in D} \uparrow d\right)$, that is, $\downarrow x \cap A \cap \bigcap_{d\in D} \uparrow d = \emptyset$, then $\uparrow \bigvee D = \bigcap_{d\in D} \uparrow d \subseteq X \setminus (\downarrow x \cap A) \in \sigma(X)$, and whence $d \in X \setminus (\downarrow x \cap A)$ for some $d \in D$, i.e., $x \notin \uparrow(A \cap \uparrow d)$. Therefore, $x \notin \bigcap_{d\in D} \uparrow(A \cap \uparrow d)$. Thus $\uparrow\left(A \cap \bigcap_{d\in D} \uparrow d\right) = \bigcap_{d\in D} \uparrow(A \cap \uparrow d)$.

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5): For $\mathcal{K} = \{\uparrow x_i : i \in I\} \subseteq \mathcal{S}^u(X)$, $\mathcal{K}$ is filtered in $\mathcal{S}^u(X)$ with the Smyth order iff $\{x_i : i \in I\} \in \mathcal{D}(X)$.

(2) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): For each $D \in \mathcal{D}(X)$, by condition (4), $\emptyset \neq D^{\uparrow} = \bigcap_{d\in D} \uparrow d = \bigcap_{d\in D} \uparrow(\overline{D} \cap \uparrow d) = \uparrow\left(\overline{D} \cap \bigcap_{d\in D} \uparrow d\right)$. By Proposition 3.3, X is a d-space. □

Theorem 3.5. *Let X be a T_0 space and $\mathbf{K}$ a full subcategory of $\mathbf{Top}_0$ containing $\mathbf{Sob}$. Then the following conditions are equivalent:*

(1) *X is a d-space.*
(2) *For every continuous mapping $f : X \longrightarrow Y$ from X to a T_0 space Y and any $D \in \mathcal{D}(X)$, $\uparrow f\left(\bigcap_{d\in D} \uparrow d\right) = \bigcap_{d\in D} \uparrow f(\uparrow d) = \bigcap_{d\in D} \uparrow f(d)$.*
(3) *For every continuous mapping $f : X \longrightarrow Y$ from X to a $\mathbf{K}$-space Y and any $D \in \mathcal{D}(X)$, $\uparrow f\left(\bigcap_{d\in D} \uparrow d\right) = \bigcap_{d\in D} \uparrow f(\uparrow d) = \bigcap_{d\in D} \uparrow f(d)$.*
(4) *For every continuous mapping $f : X \longrightarrow Y$ from X to a sober space Y and any $D \in \mathcal{D}(X)$, $\uparrow f\left(\bigcap_{d\in D} \uparrow d\right) = \bigcap_{d\in D} \uparrow f(\uparrow d) = \bigcap_{d\in D} \uparrow f(d)$.*

Proof. (1) $\Rightarrow$ (2): Since X is a d-space and f is order-preserving, we have $\uparrow f\left(\bigcap_{d\in D} \uparrow d\right) = \uparrow f(\uparrow \bigvee D) = \uparrow f(\bigvee D)$. Obviously, $\uparrow f(\bigvee D) \subseteq \bigcap_{d\in D} \uparrow f(d)$. On the other hand, if $y \in \bigcap_{d\in D} \uparrow f(d)$, then $d \in f^{-1}(\downarrow y) = f^{-1}(\overline{\{y\}}) \in \mathcal{C}(X) \subseteq \mathcal{C}(\Sigma X)$ for all $d \in D$, and hence $\bigvee D \in f^{-1}(\downarrow y)$, that is, $y \in \uparrow f(\bigvee D)$. Thus $f(\bigvee D) = \bigvee f(D)$, and whence $\uparrow f\left(\bigcap_{d\in D} \uparrow d\right) = \uparrow f(\bigvee D) = \uparrow \bigvee f(D) = \bigcap_{d\in D} \uparrow f(\uparrow d) = \bigcap_{d\in D} \uparrow f(d)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Let $\eta_X : X \to X^s$ $(= P_H(\mathsf{Irr}_c(X)))$ be the canonical topological embedding from X into its sobrification. For $D \in \mathcal{D}(X)$, by condition (4) we have $\overline{D} \in \bigcap_{d\in D} \uparrow_{\mathsf{Irr}_c(X)} \eta_X(d) = \bigcap_{d\in D} \uparrow_{\mathsf{Irr}_c(X)} \eta_X(\uparrow d) = \uparrow_{\mathsf{Irr}_c(X)} \eta_X\left(\bigcap_{d\in D} \uparrow d\right) = \uparrow_{\mathsf{Irr}_c(X)} \eta_X(D^{\uparrow})$, and whence there is a $c \in D^{\uparrow}$ such that $\overline{\{c\}} \subseteq \overline{D}$. Therefore, $\overline{D} = \overline{\{c\}}$. By Proposition 3.3, X is a d-space. □

Corollary 3.6. *Let X be a T_0 space and $\mathbf{K}$ a full subcategory of $\mathbf{Top}_0$ containing $\mathbf{Sob}$. Then the following conditions are equivalent:*

(1) *X is a d-space.*
(2) *X is a dcpo, and for every continuous mapping $f : X \longrightarrow Y$ from X to a T_0 space Y and any $D \in \mathcal{D}(X)$, $f(\bigvee D) = \bigvee f(D)$.*
(3) *X is a dcpo, and for every continuous mapping $f : X \longrightarrow Y$ from X to a $\mathbf{K}$-space Y and any $D \in \mathcal{D}(X)$, $f(\bigvee D) = \bigvee f(D)$.*
(4) *X is a dcpo, and for every continuous mapping $f : X \longrightarrow Y$ from X to a sober space Y and any $D \in \mathcal{D}(X)$, $f(\bigvee D) = \bigvee f(D)$.*

Proof. (1) $\Rightarrow$ (2): Since X is a d-space, X is a dcpo. By the proof of (1) $\Rightarrow$ (2) in Theorem 3.5, $f(\bigvee D) = \bigvee f(D)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): For every continuous mapping $f : X \longrightarrow Y$ from X to a sober space Y and any $D \in \mathcal{D}(X)$, by condition (4) we have $\uparrow f\left(\bigcap_{d\in D} \uparrow d\right) = \uparrow f(\bigvee D) = \uparrow \bigvee f(D) = \bigcap_{d\in D} \uparrow f(d) = \bigcap_{d\in D} \uparrow f(\uparrow d)$, and whence by Theorem 3.5, X is a d-space. □

Definition 3.7. A T_0 space X is called a *directed closure space*, DC *space* for short, if $\mathsf{Irr}_c(X) = \mathcal{D}_c(X)$, that is, for each $A \in \mathsf{Irr}_c(X)$, there exists a directed subset of X such that $A = \overline{D}$.

The following result follows directly from the definition of DC spaces.

Proposition 3.8. *A closed subspace of a* DC *space is a* DC *space.*

Lemma 3.9. *If $f : X \longrightarrow Y$ is continuous and $A \in \mathcal{D}_c(X)$, then $\overline{f(A)} \in \mathcal{D}_c(Y)$.*

Proof. Since $A \in \mathcal{D}_c(X)$, there is a $D \in \mathcal{D}(X)$ such that $A = \overline{D}$, and whence $f(D) \in \mathcal{D}(Y)$ and $\overline{f(A)} = \overline{f(\overline{D})} = \overline{f(D)} \in \mathcal{D}_c(Y)$. □

Proposition 3.10. *A retract of a* DC *space is a* DC *space.*

Proof. Assume X is a DC space and Y is a retract of X. Then there are continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = \mathrm{id}_Y$. Let $B \in \mathsf{Irr}_c(Y)$. Then $\overline{g(B)} \in \mathsf{Irr}_c(X)$ by Lemma 2.3. Since X is a DC space, $\overline{g(B)} \in D_c(X)$. By Lemma 3.9, $B = \overline{fg(B)} = \overline{f\left(\overline{g(B)}\right)} \in \mathcal{D}_c(Y)$. Therefore, Y is a DC space. □

By Corollary 2.7 and Lemma 3.9, we get the following result.

Proposition 3.11. *Let $\{X_i : i \in I\}$ be a family of T_0 spaces and $X = \prod_{i\in I} X_i$. If $A \in \mathcal{D}_c(X)$, then $A = \prod_{i\in I} p_i(A)$, and $p_i(A) \in \mathcal{D}_c(X_i)$ for each $i \in I$.*

Proposition 3.12. *Let $X = \prod_{i\in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of T_0 spaces and $A_i \subseteq X_i$ for each $i \in I$. Then the following two conditions are equivalent:*

(1) $\prod_{i\in I} A_i \in \mathcal{D}_c(X)$.
(2) $A_i \in \mathcal{D}_c(X_i)$ *for each* $i \in I$.

Proof. (1) ⇒ (2): By Proposition 3.11.

(2) ⇒ (1): For each $i \in I$, by $A_i \in \mathcal{D}_c(X_i)$, there is a $D_i \in \mathcal{D}(X_i)$ such that $A_i = \mathrm{cl}_{X_i} D_i$. Let $D = \prod_{i\in I} D_i$. Then $D \in \mathcal{D}(X)$. By [1, Proposition 2.3.3], $\prod_{i\in I} A_i = \prod_{i\in I} \mathrm{cl}_{X_i} D_i = \mathrm{cl}_X \prod_{i\in I} D_i = \mathrm{cl}_X D \in \mathcal{D}_c(X)$. □

Corollary 3.13. *Let $\{X_i : i \in I\}$ be a family of T_0 spaces. Then the following two conditions are equivalent:*

(1) *The product space* $\prod_{i\in I} X_i$ *is a* DC *space.*
(2) *For each* $i \in I$, X_i *is a* DC *space.*

Proof. (1) ⇒ (2): For each $i \in I$, X_i is a retract of $\prod_{i\in I} X_i$. By Proposition 3.10, X_i is a DC space.

(2) ⇒ (1): Let $X = \prod_{i\in I} X_i$. Suppose $A \in \mathsf{Irr}_c(X)$. Then for each $i \in I$, by Corollary 2.7, $A = \prod_{i\in I} p_i(A)$ and $p_i(A) \in \mathsf{Irr}_c(X_i)$, and whence $p_i(A) \in \mathcal{D}_c(X_i)$ because X_i is a DC space. By Proposition 3.12, $A = \prod_{i\in I} p_i(A) \in \mathcal{D}_c(X)$. Thus X is a DC space. □

4. Rudin's Lemma and Rudin spaces

Rudin's Lemma is a useful tool in topology and plays a crucial role in domain theory (see [3–12,18,23,28]). Rudin [21] proved her lemma by transfinite methods, using the Axiom of Choice. Heckmann and Keimel [11] presented the following topological variant of Rudin's Lemma.

Lemma 4.1 *(Topological Rudin's Lemma). Let X be a topological space and $\mathcal{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{A}$.*

Corollary 4.2. *Let X be a T_0 space. If $\mathcal{A} \in \mathsf{Irr}_c(P_S(X))$, then there exists a family $\{A_i : i \in I\}$ of minimal irreducible closed sets such that $\mathcal{A} = \bigcap_{i\in I} \Diamond A_i$.*

Applying Lemma 4.1 to the Alexandroff topology on a poset P, one obtains the original Rudin's Lemma.

Corollary 4.3 *(Rudin's Lemma). Let P be a poset, C a nonempty lower subset of P and $\mathcal{F} \in \mathbf{Fin}P$ a filtered family with $\mathcal{F} \subseteq \Diamond C$. Then there exists a directed subset D of C such that $\mathcal{F} \subseteq \Diamond{\downarrow} D$.*

For a T_0 space X and $\mathcal{K} \subseteq \mathsf{K}(X)$, let $M(\mathcal{K}) = \{A \in \mathcal{C}(X) : K \bigcap A \neq \emptyset \text{ for all } K \in \mathcal{K}\}$ (that is, $\mathcal{K} \subseteq \Diamond A$) and $m(\mathcal{K}) = \{A \in \mathcal{C}(X) : A \text{ is a minimal member of } M(\mathcal{K})\}$.

By the proof of [11, Lemma 3.1], we have the following result.

Lemma 4.4. *Let X be a T_0 space and $\mathcal{K} \subseteq \mathsf{K}(X)$. If $C \in M(\mathcal{K})$, then there is a closed subset A of C such that $C \in m(\mathcal{K})$.*

The following result shows that the reverse of Lemma 4.1 holds.

Lemma 4.5. *Let X be a T_0 space and $\mathcal{A}$ a nonempty subset of $P_S(X)$. Then the following conditions are equivalent:*

(1) *$\mathcal{A}$ is irreducible;*
(2) *$\forall A \in \mathcal{C}(X)$, if $\mathcal{A} \subseteq \Diamond A$, then there exists a minimal irreducible closed set $C \subseteq A$ such that $\mathcal{A} \subseteq \Diamond C$.*

Proof. (1) ⇒ (2): By Lemma 4.1.

(2) ⇒ (1): Let $\mathcal{A} \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$ with $\{\mathcal{A}_1, \mathcal{A}_2\} \subseteq \mathcal{C}(P_S(X))$. Then there exists $\{A_i : i \in I\} \subseteq \mathcal{C}(X)$ and $\{B_j : j \in J\} \subseteq \mathcal{C}(X)$ such that $\mathcal{A}_1 = \bigcap_{i\in I} \Diamond A_i$ and $\mathcal{A}_2 = \bigcap_{j\in J} \Diamond B_j$. Suppose, on the contrary, $\mathcal{A} \nsubseteq \mathcal{A}_1$ and $\mathcal{A} \nsubseteq \mathcal{A}_2$. Then there exists $(i,j) \in I \times J$ such that $\mathcal{A} \nsubseteq \Diamond \mathcal{A}_i$ and $\mathcal{A} \nsubseteq \Diamond B_j$. Note that $\mathcal{A}_1 \subseteq \Diamond \mathcal{A}_i$ and $\mathcal{A}_2 \subseteq \Diamond B_j$, so $\mathcal{A} \subseteq \Diamond A_i \cup \Diamond B_j = \Diamond(A_i \cup B_j)$. By (2), there exists a minimal irreducible closed set $C \subseteq A_i \cup B_j$ such that $\mathcal{A} \subseteq \Diamond C$. This implies that $C \subseteq A_i$ or $C \subseteq B_j$, so $\mathcal{A} \subseteq \Diamond C \subseteq \Diamond A_i$ or $\mathcal{A} \subseteq \Diamond C \subseteq \Diamond B_j$, a contradiction. Therefore $\mathcal{A} \subseteq \mathcal{A}_1$ or $\mathcal{A} \subseteq \mathcal{A}_2$. Thus $\mathcal{A}$ is irreducible. □

In the following, based on topological Rudin's Lemma, we introduce and investigate a new type of spaces — Rudin spaces, which lie between DC spaces and sober spaces. It is proved that closed subspaces, retracts and products of Rudin spaces are again Rudin spaces.

Definition 4.6. ([23]) Let X be a T_0 space. A nonempty subset A of X is said to have the *Rudin property*, if there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\mathsf{RD}(X) = \{A \in \mathcal{C}(X) : A \text{ has Rudin property}\}$. The sets in $\mathsf{RD}(X)$ will also be called *Rudin sets*.

The Rudin property is called the *compactly filtered property* in [23]. In order to emphasize its origin from (topological) Rudin's Lemma, here we call such a property the Rudin property. Clearly, A has Rudin property iff $\overline{A}$ has Rudin property (that is, $\overline{A}$ is a Rudin set).

Definition 4.7. A T_0 space X is called a *Rudin space*, RD *space* for short, if $\mathsf{Irr}_c(X) = \mathsf{RD}(X)$, that is, every irreducible closed set of X is a Rudin set. The category of all Rudin spaces with continuous mappings is denoted by $\mathbf{Top}_r$.

