Triangular norms

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TRIANGULAR NORMS

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1. Triangular norms and conorms

Definition 1. A *triangular norm* (*t-norm* for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T\colon [0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$ the following four axioms are satisfied:

(T1) *Commutativity*

$$T(x,y) = T(y,x)\,, \tag{1}$$

(T2) *Associativity*

$$T\big(x, T(y,z)\big) = T\big(T(x,y), z\big)\,, \tag{2}$$

(T3) *Monotonicity*

$$T(x,y) \le T(x,z) \qquad \text{whenever } y \le z\,, \tag{3}$$

(T4) *Boundary Condition*

$$T(x,1) = x\,. \tag{4}$$

Definition 2. If T is a t-norm, then its *dual t-conorm* $S\colon [0,1]^2 \to [0,1]$ is given by

$$S(x,y) = 1 - T(1-x, 1-y)\,. \tag{5}$$

Example 3. The following are the four basic t-norms together with their dual t-conorms:

(i) *Minimum* $T_{\mathbf{M}}$ and *Maximum* $S_{\mathbf{M}}$ given by

$$T_{\mathbf{M}}(x,y) = \min(x,y)\,,$$
$$S_{\mathbf{M}}(x,y) = \max(x,y)\,,$$

(ii) *Product* $T_{\mathbf{P}}$ and *Probabilistic Sum* $S_{\mathbf{P}}$ given by

$$T_{\mathbf{P}}(x,y) = x \cdot y\,,$$
$$S_{\mathbf{P}}(x,y) = x + y - x \cdot y\,,$$

(iii) *Łukasiewicz t-norm* $T_{\mathbf{L}}$ and *Łukasiewicz t-conorm* $S_{\mathbf{L}}$ given by

$$T_{\mathbf{L}}(x,y)=\max(x+y-1,0)\,,$$
$$S_{\mathbf{L}}(x,y)=\min(x+y,1)\,,$$

(iv) *Weakest t-norm (Drastic Product)* $T_{\mathbf{W}}$ and *strongest t-conorm* $S_{\mathbf{W}}$ given by

$$T_{\mathbf{W}}(x,y)=\begin{cases}\min(x,y)\,, & \text{if } \max(x,y)=1\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$
$$S_{\mathbf{W}}(x,y)=\begin{cases}\max(x,y)\,, & \text{if } \min(x,y)=0\,,\\ 1\,, & \text{otherwise}\,.\end{cases}$$

EXAMPLE 4.

(i) Another interesting t-norm is the *nilpotent Minimum* $T_{\mathbf{M}}^{\mathrm{nil}}$ given by

$$T_{\mathbf{M}}^{\mathrm{nil}}(x,y)=\begin{cases}\min(x,y)\,, & \text{if } x+y>1\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

(ii) The family $(T_{\lambda}^{\mathbf{F}}\lambda)_{\lambda\in[0,+\infty]}$ of *Frank t-norms* is given by

$$T_{\lambda}^{\mathbf{F}}\lambda(x,y)=\begin{cases}T_{\mathbf{M}}(x,y)\,, & \text{if } \lambda=0\,,\\ T_{\mathbf{P}}(x,y)\,, & \text{if } lambda=1\,,\\ T_{\mathbf{L}}(x,y)\,, & \text{if } \lambda=+\infty\,,\\ \log_{\lambda}\Big(1+\dfrac{(\lambda^{x}-1)(\lambda^{y}-1)}{\lambda-1}\Big)\,, & \text{otherwise}\,.\end{cases}$$

(iii) The family $(S_{\lambda}^{\mathbf{F}}\lambda)_{\lambda\in[0,+\infty]}$ of *Frank t-conorms* is given by

$$S_{\lambda}^{\mathbf{F}}\lambda(x,y)=\begin{cases}S_{\mathbf{M}}(x,y)\,, & \text{if } \lambda=0\,,\\ S_{\mathbf{P}}(x,y)\,, & \text{if } \lambda=1\,,\\ S_{\mathbf{L}}(x,y)\,, & \text{if } \lambda=+\infty\,,\\ 1-\log_{\lambda}\Big(1+\dfrac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\Big)\,, & \text{otherwise}\,.\end{cases}$$

(iv) The family $(T_{\lambda}^{\mathbf{Y}}\lambda)_{\lambda\in[0,+\infty]}$ of *Yager t-norms* is given by

$$T_{\lambda}^{\mathbf{Y}}\lambda(x,y)=\begin{cases}T_{\mathbf{W}}(x,y)\,, & \text{if } \lambda=0\,,\\ T_{\mathbf{M}}(x,y)\,, & \text{if } \lambda=+\infty\,,\\ \max\Big(0,1-\big((1-x)^{\lambda}+(1-y)^{\lambda}\big)^{\frac{1}{\lambda}}\Big)\,, & \text{otherwise}\,.\end{cases}$$

(v) The family $(S^{\mathbf{Y}}_\lambda \lambda)_{\lambda\in[0,+\infty]}$ of *Yager t-conorms* is given by

$$S^{\mathbf{Y}}_\lambda \lambda(x,y) = \begin{cases} S_{\mathbf{W}}(x,y)\,, & \text{if } \lambda = 0\,, \\ S_{\mathbf{M}}(x,y)\,, & \text{if } \lambda = +\infty\,, \\ \min\Big(1, \left(x^\lambda + y^\lambda\right)^{\frac{1}{\lambda}}\Big)\,, & \text{otherwise}\,. \end{cases}$$

Remark 5.

(1) Directly from Definition 1 we can deduce that, for all $x \in [0,1]$, each t-norm T satisfies the following additional boundary conditions:

$$T(0,x) = T(x,0) = 0\,, \tag{6}$$
$$T(1,x) = x\,. \tag{7}$$

This means that all t-norms coincide on the boundary of the unit square $[0,1]^2$.

(ii) The monotonicity of a t-norm T in its second component described by (T3) is, together with the commutativity (T1), equivalent to the (joint) monotonicity in both components, i.e., to

$$T(x_1,y_1) \le T(x_2,y_2) \quad \text{whenever} \quad x_1 \le x_2 \text{ and } y_1 \le y_2\,. \tag{8}$$

Definition 6.

(i) If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \le T_2(x,y)$ holds for all $(x,y) \in [0,1]^2$, then we say that T_1 is *weaker* than T_2 or, equivalently, that T_2 is *stronger* than T_1, and we write in this case $T_1 \le T_2$.

(ii) We shall write $T_1 < T_2$ whenever $T_1 \le T_2$ and $T_1 \ne T_2$, i.e., if $T_1 \le T_2$, but $T_1(x_0,y_0) < T_2(x_0,y_0)$ holds for some $(x_0,y_0) \in [0,1]^2$.

Remark 7.

(i) As a consequence of (8) we obtain for each $(x,y) \in [0,1]^2$

$$T(x,y) \le T(x,1) = x\,,$$
$$T(x,y) \le T(1,y) = y\,.$$

Since, for all $(x,y) \in]0,1[^2$ trivially $T(x,y) \ge 0 = T_{\mathbf{W}}(x,y)$, we get for an arbitrary t-norm T

$$T_{\mathbf{W}} \le T \le T_{\mathbf{M}}\,, \tag{9}$$

i.e., $T_{\mathbf{W}}$ is the weakest and $T_{\mathbf{M}}$ is the strongest of all t-norms.

(ii) As it is easy to see that $T_{\mathbf{L}} < T_{\mathbf{P}}$, we get the following ordering of the four basic t-norms

$$T_{\mathbf{W}} < T_{\mathbf{L}} < T_{\mathbf{P}} < T_{\mathbf{M}}\,. \tag{10}$$

An interesting question is whether a t-norm is determined uniquely by its values on the diagonal or some other subset of the unit square. The extremal t-norms $T_{\mathbf{W}}$ and $T_{\mathbf{M}}$ are completely determined by their values on the diagonal $\Delta = \{(x,x) \mid x \in]0,1[\}$ of the (open) unit square.

Remark 8.

(i) The Minimum $T_{\mathbf{M}}$ is the only t-norm satisfying $T(x,x) = x$ for all $x \in]0,1[$: If for a t-norm T we put $\delta_T(x) = T(x,x)$ and assume $\delta_T(x) = x$ for each $x \in]0,1[$, then whenever $y \leq x < 1$ the monotonicity (T3) implies

$$y = \delta_T(y) = T(y,y) \leq T(x,y) \leq \min(x,y) = y\,.$$

Together with (T1) and the boundary conditions this gives exactly $T = T_{\mathbf{M}}$.

