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THE NON-REFLECTIVITY OF OPEN WELL-FILTERED SPACES VIA b-TOPOLOGY

CHONG SHEN*, XIAOYONG XI, AND DONGSHENG ZHAO

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Abstract. Results have shown that Skula's b-topology plays a core role in studying sober spaces (especially in studying the sobrification). In this paper, following Skula's method of b-embedding a subcategory into the category **Top** of all topological spaces, we restrict our attention to T_0 spaces, and mainly prove that if **K** is a reflective subcategory of the category $\mathbf{Top}_0$ of all T_0 spaces such that **K** contains a non-T_1 space, then (i) each **K**-reflection is a b-dense embedding; (ii) **K** is saturated-hereditary. As a corollary, we deduce that the category **Owf** of all open well-filtered spaces is not reflective in $\mathbf{Top}_0$, which demonstrates a new approach to the reflectivity of subcategories of $\mathbf{Top}_0$.

1. Introduction

The reflective subcategories of the category **Top** of all topological spaces have been extensively studied. In the paper [9], Kennison gave three types of (full) reflective subcategories of **Top**, called simple, identifying and embedding, and provided a characterization for each of these reflections. He then asked whether these three types include all the full reflective subcategories of **Top**. Skula [11] then introduced another type, called b-embedding type, not mentioned by Kennison in [9], and showed that if **K** is a reflective subcategory of **Top** that contains a non-T_1 space, then **K** is a subcategory of one of the above-mentioned types.

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*Corresponding author.

In domain theory, many important classes of topological spaces are generally T_0 but not T_1. Sobriety is one of the most well studied topological properties. One of the heavily used structures in study sober spaces is the b-topology (also called the Skula topology, or the strong topology), which was introduced by Skula in [11]. Some major results on b-topology include:

(i) for a sober space X, the subsets of X that are sober as a subspace are exactly the b-closed subsets [3, 8];
(ii) for a sober space X and $A \subseteq X$, the sobrification of the subspace A is homeomorphic to its b-closure [3, 8];
(iii) for a T_0 space X and a sober space Y, $f : X \longrightarrow Y$ is a sobrification mapping if and only if f is an embedding and $f(X)$ is b-dense in Y [3, 5, 8];
(iv) if $f : X \longrightarrow Y$ is a continuous mapping between T_0 spaces, then f is an epimorphism in $\mathbf{Top_0}$ (the category of all T_0 spaces) if and only if $f(X)$ is b-dense in Y [1].

In this paper, we shall focus on T_0 spaces as this entails no essential loss of generality from the viewpoint of domain theory. Let $\mathbf{K}$ be a reflective subcategory of $\mathbf{Top_0}$ that contains a non-T_1 space. The main results are

(1) each $\mathbf{K}$-reflection is a b-dense embedding, hence $\mathbf{K}$ is an epireflection;
(2) $\mathbf{K}$ is saturated-hereditary: for any $X \in \mathbf{K}$, every saturated subset A of X, as a subspace, is in $\mathbf{K}$.

Johnstone's dcpo $\mathbb{J}$ and the space $\mathbb{N}_{\mathrm{cof}}$ of the set of natural numbers with the cofinite topology are commonly used for giving counterexamples. Note that $\mathbb{N}_{\mathrm{cof}}$ is homeomorphic to the maximal points space $\mathrm{Max}(\mathbb{J})$ relative to the Scott space $\Sigma\mathbb{J}$, and as an immediate corollary of the above result (2), we have the following result:

(3) If $\Sigma\mathbb{J} \in \mathbf{K}$, then $\mathbb{N}_{\mathrm{cof}} \in \mathbf{K}$.

The notion of open well-filtered spaces, proposed by Shen, Xi, Xu and Zhao [12] recently, forms a strictly larger class than that of the well-filtered spaces. One notable result on open well-filtered spaces is that every core-compact open well-filtered space is sober, which strengthens the result of Jia-Jung problem: whether core-compact well-filtered spaces are sober [6] (a positive answer is given by Lawson, Wu and Xi [10]). However, it is still unknown

- whether the category $\mathbf{Owf}$ of all open well-filtered spaces is reflective in $\mathbf{Top_0}$.