Lemma 4.8. ([23]) *Let X be a T_0 space. Then $\mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{Irr}_c(X)$.*

Proof. By Lemma 4.1 we have $\mathsf{RD}(X) \subseteq \mathsf{Irr}_c(X)$. Now we prove that the closure of a directed subset D of X is a Rudin set. Let $\mathcal{K}_D = \{\uparrow d : d \in D\}$. Then $\mathcal{K}_D \subseteq \mathsf{K}(X)$ is filtered and $\overline{D} \in M(\mathcal{K}_D)$. If $A \in M(\mathcal{K}_D)$, then $d \in A$ for every $d \in D$, and hence $\overline{D} \subseteq A$. So $\overline{D} \in m(\mathcal{K}_D)$. Therefore $\overline{D} \in \mathsf{RD}(X)$. $\square$

Proposition 4.9. *A closed subspace of a Rudin space is a Rudin space.*

Proof. Let X be a Rudin space and $A \in \mathcal{C}(X)$. For $B \in \mathsf{Irr}_c(A)$, we have $B \in \mathsf{Irr}_c(X)$ by Lemma 2.2. Since X be a Rudin space, there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $B \in m(\mathcal{K})$. Let $\mathcal{K}_B = \{\uparrow_A(K \cap B) : K \in \mathcal{K}\}$. Then $\mathcal{K}_B \subseteq \mathsf{K}(A)$ is filtered. For each $K \in \mathcal{K}$, since $K \cap B \neq \emptyset$, we have $\emptyset \neq K \cap B \subseteq \uparrow_A(K \cap B) \cap B$. So $B \in M(\mathcal{K}_B)$. If C is a closed subset of B with $C \in M(\mathcal{K}_B)$, then $C \cap \uparrow_A(K \cap B) \neq \emptyset$ for every $K \in \mathcal{K}$. So $K \cap B \cap C = K \cap C \neq \emptyset$ for all $K \in \mathcal{K}$. It follows that $B = C$ by the minimality of B, and consequently, $B \in m(\mathcal{K}_B)$. Whence A is a Rudin space. $\square$

Lemma 4.10. ([23]) *Let X, Y be two T_0 spaces and $f : X \longrightarrow Y$ a continuous mapping. If $A \in \mathsf{RD}(X)$, then $\overline{f(A)} \in \mathsf{RD}(Y)$.*

Proof. This lemma has been proved in [23]. Here we give a more direct proof. Since $A \in \mathsf{RD}(X)$, there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $A \in m(\mathcal{K})$. Let $\mathcal{K}_f = \{\uparrow f(K \cap A) : K \in \mathcal{K}\}$. Then $\mathcal{K}_f \subseteq \mathsf{K}(Y)$ is filtered. For each $K \in \mathcal{K}$, since $K \cap A \neq \emptyset$, we have $\emptyset \neq f(K \cap A) \subseteq \uparrow f(K \cap A) \cap \overline{f(A)}$. So $\overline{f(A)} \in M(\mathcal{K}_f)$. If B is a closed subset of $\overline{f(A)}$ with $B \in M(\mathcal{K}_f)$, then $B \cap \uparrow f(K \cap A) \neq \emptyset$ for every $K \in \mathcal{K}$. So $K \cap A \cap f^{-1}(B) \neq \emptyset$ for all $K \in \mathcal{K}$. It follows that $A = A \cap f^{-1}(B)$ by the minimality of A, and consequently, $\overline{f(A)} \subseteq B$. Therefore, $\overline{f(A)} = B$. Thus $\overline{f(A)} \in \mathsf{RD}(Y)$. $\square$

Corollary 4.11. *A retract of a Rudin space is a Rudin space.*

Proof. Suppose that Y is a retract of a Rudin space X. Then there are continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = id_Y$. Let $B \in \mathsf{Irr}_c(Y)$, then by Lemma 2.3, $\overline{f(\overline{g(B)})} \in \mathsf{Irr}_c(Y)$. Since X is a Rudin space, $\overline{g(B)} \in \mathsf{RD}(X)$. By Lemma 4.10, $B = \overline{f \circ g(B)} = \overline{f(\overline{g(B)})} \in \mathsf{RD}(Y)$. Thus Y is a Rudin space. $\square$

Proposition 4.12. *Let X be a T_0 space and Y a well-filtered space. If $f : X \longrightarrow Y$ is continuous and $A \subseteq X$ has Rudin property, then there exists a unique $y_A \in X$ such that $\overline{f(A)} = \overline{\{y_A\}}$.*

Proof. Since A has Rudin property, there exists a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$. Let $\mathcal{K}_f = \{\uparrow f(K \cap \overline{A}) : K \in \mathcal{K}\}$. Then $\mathcal{K}_f \subseteq \mathsf{K}(Y)$ is filtered. By the proof of Lemma 4.10, $\overline{f(A)} \in m(\mathcal{K}_f)$. Since Y is well-filtered, we have $\bigcap_{K \in \mathcal{K}} \uparrow f(K \cap \overline{A}) \cap \overline{f(A)} \neq \emptyset$. Select a $y_A \in \bigcap_{K \in \mathcal{K}} \uparrow f(K \cap \overline{A}) \cap \overline{f(A)}$. Then $\overline{\{y_A\}} \subseteq \overline{f(A)}$ and $K \cap \overline{A} \cap f^{-1}(\overline{\{y_A\}}) \neq \emptyset$ for all $K \in \mathcal{K}$. It follows that $\overline{A} = \overline{A} \cap f^{-1}(\overline{\{y_A\}})$ by the minimality of $\overline{A}$, and consequently, $\overline{f(A)} \subseteq \overline{\{y_A\}}$. Therefore, $\overline{f(A)} = \overline{\{y_A\}}$. The uniqueness of y_A follows from the T_0 separation of Y. $\square$

Lemma 4.13. ([23]) *Let $X = \prod_{i \in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of T_0 spaces and $A \in \mathsf{Irr}(X)$. Then the following conditions are equivalent:*

(1) *A is a Rudin set.*
(2) *$p_i(A)$ is a Rudin set for each $i \in I$.*

Theorem 4.14. *Let $\{X_i : i \in I\}$ be a family of T_0 spaces. Then the following two conditions are equivalent:*

(1) *The product space $\prod_{i\in I} X_i$ is a Rudin space.*
(2) *For each $i \in I$, X_i is a Rudin space.*

Proof. (1) ⇒ (2): For each $i \in I$, X_i is a retract of $\prod_{i\in I} X_i$. By Corollary 4.11, X_i is a Rudin space.

(2) ⇒ (1): Suppose $A \in \mathsf{Irr}_c(\prod_{i\in I} X_i)$. Then for each $i \in I$, since X_i is a Rudin space, $p_i(A) \in \mathsf{RD}(X_i)$ by Corollary 2.7, and consequently, by Corollary 2.7 and Lemma 4.13, $A = \prod_{i\in I} p_i(A) \in \mathsf{RD}(\prod_{i\in I} X_i)$. Therefore, $\prod_{i\in I} X_i$ is a Rudin space. □

5. Well-filtered spaces and sober spaces

In this section, we formulate and prove some equational characterizations of well-filtered spaces and sober spaces.

Theorem 5.1. *Let X be a T_0 space and $\mathbf{K}$ a full subcategory of $\mathbf{Top}_0$ containing $\mathbf{Sob}$. Then the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *For every continuous mapping $f : X \longrightarrow Y$ from X to a T_0 space Y and a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap\limits_{K\in\mathcal{K}} \uparrow f(K)$.*
(3) *For every continuous mapping $f : X \longrightarrow Y$ from X to a $\mathbf{K}$-space Y and a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap\limits_{K\in\mathcal{K}} \uparrow f(K)$.*
(4) *For every continuous mapping $f : X \longrightarrow Y$ from X to a sober space Y and a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap\limits_{K\in\mathcal{K}} \uparrow f(K)$.*

Proof. (1) ⇒ (2): It is proved in [5] for sober spaces and the proof is valid for well-filtered spaces (see [5, Lemma 8.1]). For the sake of completeness, we present the proof here. It needs only to check $\bigcap\limits_{K\in\mathcal{K}} \uparrow f(K) \subseteq \uparrow f\left(\bigcap \mathcal{K}\right)$. Let $y \in \bigcap\limits_{K\in\mathcal{K}} \uparrow f(K)$. Then for each $K \in \mathcal{K}$, $\overline{\{y\}} \cap f(K) \neq \emptyset$, that is, $K \cap f^{-1}\left(\overline{\{y\}}\right) \neq \emptyset$. Since X is well-filtered, $f^{-1}\left(\overline{\{y\}}\right) \cap \bigcap \mathcal{K} \neq \emptyset$ (otherwise, $\bigcap \mathcal{K} \subseteq X \setminus f^{-1}\left(\overline{\{y\}}\right)$, which implies that $K \subseteq X \setminus f^{-1}\left(\overline{\{y\}}\right)$ for some $K \in \mathcal{K}$, a contradiction). It follows that $\overline{\{y\}} \cap f\left(\bigcap \mathcal{K}\right) \neq \emptyset$. This implies that $y \in \uparrow f\left(\bigcap \mathcal{K}\right)$. So $\bigcap\limits_{K\in\mathcal{K}} \uparrow f(K) \subseteq \uparrow f\left(\bigcap \mathcal{K}\right)$.

(2) ⇒ (3) ⇒ (4): Trivial.

(4) ⇒ (1): Let $\eta_X : X \rightarrow X^s$ $(= P_H(\mathsf{Irr}_c(X)))$ be the canonical topological embedding from X into its sobrification. Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. If $K \not\subseteq U$ for each $K \in \mathcal{K}$, then by Lemma 4.1, $X \setminus U$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{K}$. By condition (4) we have $\uparrow_{\mathsf{Irr}_c(X)} \eta_X\left(\bigcap \mathcal{K}\right) = \bigcap\limits_{K\in\mathcal{K}} \uparrow_{\mathsf{Irr}_c(X)} \eta_X(K) \subseteq \uparrow_{\mathsf{Irr}_c(X)} \eta_X(U) = \Diamond_{\mathsf{Irr}_c(X)} U$. Clearly, $A \in \bigcap\limits_{K\in\mathcal{K}} \uparrow_{\mathsf{Irr}_c(X)} \eta_X(K)$, and whence $A \in \Diamond_{\mathsf{Irr}_c(X)} U$, that is, $A \cap U \neq \emptyset$, being in contradiction with $A \subseteq X \setminus U$. Thus X is well-filtered. □

In the above theorem, we can let $\mathbf{K}$ be the category of all d-spaces or that of all well-filtered spaces.

Lemma 5.2. ([6]) *For a nonempty family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, $\bigvee_{i\in I} K_i$ exists in $\mathsf{K}(X)$ iff $\bigcap_{i\in I} K_i \in \mathsf{K}(X)$. In this case $\bigvee_{i\in I} K_i = \bigcap_{i\in I} K_i$.*

For the well-filteredness of Smyth power space, we now prove a similar result to that of sobriety in Theorem 5.11. The following result has been first proved in [28]. The proof we present here is simpler.

Theorem 5.3. *For a T_0 space, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *$P_S(X)$ is a d-space.*
(3) *$P_S(X)$ is well-filtered.*

Proof. (1) $\Rightarrow$ (2): Suppose that X is a well-filtered space. Then by Lemma 5.2, $\mathsf{K}(X)$ is a dcpo, and $\Box U \in \sigma(\mathsf{K}(X))$ for any $U \in O(X)$. Thus $P_S(X)$ is a d-space.

(2) $\Rightarrow$ (3): Suppose that $\{\mathcal{K}_d : d \in D\} \subseteq \mathsf{K}(P_S(X))$ is filtered, $\mathcal{U} \in \mathcal{O}(P_S(X))$, and $\bigcap\limits_{d\in D} \mathcal{K}_d \subseteq \mathcal{U}$. If $\mathcal{K}_d \not\subseteq \mathcal{U}$ for each $d \in D$, then by Lemma 4.1, $\mathsf{K}(X) \setminus \mathcal{U}$ contains a minimal irreducible closed subset $\mathcal{A}$ that still meets all $\mathcal{K}_d$ $(d \in D)$. For each $d \in D$, let $K_d = \bigcup \uparrow_{\mathsf{K}(X)}(\mathcal{A} \bigcap \mathcal{K}_d)$ $(= \bigcup(\mathcal{A} \bigcap \mathcal{K}_d))$. Then by Lemma 2.13, $\{K_d : d \in D\} \subseteq \mathsf{K}(X)$ is filtered, and $K_d \in \mathcal{A}$ for all $d \in D$ since $\mathcal{A} = \downarrow_{\mathsf{K}(X)}\mathcal{A}$. Let $K = \bigcap\limits_{d\in D} K_d$. Then $K \in \mathsf{K}(X)$ and $K = \bigvee_{\mathsf{K}(X)}\{K_d : d \in D\} \in \mathcal{A}$ by Lemma 5.2 and condition (2). We claim that $K \in \bigcap\limits_{d\in D} \mathcal{K}_d$. Suppose, on the contrary, that $K \notin \bigcap\limits_{d\in D} \mathcal{K}_d$. Then there is a $d_0 \in D$ such that $K \notin \mathcal{K}_{d_0}$. Select a $G \in \mathcal{A} \bigcap \mathcal{K}_{d_0}$. Then $K \not\subseteq G$, and hence there is a $g \in K \setminus G$. It follows that $g \in K_d = \bigcup(\mathcal{A} \bigcap \mathcal{K}_d)$ for all $d \in D$ and $G \notin \Diamond_{\mathsf{K}(K)}\overline{\{g\}}$. Thus $\Diamond_{\mathsf{K}(K)}\overline{\{g\}} \bigcap \mathcal{A} \bigcap \mathcal{K}_d \neq \emptyset$ for all $d \in D$. By the minimality of $\mathcal{A}$, we have $\mathcal{A} = \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \bigcap \mathcal{A}$, and consequently, $G \in \mathcal{A} \bigcap \mathcal{K}_{d_o} = \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \bigcap \mathcal{A} \bigcap \mathcal{K}_{d_0}$, which is a contradiction with $G \notin \Diamond_{\mathsf{K}(K)}\overline{\{g\}}$. Thus $K \in \bigcap\limits_{d\in D} \mathcal{K}_d \subseteq \mathcal{U} \subseteq \mathsf{K}(X) \setminus \mathcal{A}$, being a contradiction with $K \in \mathcal{A}$. Therefore, $P_S(X)$ is well-filtered.

(3) $\Rightarrow$ (1): Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. Let $\widetilde{\mathcal{K}} = \{\uparrow_{\mathsf{K}(X)}K : K \in \mathcal{K}\}$. Then $\widetilde{\mathcal{K}} \subseteq \mathsf{K}(P_S(X))$ is filtered and $\bigcap \widetilde{\mathcal{K}} \subseteq \Box U$. By the well-filteredness of $P_S(X)$, there is a $K \in \mathcal{K}$ such that $\uparrow_{\mathsf{K}(X)}K \subseteq \Box U$, and whence $K \subseteq U$, proving that X is well-filtered. $\Box$

Definition 5.4. A T_0 space X is said to have *filtered intersection property*, **FTIP** for short, if $\bigcap \mathcal{K} \neq \emptyset$ for each filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$. X is said to have *irreducible intersection property*, **RIP** for short, if $\bigcap \mathcal{A} \neq \emptyset$ for each irreducible subset $\mathcal{A}$ of $P_S(X)$.

