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On well-filtered reflections of T_0 spaces [☆]

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ABSTRACT

Following Ershov's method of constructing the d-completion of T_0 spaces, we give a direct construction of the well-filtered reflection of T_0 spaces. Also, we obtain an elegant characterization of well-filtered spaces using KF-sets. We then show that a product of a family of T_0 spaces is well-filtered iff each factor space is well-filtered. Finally, we obtain that the well-filtered reflection of a product of a finite family of T_0 spaces is the product of the well-filtered reflections of all factor spaces. A common theme is that KF-sets, introduced by the first and the fourth authors, function prominently in all of the above proofs.

1. Introduction and preliminaries

Domain theory, initiated by Dana Scott in the late 1960s [7,8], plays a central role in computer science. It is used to specify denotational semantics, especially for programming languages. The sober spaces, well-filtered spaces and d-spaces form three of the most important and heavily studied classes of topological spaces in domain theory. One of the most basic properties people are usually concerned about a subcategory is whether it is reflective in the given category. Using different methods, various researchers showed that the

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category of all sober spaces (d-spaces) is reflective in the category of all T_0 spaces. But for quite a long time, it was not known whether the category of all well-filtered spaces is reflective in the category of all T_0 space. Recently, Wu, Xi, Xu and Zhao [9] gave a positive answer to the above problem. Their strategy is to use the criteria for the existence of K-fication suggested by Keimel and Lawson in [6]. Specifically, taking a T_0 space X as a subspace of the sober space X^s, Wu, Xi, Xu and Zhao proved that the intersection of all well-filtered subspaces of X^s that contains X is the well-filtered reflection of X. However, it is still not known how to construct the well-filtered reflection of a space X directly from itself, and we seek to fill in this gap in this paper.

For the cases of sobrification and d-completion, their constructions are clear. This is due to the simple descriptions for sober spaces and d-spaces, which we recall below:

- A T_0 space X is sober iff for each irreducible set F, there exists $x \in X$ such that $\mathrm{cl}(F) = \mathrm{cl}(\{x\})$;
- A T_0 space X is a d-space iff for each directed set D with respect to the specialization order, there exists $x \in X$ such that $\mathrm{cl}(D) = \mathrm{cl}(\{x\})$.

Results show that the sobrification of X can be obtained from X by adding the closed irreducible sets which are not singletons. The d-completion of X, as shown by Ershov's [2], can be obtained by adding the closure of directed sets onto X (and then repeating this process). So an analogue for well-filtered spaces will be of utmost importance to fill in the gap highlighted above. The question is: What conditions should the candidate satisfy? We can base our selection upon the irreducible sets (resp., directed sets) in sober spaces (resp., d-spaces). Basically, we need to select the sets of a T_0 space such that this hyperspace is well-filtered.

Guided by Ershov's method [2] of constructing the d-completion of T_0 spaces, we give a direct construction of the well-filtered reflection for any given T_0 space. At this point, we highlight that our approach relies heavily on the KF-sets of the space, which were initially introduced by the first and the fourth authors.

In Section 2, we will introduce the notion of KF-sets and establish some useful results related to well-filteredness. The following two results feature more extensively among the rest.

(1) A T_0 space X is well-filtered iff for each KF-set F, there exists a unique $x \in X$ such that $\mathrm{cl}(F) = \mathrm{cl}(\{x\})$.
(2) A product of a family of T_0 spaces is well-filtered iff each space in the family is well-filtered.

We remark that the second result strengthens an existing result in [9].

In Section 3, we construct the well-filtered reflection of T_0 spaces by using the notion of KF-sets and its related results obtained in Section 2. Additionally, we prove that the well-filtered reflection of the product of a finite family of T_0 spaces equals the product of the well-filtered reflection of the spaces in the family.

For a T_0 space X, the specialization order, written by $\sqsubseteq_X$ (or just $\sqsubseteq$), is defined as $x \sqsubseteq_X y$ iff $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator. A subset A of X is *saturated* if it equals the intersection of all open sets containing A, that is,

$$A = \bigcap\{U \subseteq X : U \text{ is open and } A \subseteq U\}.$$

In what follows, we use $\uparrow_X A$ (or just $\uparrow A$) to denote the set $\{x \in X : a \sqsubseteq_X x \text{ for some } a \in A\}$. Dually, $\downarrow_X A = \{x \in X : x \sqsubseteq_X a \text{ for some } a \in A\}$. One can easily check that A is saturated iff $A = \uparrow_X A$. As in general posets, $\downarrow_X x$ (resp., $\uparrow_X x$) denotes $\downarrow_X \{x\}$ (resp., $\uparrow_X \{x\}$). Note that for any point x, $\downarrow_X x = \mathrm{cl}(\{x\})$.

A nonempty subset A of a space X is *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called *sober* if for any irreducible closed set F, $F = \downarrow x$ for some $x \in X$.

Remark 1.1. (1) Let X be a subspace of Y. The following statements are equivalent for a subset $A \subseteq X$:

(i) A is an irreducible subset of X;

(ii) A is an irreducible subset of Y;
(iii) the closure $\overline{A}$ of A is an irreducible subset of Y.

(2) The images under continuous mappings of irreducible sets are irreducible.

Definition 1.2. [3] A T_0 space X is called *well-filtered* if for any open set U and any filtered family $\mathcal{F}$ of saturated compact subsets of X, $\bigcap \mathcal{F} \subseteq U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$.

Remark 1.3. Every sober space is well-filtered.

For a family $\{X_i : i \in I\}$ of topological spaces, their Cartesian product $\prod_{i\in I} X_i$ has a subbase of open sets of the form:

$$p_i^{-1}(U_i),$$

where $i \in I$, $U_i \in \mathcal{O}(X_i)$ and the mapping $p_i : \prod_{i\in I} X_i \longrightarrow X_i$ assigns to $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ its ith coordinate $x_i \in X_i$, called the ith *projection*.

Given a T_0 space X, denote by $\mathsf{K}(X)$ the set of all compact saturated subsets of X and $\mathcal{O}(X)$ the set of all open subsets of X. We write

$$\mathcal{K} \subseteq_{flt} \mathsf{K}(X)$$

for the case that $\mathcal{K}$ is a filtered subfamily of $\mathsf{K}(X)$ ($\forall K_1, K_2 \in \mathcal{K}$, there exists $K \in \mathcal{K}$ such that $K \subseteq K_1 \cap K_2$).

The following result can be derived from (the Topological Rudin Lemma) Lemma 3.1 in [4].

