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ON OPEN WELL-FILTERED SPACES

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ABSTRACT. We introduce and study a new class of T_0 spaces, called open well-filtered spaces. The main results we prove include (i) every well-filtered space is an open well-filtered space; (ii) every core-compact open well-filtered space is sober. As an immediate corollary, it follows that every core-compact well-filtered space is sober. This provides a different and relatively more straightforward method to answer the problem posed by Jia and Jung: is every core-compact well-filtered space sober?

1. Introduction

The sobriety is one of the most important topological properties, particularly meaningful for T_0 spaces. It has been used in the characterization of spectral spaces of commutative rings and the spaces which are determined by their open set lattices. In domain theory, it was proved that the Scott space of every domain is sober quite early on. Since then the investigation of the sobriety of Scott spaces of general directed complete posets led to many deep results. Heckmann introduced the well-filtered spaces and asked whether every well-filtered Scott space of a directed complete poset is sober [Hec90, Hec91]. This question inspired intensive studies on the relationship between sobriety and well-filteredness (see [HGJX18, JJL16, Kou01, ZXC19, XZ17, XL17, WXXZ19]). A recent problem on this topic

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is whether every core-compact well-filtered space is sober, posed by Jia and Jung [Jia18]. The problem has been answered positively by Lawson, Wu and Xi [LWX20] and Xu, Shen, Xi and Zhao [XSXZ20].

In the current paper we first introduce a new class of topological spaces, called open well-filtered spaces, which includes all well-filtered spaces. The open well-filtered spaces themselves may deserve further study that will enrich the theory of T_0 topological spaces. We prove that (i) every well-filtered space is an open well-filtered space, and (ii) every core-compact open well-filtered space is sober. As an immediate implication, we obtain that every core-compact well-filtered space is sober, thus giving a relatively more straightforward method to answer Jia and Jung's problem [Jia18].

2. Preliminaries

This section is devoted to a brief review of some basic concepts and notations that will be used in the paper. For more details, see [Eng89, GHK+03, GL13].

Let P be a poset. A nonempty subset D of P is *directed* if every two elements in D have an upper bound in D. P is called a *directed complete poset*, or *dcpo* for short, if for any directed subset $D \subseteq P$, $\bigvee D$ exists.

Let X be a T_0 space. A subset A of X is called *saturated* if A equals the intersection of all open sets containing it. The specialization order $\leq$ on X is defined by $x \leq y$ iff $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator. It is easy to show that a subset A of X is saturated if and only if $A = \uparrow A = \{x \in X : x \geq a$ for some $a \in A\}$ with respect to the specialization order.

A nonempty subset A of X is *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called *sober* if for any irreducible closed set F, $F = \downarrow x = \mathrm{cl}(\{x\})$ for some $x \in X$.

For a T_0 space X, we shall consider the following subfamilies of the power set 2^X:

$\mathcal{Q}(X)$, the set of all compact saturated subsets of X;
$\mathcal{S}(X)$, the set of all saturated subsets of X;
$\mathcal{O}(X)$, the set of all open subsets of X.

For $A, B \subseteq X$, we say that A *is relatively compact in* B, denoted by $A \ll B$, if $A \subseteq B$ and every open cover of B contains a finite subcover of A.

We write
$$\mathcal{A} \subseteq_{flt} 2^X \quad (\mathcal{Q}(X), \mathcal{S}(X), \mathcal{O}(X), \text{ resp.})$$
for that $\mathcal{A}$ is a $\ll$-filtered subfamily of 2^X ($\mathcal{Q}(X)$, $\mathcal{S}(X)$, $\mathcal{O}(X)$, resp.), that is, $\forall A_1, A_2 \in \mathcal{A}$, there exists $A_3 \in \mathcal{A}$ such that $A_3 \ll A_1, A_2$.

Remark 2.1. (1) In general, for any $A, B \subseteq X$, that each open cover of B contains a finite subcover of A does not imply $A \subseteq B$. For example, on the set $X = \{x, y\}$, consider the topology $\mathcal{O}(X) = \{\emptyset, \{x\}, X\}$. Then X is a T_0 space and $\{x\} \ll \{y\}$. Thus the requirement $A \subseteq B$ in the definition of $A \ll B$ is not redundant.