(ii) The weakest t-norm $T_{\mathbf{W}}$ is the only t-norm satisfying $T(x,x) = 0$ for all $x \in]0,1[$: Assume $\delta_T(x) = 0$ for each $x \in]0,1[$, then we obtain

$$0 \leq T(x,y) \leq T(x,x) = \delta_T(x) = 0$$

whenever $y \leq x < 1$, hence yielding $T = T_{\mathbf{W}}$.

Definition 9. If T is a t-norm, $x \in [0,1]$ and $n \in \mathbb{N}$ then we shall write

$$x_T^{(n)} = \begin{cases} x\,, & \text{if } n = 1\,, \\ T(\overbrace{x, x, \ldots, x}^{n \text{ times}})\,, & \text{if } n > 1\,. \end{cases} \tag{11}$$

2. Properties of t-norms

Proposition 10. *A t-norm T is continuous if and only if it is continuous in its first component, i.e., if for each $y \in [0,1]$ the one-place function*

$$\begin{aligned} T(\cdot,y)\colon [0,1] &\to [0,1] \\ x &\mapsto T(x,y) \end{aligned}$$

is continuous.

Proposition 11. *A t-norm T is left continuous (right continuous) if and only if it is left continuous (right continuous) in its first component, i.e., if for each $y \in [0,1]$ and for each sequence $(x_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ we have, respectively,*

$$\sup_{n\in\mathbb{N}} T(x_n, y) = T\Big(\sup_{n\in\mathbb{N}} x_n, y\Big)\,, \tag{12}$$

$$\inf_{n\in\mathbb{N}} T(x_n, y) = T\Big(\inf_{n\in\mathbb{N}} x_n, y\Big)\,. \tag{13}$$

DEFINITION 12.

(i) A t-norm T is said to be *strictly monotone* if it is strictly increasing on $]0,1]^2$ as a function from $[0,1]^2$ into $[0,1]$ or, equivalently, if (taking into account the commutativity (T1) and the boundary condition (T4))

$$T(x,y) < T(x,z) \qquad \text{whenever} \quad x \in]0,1[\text{ and } y < z\,. \tag{14}$$

(ii) A t-norm T is called *strict* if it is continuous and strictly monotone.

EXAMPLE 13.

(i) The Minimum $T_{\mathbf{M}}$ and the Łukasiewicz$t$-norm $T_{\mathbf{L}}$ are continuous but not strictly monotone.

(1) The t-norm T defined by

$$T(x,y) = \begin{cases} \frac{xy}{2}\,, & \text{if } \max(x,y) < 1\,, \\ xy\,, & \text{otherwise}\,, \end{cases}$$

is strictly monotone but not continuous.

(ii) If T is a strictly monotone t-norm then for each $x \in]0,1[$ we have $T(x,x) < x$ and, as a consequence, the sequence $\left(x_T^{(n)}\right)_{n\in\mathbb{N}}$ is strictly decreasing.

(iii) Among the four basic t-norms presented in Example 3, only the Product $T_{\mathbf{P}}$ is a strict t-norm.

PROPOSITION 14. *A t-norm T is strictly monotone if and only if the cancellation law holds, i.e., if $T(x,y) = T(x,z)$ and $x > 0$ imply $y = z$.*

DEFINITION 15. A t-norm T is called *Archimedean* if for all $(x,y) \in]0,1[^2$ there is an integer $n \in \mathbb{N}$ such that

$$x_T^{(n)} < y\,. \tag{15}$$

EXAMPLE 16. The Product $T_{\mathbf{P}}$, the Łukasiewicz$t$-norm $T_{\mathbf{L}}$ (which is not strictly monotone and, hence, not strict) and the weakest t-norm $T_{\mathbf{W}}$ are all Archimedean (observe that the latter is not continuous), the Minimum $T_{\mathbf{M}}$, however, is not an Archimedean t-norm.

PROPOSITION 17. *A t-norm T is Archimedean if and only if for each $x \in]0,1[$ we have*

$$\lim_{n\to\infty} x_T^{(n)} = 0\,. \tag{16}$$

At least for continuous t-norms it is possible to characterize the Archimedean property (16) by their diagonal mapping:

Theorem 18.

(i) *If T is an Archimedean t-norm then for each $x \in]0,1[$ we have*

$$T(x,x) < x\,. \tag{17}$$

(ii) *If T is a continuous t-norm and if for each $x \in]0,1[$ the property* (17) *is satisfied, then T is Archimedean.*

Remark 19. An immediate consequence of Theorem 18(ii) is that each strict t-norm T is Archimedean.

The continuity assumption in Theorem (18)(ii) is necessary. The following example shows that, for non-continuous t-norms, (17) for all $x \in]0,1[$ does not necessarily imply the strict monotonicity (14) or the Archimedean property (16).

Example 20. The t-norm T given by

$$T(x,y) = \begin{cases} 0\,, & \text{if } (x,y) \in [0,0.5]^2\,,\\ 2(x-0.5)(y-0.5)+0.5\,, & \text{if } (x,y) \in]0.5,1]^2\,,\\ \min(x,y)\,, & \text{otherwise}\,. \end{cases}$$

is obviously non-continuous. It satisfies (17) but it is not Archimedean.

We are now in the position to complete the presentation of the exact relationship between strict monotonicity, strictness, continuity, and the Archimedean property. A visualization of these relations is given in Figure 1.

Definition 21. A t-norm T is called *nilpotent* if it is continuous and if each element $a \in]0,1[$ is nilpotent, i.e., if there exists some $n \in \mathbb{N}$ such that $a_T^{(n)} = 0$.

Figure 2 visualizes the relationship between the different properties of t-norms studied in this section and relates them to continuity.

Definition 22. An element $x \in]0,1[$ is called a *zero divisor* of T if there exists some $y \in]0,1[$ such that $T(x,y) = 0$.

Remark 23.

(i) If a t-norm T has no zero divisors then for each $x \in]0,1]$ we have $T(x,x) > 0$.

(ii) Each nilpotent element of a t-norm T is a zero divisor of T. For the t-norm $T_{\mathbf{M}}^{\mathrm{nil}}$ each $x \in]0,1[$ is a zero divisor but no $x \in]0.5,1]$ can be a nilpotent element of $T_{\mathbf{M}}^{\mathrm{nil}}$ since in this case we always have $T(x,x) = x$.

It turns out that nilpotent t-norms can be completely characterized using Theorem 18(ii).

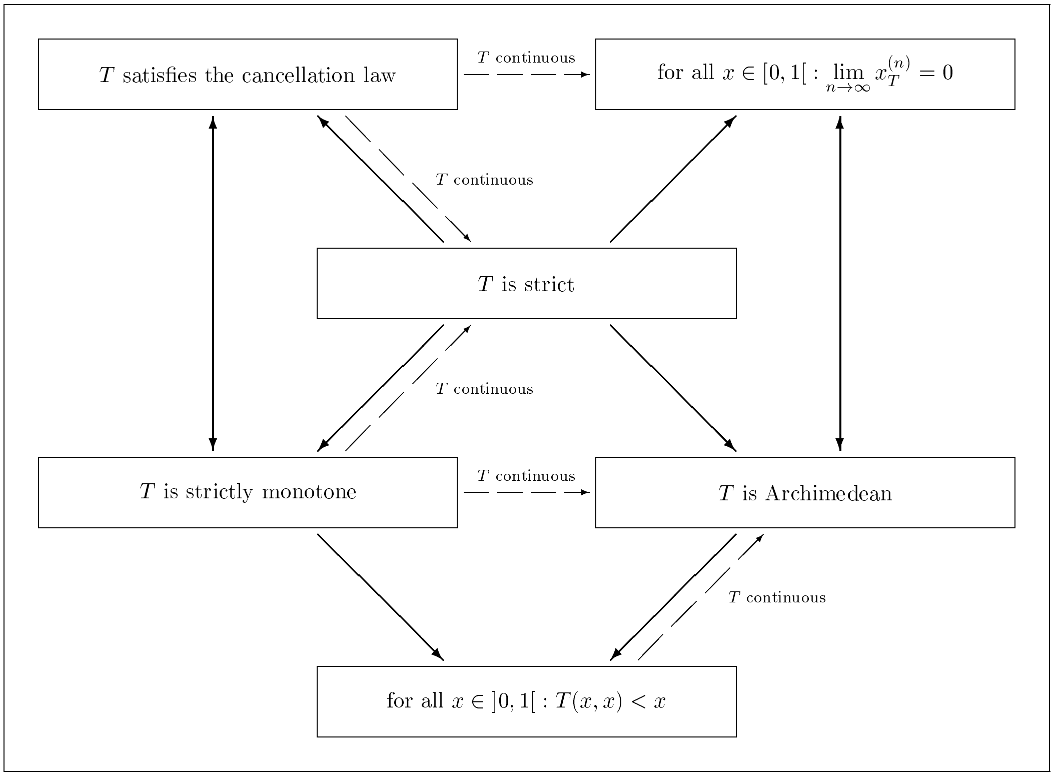

FIGURE 1. The logical relationship between various properties of t-norms

THEOREM 24. *Let T be a continuous Archimedean t-norm. Then the following are equivalent:*

