Reflectivity and reflective hull in the category of T_0 spaces

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In category theory, **reflectivity** offer a systematic way to associate each object in the larger category **C** with a best approximating object in a smaller category **D**. If a subcategory **D** is reflective in **C**, then every object in **C** can be "best approximated" by an element in the subcategory **D**.

For example, in topology theory:

- **Stone-Čech Compactification:** The category of compact Hausdorff spaces is reflective in the category of Tychonoff spaces. The reflector is the Stone-Čech compactification βX.
- **Completion of Metric Spaces:** Complete metric spaces form a reflective subcategory of metric spaces with uniformly continuous maps. The reflector is the metric completion.

This talk systematically reviews some known results on reflectivity of the category of T_0 spaces, and then show some new findings regarding the reflective hulls of the category of some T_0 spaces in domain theory.

Definition (Reflective Subcategory)

A full subcategory $\mathbf{K}$ of $\mathbf{C}$ is called **reflective** if for every $X \in \mathbf{C}$, there exists:

- An object $X^k \in \mathbf{K}$,
- A $\mathbf{C}$-morphism $\mu_X : X \longrightarrow X^k$,

such that for any $\mathbf{C}$-morphism $f : X \longrightarrow Z$ with $Z \in \mathbf{K}$, there exists a **unique** $\mathbf{K}$-morphism $g : X^k \longrightarrow Z$ such that $g \circ \mu_X = f$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "f"'] & X^k \arrow[d, dotted, "g"] \\ & Z \end{tikzcd}$$

Explanation:

Think of X^k as:

- The best approximation of X from above, but living in $\mathbf{K}$.

Reflectivity from a Poset Perspective

Example

Consider a poset $(P, \leq)$ viewed as a category, where objects are elements of P and there is a unique morphism $p \to q$ if and only if $p \leq q$. A subset $Q \subseteq P$ is **reflective** in P (as a subcategory) if and only if

- $\forall p \in P$, $\exists \hat{p} \in Q$ such that $\forall q \in Q$:

$$p \leq q \quad \Longrightarrow \quad \hat{p} \leq q.$$

- Equivalently, for each $p \in P$, $\bigwedge_Q(\uparrow p \cap Q)$ exists.

That is, $f \dashv i_Q$, where $f \colon P \to Q$, $p \mapsto \hat{p}$.

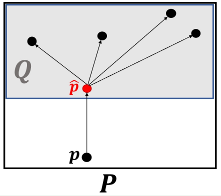

Figure: Q is reflective in P

Here $\hat{p}$ is the **best approximation of** p **from above within** Q.

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In the category of topological spaces, J.F. Kennison[1] gives three types of full reflective subcategories of all topological spaces **Top** called simple, identifying, and embedding, as follows:

Definition (Kennison 1965)

Let **P** be a full subcategory of **Top**, and $F : \mathbf{Top} \longrightarrow \mathbf{P}$ be a reflector from the category of topology. F is called

(1) simple: if $e_X : X \longrightarrow F(X)$ is bijective for all X;

(2) identifying: if $e_X(X) = F(X)$ for all X;

(3) embedding: if each object of P is a Hausdorff space and if $e_X(X)$ is a dense subset of $F(X)$ for all X.

The full category **P** is simple (resp., identifying or embedding) if there exists a simple (resp., identifying or embedding) reflector $F : \mathbf{Top} \longrightarrow \mathbf{P}$.

[1] J.F. Kennison, Reflective functors in general topology and elsewhere, Trans. Amer. Math. Soc. 118 (1965), 303–315.

In the paper, J. F. Kennison (1965) gave the characterizations of the three reflectors:

Theorem (A)

A topological property **P** *is simple iff* **P** *is hereditary, productive and contains every indiscrete space.*

Theorem (B)

A topological property **P** *is identifying iff* **P** *is hereditary and productive.*

Theorem (C)

A topological property **P** *is embedding iff* **P** *is closed-hereditary, productive and contains only Hausdorff spaces.*

In the paper, J.F. Kennison (1965) gives three types of full reflective subcategories of all topological spaces, but

- he doesn't know whether these three types include all the full reflective subcategories of **Top**.

