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THE REFLECTIVITY OF SOME CATEGORIES OF T_0 SPACES IN DOMAIN THEORY

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Keimel and Lawson (2009) proposed a set of conditions for proving the reflectivity of a category of topological spaces in the category of all T_0 spaces. Recently, these conditions were used to prove the reflectivity of the category of all well-filtered spaces. We prove that, in certain sense, these conditions are not only sufficient but also necessary for a category of T_0 spaces to be reflective. By applying this general result, we can easily deduce that several categories proposed in domain theory are not reflective, thereby answering a few open problems.

1. Introduction

Given a full subcategory $\boldsymbol{D}$ of a category $\boldsymbol{C}$, one natural and frequently asked question is whether $\boldsymbol{D}$ is reflective in $\boldsymbol{C}$. The objects in $\boldsymbol{D}$ can be viewed as "special objects", the reflectivity of $\boldsymbol{D}$ ensures that every general object in $\boldsymbol{C}$ can be "completed" to be a special object, or "densely embedded into" a special object.

Keimel and Lawson [11] proved that a full subcategory $\boldsymbol{K}$ of Top_0 of all T_0 spaces is reflective if it satisfies the following four conditions:

(K1) $\boldsymbol{K}$ contains all sober spaces.

(K2) If $X \in \boldsymbol{K}$ and Y is homeomorphic to X, then $Y \in \boldsymbol{K}$.

(K3) If $\{X_i : i \in I\} \subseteq \boldsymbol{K}$ is a family of subspaces of a sober space, then the subspace $\bigcap_{i\in I} X_i \in \boldsymbol{K}$.

(K4) If $f : X \to Y$ is a continuous mapping from a sober space X to a sober space Y, then for any subspace Y_1 of Y, $Y_1 \in \boldsymbol{K}$ implies that $f^{-1}(Y_1) \in \boldsymbol{K}$.

It has been proved that the categories of d-spaces, well-filtered spaces and sober spaces all satisfy the aforementioned four conditions, as shown in [11; 21; 22]. Therefore, they are all reflective subcategories of Top_0.

For a full subcategory $\boldsymbol{K}$ of Top_0, we say that $\boldsymbol{K}$

(1) is *productive*, if the product $\prod_{i\in I} X_i \in \boldsymbol{K}$ whenever $\{X_i : i \in I\} \subseteq \boldsymbol{K}$, and

(2) is *b-closed-hereditary*, if $Y \in \boldsymbol{K}$ whenever Y is a b-closed subspace of some $X \in \boldsymbol{K}$.

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The four conditions (K1)–(K4) can, however, usually only be used to confirm the reflectivity of subcategories of Top_0. In this paper we shall prove that, in certain sense, they are also necessary conditions, and can therefore be used to disprove the reflectivity of some subcategories of Top_0. In particular, we shall use this result to solve several open problems that were posed in [24]. Our main results are as follows.

Theorem A. *For a full subcategory* $\boldsymbol{K}$ *of* Top_0 *with* $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$, *if* $\boldsymbol{K}$ *satisfies* (K2), *then the following four statements are equivalent*:

(1) $\boldsymbol{K}$ *is reflective in* Top_0.

(2) $\boldsymbol{K}$ *satisfies conditions* $(K1)$–$(K4)$.

(3) $\boldsymbol{K}$ *is productive and b-closed-hereditary.*

(4) $\boldsymbol{K}$ *is productive and has equalizers.*

Theorem B. *The categories of cosober spaces, strongly k-bounded sober spaces, strongly d-spaces, and consonant* T_0 *spaces, are not reflective in* Top_0.

2. Preliminaries

Let P be a poset. For any $A \subseteq P$, let

$$\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\} \quad \text{and} \quad \uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}.$$

For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$, respectively. A subset A of P is called a *lower set* (resp. *upper set*) if $A = \downarrow A$ (resp. $A = \uparrow A$).

For a T_0 space X, the specialization order $\leq$ on X is defined as $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator on X. In the following, when we consider a T_0 space X as a poset, it is always equipped with the specialization order.

For a T_0 space X, we use $\mathcal{O}(X)$ to denote the topology of X. For any subset A of X, the *saturation* of A, denoted by $\mathrm{Sat}(A)$, is defined to be

$$\mathrm{Sat}(A) = \bigcap \{U \in \mathcal{O}(X) : A \subseteq U\}.$$

A subset A of a T_0 space X is *saturated* if $A = \mathrm{Sat}(A)$.

Remark 2.1 [6; 7]. Let X be a T_0 space.

(1) For any subset A of X, $\mathrm{Sat}(A) = \uparrow A$.

(2) For any $x \in X$, $\downarrow x = \mathrm{cl}(\{x\})$, and $x \in \mathrm{Sat}(A)$ if and only if $\downarrow x \cap A \neq \varnothing$.

(3) For any open subset U of X, $U = \uparrow U$, and for any closed subset F of X, $F = \downarrow F$.

A nonempty subset A of a T_0 space is called *irreducible* if for any closed sets F_1, F_2, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called *sober* if for any irreducible closed set F of X there is a (unique) point $x \in X$ such that $F = \mathrm{cl}(\{x\})$.

A very effective tool for studying sober spaces is the b-topology introduced by Skula [19] (see also [3]).

Definition 2.2 [3; 19]. Let X be a T_0 space. The *b-topology* (also called *Skula topology* [19] or *strong topology* [6, Exercise V-5.31]) associated with X is the topology having

$$\{U \cap \downarrow x : x \in U \in \mathcal{O}(X)\}$$

as a base. The resulting space will be denoted by bX. A subset B of X is *b-dense* in X, if it is dense in X with respect to the b-topology.

The following properties on b-topology will be used later. For further information, one can refer to [9; 11] and Exercise V-5.31 in [6].

Remark 2.3. Let X be a T_0 space.

(1) The b-topology on X is finer than the original topology on X. This follows trivially from the fact that for any open set U in X, we have $U = \bigcup_{x \in U} U \cap \downarrow x$.

(2) Let X be a T_0 space. For each $x \in X$, we have that $\downarrow x = X \cap \downarrow x$, so $\downarrow x$ is b-open, and it is also b-closed since $X \setminus \downarrow x$ is b-open. Thus, the b-topology of X is always Hausdorff.

(3) Every saturated set A in a T_0 space X is b-closed. In fact, we have that

$$X \setminus A = \downarrow(X \setminus A) = \bigcup_{x \in X \setminus A} \downarrow x,$$

which is b-open by (2). Therefore, $A = \uparrow A$ is b-closed.

(4) For each b-closed set E of X, $E = \bigcap_{i \in I} U_i \cup (X \setminus V_i)$, where $U_i, V_i \in \mathcal{O}(X)$ for any $i \in I$. In fact, since $X \setminus E$ is b-open, for each $x \notin E$, there exists an open neighborhood V_x of x such that $V_x \cap \downarrow x \subseteq X \setminus E$, which implies that $X \setminus E = \bigcup_{x \notin E} V_x \cap \downarrow x$; thus $E = \bigcap_{x \notin E} (X \setminus \downarrow x) \cap (X \setminus V_x)$, completing the proof.

Definition 2.4. (1) A space X is a *retract* of space Y, if there exist two continuous maps $s : X \to Y$ (the *section*) and $r : Y \to X$ (the *retraction*) such that $r \circ s = \mathrm{id}_X$, the identity mapping on X [7].