To solve this problem, we prove that $\Sigma\mathbb{J} \in \mathbf{Owf}$ but $\mathbb{N}_{\mathrm{cof}} \notin \mathbf{Owf}$, and then by (3), we deduce the following result:

(4) The category **Owf** is not reflective in $\mathbf{Top}_0$.

2. Preliminaries

Let P be a poset. For $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a$ for some $a \in A\}$ and $\uparrow A = \{x \in P : x \geq a$ for some $a \in A\}$. For $x \in P$, $\downarrow\{x\}$ and $\uparrow\{x\}$ will be abbreviated as $\downarrow x$ and $\uparrow x$, respectively.

For a T_0 space X, the specialization order $\leq$ is defined as $x \leq y$ iff $x \in \mathrm{cl}_X(\{y\})$, where cl_X is the closure operator of X. In the following, when we consider a T_0 space X as a poset, it is always equipped with the specialization order.

For a T_0 space X, we use $\mathcal{O}(X)$ to denote the topology of X. For any subset A of X, the *saturation* of A, denoted by $Sat_X(A)$, is defined to be

$$Sat_X(A) = \bigcap\{U \subseteq \mathcal{O}(X) : A \subseteq U\}.$$

A subset A of a T_0 space X is called *saturated* if $A = Sat_X(A)$. In the following, we use $Sat(A)$ and $\mathrm{cl}(A)$ for $Sat_X(A)$ and $\mathrm{cl}_X(A)$ respectively if no ambiguity occurs.

Remark 2.1 ([3, 4]). Let X be a T_0 space.

(1) For any subset A of X, $Sat(A) = \uparrow A$ with respect to the specialization order.
(2) For any $x \in X$, $\downarrow x = \mathrm{cl}(\{x\})$, and $x \in Sat(A)$ if and only if $\downarrow x \cap A \neq \emptyset$.
(3) For any open subset U of X, $U = Sat(U) = \uparrow U$, and dually for any closed subset F of X, $F = \downarrow F$.

Definition 2.2 ([3, 4]). Let X be a T_0 space.

(1) A nonempty subset $A \subseteq X$ is called *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$.
(2) The space X is called *sober*, if for any irreducible closed set F of X there is a unique point $x \in X$ such that $F = \downarrow x$.

An essential tool for studying sober spaces is the b-topology introduced by L. Skula [11] (see also [2]).

Definition 2.3 ([2, 11]). Let X be a T_0 space. The *b-topology* associated with X is the topology which has the family

$$\{U \cap \downarrow x : x \in U \in \mathcal{O}(X)\}$$

as a base, and the resulting space is denoted by bX. We call a subset B of X *b-dense* in X, if it is dense in bX.

Lemma 2.4. *Let X be a T_0 space, and Y be a b-dense subspace of X.*

(1) For each $V \in \mathcal{O}(X)$, $V = \uparrow(V \cap Y)$.
(2) For each $U \in \mathcal{O}(Y)$, $\uparrow U \in \mathcal{O}(X)$.

Proof. (1) Since every open set is saturated, it follows that $\uparrow(V \cap Y) \subseteq V$. Let $x \in V$. Then $\downarrow_X x \cap V$ is b-open in X, and since Y is b-dense in X, it follows that $\downarrow_X x \cap V \cap Y \neq \emptyset$, which implies that $x \in \uparrow(V \cap Y)$. Hence, $V \subseteq \uparrow(V \cap Y)$.

(2) Since U is open in Y, there exists an open set V in X such that $U = V \cap Y$. By (1), we have that $V = \uparrow(V \cap Y) = \uparrow U$. Hence, $\uparrow U$ is open in X. $\square$

The following theorem is trivial by Lemma 2.4.

Theorem 2.5. *Let X be a T_0 space, and Y be a b-dense subspace of X. Define $\varphi : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ and $\phi : \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ by*

$$\varphi(U) = \uparrow U \text{ and } \phi(V) = V \cap Y,$$

for any $U \in \mathcal{O}(Y)$ and $V \in \mathcal{O}(X)$. Then φ and ϕ are order-isomorphisms under the inclusion order, and they are inverse to each other.

In what follows, we denote by $\mathbf{Top}_0$ (resp., $\mathbf{Top_1}$, $\mathbf{Sob}$) the category of all T_0 spaces (resp., T_1 spaces, sober spaces) with continuous mappings as morphisms, and when we say $\mathbf{K}$ a subcategory of $\mathbf{Top}_0$ it is always assumed to be full and replete (i.e., $\mathbf{K}$ is closed under the formation of homeomorphic objects).