By Remark 2.12, X has **RIP** iff $\bigcap \mathcal{A} \neq \emptyset$ for all irreducible closed subset $\mathcal{A}$ of $P_S(X)$. For a T_0 space X, by Lemma 5.2, Theorem 5.3 and Theorem 6.6, we have the following implications:

$$\text{sobriety} \Rightarrow \text{irreducible completeness of } P_S(X) \Rightarrow \textbf{RIP} \Rightarrow \textbf{FTIP};$$

$$\text{sobriety} \Rightarrow \text{well-filteredness} \Rightarrow d\text{-space of } P_S(X) \Rightarrow \text{directed completeness of } \mathsf{K}(X) \Rightarrow \textbf{FTIP}.$$

Theorem 5.5. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *$\mathsf{K}(X)$ is a dcpo, and $\uparrow(A \cap \bigcap \mathcal{K}) = \bigcap\limits_{K\in\mathcal{K}} \uparrow(A \cap K)$ for every filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ and $A \in \mathcal{C}(X)$.*
(3) *$\mathsf{K}(X)$ is a dcpo, and $\uparrow(A \cap \bigcap \mathcal{K}) = \bigcap\limits_{K\in\mathcal{K}} \uparrow(A \cap K)$ for every filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ and $A \in \mathsf{Irr}_c(X)$.*

(4) *X has* **FTIP**, *and* $\uparrow(A\cap\bigcap\mathcal{K})=\bigcap_{K\in\mathcal{K}}\uparrow(A\cap K)$ *for every filtered family* $\mathcal{K}\subseteq\mathsf{K}(X)$ *and* $A\in\mathcal{C}(X)$.

(5) *X has* **FTIP**, *and* $\uparrow(A\cap\bigcap\mathcal{K})=\bigcap_{K\in\mathcal{K}}\uparrow(A\cap K)$ *for every filtered family* $\mathcal{K}\subseteq\mathsf{K}(X)$ *and* $A\in\mathsf{Irr}_c(X)$.

Proof. We directly have (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5).

(1) $\Rightarrow$ (2): By Theorem 5.3, $\mathsf{K}(X)$ is a dcpo. Suppose $\mathcal{K}\subseteq\mathsf{K}(X)$ and $A\in\mathcal{C}(X)$. Obviously, $\uparrow(A\cap\bigcap\mathcal{K})\subseteq\bigcap_{K\in\mathcal{K}}\uparrow(A\cap K)$. On the other hand, if $x\in\bigcap_{K\in\mathcal{K}}\uparrow(A\cap K)$, then for each $K\in\mathcal{K}$, $\downarrow x\cap A\cap K\neq\emptyset$, and hence $K\nsubseteq X\setminus(\downarrow x\cap A)$. It follows by the well-filteredness of X that $\bigcap\mathcal{K}\nsubseteq X\setminus(\downarrow x\cap A)$, that is, $\downarrow x\cap A\cap\bigcap\mathcal{K}\neq\emptyset$. Therefore $x\in\uparrow(A\cap\bigcap\mathcal{K})$. The equation $\uparrow(A\cap\bigcap\mathcal{K})=\bigcap_{K\in\mathcal{K}}\uparrow(A\cap K)$ thus holds.

(5) $\Rightarrow$ (1): Suppose that $\mathcal{K}\subseteq\mathsf{K}(X)$ is filtered, $U\in\mathcal{O}(X)$, and $\bigcap\mathcal{K}\subseteq U$. If $K\nsubseteq U$ for each $K\in\mathcal{K}$, then by Lemma 4.1, $X\setminus U$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{K}$. By condition (5) we have $\emptyset\neq\bigcap_{K\in\mathcal{K}}\uparrow(A\cap K)=\uparrow(A\cap\bigcap\mathcal{K})=\emptyset$ since $\bigcap\mathcal{K}\subseteq U\subseteq X\setminus A$, a contradiction. Thus X is well-filtered. $\square$

By Lemma 2.16 and Theorem 5.5, we get the following corollary.

Corollary 5.6. ([24]) *Suppose that X is a well-filtered space and $\mathcal{K}\subseteq\mathsf{K}(X)$ is a filtered family. Let $C=\bigcap\mathcal{K}$. Then $C\in\mathsf{K}(X)$, and for each $c\in min(C)$, $\bigcap_{K\in\mathcal{K}}\uparrow(\downarrow c\cap K)=\uparrow(\downarrow c\cap\bigcap\mathcal{K})=\uparrow c$.*

Proposition 5.7. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is a sober space.*
(2) *For any $A\in\mathsf{Irr}(X)$, $\overline{A}\cap\bigcap_{a\in A}\uparrow a\neq\emptyset$.*
(3) *For any $A\in\mathsf{Irr}_c(X)$, $A\cap\bigcap_{a\in A}\uparrow a\neq\emptyset$.*
(4) *For any $A\in\mathsf{Irr}(X)$ and $U\in\mathcal{O}(X)$, $\bigcap_{a\in A}\uparrow a\subseteq U$ implies $\uparrow a\subseteq U$ (i.e., $a\in U$) for some $a\in A$.*
(5) *For any $A\in\mathsf{Irr}_c(X)$ and $U\in\mathcal{O}(X)$, $\bigcap_{a\in A}\uparrow a\subseteq U$ implies $\uparrow a\subseteq U$ (i.e., $a\in U$) for some $a\in A$.*
(6) *For any $\mathcal{A}\subseteq\mathsf{Irr}(P_S(X))$ with $\mathcal{A}\subseteq\mathcal{S}^u(X)$ and $U\in\mathcal{O}(X)$, $\bigcap\mathcal{A}\subseteq U$ implies $K\subseteq U$ for some $K\in\mathcal{A}$.*
(7) *For any $\mathcal{A}\subseteq\mathsf{Irr}_c(P_S(\mathcal{S}^u(X))$ and $U\in\mathcal{O}(X)$, $\bigcap\mathcal{A}\subseteq U$ implies $K\subseteq U$ for some $K\in\mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2): If X is sober and $A\in\mathsf{Irr}(X)$, then there is an $x\in X$ such that $\overline{A}=\overline{\{x\}}=\downarrow x$, and whence $x\in\overline{A}\cap\bigcap_{a\in A}\uparrow a$.

(2) $\Leftrightarrow$ (3): Clearly, we have (2) $\Rightarrow$ (3). Conversely, if condition (3) is satisfied, then for $A\in\mathsf{Irr}(X)$, $\overline{A}\in\mathsf{Irr}_c(X)$ by Lemma 2.2, and $\emptyset\neq\overline{A}\cap\bigcap_{b\in\overline{A}}\uparrow b=\overline{A}\cap\bigcap_{a\in A}\uparrow a$ by Remark 2.1.

(2) $\Rightarrow$ (4): If $\uparrow a\nsubseteq U$ for each $a\in A$, then $A\subseteq X\setminus U$, and hence $\overline{A}\subseteq X\setminus U$. By condition (2), $\emptyset\neq\overline{A}\cap\bigcap_{a\in A}\uparrow a\subseteq(X\setminus U)\cap U=\emptyset$, a contradiction.

(4) $\Leftrightarrow$ (5): Obviously, (4) $\Rightarrow$ (5). Conversely, if condition (5) is satisfied, then for $A\in\mathsf{Irr}(X)$ and $U\in\mathcal{O}(X)$ with $\bigcap_{a\in A}\uparrow a\subseteq U$, we have $\overline{A}\in\mathsf{Irr}_c(X)$ and $\bigcap_{b\in\overline{A}}\uparrow b=\bigcap_{a\in A}\uparrow a\subseteq U$ by Remark 2.2 and Lemma 2.1. By condition (5), $b\in U$ for some $b\in\overline{A}$, and whence $A\cap U\neq\emptyset$. Condition (4) is thus satisfied.

(4) $\Leftrightarrow$ (6) and (5) $\Leftrightarrow$ (7): By Remark 2.11.

(5) $\Rightarrow$ (1): Suppose $A\in\mathsf{Irr}_c(X)$. Then $A\cap\bigcap_{a\in A}\uparrow a\neq\emptyset$ (otherwise, by condition (5), $A\cap\bigcap_{a\in A}\uparrow a=\emptyset\Rightarrow\bigcap_{a\in A}\uparrow a\subseteq X\setminus A\Rightarrow\uparrow a\subseteq X\setminus A$ for some $a\in A$, a contradiction). Select an $x\in A\cap\bigcap_{a\in A}\uparrow a$. Then $A\subseteq\downarrow x=\overline{\{x\}}\subseteq A$, and hence $A=\overline{\{x\}}$. Thus X is sober. $\square$

The single most important result about sober spaces is the Hofmann-Mislove Theorem (see [13] or [6, Theorem II-1.20 and Theorem II-1.21]).

Theorem 5.8 *(The Hofmann-Mislove Theorem). For a T_0 space X, the following conditions are equivalent:*

(1) *X is a sober space.*
(2) *For any $\mathcal{F} \in \mathrm{OFilt}(\mathcal{O}(X))$, there is a $K \in \mathsf{K}(X)$ such that $\mathcal{F} = \Phi(K)$.*
(3) *For any $\mathcal{F} \in \mathrm{OFilt}(\mathcal{O}(X))$, $\mathcal{F} = \Phi(\bigcap \mathcal{F})$.*

Lemma 5.9. *Suppose that X is a T_0 space and $\mathcal{A} \in \mathsf{Irr}(P_S(X))$. Then $\mathcal{F}_{\mathcal{A}} = \bigcup_{K\in\mathcal{A}} \Phi(K) \in \mathrm{OFilt}(\mathcal{O}(X))$.*

Proof. Clearly, $\mathcal{F}_{\mathcal{A}} \in \sigma(\mathcal{O}(X))$ since $\Phi(K) \in \sigma(\mathcal{O}(X))$ for all $K \in \mathsf{K}(X)$. Now we show that $\mathcal{F}_{\mathcal{A}} \in \mathrm{Filt}(\mathcal{O}(X))$. Suppose $U, V \in \mathcal{F}_{\mathcal{A}}$. Then $\mathcal{A} \bigcap \Box U \neq \emptyset$ and $\mathcal{A} \bigcap \Box V \neq \emptyset$, and hence $\mathcal{A} \bigcap \Box(U \cap V) = \mathcal{A} \bigcap \Box U \bigcap \Box V \neq \emptyset$ by $\mathcal{A} \in \mathsf{Irr}(P_S(X))$. Therefore, $U \cap V \in \mathcal{F}_{\mathcal{A}}$. $\Box$

Remark 5.10. For a T_0 space X and $\mathcal{A} \in \mathsf{Irr}(P_S(X))$, $\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathrm{cl}\mathcal{A}}$. In fact, if $U \in \mathcal{O}(X)$ and $U \in \mathcal{F}_{\mathrm{cl}\mathcal{A}}$, then $\mathrm{cl}\mathcal{A} \bigcap \Box U \neq \emptyset$, and whence $\mathcal{A} \bigcap \Box U \neq \emptyset$. It follows $U \in \mathcal{F}_{\mathcal{A}}$.

Using the Hofmann-Mislove Theorem and Lemma 5.9, we present an alternative proof of the following result of Heckmann and Keimel.

Theorem 5.11. ([11]) *For a T_0 space X, the following conditions are equivalent:*

(1) *X is a sober space.*
(2) *For any $\mathcal{A} \subseteq \mathsf{Irr}(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.*
(3) *For any $\mathcal{A} \subseteq \mathsf{Irr}_c(P_S(X))$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.*
(4) *$P_S(X)$ is sober.*

Proof. (1) $\Rightarrow$ (2): By Lemma 5.9, $\mathcal{F}_{\mathcal{A}} \in \mathrm{OFlit}(\mathcal{O}(X))$, and hence by the Hofmann-Mislove Theorem, $\mathcal{F}_{\mathcal{A}} = \Phi(\bigcap \mathcal{F}_{\mathcal{A}}) = \Phi(\bigcap \mathcal{A})$. Therefore, $K \subseteq U$ for some $K \in \mathcal{A}$.

(2) $\Leftrightarrow$ (3): By Remark 2.12 and Remark 5.10.

(3) $\Rightarrow$ (4): Suppose $\mathcal{A} \subseteq \mathsf{Irr}_c(P_S(X))$. Let $H = \bigcap \mathcal{A}$. Then $H \neq \emptyset$ by condition (3). Now we prove that $H \in \mathsf{K}(X)$. If $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ such that $H \subseteq \bigcup\limits_{i\in I} U_i$, then by condition (3), there is a $K \in \mathcal{A}$ such that $K \subseteq \bigcup\limits_{i\in I} U_i$. Since $K \in \mathsf{K}(X)$, there is a $J \in I^{(<\omega)}$ such that $\bigcup\limits_{i\in J} U_i \supseteq K \supseteq H$. Thus $H \in \mathsf{K}(X)$. For each $U \in \mathcal{O}(X)$, by condition (3), we have that $H \in \Box U \Leftrightarrow \mathcal{A} \bigcap \Box U \neq \emptyset$, proving $\mathcal{A} = \overline{\{H\}}$. Thus $P_S(X)$ is sober.

(4) $\Rightarrow$ (1): For any $A \in \mathsf{Irr}(X)$ and $U \in \mathcal{O}(X)$ with $\bigcap\limits_{a\in A} \uparrow a \subseteq U$, $\xi_X(A) \in \mathsf{Irr}(P_S(X))$ and $\bigcap\limits_{a\in A} \uparrow_{\mathsf{K}(X)} \xi_X(a) \subseteq \Box U$. By Proposition 5.7, $\uparrow_{\mathsf{K}(X)} \xi_X(a) \subseteq \Box U$ for some $a \in A$, and hence $a \in U$. By Proposition 5.7 again, X is sober. $\Box$

Definition 5.12. A T_0 space X is called *irreducible bounded*, *r-bounded* for short, if for any $A \in \mathsf{Irr}(X)$, A has an upper bound in X, that is, there is an $x \in X$ such that $A \subseteq \downarrow x = \overline{\{x\}}$, or equivalently, $A^{\uparrow} = \bigcap_{a\in A} \uparrow a \neq \emptyset$.

By Remark 2.1, X is r-bounded iff A has an upper bound in X for each $A \in \mathsf{Irr}(X)$. Clearly, we have the following implications:

$$\text{sobriety} \Rightarrow \text{irreducible completeness} \Rightarrow r\text{-boundedness}.$$

For a poset P with a largest element $\top$, any order compatible topology τ on P is r-bounded.

Proposition 5.13. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*

(2) *X is r-bounded (especially, X is r-complete), and $\uparrow\left(C \cap \bigcap_{a\in A} \uparrow a\right) = \bigcap_{a\in A} \uparrow(C \cap \uparrow a)$ for any $A \in \mathsf{Irr}(X)$ and $C \in \mathcal{C}(X)$.*

(3) *X is r-bounded (especially, X is r-complete), and $\uparrow\left(C \cap \bigcap_{a\in A} \uparrow a\right) = \bigcap_{a\in A} \uparrow(C \cap \uparrow a)$ for any $A \in \mathsf{Irr}(X)$ and $C \in \mathsf{Irr}_c(X)$.*

(4) *X is r-bounded (especially, X is r-complete), and $\uparrow(C \cap \bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow(C \cap K)$ for any $\mathcal{A} \in \mathsf{Irr}(P_S(X))$ with $\mathcal{A} \subseteq \mathcal{S}^u(X)$ and $C \in \mathcal{C}(X)$.*

(5) *X is r-bounded (especially, X is r-complete), and $\uparrow(C \cap \bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow(C \cap K)$ for any $\mathcal{A} \in \mathsf{Irr}(P_S(X))$ with $\mathcal{A} \subseteq \mathcal{S}^u(X)$ and $C \in \mathsf{Irr}_c(X)$.*

Proof. (1) $\Rightarrow$ (2): Since X is sober, X is r-complete by Remark 2.8. For $A \in \mathsf{Irr}(X)$ and $C \in \mathcal{C}(X)$, clearly, $\uparrow\left(C \cap \bigcap_{a\in A} \uparrow a\right) \subseteq \bigcap_{a\in A} \uparrow(C \cap \uparrow a)$. Conversely, if $x \notin \uparrow\left(C \cap \bigcap_{a\in A} \uparrow a\right)$, that is, $\downarrow x \cap C \cap \bigcap_{a\in A} \uparrow a = \emptyset$, then $\bigcap_{a\in A} \uparrow a \subseteq X \setminus (\downarrow x \cap C)$, and whence by Theorem 5.7, $a \in X \setminus (\downarrow x \cap C)$ for some $a \in A$, i.e., $x \notin \uparrow(C \cap \uparrow a)$. Therefore, $x \notin \bigcap_{a\in A} \uparrow(C \cap \uparrow a)$. Thus $\uparrow\left(C \cap \bigcap_{a\in A} \uparrow d\right) = \bigcap_{a\in A} \uparrow(C \cap \uparrow a)$.