In the paper[2], L. Skula (1969) gave an Negative answer, and proposed another type called *b*-embedding NOT mentioned by Kennison. Then he show that

Theorem (Skula-1969)

If **P** *is a full reflective subcategory of the category of* **Top** *containing at least one non-*T_1 *space, then* **P** *is a subcategory of one of the above-mentioned 4 types.*

[2] L. Skula, On a reflective subcategory of the category of all topological spaces, Trans. Amer. Math. Soc. 142 (1969) 37–41.

Why Skula topology?

Let $A \subseteq X \in \mathbf{Top}$, $Y \in \mathbf{Top}_0$, and $f\colon A \longrightarrow Y$ be a continuous map.

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

at most one

It requires that there is **at most one** continuous extension g of f. Otherwise, $\exists$ conti. map $g_1 \neq g_2$ s.t. $g_1|_A = g_2|_A = f$. Then $\exists x_0 \in X$ s.t.

- $g_1(x_0) \neq g_2(x_0)$ in Y.
- $\exists V_0 \in \mathcal{O}(Y)$, $g_1(x_0) \in V_0$ and $g_2(x_0) \notin V_0$ (without loss of generality).
- $\exists U_1, U_2 \in \mathcal{O}(X)$, $x_0 \in U_1 \setminus U_2$ and $U_1 \cap A = U_2 \cap A$. (take $U_i = g_i^{-1}(V_0)$). Consequently,

Proposition (Skula-1969)

The extension is at most one for all $Y \in \mathbf{Top}_0$ iff $\forall x \in X$, $\nexists U_1, U_2 \in \mathcal{O}(X)$ s.t. $x \in U_1 \setminus U_2$ and $U_1 \cap A = U_2 \cap A$.

Let $A \subseteq X \in \mathbf{Top}$, and define

- $x \in \widehat{A} \;\Leftrightarrow\; \nexists U_1, U_2 \in \mathcal{O}(X)$ s.t. $x \in U_1 \setminus U_2$ and $U_1 \cap A = U_2 \cap A$.

Proposition (Skula-1969)

The following assertion holds:

1. $A \subseteq \widehat{A} \subseteq \overline{A}$;
2. $A \subseteq B \;\Rightarrow\; \widehat{A} \subseteq \widehat{B}$;
3. $\widehat{A} = \widehat{\widehat{A}}$;
4. $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$.

This new topology is the so called b-topology, denoted by bX.

Corollary (Skula-1969)

The extension of the inclusion $i : A \to X$ is at most one for all $Y \in \mathbf{Top}_0$ iff A is b-dense in X (i.e., dense in bX).

b-embedding reflector

Definition (Skula 1969)

- Let $F : \mathbf{Top} \longrightarrow \mathbf{P}$ be a reflector. F is called a b-embedding reflector iff $\mathbf{P} \subseteq \mathbf{Top}_0$ and if $e_X(X)$ is a b-dense subset of $F(X)$ for all $X \in \mathbf{Top}$.
- A topological property $\mathbf{P}$ is called a b-embedding iff there exists a b-embedding reflector $F : \mathbf{Top} \longrightarrow \mathbf{P}$.
- A topological property $\mathbf{P}$ is b-closed-hereditary if $Y \in \mathbf{P}$ whenever Y is a b-closed subspace of some $X \in \mathbf{P}$.

Theorem (D)

A topological property $\mathbf{P}$ *is b-embedding iff* $\mathbf{P}$ *is productive, b-closed-hereditary and* $\mathbf{P} \subseteq \mathbf{Top}_0$.

Skula's example

The Sierpinski space $\mathbb{S} = \Sigma 2$, where $2 = \{0, 1\}$ with open sets $\emptyset, 2, \{1\}$. Then the power $\prod_{i \in I} \mathbb{S} = \Sigma(2^I, \subseteq)$, since $\uparrow\chi_F = \bigcap_{i \in F} p^{-1}(\{1\})$.

S The class of spaces that is homeomorphic to a b-closed subspace of a product $\prod_{i \in I} \mathbb{S}$ for some non-empty set I.