Definition 2.6. A subcategory $\mathbf{K}$ of $\mathbf{Top}_0$ is *reflective*, if for each $X \in \mathbf{Top}_0$, there exists $X^k \in \mathbf{K}$ (the $\mathbf{K}$-*completion* for X) and a continuous mapping $\mu_X : X \longrightarrow X^k$ (the $\mathbf{K}$-*reflection* for X) satisfying the universal property: for any continuous mapping $f : X \longrightarrow Z$ to any space $Z \in \mathbf{K}$, there exists a unique continuous mapping $g : X^k \longrightarrow Z$ such that $g \circ \mu_X = f$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "f"'] & X^k \arrow[d, dotted, "g"] \\ & Z \end{tikzcd}$$

Definition 2.7. Let $e : X \longrightarrow Y$ be a mapping between topological spaces. We call e a *b-dense embedding*, if it is an embedding such that $e(X)$ is b-dense in Y.

The following result, due to Skula [11], will be used. Here, for the sake of completeness, we provide a proof by using Definition 2.3 directly.

Lemma 2.8 (compare to [11, I-2.5]). *Let $X, Y \in \mathbf{Top}_0$ and $e : X \longrightarrow Y$ be a b-dense embedding. Then for any continuous mapping $f : X \longrightarrow Z$ to a T_0 space Z, there exists at most one continuous mapping $g : Y \longrightarrow Z$ such that $g \circ e = f$.*

Proof. Suppose there exist two continuous mappings $g_1, g_2 : Y \longrightarrow Z$ such that $g_1 \circ e = g_2 \circ e = f$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "f"'] & Y \arrow[d, "g_1, g_2"] \\ & Z \end{tikzcd}$$

Let $y \in Y$. Suppose $V \in \mathcal{O}(Z)$ such that $g_1(y) \in V$. Then $y \in g_1^{-1}(V) \in \mathcal{O}(Y)$, which follows that $\downarrow_Y y \cap g_1^{-1}(V)$ is b-open in Y. Since $e(X)$ is b-dense in Y, $\downarrow_Y y \cap g_1^{-1}(V) \cap e(X) \neq \emptyset$, and from $g_1 \circ e = g_2 \circ e = f$, we deduce that $g_1^{-1}(V) \cap e(X) = g_2^{-1}(V) \cap e(X)$, and hence $\downarrow_Y y \cap g_2^{-1}(V) \cap e(X) \neq \emptyset$. It follows that $y \in \uparrow_Y g_2^{-1}(V) = g_2^{-1}(V)$, so $g_2(y) \in V$. All this shows that each open neighborhood of $g_1(y)$ contains $g_2(y)$, that is, $g_1(y) \in \mathrm{cl}_Z(\{g_2(y)\})$. Dually, $g_2(y) \in \mathrm{cl}_Z(\{g_1(y)\})$. Since Z is a T_0 space, we have that $g_1(y) = g_2(y)$. Therefore, $g_1 = g_2$. $\square$

Definition 2.9 ([3, 4])**.** Let X and Y be two topological spaces. X is called a *retract* of Y if there are two continuous mappings $s : X \longrightarrow Y$ (the *section*) and $r : Y \longrightarrow X$ (the *retraction*) such that $r \circ s = \mathrm{id}_X$, the identity mapping on X.

We call X a *b-retract* of Y, if X is a retraction of Y such that $s(X)$ is b-dense in Y.

Remark 2.10 ([3, 4]). (1) Every section $s : X \longrightarrow Y$ is an embedding.
(2) Every retraction $r : Y \longrightarrow X$ is quotient.