(2) $\Rightarrow$ (3): Trivial.

(2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5): For $\mathcal{A} = \{\uparrow x_i : i \in I\} \subseteq \mathcal{S}^u(X)$, $\mathcal{A} \in \mathsf{Irr}(P_S(X))$ iff $\{x_i : i \in I\} \in \mathsf{Irr}(X)$.

(3) $\Rightarrow$ (1): For each $A \in \mathsf{Irr}_c(X)$, by condition (3), $\emptyset \neq A^{\uparrow} = \bigcap_{a\in A} \uparrow a = \bigcap_{a\in A} \uparrow(A \cap \uparrow a) = \uparrow\left(A \cap \bigcap_{a\in A} \uparrow d\right)$. By Theorem 5.7, X is sober. $\square$

Remark 5.14. For a T_0 space X, by the proof of Proposition 5.13, we see that the following conditions are equivalent:

(a) X is sober.

(b) X is r-bounded (especially, X is r-complete), and $\uparrow\left(C \cap \bigcap_{a\in A} \uparrow a\right) = \bigcap_{a\in A} \uparrow(C \cap \uparrow a)$ for any $A \in \mathsf{Irr}_c(X)$ and $C \in \mathcal{C}(X)$.

(c) X is r-bounded (especially, X is r-complete), and $\uparrow\left(C \cap \bigcap_{a\in A} \uparrow a\right) = \bigcap_{a\in A} \uparrow(C \cap \uparrow a)$ for any $A \in \mathsf{Irr}_c(X)$ and $C \in \mathsf{Irr}_c(X)$.

Theorem 5.15. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*

(2) *X has* RIP *(especially, $P_S(X)$ is r-complete) and $\uparrow(C \cap \bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow(C \cap K)$ for every $\mathcal{A} \subseteq \mathsf{Irr}(P_S(X))$ and $C \in \mathcal{C}(X)$.*

(3) *X has* RIP *(especially, $P_S(X)$ is r-complete) and $\uparrow(C \cap \bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow(C \cap K)$ for every $\mathcal{A} \subseteq \mathsf{Irr}(P_S(X))$ and $C \in \mathsf{Irr}_c(X)$.*

(4) *X has* RIP *(especially, $P_S(X)$ is r-complete) and* $\uparrow(C \cap \bigcap \mathcal{A}) = \bigcap\limits_{K\in\mathcal{A}} \uparrow(C \cap K)$ *for every* $\mathcal{A} \subseteq \mathsf{Irr}_c(P_S(X))$ *and* $C \in \mathcal{C}(X)$.

(5) *X has* RIP *(especially, $P_S(X)$ is r-complete) and* $\uparrow(C \cap \bigcap \mathcal{A}) = \bigcap\limits_{K\in\mathcal{A}} \uparrow(C \cap K)$ *for every* $\mathcal{A} \subseteq \mathsf{Irr}_c(P_S(X))$ *and* $C \in \mathsf{Irr}_c(X)$.

Proof. We directly have (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5).

(1) $\Rightarrow$ (2): By Remark 2.8 and Corollary 5.11, X is r-complete. Suppose $\mathcal{A} \subseteq \mathsf{Irr}(P_S(X))$ and $C \in \mathcal{C}(X)$. Obviously, $\uparrow(C \cap \bigcap \mathcal{A}) \subseteq \bigcap_{K\in\mathcal{A}} \uparrow(C\cap K)$. On the other hand, if $x \in \bigcap_{K\in\mathcal{A}} \uparrow(C\cap K)$, then for each $K \in \mathcal{A}$, $\downarrow x \cap C \cap K \neq \emptyset$, and hence $K \nsubseteq X \setminus (\downarrow x \cap C)$. By Corollary 5.11, we have $\bigcap \mathcal{K} \nsubseteq X \setminus (\downarrow x \cap C)$, that is, $\downarrow x \cap C \cap \bigcap \mathcal{K} \neq \emptyset$. Therefore $x \in \uparrow(C \cap \bigcap \mathcal{A})$. The equation $\uparrow(C \cap \bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow(C\cap K)$ thus holds.

(5) $\Rightarrow$ (1): Suppose that $\mathcal{A} \subseteq \mathsf{Irr}_c(P_S(X))$, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{A} \subseteq U$. If $K \nsubseteq U$ for each $K \in \mathcal{A}$, then by Lemma 4.1, $X \setminus U$ contains a minimal irreducible closed subset C that still meets all members of $\mathcal{A}$. Let $\mathcal{A}_C = \{\uparrow(C\cap K) : K \in \mathcal{A}\}$. Then $\mathcal{A}_C \subseteq \mathsf{K}(X)$. Now we show that $\mathcal{A}_C \in \mathsf{Irr}(P_S(X))$. Suppose $V, W \in \mathcal{O}(X)$ such that $\mathcal{A}_C \bigcap \Box V \neq \emptyset$ and $\mathcal{A}_C \bigcap \Box W \neq \emptyset$. Then $\mathcal{A} \bigcap \Box(V \cup (X\setminus C)) \neq \emptyset$ and $\mathcal{A} \bigcap \Box(W \cup (X\setminus C)) \neq \emptyset$, and whence $\mathcal{A} \bigcap \Box((V\cap W) \cup (X\setminus C)) = \mathcal{A} \bigcap \Box(V \cup (X\setminus C)) \bigcap \Box(W \cup (X\setminus C)) \neq \emptyset$ by the irreducibility of $\mathcal{A}$. It follows that $\mathcal{A}_C \bigcap \Box V \bigcap \Box W = \mathcal{A}_C \bigcap \Box(V\cap W) \neq \emptyset$, proving the irreducibility of $\mathcal{A}_C$. By condition (5) we have $\emptyset \neq \bigcap \mathcal{A}_C = \uparrow(C \cap \bigcap \mathcal{K}) = \emptyset$ since $\bigcap \mathcal{A} \subseteq U \subseteq X \setminus C$, a contradiction. Thus X is sober by Theorem 5.11. $\square$

As a corollary of Theorem 5.15, we get a similar result to Corollary 5.6.

Corollary 5.16. *Let X be a sober space and $\mathcal{A} \subseteq \mathsf{Irr}(P_S(X))$. Then $C = \bigcap \mathcal{A} \in \mathsf{K}(X)$, and for each $c \in min(C)$, $\bigcap_{K\in\mathcal{A}} \uparrow(\downarrow c \cap K) = \uparrow(\downarrow c \cap \bigcap \mathcal{A}) = \uparrow c$.*

Theorem 5.17. *Let X be a T_0 space and $\mathbf{K}$ a full subcategory of $\mathbf{Top}_0$ containing $\mathbf{Sob}$. Then the following conditions are equivalent:*

(1) *X is sober.*
(2) *For every continuous mapping $f : X \longrightarrow Y$ from X to a T_0 space Y and any $\mathcal{A} \in \mathsf{Irr}(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*
(3) *For every continuous mapping $f : X \longrightarrow Y$ from X to a T_0 space Y and any $\mathcal{A} \in \mathsf{Irr}_c(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*
(4) *For every continuous mapping $f : X \longrightarrow Y$ from X to a $\mathbf{K}$-space Y and any $\mathcal{A} \in \mathsf{Irr}(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*
(5) *For every continuous mapping $f : X \longrightarrow Y$ from X to a $\mathbf{K}$-space Y and any $\mathcal{A} \in \mathsf{Irr}_c(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*
(6) *For every continuous mapping $f : X \longrightarrow Y$ from X to a sober space Y and any $\mathcal{A} \in \mathsf{Irr}(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*
(7) *For every continuous mapping $f : X \longrightarrow Y$ from X to a sober space Y and any $\mathcal{A} \in \mathsf{Irr}_c(P_S(X))$, $\uparrow f(\bigcap \mathcal{A}) = \bigcap_{K\in\mathcal{A}} \uparrow f(K)$.*

Proof. We only need to prove the equivalences of conditions (1), (2), (3), (6), and (7).

(1) $\Rightarrow$ (2): It needs only to check $\bigcap_{K\in\mathcal{A}} \uparrow f(K) \subseteq \uparrow f(\bigcap \mathcal{A})$. Let $y \in \bigcap_{K\in\mathcal{A}} \uparrow f(K)$. Then for each $K \in \mathcal{A}$, $\overline{\{y\}} \cap f(K) \neq \emptyset$, that is, $K \cap f^{-1}\left(\overline{\{y\}}\right) \neq \emptyset$. Since X is sober, $f^{-1}\left(\overline{\{y\}}\right) \cap \bigcap \mathcal{A} \neq \emptyset$ (otherwise, $\bigcap \mathcal{A} \subseteq X \setminus f^{-1}\left(\overline{\{y\}}\right)$, and consequently, by Theorem 5.11, $K \subseteq X \setminus f^{-1}\left(\overline{\{y\}}\right)$ for some $K \in \mathcal{A}$, a contradiction). It follows that $\overline{\{y\}} \cap f(\bigcap \mathcal{A}) \neq \emptyset$. This implies that $y \in \uparrow f(\bigcap \mathcal{K})$. So $\bigcap_{K\in\mathcal{A}} \uparrow f(K) \subseteq \uparrow f(\bigcap \mathcal{A})$.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (6), (3) $\Rightarrow$ (7) and (6) $\Rightarrow$ (7): Trivial.

(7) $\Rightarrow$ (1): Let $\eta_X : X \to X^s$ $(= P_H(\mathsf{Irr}_c(X)))$ be the canonical topological embedding from X into its sobrification and $\xi_X : X \to P_S(X)$ the canonical topological embedding from X into the Smyth power space of X. Suppose that $A \in \mathsf{Irr}_c(X)$. Then $\Diamond_{\mathsf{K}(X)} A = \mathrm{cl}_{P_S(X)}\xi_X(A) \in \mathsf{Irr}_c(P_S(X))$. By Remark 2.12 and condition (7), we have $\uparrow_{\mathsf{Irr}_c(X)}\eta_X(A^\uparrow) = \uparrow_{\mathsf{Irr}_c(X)}\eta_X(\bigcap \xi_X(A)) = \uparrow_{\mathsf{Irr}_c(X)}\eta_X(\bigcap \mathrm{cl}_{P_S(X)}\xi_X(A)) = \uparrow_{\mathsf{Irr}_c(X)}\eta_X(\bigcap \Diamond_{\mathsf{K}(X)} A) = \bigcap\limits_{K \in \Diamond_{\mathsf{K}(X)} A} \uparrow_{\mathsf{Irr}_c(X)}\eta_X(K) = \bigcap\limits_{a\in A} \uparrow_{\mathsf{Irr}_c(X)}\overline{\{a\}}$. It follows that $A \in \uparrow_{\mathsf{Irr}_c(X)}\eta_X(A^\uparrow)$ by $A \in \bigcap\limits_{a\in A} \uparrow_{\mathsf{Irr}_c(X)}\overline{\{a\}}$. Therefore, there is an $x \in A^\uparrow$ such that $\overline{\{x\}} \subseteq A$, and consequently, $A = \overline{\{x\}}$. Thus X is sober. $\square$

6. Well-filtered determined spaces

In this section, we introduce another new type of subsets in a T_0 topological space — well-filtered determined sets (WD sets for short), which are closely related to Rudin sets. Using WD sets, we introduce and investigate another new kind of spaces — well-filtered determined spaces (WD spaces for short). The Rudin spaces lie between WD spaces and DC spaces, and DC spaces lie between Rudin spaces and sober spaces. For a T_0 space X, it is proved that X is sober iff X is a well-filtered Rudin space iff X is a well-filtered WD space.

In [4], it is shown that for a locally hypercompact T_0 space X, every irreducible closed subset A of X is the closure of a directed subset of X. Therefore, locally hypercompact spaces are DC spaces. Further, we prove that every locally compact T_0 space is a Rudin space and every core compact T_0 space is a WD space. As a corollary we deduce that every core compact well-filtered space is sober, giving a positive answer to Jia-Jung problem [15], which was first answered by Lawson and Xi [20] using a different method.

Firstly, motivated by Proposition 4.12, we give the following definition.

Definition 6.1. A subset A of a T_0 space X is called a *well-filtered determined set*, WD *set* for short, if for any continuous mapping $f : X \longrightarrow Y$ to a well-filtered space Y, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}(X)$ the set of all closed well-filtered determined subsets of X. X is called a *well-filtered determined* space, WD *space* for short, if all irreducible closed subsets of X are well-filtered determined, that is, $\mathsf{Irr}_c(X) = \mathsf{WD}(X)$.

Obviously, a subset A of a space X is well-filtered determined iff $\overline{A}$ is well-filtered determined.

Proposition 6.2. *Let X be a T_0 space. Then $\mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$.*

Proof. By Lemma 4.8 and Proposition 4.12, $\mathcal{D}_c(X) \subseteq \mathsf{RD}(X) \subseteq \mathsf{WD}(X)$. We need to show $\mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$. Let $A \in \mathsf{WD}(X)$. Since $\eta_X : X \longrightarrow X^s$, $x \mapsto \downarrow x$, is a continuous mapping to a well-filtered space (X^s is sober), there exists $C \in \mathsf{Irr}_c(X)$ such that $\overline{\eta_X(A)} = \overline{\{C\}}$. Let $U \in \mathcal{O}(X)$. Note that

$$\begin{aligned} A \cap U \neq \emptyset \quad &\Leftrightarrow \quad \eta_X(A) \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \quad \{C\} \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \quad C \in \Diamond U \\ &\Leftrightarrow \quad C \cap U \neq \emptyset. \end{aligned}$$

This implies that $A = C$, and hence $A \in \mathsf{Irr}_c(X)$. $\square$

Corollary 6.3. *Sober* $\Rightarrow$ DC $\Rightarrow$ RD $\Rightarrow$ WD.

By Theorem 5.1 and Corollary 6.3, we have the following corollary.

Corollary 6.4. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) *For every continuous mapping $f : X \longrightarrow Y$ from X to a WD space Y and a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap_{K\in\mathcal{K}} \uparrow f(K)$.*
(3) *For every continuous mapping $f : X \longrightarrow Y$ from X to a RD space Y and a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap_{K\in\mathcal{K}} \uparrow f(K)$.*
(4) *For every continuous mapping $f : X \longrightarrow Y$ from X to a DC space Y and a filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\uparrow f\left(\bigcap \mathcal{K}\right) = \bigcap_{K\in\mathcal{K}} \uparrow f(K)$.*

It is easy to verify the following result (cf. [29, Theorem 5.7]).

Proposition 6.5. *For any poset P, the Alexandroff space $(P, \alpha(P))$ is a DC space and the following conditions are equivalent:*

(1) *$(P, \alpha(P))$ is sober.*
(2) *$(P, \alpha(P))$ is well-filtered.*
(3) *$(P, \alpha(P))$ is a d-space.*
(4) *P satisfies the ACC condition.*
(5) *P is a dcpo such that every element of P is compact (i.e., $x \ll x$ for all $x \in P$).*
(6) *P is a dcpo such that $\alpha(P) = \sigma(P)$.*

Theorem 6.6. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *X is a DC d-space.*
(3) *X is a well-filtered DC space.*
(4) *X is a well-filtered Rudin space.*
(5) *X is a well-filtered WD space.*

Proof. By Corollary 6.3 we only need to check (5) $\Rightarrow$ (1). Assume X is a well-filtered WD space. Let $A \in \mathsf{Irr}_c(X)$. Since the identity $id_X : X \longrightarrow X$ is continuous, there is a unique $x \in X$ such that $\overline{A} = \overline{\{x\}}$. So X is sober. $\square$

Lemma 6.7. ([4]) *Let X be a locally hypercompact T_0 space and $A \in \mathsf{Irr}(X)$. Then there exists a directed subset $D \subseteq \downarrow A$ such that $\overline{A} = \overline{D}$.*

Remark 6.8. For $C \subseteq X$, we have $C^\delta = \bigcap\{\downarrow x : C \subseteq \downarrow x\} = \bigcap\{\downarrow x : \overline{C} \subseteq \downarrow x\} = \overline{C}^\delta$. So in Lemma 6.7 we also have $A^\delta = D^\delta$.