(2) We call X a *b-retract* of Y if X is a retraction of Y such that $s(X)$ is b-dense in Y.

Remark 2.5 [7]. Every section $s : X \to Y$ is an embedding and every retraction $r : Y \to X$ is a quotient mapping.

Proposition 2.6 (19, 2.6; 13, Proposition 2.11). *If X and Y are T_0 spaces and X is a b-retract of Y, then X is homeomorphic to Y.*

In what follows, we shall denote by Top_0 (resp. Top_1, Sob) the category of all T_0 spaces (resp. T_1 spaces, sober spaces) with continuous mappings as morphisms. All subcategories of Top_0 are assumed to be full and closed under the formation of homeomorphic objects (i.e., satisfy (K2)).

Definition 2.7 [15]. A full subcategory $\boldsymbol{K}$ of Top_0 is *reflective* if, for each $X \in \mathrm{Top}_0$, there exists $X^k \in \boldsymbol{K}$ (the $\boldsymbol{K}$*-completion* for X) and a continuous mapping $\mu_X : X \to X^k$ (the $\boldsymbol{K}$*-reflection* for X) satisfying the universal property: for any continuous mapping $f : X \to Z$ to a space $Z \in \boldsymbol{K}$, there exists a unique continuous mapping $g : X^k \to Z$ such that $g \circ \mu_X = f$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "f"'] & X^k \arrow[d, dashed, "g"] \\ & Z \end{tikzcd}$$

Equivalently, $\boldsymbol{K}$ is reflective if the inclusion functor $I : \boldsymbol{K} \to \mathrm{Top}_0$ has a left adjoint (see IV-3 in [15]). The category Sob is a full reflective subcategory of Top_0. The Sob-completion for X is usually called the *sobrification* of X.

The following lemma can be easily verified by using the definition of $\boldsymbol{K}$-reflection.

Lemma 2.8. *Let $\mu_1 : X \to Y_1$ be a $\boldsymbol{K}$-reflection for X. Then, the following conditions are equivalent:*

(1) *$\mu_2 : X \to Y_2$ is a $\boldsymbol{K}$-reflection.*

(2) *There exists a (unique) homeomorphism $h : Y_1 \to Y_2$ such that $h \circ \mu_1 = \mu_2$.*

Definition 2.9. A mapping $e : X \to Y$ between topological spaces is called a *b-dense embedding*, if it is a topological embedding such that $e(X)$ is b-dense in Y.

Theorem 2.10 [11, Proposition 3.4, Corollary 3.5]. *Let X be a sober space and $Y \subseteq X$.*

(1) *The subspace Y is sober if and only if Y is b-closed.*

(2) *The inclusion mapping $e : Y \to Y^s$, $x \mapsto x$, is a sober reflection for Y, where Y^s is the b-closure of Y in X.*

Theorem 2.11 [11, Proposition 3.2]. *Let X be a T_0 space, Y a sober space and $f : X \to Y$ a continuous mapping. Then, f is a sober reflection for X if and only if it is a b-dense embedding.*

Theorem 2.12 [13, Theorem 3.2]. *Let $\boldsymbol{K}$ be a reflective subcategory of Top_0 such that $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. Then, each $\boldsymbol{K}$-reflection is a b-dense embedding.*

3. Main results

In this section, we present the main results, starting with a simple yet useful topological space in domain theory.

Definition 3.1 [6; 7]. The *Sierpiński space* is the Scott space $\Sigma 2$, where the underlying set 2 is the two-element chain $2 = \{0, 1\}$ with the order defined by $0 \leq 1$. Note that the open sets in this space are $\varnothing$, $\{0, 1\}$, and $\{1\}$.

Remark 3.2 [6; 7]. (1) For any set M, $(\Sigma 2)^M = \Sigma(2^M, \subseteq)$.

(2) Let X be a T_0 space and $M = \mathcal{O}(X)$. Then, the mapping $e : X \to (\Sigma 2)^M$, $x \mapsto (\chi_U(x))_{U \in M}$, is an embedding. Hence, by Theorem 2.10, X is a sober space if and only if $e(X)$ is a b-closed subset of $(\Sigma 2)^M$.

Lemma 3.3. *Let X be a T_0 space. Then, the following statements are equivalent:*

(1) *X is non-T_1.*

(2) *$\Sigma 2$ is a retract of X.*

(3) *$\Sigma 2$ is homeomorphic to a b-closed subspace of X.*

(4) *$\Sigma 2$ is homeomorphic to a subspace of X.*

Proof. (1) $\Rightarrow$ (2): Suppose X is non-T_1. Then, there exist $x_0, x_1 \in X$ such that $x_0 < x_1$. We define two mappings $s : \Sigma 2 \to X$ by $s(0) = x_0$ and $s(1) = x_1$, and $r : X \to \Sigma 2$ by

$$r(x) = \begin{cases} 0, & x \leq x_0, \\ 1, & \text{else} \end{cases}$$

for any $x \in X$. It is trivial to verify that both r and s are continuous mappings such that $r \circ s = \mathrm{id}_{\Sigma 2}$, where $\mathrm{id}_{\Sigma 2}$ is the identity mapping on $\Sigma 2$. Therefore, $\Sigma 2$ is a retract of X.

(2) $\Rightarrow$ (3): If $\Sigma 2$ is a retract of X, then by Remark 2.5, $\Sigma 2$ is homeomorphic to a subspace $\{x_1, x_2\}$ of X. In addition, by Remark 2.3(2), we know that bX is Hausdorff; thus $\{x_1, x_2\}$ is b-closed.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (1): Note that the T_1-separation property is hereditary. Then, since $\Sigma 2$ is a non-T_1 subspace of X up to homeomorphism, it follows that X is also a non-T_1 space. $\square$

As an immediate result of Lemma 3.3, the following corollary is clear.

Corollary 3.4. *Let $\boldsymbol{K}$ be a full subcategory of* Top_0. *Then, the following statements are equivalent*:

(1) $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$.

(2) *The space $\Sigma 2$ can be topologically embedded into some space Y that belongs to $\boldsymbol{K}$.*

The following lemma extends Result 2.5 in [19].

Lemma 3.5. *Let $X, Y, Z \in \mathrm{Top}_0$, $k : X \to Y$ be a continuous mapping such that $k(X)$ is b-dense in Y, and $f : X \to Z$ a continuous mapping.*

(1) *There exists at most one continuous mapping $g : Y \to Z$ such that $f = g \circ k$.*

(2) *If $g : Y \to Z$ is a continuous mapping such that $f = g \circ k$, then $g(Y) \subseteq \mathrm{cl}_b(f(X))$, where $\mathrm{cl}_b(f(X))$ is the b-closure of $f(X)$ in Z.*

Proof. (1) Suppose that there exist two continuous mappings $g_1, g_2 : Y \to Z$ such that $g_1 \circ k = g_2 \circ k = f$:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "g_1,\ g_2"] \\ & Z \end{tikzcd}$$

Let $y \in Y$. Suppose $V \in \mathcal{O}(Z)$ such that $g_1(y) \in V$. Then $y \in g_1^{-1}(V) \in \mathcal{O}(Y)$. Since $k(X)$ is b-dense in Y and $g_1^{-1}(V) \cap \downarrow y$ is b-open, $k(X) \cap g_1^{-1}(V) \cap \downarrow y \neq \varnothing$. In addition, since $g_1 \circ k = g_2 \circ k = f$, we deduce that $k(X) \cap g_1^{-1}(V) = k(X) \cap g_2^{-1}(V) \subseteq g_2^{-1}(V)$. It follows that $g_2^{-1}(V) \cap \downarrow y \neq \varnothing$, which implies that $y \in g_2^{-1}(V)$, i.e., $g_2(y) \in V$. These show that each open neighborhood of $g_1(y)$ contains $g_2(y)$; thus $g_1(y) \in \mathrm{cl}(\{g_2(y)\})$. Dually, it holds that $g_2(y) \in \mathrm{cl}(\{g_1(y)\})$. Since Z is a T_0 space, we have that $g_1(y) = g_2(y)$. Therefore, $g_1 = g_2$.