Example (Skula-1969)

Let $X = \prod_{i=1}^{+\infty} \mathbb{S} = \Sigma 2^N$ and $A = X - \{N\}$. Note that each open set in X contains N, so $\widehat{A} = X$, which implies that X is the reflection of A in **S**. Also, **S** is a topological property such that

- **S** is b-embedding.
- e_A is not surjective, so **S** is *not simple*.
- $e_A(A) \neq X$, so **S** is *not identifying*.
- **S** $\not\subseteq$ $\mathbf{Top}_2$, so **S** is *not embedding*.

Theorem (Skula-1969)

If **P** *is a full reflective subcategory of the category of* **Top** *containing at least one non-*T_1 *space, then* **P** *is a subcategory of one of the above-mentioned 4 types:*

- *simple;*
- *identifying;*
- *embedding;*
- *b-embedding*

Content

Some properties on b-topology

In the paper[3], Hoffmann (1979) proved some properties on b-topology.

Theorem (Hoffmann-1979)

Let X be a topological space.

- *The topology of bX is equivalently generated by any of the following:*
  - *The collection $\{{\downarrow}x \cap U : x \in U \in \mathcal{O}(X)\}$;*
  - *The union $\mathcal{O}(X) \cup \mathcal{C}(X)$ (hence the b-topology is **finer** than the original);*
  - *The union $\mathcal{O}(X) \cup \{A \subseteq X : A = {\downarrow}A\}$.*
- *For each $x \in U \in \mathcal{O}(X)$, ${\downarrow}x \cap U$ is both b-closed and b-open, so bX is zero-dimensional, hence is completely regular.*
- *If $X \in \mathbf{Top}_0$, then bX is Hausdorff, hence is a Tychonoff space.*
- *A T_0 space X is Noetherian and sober iff bX is a compact T_2 space.*

[3] R.E Hoffmann, On the sobrification remainder $X^s - X$, Pacific J. Math. 83(1), 1979

Characterize sobriety via b-topology

The b-topology also proves to be a effective tool for describing (or characterizing) the sobrification of T_0 spaces[4].

Definition

A nonempty subset A of a T_0 space is called irreducible if for any closed sets F_1, F_2, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called sober, if for any irreducible closed set F of X there is a (unique) point $x \in X$ such that $F = \downarrow x$.

Theorem (Keimel and Lawson 2009)

*Let $A \subseteq X \in$ **Sober**.*

(1) *A is sober iff A is b-closed;*

(2) *$A^s \cong \mathrm{cl}_b(A)$, where A^s is the sobrification of A.*

[4] K. Keimel, J.D. Lawson, D-completions and the d-topology, Ann. Pure Appl. Logic 159 (2009) 292–306.

Transition to $\mathbf{Top}_0$: why T_0 spaces?

In the remainder of this talk, we **shift our focus** from the general category **Top** to the category $\mathbf{Top}_0$ of T_0 spaces.

Motivation from Domain Theory

From the perspective of domain theory, $\mathbf{Top}_0$ captures **most structures of interest**, including:

- **sober spaces**, **well-filtered spaces**, *d*-**spaces**, et al.

Why not T_1?

Many important topologies in domain theory are naturally T_0 but **not** T_1, such as:

- the **Scott topology**, the **Alexandroff topology**, and the **upper topology**.

The T_0 separation axiom thus provides the **natural setting** for our study.

In domain theory, the mostly concerned topological spaces are usually just T_0. We use

- **Top₀**: all T_0 spaces + continuous maps.

Definition

A subcategory $\mathbf{K}$ of $\mathbf{Top_0}$ is called reflective, if $\forall X \in \mathbf{Top_0}$, $\exists X^k \in \mathbf{K}$ (the $\mathbf{K}$-completion) and a continuous map $\mu_X : X \longrightarrow X^k$ (the $\mathbf{K}$-reflection) s.t. for any conti. map $f : X \longrightarrow Z \in \mathbf{K}$, $\exists$ a unique conti. map $g : X^k \longrightarrow Z$ such that $g \circ \mu_X = f$.

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "f"'] & X^k \arrow[d, dotted, "g"] \\ & Z. \end{tikzcd}$$

In 2009, Keimel and Lawson[5] proved that a full subcategory **K** of T_0 spaces is reflective in the category $\mathbf{Top_0}$ of all T_0 spaces if it satisfies the following **four conditions**:

(K1) **K** contains all sober spaces;

(K2) If $X \in \mathbf{K}$ and Y is homeomorphic to X, then $Y \in \mathbf{K}$;

(K3) If $\{X_i : i \in I\} \subseteq \mathbf{K}$ is a family of subspaces of a sober space, then the subspace $\bigcap_{i\in I} X_i \in \mathbf{K}$.