(i) *T is nilpotent.*

(ii) *There exists some nilpotent element of T.*

(iii) *T is not strict.*

(iv) *T has zero divisors.*

3. Ordinal sums

THEOREM 25. *Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of pairwise disjoint open subintervals of $[0,1]$. Then the function $T : [0,1]^2 \to [0,1]$ defined by*

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\frac{x-a_\alpha}{e_\alpha - a_\alpha}, \frac{y-a_\alpha}{e_\alpha - a_\alpha}\right), & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2\,, \\ \min(x,y)\,, & \text{otherwise}\,, \end{cases}$$

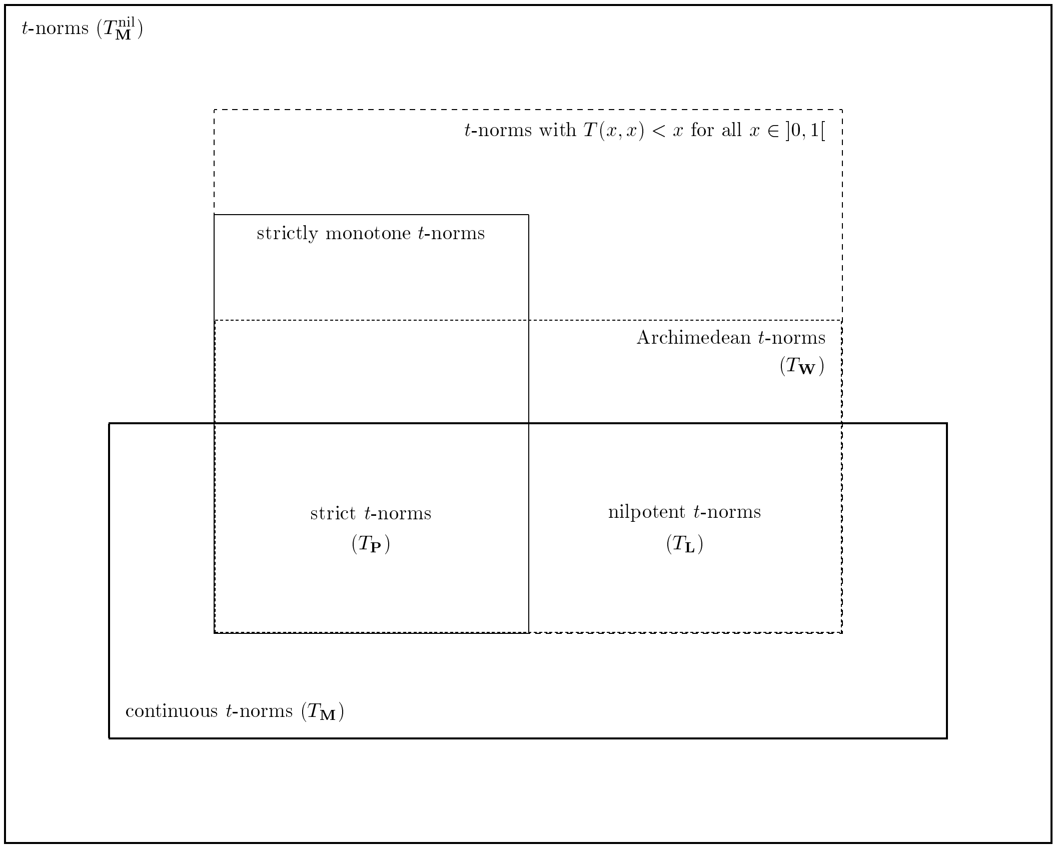

FIGURE 2. Different classes of t-norms

is a t-norm. It is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha\rangle$, $\alpha \in A$, and we shall write

$$T \approx \big(\langle a_\alpha, e_\alpha, T_\alpha\rangle\big)_{\alpha\in A}\,.$$

EXAMPLE 26.

(i) An empty ordinal sum of t-norms, i.e., an ordinal sum of t-norms with index set $\emptyset$, yields the Minimum $T_{\mathbf{M}}$:

$$T_{\mathbf{M}} \approx (\emptyset) = \big(\langle a_\alpha, e_\alpha, T_\alpha\rangle\big)_{\alpha\in\emptyset}\,.$$

(ii) Each t-norm T can be read as a trivial ordinal sum with one summand $\langle 0,1,T\rangle$:

$$T \approx \big(\langle 0,1,T\rangle\big)\,.$$

(iii) The ordinal sum T of the two summands $\langle \frac{1}{4}, \frac{1}{2}, T_{\mathbf{P}}\rangle$ and $\langle \frac{2}{3}, \frac{3}{4}, T_{\mathbf{L}}\rangle$, i.e.,

$$T \approx \left(\Big\langle \frac{1}{4}, \frac{1}{2}, T_{\mathbf{P}}\rangle, \langle \frac{2}{3}, \frac{3}{4}, T_{\mathbf{L}}\Big\rangle\right),$$

is given by

$$T(x,y)=\begin{cases}\frac{1}{4}\big(1+(4x-1)(4y-1)\big)\,, & \text{if } (x,y)\in[\frac{1}{4},\frac{1}{2}]^2\,,\\ \frac{2}{3}+\max(0,x+y-\frac{17}{12})\,, & \text{if } (x,y)\in[\frac{2}{3},\frac{3}{4}]^2\,,\\ \min(x,y)\,, & \text{otherwise}\,.\end{cases}$$

(iv) An ordinal sum of t-norms may have infinitely many summands. For instance,

$$T\approx\left(\left\langle\frac{1}{2^n}\,,\frac{1}{2^{n-1}},T_{\mathbf{P}}\right\rangle\right)_{n\in\mathbb{N}}$$

means that

$$T(x,y)=\begin{cases}\frac{1}{2^n}\big(1+(2^nx-1)(2^ny-1)\big)\,, & \text{if } (x,y)\in\left[\frac{1}{2^n},\frac{1}{2^{n-1}}\right]^2,\\ \min(x,y)\,, & \text{otherwise}\,.\end{cases}$$

Corollary 27. *Let $(S_\alpha)_{\alpha\in A}$ be a family of t-conorms and $(]a_\alpha,e_\alpha[)_{\alpha\in A}$ be a family of pairwise disjoint open subintervals of $[0,1]$. Then the function $S:[0,1]^2\to[0,1]$ defined by*

$$S(x,y)=\begin{cases}a_\alpha+(e_\alpha-a_\alpha)\cdot S_\alpha\,,\left(\frac{x-a_\alpha}{e_\alpha-a_\alpha},\frac{y-a_\alpha}{e_\alpha-a_\alpha}\right), & \textit{if } (x,y)\in[a_\alpha,e_\alpha]^2\,,\\ \max(x,y) & \textit{otherwise}\,,\end{cases}\tag{18}$$

is a t-conorm which is called the ordinal sum of the summands $\langle a_\alpha,e_\alpha,S_\alpha\rangle$, $\alpha\in A$, and we shall write

$$S\approx(\langle a_\alpha,e_\alpha,S_\alpha\rangle)_{\alpha\in A}\,.$$

All results for ordinal sums of t-norms remain valid for t-conorms with the obvious changes where necessary. In particular, concerning the duality of ordinal sums we get the following:

Remark 28. Let $T\approx(\langle a_\alpha,e_\alpha,T_\alpha\rangle)_{\alpha\in A}$ be an ordinal sum of t-norms. Then the dual t-conorm S given by (18) can be be written as an ordinal sum of t-conorms:

$$S\approx(\langle 1-e_\alpha,1-a_\alpha,S_\alpha\rangle)_{\alpha\in A}\,,$$

where the t-conorm S_α is the dual of the t-norm T_α. Note, however, that the t-norm T_α and the t-conorm S_α in general act on different intervals.