(2) Let $y \in Y$ and $V \in \mathcal{O}(Z)$ such that $g(y) \in V$. Then $y \in g^{-1}(V) \in \mathcal{O}(Y)$, and since $k(X)$ is b-dense in Y, $k(X) \cap g^{-1}(V) \cap \downarrow y \neq \varnothing$. Then there exists $x_0 \in X$ such that $k(x_0) \in g^{-1}(V) \cap \downarrow y$, which implies that $g(y) \geq g(k(x_0)) \in V$ (note that g is monotone since it is continuous); thus

$$f(x_0) = g(k(x_0)) \in f(X) \cap V \cap \downarrow g(y) \neq \varnothing.$$

This shows that $g(y) \in \mathrm{cl}_b(f(X))$. Hence, $g(Y) \subseteq \mathrm{cl}_b(f(X))$. $\square$

Theorem 3.6. *Let $\boldsymbol{K}$ be a reflective subcategory of* Top_0 *such that $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. Then, the following statements hold.*

(1) *$\boldsymbol{K}$ is b-closed-hereditary.*

(2) *The Sierpiński space $\Sigma 2 \in \boldsymbol{K}$. Hence, for any set M, the product $(\Sigma 2)^M \in \boldsymbol{K}$.*

(3) $\mathrm{Sob} \subseteq \boldsymbol{K}$.

Proof. (1) Let $X \in \boldsymbol{K}$, A be a b-closed subspace of X, and $\mu_A : A \to A^k$ be the $\boldsymbol{K}$-reflection for A. Then, $\mu_A(A)$ is a b-dense subset of A^k by Theorem 2.12. Consider the inclusion mapping $e : A \to X,\ x \mapsto x$. Then there exists a unique continuous mapping $f : A^k \to X$ such that $f \circ \mu_A = e$:

$$\begin{tikzcd} A \arrow[r, "\mu_A"] \arrow[dr, "e"'] & A^k \arrow[d, dashed, "f"] \\ & X \end{tikzcd}$$

Then by Lemma 3.5, we have $f(A^k) \subseteq \mathrm{cl}_b(e(A)) = A$, which shows that A is a b-dense retract of A^k. By Proposition 2.6, A is homeomorphic to A^k, and since $A^k \in \boldsymbol{K}$, it follows that $A \in \boldsymbol{K}$.

(2) Since $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$, there exists a T_0 and non-T_1 space $X \in \boldsymbol{K}$. By Lemma 3.3, $\Sigma 2$ is a b-closed subspace of X up to homeomorphism, and from result (1) it follows that $\Sigma 2 \in \boldsymbol{K}$. Since $\boldsymbol{K}$ is reflective, $\boldsymbol{K}$ is productive (see V-6 in [15]), and hence $(\Sigma 2)^M \in \boldsymbol{K}$.

(3) Let $X \in \mathrm{Sob}$. By Remark 3.2, there is an embedding $e : X \to (\Sigma 2)^M$ such that $e(X)$ is a b-closed subspace of $(\Sigma 2)^M$, where $M = \mathcal{O}(X)$. By (2), $(\Sigma 2)^M \in \boldsymbol{K}$ and since $\boldsymbol{K}$ is b-closed-hereditary, we have that $e(X) \in \boldsymbol{K}$, and since X is homeomorphic to $e(X)$, it follows that $X \in \boldsymbol{K}$. Hence, $\mathrm{Sob} \subseteq \boldsymbol{K}$. $\square$

Note that every saturated subset of a T_0 space is b-closed by Remark 2.3(3). Thus, the following corollary follows directly from Theorem 3.6(1).

Corollary 3.7. *Let $\boldsymbol{K}$ be a reflective subcategory of* Top_0 *such that $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. If $X \in \boldsymbol{K}$ and Y is a saturated subspace of X, then Y belongs to $\boldsymbol{K}$.*

Since there exist sober but non-T_1 spaces (such as $\Sigma 2$), the following corollary follows directly from Theorem 3.6(3).

Corollary 3.8. *Let $\boldsymbol{K}$ be a reflective subcategory of* Top_0. *Then, $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$ if and only if* $\mathrm{Sob} \subseteq \boldsymbol{K}$.

Recall that the reflective hull of a subcategory $\boldsymbol{C}$ of Top_0 is the smallest reflective subcategory of Top_0 containing $\boldsymbol{C}$. Let Sier be the full subcategory of Top_0 consisting of all T_0 spaces X which are homeomorphic to $\Sigma 2$.

Corollary 3.9 [16, Theorem 3.4]. *The reflective hull of* Sier *in* Top_0 *is* Sob.

Proof. Suppose $\boldsymbol{K}$ is a reflective subcategory of Top_0 such that $\mathrm{Sier} \subseteq \boldsymbol{K}$. Note that $\Sigma 2$ is a T_0 but non-T_1 space; thus $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. By Theorem 3.6(3), $\mathrm{Sob} \subseteq \boldsymbol{K}$. Since Sob is reflective, it is the smallest reflective subcategory of Top_0 having Sier as a subcategory. Therefore, Sob is the reflective hull of Sier in Top_0. $\square$

Lemma 3.10 [12, Lemma 5, p. 116]. *If $\{f_i : X \to Y_i\}_{i\in I}$ is a family of continuous mappings between T_0 spaces, then the diagonal $\Delta_{i\in I} f_i : X \to \prod_{i\in I} Y_i$ is a continuous mapping, where*

$$\forall x \in X, \quad (\Delta_{i\in I} f_i)(x) = (f_i(x))_{i\in I}.$$

A *skeleton* of a category $\boldsymbol{C}$ is a full subcategory, denoted by skC, such that each object of $\boldsymbol{C}$ is isomorphic to exactly one object of skC.

Remark 3.11. Some properties on the skeleton are listed below (see [1, Proposition 4.14, p. 51]):

(1) Every category has a skeleton.

(2) Any two skeletons of a category are isomorphic.

A category is a *small category* if its class of objects is a set.