(K4) If $f : X \longrightarrow Y$ is a continuous map from a sober space X to a sober space Y, then for any subspace Y_1 of Y, $Y_1 \in \mathbf{K}$ implies that $f^{-1}(Y_1) \in \mathbf{K}$.

Theorem (Wu-Xi-Xu-Zhao, 2019)

The category of well-filtered spaces satisfies (K1) – (K4), thus is reflective in $\mathbf{Top_0}$[a].

[a] G. Wu, X. Xi, X. Xu and D. Zhao, Existence of well-filterification, Topol. Appl. 267 (2019) 107044.

[5] K. Keimel, J.D. Lawson, *D*-completions and the *d*-topology, Ann. Pure Appl. Logic 159 (2009) 292–306.

Definition

A category **K** has equalizers if for any morphisms $f, g : X \longrightarrow Y$ in **K**, the equalizer $E_{f,g} = \{x \in X : f(x) = g(x)\}$ of f and g belongs to **K**.

We proved that the Keimel-Lawson condition is not only **sufficient** but also **necessary**.

Theorem (Shen-Xi-Zhao 2024)

Let **K** *be a subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{Top_1}$. *Then the following statements are equivalent:*

(1) **K** *is reflective in* $\mathbf{Top_0}$*;*

(2) **K** *satisfies conditions (K1)–(K4);*

(3) **K** *is productive and b-closed-hereditary*[a]*;*

(4) **K** *is productive and has equalizers*[b]*.*

[a] C. Shen, X. Xi, D. Zhao, the reflectivity of some categories of T_0 spaces in domain theory, Rocky Mountain Journal of Mathematics, 54 (2024), 1149 – 1166.

[b] L. Nel, R. Wilson, Epireflections in the category of T_0-spaces, Fund. Math. 75 (1972) 69-74.

By using the above result, we can easily prove the following:

- well-filtered spaces ($\checkmark$), by Wu, Xu, Xi, Zhao (2019)
- $\mathcal{U}_s$-admitting spaces ($\checkmark$), by Shen, Xi, Zhao (2025)
- k-bounded sober spaces ($\times$), by Lu, Wang, Wu, Zhao (2020)
  — *(K3) does not hold.*
- strong-d-spaces ($\times$), by Jin, Miao, Li (2021)
  — *(K3) does not hold.*
- open well-filtered spaces ($\times$), , by Shen, Xi, Zhao (2025)
  — *Not b-closed hereditary.*
- co-sober spaces ($\times$), by Shen, Xi, Zhao (2025)
  — *(K3) does not hold..*
- T_D spaces ($\times$), by Hou, Li, Miao, Zhao (2023)
  — *Not productive.*
- et al.

Related papers of the last page

- G. Wu, X. Xi, X. Xu, D. Zhao, Existence of well-filtered reflections of T_0 topological spaces, Topol. Appl. 267 (2019) 107044.
- X. Xu, D. Zhao, Some open problems on well-filtered spaces and sober spaces, Topol. Appl. (2020) 107540.
- Q. Li, M. Jin, H. Miao, S. Chen, On some results related to sober spaces, Acta Mathematica Scientia, 2023, 43B(4): 1477 -1490.
- C. Shen, X. Xi, X. Xu, D. Zhao, On open well-filtered spaces, Logic Meth. Computer Sci. 16 (4) (2020) 4–18.
- X. Xu, D. Zhao, On topological Rudin's lemma, well-filtered spaces and sober spaces, Topol. Appl. 272 (2020) 107080.
- H. Hou, Q. Li, H. Miao, D. Zhao, The reflective hulls of some subcategories in the category of all T0 spaces. Houston Journal of Mathematics, 49(2)(2023) 381–395.

Remark 1: Condition $\mathbf{K} \not\subseteq \mathbf{Top_1}$ and Ershov's analogous result

Definition (Sierpiński space)

The Scott space $\Sigma 2$, where the underlying set $2 = \{0, 1\}$ with $0 \leq 1$, is called the Sierpiński space. Its open sets are $\varnothing$, $\{0, 1\}$, and $\{1\}$.