Lemma 1.4. *Let X be a T_0 space, C a closed subset of X and $\mathcal{K} \subseteq_{flt} \mathsf{K}(X)$. If C intersects all members of $\mathcal{A}$, then there exists a minimal (irreducible) closed subset F of C that intersects all members of $\mathcal{A}$.*

2. KF-sets and their properties

Obviously, not every T_0 space is well-filtered. So to construct a well-filtered reflection for a non-well-filtered space X, we need to look deeper and identify the behavior of certain sets analogous to the irreducible sets (resp., directed sets) in a sober space (resp., d-space). Let us consider the following discussion.

For the non-well-filtered space X, there exist $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$ such that

$$\bigcap_{i\in I} K_i \subseteq U, \text{ but } K_i \cap (X \setminus U) \neq \emptyset, \forall i \in I.$$

Using Lemma 1.4, there exists a minimal closed set A such that

$$A \subseteq X \setminus U \text{ such that } A \cap K_i \neq \emptyset \text{ for all } i \in I.$$

Observe that A can not be of the form $\mathrm{cl}(\{x\}) = {\downarrow_X} x$ for some $x \in X$ (Otherwise, $A = {\downarrow_X} x$, thus $\forall i \in I$, ${\downarrow_X} x \cap K_i \neq \emptyset$, which implies $x \in \bigcap_{i\in I} K_i \subseteq U$, contradicting $A \cap U = \emptyset$). These roughly explain the causes for a T_0 space to be non-well-filtered: there is such a set 'A' (what we called KF-sets) that is not the closure of a point.

The above motivates the following notion of KF-sets.

Definition 2.1. Let X be a T_0 space. A nonempty subset A of X is said to have the *compactly filtered property* (*KF property*), if there exists $\mathcal{K} \subseteq_{flt} \mathsf{K}(X)$ such that $\mathrm{cl}(A)$ is a minimal closed set that intersects all members of $\mathcal{K}$.

We call such a set *KF*, or a *KF-set*. Denote by $\mathsf{KF}(X)$ the set of all closed KF subsets of X.

Remark 2.2. At first glance, the notion of KF-sets seems to be heavily dependent on the other objects of the space (i.e., the filtered families of $\mathsf{K}(X)$).

However, it turns out that this notion is situated 'between' the familiar notions of directed sets and irreducible sets, as these three notions are connected by the following chain of implications.

$$\text{directed set} \quad \Rightarrow \quad \text{KF-set} \quad \Rightarrow \quad \text{irreducible set}$$

The first and second implications follow from Corollary 2.4 and Lemma 2.9, respectively.

The following result shows that KF-sets can be characterized by a specific formula, which will be used frequently in the subsequent proof.

Proposition 2.3. *Let X be a T_0 space and $A \subseteq X$. Then the following statements are equivalent.*

(1) A is a KF-set.
(2) $\mathrm{cl}(A)$ is a KF-set.
(3) There exists $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ satisfying the following conditions:

  (i) $\forall i \in I$, $K_i \cap \mathrm{cl}(A) \neq \emptyset$;
  (ii) $\forall (x_i)_{i\in I} \in \prod_{i\in I} (K_i \cap \mathrm{cl}(A))$, $\mathrm{cl}(\{x_i : i \in I\}) = \mathrm{cl}(A)$.

Proof. (1) $\Leftrightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Assume that $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ is a filtered family such that $\mathrm{cl}(A)$ is the minimal closed set that intersects all K_i ($i \in I$). We only need to verify condition (ii).

Let $(x_i)_{i\in I} \in \prod_{i\in I}(K_i \cap \mathrm{cl}(A))$. Note that $\mathrm{cl}(\{x_i : i \in I\}) \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(\{x_i : i \in I\}) \cap K_i \neq \emptyset$ for all $i \in I$. By the minimality of $\mathrm{cl}(A)$, we must have $\mathrm{cl}(\{x_i : i \in I\}) = \mathrm{cl}(A)$.

(3) $\Rightarrow$ (2). It suffices to show that $\mathrm{cl}(A)$ is the minimal closed set that intersects all K_i ($i \in I$). Suppose that B is a closed set that intersects all K_i ($i \in I$) and $B \subseteq \mathrm{cl}(A)$. Then $\prod_{i\in I}(K_i \cap B) \neq \emptyset$ and let $(x_i)_{i\in I} \in \prod_{i\in I}(K_i \cap B)$, thus $(x_i)_{i\in I} \in \prod_{i\in I}(K_i \cap \mathrm{cl}(A))$. By (ii), we have

$$\mathrm{cl}(A) = \mathrm{cl}(\{x_i : i \in I\}) \subseteq B \subseteq \mathrm{cl}(A),$$

implying $B = \mathrm{cl}(A)$. So $\mathrm{cl}(A)$ is minimal and thus KF. $\square$

Let D be a subset of a T_0 space X. If D is directed with respect to the specialization order, then one can easily verify that $\{\uparrow d : d \in D\} \subseteq_{flt} \mathsf{K}(X)$ and satisfies the conditions (i)-(ii) of Proposition 2.3. This yields the following.

Corollary 2.4. *Every subset of a T_0 space that is directed with respect to the specialization is KF.*

Lemma 2.5. *Let X, Y be two T_0 spaces, $f : X \longrightarrow Y$ a continuous mapping and $A \subseteq X$. If A is a KF-set, then $f(A)$ is a KF-set.*

Proof. Assume that A is a KF-set. By Proposition 2.3, there exists $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ satisfying the following conditions:

(i) $\forall i \in I$, $K_i \cap \mathrm{cl}_X(A) \neq \emptyset$;
(ii) $\forall (x_i)_{i\in I} \in \prod_{i\in I} (K_i \cap \mathrm{cl}_X(A))$, $\mathrm{cl}_X(\{x_i : i \in I\}) = \mathrm{cl}_X(A)$.

For each $i \in I$, let $Q_i = \uparrow_Y f(K_i \cap \mathrm{cl}_X(A))$. Since f is continuous, the family $\{Q_i : i \in I\} \subseteq_{flt} \mathsf{K}(Y)$. Note that $f(K_i \cap \mathrm{cl}_X(A)) \subseteq f(\mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A))$, which implies $Q_i \cap \mathrm{cl}_Y(f(A)) \neq \emptyset$ for all $i \in I$. By Proposition 2.3, we need to prove $\mathrm{cl}_Y(\{y_i : i \in I\}) = \mathrm{cl}_Y(f(A))$ for $(y_i)_{i\in I} \in \prod_{i\in I}(Q_i \cap \mathrm{cl}_Y(f(A)))$.