(2) For any $A \subseteq X$ and $B \in \mathcal{O}(X)$, $A \ll B$ if and only if each open cover of B contains a finite subcover of A. This is because $\{B\}$ is an open cover of A.

(3) For any $A \subseteq X$ and $Q \in \mathcal{Q}(X)$, $A \ll Q$ if and only if $A \subseteq Q$. Hence, $\mathcal{A} \subseteq_{flt} \mathcal{Q}(X)$ if and only if $(\mathcal{A}, \supseteq)$ is a directed family.

A T_0 space X is called *well-filtered* if for any $\mathcal{K} \subseteq_{flt} \mathcal{Q}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$. We note that every sober space is well-filtered [GHK+03].

In what follows, the symbol ω will denote the smallest infinite ordinal, and for any set X, the family of all finite subsets of X is denoted by $X^{(<\omega)}$.

Definition 2.2. A T_0 space X is called *ω-well-filtered*, if for any $\{K_n : n < \omega\} \subseteq_{flt} \mathcal{Q}(X)$ and $U \in \mathcal{O}(X)$,

$$\bigcap_{n<\omega} K_n \subseteq U \ \Rightarrow\ \exists n_0 < \omega, K_{n_0} \subseteq U.$$

Proposition 2.3. *A T_0 space X is ω-well-filtered if and only if for any descending chain $\{K_n : n < \omega\} \subseteq \mathcal{Q}(X)$, that is,*

$$K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq K_{n+1} \supseteq \ldots,$$

and $U \in \mathcal{O}(X)$,

$$\bigcap_{n<\omega} K_n \subseteq U \ \Rightarrow\ \exists n_0 < \omega,\ K_{n_0} \subseteq U.$$

Proof. We only need to prove the sufficiency. Let $\mathcal{K} \subseteq_{flt} \mathcal{Q}(X)$ be a countable family and $U \in \mathcal{O}(X)$ such that $\bigcap \mathcal{K} \subseteq U$.

If the cardinality $|\mathcal{K}| < \omega$, i.e., $\mathcal{K}$ is a finite family, then $\mathcal{K}$ contains a smallest element Q, and hence $Q = \bigcap \mathcal{K} \subseteq U$, completing the proof.

Now assume $|\mathcal{K}| = \omega$. We may let $\mathcal{K} = \{K_n : n < \omega\}$. We use induction on $n < \omega$ to define a descending chain $\widehat{\mathcal{K}} = \left\{\widehat{K}_n : n < \omega\right\}$. Specifically, let $\widehat{K}_0 = K_0$ and let $\widehat{K}_{n+1} \in \mathcal{K}$ be a lower bound of $\left\{K_{n+1}, \widehat{K}_0, \widehat{K}_1, \widehat{K}_2 \ldots, \widehat{K}_n\right\}$ under the inclusion order. Then $\widehat{\mathcal{K}} \subseteq \mathcal{K}$ is a descending chain and $\widehat{K}_n \subseteq K_n$ for all $n < \omega$, implying that $\bigcap \widehat{\mathcal{K}} = \bigcap \mathcal{K} \subseteq U$. Then by assumption, there exists $n_0 < \omega$ such that $\widehat{K}_{n_0} \subseteq U$, completing the proof. □

Lemma 2.4. *Let X be a T_0 space and $\mathcal{A} \subseteq_{flt} 2^X$. Each closed set $C \subseteq X$ that intersects all members of $\mathcal{A}$ contains a minimal (irreducible) closed subset F that still intersects all members of $\mathcal{A}$.*

Proof. Let $\mathcal{B} := \{B \in \mathcal{C}(X) : \forall A \in \mathcal{A}, B \cap A \neq \emptyset\}$, where $\mathcal{C}(X)$ is the set of all closed subsets of X.

(i) $\mathcal{B} \neq \emptyset$ because $C \in \mathcal{B}$.

(ii) Let $\mathcal{H} \subseteq \mathcal{B}$ be a chain. We claim that $\bigcap \mathcal{H} \in \mathcal{B}$. Otherwise, there exists $A_0 \in \mathcal{A}$ such that $A_0 \cap \bigcap \mathcal{H} = \emptyset$. As $\mathcal{A}$ is $\ll$-filtered, there exists $A_1 \in \mathcal{A}$ such that $A_1 \ll A_0$. Since $\{X \setminus B : B \in \mathcal{H}\}$ is a directed open cover of A_0, there exists $B_0 \in \mathcal{H}$ such that $A_1 \subseteq X \setminus B_0$ implying $A_1 \cap B_0 = \emptyset$. This means that $B_0 \notin \mathcal{B}$, contradicting $B_0 \in \mathcal{H} \subseteq \mathcal{B}$.