Lemma 3.12. *For any cardinal number α, let $\mathbf{T}_\alpha$ be the full subcategory of* Top_0 *consisting of all T_0 spaces whose cardinality is less than or equal to α. Then, every skeleton of $\boldsymbol{T}_\alpha$ is a small category.*

Proof. Let skT_α be the full subcategory of $\boldsymbol{T}_\alpha$ consisting of all T_0 spaces of the form $(\beta, \mathcal{T})$, where β is a cardinal number such that $\beta \leq \alpha$, and $\mathcal{T}$ is an arbitrary T_0 topology on β. Then, it is clear that skT_α is a skeleton of $\boldsymbol{T}_\alpha$, and $|\mathrm{skT}_\alpha| \leq |\bigcup_{\beta\leq\alpha} 2^\beta|$, where $|\mathrm{skT}_\alpha|$ is the cardinality of the class of all objects of skT_α. Thus, the class of objects of skT_α is a set. Therefore, skT_α is a small category. □

Theorem 3.13. *Let $\boldsymbol{K}$ be a full subcategory of* Top_0 *such that $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. Then, the following statements are equivalent*:

(1) *$\boldsymbol{K}$ is reflective.*

(2) *$\boldsymbol{K}$ is productive and b-closed-hereditary.*

Proof. (1) ⇒ (2): It is well known that if $\boldsymbol{K}$ is reflective, then it is productive (see V-6 in [15]), and by Theorem 3.6, it is b-closed-hereditary.

(2) ⇒ (1): Let $X \in \mathrm{Top}_0$. We will complete the proof in a few steps.

Step 1: We define the full subcategory $\boldsymbol{C}(X)$ of $\boldsymbol{K}$ to consist of all objects Y such that there exists a continuous mapping $f : X \to Y$ with the property that $f(X)$ is b-dense in Y. Then, for each $Y \in \boldsymbol{C}(X)$, the sobrification $f(X)^s$ of $f(X)$ and the sobrification Y^s of Y are homeomorphic (see [11, Proposition 3.4]), which implies that

$$|Y| \leq |Y^s| = |f(X)^s| = |\mathrm{Irr}(f(X))| \leq 2^{|f(X)|},$$

where $\mathrm{Irr}(f(X))$ is the set of all irreducible closed sets in the subspace $f(X)$ of Y. Note that $|f(X)| \leq |X|$ (because f is a mapping), so $|Y| \leq 2^{|X|}$.

Let $\mathrm{skC}(X)$ be a skeleton of $\boldsymbol{C}(X)$. Since the cardinality of each space in $\mathrm{skC}(X)$ is less than or equal to $2^{|X|}$, by Lemma 3.12, $\mathbf{skC}(X)$ is a small category, so there is a cardinal number α such that $|\mathrm{skC}(X)| \leq \alpha$.

Step 2: Denote by $\Phi(X)$ the family of all pairs (Y, f), where $Y \in \mathbf{skC}(X)$ and $f : X \to Y$ is a continuous mapping such that $f(X)$ is b-dense in Y. For each $Y \in \mathbf{skC}(X)$, since the cardinality of the set of all continuous mappings from X to Y is less than or equal to $|Y|^{|X|}$, we have that

$$|\Phi(X)| \leq \left| \bigcup_{Y\in\mathrm{skC}(X)} |Y|^{|X|} \right|.$$

Thus, $\Phi(X)$ is a set, and then we may assume that $\Phi(X)=\{(Y_i, f_i): i\in I\}$, where I is a set. Therefore, for each $i\in I$, $Y_i\in \boldsymbol{K}$ and $f_i : X\to Y_i$ is a continuous mapping such that $f_i(X)$ is b-dense in Y_i.

Step 3: Let $X^k=\mathrm{cl}_b((\Delta_{i\in I} f_i)(X))$ be the b-closure of $(\Delta_{i\in I} f_i)(X)$ in the product space $\prod_{i\in I} Y_i$, where $\Delta_{i\in I} f_i : X\to \prod_{i\in I} Y_i$ is the diagonal (i.e., $x\mapsto (f_i(x))_{i\in I}$). Since $\{Y_i : i\in I\}\subseteq \boldsymbol{K}$ and $\boldsymbol{K}$ is productive, $\prod_{i\in I} Y_i\in \boldsymbol{K}$, and since $\boldsymbol{K}$ is b-closed-hereditary, $X^k\in \boldsymbol{K}$. Let $k : X\to X^k$ be the restriction of the diagonal $\Delta_{i\in I} f_i$, that is, $k(x)=(f_i(x))_{i\in I}$ for each $x\in X$. It is clear that k is continuous, and since $X^k=\mathrm{cl}_b((\Delta_{i\in I} f_i)(X))$, it follows that $k(X)=(\Delta_{i\in I} f_i)(X)$ is b-dense in X^k.

Step 4: Now, we prove that the subspace X^k of $\prod_{i\in I} Y_i$ with the mapping k is the $\boldsymbol{K}$-reflection for X. To see this, suppose $Y\in \boldsymbol{K}$ and $f : X\to Y$ is a continuous mapping. We consider the following two cases:

(c1) $f(X)$ is b-dense in Y. Then, $Y\in \boldsymbol{C}(X)$, and there is a homeomorphism h from Y to a unique space Z in skC(X). It is trivial to check that $h\circ f : X\to Z$ is a continuous mapping such that $h(f(X))$ is b-dense in Z, so $(Z, h\circ f)\in \Phi(X)$. Assume $(Z, h\circ f)=(Y_j, f_j)$ for some $j\in I$. Let $p_j : X^k\to Y_j$ be the restriction of the projection from $\prod_{i\in I} Y_i$ to Y_j (i.e., $(x_i)_{i\in I}\mapsto x_j$). Then p_j is a continuous mapping, and clearly $p_j\circ k=f_j$:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[dr, "f_j"'] & X^k \arrow[d, dashed, "p_j"] \\ & Y_j \end{tikzcd}$$

Let $\hat{f}=h^{-1}\circ p_j$. Then $\hat{f} : X^k\to Y$ is a continuous mapping such that

$$\hat{f}\circ k=(h^{-1}\circ p_j)\circ k=h^{-1}\circ (p_j\circ k)=h^{-1}\circ f_j=h^{-1}\circ (h\circ f)=(h^{-1}\circ h)\circ f=f.$$

$$\begin{tikzcd} X \arrow[rr, "k"] \arrow[dr, "f_j"] \arrow[ddr, "f"'] & & X^k \arrow[dl, "p_j"'] \arrow[ddl, dashed, "h^{-1}\circ p_j=\hat{f}"] \\ & Y_j \arrow[d, shift left, "h^{-1}"] & \\ & Y \arrow[u, shift left, "h"] & \end{tikzcd}$$

Recall that $k : X\to X^k$ is a continuous mapping such that $k(X)$ is b-dense in X^k. Then, by Lemma 3.5, $\hat{f}$ is the unique continuous mapping such that $\hat{f}\circ k=f$.

(c2) $f(X)$ is not b-dense in Y. Let $\mathrm{cl}_b(f(X))$ be the b-closure of $f(X)$ in Y with the relative topology. Then the corestriction $f^* : X\to \mathrm{cl}_b(f(X))$ of f (i.e., $\forall x\in X,\ f^*(x)=f(x)$) is a continuous mapping such that $f^*(X)$ is b-dense in $\mathrm{cl}_b(f(X))$. Since $Y\in \boldsymbol{K}$ and $\boldsymbol{K}$ is b-closed-hereditary, it follows that $\mathrm{cl}_b(f(X))\in \boldsymbol{K}$. Then using the argument of (c1), there is a continuous mapping $g : X^k\to \mathrm{cl}_b(f(X))$ such that $g\circ k=f^*$:

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[dr, "f^*"'] & X^k \arrow[d, dashed, "g"] \\ & \mathrm{cl}_b(f(X)). \end{tikzcd}$$

Let $e : \mathrm{cl}_b(f(X)) \to Y$ be the inclusion mapping. Then, $e \circ f^* = f$. Let $\hat{f} = e \circ g$. Then $\hat{f} : X^k \to Y$ is a continuous mapping such that $\hat{f} \circ k = (e \circ g) \circ k = e \circ (g \circ k) = e \circ f^* = f$:

$$\begin{tikzcd} X \arrow[rr, "k"] \arrow[dr, "f^*"] \arrow[ddr, "f"'] & & X^k \arrow[dl, dashed, "g"'] \arrow[ddl, dashed, "e\circ g=\hat{f}"] \\ & \mathrm{cl}_b(f(X)) \arrow[d, "e"] & \\ & Y & \end{tikzcd}$$

The uniqueness of $\hat{f}$ follows from Lemma 3.5.