For each $i \in I$, $y_i \in Q_i = \uparrow_Y f(K_i \cap \mathrm{cl}_X(A))$, then there is $x_i \in K_i \cap \mathrm{cl}_X(A)$ such that $f(x_i) \sqsubseteq_Y y_i$. It follows that $\{f(x_i) : i \in I\} \subseteq \downarrow_Y\{y_i : i \in I\} \subseteq \mathrm{cl}_Y(\{y_i : i \in I\})$, and we have

$$\mathrm{cl}_Y(f(A)) = \mathrm{cl}_Y(f(\{x_i : i \in I\})) \subseteq \mathrm{cl}_Y(\{y_i : i \in I\}) \subseteq \mathrm{cl}_Y(f(A)).$$

This implies that $\mathrm{cl}_Y(\{y_i : i \in I\}) = \mathrm{cl}_Y(f(A))$. By Proposition 2.3, $f(A)$ is a KF-set. □

The next result provides a new characterization for the well-filtered spaces, which plays a crucial role in constructing the well-filtered reflection of T_0 spaces.

Theorem 2.6. *Let X be a T_0 space. Then the following statements are equivalent:*

(1) X is well-filtered;
(2) $\forall A \in \mathsf{KF}(X)$, there exists a $x \in X$ such that $A = \downarrow x$.

Proof. (1) ⇒ (2). Assume that X is well-filtered and $A \in \mathsf{KF}(X)$. There exists $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ such that A is the minimal closed subset of X that intersects all K_i $(i \in I)$.

First, we claim $\left(\bigcap_{i\in I} K_i\right) \cap A \neq \emptyset$. Otherwise, $\bigcap_{i\in I} K_i \subseteq X \setminus A$. Since X is well-filtered, there is $i \in I$ such that $K_i \subseteq X \setminus A$, i.e., $A \cap K_i = \emptyset$, a contradiction. Take one $x \in \left(\bigcap_{i\in I} K_i\right) \cap A$. From the minimality of A, it follows that $A = \downarrow x$. It is easily observed that $\left(\bigcap_{i\in I} K_i\right) \cap A = \{x\}$.

(2) ⇒ (1). Assume, on the contrary, that X is not well-filtered. Then there exist $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$ such that

$$\bigcap\nolimits_{i\in I} K_i \subseteq U \text{ and } K_i \nsubseteq U,\ \forall i \in I.$$

Then by Lemma 1.4, there exists a minimal closed set $A \subseteq X \setminus U$ such that $A \cap K_i \neq \emptyset$, $\forall i \in I$. This means $A \in \mathsf{KF}(X)$. Then by condition (2), there exists $x \in X$ such that $A = \downarrow x$. For each $i \in I$, $\downarrow x \cap K_i \neq \emptyset$, so $x \in \bigcap_{i\in I} K_i \subseteq U$, contradicting $\downarrow x \subseteq X \setminus U$. Hence, X is well-filtered. □

In the proof of the implication (1) ⇒ (2) in the above theorem, we can obtain the following result.

Corollary 2.7. *Let X be a well-filtered space and $A \in \mathsf{KF}(X)$. Then there exists $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ such that $\left(\bigcap_{i\in I} K_i\right) \cap A = \{x\}$ for some $x \in X$. In addition, if in this case, $A = \downarrow x$.*

Proposition 2.8. *Let $\{X_i : i \in I\}$ be a family of T_0 spaces. If A is an irreducible subset of $X = \prod_{i\in I} X_i$, then*

$$\mathrm{cl}_X(A) = \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A)).$$

Proof. Recall that $\prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A)) = \mathrm{cl}_X\left(\prod_{i\in I} p_i(A)\right)$ (see Proposition 2.3.3 of [1]). Since $A \subseteq \prod_{i\in I} p_i(A)$, it holds that $\mathrm{cl}_X(A) \subseteq \mathrm{cl}_X\left(\prod_{i\in I} p_i(A)\right) = \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A))$. For the reverse inclusion, let $(x_i)_{i\in I} \in \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A))$. Suppose that U is an open neighborhood of $(x_i)_{i\in I}$. Then there exists finite $I_0 \subseteq I$ and $U_i \in \mathcal{O}(X_i)$ $(i \in I_0)$, such that $(x_i)_{i\in I} \in \bigcap_{i\in I_0} p_i^{-1}(U_i) \subseteq U$. For each $i \in I_0$, since $x_i \in \mathrm{cl}_{X_i}(p_i(A))$, it follows that $U_i \cap p_i(A) \neq \emptyset$, that is, $A \cap p_i^{-1}(U_i) \neq \emptyset$. Since A is irreducible and I_0 is finite, $A \cap \bigcap_{i\in I_0} p_i^{-1}(U_i) \neq \emptyset$, thus $A \cap U \neq \emptyset$. This implies $(x_i)_{i\in I} \in \mathrm{cl}_X(A)$. Thus $\prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A)) \subseteq \mathrm{cl}_X(A)$. So $\mathrm{cl}_X(A) = \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A))$. □

Lemma 2.9. *Let X be a T_0 space and A a subset of X. If A is a KF-set, then A is irreducible.*

Proof. It suffices to prove $\mathrm{cl}(A)$ is irreducible. Suppose $\mathrm{cl}(A) = B \cup C$ with closed sets $B, C \subseteq X$. Since A is KF, there exists $\{K_i : i \in I\} \subseteq_{flt} \mathsf{K}(X)$ such that $\mathrm{cl}(A)$ is the minimal closed set such that $\mathrm{cl}(A) \cap K_i \neq \emptyset$, $\forall i \in I$. Let

$$I_B = \{i \in I : K_i \cap B \neq \emptyset\} \text{ and } I_C = \{i \in I : K_i \cap C \neq \emptyset\}.$$

Claim. *$I = I_B$ or $I = I_C$.*

Assume, on the contrary, $I \neq I_B$ and $I \neq I_C$. Then there exist $i_1 \in I \setminus I_B$ and $i_2 \in I \setminus I_C$, which implies that $K_{i_1} \cap B = \emptyset$ and $K_{i_2} \cap C = \emptyset$. By the filteredness of $\{K_i : i \in I\}$, there exists $i_3 \in I$ such that $K_{i_3} \subseteq K_{i_1} \cap K_{i_2}$. It follows that $K_{i_3} \cap B = \emptyset$ and $K_{i_3} \cap C = \emptyset$, implying that $K_{i_3} \cap A = (K_{i_3} \cap B) \cup (K_{i_3} \cap C) = \emptyset$, which is a contradiction.