4. Representations of t-norms

Theorem 29. *A function $T\colon[0,1]^2\to[0,1]$ is a continuous Archimedean triangular norm if and only if there exists a continuous, strictly decreasing function $f\colon[0,1]\to[0,+\infty]$ with $f(1)=0$ such that for all $x,y\in[0,1]$*

$$T(x,y)=f^{-1}\big(\min(f(x)+f(y),f(0))\big)\,.\tag{19}$$

The function f is then called an additive generator of T; it is uniquely determined by T up to a positive multiplicative constant.

Corollary 30. *A function $T\colon [0,1]^2 \to [0,1]$ is a continuous Archimedean triangular norm if and only if there exists a continuous, strictly increasing function $g\colon [0,1] \to [0,1]$ with $g(1) = 1$ such that for all $x, y \in [0,1]$*

$$T(x,y) = g^{-1}\big(\max(g(x)\cdot g(y), g(0))\big)\,. \tag{20}$$

The function g is then called an multiplicative generator of T; it is uniquely determined by T up to a positive constant exponent.

Remark 31.

(i) If $f\colon [0,1] \to [0,+\infty]$ is an additive generator of a continuous Archimedean t-norm T then $g\colon [0,1] \to [0,1]$ given by $g(x) = e^{-f(x)}$ is a multiplicative generator of T.

(ii) Conversely, if $g\colon [0,1] \to [0,1]$ is a multiplicative generator of a continuous Archimedean t-norm T then $f\colon [0,1] \to [0,+\infty]$ given by $f(x) = -\log\big(g(x)\big)$ is an additive generator of T.

(iii) A t-norm T is strict if and only if for each addititive generator f of T we have $f(0) = +\infty$.

(iv) A t-norm T is nilpotent if and only if for each addititive generator f of T we have $f(0) < +\infty$.

Example 32.

(i) A family of additive generators $\big(f^{\mathbb{F}}_\lambda\colon [0,1] \to [0,+\infty]\big)_{\lambda\in]0,+\infty]}$ for the family $(T^{\mathbf{F}}_\lambda\lambda)_{\lambda\in]0,+\infty]}$ of Frank t-norms is given by

$$f^{\mathbb{F}}_\lambda(x) = \begin{cases} -\log x\,, & \text{if } \lambda = 1\,,\\ 1-x\,, & \text{if } \lambda = +\infty\,,\\ -\log\frac{\lambda^x-1}{\lambda-1}\,, & \text{otherwise}\,. \end{cases}$$

(ii) A family of additive generators $\big(f^{\mathbb{Y}}_\lambda\colon [0,1] \to [0,1]\big)_{\lambda\in]0,+\infty[}$ for the family $(T^{\mathbf{Y}}_\lambda\lambda)_{\lambda\in]0,+\infty[}$ of Yager t-norms is given by

$$f^{\mathbb{Y}}_\lambda(x) = (1-x)^\lambda\,.$$

Theorem 33. *A function $T\colon [0,1]^2 \to [0,1]$ is a strict t-norm if and only if T is order isomorphic to the Product $T_{\mathbf{P}}$, i.e., if there is a strictly increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that for all $x, y \in [0,1]$*

$$T(x,y) = \varphi^{-1}\big(T_{\mathbf{P}}(\varphi(x),\varphi(y))\big)\,.$$

Theorem 34. *A function $T\colon [0,1]^2 \to [0,1]$ is a nilpotent t-norm if and only if T is order isomorphic to the Łukasiewicz$t$-norm $T_{\mathbf{L}}$, i.e., if there is a strictly increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that for all $x, y \in [0,1]$*

$$T(x,y) = \varphi^{-1}\big(T_{\mathbf{L}}(\varphi(x), \varphi(y))\big) .$$

Theorem 35. *A function $T\colon [0,1]^2 \to [0,1]$ is a continuous t-norm if and only if T is an ordinal sum of continuous Archimedean t-norms.*

5. Many-valued logical connectives

The study of many-valued logic was highly influenced by J. Łukasiewicz who began his investigations on a three-valued logic around 1920. We shall indicate here some ways how to construct logical connectives on the basis of t-norms in some many-valued logics, which have the interval $[0,1]$, i.e., a continuum, as truth values.

Starting with a t-norm T, the standard strong negation $N_{\mathbf{s}}$ given by

$$N_{\mathbf{s}}(x) = 1 - x$$

and, implicitly, with the t-conorm S given by

$$S(x,y) = N_{\mathbf{s}}\big(T(N_{\mathbf{s}}(x), N_{\mathbf{s}}(y)\big) ,$$

we can introduce the basic connectives in a $[0,1]$-valued logic as follows:

(i) *conjunction*: $\quad x \wedge_T y = T(x,y)$,

(ii) *disjunction*: $\quad x \vee_T y = S(x,y)$.

If x and y are the truth values of two propositions A and B, respectively, then $x \wedge_T y$ is the truth value of

‘A AND B’,

$x \vee_T y$ is the truth value of

‘A OR B’,

and $N(x)$ is the truth value of

‘NOT A’.

Obviously, when restricting ourselves to Boolean (i.e., two-valued) logic with truth values 0 and 1 only, then we obtain the classical logical connectives. However, $\big([0,1], T, S, N_{\mathbf{s}}, 0, 1\big)$ never yields a Boolean algebra. To see this, note that $\big([0,1], T_{\mathbf{M}}, S_{\mathbf{M}}\big)$ violates the *law of excluded middle* $x \vee_T y = 1$, and all the other dual pairs (T, S) are not distributive.

As in classical logic, it is possible to construct implication, bi-implication and so on by means of negation, conjunction and disjunction. Taking into account that in Boolean logic 'NOT A OR B' is equivalent to 'IF A THEN B', one possibility of modelling the implication in a $[0,1]$-valued logic (based on T, $N_{\mathbf{s}}$ and S) is to define the function $I_T\colon [0,1]^2 \to [0,1]$ by

$$I_T(x,y) = S\big(N_{\mathbf{s}}(x), y\big) = N_{\mathbf{s}}\big(T(x, N_{\mathbf{s}}(y))\big)\,.$$

It is clear that in this case the *law of contraposition*

$$I_T(x,y) = I_T\big(N_{\mathbf{s}}(y), N_{\mathbf{s}}(x)\big)$$

is always valid.

For the four basic t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$, $T_{\mathbf{L}}$ and $T_{\mathbf{W}}$ we obtain the following implications:

$$\begin{aligned}
I_{T_{\mathbf{M}}}(x,y) &= \max(1-x, y)\,,\\
I_{T_{\mathbf{P}}}(x,y) &= 1 - x + x\cdot y\,,\\
I_{T_{\mathbf{L}}}(x,y) &= \min(1-x+y, 1)\,,\\
I_{T_{\mathbf{W}}}(x,y) &= \begin{cases} 1-x\,, & \text{if } y = 0\,,\\ y\,, & \text{if } x = 1\,,\\ 1\,, & \text{otherwise}\,. \end{cases}
\end{aligned}$$

Another way of extending the classical binary implication operator (acting on $\{0,1\}$) to the unit interval $[0,1]$ uses the *residuation* R_T with respect to a left continuous t-norm T

$$R_T(x,y) = \sup\big\{z \in [0,1] \mid T(x,z) \le y\big\}\,. \tag{21}$$

For the left continuous t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ we obtain the following residuations:

$$\begin{aligned}
R_{T_{\mathbf{M}}}(x,y) &= \begin{cases} 1\,, & \text{if } x \le y\,,\\ y\,, & \text{otherwise}\,, \end{cases}\\
R_{T_{\mathbf{P}}}(x,y) &= \min\left(\frac{y}{x}, 1\right)\,,\\
R_{T_{\mathbf{L}}}(x,y)\,, &= \min(1-x+y, 1)\,,
\end{aligned}$$

The fact that $I_{T_{\mathbf{L}}} = R_{T_{\mathbf{L}}}$ is just the implication introduced by Łukasiewicz justifies it to call $T_{\mathbf{L}}$ and $S_{\mathbf{L}}$ the Łukasiewicz$t$-norm and t-conorm, respectively, although these operations nowhere appear explicitly in the work of Łukasiewicz. Observe that, in general, I_T and R_T are different (although both are extensions of the Boolean implication), but, as already mentioned, we have $I_{T_{\mathbf{L}}} = R_{T_{\mathbf{L}}}$.