Lemma

Let X be a T_0 space. Then, the following statements are equivalent:

(1) *X is non-T_1.*

(2) *$\Sigma 2$ is a retract of X.*

(3) *$\Sigma 2$ is homeomorphic to a b-closed subspace of X.*

(4) *$\Sigma 2$ is homeomorphic to a subspace of X.*

Corollary (Shen-Xi-Zhao, 2024)

Let $\mathbf{K}$ *be a full subcategory of* $\mathbf{Top}_0$*. Then, the following statements are equivalent:*

(1) $\mathbf{K} \not\subseteq \mathbf{Top}_1$.

(2) *The space* $\Sigma 2$ *can be topologically embedded into some space* $Y \in \mathbf{K}$.

Definition (Ershov-2022)

A full subcategory $\mathbf{K}$ of $\mathbf{Top}$ is called wide if every T_0 space X can be topologically embedded into some space $Y \in \boldsymbol{K}$[a].

[a] Y. L. Ershov, "$\mathbf{K}$-completions of T_0-spaces", Algebra Logic 61:3 (2022), 177–187.

Lemma (Ershov-2022)

If $\mathbf{K}$ *is a wide subcategory of* $\mathbf{Top_0}$*, then* $\mathbf{K} \not\subseteq \mathbf{Top_1}$*. Not vice versa.*

Ershov's result is **later** and **weaker** than ours

Corollary (Ershov-2022; Shen,Xi,Zhao-2021)

A **wide** *subcategory* $\boldsymbol{K}$ *of* $\mathbf{Top}_0$ *is reflective iff* $\boldsymbol{K}$ *satisfies Keimel-Lawson conditions (K1)-(K4).*

arXiv > math > arXiv:2110.01138

Mathematics > General Topology

4 Oct 2021

[Submitted on 4 Oct 2021]

The reflectivity of some categories of T0 spaces in domain theory

Chong Shen, Xiaoyong Xi, Dongsheng Zhao

Keimel and Lawson proposed a set of conditions for proving a category of topological spaces to be reflective in the categ
we prove that, in certain sense, these conditions are not just sufficient but also necessary for a category of T0 spaces to b
answered a few open problems.

Subjects:	**General Topology (math.GN)**; Category Theory (math.CT)
Cite as:	arXiv:2110.01138 **[math.GN]**
	(or arXiv:2110.01138v1 **[math.GN]** for this version)
	https://doi.org/10.48550/arXiv.2110.01138
Journal reference:	Rocky Mountain J. Math. 54(4), 1149-1166 (2024)
Related DOI:	https://doi.org/10.1216/rmj.2024.54.1149

Remark 2: Xu's approach to **K**-reflection

Definition (Wright-Wagner-Thatcher-1978, Xu-2021)

A covariant functor $\mathrm{H} : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ is called a R-subset system[a][b] on $\mathbf{Top}_0$ provided that the following two conditions are satisfied:

(1) $\forall X \in \mathbf{Top_0}$, $\mathcal{S}(X) \subseteq \mathrm{H}(X) \subseteq Irr(X)$;

(2) $\forall f : X \longrightarrow Y \in \mathbf{Top}_0$, $\mathrm{H}(f)(A) = f(A) \in \mathrm{H}(Y)$ for all $A \in \mathrm{H}(X)$.

[a] J. Wright, E. Wagner, J. Thatcher. A uniform approach to inductive posets and inductive closure, Theoret. Comput. Sci. 7(1)(1978) 57-77.

[b] X. Xu, On H-sober spaces and H-sobrifications of T_0 spaces, Topol. Appl. 289 (2021) 107548.

Definition (Xu - 2021)

Let $H : \mathbf{Top}_0 \longrightarrow \mathbf{Set}$ be an R-subset system.

- A T_0 space X is called *H*-sober if for any $A \in H(X)$, there is a (unique) point $x \in X$ such that $\overline{A} = \overline{\{x\}}$.
- The category of all H-sober spaces with continuous mappings is denoted by **H-Sob**.