Without loss of generality, assume $I_B = I$, i.e., $K_i \cap B \neq \emptyset$, $\forall i \in I$. Since $B \subseteq \mathrm{cl}(A)$ and $\mathrm{cl}(A)$ is minimal, we have $\mathrm{cl}(A) = B$. So $\mathrm{cl}(A)$ is irreducible. □

Theorem 2.10. *Let $\{X_i : i \in I\}$ be a family of T_0 spaces and $A \subseteq \prod_{i \in I} X_i$. Then the following statements are equivalent:*

(1) $A \in \mathsf{KF}\left(\prod_{i \in I} X_i\right)$;
(2) For each $i \in I$, $\mathrm{cl}_{X_i}(p_i(A)) \in \mathsf{KF}(X_i)$.

Proof. (1) $\Rightarrow$ (2). This follows from Lemma 2.5 and Proposition 2.3.

(2) $\Rightarrow$ (1). Assume for each $i \in I$, $\mathrm{cl}_{X_i}(p_i(A)) \in \mathsf{KF}(X_i)$. By Proposition 2.8 and Lemma 2.9, we need to show $A = \prod_{i \in I} \mathrm{cl}_{X_i}(p_i(A)) \in \mathsf{KF}\left(\prod_{i \in I} X_i\right)$. Then for each $i \in I$, there exists $\mathcal{K}_i \subseteq_{flt} \mathsf{K}(X_i)$ such that $\mathrm{cl}_{X_i}(p_i(A))$ is the minimal closed set that intersects all members of $\mathcal{K}_i$. Let

$$\mathcal{K} = \left\{K_f = \prod_{i \in I} f(i) : f \in \prod_{i \in I} \mathcal{K}_i\right\},$$

where $f(i) = p_i(f) \in \mathcal{K}_i$. Then by Tychonoff's Theorem (see Theorem 3.2.4 of [1]), $\mathcal{K} \subseteq \mathsf{K}(\prod_{i \in I} X_i)$. Since $\mathcal{K}_i$ is filtered for each $i \in I$, one can easily verify that $\mathcal{K}$ is filtered.

Claim 1. *$\forall f \in \prod_{i \in I} \mathcal{K}_i$, $K_f \cap A \neq \emptyset$.*

Note that $f(i) \cap \mathrm{cl}_{X_i}(p_i(A)) \neq \emptyset$ for all $i \in I$ and thus by Lemma 2.8, we have

$$K_f \cap A = \left(\prod_{i \in I} f(i)\right) \cap \left(\prod_{i \in I} \mathrm{cl}_{X_i}(p_i(A))\right) = \prod_{i \in I} \left(f(i) \cap \mathrm{cl}_{X_i}(p_i(A))\right) \neq \emptyset.$$

Claim 2. *$\forall \varphi \in \prod_{f \in \prod_{i \in I} \mathcal{K}_i} (K_f \cap A)$, $\mathrm{cl}_X\left(\{\varphi(f) : f \in \prod_{i \in I} \mathcal{K}_i\}\right) = A$.*

Let $\varphi \in \prod_{f \in \prod_{i \in I} \mathcal{K}_i} (K_f \cap A) = \prod_{f \in \prod_{i \in I} \mathcal{K}_i} \left(\prod_{i \in I} f(i) \cap \mathrm{cl}_{X_i}(p_i(A))\right)$. So for each $f \in \prod_{i \in I} \mathcal{K}_i$ and $i \in I$, $\varphi(f)(i) \in f(i) \cap \mathrm{cl}_{X_i}(p_i(A))$. Clearly, we only need to show $A \subseteq \mathrm{cl}_X\left(\{\varphi(f) : f \in \prod_{i \in I} \mathcal{K}_i\}\right)$. Let $x \in A$ and U an open neighborhood of x. Then there exists a finite $I_0 \subseteq I$ and $U_j \in \mathcal{O}(X_j)$ $(\forall j \in I_0)$ such that $x \in \bigcap_{j \in I_0} p_i^{-1}(U_j) \subseteq U$. For each $j \in I_0$, let $x_j := p_j(x) \in U_j \cap p_j(A) \neq \emptyset$, and we observe that

$$\mathrm{cl}_{X_j}(p_j(A)) = \mathrm{cl}_X\left(\left\{\varphi(f)(j) : f \in \prod_{i \in I} \mathcal{K}_i\right\}\right),$$

hence $U_j \cap \{\varphi(f)(j) : f \in \prod_{i\in I} \mathcal{K}_i\} \neq \emptyset$. Thus for each $j \in I_0$, there exists $f_j \in \prod_{i\in I} \mathcal{K}_i$ such that $\varphi(f_j)(j) \in U_j$. Choose $f^* \in \prod_{i\in I} \mathcal{K}_i$ such that for each $j \in I_0$, $f^*(j) = f_j(j)$, implying that $\varphi(f^*)(j) = \varphi(f_j)(j) \in U_j$, so $\varphi(f^*) \in \bigcap_{i\in I_0} p^{-1}(U_i) \subseteq U$. It follows that $\varphi(f^*) \in U \cap \{\varphi(f)(i) : f \in \prod_{i\in I} \mathcal{K}_i\} \neq \emptyset$. Thus $x \in \mathrm{cl}_X\left(\{\varphi(f) : f \in \prod_{i\in I} \mathcal{K}_i\}\right)$. So $A \subseteq \mathrm{cl}_X\left(\{\varphi(f) : f \in \prod_{i\in I} \mathcal{K}_i\}\right)$.

All these show that $A \in \mathsf{KF}\left(\prod_{i\in I} X_i\right)$. □

Wu, Xi, Xu and Zhao [9] have shown that the cartesian product of finite many well-filtered spaces is well-filtered iff each of them is well-filtered. This result can be strengthened to the general cases.

Theorem 2.11. *Let $\{X_i : i \in I\}$ be a family of T_0 spaces. Then the following statements are equivalent.*

(1) $\prod_{i\in I} X_i$ is well-filtered.
(2) For each $i \in I$, X_i is well-filtered.