By Zorn's Lemma, it turns out that there exists a minimal closed subset $F \subseteq C$ such that $F \cap A \neq \emptyset$ for all $A \in \mathcal{A}$.

Now we show that F is irreducible. If F is not irreducible, then there exist closed sets F_1, F_2 such that $F = F_1 \cup F_2$ but $F \neq F_1$ and $F \neq F_2$. Since F_1, F_2 are proper subsets of F, by the minimality of F, there exist $A_1, A_2 \in \mathcal{A}$ such that $F_1 \cap A_1 = \emptyset$ and $F_2 \cap A_2 = \emptyset$. Since $\mathcal{A}$ is a $\ll$-filtered family, there exists $A_3 \in \mathcal{A}$ such that $A_3 \ll A_1, A_2$ (hence $A_3 \subseteq A_1, A_2$). It then follows that $F_1 \cap A_3 \subseteq F_1 \cap A_1 = \emptyset$ and $F_2 \cap A_3 \subseteq F_2 \cap A_2 = \emptyset$, which implies that $F_1 \cap A_3 = F_2 \cap A_3 = \emptyset$. Thus $F \cap A_3 = (F_1 \cup F_2) \cap A_3 = (F_1 \cap A_3) \cup (F_2 \cap A_3) = \emptyset$, contradicting the assumption on F. Therefore, F is irreducible. □

3. Saturated well-filtered spaces

In this section, we show that well-filteredness can be characterized by means of saturated sets, instead of compact saturated sets.

Definition 3.1. A T_0 space X is called *saturated well-filtered*, if for any $\{A_i : i \in I\} \subseteq_{flt} \mathcal{S}(X)$ and $U \in \mathcal{O}(X)$,

$$\bigcap_{i \in I} A_i \subseteq U \;\Rightarrow\; \exists i_0 \in I, A_{i_0} \subseteq U.$$

Definition 3.2. A T_0 space X is called *saturated ω-well-filtered*, if for any $\{A_n : n < \omega\} \subseteq_{flt} \mathcal{S}(X)$ and $U \in \mathcal{O}(X)$,

$$\bigcap_{n<\omega} A_n \subseteq U \;\Rightarrow\; \exists n_0 < \omega, A_{n_0} \subseteq U.$$

A countable family $\{A_n : n < \omega\} \subseteq \mathcal{S}(X)$ is called a *descending $\ll$-chain* if

$$A_0 \gg A_1 \gg A_2 \gg \ldots \gg A_n \gg A_{n+1} \gg \ldots .$$

Analogous to Proposition 2.3, we can prove the following.

Proposition 3.3. *A T_0 space X is saturated ω-well-filtered if and only if for any countable descending $\ll$-chain $\{A_n : n < \omega\} \subseteq \mathcal{S}(X)$ and $U \in \mathcal{O}(X)$,*

$$\bigcap_{n<\omega} A_n \subseteq U \;\Rightarrow\; \exists n_0 < \omega,\; A_{n_0} \subseteq U.$$

Proposition 3.4. *Let X be a saturated well-filtered space. Then for any $\{A_i : i \in I\} \subseteq_{flt} \mathcal{S}(X) \setminus \{\emptyset\}$, $\bigcap_{i\in I} A_i$ is a nonempty compact saturated set.*

Proof. It is clear that $\bigcap_{i\in I} A_i$ is saturated. Now suppose that $\bigcap_{i\in I} A_i = \emptyset$. Since X is saturated well-filtered and $\emptyset$ is open, we have that $A_{i_0} \subseteq \emptyset$ for some $i_0 \in I$, which contradicts that $A_{i_0} \neq \emptyset$. Thus $\bigcap_{i\in I} A_i \neq \emptyset$.