All these show that $k : X \to X^k$ is a $\boldsymbol{K}$-reflection for X. Therefore, $\boldsymbol{K}$ is a reflective subcategory of Top_0. $\square$

Definition 3.14. We say that a full subcategory $\boldsymbol{K}$ of Top_0 *has equalizers* if it has equalizers in the sense of category theory. Specifically, for any continuous mappings $f, g : X \to Y$ in $\boldsymbol{K}$, the set $\{x \in X : f(x) = g(x)\}$ equipped with the subspace topology of X belongs to $\boldsymbol{K}$.

Lemma 3.15. *Let $X \in \mathrm{Top}_0$ and $E \subseteq X$. Then, the following statements are equivalent*:

(1) *E is b-closed in X.*

(2) *There exist continuous mappings $f, g : X \to (\Sigma 2)^M$ for some set M such that $E = \{x \in X : f(x) = g(x)\}$.*

(3) *There exist continuous mappings $f, g : X \to Y$ for some $Y \in \mathrm{Top}_0$ such that $E = \{x \in X : f(x) = g(x)\}$.*

Proof. (1) $\Rightarrow$ (2): Since E is b-closed, by Remark 2.3(4), we have that

$$E = \bigcap_{i \in M} (U_i \cup (X \setminus V_i)),$$

where $U_i, V_i \in \mathcal{O}(X)$ for all $i \in M$. Define $f, g : X \to (\Sigma 2)^M$ by

$$f(x)(i) = \chi_{U_i}(x) \quad \text{and} \quad g(x)(i) = \chi_{U_i \cup V_i}(x)$$

for any $x \in X$ and $i \in M$. It is easy to verify that both f and g are continuous, and for each $x \in X$, $f(x)(i) = g(x)(i)$ if and only if $x \in U_i \cup (X \setminus V_i)$ for all $i \in M$. It follows that $E = \{x \in X : f(x) = g(x)\}$.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Let $x \notin E$. That is, $f(x) \neq g(x)$. Since Y is T_0, we may assume $f(x) \not\leq g(x)$ without loss of generality. Then, there exists $V \in \mathcal{O}(Y)$ such that $f(x) \in V$ and $g(x) \notin V$. It follows that $x \in f^{-1}(V)$ and $x \notin g^{-1}(V)$. We claim that $E \cap f^{-1}(V) \cap {\downarrow} x = \varnothing$. In fact, if $y \in {\downarrow} x \cap f^{-1}(V) \cap E$, then $g(y) = f(y) \in V$ and $g(y) \leq g(x)$, and hence $g(x) \in V$, a contradiction. This shows that E is b-closed. $\square$

By Lemma 3.15, the following proposition is clear.

Proposition 3.16. *Let $\boldsymbol{K}$ be a full subcategory of* Top_0. *If* $\{(\Sigma 2)^M : M \text{ is a set}\} \subseteq \boldsymbol{K}$, *then the following statements are equivalent*:

(1) *$\boldsymbol{K}$ has equalizers.*

(2) *$\boldsymbol{K}$ is b-closed-hereditary.*

As an immediate result of Theorem 3.13 and Proposition 3.16, we have the following theorem.

Theorem 3.17 [8, 9.33 and 10.2.1]. *Let $\boldsymbol{K}$ be a full subcategory of* Top_0 *such that* $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. *Then, the following statements are equivalent*:

(1) *$\boldsymbol{K}$ is reflective.*

(2) *$\boldsymbol{K}$ is productive and has equalizers.*

Theorem 3.18. *Let $\boldsymbol{K}$ be a reflective subcategory of* Top_0 *such that* $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$, *and Z a sober space. Then, the following statements hold.*

(1) *If $\{X_i : i \in I\} \subseteq \boldsymbol{K}$ is a family of subspaces of Z, then the subspace $\bigcap_{i \in I} X_i$ of Z belongs to $\boldsymbol{K}$.*

(2) *For each subspace X of Z, the inclusion mapping $e^k : X \to \mathrm{cl}_k(X)$ is a $\boldsymbol{K}$-reflection for X, where* $\mathrm{cl}_k(X) = \bigcap\{A \in \boldsymbol{K} : X \subseteq A \subseteq Z\}$.

Proof. (1) We prove this in a few steps.

Step 1: Let $X = \bigcap_{i \in I} X_i$. Then, by Theorem 2.10(2), the inclusion mapping $e^s : X \to X^s$ is a sober reflection for X, where $X^s = \mathrm{cl}_b(X)$ is the b-closure of X in Z. Assume $\mu_X : X \to X^k$ is a $\boldsymbol{K}$-reflection for X. By Theorem 3.6(3), $X^s \in \boldsymbol{K}$, and thus there exists a unique continuous mapping $f : X^k \to X^s$ such that $f \circ \mu_X = e^s$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "e^s"'] & X^k \arrow[d, dashed, "f"] \\ & X^s \end{tikzcd}$$

Step 2: We prove that $f(X^k) = X$. Note that $X = e^s(X) = f(\mu_X(X)) \subseteq f(X^k)$. It remains to prove that $f(X^k) \subseteq X_i$ for each $i \in I$.

(c1) Let $e^i : X \to X_i$ be the inclusion mapping (note that X is a subspace of X_i). Since $X_i \in \boldsymbol{K}$, there exists a unique continuous mapping $f_i : X^k \to X_i$ such that $f_i \circ \mu_X = e^i$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "e^i"'] & X^k \arrow[d, dashed, "f_i"] \\ & X_i \end{tikzcd}$$

(c2) Let $(X_i)^s = \mathrm{cl}_b(X_i)$ be the b-closure of X_i in Z, which belongs to $\boldsymbol{K}$ by Theorem 3.6(3). Let $e^{is} : X \to (X_i)^s$ be the inclusion mapping (note that X is a subspace of X_i and X_i is a subspace of $(X_i)^s$). Then, there exists a unique continuous mapping $g : X^k \to (X_i)^s$ such that $g \circ \mu_X = e^{is}$:

$$\begin{tikzcd}
X \arrow[r, "\mu_X"] \arrow[dr, "e^{is}"'] & X^k \arrow[d, dashed, "g"] \\
& (X_i)^s
\end{tikzcd}$$

(c3) Let $e_i^{is} : X_i \to (X_i)^s$ be the inclusion mapping. Then, for each $x \in X$, by (c1) and (c2), we have that

$$(g \circ \mu_X)(x) \overset{\text{(c2)}}{=} e^{is}(x) = x = e^i(x) \overset{\text{(c1)}}{=} (f_i \circ \mu_X)(x) = ((e_i^{is} \circ f_i) \circ \mu_X)(x).$$