By Corollary 6.3 and Lemma 6.7, we get the following corollary.

Corollary 6.9. *If X is a locally hypercompact T_0 space, then it is a DC space. Therefore, it is a Rudin space and a WD space.*

Theorem 6.10. *Every locally compact T_0 space is a Rudin space.*

Proof. Suppose that X is a locally compact T_0 space and $A \in \mathsf{Irr}_c(X)$. Let $\mathcal{K}_A = \{K \in \mathsf{K}(X) : A \cap \mathrm{int}\, K \neq \emptyset\}$.

Claim 1. $\mathcal{K}_A \neq \emptyset$.

Let $a \in A$. Since X is locally compact, there exists a $K \in \mathsf{K}(X)$ such that $a \in \operatorname{int} K$. So $a \in A \cap \operatorname{int} K$ and $K \in \mathcal{K}_A$.

Claim 2. *$\mathcal{K}_A$ is filtered.*

Let $K_1, K_2 \in \mathcal{K}_A$, that is, $A \cap \operatorname{int} K_1 \neq \emptyset$ and $A \cap \operatorname{int} K_2 \neq \emptyset$. Since A is irreducible, $A \cap \operatorname{int} K_1 \cap \operatorname{int} K_2 \neq \emptyset$. Let $x \in A \cap \operatorname{int} K_1 \cap \operatorname{int} K_2$. By the local compactness of X again, there exists a $K_3 \in \mathsf{K}(X)$ such that $x \in \operatorname{int} K_3 \subseteq K_3 \subseteq \operatorname{int} K_1 \cap \operatorname{int} K_2$. Thus $K_3 \in \mathcal{K}_A$ and $K_3 \subseteq K_1 \cap K_2$. So $\mathcal{K}_A$ is filtered.

Claim 3. $A \in m(\mathcal{K}_A)$.

Clearly, $\mathcal{K}_A \subseteq \Diamond A$. If B is a proper closed subset of A, then there is $a \in A \setminus B$. Since X is locally compact, there is $K_a \in \mathsf{K}(X)$ such that $a \in \operatorname{int} K_a \subseteq K_a \subseteq X \setminus B$. Then $K_a \in \mathcal{K}_A$ but $K_a \cap B = \emptyset$, and whence $B \notin M(\mathcal{K}_A)$, proving $A \in m(\mathcal{K}_A)$. Thus X is a Rudin space. $\square$

Definition 6.11. For a T_0 space and $A, B \subseteq X$, we say A is *way below* B, or A is *compact relative to* B, written as $A \ll B$, if for each $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, $B \subseteq \bigcup_{i \in I} U_i$ implies $A \subseteq \bigcup_{i \in J} U_i$ for some finite subset J of I.

Clearly, we have $A \ll B \Rightarrow \uparrow A \subseteq \uparrow B$, and if $A, B, G, H \in \mathbf{up}(X)$, then $G \subseteq A \ll B \subseteq H \Rightarrow G \ll H$.

Definition 6.12. Let X be a T_0 space and $\mathcal{F} = \{F_\infty, ..., F_n, ..., F_2, F_1\} \subseteq \mathbf{up}(X)$. $\mathcal{F}$ is called a *bounded decreasing $\ll$-sequence* in X if $F_\infty \ll ... \ll F_n \ll ... \ll F_2 \ll F_1$. Denote the minimal set F_∞ in $\mathcal{F}$ by $min\,\mathcal{F}$ and the maximal set F_1 in $\mathcal{F}$ by $max\,\mathcal{F}$.

Lemma 6.13. *Let $f : X \longrightarrow Y$ be a continuous mapping and $A, B \subseteq X$. If $A \ll B$, then $f(A) \ll f(B)$.*

Proof. Suppose $\{V_i : i \in I\} \subseteq \mathcal{O}(Y)$. If $f(B) \subseteq \bigcup_{i \in I} V_i$, then $B \subseteq f^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} f^{-1}(V_i)$. Since f is continuous and $A \ll B$, there is a $J \in I^{(<\omega)}$ such that $A \subseteq \bigcup_{i \in J} f^{-1}(V_i)$, and whence $f(A) \subseteq \bigcup_{i \in J} V_i$. Thus $f(A) \ll f(B)$. $\square$

Corollary 6.14. *Let X and Y be T_0 spaces and $f : X \longrightarrow Y$ a continuous mapping. If $\mathcal{F} \subseteq \mathbf{up}(X)$ is a bounded decreasing $\ll$-sequence in X, then $\{\uparrow f(F) : F \in \mathcal{F}\}$ is a bounded decreasing $\ll$-sequence in Y.*

Theorem 6.15. *Every core compact T_0 space is well-filtered determined.*

Proof. Let X be a core compact T_0 space and $A \in \mathsf{Irr}_c(X)$. We need to show $A \in \mathsf{WD}(X)$. Suppose that $f : X \longrightarrow Y$ is a continuous mapping from X to a well-filtered space Y. Let $\mathfrak{F}_A = \{\mathcal{F} : \mathcal{F} \subseteq \mathcal{O}(X)$ is a bounded decreasing $\ll$-sequence in X with $A \cap min\,\mathcal{F} \neq \emptyset\}$. Define a partial order $\prec$ on $\mathfrak{F}_A$ by $\mathcal{F}_1 \prec \mathcal{F}_2$ iff $max\,\mathcal{F}_1 \subseteq min\,\mathcal{F}_2$. For each $\mathcal{F} \in \mathfrak{F}_A$, let $K_{\mathcal{F}} = \bigcap_{U \in \mathcal{F} \setminus \{min\,\mathcal{F}\}} \uparrow f(U)$.

Claim 1. $\mathfrak{F}_A \neq \emptyset$.

Select a point $a \in A$ and a $U \in \mathcal{O}(X)$. Then by the core compactness of X, there is a sequence $\mathcal{F}_a = \{U_\infty, ..., U_n, ..., U_2, U_1\} \subseteq \mathcal{O}(X)$ such that $a \in U_\infty$ and $U_\infty \ll ... \ll U_n \ll ... \ll U_2 \ll U_1 = U$. Then $\mathcal{F}_a \in \mathfrak{F}_A$.

Claim 2. *$\mathfrak{F}_A$ is $\prec$-filtered.*

Suppose that $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}_A$. Then $A \cap min\, \mathcal{F}_1 \neq \emptyset \neq A \cap min\, \mathcal{F}_2$, and hence $A \cap min\, \mathcal{F}_1 \cap min\, \mathcal{F}_2 \neq \emptyset$ by the irreducibility of A. Let $W_1 = min\, \mathcal{F}_1 \cap min\, \mathcal{F}_2$ and select a point $b \in A \cap W_1$. Then by the core compactness of X, there is a sequence $\mathcal{F}_3 = \{W_\infty, ..., W_n, ..., W_2, W_1\} \subseteq \mathcal{O}(X)$ such that $b \in W_\infty$ and $W_\infty \ll ... \ll W_n \ll ... \ll W_2 \ll W_1 = W$. Then $\mathcal{F}_3 \in \mathfrak{F}_A$, $\mathcal{F}_3 \prec \mathcal{F}_1$ and $\mathcal{F}_3 \prec \mathcal{F}_2$.

Claim 3. *For $\mathcal{F} \in \mathfrak{F}_A$ and $W \in \mathcal{O}(Y)$, if $K_\mathcal{F} \subseteq W$, then $\uparrow f(U) \subseteq W$ for some $U \in \mathcal{F} \setminus \{min\, \mathcal{F}\}$.*

Let $\mathcal{F} = \{U_\infty, ..., U_n, ..., U_2, U_1\} \subseteq \mathcal{O}(X)$ and $U_\infty \ll ... \ll U_n \ll ... \ll U_2 \ll U_1$. Assume, on the contrary, that $\uparrow f(U_n) \not\subseteq W$ for all $n \in N$. Let $\mathcal{B} = \{B \in \mathcal{C}(Y) : B \subseteq Y \setminus W$ and $\uparrow f(U_n) \cap B \neq \emptyset$ for all $n \in N\}$. Then we have the following two facts.

(b1) $\mathcal{B} \neq \emptyset$ because $Y \setminus W \in \mathcal{B}$.

(b2) For any filtered family $\mathcal{Z} \subseteq \mathcal{B}$, $\bigcap \mathcal{Z} \in \mathcal{B}$.

Let $Z = \bigcap \mathcal{Z}$. Then $Z \in \mathcal{C}(Y)$ and $Z \subseteq Y \setminus W$. Assume $Z \notin \mathcal{B}$. Then there exists $n \in N$ such that $f(U_n) \cap Z = \emptyset$. Then $U_{n+1} \ll U_n \subseteq \bigcup_{C \in \mathcal{Z}} f^{-1}(Y \setminus C)$, and consequently, there is a $C \in \mathcal{Z}$ such that $U_{n+1} \subseteq f^{-1}(Y \setminus C)$, that is, $f(U_{n+1}) \cap C = \emptyset$, which is a contradiction with $C \in \mathcal{Z} \subseteq \mathcal{B}$. Therefore, $\bigcap \mathcal{Z} \in \mathcal{B}$.

By Zorn's Lemma, there exists a minimal element E in $\mathcal{B}$. Since $E = \downarrow E$, E intersects all $f(U_n)$. For each $n \in N$, select an $e_n \in f(U_n) \cap E$ and let $H_n = \{e_m : n \leq m\}$. Now we prove that $\uparrow H_n \in \mathsf{K}(Y)$ for all $n \in N$. Suppose that $\{V_d : d \in D\} \subseteq \mathcal{O}(Y)$ is a directed open cover of $\uparrow H_n$.

(c1) If for some $d_1 \in D$, $H_n \cap (Y \setminus V_{d_1}) = H_n \setminus V_{d_1}$ is finite, then $H_n \setminus V_{d_1} \subseteq V_{d_2}$ for some $d_2 \in D$ because $H_n \subseteq \bigcup\limits_{d \in D} V_d$. By the directness of $\{V_d : d \in D\}$, $V_{d_1} \cup V_{d_2} \subseteq V_{d_3}$ for some $d_3 \in D$. Then $H_n \subseteq V_{d_3}$.

(c2) If for all $d \in D$, $H_n \cap (Y \setminus V_d)$ is infinite, then $f(U_n) \cap E \cap (Y \setminus V_d) \neq \emptyset$ since $\mathcal{F} = \{U_\infty, ..., U_n, ..., U_2, U_1\}$ is a bounded decreasing $\ll$-sequence in X, and whence $E \cap (Y \setminus V_d) \in \mathcal{B}$. By the minimality of E, $E \cap (Y \setminus V_d) = E$ for all $d \in D$. Therefore, $H_n \subseteq E \subseteq \bigcap_{d \in D} (Y \setminus V_d) = Y \setminus \bigcup_{d \in D} V_d$, which is a contradiction with $H_n \subseteq \bigcup_{d \in D} V_d$.

By (c1) and (c2), $\uparrow H_n \in \mathsf{K}(Y)$. Clearly, $\{\uparrow H_n : n \in N\} \subseteq \mathsf{K}(Y)$ is filtered and $\bigcap_{n \in N} \uparrow H_n \subseteq \bigcap_{n \in N} \uparrow f(U_n) \subseteq W$. By the well-filteredness of X, $\uparrow H_m \subseteq W$ for some $m \in N$, which is a contradiction with $H_m \subseteq E \subseteq Y \setminus W$, proving Claim 3.

Claim 4. *$K_\mathcal{F} \in \mathsf{K}(Y)$ for each $\mathcal{F} \in \mathfrak{F}_A$.*

Suppose $\{V_i : i \in I\} \subseteq \mathcal{O}(Y)$ and $K_\mathcal{F} \subseteq \bigcup_{i \in I} V_i$. Then by Claim 3, $\uparrow f(U) \subseteq \bigcup_{i \in I} V_i$ for some $U \in \mathcal{F} \setminus \{min\, \mathcal{F}\}$, and whence $U \subseteq \bigcup_{i \in I} f^{-1}(V_i)$. Since $\mathcal{F}$ is a bounded decreasing $\ll$-sequence and $U \neq min\, \mathcal{F}$, there is a $U^* \in \mathcal{F}$ such that $U^* \ll U$. It follows that $U^* \subseteq \bigcup_{i \in J} f^{-1}(V_i)$ for some $J \in I^{(<\omega)}$, and consequently, $K_\mathcal{F} = \bigcap\limits_{U \in \mathcal{F} \setminus \{min\, \mathcal{F}\}} \uparrow f(U) \subseteq \uparrow f(U^*) \subseteq \bigcup_{i \in J} V_i$. Thus $K_\mathcal{F} \in \mathsf{K}(Y)$.

Claim 5. $\bigcap_{\mathcal{F} \in \mathfrak{F}_A} K_\mathcal{F} \in \mathsf{K}(Y)$.

By Claim 2 and Claim 4, $\{K_\mathcal{F} : \mathcal{F} \in \mathfrak{F}_A\} \in \mathsf{K}(Y)$ if filtered, and whence $\bigcap_{\mathcal{F} \in \mathfrak{F}_A} K_\mathcal{F} \in \mathsf{K}(Y)$ by the well-filteredness of Y.

Claim 6. $A \in \mathsf{WD}(X)$.

We first show that $\bigcap\limits_{\mathcal{F} \in \mathfrak{F}_A} K_\mathcal{F} \cap \overline{f(A)} \neq \emptyset$. Assume, on the contrary, $\bigcap\limits_{\mathcal{F} \in \mathfrak{F}_A} K_\mathcal{F} \subseteq Y \setminus \overline{f(A)}$, then by Claim 3, Claim 5 (and its proof) and the well-filteredness of Y, there is an $\mathcal{F} \in \mathfrak{F}_A$ such that $\uparrow f(U) \subseteq Y \setminus \overline{f(A)}$ for some $U \in \mathcal{F} \setminus \{min\, \mathcal{F}\}$, and hence $\emptyset \neq A \cap U \subseteq A \cap f^{-1}(Y \setminus \overline{f(A)}) = \emptyset$, a contraction. Therefore, $\bigcap\limits_{\mathcal{F} \in \mathfrak{F}_A} K_\mathcal{F} \cap \overline{f(A)} \neq \emptyset$. Select a point $y_A \in \bigcap\limits_{\mathcal{F} \in \mathfrak{F}_A} K_\mathcal{F} \cap \overline{f(A)}$. Then $\overline{\{y_A\}} \subseteq \overline{f(A)}$. On the other hand, for

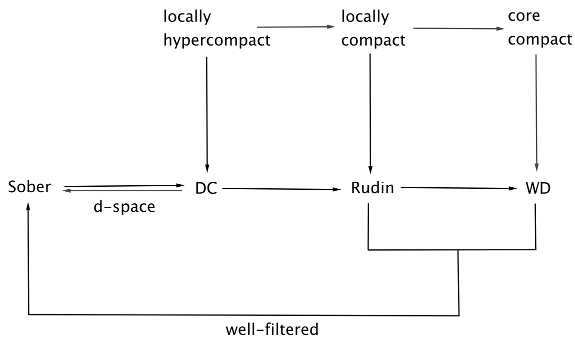

Fig. 1. Certain relations among some kinds of spaces.