Let $\{V_j : j \in J\}$ be an open cover of $\bigcap_{i\in I} A_i$. As X is saturated well-filtered, there exists $i_0 \in I$ such that $A_{i_0} \subseteq \bigcup_{j\in J} V_j$. Since $\{A_i : i \in I\} \subseteq \mathcal{S}(X)$ is a $\ll$-filtered family, there exists $i_1 \in I$ such that $A_{i_1} \ll A_{i_0} \subseteq \bigcup_{j\in J} V_j$. Then there exists $J_0 \subseteq J^{(<\omega)}$ such that $A_{i_1} \subseteq \bigcup_{j\in J_0} V_j$. It follows that $\bigcap_{i\in I} A_i \subseteq \bigcup_{j\in J_0} V_j$. Therefore, $\bigcap_{i\in I} A_i$ is compact. □

Using a similar proof to that of Proposition 3.4, we deduce the following.

Proposition 3.5. *Let X be a saturated ω-well-filtered space. Then for any $\{A_n : n < \omega\} \subseteq_{flt} \mathcal{S}(X) \setminus \{\emptyset\}$, $\bigcap_{n<\omega} A_n$ is a nonempty compact saturated set.*

Theorem 3.6. *The saturated ω-well-filtered spaces are exactly the ω-well-filtered spaces.*

Proof. Note that every descending chain in $\mathcal{Q}(X)$ is a descending $\ll$-chain in $\mathcal{S}(X)$. Hence every saturated ω-well-filtered space is an ω-well-filtered space.

Now let X be an ω-well-filtered space. Suppose that $\{A_n : n < \omega\} \subseteq \mathcal{S}(X)$ is a descending $\ll$-chain, i.e.,

$$A_0 \gg A_1 \gg A_2 \gg \ldots \gg A_n \gg A_{n+1} \gg \ldots,$$

and $U \in \mathcal{O}(X)$ such that $\bigcap_{n<\omega} A_n \subseteq U$. We need to prove that $A_{n_0} \subseteq U$ for some $n_0 < \omega$. Otherwise, $A_n \nsubseteq U$ for all $n < \omega$, that is, $A_n \cap (X \setminus U) \neq \emptyset$. Then by Lemma 2.4, there exists a minimal (irreducible) closed set $F \subseteq X \setminus U$ such that $F \cap A_n \neq \emptyset$ for all $n < \omega$. Choose one $x_n \in F \cap A_n$ for each $n < \omega$, and let $H := \{x_n : n < \omega\}$.

Claim: H is compact.

Let $\{C_i : i \in I\}$ be a family of closed subsets of X such that for any $J \in I^{(<\omega)}$, $H \cap \bigcap_{i\in J} C_i \neq \emptyset$. It needs to prove that $H \cap \bigcap_{i\in I} C_i \neq \emptyset$. We complete the proof by considering two cases.

(c1) $C_i \cap H$ is infinite for all $i \in I$.

In this case, for each $n < \omega$, there exists $k_n \geq n$ such that $x_{k_n} \in C_i$. Since $A_{k_n} \subseteq A_n$ and $x_{k_n} \in F \cap A_{k_n}$, we have that $x_{k_n} \in C_i \cap F \cap A_{k_n} \subseteq C_i \cap F \cap A_n \neq \emptyset$. Thus $C_i \cap F$ is a closed set that intersects all A_n $(n < \omega)$. By the minimality of F, we have $F = C_i \cap F$, that is, $F \subseteq C_i$. By the arbitrariness of $i \in I$, it follows that $F \subseteq \bigcap_{i\in I} C_i$. Note that $H \subseteq F$, so $H \cap \bigcap_{i\in I} C_i = H \neq \emptyset$.

(c2) $C_i \cap H$ is finite for some $i \in I$.

Let $i_0 \in I$ such that $C_{i_0} \cap H$ is finite (hence compact). Note that the family $\{C_i : i \in I\}$ satisfies that for any $J \in I^{(<\omega)}$, $H \cap C_{i_0} \cap \bigcap_{i\in J} C_i \neq \emptyset$. Since $H \cap C_{i_0}$ is compact, we conclude $H \cap \bigcap_{i\in I} C_i = (H \cap C_{i_0}) \cap \bigcap_{i\in I} C_i \neq \emptyset$.