Hence, $g \circ \mu_X = (e_i^{is} \circ f_i) \circ \mu_X$. By the uniqueness of g, we deduce that $g = e_i^{is} \circ f_i$, i.e., the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "\mu_X"] \arrow[dr, "e^i"] \arrow[ddr, "e^{is}"'] & & X^k \arrow[dl, dashed, "f_i"'] \arrow[ddl, dashed, "g"] \\
& X_i \arrow[d, "e_i^{is}"] & \\
& (X_i)^s &
\end{tikzcd}.$$

(c4) Let $e_s^{is} : X^s \to (X_i)^s$ be the inclusion mapping (by noting that $X^s = \mathrm{cl}_b(X) \subseteq \mathrm{cl}_b(X_i) = (X_i)^s$). Then, for each $x \in X$, we have that

$$(g \circ \mu_X)(x) \overset{\text{(c2)}}{=} e^{is}(x) = x = e^s(x) \overset{\text{Step 1}}{=} (f \circ \mu_X)(x) = ((e_s^{is} \circ f) \circ \mu_X)(x).$$

Hence, $g \circ \mu_X = (e_s^{is} \circ f) \circ \mu_X$. By the uniqueness of g, we deduce that $g = e_s^{is} \circ f$, i.e., the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "\mu_X"] \arrow[dr, "e^s"] \arrow[ddr, "e^{is}"'] & & X^k \arrow[dl, dashed, "f"'] \arrow[ddl, dashed, "g"] \\
& X^s \arrow[d, "e_s^{is}"] & \\
& (X_i)^s &
\end{tikzcd}.$$

(c5) For each $y \in X^k$, we have that

$$f(y) = e_s^{is}(f(y)) \overset{\text{(c4)}}{=} g(y) \overset{\text{(c3)}}{=} e_i^{is}(f_i(y)) = f_i(y) \in X_i.$$

Thus, $f(X^k) \subseteq X_i$.

Therefore, $f(X^k) = \bigcap_{i \in I} X_i = X$.

Step 3: Now we have proved that the codomain of $f : X^k \to X^s$ is X, that is, $f(X^k) = X$. Then, we define the corestriction $\hat{f} : X^k \to f(X^k) = X$ of f, which is a continuous mapping such that $\hat{f} \circ \mu_X = \mathrm{id}_X$, the identity mapping on X. From Theorem 2.12, μ_X is a b-dense embedding, which implies that X is a b-retract of X^k; then by Proposition 2.6, X is homeomorphic to $X^k \in \boldsymbol{K}$. Therefore, $X = \bigcap_{i \in I} X_i \in \boldsymbol{K}$.

(2) We prove the conclusion in the following steps.

Step 1: Suppose that $\mu_X : X \to X^k$ is a $\boldsymbol{K}$-reflection for X. Applying result (1), we have that $\mathrm{cl}_k(X) \in \boldsymbol{K}$. Thus, there exists a unique continuous mapping $f : X^k \to \mathrm{cl}_k(X)$ such that $f \circ \mu_X = e^k$:

$$\begin{tikzcd}
X \arrow[r, "\mu_X"] \arrow[dr, "e^k"'] & X^k \arrow[d, dashed, "f"] \\
& \mathrm{cl}_k(X)
\end{tikzcd}$$

Step 2: Suppose that $\eta_{X^k} : X^k \to Y$ is a sober reflection for X^k. Let $X^s = \mathrm{cl}_b(X)$ be the b-closure of X in Z. Then, $X^s \in \mathrm{Sob} \subseteq \boldsymbol{K}$ by Theorem 3.6(3), which implies that $\mathrm{cl}_k(X) \subseteq X^s$. Let $e_k^s : \mathrm{cl}_k(X) \to X^s$ be the inclusion mapping. Then, there exists a unique continuous mapping $h : Y \to X^s$ such that $h \circ \eta_{X^k} = e_k^s \circ f$:

$$\begin{tikzcd}
X^k \arrow[r, "\eta_{X^k}"] \arrow[d, dashed, "f"] & Y \arrow[d, dashed, "h"] \\
\mathrm{cl}_k(X) \arrow[r, "e_k^s"] & X^s .
\end{tikzcd}$$

Step 3: Let $e^s = e_k^s \circ e^k : X \to X^s$ be the inclusion mapping. Using results of Step 2 and Step 3, we have that $h \circ \eta_{X^k} \circ \mu_X = (e_k^s \circ f) \circ \mu_X = e_k^s \circ (f \circ \mu_X) = e_k^s \circ e^k = e^s$:

$$\begin{tikzcd}
X \arrow[r, "\mu_X"] \arrow[dr, "e^k"] \arrow[ddr, "e^s"'] & X^k \arrow[r, "\eta_{X^k}"] \arrow[d, dashed, "f"] & Y \arrow[ddl, dashed, "h"] \\
& \mathrm{cl}_k(X) \arrow[d, "e_k^s"] & \\
& X^s &
\end{tikzcd}$$

By Theorems 2.11 and 2.12, both μ_X and η_{X^k} are b-dense embeddings, so is $\eta_{X^k} \circ \mu_X : X \to Y$, their composition. Then, by Theorem 2.11, $\eta_{X^k} \circ \mu_X$ is a sober reflection for X, and by Theorem 2.10, the inclusion mapping $e^s = e_k^s \circ e^k : X \to X^s$ is also a sober reflection for X. Applying Lemma 2.8, we deduce that h is a homeomorphism.

Step 4: We prove that $f(X^k) = h(\eta_{X^k}(X^k)) = \mathrm{cl}_k(X)$. On the one hand, by Step 3, it is clear that $X = e^s(X) = h(\eta_{X^k}(\mu_X(X))) \subseteq h(\eta_{X^k}(X^k)) \subseteq X^s \subseteq Z$. On the other hand, since η_{X^k} is an embedding and h is a homeomorphism, $h(\eta_{X^k}(X^k))$ is homeomorphic to $X^k \in \boldsymbol{K}$, so $h(\eta_{X^k}(X^k)) \in \boldsymbol{K}$. Recall that $\mathrm{cl}_k(X) = \bigcap\{K \in \boldsymbol{K} : X \subseteq K \subseteq Z\}$, so we have that

$$\mathrm{cl}_k(X) \subseteq h(\eta_{X^k}(X^k)) \overset{\text{Step 2}}{=} e_k^s(f(X^k)) = f(X^k) \subseteq \mathrm{cl}_k(X).$$

Therefore, $f(X^k) = h(\eta_{X^k}(X^k)) = \mathrm{cl}_k(X)$.