Proof. (1) ⇒ (2). Let $i_0 \in I$ and $A_{i_0} \in \mathsf{KF}(X_{i_0})$. Choose one $a_i \in X_i$ for each $i \in I \setminus \{i_0\}$. Let $A = \prod_{i\in I} A_i$, where

$$A_i = \begin{cases} A_{i_0}, & \text{if } \ i = i_0, \\ \downarrow_{X_i} a_i, & \text{if } \ i \neq i_0. \end{cases}$$

Note that $A_i \in \mathsf{KF}(X_i)$ for all $i \in I$. By Theorem 2.10, $A \in \mathsf{KF}\left(\prod_{i\in I} X_i\right)$. By condition (1) and Theorem 2.6, there exists $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $A = \prod_{i\in I} A_i = \downarrow_X x$, which implies $A_{i_0} = \downarrow_{X_{i_0}} x_{i_0}$. Thus, by Theorem 2.6, X_{i_0} is well-filtered.

(2) ⇒ (1). Let $A \in \mathsf{KF}\left(\prod_{i\in I} X_i\right)$. For each $i \in I$, by Theorem 2.10, $\mathrm{cl}_{X_i}(p_i(A)) \in \mathsf{KF}(X_i)$, and since X_i is well-filtered, by Theorem 2.6, there exists $x_i \in X_i$ such that $\mathrm{cl}_{X_i}(p_i(A)) = \downarrow_{X_i} x_i$. By Proposition 2.8,

$$A = \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A)) = \prod_{i\in I} \downarrow_{X_i} x_i = \downarrow_X x,$$

where $x = (x_i)_{i\in I}$. Hence, by Theorem 2.6, $\prod_{i\in I} X_i$ is well-filtered. □

3. Well-filtered reflections of T_0 spaces

In this section we use the KF-sets to construct the well-filtered reflection of T_0 spaces. Theorem 2.6 informs that a space X is non-well-filtered iff it contains KF-sets that are not pointed closures. So our strategy is to add such sets, as points, to X, and repeat this process until it stabilizes. This approach is similar to that of Ershov [2] where he performed the d-completion. It is proved that the resulting space is exactly its well-filtered reflection. As a consequence, we obtain another proof for the existence of well-filtered reflection given in [9].

We begin by introducing the notion of a KF-base, which involves the KF-sets.

Definition 3.1. Let X be a T_0 space. A subspace X_0 of X is called a *KF-base* for X if for each $x \in X$, there is $F \in \mathsf{KF}(X_0)$ such that $\mathrm{cl}_X(F) = \downarrow_X x$.

The following properties on a KF-base are crucial for proving our main results.

Proposition 3.2. *Let X be a T_0 space and X_0 a KF-base for X.*

(1) For each $x \in X$,

(i) $\downarrow_X x \cap X_0 \in \mathsf{KF}(X_0)$ *and*
(ii) $\mathrm{cl}_X(\downarrow_X x \cap X_0) = \downarrow_X x$.

(2) $X = \uparrow_X X_0$.
(3) For each $U \in \mathcal{O}(X)$, $U = \uparrow_X(U \cap X_0)$. *Hence, if* $U_1, U_2 \in \mathcal{O}(X)$ *such that* $U_1 \cap X_0 = U_2 \cap X_0$, *then* $U_1 = U_2$.
(4) For each $V \in \mathcal{O}(X_0)$, $\uparrow_X V \in \mathcal{O}(X)$. *Hence,* $\mathcal{O}(X) = \{\uparrow_X U : U \in \mathcal{O}(X_0)\}$.
(5) The lattices $(\mathcal{O}(X), \subseteq)$ *and* $(\mathcal{O}(X_0), \subseteq)$ *are order isomorphic under the inclusion.*

Proof. (1) Let $x \in X$. Then there exists $F \in \mathsf{KF}(X_0)$ such that $\mathrm{cl}_X(F) = \downarrow_X x$. Note that $F = \mathrm{cl}_{X_0}(F) = \mathrm{cl}_X(F) \cap X_0 = \downarrow_X x \cap X_0$. So $\downarrow_X x \cap X_0 = F \in \mathsf{KF}(X_0)$, showing (i). In addition, $\mathrm{cl}_X(\downarrow_X x \cap X_0) = \mathrm{cl}_X(F) = \downarrow_X x$, thus (ii) holds.

(2) We only need to show $X \subseteq \uparrow_X X_0$. Let $x \in X$. By (i), $\downarrow_X x \cap X_0 \in \mathsf{KF}(X_0)$, it follows that $\downarrow_X x \cap X_0 \neq \emptyset$, and thus $x \in \uparrow_X X_0$. This implies $X \subseteq \uparrow_X X_0$. So $X = \uparrow_X X_0$.

(3) Let $U \in \mathcal{O}(X)$. Then $U \cap X_0 \subseteq U$, implying that $\uparrow_X(U \cap X_0) \subseteq \uparrow_X U = U$. Let $x \in U$. By (ii), $\mathrm{cl}_X(\downarrow_X x \cap X_0) = \downarrow_X x$. It follows that $\downarrow_X x \cap X_0 \cap U \neq \emptyset$, so $x \in \uparrow_X(U \cap X_0)$. Hence, $U \subseteq \uparrow_X(U \cap X_0)$. Therefore, $U = \uparrow_X(U \cap X_0)$.

(4) Let $U \in \mathcal{O}(X_0)$. Then there exists $V \in \mathcal{O}(X)$ such that $U = V \cap X_0$. By (3), $V = \uparrow_X(V \cap X_0) = \uparrow_X U$, showing that $\uparrow_X U \in \mathcal{O}(X)$.

(5) Define $\phi : (\mathcal{O}(X), \subseteq) \longrightarrow (\mathcal{O}(X_0), \subseteq)$ by $\phi(U) = U \cap X_0$ $(U \in \mathcal{O}(X))$ and $\psi : (\mathcal{O}(X_0), \subseteq) \longrightarrow (\mathcal{O}(X), \subseteq)$ by $\psi(V) = \uparrow_X V$ $(V \in \mathcal{O}(X_0))$. Then by (3) and (4), ϕ and ψ are monotone and $\phi = \psi^{-1}$. Hence ϕ is an order-isomorphism between $(\mathcal{O}(X), \subseteq)$ and $(\mathcal{O}(X_0), \subseteq)$. $\square$

Theorem 3.3. *Let* X_0 *be a KF-base for* X *and* Y *a well-filtered space. If* $f : X_0 \longrightarrow Y$ *is a continuous mapping, then there exists a unique continuous mapping* $f^* : X \longrightarrow Y$ *that extends* f *(i.e.,* $\forall x \in X_0$, $f^*(x) = f(x)$*).*

Proof. Let $x \in X$. Then $\downarrow_X x \cap X_0 \in \mathsf{KF}(X_0)$ by Proposition 3.2. Since $f : X_0 \longrightarrow Y$ is continuous, by Lemma 2.5 and Proposition 2.9, $\mathrm{cl}_Y(f(\downarrow_X x \cap X_0)) \in \mathsf{KF}(Y)$. Since Y is well-filtered, by Theorem 2.6, there exists a unique $y_x \in Y$ such that $\mathrm{cl}_Y(f(\downarrow_X x \cap X_0)) = \downarrow_Y y_x$. Put

$$f^*(x) = y_x.$$

Claim 1. f^* *is an extension of* f*.*

Let $x \in X_0$. Then

$$\mathrm{cl}_Y(f(x)) = \mathrm{cl}_Y(f(\downarrow_{X_0} x)) = \mathrm{cl}_Y(f(\downarrow_X x \cap X_0)) = \downarrow_Y f^*(x),$$

implying that $f(x) = f^*(x)$. So f^* extends f.