Now for each $n < \omega$, let $H_n := \{x_k : k \geq n\}$, which is compact by using a similar proof for H. Then $\{\uparrow H_k : k < \omega\} \subseteq_{flt} \mathcal{Q}(X)$ such that $\bigcap_{n<\omega} \uparrow H_n \subseteq \bigcap_{n<\omega} A_n \subseteq U$. As X is an ω-well-filtered space, there exists $n_0 < \omega$ such that $\uparrow H_{n_0} \subseteq U$, which contradicts that $H_{n_0} \subseteq F \subseteq X \setminus U$. $\square$

Theorem 3.7. *The saturated well-filtered spaces are exactly the well-filtered spaces.*

Proof. Clearly, every saturated well-filtered space is a well-filtered space.

Now assume that X is a well-filtered space. Let $\mathcal{A} \subseteq_{flt} \mathcal{S}(X)$ and $U \in \mathcal{O}(X)$ such that $\bigcap \mathcal{A} \subseteq U$.

Define

$$\widehat{\mathcal{A}} = \left\{ \bigcap_{n<\omega} A_n : \forall n < \omega, A_n \in \mathcal{A} \text{ and } A_n \gg A_{n+1} \right\}.$$

By Theorem 3.6 and the fact that every well-filtered space is ω-well-filtered, we deduce that X is saturated ω-well-filtered. Thus by Proposition 3.5, every member of $\widehat{\mathcal{A}}$ is nonempty compact saturated.

Claim: $\widehat{\mathcal{A}}$ is a filtered family.

Let $\bigcap_{n<\omega} A_n, \bigcap_{n<\omega} B_n \in \widehat{\mathcal{A}}$, that is,

$$A_0 \gg A_1 \gg A_2 \gg \cdots \gg A_n \gg A_{n+1} \gg \ldots$$

and

$$B_0 \gg B_1 \gg B_2 \gg \cdots \gg B_n \gg B_{n+1} \gg \cdots .$$

(i) There exists $C_0 \in \mathcal{A}$ such that $C_0 \ll A_0, B_0$ because $\mathcal{A}$ is $\ll$-filtered.

(ii) If we have defined $\{C_0, C_1, \cdots C_n\}$, then there exists $A \in \mathcal{A}$ such that $A \ll C_n, A_{n+1}, B_{n+1}$. Put $C_{n+1} := A$.

By Induction, we obtain a family $\{C_n : n < \omega\} \subseteq \mathcal{A}$ such that

$$C_0 \gg C_1 \gg C_2 \gg \cdots C_n \gg C_{n+1} \gg \cdots ,$$

and that $C_n \ll A_n, B_n$ (hence $C_n \subseteq A_n, B_n$) for all $n < \omega$. It follows that $\bigcap_{n<\omega} C_n \in \widehat{\mathcal{A}}$ and it is a lower bound of $\left\{\bigcap_{n<\omega} A_n,\ \bigcap_{n<\omega} B_n\right\}$. Hence $\widehat{\mathcal{A}}$ is filtered.

Since X is well-filtered and $\bigcap \widehat{\mathcal{A}} = \bigcap \mathcal{A} \subseteq U$, there is $\bigcap_{n<\omega} A_n \in \widehat{\mathcal{A}}$, where

$$A_0 \gg A_1 \gg A_2 \gg \cdots \gg A_n \gg A_{n+1} \gg \ldots ,$$

such that $\bigcap_{n<\omega} A_n \subseteq U$. By Theorem 3.6, X is saturated ω-well-filtered. Then there exists $n_0 < \omega$ such that $A_{n_0} \subseteq U$ (note that $A_{n_0} \in \mathcal{A}$). Therefore, X is well-filtered. $\square$

4. Open well-filtered spaces

In this section, we define another class of T_0 spaces, called open well-filtered spaces. It turns out that every well-filtered space is open well-filtered, and every core-compact open well-filtered space is sober. As an immediate consequence, we have that every core-compact well-filtered space is sober.

Definition 4.1. A T_0 space is called *open well-filtered*, if for any $\{U_i : i \in I\} \subseteq_{flt} \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$,

$$\bigcap_{i\in I} U_i \subseteq U \ \Rightarrow \ \exists i_0 \in I,\ U_{i_0} \subseteq U.$$

Note that every open set is saturated. By Theorem 3.7, we obtain the following result.

Remark 4.2. Every well-filtered space is open well-filtered.

Definition 4.3. A T_0 space X is called *open ω-well-filtered*, if for any $\{U_n : n < \omega\} \subseteq_{flt} \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$,

$$\bigcap_{n<\omega} U_n \subseteq U \ \Rightarrow \ \exists n_0 < \omega,\ U_{n_0} \subseteq U.$$

Using a similar proof to that of Proposition 2.3, we deduce the following result.