Theorem (Xu-2021)

For a full subcategory $\mathbf{K}$ *of* $\mathbf{Top}_0$ *with* $\mathbf{K} \not\subseteq \mathbf{Top_1}$*, the following conditions are equivalent:*

(1) $\mathbf{K}$ *is reflective in* $\mathbf{Top}_0$*.*

(2) *There exists an R-subset system H:* $\mathbf{Top}_0 \longrightarrow \mathbf{Set}$ *such that* $\mathbf{K} = \textbf{H-sob}$*.*

Definition (Xu-2020)

Let $\mathbf{K}$ be a full subcategory of $\mathbf{Top}_0$ with $\mathbf{Sob} \subseteq \mathbf{K}$.

- A subset A of a T_0 space X is called $\mathbf{K}$-determined provided for any continuous mapping $f : X \longrightarrow Y \in \mathbf{K}$, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$.
- Denote by $\mathbf{K}(X)$ the set of all closed $\mathbf{K}$-determined sets of X.
- The lower Vietoris topology on $\mathbf{K}(X)$ is the topology $\{\diamond U : U \in \mathcal{O}(X)\}$, where

$$\diamond U = \{A \in \mathbf{K}(X) : A \cap U \neq \emptyset\},$$

and the resulting space is denoted by $P_H(\mathbf{K}(X))$ [a].

[a] X. Xu, A direct approach to K-reflections of T_0 spaces, Topol. Appl. 272 (2020) 107076.

Definition (Xu-2020)

$\mathbf{K}$ is called adequate if for any T_0 space X, $P_H(\mathbf{K}(X))$ is a $\mathbf{K}$-space.

Theorem (Xu-2020)

For a full subcategory $\mathbf{K}$ of $\mathbf{Top}_0$ with $\mathbf{K} \not\subseteq \mathbf{Top_1}$, the following conditions are equivalent:

(1) *$\mathbf{K}$ is reflective in $\mathbf{Top}_0$.*

(2) *$\mathbf{K}$ is adequate.*

Theorem (Shen,Xi,Zhao-2024; Xu-2020-2021)

For a full subcategory $\mathbf{K}$ *of* $\mathbf{Top_0}$ *with* $\mathbf{K} \not\subseteq \mathbf{Top_1}$, *the following conditions are equivalent:*

(1) $\mathbf{K}$ *is reflective in* $\mathbf{Top_0}$.

(2) $\mathbf{K}$ *is adequate.*

(3) $\mathbf{K} = \mathbf{H\text{-}Sob}$ *for some R-subset system* $H : \mathbf{Top_0} \longrightarrow \mathbf{Set}$.

(4) $\mathbf{K}$ *is a Keimel-Lawson category.*

(5) $\mathbf{K}$ *is productive and b-closed-hereditary.*

(6) $\mathbf{K}$ *is productive and has equalizers.*

Remark 3: T_D-space and **K**-remainder $X^k \setminus X$

Observing that

- T_0-separation: $\mathrm{cl}(\{x\}) \setminus \{x\} = \bigcup_{y \in \mathrm{cl}(\{x\}) \setminus \{x\}} \mathrm{cl}(\{y\})$, the **union of closed** sets.
- T_1-separation: $\mathrm{cl}(\{x\}) \setminus \{x\} = \emptyset$, a **closed** set.

Definition (Aull-Thron, 1963)

A space X is T_D if for every $x \in X$, $\mathrm{cl}(\{x\}) \setminus \{x\}$ is closed [a].

[a] C.E. Aull, W.J. Thron. Separation axioms between T_0 and T_1, Indag. Math., 24(1963), 26 – 37.

Remark

$$\begin{tikzcd}
\textit{Finite } T_0 \textit{ space} \arrow[r] & \textit{Alexandorff } T_0 \textit{ spaces} \arrow[dr] & & \\
\textit{Hausdorff} \arrow[r] \arrow[dr] & T_1 \arrow[r] & T_D \arrow[r] & T_0 \\
& \textit{sober} \arrow[urr] & &
\end{tikzcd}$$

Theorem (Hoffmann-1977)

[a] *The following are equivalent:*

(1) *X is T_D;*

(2) *For any subset $A \subseteq X$, the derived set A' of A is closed.*

(3) *bX is discrete; that is, $\forall x \in X$, $\exists U \in \mathcal{O}(X)$ s.t. $U \cap \mathrm{cl}(\{x\}) = \{x\}$.*

[a] R.E. Hoffmann, Irreducible filters and spaces, Manuscripta Math. 22 (1977) 365 - 380.