Claim 2. f^* *is continuous.*

Let $V \in \mathcal{O}(Y)$. Since f is continuous, $f^{-1}(V) \in \mathcal{O}(X_0)$. Then there exists $U \in \mathcal{O}(X)$ such that $U \cap X_0 = f^{-1}(V)$. We show that $U = (f^*)^{-1}(V)$. Let $x \in U$. Note that $U = \uparrow_X(U \cap X_0) = \uparrow_X f^{-1}(V)$ by Proposition 3.2. Then there exists $x' \in f^{-1}(V)$ such that $x' \in \downarrow_X x$. Note that $x' \in \downarrow_X x \cap X_0$, we have

$$f(x') \in \mathrm{cl}_Y(f(\downarrow_X x \cap X_0)) = \downarrow_Y f^*(x).$$

It follows that $f^*(x) \in \uparrow_Y f(x') \subseteq V$, showing that $x \in (f^*)^{-1}(V)$. Hence, $U \subseteq (f^*)^{-1}(V)$. Conversely, assume $x \in (f^*)^{-1}(V)$, i.e., $f^*(x) \in V$. Since $f^*(x) \in \downarrow_Y f^*(x) = \mathrm{cl}_Y(f(\downarrow_X x \cap X_0))$, we have that $f(\downarrow_X x \cap X_0) \cap V \neq \emptyset$, and thus

$$\downarrow_X x \cap X_0 \cap f^{-1}(V) = \downarrow_X x \cap X_0 \cap U \neq \emptyset,$$

implying that $x \in U$ (note that $U = \uparrow_X U$). This implies $(f^*)^{-1}(V) \subseteq U$. Consequently, $(f^*)^{-1}(V) = U$. So f^* is continuous.

Claim 3. *f^* is the unique extension of f.*

Suppose that $g : X \longrightarrow Y$ is a continuous mapping that extends f. Let $x \in X$. By Proposition 3.2, $\downarrow_X x = \mathrm{cl}_X(\downarrow_X x \cap X_0)$ and

$$\begin{aligned} \mathrm{cl}_Y(g(x)) &= \mathrm{cl}_Y(g(\downarrow_X x)) = \mathrm{cl}_Y(g(\mathrm{cl}_X(\downarrow_X x \cap X_0))) \\ &= \mathrm{cl}_Y(g(\downarrow_X x \cap X_0)) = \mathrm{cl}_Y(f(\downarrow_X x \cap X_0)) \\ &= \mathrm{cl}_Y(f^*(x)), \end{aligned}$$

where the third equation follows from the assumption that g is continuous. So $g(x) = f^*(x)$. $\square$

The following definition is a generalization of the notion of KF-bases, motivated by the properties of the KF-base in Theorem 3.3.

Definition 3.4. A subspace X_0 of X is called a *KF^*-base* if for any continuous mapping $f : X_0 \longrightarrow Y$ into a well-filtered space Y, there exists a unique continuous mapping $f^* : X \longrightarrow Y$ that extends f, that is, the following diagram commutes.

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}_{X_0}"] \arrow[dr, "f"'] & X \arrow[d, dotted, "f^*"] \\ & Y \end{tikzcd}$$

As a consequence of Theorem 3.3, we deduce the following.

Corollary 3.5. *If X_0 is a KF-base for X, then X_0 is a KF^*-base for X.*

Remark 3.6. If X_0 is a KF^*-base for a well-filtered space X, then the pair $\langle X, \mathrm{id}_{X_0}\rangle$ is a well-filtered reflection of X_0, where $\mathrm{id}_{X_0} : X_0 \longrightarrow X$, $x \mapsto x$.

Theorem 3.7. *Let X_0 be a subspace of X. Assume that there exists an increasing transfinite sequence $\{X_\beta : \beta \leq \alpha\}$ of subspaces of X (starting from X_0 and terminating with $X = X_\alpha$) such that*

(K1) X_β is a KF-base for $X_{\beta+1}$ for $\beta < \alpha$;
(K2) $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$ for each limit ordinal $\beta \leq \alpha$.

Then X_0 is a KF^*-base for the space X.

Proof. The following claim is necessary.

Claim. *For each $\beta \leq \alpha$ and for each $V \in \mathcal{O}(X_\beta)$, $V = \uparrow_{X_\beta}(V \cap X_0)$.*

We prove this inductively.

(i) By Proposition 3.2, $V = \uparrow_{X_1}(V \cap X_0)$ for all $V \in \mathcal{O}(X_1)$.

(ii) Let $\beta < \alpha$. Assume $V = \uparrow_{X_\beta}(V \cap X_0)$ for all $V \in \mathcal{O}(X_\beta)$. Then for each $W \in \mathcal{O}(X_{\beta+1})$, by Proposition 3.2, we have

$$\begin{aligned} W &= \uparrow_{X_{\beta+1}}(W \cap X_\beta) \\ &= \uparrow_{X_{\beta+1}}(\uparrow_{X_\beta}(W \cap X_0)) \\ &= \uparrow_{X_{\beta+1}}(W \cap X_0). \end{aligned}$$

(iii) Let $\beta \leq \alpha$ be a limit ordinal. Assume for any $\gamma < \beta$ and for any $V \in \mathcal{O}(X_\gamma)$, $V = \uparrow_{X_\gamma}(V \cap X_0)$. The following straightforward observation is useful:

$$\forall A \subseteq X_0,\ \uparrow_{X_\beta} A = \bigcup_{\gamma<\beta} \uparrow_{X_\gamma} A.$$

Then for each $W \in \mathcal{O}(X_\beta)$, we have

$$W = \bigcup_{\gamma<\beta} (W \cap X_\gamma) = \bigcup_{\gamma<\beta} \uparrow_{X_\gamma}(W \cap X_0) = \uparrow_{X_\beta}(W \cap X_0).$$

By transfinite induction, the claim holds.