The same is true for arbitrary nilpotent t-norms when replacing the standard negation $N_{\mathbf{s}}$ by the negation $N_T\colon [0,1] \to [0,1]$ given by

$$N_T(x) = \sup\big\{z \in [0,1] \mid T(x,z) = 0\big\} = R_T(x,0)\,.$$

Another t-norm T fulfilling $I_T = R_T$ is the nilpotent Minimum $T_{\mathbf{M}}^{\mathrm{nil}}$.

The so-called Φ-*operator* $\varphi\colon [0,1]^2 \to [0,1]$ with respect to T was introduced axiomatically:

(Φ1) $\varphi(x,y) \le \varphi(x,z)$ whenever $y \le z$,

(Φ2) $T\big(x, \varphi(x,y)\big) \le y$,

(Φ3) $y \le \varphi\big(x, T(x,y)\big)$.

Moreover, it was shown that a (unique) Φ-operator with respect to T exists ifand only if T is left continuous, in which case it coincides with R_T.

The original axioms of the Φ-operator with respect to T involve the t-norm T. The following axioms for an implication operator $I\colon [0,1]^2 \to [0,1]$ are given:

(I1) $I\big(x, I(y,z)\big) = I\big(y, I(x,z)\big)$, (*exchange principle*)

(I2) $I(1,y) = y$,

(I3) $I(x,y) = 1$ if and only if $x \le y$,

(I4) I is right continuous in its second component,

(I5) I is non-decreasing in its second component.

There is a one-to-one correspondence between implication operators fulfilling the axioms (I1) – (I5) and Φ-operators, where the corresponding left continuous t-norm T is given by

$$T(x,y) = \inf\ \big\{z \in [0,1] \mid I(x,z) \ge y\big\}\,.$$

Keeping (I1) and (I2) and replacing (I3), (I4) and (I5) by (I6) and (I7) as follows,

(I6) I is non-increasing in its first component,

(I7) $I\big(I(x,y), y\big) = x$ if and only if $x \ge y$,

then the corresponding t-norm is nilpotent. Requiring the validity of (I7) only for positive y, i.e., for $y > 0$, yields a continuous Archimedean t-norm.

Another way of constructing connevtives in $[0,1]$-valued logics is to start with a left continuous t-norm T, to use the residuation R_T as implication, and to define the negation N_T as an implication with consequence 0, and the disjunction S_T using the De Morgan formula:

$$N_T(x) = R_T(x,0)\,,$$
$$S_T(x,y) = N_T\big(T(N_T(x), N_T(y))\big)\,.$$

Along these lines we have the following three main examples:

EXAMPLE 36.

(i) *Łukasiewiczlogic*: Choosing the Łukasiewicz$t$-norm $T_{\mathbf{L}}$ as conjunction operator, we obtain

$$\begin{aligned} R_{T_{\mathbf{L}}}(x,y) &= \min(1-x+y,1)\,, \\ N_{T_{\mathbf{L}}}(x,y) &= 1-x\,, \\ S_{T_{\mathbf{L}}}(x,y) &= \min(x+y,1)\,. \end{aligned}$$

Moreover, it is also possible to express the lattice operations max and min by means of the implication and the negation:

$$\begin{aligned} \max(x,y) &= R_{T_{\mathbf{L}}}\left(R_{T_{\mathbf{L}}}(x,y),y\right)\,, \\ \min(x,y) &= N_{T_{\mathbf{L}}}\left(R_{T_{\mathbf{L}}}\left(R_{T_{\mathbf{L}}}(N_{T_{\mathbf{L}}}(x),N_{T_{\mathbf{L}}}(y))\,,N_{T_{\mathbf{L}}}(y)\right)\right)\,. \end{aligned}$$

The Łukasiewiczpropositional logic is sound, complete and axiomatizable; the corresponding algebraic model is an MV-algebra. However, the classical deduction theorem does not hold, and the Łukasiewiczpredicate calculus is not axiomatizable by a recursive set of axioms.

(ii) *Gödel logic*: In this case the conjunction is modelled by the Minimum $T_{\mathbf{M}}$, and consequently we have

$$\begin{aligned} R_{T_{\mathbf{M}}}(x,y) &= \begin{cases} 1\,, & \text{if } x \le y\,, \\ y\,, & \text{otherwise}\,, \end{cases} \\ N_{T_{\mathbf{M}}}(x,y) &= \begin{cases} 0\,, & \text{if } x > 0\,, \\ 1\,, & \text{if } x = 0\,, \end{cases} \\ S_{T_{\mathbf{M}}}(x,y) &= \begin{cases} 0\,, & \text{if } x = y = 0\,, \\ 1\,, & \text{otherwise}\,. \end{cases} \end{aligned}$$

Note that in this case the range of the negation $N_{T_{\mathbf{M}}}$ and, consequently, of the disjunction $S_{T_{\mathbf{M}}}$ only consists of the two extremal values 0 and 1 only. Therefore the disjunction $S_{T_{\mathbf{M}}}$ is no t-conorm, so it cannot coincide with the dual t-conorm of $T_{\mathbf{M}}$, i.e., with $S_{\mathbf{M}}$. The operation max can be expressed in terms of conjunction, implication and disjunction only:

$$\max(x,y) = T_{\mathbf{M}}\left[R_{T_{\mathbf{M}}}\left(R_{T_{\mathbf{M}}}(x,y),y\right), R_{T_{\mathbf{M}}}\left(R_{T_{\mathbf{M}}}(y,x),x\right)\right]\,.$$

The Gödel propositional logic is sound, complete and axiomatizable; the corresponding algebraic model is a Heyting algebra. The Gödel logic is the only t-norm-based $[0,1]$-valued logic where the classical deduction theorem holds. Moreover, the Gödel predicate calculus is axiomatizable by a recursive set of axioms.

(iii) *Product logic*: Taking the Product t-norm $T_{\mathbf{P}}$ for the conjunction, we get

$$R_{T_{\mathbf{P}}}(x,y) = \min\left(\frac{y}{x}, 1\right) ,$$

$$N_{T_{\mathbf{P}}}(x,y) = \begin{cases} 0\,, & \text{if } x > 0\,, \\ 1\,, & \text{if } x = 0\,, \end{cases}$$

$$S_{T_{\mathbf{P}}}(x,y) = \begin{cases} 0\,, & \text{if } x = y = 0\,, \\ 1\,, & \text{otherwise}\,. \end{cases}$$

Observe that the negation and disjunction in the Product logic are the same as in the Gödel logic. Again, the disjunction $S_{T_{\mathbf{P}}}$ does not coincide with the dual t-conorm of $T_{\mathbf{P}}$, i.e., with $S_{\mathbf{P}}$. Again, it is possible to compute the lattice operations min and max:

$$\min(x,y) = T_{\mathbf{P}}\big(x, R_{T_{\mathbf{P}}}(x,y)\big) ,$$
$$\max(x,y) = \min\big[R_{T_{\mathbf{P}}}\big(R_{T_{\mathbf{P}}}(x,y), y\big), R_{T_{\mathbf{P}}}\big(R_{T_{\mathbf{P}}}(y,x), x\big)\big] .$$

The Product propositional logic is sound, complete and axiomatizable; the corresponding algebraic model is a Product algebra. However, the classical deduction theorem does not hold, and the problem whether the Product predicate calculus is axiomatizable by a recursive set of axioms is still unsolved.