Proposition 4.4. *A T_0 space X is open ω-well-filtered if and only if for any countable descending $\ll$-chain $\{U_n : n < \omega\} \subseteq \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$,*

$$\bigcap_{n<\omega} U_n \subseteq U \ \Rightarrow \ \exists n_0 < \omega,\ U_{n_0} \subseteq U.$$

Analogous to Proposition 3.4, we have the following two results.

Proposition 4.5. *Let X be an open well-filtered space. Then for any $\{U_i : i \in I\} \subseteq_{flt} \mathcal{O}(X)$, $\bigcap_{i\in I} U_i$ is a nonempty compact saturated set.*

Proposition 4.6. *Let X be an open ω-well-filtered space. Then for any $\{U_n : n < \omega\} \subseteq_{flt} \mathcal{O}(X)$, $\bigcap_{n<\omega} U_n$ is a nonempty compact saturated set.*

Theorem 4.7. *Every core-compact open well-filtered space is sober.*

Proof. Assume X is a core-compact open well-filtered space. Let A be an irreducible closed subset of X. Define

$$\mathcal{F}_A := \{U \in \mathcal{O}(X) : U \cap A \neq \emptyset\}.$$

Claim: $\mathcal{F}_A$ is a $\ll$-filtered family.

Let $U_1, U_2 \in \mathcal{F}_A$. Then $U_1 \cap A \neq \emptyset \neq U_2 \cap A$. As A is irreducible, $U_1 \cap U_2 \cap A \neq \emptyset$, so there is an $x \in A \cap U_1 \cap U_2$. Since X is core-compact, there exists $U_3 \in \mathcal{O}(X)$ such that $x \in U_3 \ll U_1 \cap U_2$. Note that $x \in U_3 \cap A \neq \emptyset$, so $U_3 \in \mathcal{F}_A$. Hence, $\mathcal{F}_A$ is a $\ll$-filtered family.

Since X is open well-filtered, $A \cap \bigcap_{i\in I} \mathcal{F}_A \neq \emptyset$. Let $x_0 \in A \cap \bigcap \mathcal{F}_A$. We show that $A = \mathrm{cl}(\{x_0\})$. Otherwise, $A \setminus \mathrm{cl}(\{x_0\}) = A \cap (X \setminus \mathrm{cl}(\{x_0\})) \neq \emptyset$, implying that $X \setminus \mathrm{cl}(\{x_0\}) \in \mathcal{F}_A$. It follows that $x_0 \in \bigcap \mathcal{F}_A \subseteq X \setminus \mathrm{cl}(\{x_0\})$, a contradiction. Thus $A = \mathrm{cl}(\{x_0\})$. Since X is a T_0 space, $\{x_0\}$ is unique. So X is sober. $\square$

As a consequence of Remark 4.2 and Theorem 4.7, we obtain the following result.

Corollary 4.8. *Every core-compact well-filtered space is sober.*

Theorem 4.9. *Every core-compact open ω-well-filtered space is locally compact.*

Proof. Assume that X is a core-compact open ω-well-filtered space. Let $x \in X$ and $U \in \mathcal{O}(X)$ such that $x \in U$. Since X is core-compact, there exists an open set $W \ll U$ such that $x \in W$ and a sequence of open sets $\{U_n : n < \omega\}$ such that

$$U = U_0 \gg U_1 \gg U_2 \gg U_3 \gg \ldots \gg W.$$

Let $Q = \bigcap_{n<\omega} U_n$. Since X is open ω-well-filtered and by Proposition 4.5, $Q \in \mathcal{Q}(X)$ satisfies that $x \in W \subseteq Q \subseteq U$. Thus X is locally compact. $\square$

Remark 4.10. In [GL19], J. Goubault-Larrecq gives a slighly different proof for the above theorem.

As a corollary of Theorem 4.9, we deduce the following result.

Corollary 4.11. *A well-filtered space is core-compact if and only if it is locally compact.*

The following result is a small variant of Kou's result that every well-filtered space is a d-space (see Proposition 2.4 in [Kou01]).