Step 5: Let $\hat{f} : X^k \to \mathrm{cl}_k(X)$ be the corestriction of $h \circ \eta_{X^k}$, i.e., $\hat{f}(y) = h(\eta_{X^k}(y))$ for any $y \in X^k$. Then, $\hat{f}$ is a homeomorphism, since $h \circ \eta_{X^k}$ is a topological embedding. In addition, by Step 4, it satisfies that $\hat{f}(\mu_X(x)) = h(\eta_{X^k}(\mu_X(x))) = f(\mu_X(x))$ for each $x \in X$. Hence, $\hat{f} \circ \mu_X = f \circ \mu_X$. By the uniqueness of f, we deduce that $f = \hat{f}$ is a homeomorphism:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "e^k"'] & X^k \arrow[d, dashed, "\hat{f}=f \text{ (a homeomorphism)}"] \\ & \mathrm{cl}_k(X) \end{tikzcd}$$

Therefore, by Lemma 2.8, $e^k : X \to \mathrm{cl}_k(X)$ is also a $\boldsymbol{K}$-reflection for X. $\square$

Theorem 3.19. *Let $\boldsymbol{K}$ be a reflective subcategory of* Top_0 *such that $\boldsymbol{K} \not\subseteq$* Top_1. *If $f : X \to Y$ is a continuous mapping from a sober space X to a sober space Y, then for any subspace Y_1 of Y, $Y_1 \in \boldsymbol{K}$ implies that the subspace $f^{-1}(Y_1)$ of X belongs to $\boldsymbol{K}$.*

Proof. Let $X_1 = f^{-1}(Y)$ and $(X_1)^k = \bigcap\{K \in \boldsymbol{K} : X_1 \subseteq K \subseteq X\}$ be the subspace of X. By Theorem 3.18, the inclusion mapping $e_1 : X_1 \to (X_1)^k$ is a $\boldsymbol{K}$-reflection for X_1. Consider the restriction $f_1 : X_1 \to Y_1$ $(x \mapsto f(x))$ of f, then there exists a unique continuous mapping $g_1 : (X_1)^k \to Y_1$ such that $g_1 \circ e_1 = f_1$:

$$\begin{tikzcd} X_1 \arrow[r, "e_1"] \arrow[dr, "f_1"'] & (X_1)^k \arrow[d, dashed, "g_1"] \\ & Y_1 \end{tikzcd}$$

Consider the composition $e_{Y_1} \circ f_1 : X_1 \to Y$, where $e_{Y_1} : Y_1 \to Y$ is the inclusion mapping. Since Y is a sober space, by Theorem 3.6(3), $Y \in \boldsymbol{K}$. Then, there exists a unique continuous mapping $g_2 : (X_1)^k \to Y$ such that $g_2 \circ e_1 = e_{Y_1} \circ f_1$:

$$\begin{tikzcd} X_1 \arrow[r, "e_1"] \arrow[dr, "f_1"'] & (X_1)^k \arrow[dr, dashed, "g_2"] & \\ & Y_1 \arrow[r, "e_{Y_1}"'] & Y \end{tikzcd}$$

Let $f_2 : (X_1)^k \to Y$ $(x \mapsto f(x))$ be the restriction of f. On the one hand, for each $x \in X_1$, we have $(f_2 \circ e_1)(x) = f(x) = (e_{Y_1} \circ f_1)(x) = (g_2 \circ e_1)(x)$, it follows that $f_2 \circ e_1 = g_2 \circ e_1$, which implies $g_2 = f_2$ by the uniqueness of g_2. On the other hand, $g_2 \circ e_1 = e_{Y_1} \circ f_1 = e_{Y_1} \circ (g_1 \circ e_1) = (e_{Y_1} \circ g_1) \circ e_1$, which implies that $e_{Y_1} \circ g_1 = g_2 = f_2$ by the uniqueness of g_2, i.e., the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[r, "e_1"] \arrow[dr, "f_1"'] & (X_1)^k \arrow[d, dashed, "g_1"] \arrow[dr, dashed, "g_2=f_2"] & \\ & Y_1 \arrow[r, "e_{Y_1}"] & Y \end{tikzcd}$$

Then for each $x \in (X_1)^k$, we have $f(x) = f_2(x) = g_2(x) = (e_{Y_1} \circ g_1)(x) = g_1(x) \in Y_1$, which implies $x \in f^{-1}(Y_1) = X_1$. Hence, $(X_1)^k \subseteq X_1$, and so $X_1 = (X_1)^k \in \boldsymbol{K}$. □

Using Theorems 3.6, 3.18 and 3.19, and Keimel and Lawson's result in [11], we obtain the main result in this paper.

Theorem 3.20. *Let $\boldsymbol{K}$ be a full subcategory of* Top_0 *such that $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. Then, the following statements are equivalent*:

(1) *$\boldsymbol{K}$ is reflective.*

(2) *$\boldsymbol{K}$ satisfies conditions* $(K1)$–$(K4)$.

By Theorems 3.13, 3.17 and 3.20, several equivalent conditions for the reflectivity of $\boldsymbol{K}$ are summarized as follows.

Theorem 3.21. *Let $\boldsymbol{K}$ be a full subcategory of* Top_0 *such that $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. Then, the following statements are equivalent*:

(1) *$\boldsymbol{K}$ is reflective in* Top_0.

(2) *$\boldsymbol{K}$ satisfies conditions* $(K1)$–$(K4)$.

(3) *$\boldsymbol{K}$ is productive and b-closed-hereditary.*

(4) *$\boldsymbol{K}$ is productive and has equalizers.*

Remark 3.22. Ershov [4] proved that $\boldsymbol{K}$ is reflective in Top_0 if and only if $\boldsymbol{K}$ satisfies the conditions (K1)–(K4) for every wide category $\boldsymbol{K}$, where a *wide category* $\mathbf{K}$ is a full subcategory of Top_0 such that every T_0 space X can be topologically embedded into some space Y belonging to $\boldsymbol{K}$. By Corollary 3.4, it is clear that every wide category $\boldsymbol{K}$ satisfies $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$. However, the converse is not true. For example, the full subcategory Sier of Top_0, consisting of all topological spaces that are homeomorphic to $\Sigma 2$, satisfies Sier $\nsubseteq \mathrm{Top}_1$ but is not a wide category. Consequently, Ershove's result can be regarded as a corollary of Theorem 3.21. Furthermore, the condition $\boldsymbol{K} \nsubseteq \mathrm{Top}_1$ of Theorem 3.21 is a common and easily checkable condition in domain theory. Additionally, the approach presented in this paper differs significantly from that employed in [4].

4. Some applications

By using the results in the last section, we investigate the reflectivity of several categories of T_0 spaces, including cosober spaces, strong d-spaces, k-bounded sober spaces, and consonant T_0 spaces. It is worth noting that all these classes of spaces are closed under the formation of homeomorphic objects.

Cosober spaces. In order to study the dual Hofmann–Mislove theorem, Escardó, Lawson and Simpson [5] introduced the cosober spaces [5], which are defined below.

Definition 4.1 [5]. Let X be a T_0 space, and Q a nonempty compact saturated subset of X.

(1) Q is called *k-irreducible* if for any compact saturated subsets Q_1, Q_2 of X, $Q = Q_1 \cup Q_2$ implies $Q = Q_1$ or $Q = Q_2$.

(2) X is called *cosober* if for each k-irreducible set Q, there exists a unique $x \in X$ such that $Q = {\uparrow} x$.

For a poset P, the family of all upper sets of P forms a topology, called the *Alexandroff topology* on P [7].

Lemma 4.2. (1) *Every poset equipped with the Alexandroff topology is cosober.*

(2) *A poset equipped with the Alexandroff topology is sober if and only if the poset is a dcpo. Hence, cosober spaces need not be sober.*

Proof. Let P be a poset equipped with the Alexandroff topology.

(1) Note that every nonempty compact saturated set in P is of the form $\uparrow F$, where F is a finite subset of P. Thus, every k-irreducible compact saturated set is of the form $\uparrow x$, where $x \in P$. Therefore, P is cosober.