TABLE 1. Three important t-norm based $[0,1]$-valued logics

	conjunction	axiomatizable sound complete	algebra	classical deduction theorem	predicate calc ulus axiomatizable
Łukasiewicz logic	$T_{\mathbf{L}}$	yes	MV-algebra	no	no
Gödel logic	$T_{\mathbf{M}}$	yes	Heyting algebra	yes	yes
Product logic	$T_{\mathbf{P}}$	yes	Product algebra	no	oppen

6. Fuzzy subsets of a universe

The earliest traces of t-norms in fuzzy set theory can be found in the very first paper by Zadeh who suggested to model the intersection of fuzzy sets using the t-norms $T_{\mathbf{M}}$ and $T_{\mathbf{P}}$ and the union of fuzzy sets using the t-conorms $S_{\mathbf{M}}$ and $S_{\mathbf{L}}$ (the latter in a restricted form).

From the mathematical point of view, the investigation of triangular norms and conorms has built bridges between many-valued logics and the theory of functional equations. Actually, most results in these logics are nothing but a reinterpretation of results on functional equations (here the associativity of triangular norms and conorms plays the key role in terms of logic).

Given a (crisp) universe of discourse X, a *fuzzy subset* A of X is characterized by its *membership function*

$$\mu_A\colon X \to [0,1]\,, \tag{22}$$

where for $x \in X$ the number $\mu_A(x)$ is interpreted as the *degree of membership* of x in the fuzzy set A or, equivalently, as the *truth value* of the statement

$$\text{‘}x \textit{ is element of } A\text{’}\,.$$

The membership function μ_A of a fuzzy subset A of X is a quite natural generalization of the *characteristic function* $\mathbf{1}_B$ of a crisp subset B of X, assigning the value 1 to all elements of X which belong to B, and the value 0 to all remaining elements of X:

$$\mathbf{1}_B\colon X \to \{0,1\} \tag{23}$$

$$x \mapsto \begin{cases} 1\,, & \text{if } x \in B\,,\\ 0\,, & \text{if } x \notin B\,. \end{cases} \tag{24}$$

A fuzzy subset A of X is called *normalized* if there is an element $x_0 \in X$ with

$$\mu_A(x_0) = 1\,. \tag{25}$$

If A is a fuzzy subset X, then for each $\alpha \in [0,1]$ we can associate with A a crisp subset of X, the so-called α-*cut* $[A]_\alpha$, given by

$$[A]_\alpha = \bigl\{x \in X \mid \mu_A(x) \ge \alpha\bigr\}\,. \tag{26}$$

It is immediately clear that the family $(A_\alpha)_{\alpha\in[0,1]}$ of α-cuts of a fuzzy subset A of X has the following properties:

$$A_0 = X\,, \tag{27}$$

$$A_\alpha \subseteq A_\beta \qquad \text{whenever } \alpha \ge \beta\,, \tag{28}$$

$$A_\alpha = \bigcap_{\beta\in[0,\alpha[} A_\beta\,. \tag{29}$$

The family $(A_\alpha)_{\alpha\in[0,1]}$ of α-cuts of a fuzzy subset A contains all the information concerning A, since it is possible to reconstruct the membership function of A from the α-cuts: for each $x \in X$ we have

$$\mu_A(x) = \sup\bigl\{\min\bigl(\alpha, \mathbf{1}_A(x)\bigr) \mid \alpha \in]0,1]\bigr\}\,. \tag{30}$$

This means that each fuzzy subset A of X can be identified with its family $(A_\alpha)_{\alpha\in[0,1]}$ of α-cuts. Moreover, a family $(A_\alpha)_{\alpha\in]0,1]}$ of crisp subsets of X constitutes the family of α-cuts of some fuzzy subset A of X if and only if it satisfies (28) and (29).

In order to generalize the Boolean set-theoretical operations like *intersection* and *union*, respectively, it is quite natural to use triangular norms and conorms.

Given a t-norm T, a t-conorm S and the standard strong negation $N_{\mathbf{s}}$, then for fuzzy subsets A, B of the universe X the membership functions of the *intersection* $A \cap_T B$, the *union* $A \cup_S B$ and the *complement* $A^{\mathbf{c}}$ are given by:

$$\mu_{A\cap_T B}(x) = T\big(\mu_A(x), \mu_B(x)\big)\,, \tag{31}$$

$$\mu_{A\cup_S B}(x) = S\big(\mu_A(x), \mu_B(x)\big)\,, \tag{32}$$

$$\mu_{A^{\mathbf{c}}}(x) = N_{\mathbf{s}}\big(\mu_A(x)\big)\,. \tag{33}$$

The values $\mu_{A\cap_T B}(x)$, $\mu_{A\cup_S B}(x)$ and $\mu_{A^{\mathbf{c}}}(x)$ describe the truth values of the statements

'*x is element of A* AND *x is element of B*',

'*x is element of A* OR *x is element of B*',

and

'*x is* NOT *element of A*',

respectively.

An interesting and natural question is now whether the α-cuts of the intersection $A \cap_T B$ and the union $A \cup_S B$ of two fuzzy subsets A, B coincide with the intersection and union, respectively, of the corresponding α-cuts $[A]_\alpha$ and $[B]_\alpha$.

In general, this is not the case, since we have the following result: given a t-norm T, a t-conorm S and $\alpha \in [0,1]$, then the equalities

$$[A \cap_T B]_\alpha = [A]_\alpha \cap [B]_\alpha\,, \tag{34}$$

$$[A \cup_S B]_\alpha = [A]_\alpha \cup [B]_\alpha\,, \tag{35}$$

hold for all fuzzy subsets A, B of X if and only if α is an idempotent of T and S, respectively.

In particular, this means that equations (34) and (35) hold for all $\alpha \in [0,1]$ and for all fuzzy subsets A, B of X if and only if we have $T = T_{\mathbf{M}}$ and $S = S_{\mathbf{M}}$, respectively.

7. Equality relations

An important theoretical basis for fuzzy subsets is given by the concept of equality relations which measure the *degree of indistinguishability* of any two points of the universe, and which generalize and somehow relax the concept of *equivalence relations*. A strong motivation for this notion came from the so-called *Poincaré Paradox*: if for three (real world) objects A, B and C we have that A is indistinguishible from B and B is indistinguishible from C, we cannot always conclude that A is indistinguishible from C too.

Definition 37. An *equality relation* on X (with respect to a t-norm T) is a function $E\colon X^2 \to [0,1]$ such that the following properties are satisfied:

(E1) *Reflexivity*

$$E(x,x) = 1\,, \tag{36}$$

(E2) *Symmetry*

$$E(x,y) = E(y,x)\,, \tag{37}$$

(E3) *T-transitivity*

$$T\big(E(x,y),E(y,z)\big) \leq E(x,z)\,. \tag{38}$$

The value $E(x,y)$ can be interpreted as the *degree of equality* of x and y or, equivalently, as the truth value of the statement

'*x is equal to y*'.

The T-transitivity (38) is a many-valued model of the proposition

'IF *x is equal to y* AND *y is equal to z* THEN *x is equal to z*'.

Note that if E is a T^*-equality relation, then it is a T-equality relation with respect to any t-norm T weaker than T^*.

Equality relations with respect to the Łukasiewicz t-norm $T_{\mathbf{L}}$ are closely related to pseudo-metrics on X. If $d\colon X^2 \to [0,+\infty[$ is a pseudo-metric on X, then $E_d\colon X^2 \to [0,1]$ given by

$$E_d(x,y) = 1 - \min\big(d(x,y),1\big) \tag{39}$$

is an equality relation with respect to $T_{\mathbf{L}}$. Conversely, if $E\colon X^2 \to [0,1]$ is an equality relation on X with respect to $T_{\mathbf{L}}$, then $d_E\colon X^2 \to [0,1]$ given by

$$d_E(x,y) = 1 - E(x,y) \tag{40}$$

is a pseudo-metric on X. This fact emphasizes both the geometric nature of equality relations and the special role of $T_{\mathbf{L}}$.

These statements can be generalized for an arbitrary continuous Archimedean t-norm T with additive generator f. If E_T is a T-equality relation on X then $d_{E_T}\colon X^2 \to [0,1]$ given by

$$d_{E_T}(x,y) = f\big(E(x,y)\big)$$

is a pseudo-metric on X. Conversely, if d is a pseudo-metric on X, then $E_{d,T}\colon X^2 \to [0,1]$ given by

$$E_{d,T}(x,y) = f^{-1}\big(\min\big(f(0), d(x,y)\big)\big)$$

always defines a T-equality relation on X.