$a \in A$, if $f(a) \notin \overline{\{y_A\}}$, then $a \in f^{-1}(Y \setminus \overline{\{y_A\}})$. By the core compactness of X, there is a sequence $\mathcal{F}_a = \{U^a_\infty, ..., U^a_n, ..., U^a_2, U^a_1\} \subseteq \mathcal{O}(X)$ such that $a \in U^a_\infty$ and $U^a_\infty \ll ... \ll U^a_n \ll ... \ll U^a_2 \ll U^a_1 = f^{-1}(Y \setminus \overline{\{y_A\}})$. Then $\mathcal{F}_a \in \mathfrak{F}_A$, and whence

$$y_A \in K_{\mathcal{F}_a} \subseteq \uparrow f(f^{-1}(Y \setminus \overline{\{y_A\}})) \subseteq Y \setminus \overline{\{y_A\}}),$$

a contradiction. Therefore, $f(A) \subseteq \overline{\{y_A\}})$. Thus $\overline{f(A)} = \overline{\{y_A\}}$, proving $A \in \mathsf{WD}(X)$. $\square$

By Theorem 6.6 and Theorem 6.15, we get the following result, which has been first obtained by Lawson and Xi (see [20, Theorem 3.1]) using a different method.

Theorem 6.16. *Every core compact well-filtered space is sober.*

Theorem 6.16 gives a positive answer to Jia-Jung problem [15] (see [15, Question 2.5.19]) and improves a well-known result that every locally compact well-filtered space is sober (see, e.g., [6,19]).

By Theorem 2.15 and Theorem 6.16, we get the following corollary.

Corollary 6.17. *Let X be a well-filtered space. Then X is locally compact iff X is core compact.*

Fig. 1 shows certain relations among some kinds of spaces.

Theorem 6.18. *Let X be a T_0 space. Consider the following conditions:*

(1) *X is sober.*
(2) *For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$, $max(A) \neq \emptyset$ and $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(3) *X is well-filtered.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*, and all three conditions are equivalent if X is core compact.*

Proof. (1) $\Rightarrow$ (2): Suppose that X is sober and $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$. Then there is an $x \in X$ such that $A = \overline{\{x\}}$, and hence $max(A) = \{x\} \neq \emptyset$. Now we show that $\downarrow (A \cap K) = \downarrow (\downarrow x \cap K)$ is closed. If $\downarrow (\downarrow x \cap K) \neq \emptyset$ (i.e., $\downarrow x \cap K \neq \emptyset$), then $x \in K$ since K is saturated (that is, K is an upper set). It follows that $\downarrow (\downarrow x \cap K) = \downarrow x \in \mathcal{C}(X)$.

(2) $\Rightarrow$ (3): Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. If $K \nsubseteq U$ for each $K \in \mathcal{K}$, then by Lemma 4.1, $X \setminus U$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{K}$. For any $\{K_1, K_2\} \subseteq \mathcal{K}$, we can find $K_3 \in \mathcal{K}$ with $K_3 \subseteq K_1 \cap K_2$. It follows that $\downarrow (A \cap K_1) \in \mathcal{C}(X)$ and $\emptyset \neq A \cap K_3 \subseteq \downarrow (A \cap K_1) \cap K_2 \neq \emptyset$, and hence $\downarrow (A \cap K_1) = A$ by the minimality of A. Select an $x \in max(A)$.

Then for each $K \in \mathcal{K}$, $x \in \downarrow (A \cap K)$, and consequently, there is $a_k \in A \cap K$ such that $x \le a_k$. By the maximality of x we have $x = a_k$. Therefore, $x \in K$ for all $K \in \mathcal{K}$, and whence $x \in \bigcap \mathcal{K} \subseteq U \subseteq X \setminus A$, a contradiction. Thus X is well-filtered.

Finally assume that X is core compact and well-filtered, then by Theorem 6.16, X is sober. $\square$

If X is a d-space and A a nonempty closed subset of X, then by Zorn's Lemma there is a maximal chain C in A. Let $c = \vee C$. Then $c \in max(A)$. So by Theorem 6.18 we get the following corollary.

Corollary 6.19. *Let X be a d-space. Consider the following conditions:*

(1) *X is sober.*
(2) *For each $(A, K) \in \mathsf{Irr}_c(X) \times \mathsf{K}(X)$, $\downarrow (A \cap K) \in \mathcal{C}(X)$.*
(3) *X is well-filtered.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), *and all three conditions are equivalent if X is core compact.*

Corollary 6.20. ([26]) *Let X be a d-space with the property that $\downarrow (A \cap K)$ is closed whenever $A \in \mathcal{C}(X)$ and $K \in \mathsf{K}(X)$. Then X is well-filtered.*

Example 6.21. Let X be a countable infinite set and endow X with the cofinite topology (having the complements of the finite sets as open sets). The resulting space is denoted by X_{cof}. Then $\mathsf{K}(X_{cof}) = 2^X \setminus \{\emptyset\}$ (that is, all nonempty subsets of X), and hence X_{cof} is a locally compact and first countable T_1 space. By Theorem 6.10, X_{cof} is a Rudin space (and hence a WD-space). Let $\mathcal{K} = \{X \setminus F : F \in X^{(<\omega)}\}$. It is easy to check that $\mathcal{K} \subseteq \mathsf{K}(X_{cof})$ is filtered and $X \in m(\mathcal{K})$. Therefore, $X \in \mathsf{RD}(X)$ but $X \notin \mathsf{D}_c(X)$. Thus $\mathsf{RD}(X) \neq \mathsf{D}_c(X)$. X_{cof} is not sober, and hence X_{cof} is not well-filtered by Theorem 6.16.

Example 6.22. Let L be the non-sober complete lattice constructed by Isbell [14]. Then by [26, Corollary 3.2] and Theorem 6.6, ΣL is a well-filtered space but not a WD space (hence not a Rudin space). So $\mathsf{WD}(X) \neq \mathsf{Irr}_c(X)$ and $\mathsf{RD}(X) \neq \mathsf{Irr}_c(X)$.

Lemma 6.23. *Let X, Y be two T_0 spaces. If $f : X \longrightarrow Y$ is a continuous mapping and $A \in \mathsf{WD}(X)$, then $\overline{f(A)} \in \mathsf{WD}(Y)$.*

Proof. Let Z be a well-filtered space and $g : Y \longrightarrow Z$ is a continuous mapping. Since $g \circ f : X \longrightarrow Z$ is continuous and $A \in \mathsf{WD}(X)$, there is $z \in Z$ such that $\overline{g(\overline{f(A)})} = \overline{g \circ f(A)} = \overline{\{z\}}$. Thus $\overline{f(A)} \in \mathsf{WD}(Y)$. $\square$

Proposition 6.24. *A retract of a well-filtered determined space is well-filtered determined.*

Proof. Assume X is a well-filtered determined space and Y a retract of X. Then there are continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = id_Y$. Let $B \in \mathsf{Irr}_c(Y)$. Then $\overline{g(B)} \in \mathsf{Irr}_c(X)$ by Lemma 2.2 and Lemma 2.3. As X is well-filtered determined, $\overline{g(B)} \in \mathsf{WD}(X)$). By Lemma 6.23, $B = \overline{f(g(B))} = \overline{f(\overline{g(B)})} \in \mathsf{WD}(Y)$. Hence, Y is well-filtered determined. $\square$

Definition 6.25. For a T_0 space X, take a point ∞ such that $\infty \notin X$. Then $\mathcal{C}(X) \cup \{X \cup \{\infty\}\}$ is the set of all closed sets of a topology on $X \cup \{\infty\}$. The resulting space is denoted by X_∞.

Lemma 6.26. *If X is a well-filtered space, then X_∞ is a well-filtered space.*

Proof. We first show that X_∞ is T_0. Let $x, y \in X_\infty$ with $x \neq y$. There are two cases:

Case 1. $x, y \in X$. Then we have $\mathrm{cl}_{X_\infty}\{x\} = \mathrm{cl}_X\{x\} \neq \mathrm{cl}_X\{y\} = \mathrm{cl}_{X_\infty}\{y\}$.

Case 2. $x \in X$ and $y = \infty$. Note that $\mathrm{cl}_{X_\infty}\{\infty\} = X_\infty$ and $\mathrm{cl}_{X_\infty}\{x\} \subseteq X$. It follows that $\mathrm{cl}_{X_\infty}\{\infty\} \neq \mathrm{cl}_{X_\infty}\{x\}$.

Thus X_∞ is T_0. Let $\{K_i : i \in I\} \subseteq \mathsf{K}(X_\infty)$ be a filtered family and $U \in \mathcal{O}(X_\infty)$ such that $\bigcap_{i\in I} K_i \subseteq U$. Note that ∞ is the largest element in X with respect to the specialization order, so $\infty \in \bigcap_{i\in I} K_i \subseteq U$. Let $V = U \setminus \{\infty\} = X \setminus (X_\infty \setminus U)$. Then $V \in \mathcal{O}(X)$ and $U = V \cup \{\infty\}$. For each $i \in I$, let $K_i^* = K_i \setminus \{\infty\}$. One can easily check that $\{K_i^* : i \in I\} \subseteq \mathsf{K}(X)$ is a filtered family and $\bigcap_{i\in I} K_i^* \subseteq V$. Since X is well-filtered, there exists $i_0 \in I$ such that $K_{i_0}^* \subseteq V$, which implies that $K_{i_0} \subseteq U$. Thus X_∞ is well-filtered. $\square$

Proposition 6.27. *Every closed subspace of a well-filtered determined space is well-filtered determined.*

Proof. Let X be a well-filtered determined space and $A \in \mathcal{C}(X)$. We need to show A, as a subspace of X, is well-filtered determined. Let $B \in \mathsf{Irr}_c(A)$ and $f : A \longrightarrow Y$ a continuous mapping to a well-filtered space Y. Then by Lemma 6.26, Y_∞ is well-filtered. Define a mapping $f_\infty : X \longrightarrow Y_\infty$ as follows:

$$f_\infty(x) = \begin{cases} f(x) & x \in A \\ \infty & x \notin A. \end{cases}$$

Then f_∞ is continuous since for each $C \in \mathcal{C}(Y_\infty)$, it holds that

$$f_\infty^{-1}(C) = \begin{cases} f^{-1}(C) & \infty \notin C \\ X & \infty \in C. \end{cases}$$

Since X is well-filtered determined, there exists $y_B \in Y_\infty$ such that $\overline{f_\infty(B)} = \overline{f(B)} = \overline{\{y_B\}}$. Clearly, $y_B \in Y$. So $B \in \mathsf{WD}(A)$. Thus A is well-filtered determined. $\square$

Lemma 6.28. *Let $\{X_i : 1 \leq i \leq n\}$ be a finite family of T_0 spaces and $X = \prod_{i=1}^{n} X_i$ the product space. For $A \in \mathsf{Irr}(X)$, the following conditions are equivalent:*

(1) *A is a* WD *set.*
(2) *$p_i(A)$ is a* WD *set for each $1 \leq i \leq n$.*

Proof. (1) $\Rightarrow$ (2): By Lemma 6.23.

(2) $\Rightarrow$ (1): By induction, we need only to prove the implication for the case of $n = 2$. Let $A_1 = \mathrm{cl}_{X_1} p_1(A)$ and $A_2 = \mathrm{cl}_{X_2} p_2(A)$. Then by condition (2), $(A_1, A_2) \in \mathsf{WD}(X_1) \times \mathsf{WD}(X_2)$. Now we show that the product $A_1 \times A_2 \in \mathsf{WD}(X)$. Let $f : X_1 \times X_2 \longrightarrow Y$ a continuous mapping from $X_1 \times X_2$ to a well-filtered space Y. For each $b \in X_2$, X_1 is homeomorphic to $X_1 \times \{b\}$ (as a subspace of $X_1 \times X_2$) via the homeomorphism $\mu_b : X_1 \longrightarrow X_1 \times \{b\}$ defined by $\mu_b(x) = (x, b)$. Let $i_b : X_1 \times \{b\} \longrightarrow X_1 \times X_2$ be the embedding of $X_1 \times \{b\}$ in $X_1 \times X_2$. Then $f_b = f \circ i_b \circ \mu_b : X_1 \longrightarrow Y$, $f_b(x) = f((x, b))$, is continuous. Since $A_1 \in \mathsf{WD}(X_1)$, there is a unique $y_b \in Y$ such that $\overline{f(A_1 \times \{b\})} = \overline{f_b(A_1)} = \overline{\{y_b\}}$. Define a mapping $g_A : X_2 \longrightarrow Y$ by $g_A(b) = y_b$. For each $V \in \mathcal{O}(Y)$,

$$\begin{aligned} g_A^{-1}(V) &= \{b \in X_2 : g_A(b) \in V\} \\ &= \{b \in X_2 : \overline{f_b(A_1)} \cap V \neq \emptyset\} \\ &= \{b \in X_2 : \overline{f(A_1 \times \{b\})} \cap V \neq \emptyset\} \end{aligned}$$

$$
\begin{aligned}
&= \{b \in X_2 : f(A_1 \times \{b\}) \cap V \neq \emptyset\} \\
&= \{b \in X_2 : (A_1 \times \{b\}) \cap f^{-1}(V) \neq \emptyset\}.
\end{aligned}
$$

Therefore, for each $b \in g_A^{-1}(V)$, there is an $a_1 \in A_1$ such that $(a_1, b) \in f^{-1}(V) \in \mathcal{O}(X_1 \times X_2)$, and hence there is $(U_1, U_2) \in \mathcal{O}(X_1) \times \mathcal{O}(X_2)$ such that $(a_1, b) \in U_1 \times U_2 \subseteq f^{-1}(V)$. It follows that $b \in U_2 \subseteq g_A^{-1}(V)$. Thus $g_A : X_2 \longrightarrow Y$ is continuous. Since $A_2 \in \mathsf{WD}(X_2)$, there is a unique $y_A \in Y$ such that $\overline{g_A(A_2)} = \overline{\{y_A\}}$. Therefore, by Lemma 2.5, we have

$$
\begin{aligned}
\overline{f(\mathrm{cl}_X A)} &= \overline{f(A_1 \times A_2)} \\
&= \overline{\bigcup_{a_2 \in A_2} f(A_1 \times \{a_2\})} \\
&= \overline{\bigcup_{a_2 \in A_2} \overline{f(A_1 \times \{a_2\})}} \\
&= \overline{\bigcup_{a_2 \in A_2} \overline{\{g_A(a_2)\}}} \\
&= \overline{\bigcup_{a_2 \in A_2} \{g_A(a_2)\}} \\
&= \overline{g_A(A_2)} \\
&= \overline{\{y_A\}}.
\end{aligned}
$$

Thus $\mathrm{cl}_X A \in \mathsf{WD}(X)$, and hence A is a WD set. $\square$

By Corollary 2.7 and Lemma 6.28, we get the following result.

Corollary 6.29. *Let $X = \prod_{i=1}^{n} X_i$ be the product of a finite family $\{X_i : 1 \leq i \leq n\}$ of T_0 spaces. If $A \in \mathsf{WD}(X)$, then $A = \prod_{i=1}^{n} p_i(X_i)$, and $p_i(A) \in \mathsf{WD}(X_i)$ for all $1 \leq i \leq n$.*

Theorem 6.30. *Let $\{X_i : 1 \leq i \leq n\}$ be a finite family of T_0 spaces. Then the following two conditions are equivalent:*

(1) *The product space $\prod_{i=1}^{n} X_i$ is a well-filtered determined space.*
(2) *For each $1 \leq i \leq n$, X_i is a well-filtered determined space.*

Proof. (1) $\Rightarrow$ (2): For each $1 \leq i \leq n$, X_i is a retract of $\prod_{i=1}^{n} X_i$. By Proposition 6.24, X_i is a well-filtered determined space.

(2) $\Rightarrow$ (1): Let $X = \prod_{i=1}^{n} X_i$. For any $A \in \mathsf{Irr}_c(X)$, by Corollary 2.7 and Lemma 6.28, we have $A \in \mathsf{WD}(X)$, proving that X is a well-filtered determined space. $\square$

7. A direct construction of well-filtered reflections of T_0 spaces

Section 7 is devoted to the reflection of the category of well-filtered spaces in that of T_0 spaces. Using WD sets, we present a direct construction of the well-filtered reflections of T_0 spaces, and show that the product of any family of well-filtered spaces is well-filtered. Some important properties of well-filtered reflections of T_0 spaces are investigated.