Now we proceed the proof of the theorem. Let Y be a well-filtered space and $f_0 : X_0 \longrightarrow Y$ a continuous mapping.

(1) For $\beta < \alpha$, if we have defined $f_\beta : X_\beta \longrightarrow Y$, then by Theorem 3.3, there is a unique continuous mapping $f_{\beta+1} : X_{\beta+1} \longrightarrow Y$ that extends f_β, that is, the following diagram commutes.

$$\begin{tikzcd} X_\beta \arrow[r, "\mathrm{id}_{X_\beta}"] \arrow[dr, "f_\beta"'] & X_{\beta+1} \arrow[d, dotted, "f_{\beta+1}"] \\ & Y \end{tikzcd}$$

(2) Assume $\beta \leq \alpha$ is a limit ordinal and we have defined $\{f_\gamma : \gamma < \beta\}$ such that $\forall \gamma < \beta$, $f_{\gamma+1}$ is the unique continuous mapping that extends f_γ. Let $x \in X_\beta = \bigcup_{\gamma<\beta} X_\gamma$. Then there exists $\gamma_0 < \beta$ such that $x \in X_{\gamma_0}$. Put

$$f_\beta(x) = f_{\gamma_0}(x).$$

(a1) f_β is well-defined.

Let $x \in X$. Suppose there are $\gamma_1, \gamma_2 < \beta$ such that $x \in X_{\gamma_1} \cap X_{\gamma_2}$. We need to show $f_{\gamma_1}(x) = f_{\gamma_2}(x)$. Without of loss of generality, assume $\gamma_1 < \gamma_2$. Then f_{γ_2} is an extension of f_{γ_1}, so $f_{\gamma_1}(x) = f_{\gamma_2}(x)$. Hence, f_β is well-defined.

(a2) f_β is an extension of f_0.

Let $x \in X_0$. Fix $\gamma < \beta$. Then we have $f_\beta(x) = f_\gamma(x) = f_0(x)$. So f_β extends f_0.

(a3) f_β is continuous.

Let $V \in \mathcal{O}(Y)$. By the proceeding claim, we have

$$\begin{aligned} f_\beta^{-1}(V) &= \bigcup_{\gamma<\beta} f_\beta^{-1}(V) \cap X_\gamma \\ &= \bigcup_{\gamma<\beta} f_\gamma^{-1}(V) \\ &= \bigcup_{\gamma<\beta} \uparrow_{X_\gamma}(f_\gamma^{-1}(V) \cap X_0) \\ &= \bigcup_{\gamma<\beta} \uparrow_{X_\gamma} f_0^{-1}(V) \\ &= \uparrow_{X_\beta} f_0^{-1}(V). \end{aligned}$$

Since X_0 is a subspace of X_β and $f_0^{-1}(V) \in \mathcal{O}(X_0)$, there exists $W \in \mathcal{O}(X_\beta)$ such that $f_0^{-1}(V) = W \cap X_0$. From the claim, it follows that $\uparrow_{X_\beta} f_\beta^{-1}(V) = \uparrow_{X_\beta}(W \cap X_0) = W$, which is an open subset of X_β. So f_β is continuous.

(a4) f_β is the unique continuous mapping that extends f_0.

Suppose $g : X_\beta \longrightarrow Y$ is a continuous mapping that extends f_0. Let $x \in X_\beta$. Then there is $\gamma_0 < \beta$ such that $x \in X_{\gamma_0}$. Since f_{γ_0} is unique and $g|_{X_{\gamma_0}} : X_{\gamma_0} \longrightarrow Y$, $x \mapsto g(x)$, is a continuous mapping that extends f_0, we have $f_\beta(x) = f_{\gamma_0}(x) = g|_{X_{\gamma_0}}(x) = g(x)$. Hence, $g = f_\beta$.

By the above construction, we obtain a continuous mapping $f_\alpha : X \longrightarrow Y$ that uniquely extends f_0. Hence, X_0 is a KF^*-base for the space X. □

Recall that every T_0 space X can be topologically embedded into a sober space (the sobrification of X). So every T_0 space can be regarded as a subspace of a sober space.

Proposition 3.8. *For any T_0 space X_0, there exists a well-filtered space $W(X_0)$ such that X_0 embeds into $W(X_0)$ as a KF^*-base.*

Proof. Assume that X is a sober space that has X_0 as a subspace. For each ordinal β, define

(i) $X_{\beta+1} = \{x \in X : \exists F \in \mathsf{KF}(X_\beta), \mathrm{cl}_X(F) = \downarrow_X x\}$;
(ii) $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$ for a limit ordinal β.

Then due to the cardinality reason, there exists α such that $X_\alpha = X_{\alpha+1}$.

Let

$$W(X_0) = X_\alpha.$$

From the definition, we see that the transfinite sequence $\{X_\beta : \beta \le \alpha\}$ satisfies (K1) and (K2) of Theorem 3.7. So we have

(c1) X_0 is a KF^*-base for $W(X_0)$.

Now we show:

(c2) $W(X_0)$ is well-filtered.

Let $A \in \mathsf{KF}(X_\alpha)$. By Lemma 2.9, A is irreducible in X_α hence irreducible in X. Since X is sober, there exists $a \in X$ such that $\mathrm{cl}_X(A) = \downarrow_X a$. Thus $a \in X_{\alpha+1} = X_\alpha$. It follows that $A = \mathrm{cl}_{X_\alpha}(A) = \mathrm{cl}_X(A) \cap X_\alpha = \downarrow_X a \cap X_\alpha = \downarrow_{X_\alpha} a$. By Theorem 2.6, $W(X_0) = X_\alpha$ is a well-filtered space. □

By Remark 3.6 and Proposition 3.8, we obtain our main result.

Corollary 3.9. *The well-filtered reflection exists for every T_0 space.*

Hoffmann [5, 1.4] proved that the sobrification of a product space is the product of the sobrification of its factors. Recently, Keimel and Lawson [6, 6.12] established a similar result for the D-completion: the D-completion of a product of finitely many spaces is the product of the D-completion of each factor space.

It is then natural to ask the following:

Is the well-filtered reflection of a product space the product of the well-filtered reflection of its factors?

We answer the question in the positive for finite products. To do that, we need to have the following lemma.