In applications such as fuzzy control we shall quite often use something like fuzzy points (e.g., '*approximately five*') when formulating the rules in a rule base. In order to extend the concept of a point, i.e., a set which contains exactly one element, we need the subsequent notions:

Definition 38. Let E be an equality relation on X with respect to some t-norm T.

(i) A fuzzy subset A of X is called *extensional* if

$$T\big(\mu_A(x), E(x,y)\big) \le \mu_A(y). \tag{41}$$

(ii) A normalized extensional fuzzy subset P of X is called a *fuzzy point* if

$$T\big(\mu_P(x), \mu_P(y)\big) \le E(x,y). \tag{42}$$

The extensionality of a fuzzy set A describes a certain degree of compatibility between the t-norm T, the equality relation E, and the membership function μ_A. It is a many-valued formulation of the statement

'IF *x is element of A* AND *x is equal to y* THEN *y is element of A*'.

Equation (42), on the other hand, is the proper extension of the statement

'IF *x is element of P* AND *y is element of P* THEN *x is equal to y*',

indicating that P contains *at most one element*, which, together with the normality (25) means that P contains *exactly one element* of X.

The following result shows that fuzzy points always can be expressed in terms of equality relations, illustrating the close relationship between the two concepts: An extensional fuzzy subset P of X is a fuzzy point if and only if there exists an $x_0 \in X$ such that $P = P_{x_0}$ with

$$\mu_{P_{x_0}}(x) = E(x_0, x). \tag{43}$$

Moreover, P_{x_0} is the extensional hull of the crisp point $\{x_0\}$, i.e., the smallest extensional fuzzy subset A of X such that $\mu_A(x_0) = 1$.

For special pseudo-metrics we obtain very simple and familiar shapes for the membership functions of fuzzy points:

EXAMPLE 39.

(i) If d is a pseudo-metric on X, then for each $x_0 \in X$ the membership function of the fuzzy point P_{x_0} (relative to the equality relation E_d with respect to $T_{\mathbf{L}}$) is given by

$$\mu_{P_{x_0}}(x) = \max\bigl(1 - d(x, x_0), 0\bigr) . \tag{44}$$

(ii) In the case $X = \mathbb{R}$ and the usual, i.e., Euclidean metric on $\mathbb{R}$, the membership function of a fuzzy point P_{x_0} is just represented by the isosceles triangle of height 1, whose base is the interval $[x_0 - 1, x_0 + 1]$.

In real control applications, we usually have an input space of dimension higher than one. It is therefore necessary to consider Cartesian products of fuzzy sets and of equality relations. Again, this can be done using an appropriate t-norm:

DEFINITION 40. Let X and Y be two (crisp) universes of discourse and let T be a t-norm.

(i) If A and B are fuzzy subsets of X and Y, respectively, then the *Cartesian product* $A \times B$ is the fuzzy subset of $X \times Y$ with the following membership function:

$$\mu_{A\times B}(x, y) = T\bigl(\mu_A(x), \mu_B(y)\bigr) . \tag{45}$$

(ii) If E and F are equality relations (with respect to T) on X and Y, respectively, then the equality relation $E \times F$ on $X \times Y$ (again with respect to T) is defined by

$$E \times F\bigl((x_1, y_1), (x_2, y_2)\bigr) = T\bigl(E(x_1, x_2), F(y_1, y_2)\bigr) . \tag{46}$$

In accordance with the other interpretations of membership functions, the value $\mu_{A\times B}(x, y)$ stands for the truth value of the statement

'*x is element of A* AND *y is element of B*',

and $E \times F\bigl((x_1, y_1), (x_2, y_2)\bigr)$ is the truth value of

'*x_1 is equal to x_2* AND *y_1 is equal to y_2*'.

The product of equality relations with respect to $T_{\mathbf{L}}$ induced by pseudo-metrics is again an equality relation induced by some pseudo-metric on the product space, and in the Euclidean case the fuzzy points again have membership functions with a simple geometric form:

EXAMPLE 41.

(i) If d_1, d_2 are pseudo-metrics on X and Y, respectively, and E_{d_1} and E_{d_2} are the corresponding equality relations on X and Y (with respect to $T_{\mathbf{L}}$), then the product equality relation $E_{d_1} \times E_{d_2}$ is just the equality relation E_d on $X \times Y$ induced by the pseudo-metric

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\big\{d_1(x_1, y_1), d_2(x_2, y_2)\big\}. \tag{47}$$

(ii) If $X = Y = \mathbb{R}$ and if d_1 and d_2 are both the Euclidean metric on $\mathbb{R}$ this means that the membership function of a fuzzy point $P_{\mathbf{x}_0}$ is a regular pyramid of height 1, whose base is the square of side length 2 with center $\mathbf{x}_0 \in \mathbb{R}^2$.

We shall later need the so-called (FP)-*property*, which is a kind of disjointness property for fuzzy subsets $A_1, \ldots, A_n$ of X:

$$\sup\big\{T_{\mathbf{L}}\big(\mu_{A_i}(x), \mu_{A_j}(x)\big) \,\big|\, x \in X\big\} \le \inf\big\{1 - |\mu_{A_i}(x) - \mu_{A_j}(x)| \,\big|\, x \in X\big\}. \tag{48}$$

Roughly speaking, the (FP)-property states that, for any two fuzzy subsets A_i, A_j of X, their degree of being *not disjoint* is not larger than their degree of being *equal* (which, in the case of crisp subsets, is equivalent to the fact that they are either disjoint or equal). The (FP)-property can be defined for arbitrary left continuous t-norm T. A system $\mathcal{A} = \{A_1, \ldots, A_n\}$ of fuzzy subsets of X is a T-partition of X if and only if $\sup\limits_i A_i(x) = 1$ for all $x \in X$ and for each $i, j \in \{1, \ldots, n\}$ the fuzzy subsets A_i and A_j fulfill the (FP)-property (with respect to T).

In a typical rulebase of a fuzzy controller, the possible input values are somehow partitioned into a finite number of (usually linguistically described) classes or types, and a rule assigns to a certain type of input its corresponding output value. The following result relates equality relations and partitions:

THEOREM 42.

Let P_1, P_2, ..., P_n be normalized fuzzy subsets of X. The following are equivalent:

(i) *There exists an equality relation E on X with respect to $T_{\mathbf{L}}$ such that all the fuzzy sets P_1, P_2, ..., P_n are fuzzy points.*

(ii) *For all i, $j = 1, 2, \ldots, n$, the fuzzy subsets P_i and P_j fulfill the (FP)-property* (48).

Property (48) has indeed a strong connection with the concept of a *partition*: it essentially requires that the *degree of overlapping*, described by the left hand side of (48), is not larger than the *degree of equality* of P_i and P_j, which could be an interpretation of the right hand side. It generalizes the Boolean statement

'IF P_i *and* P_j *are not disjoint* THEN $P_i = P_j$'.

8. Fuzzy control

Fuzzy controllers are essentially rule-based systems assigning (with the help of the *fuzzification*, a suitable *inference* and the so-called *defuzzification*) to each input value the corresponding output value, therefore producing an input-output function.

In this survey, we shall restrict ourselves to the two most important and most widely used fuzzy controllers, the Mamdani and the Sugeno controller. We start with the Mamdani controller which uses fuzzy sets both for input and output and, therefore, needs a defuzzification in order to produce an input-output function.

Definition 43.

Let X be an arbitrary input space, let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be normalized fuzzy subsets of X and $\mathbb{R}^m$ with Borel measurable membership functions, respectively, let T be a Borel measurable t-norm, and consider the rulebase $(i = 1, 2, \ldots, n)$

$$\text{IF } x \text{ is } A_i \text{ THEN } \mathbf{u} \text{ is } B_i .$$

Then the *Mamdani controller* defines the following input-output function $F_M\colon X \to \mathbb{R}^m$

$$F_M(x) = \frac{\int\limits_{\mathbb{R}^m} \mu_R(x,\mathbf{u}) \cdot \mathbf{u}\, d\mathbf{u}}{\int\limits_{\mathbb{R}^m} \mu_R(x,\mathbf{u})\, d\mathbf{u}}\,, \tag{49}$$

provided that $\int \mu_R(x,\mathbf{u})\, d\mathbf{u} > 0$, where the membership function μ_R of the fuzzy relation R on $X \times \mathbb{R}^m$ is given by

$$\mu_R(x,\mathbf{u}) = \max\big[T\big(\mu_{A_1}(x), \mu_{B_1}(\mathbf{u})\big), \ldots, T\big(\mu_{A_n}(x), \mu_{B_n}(\mathbf{u})\big)\big]\,. \tag{50}$$

In a strict mathematical sense, the measurability requirements are necessary for (49) being well-defined; in practical situations, these hypotheses, however, are usually satisfied.