Proposition 4.12 [Kou01]**.** *Let X be an ω-well-filtered space. If D is a directed (under the specialization order) subset of X with the cardinality $|D| \leq \omega$, then $\bigvee D$ exists.*

Let P be a poset. A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, called the *Scott topology* on P and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

Example 4.13. Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ be the Johnstone's dcpo and $\mathbb{N} = \{1, 2, 3 \ldots\}$ with the usual order. Let $P = \mathbb{J} \cup \mathbb{N}$. For any $x, y \in P$, define $x \leq y$ (refer to Figure 1) if one of the following conditions holds:

(i) $x, y \in \mathbb{N}$ and $x \leq y$ in $\mathbb{N}$;
(ii) $x, y \in \mathbb{J}$ and $x \leq y$ in $\mathbb{J}$;
(iii) $x \in \mathbb{N}, y \in \mathbb{J}$ and $y = (x, \omega)$.

Next, we show that $\Sigma\, P$ is an open well-filtered space. The following conclusions on P will be used later.

(c1) For any directed subset D of P, if $\bigvee D$ exists, then $\bigvee D \in D$ or $D \subseteq \mathbb{J}$.

In fact, assume $D \nsubseteq \mathbb{J}$, that is, $D \cap \mathbb{N} \neq \emptyset$. We prove $\bigvee D \in D$ by considering the following two cases:

Case 1: $D \subseteq \mathbb{N}$. In this case, it is trivial that $\bigvee D \in D$.

Case 2: $D \nsubseteq \mathbb{N}$, that is, $D \cap \mathbb{N} \neq \emptyset$ and $D \cap \mathbb{J} \neq \emptyset$. Let $k \in D \cap \mathbb{N}$ and $(m, n) \in D \cap \mathbb{J}$. Since D is directed, there exists $d \in D$ such that $k, (m, n) \leq d$. It forces that $d = (n_0, \omega)$ for some $n_0 \in \mathbb{N}$, which is a maximal point of P, so $d = \bigvee D \in D$ (note that the maximal point of a directed set is exactly the least upper bound of the set).

(c2) For any $U \in \sigma(P) \setminus \{\emptyset\}$, there exists a minimal $n_U \in \mathbb{N}$ such that $(n, \omega) \in U$ for all $n \geq n_U$, that is, $n_U = \min\{k \in \mathbb{N} : \forall n \geq k, (n, w) \in U\}$ exists in $\mathbb{N}$ (refer to Figure 1).

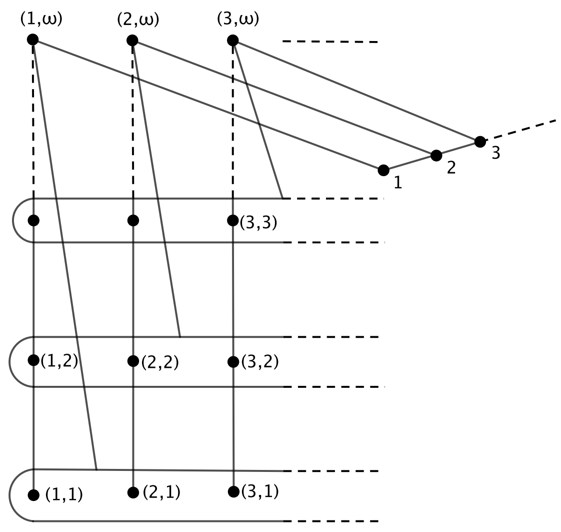

Figure 1: The poset P

Assume, on the contrary, that n_U does not exist in $\mathbb{N}$. Then there exist infinitely countable numbers $n_1, n_2, n_3, \ldots$ in $\mathbb{N}$ such that $(n_k, \omega) \notin U$ for all $k \in \mathbb{N}$. Since U is a nonempty upper set, there exists $m \in \mathbb{N}$ such that $(m, \omega) \in U$. Note that $\{(m, n_k) : k \in \mathbb{N}\}$ is a directed set such that $\bigvee\{(m, n_k) : k \in \mathbb{N}\} = (m, \omega) \in U$. Then there exists $k_0 \in \mathbb{N}$ such that $(m, n_{k_0}) \in U$. Since $(n_{k_0}, \omega) \geq (m, n_{k_0}) \in U$ and U is an upper set, it holds that $(n_{k_0}, \omega) \in U$, a contradiction.