Theorem (Thron-1962)

[a] *Let X and Y are T_D spaces. If $(\mathcal{O}(X), \subseteq) \cong (\mathcal{O}(Y), \subseteq)$, then $X \cong Y$.*

[a] W.J. Thron, Lattice-equivalence of topological spaces, Duke Math. J., 29 (1962) 671–679.

Let X be a T_0 space, and $\mathbb{N}$ be the set of natural numbers with Alexandorff topology. Let

$$\mathbb{N}_X := (\mathbb{N}^s \times X^s) - (\{\infty\} \times X)$$

with the topology induced from $\mathbb{N}^s \times X^s$ (X is to be considered as a subspace of X^s).

Theorem (Hoffmann-1979)

[a] *For every T_0-space X holds $X \cong {}^s N_X - N_X$, i.e., every T_0-space is a sobrification remainder.*

[a] R.E Hoffmann, On the sobrification remainder $X^s - X$, Pacific J. Math. 83(1), 1979.

Lemma (Hoffmann-1979)

(i) *If Y is a T_D-space, then ${}^s Y - Y$ is sober.*

(ii) *N_X is T_D iff X is both sober and T_D.*

Proposition

Let Y be a b-dense subspace of a topological space X. If Y is a T_D space, then Y is a b-open subset of X, or equivalently, $X \setminus Y$ is b-closed.

Theorem (Remainder of $\mathbf{K}$-reflection)

Let $\mathbf{K}$ be a reflective subcategory of $\mathbf{Top}_0$. If X is a T_D space, then $X^k \setminus X$ (as a subspace of X^k) belongs to $\mathbf{K}$.

Corollary

If Y is a T_D-space, then

(1) $Y^s \setminus Y$ is sober;

(2) $Y^{wf} \setminus Y$ is well-filtered;

(3) $Y^d \setminus Y$ is a d-space;

(4) $Y^{us} \setminus Y$ is $\mathcal{U}_s$-admitting.

Thank you for your attention.

I am happy to take any questions you might have.

Content

Determine the reflective hull via Xi-Zhao model

Definition (reflective hull)

Let $\mathbf{P} \subseteq \mathbf{C}$ be a full subcategory. The reflective hull of $\mathbf{P}$ (denoted by $\overline{\mathbf{P}}$) in $\mathbf{C}$ is, if it exists, the smallest reflective subcategory of $\mathbf{C}$ containing $\mathbf{P}$.

Proposition (J. Martin Harvey-1985)

[a] *If the intersection of a family of reflective subcategories of* $\mathbf{C}$ *is cowell-powering, then it is reflective.*

[a] J. Martin Harvey. Reflective subcategories, Illinois J. Math. 29 (1985) 365–369.

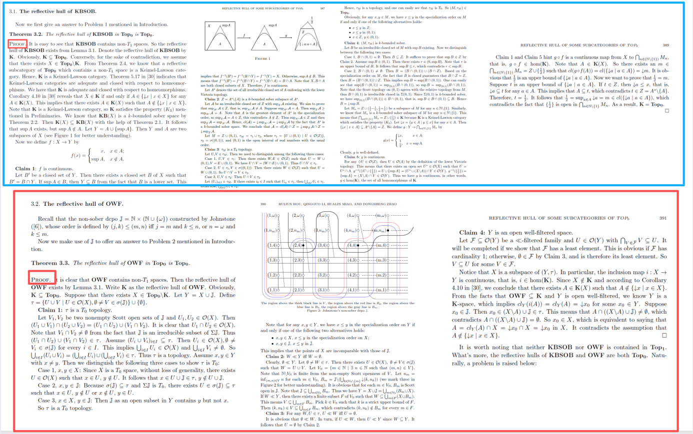
Theorem (J.M. Harvey,1985; Hou-Li-Miao-Zhao, 2023)

Let $\mathbf{P}$ *be a full subcategory of* $\mathbf{Top_0}$ *with* $\mathbf{P} \not\subseteq \mathbf{Top_1}$. *Then the reflective hull of* $\mathbf{P}$ *exists*[a].