In Definition 43 we have implicitly included a rather special defuzzification method, namely, the so-called *center of gravity*, which is basically contained in equation (49). We only mention that there are also other methods of defuzzification, e.g., the *center of maximum.*

In most practical examples, the t-norm used for the Mamdani controller is either $T_{\mathbf{M}}$ or $T_{\mathbf{P}}$; in the first case this is also referred to as *max-min-inference*, in the latter case as *max-prod-inference* or *max-dot-inference.*

The second important type of fuzzy controllers is the so-called Sugeno controller which uses crisp values in the output space. In a way this means that the inference has a built-in defuzzification.

Definition 44. Let X be an input space, let A_1, A_2, ..., A_n be normalized fuzzy subsets of X with $\sum \mu_{A_i}(x) > 0$ for all $x \in X$, and f_1, f_2, ..., f_n be functions from X to $\mathbb{R}^m$, and consider the rulebase $(i = 1, 2, \ldots, n)$

$$\text{IF } x \text{ is } A_i \text{ THEN } \mathbf{u} = f_i(x).$$

Then the *Sugeno controller* defines the following input-output function $F_S\colon X \to \mathbb{R}^m$

$$F_S(x) = \frac{\sum \mu_{A_i}(x) \cdot f_i(x)}{\sum \mu_{A_i}(x)}. \tag{51}$$

In the special situation, when for $i = 1, 2, \ldots, n$ the functions f_i are constant, i.e., $f_i(x) = \mathbf{u}_i$, the Sugeno controller can be considered as a special case of the Mamdani controller:

Theorem 45. *Let X be an input space, let A_1, A_2, ..., A_n be normalized fuzzy subsets of X with Borel measurable membership functions and with $\sum \mu_{A_i}(x) > 0$ for all $x \in X$, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be different elements of $\mathbb{R}^m$, consider the rulebase $(i = 1, 2, \ldots, n)$*

$$\text{IF } x \textit{ is } A_i \text{ THEN } \mathbf{u} = \mathbf{u}_i,$$

and let F_S be the input-output function of the corresponding Sugeno controller. Then there exists a Mamdani controller with input space X such that its corresponding input-output function F_M coincides with F_S.

In $\mathbb{R}^m$ we can consider crisp points $\mathbf{x}_0$, i.e., one-point sets $\{\mathbf{x}_0\}$, as limits of closed ε-balls $B_i^{(\varepsilon)}$ with center $\mathbf{x}_0$ as ε goes to zero. Therefore Theorem 45 also says that Sugeno controllers (with constant functions f_i) are limits of suitable Mamdani controllers (a result which holds for other defuzzification methods too).

If all the fuzzy subsets of both the input space X and the output space $\mathbb{R}^m$ involved in the rulebase are fuzzy points, then the fuzzy relation R on $X \times \mathbb{R}^m$ induced by the Mamdani controller can be viewed as the extensional hull of a partial function from X into $\mathbb{R}^m$:

Theorem 46. *Let x_1, x_2, ..., x_n be different elements of the input space X, $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n \in \mathbb{R}^m$, let A_1, A_2, ..., A_n and B_1, B_2, ..., B_n be normalized fuzzy subsets of X and $\mathbb{R}^m$, respectively, let T be a t-norm, and consider the rulebase $(i = 1, 2, \ldots, n)$*

$$\text{IF } x \textit{ is } A_i \text{ THEN } \mathbf{u} \textit{ is } B_i.$$

If the two families $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ both satisfy (48) and if $\mu_{A_i}(x_i) = 1$ and $\mu_{B_i}(\mathbf{u}_i) = 1$ for all $i = 1, 2, \ldots, n$, then the fuzzy relation R on $X \times \mathbb{R}^m$ induced by the Mamdani controller is the extensional hull of the *crisp set* $\{(x_1,\mathbf{u}_1),(x_2,\mathbf{u}_2),\dots,(x_n,\mathbf{u}_n)\}$ *with respect to the equality relation* $E\times F$, *where* E *and* F *are equality relations on* X *and* $\mathbb{R}^m$ *existing because of Theorem 42*

Indeed, the crisp set $\{(x_1,\mathbf{u}_1),(x_2,\mathbf{u}_2),\dots,(x_n,\mathbf{u}_n)\}$ can be regarded as a *partial function* from X into $\mathbb{R}^m$ assigning only to the elements x_i (rather than to the whole domain X) a corresponding value, namely, $\mathbf{u}_i$. The fuzzy relation R of the corresponding Mamdani controller then can be viewed as a *fuzzy graph* with the property that the points of the graph of the partial function have degree of membership 1. In this spirit, the rulebase of the Mamdani controller in Theorem 46 could be interpreted as $(i=1,2,\dots,n)$

$$\text{IF } x \textit{ is approximately } x_i \text{ THEN } \mathbf{u} \textit{ is approximately } \mathbf{u}_i$$

A fuzzy relation R on $X\times Y$ can be used to associate with each fuzzy subset A of X a certain fuzzy subset of Y, using the following *compositional rule of inference*:

DEFINITION 47. Let X and Y be crisp sets, T be a left continuous t-norm, A a normalized fuzzy subset of X, and R be a fuzzy relation on $X\times Y$. The membership function of the fuzzy subset $A\circ R$ of Y is given by

$$\mu_{A\circ R}(y)=\sup\bigl\{T\bigl(\mu_A(x),\mu_R(x,y)\bigr)\bigm| x\in X\bigr\}. \tag{52}$$

The compositional rule of inference has two nice interpretations, a logical one and a geometrical one. The logical interpretation quite naturally generalizes the classical implication, which is an immediate consequence of the following result:

THEOREM 48. *Let* A, B *be fuzzy subsets of the crisp sets* X *and* Y, *respectively, and let* T *be a left continuous* t*-norm. There exists a fuzzy relation* R *on* $X\times Y$ *which solves the relational equation*

$$B=A\circ R \tag{53}$$

if and only if the fuzzy relation $R_T(A,B)$ *on* $X\times Y$ *solves* (53), *whose membership function is given by*

$$\mu_{R_T(A,B)}(x,y)=R_T\bigl(\mu_A(x),\mu_B(y)\bigr) \tag{54}$$

and where $R_T\colon[0,1]^2\to[0,1]$ *is the residuation with respect to* T *defined in* (21). *Moreover, if* (53) *is solvable, then* $R_T(A,B)$ *is the largest solution.*

It is quite natural to interpret the value $\mu_{R_T(A,B)}(x,y)$ as the truth value of the proposition

$$\text{`IF } x \textit{ is } A \text{ THEN } y \textit{ is } B\text{'}.$$

From a geometric point of view, the compositional rule of inference essentially consists of three elementary geometric operations:

(i) *cylindric extension* of A to $A \times Y$, i.e.,

$$\mu_{A\times Y}(x,y) = T\big(\mu_A(x), \mu_Y(y)\big) = T\big(\mu_A(x), 1\big) = \mu_A(x)\,; \tag{55}$$

(ii) *intersection* of $A \times Y$ with R, i.e.,

$$\begin{aligned} \mu_{(A\times Y)\cap_T R}(x,y) &= T\big(\mu_{A\times Y}(x,y), \mu_R(x,y)\big) \\ &= T\big(\mu_A(x), \mu_R(x,y)\big)\,; \end{aligned} \tag{56}$$

(iii) *projection* of $(A \times Y) \cap_T R$ onto Y, i.e.,

$$\begin{aligned} \mu_B(y) &= \sup\big\{\mu_{(A\times Y)\cap_T R}(x,y) \,\big|\, x \in X\big\} \\ &= \sup\big\{T\big(\mu_A(x), \mu_R(x,y)\big) \,\big|\, x \in X\big\}\,. \end{aligned} \tag{57}$$

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