Lemma 3.10. *Let X_i be a subspace of Y_i $(i \in I)$. Then the following statements are equivalent.*

(1) $\forall i \in I$, X_i is a KF-base for Y_i.
(2) $\prod_{i\in I} X_i$ is a KF-base for $\prod_{i\in I} Y_i$.

Proof. For convenience, let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$.

(1) $\Rightarrow$ (2). Let $y = (y_i)_{i\in I} \in Y$. Then for each $i \in I$, since X_i is a KF-base for Y_i, there exists $F_i \in \mathsf{KF}(X_i)$ such that $\mathrm{cl}_{Y_i}(F_i) = {\downarrow_{Y_i}} y_i$. Let $F = \prod_{i\in I} F_i$. By Theorem 2.10, $F \in \mathsf{KF}(X)$ and since

$$\mathrm{cl}_Y(F) = \mathrm{cl}_Y\left(\prod_{i\in I} F_i\right) = \prod_{i\in I} \mathrm{cl}_{Y_i}(F_i) = \prod_{i\in I} {\downarrow_{Y_i}} y_i = {\downarrow_Y} y,$$

we have that X is a KF-base for Y.

(2) $\Rightarrow$ (1). Let $i_0 \in I$ and $y_{i_0} \in Y_{i_0}$. For each $i \neq i_0$, choose $y_i \in Y_i$. Then $y = (y_i)_{i\in I} \in Y$ and by (2) there exists $F \in \mathsf{KF}(\prod_{i\in I} X_i)$ such that $\mathrm{cl}_Y(F) = {\downarrow_Y} y$. By Proposition 2.8 and Lemma 2.9, $F = \prod_{i\in I} F_i$, where $F_i = \mathrm{cl}_{X_i}(p_i(F))$. By Theorem 2.10, $F_{i_0} \in \mathsf{KF}(X_{i_0})$. It remains to show that $\mathrm{cl}_{Y_{i_0}}(F_{i_0}) = {\downarrow_{Y_{i_0}}} y_{i_0}$. On the one hand,

$${\downarrow_{Y_{i_0}}} y_{i_0} = p_{i_0}({\downarrow_Y} y) = p_{i_0}(\mathrm{cl}_Y(F)) \subseteq \mathrm{cl}_{Y_{i_0}}(p_{i_0}(F)) \subseteq \mathrm{cl}_{Y_{i_0}}(F_{i_0}).$$

On the other hand, since $F_{i_0} = p_{i_0}(F) \subseteq {\downarrow_{Y_{i_0}}} y_{i_0}$ and ${\downarrow_{Y_{i_0}}} y_{i_0}$ is closed, we have $\mathrm{cl}_{Y_{i_0}}(F_{i_0}) \subseteq {\downarrow_{Y_{i_0}}} y_{i_0}$. Hence, $\mathrm{cl}_{Y_{i_0}}(F_{i_0}) = {\downarrow_{Y_{i_0}}} y_{i_0}$. So X_{i_0} is a KF-base for Y_{i_0}. $\square$

Now we are in the position to prove the last major result.

Theorem 3.11. *For any finitely collection of T_0 spaces $X_1, \ldots, X_n$, $W(\prod_{1\leq i\leq n} X_i) = \prod_{1\leq i\leq n} W(X_i)$.*

Proof. Let $i \in \{1, 2, \ldots, n\}$. For each ordinal β, define $(X_i)_\beta$ inductively by

(i) $(X_i)_0 = X_i$;
(ii) $(X_i)_{\beta+1} = \left\{x_i \in W(X_i) : \exists F \in \mathsf{KF}((X_i)_\beta), \mathrm{cl}_{W(X_i)}(F) = {\downarrow_{W(X_i)}} x_i\right\}$;
(iii) $(X_i)_\beta = \bigcup_{\gamma<\beta} (X_i)_\gamma$ for each limit ordinal β.

By Proposition 3.8, there exists $\alpha(i)$ such that $W(X_i) = (X_i)_{\alpha(i)}$. Let

$$\alpha = \max\{\alpha(i) : 1 \leq i \leq n\}.$$

Clearly, for any $i \in \{1, 2, \ldots, n\}$, $W(X_i) = (X_i)_\alpha$. Let $X = \prod_{1\leq i\leq n} X_i$. For each ordinal β, define

$$X_\beta = \prod_{1\leq i\leq n} (X_i)_\beta.$$

(K1) $\forall \beta < \alpha$, X_β is a KF-base for $X_{\beta+1}$.

Since for each $i \in \{1, 2 \ldots, n\}$, $(X_i)_\beta$ is a KF-base for $(X_i)_{\beta+1}$, by Lemma 3.10, X_β is a KF-base for $X_{\beta+1}$.

(K2) $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$ for each limit ordinal $\beta \leq \alpha$.

It suffices to show $\prod_{1\leq i\leq n} \bigcup_{\gamma<\beta} (X_i)_\gamma \subseteq \bigcup_{\gamma<\beta} \prod_{1\leq i\leq n} (X_i)_\gamma$. Let $(x_i)_{i\in I} \in \prod_{1\leq i\leq n} \bigcup_{\gamma<\beta} (X_i)_\gamma$. Then for each $i \in \{1, 2, \ldots, n\}$, there exists $\gamma(i) < \beta$ such that $x_i \in (X_i)_{\gamma(i)}$. Let $\gamma^* = \max\{\gamma(i) : 1 \leq i \leq n\}$. Note that β is a limit ordinal, so $\gamma^* < \beta$ and

$$x_i \in (X_i)_{\gamma(i)} \subseteq (X_i)_{\gamma^*}, \ \forall i \in I,$$

showing that $(x_i)_{i\in I} \in \prod_{1\leq i\leq n} (X_i)_{\gamma^*} \subseteq \bigcup_{\gamma<\beta} \prod_{1\leq i\leq n} (X_i)_\gamma$. Hence, $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$.

Thus $\{X_\beta : \beta \leq \alpha\}$ is an increasing transfinite sequence satisfying (K1) and (K2) of Theorem 3.7. This implies X is a KF^*-base for X_α. Note that $X_\alpha = \prod_{1\leq i\leq n} (X_i)_\alpha = \prod_{1\leq i\leq n} W(X_i)$ is well-filtered by Theorem 2.11. Then by Remark 3.6, $\prod_{1\leq i\leq n} W(X_i)$ is the well-filtered reflection of $\prod_{1\leq i\leq n} X_i$, that is, $W(\prod_{1\leq i\leq n} X_i) = \prod_{1\leq i\leq n} W(X_i)$. $\square$

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