Definition 7.1. Let X be a T_0 space. A *well-filtered reflection* of X is a pair $\langle \widetilde{X}, \mu\rangle$ consisting of a well-filtered space $\widetilde{X}$ and a continuous mapping $\mu : X \longrightarrow \widetilde{X}$ satisfying that for any continuous mapping $f : X \longrightarrow Y$ to a well-filtered space, there exists a unique continuous mapping $f^* : \widetilde{X} \longrightarrow Y$ such that $f^* \circ \mu = f$, that is, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, dotted, "f^*"] \\ & Y \end{tikzcd}$$

Well-filtered reflections, if they exist, are unique up to homeomorphism. We shall use X^w to denote the space of the well-filtered reflection of X if it exists.

Let X be a T_0 space. Then by Proposition 6.2, $\mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$, and whence the space $P_H(\mathsf{WD}(X))$ has the topology $\{\Diamond U : U \in \mathcal{O}(X)\}$, where $\Diamond U = \{A \in \mathsf{WD}(X) : A \cap U \neq \emptyset\}$. The closed subsets of $P_H(\mathsf{WD}(X))$ are exactly the set of forms $\Box C = \downarrow_{\mathsf{WD}(X)} C$ with $C \in \mathcal{C}(X)$.

Lemma 7.2. *Let X be a T_0 space and $A \subseteq X$. Then $\overline{\eta_X(A)} = \overline{\eta_X\left(\overline{A}\right)} = \overline{\Box A} = \Box\overline{A}$ in $P_H(\mathsf{WD}(X))$.*

Proof. Clearly, $\eta_X(A) \subseteq \Box A \subseteq \Box\overline{A}$, $\eta_X\left(\overline{A}\right) \subseteq \Box\overline{A}$ and $\Box\overline{A}$ is closed in $P_H(\mathsf{WD}(X))$. It follows that

$$\overline{\eta_X(A)} \subseteq \overline{\Box A} \subseteq \Box\overline{A} \ \text{ and } \ \overline{\eta_X(A)} \subseteq \overline{\eta_X\left(\overline{A}\right)} \subseteq \Box\overline{A}.$$

To complete the proof, we need to show $\Box\overline{A} \subseteq \overline{\eta_X(A)}$. Let $F \in \Box\overline{A}$. Suppose $U \in \mathcal{O}(X)$ such that $F \in \Diamond U$, that is, $F \cap U \neq \emptyset$. Since $F \subseteq \overline{A}$, we have $A \cap U \neq \emptyset$. Let $a \in A \cap U$. Then $\downarrow a \in \Diamond U \cap \eta_X(A) \neq \emptyset$. This implies that $F \in \overline{\eta_X(A)}$. Whence $\Box\overline{A} \subseteq \overline{\eta_X(A)}$. $\Box$

Lemma 7.3. *The mapping $\eta_X : X \longrightarrow P_H(\mathsf{WD}(X))$ defined by*

$$\forall x \in X,\ \eta_X(x) = \downarrow x,$$

is a topological embedding.

Proof. For $U \in \mathcal{O}(X)$, we have

$$\eta_X^{-1}(\Diamond U) = \{x \in X : \downarrow x \in \Diamond U\} = \{x \in X : x \in U\} = U,$$

so η_X is continuous. In addition, we have

$$\eta_X(U) = \{\downarrow x : x \in U\} = \{\downarrow x : \downarrow x \in \Diamond U\} = \Diamond U \cap \eta_X(X),$$

which implies that η_X is an open mapping to $\eta_X(X)$, as a subspace of $P_H(\mathsf{WD}(X))$. As η_X is an injection, η_X is a topological embedding. $\Box$

Lemma 7.4. *Let X be a T_0 space and A a nonempty subset of X. Then the following conditions are equivalent:*

(1) *A is irreducible in X.*
(2) *$\Box A$ is irreducible in $P_H(\mathsf{WD}(X))$.*
(3) *$\Box\overline{A}$ is irreducible in $P_H(\mathsf{WD}(X))$.*

Proof. (1) $\Rightarrow$ (3): Assume A is irreducible. Then $\eta_X(A)$ is irreducible in $P_H(\mathsf{WD}(X))$ by Lemma 2.3 and Lemma 7.3. By Lemma 2.2 and Lemma 7.2, $\Box\overline{A} = \overline{\eta_X(A)}$ is irreducible in $P_H(\mathsf{WD}(X))$.

(3) $\Rightarrow$ (1): Assume $\Box\overline{A}$ is irreducible. Let $A \subseteq B \cup C$ with $B, C \in \mathcal{C}(X)$. By Proposition 6.2, $\mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$, and consequently, we have $\Box\overline{A} \subseteq \Box B \cup \Box C$. Since $\Box\overline{A}$ is irreducible, $\Box\overline{A} \subseteq \Box B$ or $\Box\overline{A} \subseteq C$, showing that $\overline{A} \subseteq B$ or $\overline{A} \subseteq C$, and consequently, $A \subseteq B$ or $A \subseteq C$, proving A is irreducible.

(2) $\Leftrightarrow$ (3): By Lemma 2.2 and Lemma 7.2. $\Box$

Lemma 7.5. *Let X be a T_0 space and $f : X \longrightarrow Y$ a continuous mapping from X to a well-filtered space Y. Then there exists a unique continuous mapping $f^* : P_H(\mathsf{WD}(X)) \longrightarrow Y$ such that $f^* \circ \eta_X = f$, that is, the following diagram commutes.*

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & P_H(\mathsf{WD}(X)) \arrow[d, dotted, "f^*"] \\
 & Y
\end{tikzcd}$$

Proof. For each $A \in \mathsf{WD}(X)$, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Then we can define a mapping $f^* : P_H(\mathsf{WD}(X)) \longrightarrow Y$ by

$$\forall A \in \mathsf{WD}(X), \quad f^*(A) = y_A.$$

Claim 1. $f^* \circ \eta_X = f$.

Let $x \in X$. Since f is continuous, we have $\overline{f\left(\overline{\{x\}}\right)} = \overline{f(\{x\})} = \overline{\{f(x)\}}$, so $f^*\left(\overline{\{x\}}\right) = f(x)$. Thus $f^* \circ \eta_X = f$.

Claim 2. *f^* is continuous.*

Let $V \in \mathcal{O}(Y)$. Then

$$\begin{aligned}
(f^*)^{-1}(V) &= \{A \in \mathsf{WD}(X) : f^*(A) \in V\} \\
&= \{A \in \mathsf{WD}(X) : \overline{\{f^*(A)\}} \cap V \neq \emptyset\} \\
&= \{A \in \mathsf{WD}(X) : \overline{f(A)} \cap V \neq \emptyset\} \\
&= \{A \in \mathsf{WD}(X) : f(A) \cap V \neq \emptyset\} \\
&= \{A \in \mathsf{WD}(X) : A \cap f^{-1}(V) \neq \emptyset\} \\
&= \Diamond f^{-1}(V),
\end{aligned}$$

which shows that $(f^*)^{-1}(V)$ is open in $P_H(\mathsf{WD}(X))$. Thus f^* is continuous.

Claim 3. *The mapping f^* is unique such that $f^* \circ \eta_X = f$.*

Assume $g : P_H(\mathsf{WD}(X)) \longrightarrow Y$ is a continuous mapping such that $g \circ \eta_X = f$. Let $A \in \mathsf{WD}(X)$. We need to show $g(A) = f^*(A)$. Let $a \in A$. Then $\overline{\{a\}} \subseteq A$, implying that $g(\overline{\{a\}}) \leq_Y g(A)$, that is, $g\left(\overline{\{a\}}\right) = f(a) \in \overline{\{g(A)\}}$. Thus $\overline{\{f^*(A)\}} = \overline{f(A)} \subseteq \overline{\{g(A)\}}$. In addition, since $A \in \overline{\eta_X(A)}$ and g is continuous, $g(A) \in g\left(\overline{\eta_X(A)}\right) \subseteq \overline{g(\eta_X(A))} = \overline{f(A)} = \overline{\{f^*(A)\}}$, which implies that $\overline{\{g(A)\}} \subseteq \overline{\{f^*(A)\}}$. So $\overline{\{g(A)\}} = \overline{\{f^*(A)\}}$. Since Y is T_0, $g(A) = f^*(A)$. Thus $g = f^*$. $\Box$

Lemma 7.6. *Let X be a T_0 space and $C \in \mathcal{C}(X)$. Then the following conditions are equivalent:*

(1) *C is well-filtered determined in X.*
(2) *$\Box C$ is well-filtered determined in $P_H(\mathsf{WD}(X))$.*

Proof. (1) $\Rightarrow$ (2): By Propositions 6.23, Lemma 7.2 and Lemma 7.3.

(2) $\Rightarrow$ (1). Let Y be a well-filtered space and $f : X \longrightarrow Y$ a continuous mapping. By Lemma 7.5, there exists a continuous mapping $f^* : P_H(\mathsf{WD}(X)) \longrightarrow Y$ such that $f^* \circ \eta_X = f$. Since $\Box C = \overline{\eta_X(C)}$ is well-filtered determined and f^* is continuous, there exists a unique $y_C \in Y$ such that $\overline{f^*\left(\overline{\eta_X(C)}\right)} = \overline{\{y_C\}}$. Furthermore, we have

$$\overline{\{y_C\}} = \overline{f^*\left(\overline{\eta_X(C)}\right)} = \overline{f^*(\eta_X(C))} = \overline{f(C)}.$$

So C is well-filtered determined. $\Box$

Theorem 7.7. *Let X be a T_0 space. Then $P_H(\mathsf{WD}(X))$ is a well-filtered space.*

Proof. Since X is T_0, one can deduce that $P_H(\mathsf{WD}(X))$ is T_0. Let $\{\mathcal{K}_i : i \in I\} \subseteq \mathsf{K}(P_H(\mathsf{WD}(X)))$ be a filtered family and $U \in \mathcal{O}(X)$ such that $\bigcap_{i\in I} \mathcal{K}_i \subseteq \Diamond U$. We need to show $\mathcal{K}_i \subseteq \Diamond U$ for some $i \in I$. Assume, on the contrary, $\mathcal{K}_i \nsubseteq \Diamond U$, i.e., $\mathcal{K}_i \cap \Box(X \setminus U) \neq \emptyset$, for any $i \in I$.

Let $\mathcal{A} = \{C \in \mathcal{C}(X) : C \subseteq X \setminus U \text{ and } \mathcal{K}_i \cap \Box C \neq \emptyset \text{ for all } i \in I\}$. Then we have the following two facts.

(a1) $\mathcal{A} \neq \emptyset$ because $X \setminus U \in \mathcal{A}$.

(a2) For any filtered family $\mathcal{F} \subseteq \mathcal{A}$, $\bigcap \mathcal{F} \in \mathcal{A}$.

Let $F = \bigcap \mathcal{F}$. Then $F \in \mathcal{C}(X)$ and $F \subseteq X \setminus U$. Assume, on the contrary, $F \notin \mathcal{A}$. Then there exists $i_0 \in I$ such that $\mathcal{K}_{i_0} \cap \Box F = \emptyset$. Note that $\Box F = \bigcap_{C\in\mathcal{F}} \Box C$, implying that $\mathcal{K}_{i_0} \subseteq \bigcup_{C\in\mathcal{F}} \Diamond(X \setminus C)$ and $\{\Diamond(X \setminus C) : C \in \mathcal{F}\}$ is a directed family since $\mathcal{F}$ is filtered. Then there is $C_0 \in \mathcal{F}$ such that $\mathcal{K}_{i_0} \subseteq \Diamond(X \setminus C_0)$, i.e., $\mathcal{K}_{i_0} \cap \Box C_0 = \emptyset$, contradicting $C_0 \in \mathcal{A}$. Hence $F \in \mathcal{A}$.

By Zorn's Lemma, there exists a minimal element C_m in $\mathcal{A}$ such that $\Box C_m$ intersects all $\mathcal{K}_i$ $(i \in I)$. Clearly, $\Box C_m$ is also a minimal closed set that intersects all $\mathcal{K}_i$ $(i \in I)$, hence is a Rudin set in $P_H(\mathsf{WD}(X))$. By Proposition 6.2 and Lemma 7.6, C_m is well-filtered determined. So $C_m \in \Box C_m \cap \bigcap \mathcal{K} \neq \emptyset$. It follows that $\bigcap \mathcal{K} \nsubseteq \Diamond(X \setminus C_m) \supseteq \Diamond U$, which implies that $\bigcap \mathcal{K} \nsubseteq \Diamond U$, a contradiction. $\Box$

By Lemma 7.5 and Theorem 7.7, we have the following result.

Theorem 7.8. *Let X be a T_0 space and $X^w = P_H(\mathsf{WD}(X))$. Then the pair $\langle X^w, \eta_X\rangle$, where $\eta_X : X \longrightarrow X^w$, $x \mapsto \overline{\{x\}}$, is the well-filtered reflection of X.*

Corollary 7.9. *The category* $\mathbf{Top}_w$ *of all well-filtered spaces is a reflective full subcategory of* $\mathbf{Top}_0$.

Corollary 7.10. *Let X, Y be two T_0 spaces and $f : X \longrightarrow Y$ a continuous mapping. Then there exists a unique continuous mapping $f^w : X^w \longrightarrow Y^w$ such that $f^w \circ \eta_X = \eta_Y \circ f$, that is, the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & X^w \arrow[d, "f^w"] \\ Y \arrow[r, "\eta_Y"] & Y^w \end{tikzcd}$$

For each $A \in \mathsf{WD}(X)$, $f^w(A) = \overline{f(A)}$.

Corollary 7.10 defines a functor $W : \mathbf{Top}_0 \longrightarrow \mathbf{Top}_w$, which is the left adjoint to the inclusion functor $I : \mathbf{Top}_w \longrightarrow \mathbf{Top}_0$.

Corollary 7.11. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is well-filtered.*
(2) $\mathsf{RD}(X) = \mathcal{S}_c(X)$.
(3) $\mathsf{WD}(X) = \mathcal{S}_c(X)$, *that is, for each $A \in \mathsf{WD}(X)$, there exists a unique $x \in X$ such that $A = \overline{\{x\}}$.*
(4) $X \cong X^w$.

Proof. (1) $\Rightarrow$ (2): Applying Proposition 4.12 to the identity $id_X : X \longrightarrow X$.

(2) $\Rightarrow$ (3): By Proposition 6.2.

(3) $\Rightarrow$ (4): By assumption, $\mathsf{WD}(X) = \left\{\overline{\{x\}} : x \in X\right\}$, so $X^w = P_H(\mathsf{WD}(X)) = P_H(\{\overline{\{x\}} : x \in X\})$, and whence $X \cong X^w$.

(4) $\Rightarrow$ (1): By Theorem 7.7 or by Proposition 6.2 and Corollary 7.11. $\square$

The equivalence of (1) and (2) in Corollary 7.11 has been proved in [23] in a different way.