[a] H. Hou, Q. Li, H. Miao, D. Zhao, The reflective hulls of some subcategories in the category of all T_0 spaces. Houston Journal of Mathematics, 49 (2023), 381 – 395.

Theorem (Hou-Li-Miao-Zhao, 2023)

The reflective hull of **k-bounded sober spaces** *and* **open well-filtered spaces** *is* $\mathbf{Top}_0$.



Remark

Let $\mathbf{P}$ *be a full subcategory of* $\mathbf{Top}_0$, *then* $\overline{\mathbf{P}}$ *contains:*

- *all sober spaces;*
- $\prod_{i \in I} X_i$, *for* $X_i \in \mathbf{P}$, $i \in I$;
- Y, *if* Y *is a b-closed (hence, saturated or closed) subspace of some* $X \in \mathbf{P}$.
- Z, *if* Z *is a retract of some* $X \in \mathbf{P}$.

Corollary

If every T_0 *space is a saturated (b-closed) subspace of some space in* $\mathbf{P}$, *then* $\overline{\mathbf{P}} = \mathbf{Top}_0$.

Proposition (Shen-Xi-Zhao, 2022)

For each T_0 space X, the product $X \times \Sigma\mathbb{J}$ is open well-filtered[a].

[a] C. Shen, X. Xi, D. Zhao. Further studies on open well-filtered spaces, Electronic Notes in Theoretical Informatics and Computer Science 2(2022), 1–12

Proposition

Every T_0 space is a retract of some open well-filtered space.

Using the above method, we can therefore recover the result established by Hou et al. (2023).

Corollary (Hou et al, 2023)

The reflective hull of open well-filtered spaces in $\mathbf{Top}_0$ is $\mathbf{Top}_0$.

Definition

A T_0 space X is k-bounded sober (resp., bounded sober) if for any irreducible closed set A with $\sup A$ existing (resp., A being upper bounded), there is a unique point $x \in X$ such that $A = \downarrow x$.

Lemma

For each T_0 space X, the product $X \times \mathbb{N}_{cof}$ is k-bounded (bounded) sober.

Proposition

Every T_0 space X is a retract of some k-bounded (bounded) sober space.

We can therefore easily recover the result of Hou et al. (2023) using a relatively simple method.

Corollary (Hou et al, 2023)

The reflective hull of k-bounded (bounded) sober spaces in $\mathbf{Top}_0$ is $\mathbf{Top}_0$.

Proposition

For each d-space X, there is a dcpo P such that

(1) *X is a subspace of ΣP;*

(2) *X is a saturated subspace of ΣP.*

(3) *X is sober if and only if ΣP is sober.*

Theorem (A)

(1) *The reflective hull of dcpos in $\mathbf{Top}_0$ is d-spaces.*

(2) *The reflective hull of sober dcpos in $\mathbf{Top}_0$ is sober spaces.*

Theorem (B)

The reflective hull of Rudin space is $\mathbf{Top_0}$.

Lemma

Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Then, the topology of $b\prod_{i\in I} X_i$ is finer than that of $\prod_{i\in I} bX_i$.

Lemma

Let X be a topological space and Y a subspace of X. Then, bY is a subspace of bX.

Direct construction of the reflective hull

Theorem

Let $\mathbf{P}$ *be a* T_0 *topological property. Then, the reflective hull* $\overline{\mathbf{P}}$ *of* $\mathbf{P}$ *in* $\mathbf{Top}_0$ *is given by*

$$\overline{\mathbf{P}} = \left\{ X \in \mathbf{Top}_0 : X \hookrightarrow \prod_{i \in I} X_i \textit{ as a b-closed subspace, with } X_i \in \mathbf{P} \right\}.$$

Thus, the class $\overline{\mathbf{P}}$ can be constructed in two stages:

- First, form arbitrary products of members of $\mathbf{P}$;
- Second, take all b-closed subspaces (within the T_0 category) of these products.

Current Work

Research Topic

Determine the reflective hull (in the category of $\mathbf{Top_0}$) of T_D spaces and other non-Hausdorff spaces (e.g. co-sober spaces, consonant space, strong-d-spaces, et al)?