(2) This follows immediately from the fact that the irreducible subsets of P are exactly the directed sets. □

Let co-Sob be the full subcategory of Top_0 consisting of all cosober spaces.

It is worth noting that the topology of the Sierpiński space $\Sigma 2$ coincides with the Alexandroff topology on the two-point chain $2 = \{0, 1\}$. Thus, by Lemma 4.2, $\Sigma 2$ is cosober, and since it is not T_1, we can conclude that co-Sob $\nsubseteq \mathrm{Top}_1$. The question of whether every sober space is cosober was raised in [5]. A negative answer was given by Wen and Xu in [20], where they proved that Isbell's complete lattice (see [10]) equipped with the lower topology is sober but not cosober. Furthermore, it has been proved in [18] that there exists a dcpo that is sober but not cosober with respect to the Scott topology. Therefore, we have that Sob $\nsubseteq$ co-Sob. Then, by applying Theorem 3.6(3), we obtain the following result.

Corollary 4.3. *The category co-*Sob *is not reflective in* Top_0.

Strong d-spaces. The class of strong d-spaces was introduced by Xu and Zhao [23], which lies between the classes of T_1 spaces and d-spaces.

Definition 4.4 [23]. A T_0 space X is called a *strong d-space* if for any $x \in X$, any directed subset D of X, and any open subset U of X, $\bigcap_{d\in D} \uparrow d \cap \uparrow x \subseteq U$ implies $\uparrow d_0 \cap \uparrow x \subseteq U$ for some $d_0 \in D$.

Let SD be the full subcategory of Top_0 consisting of all strong d-spaces.

In [23, Example 3.34], it was shown that there exists a continuous dcpo P whose Scott topology is not a strong d-space. However, it is well known that the Scott topology on any continuous dcpo is always sober. In addition, it has been noted in [23, Remark 3.21] that the Scott topology on every continuous lattice is a strong d-space. Therefore, Sob $\nsubseteq$ SD and SD $\nsubseteq \mathrm{Top}_1$. By applying Theorem 3.6(3), we deduce the following result.

Corollary 4.5. *The category SD is not reflective in* Top_0.

k-Bounded sober spaces. In [25], Zhao and Ho introduced another weaker notion of sobriety, called *k-bounded sobriety*. This notion is defined as follows.

Definition 4.6 [25]. A T_0 space X is *k-bounded sober* if for any irreducible closed subset F of X with $\bigvee F$ existing, there is a unique point $x \in X$ such that $F = \downarrow x$.

Let KSob be the full subcategory of Top_0 consisting of all k-bounded sober spaces. It is clear that Sob $\subseteq$ KSob and Sob $\nsubseteq \mathrm{Top}_1$. Thus, we conclude that KSob $\nsubseteq \mathrm{Top}_1$.

Example 4.7. Let $X = [0, 3]$ equipped with the Scott topology (i.e., the open sets are $\varnothing$, $[0, 3]$ and all sets of the form $(x, 3]$, where $x \in [0, 3]$). Since $[0, 3]$ is a continuous lattice, we know that X is a sober space, and hence it is also k-bounded sober. For each integer $n \geq 2$, let $X_n = [0, 1) \cup \big(2 - \frac{1}{n}, 2 + \frac{1}{n}\big)$. We have the following facts.

(1) Each subspace X_n of X is k-bounded sober. To show this, let F be an irreducible closed set in X_n and $x \in X_n$ such that $\bigvee_{X_n} F = x$. There are two cases:

(c1) $x \in [0, 1)$. Then, $F \subseteq {\downarrow} x \subseteq [0, 1)$, which follows that $\mathrm{cl}_{X_n}(F) = \mathrm{cl}_{X_n}(\{x\})$.

(c2) $x \in \big(2 - \frac{1}{n}, 2 + \frac{1}{n}\big)$. Then, $F \cap \big(2 - \frac{1}{n}, 2 + \frac{1}{n}\big) \neq \varnothing$, which implies that

$$x = \bigvee_{X_n} F = \bigvee_{X_n} F \cap \Big(2 - \frac{1}{n}, 2 + \frac{1}{n}\Big) = \bigvee_{X} F \cap \Big(2 - \frac{1}{n}, 2 + \frac{1}{n}\Big) = \bigvee_{X} F \cap X_n = \bigvee_{X} F.$$

Since X is sober, we have $\mathrm{cl}_X(F) = \mathrm{cl}_X(\{x\})$, and thus $\mathrm{cl}_{X_n}(F) = \mathrm{cl}_X(F) \cap X_n = \mathrm{cl}_X(\{x\}) \cap X_n = \mathrm{cl}_{X_n}(\{x\})$, where the last equality holds because $x \in X_n$.

All these show that X_n is k-bounded sober.

(2) The intersection $Y = \bigcap_{n \geq 2} X_n = [0, 1) \cup \{2\}$ equipped with the subspace topology of X is not k-bounded sober. In fact, the set $F := [0, 1)$ is irreducible since it is directed, and $\bigvee_Y F = 2$. In addition, since $[0, 1]$ is a closed set in X and $F = [0, 1] \cap Y$, we have that F is a closed set in Y. For each $x \in [0, 1)$, we have that $\mathrm{cl}_Y(\{x\}) = [0, x] \neq F$, and $\mathrm{cl}_Y(\{2\}) = Y \neq F$. Therefore, Y is not a k-bounded sober space.

The above example shows that KSob does not satisfy (K3). Thus, by Theorem 3.21, we obtain the following corollary.

Corollary 4.8 [14]. *The category* KSob *is not reflective in* Top_0.

Consonant spaces. The class of consonant spaces was introduced by Dolecki, Greco and Lechicki in [2], which plays an important role in discussion of the equality of the Isbell topology and the compact-open topology on function spaces [17]. The definition is given as follows.

Definition 4.9 [2]. A topological space X is called *consonant* if for every Scott open subset $\mathcal{U}$ of $\mathcal{O}(X)$, there exists a family $\{K_i : i \in I\}$ of compact subsets of X such that $\mathcal{U} = \bigcup_{i \in I} \mathcal{N}(K_i)$, where

$$\mathcal{N}(K_i) := \{U \in \mathcal{O}(X) : K_i \subseteq U\} \quad \text{for all } i \in I.$$

Let Const be the full subcategory of Top_0 consisting of all consonant T_0 spaces. We note that:

(1) Every finite topological space X is consonant, since every subset of X is compact. As a consequence, $\Sigma 2$ is a consonant T_0 but non-T_1 space, so we have that Const $\nsubseteq \mathrm{Top}_1$.

(2) Nogura and Shakhmatov [17] have shown that there exists a metric space (hence is sober) that is not consonant, so we have that Sob $\nsubseteq$ Const.

Therefore, by Theorem 3.6(3), we obtain the following corollary.

Corollary 4.10. *The category* Const *is not reflective in* Top_0.

5. Conclusion

In this paper we proved that if a reflective subcategory of Top_0 contains a non-T_1 space and satisfies condition (K2) proposed by Lawson and Keimel, then it also satisfies the remaining conditions (K1), (K3) and (K4). Based on this result, we concluded that several subcategories are not reflective, thus giving negative answers to some open problems. We expect that this result might also serve as a tool for verifying the reflectivity of other subcategories of Top_0.

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