By Proposition 3.3, Proposition 6.2 and Corollary 7.11, we get the following known result (see, e.g., [26, Proposition 2.1])

Corollary 7.12. *A well-filtered space is a d-space.*

Corollary 7.13. ([27]) *A retract of a well-filtered space is well-filtered.*

Proof. Suppose that Y is a retract of a well-filtered space X. Then there are continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = id_Y$. Let $B \in \mathsf{WD}(Y)$. Then by Lemma 6.23 and Corollary 7.11, there exists a unique $x_B \in X$ such that $\overline{g(B)} = \overline{\{x_B\}}$. Therefore, $B = \overline{f \circ g(B)} = \overline{f(\overline{g(B)})} = \overline{f(\overline{\{x_B\}})} = \overline{\{f(x_B)\}}$. By Corollary 7.11, Y is well-filtered. $\square$

Theorem 7.14. *Let $\{X_i : 1 \le i \le n\}$ be a finite family of T_0 spaces. Then $(\prod_{i=1}^{n} X_i)^w = \prod_{i=1}^{n} X_i^w$ (up to homeomorphism).*

Proof. Let $X = \prod_{i=1}^{n} X_i$. By Corollary 6.29, we can define a mapping $\gamma : P_H(\mathsf{WD}(X)) \longrightarrow \prod_{i=1}^{n} P_H(\mathsf{WD}(X_i))$ by

$$\forall A \in \mathsf{WD}(X),\ \gamma(A) = (p_1(A), p_2(A), ..., p_n(A)).$$

By Lemma 6.28 and Corollary 6.29, γ is bijective. Now we show that γ is a homeomorphism. For any $(U_1, U_2, ..., U_n) \in \mathcal{O}(X_1) \times \mathcal{O}(X_2) \times ... \times \mathcal{O}(X_n)$, by Lemma 6.28 and Corollary 6.29, we have

$$\begin{aligned}
\gamma^{-1}(\Diamond U_1 \times \Diamond U_2 \times ... \times \Diamond U_n) &= \{A \in \mathsf{WD}(X) : \gamma(A) \in \Diamond U_1 \times \Diamond U_2 \times ... \times \Diamond U_n\} \\
&= \{A \in \mathsf{WD}(X) : p_1(A) \cap U_1 \neq \emptyset, p_2(A) \cap U_2 \neq \emptyset, ..., p_n(A) \cap U_n \neq \emptyset\} \\
&= \{A \in \mathsf{WD}(X) : A \cap U_1 \times U_2 \times ... \times U_n \neq \emptyset\} \\
&= \Diamond U_1 \times U_2 \times ... \times U_n \in \mathcal{O}(P_H(\mathsf{WD}(X)),\ \text{and} \\
\gamma(\Diamond U_1 \times U_2 \times ... \times U_n) &= \{\gamma(A) : A \in \mathsf{WD}(X) \text{ and } A \cap U_1 \times U_2 \times ... \times U_n \neq \emptyset\} \\
&= \{\gamma(A) : A \in \mathsf{WD}(X),\ \text{and } p_1(A) \cap U_1 \neq \emptyset, ..., p_n(A) \cap U_n \neq \emptyset\}
\end{aligned}$$

$$= \Diamond U_1 \times \Diamond U_2 \times ... \times \Diamond U_n \in O(\prod_{i=1}^{n} P_H(\mathsf{WD}(X_i))).$$

Therefore, $\gamma : P_H(\mathsf{WD}(X)) \longrightarrow \prod_{i=1}^{n} P_H(\mathsf{WD}(X_i))$ is a homeomorphism, and hence X^w $(= P_H(\mathsf{WD}(X)))$ and $\prod_{i=1}^{n} X_i^w$ $(= \prod_{i=1}^{n} P_H(\mathsf{WD}(X_i)))$ are homeomorphic. $\square$

Using WD sets and Corollary 7.11, we can present a simple proof the following result, which is proved in [23] by using Rudin sets.

Theorem 7.15. ([23]) *Let $\{X_i : i \in I\}$ be a family of T_0 spaces. Then the following two conditions are equivalent:*

(1) *The product space $\prod_{i\in I} X_i$ is well-filtered.*
(2) *For each $i \in I$, X_i is well-filtered.*

Proof. (1) $\Rightarrow$ (2): For each $i \in I$, X_i is a retract of $\prod_{i\in I} X_i$. By Corollary 7.13, X_i is well-filtered.

(2) $\Rightarrow$ (1): Let $X = \prod_{i\in I} X_i$. Suppose $A \in \mathsf{WD}(X)$. Then by Proposition 6.2 and Lemma 6.23, $A \in \mathsf{Irr}_c(X)$ and for each $i \in I$, $\mathrm{cl}_{X_i}(p_i(A)) \in \mathsf{WD}(X_i)$, and consequently, there is a $u_i \in X_i$ such that $\mathrm{cl}_{X_i}(p_i(A)) = cl_{X_i}\{u_i\}$ by condition (2) and Corollary 7.11. Let $u = (u_i)_{i\in I}$. Then by Lemma 2.5 and [1, Proposition 2.3.3]), we have $A = \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A)) = \prod_{i\in I} \mathrm{cl}_{u_i}\{u_i\} = \mathrm{cl}_X\{u\}$. Whence X is well-filtered by Corollary 7.11. $\square$

Theorem 7.16. *For a T_0 space X, the following conditions are equivalent:*

(1) *X^w is the sobrification of X, in other words, the well-filtered reflection of X and sobrification of X are the same.*
(2) *X^w is sober.*
(3) *X is well-filtered determined, that is, $\mathsf{WD}(X) = \mathsf{Irr}_c(X)$.*

Proof. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): Let $\eta_X^w : X \longrightarrow X^w$ be the canonical topological embedding defined by $\eta_X^w(x) = \overline{\{x\}}$ (see Theorem 7.8). Since the pair $\langle X^s, \eta_X^s\rangle$, where $\eta_X^s : X \longrightarrow X^s = P_H(\mathsf{Irr}_c(X))$, $x \mapsto \overline{\{x\}}$, is the sobrification of X and X^w is sober, there exists a unique continuous mapping $\eta_X^{w*} : X^s \longrightarrow X^w$ such that $\eta_X^{w*} \circ \eta_X^s = \eta_X^w$, that is, the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\eta_X^s"] \arrow[dr, "\eta_X^w"'] & X^s \arrow[d, dotted, "\eta_X^{w*}"] \\
& X^w
\end{tikzcd}$$

So for each $A \in \mathsf{Irr}_c(X)$, there exists a unique $B \in \mathsf{WD}(X)$ such that $\downarrow_{\mathsf{WD}(X)} A = \overline{\eta_X^w(A)} = \overline{\{B\}} = \downarrow_{\mathsf{WD}(X)} B$. Clearly, we have $B \subseteq A$. On the other hand, for each $a \in A$, $\overline{\{a\}} \in \downarrow_{\mathsf{WD}(X)} A = \downarrow_{\mathsf{WD}(X)} B$, and whence $\overline{\{a\}} \subseteq B$. Thus $A \subseteq B$, and consequently, $A = B$. Thus $A \in \mathsf{WD}(X)$.

(3) $\Rightarrow$ (1): If $\mathsf{WD}(X) = \mathsf{Irr}_c(X)$, then $X^w = P_H(\mathsf{WD}(X)) = P_H(\mathsf{Irr}_c(X)) = X^s$, with $\eta_X^w = \eta_X^s : X \longrightarrow X^w$, is the sobrification of X. $\square$

Proposition 7.17. *A T_0 space X is compact iff X^w is compact.*

Proof. By Proposition 6.2, we have $\mathcal{S}_c(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$. Suppose that X is compact. For $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, if $\mathsf{WD}(X) \subseteq \bigcup_{i\in I} \Diamond U_i$, then $X \subseteq \bigcup_{i\in I} U_i$ since $\mathcal{S}_c(X) \subseteq \mathsf{WD}(X)$, and consequently, $X \subseteq \bigcup_{i\in I_0} U_i$ for some $I_0 \in I^{(<\omega)}$. It follows that $\mathsf{WD}(X) \subseteq \bigcup_{i\in I_0} \Diamond U_i$. Thus X^w is compact by Alexander's Subbase Lemma (see, e.g., [6, Proposition I-3.22]). Conversely, if X^w is compact and $\{V_j : j \in J\}$ is a open cover of X, then $\mathsf{WD}(X) \subseteq \bigcup_{j\in J} \Diamond V_j$. By the compactness of X^w, there is a finite subset $J_0 \subseteq J$ such that $\mathsf{WD}(X) \subseteq \bigcup_{j\in J_0} \Diamond V_j$, and whence $X \subseteq \bigcup_{j\in J_0} V_j$, proving the compactness of X. $\square$

Since $\mathcal{S}_c(X) \subseteq \mathsf{WD}(X) \subseteq \mathsf{Irr}_c(X)$ (see Proposition 6.2), the correspondence $U \leftrightarrow \Diamond_{\mathsf{WD}(X)} U$ is a lattice isomorphism between $\mathcal{O}(X)$ and $\mathcal{O}(X^w)$, and whence we have the following proposition.

Proposition 7.18. *Let X be a T_0 space. Then*

(1) *X is locally hypercompact iff X^w is locally hypercompact.*
(2) *X is a C-space iff X^w is a C-space.*

Proposition 7.19. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is core compact.*
(2) *X^w is core compact.*
(3) *X^w is locally compact.*

Proof. (1) $\Leftrightarrow$ (2): Since $\mathcal{O}(X)$ and $\mathcal{O}(X^w)$ are lattice-isomorphic.

(2) $\Rightarrow$ (3): By Theorem 7.7, X^w is well-filtered. If X^w is core compact, then X^w is locally compact by Corollary 6.17.

(3) $\Rightarrow$ (2): Trivial. $\square$

Remark 7.20. In [12] (see also [6, Exercise V-5.25]) Hofmann and Lawson given a core compact T_0 space X but not locally compact. By Proposition 7.19, X^w is locally compact. So the local compactness of X^w does not imply the local compactness of X.

Theorem 7.21. *Let X be a T_0 space. If $P_S(X)$ is well-filtered determined, then X is well-filtered determined.*

Proof. Let $A \in \mathsf{Irr}_c(X)$, Y a well-filtered space and $f : X \longrightarrow Y$ a continuous mapping. Then $\overline{\xi_X(A)} = \Diamond A \in \mathsf{Irr}_c(P_S(X)) = \mathsf{WD}(P_S(X))$ since $P_S(X)$ is well-filtered determined, where $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$. Define a mapping $P_S(f) : P_S(X) \longrightarrow P_S(Y)$ by

$$\forall K \in \mathsf{K}(X),\ P_S(f)(K) = \uparrow f(K).$$

Claim 1. $P_S(f) \circ \xi_X = \xi_Y \circ f$.

For each $x \in X$, we have

$$P_S(f) \circ \xi_X(x) = P_S(f)(\uparrow x) = \uparrow f(x) = \xi_Y \circ f(x),$$

that is, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\xi_X"] \arrow[d, "f"'] & P_S(X) \arrow[d, "P_S(f)"] \\ Y \arrow[r, "\xi_Y"] & P_S(Y) \end{tikzcd}$$

Claim 2. *$P_S(f): P_S(X) \longrightarrow P_S(Y)$ is continuous.*

Let $V \in \mathcal{O}(Y)$. We have

$$\begin{aligned} P_S(f)^{-1}(\Box V) &= \{K \in \mathsf{K}(X) : P_S(f)(K) = \uparrow f(K) \subseteq V\} \\ &= \{K \in \mathsf{K}(X) : K \subseteq f^{-1}(V)\} \\ &= \Box f^{-1}(V), \end{aligned}$$

which is open in $P_S(X)$. This implies that $P_S(f)$ is continuous.

By Theorem 5.3, $P_S(Y)$ is well-filtered. Since $P_S(f)$ is continuous and $\Diamond A \in \mathsf{WD}(P_S(X))$, there exists a unique $Q \in \mathsf{K}(Y)$ such that $\overline{P_S(f)(\Diamond A)} = \overline{\{Q\}}$.

Claim 3. *Q is supercompact.*

Let $\{U_j : j \in J\} \subseteq \mathcal{O}(X)$ with $Q \subseteq \bigcup_{j\in J} U_j$, i.e., $Q \in \Box \bigcup_{j\in J} U_j$. Note that $\overline{P_S(f)(\Diamond A)} = \overline{\{\uparrow f(a) : a \in A\}}$, thus $\{\uparrow f(a) : a \in A\} \cap \Box \bigcup_{j\in J} U_j \neq \emptyset$. Then there exists $a_0 \in A$ and $j_0 \in J$ such that $Q \subseteq \uparrow f(a_0) \subseteq U_{j_0}$.

Hence, by [11, Fact 2.2], there exists $y_Q \in Y$ such that $Q = \uparrow y_Q$.

Claim 4. $\overline{f(A)} = \overline{\{y_Q\}}$.

Note that $\overline{\{\uparrow f(a) : a \in A\}} = \overline{\{\uparrow y_Q\}}$. Thus for each $y \in f(A)$, $\uparrow y \in \overline{\{\uparrow y_Q\}}$, showing that $\uparrow y_Q \subseteq \uparrow y$, i.e., $y \in \overline{\{y_Q\}}$. This implies that $f(A) \subseteq \overline{\{y_Q\}}$. In addition, since $\uparrow y_Q \in \overline{\{\uparrow f(a) : a \in A\}} = \Diamond \overline{f(A)}$, $\uparrow y_Q \cap \overline{f(A)} \neq \emptyset$. This implies that $y_Q \in \overline{f(A)}$. Therefore, $\overline{f(A)} = \overline{\{y_Q\}}$. $\Box$

8. Conclusions and further work

In this paper, we introduced and investigated two new classes of subsets in T_0 spaces — Rudin sets and WD sets lying between the class of all closures of directed subsets and that of irreducible closed subsets, as well as three new types of spaces — DC spaces, Rudin spaces and WD spaces. Rudin spaces lie between WD spaces and DC spaces, and DC spaces lie between Rudin spaces and sober spaces. Through such spaces, the sobriety can be decomposed as the combinations of some weaker properties. More precisely, for a T_0 space X, it is proved that the following conditions are equivalent: (1) X is sober; (2) X is a DC d-space; (3) X is a well-filtered DC space; (4) X is a well-filtered Rudin space; and (5) X is a well-filtered WD space. It is shown that locally hypercompact T_0 spaces are DC spaces, locally compact T_0 spaces are Rudin spaces, and core compact T_0 spaces are WD spaces. As a corollary we have that every core compact well-filtered space is sober, giving a positive answer to Jia-Jung problem [15]. Using Rudin sets and WD sets, we formulate and prove a number of new characterizations of well-filtered spaces and sober spaces.

Recently, following Keimel and Lawson's method [18], which originated from Wyler's method [25], Wu, Xi, Xu and Zhao [9] gave a positive answer to the above problem. Following Ershov's method of constructing the d-completion of T_0 spaces, Shen, Xi, Xu and Zhao have presented a construction of the well-filtered reflection of T_0 spaces. In this paper, using WD sets, we give a direct approach to well-filtered reflections of T_0 spaces, and show that products of well-filtered spaces are well-filtered. Some important properties of well-filtered reflections of T_0 spaces are investigated. Comparatively, the technique presented in the paper is not just more direct, but also simpler. Furthermore, it can be also applied to the general K-ifications considered by Keimel and Lawson [18].

Our work shows that DC spaces, Rudin spaces and WD spaces may deserve further investigation. Our study also leads to a number of problems, whose answers will deepen our understanding of the related spaces and structures.

We now close our paper with the following questions about Rudin spaces, WD spaces, products of WD spaces and well-filtered reflections of products of T_0 spaces.

Question 8.1. Does $\mathsf{RD}(X) = \mathsf{WD}(X)$ hold for every T_0 space X?

Question 8.2. Is every well-filtered determined space a Rudin space?

Question 8.3. Let $X = \prod_{i \in I} X_i$ be the product space of a family $\{X_i : i \in I\}$ of T_0 spaces. If each $A_i \subseteq X_i (i \in I)$ is a WD set, must the product set $\prod_{i \in I} A_i$ be a WD set of X?

Question 8.4. Is the product space of an arbitrary collection of WD spaces well-filtered determined?

Question 8.5. Does $(\prod\limits_{i \in I} X_i)^w = \prod\limits_{i \in I} X_i^w$ (up to homeomorphism) hold for any family $\{X_i : i \in I\}$ of T_0 spaces?

Question 8.6. Is the Smyth power space $P_S(X)$ of a well-filtered determined T_0 space X again well-filtered determined?

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