(c3) For any $U \in \sigma(P) \setminus \{\emptyset\}$ and $\forall (n, \omega) \in U$, there exists a minimal $\phi_U(n) \in \mathbb{N}$ such that $(n, \phi_U(n)) \in U$, i.e., $\phi_U(n) = \min\{m \in \mathbb{N} : (n, m) \in U\}$ exists.

Suppose $(n, \omega) \in U$. Then $\{(n, m) : m \in \mathbb{N}\}$ is a directed subset of P such that $\bigvee\{(n, m) : m \in \mathbb{N}\} = (n, \omega) \in U$. Since U is Scott open, there exists $m_0 \in \mathbb{N}$ such that $(n, m_0) \in U$. Thus $m_0 \in \{m \in \mathbb{N} : (n, m) \in U\} \neq \emptyset$, which means that $\varphi_U(n) = \min\{m \in \mathbb{N} : (n, m) \in U\}$ exists.

(c4) For any $U, V \in \sigma(P)$, $U \ll V$ if and only if $U = \emptyset$.

If $U = \emptyset$, then trivially $U = \emptyset \ll V$. Now assume $U \ll V$ and $U \neq \emptyset$. By using (c2) and (c3), for each $n \geq n_U$, define $U_n = P \setminus \bigcup_{k \geq n} \downarrow(k, \phi_U(k))$. It is trivial that $\bigcup_{k \geq n} \downarrow(k, \phi_U(k))$ is a Scott closed subset of P, which means that $U_n \in \sigma(P)$. Since $\bigcup_{n \geq n_U} P \setminus \bigcup_{k \geq n} \downarrow(k, \phi_U(k)) = P \setminus \bigcap_{n \geq n_U} \bigcup_{k \geq n} \downarrow(k, \phi_U(k)) = P \setminus \emptyset = P$. Thus the family $\{U_n : n \geq n_U\}$ is a directed open cover of P, hence a directed open cover of V. By assumption that $U \ll V$, there exists $n_0 \geq n_U$ such that $U \subseteq U_{n_0}$. By the definition of n_U, it follows that $(n_0, \omega) \in U$. Thus by the definition of ϕ_U in (c3), we have that $(n_0, \phi_U(n_0)) \in U$, but $(n_0, \phi_U(n_0)) \notin P \setminus \downarrow(n_0, \phi_U(n_0)) \supseteq U_{n_0}$, which implies $(n_0, \phi_U(n_0)) \notin U_{n_0}$. It follows that $(n_0, \phi_U(n_0)) \in U \setminus U_{n_0}$, contradicting that $U \subseteq U_{n_0}$.

By (c4), we conclude that every $\ll$-filtered family $\mathcal{F}$ of Scott open subsets of P contains $\emptyset$, the minimal element in $\mathcal{F}$. This implies that ΣP is an open well-filtered space (hence an open ω-well-filtered space). Since $\bigvee \mathbb{N}$ does not exist in P and by Proposition 4.12, ΣP is not an ω-well-filtered space.

Recall that a T_0 space X is called a *d-space* if X is a dcpo in its specialization order and each open subset of X is Scott open in the specialization order. It is well-known that every well-filtered space is a d-space [GL13]. From Example 4.13, we have the following conclusions.

(1) An open well-filterd space need not be a d-space.
(2) An open well-filtered space need not be an ω-well-filtered space.

A summary on the relations among kinds of well-filtered spaces is shown in Figure 2.

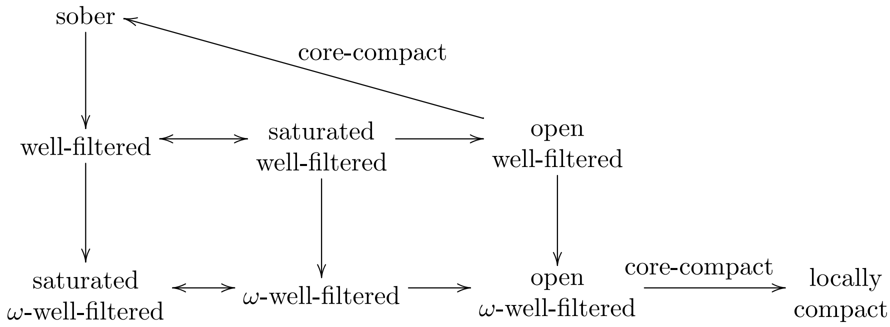

Figure 2: The relations among various types of well-filtered spaces

5. Acknowledgment

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