Proposition 2.11. *Let $X, Y \in \mathbf{Top}_0$. If X is a b-retract of Y, then X is homeomorphic to Y.*

Proof. Suppose X is a b-retract of Y. Then there are two continuous mappings $s : X \longrightarrow Y$ and $r : Y \longrightarrow X$ such that $r \circ s = \mathrm{id}_X$. We have that $(s \circ r) \circ s = s \circ (r \circ s) = s \circ \mathrm{id}_X = s = \mathrm{id}_Y \circ s$, i.e., the following two diagrams commute:

$$\begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "s"'] & & Y \arrow[dl, "\mathrm{id}_Y"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, dotted, "\mathrm{id}_X = r\circ s"] \arrow[ddr, "s"'] & & Y \arrow[dl, dotted, "r"'] \arrow[ddl, "s\circ r"] \\ & X \arrow[d, dotted, "s"] & \\ & Y & \end{tikzcd}$$

By Remark 2.10, s is a b-dense embedding, and by Lemma 2.8, we have $s \circ r = \mathrm{id}_Y$. Since $r \circ s = \mathrm{id}_X$, s is a homeomorphism. Therefore, X is homeomorphic to Y. $\square$

3. Main results

Lemma 3.1. *Let X, Y be T_0 spaces, U be an open subset of X, and $y_1, y_2 \in Y$. If $y_1 < y_2$ (i.e., $y_1 \leq y_2$ and $y_1 \neq y_2$) in Y, then the mapping $f : X \longrightarrow Y$ defined by*

$$f(x) = \begin{cases} y_1 & x \notin U \\ y_2 & x \in U \end{cases}$$

is continuous.

Proof. In fact, as $y_1 < y_2$, every open set in Y containing y_1 contains y_2. Then, for each open subset V of Y, we have that

$$f^{-1}(V) = \begin{cases} X & y_1 \in V \ (\Rightarrow y_2 \in V) \\ U & y_1 \notin V, y_2 \in V \\ \emptyset & \text{otherwise,} \end{cases}$$

which implies that $f^{-1}(V)$ is open in X. Therefore, f is continuous. □

Theorem 3.2. *Let $\mathbf{K}$ be a reflective subcategory of $\mathbf{Top}_0$ such that $\mathbf{K} \nsubseteq \mathbf{Top_1}$. Then each $\mathbf{K}$-reflection is a b-dense embedding.*

Proof. Let $X \in \mathbf{Top}_0$ and $\mu_X : X \longrightarrow X^k$ be a $\mathbf{K}$-reflection for X. Since $\mathbf{K} \subseteq \mathbf{Top}_0$ and $\mathbf{K} \nsubseteq \mathbf{Top_1}$, there exists a T_0 space $Y \in \mathbf{K}$ that is not T_1. Then there exists a point $y_2 \in Y$ such that $\{y_2\}$ is not closed, i.e., $\downarrow y_2 \neq \{y_2\}$. Choose a point $y_1 \in \downarrow y_2 \setminus \{y_2\}$, i.e., $y_1 < y_2$. Note that $y_2 \in Y \setminus \downarrow y_1$. Next, we prove our conclusion in a few steps.

Step 1: μ_X is injective.

Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since X is T_0, either $x_1 \nleq x_2$ or $x_2 \nleq x_1$. Without loss of generality, we assume $x_2 \nleq x_1$. Then there exists an open subset U of X such that $x_2 \in U$ and $x_1 \notin U$. Define $f : X \longrightarrow Y$ by

$$f(x) = \begin{cases} y_1 & x \notin U \\ y_2 & x \in U. \end{cases}$$

Then by Lemma 3.1, f is a continuous mapping, and clearly $f(x_1) = y_1$ and $f(x_2) = y_2$. Since μ_X is a $\mathbf{K}$-reflection, there exists a unique continuous mapping $g : X^k \longrightarrow Y$ such that $g \circ \mu_X = f$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "f"'] & X^k \arrow[d, dotted, "g"] \\ & Y \end{tikzcd}$$

We have that $g(\mu_X(x_1)) = f(x_1) = y_1$ and $g(\mu_X(x_2)) = f(x_2) = y_2$, so that $g(\mu_X(x_1)) \neq g(\mu_X(x_2))$, which implies that $\mu_X(x_1) \neq \mu_X(x_2)$. Hence, μ_X is injective.

Step 2: μ_X is an embedding.

Let V be any open subset of X. Define a mapping $\widetilde{f} : X \longrightarrow Y$ as follows:

$$\widetilde{f}(x) = \begin{cases} y_1 & x \notin V \\ y_2 & x \in V. \end{cases}$$

By Lemma 3.1, $\widetilde{f}$ is a continuous mapping. Then there exists a unique continuous mapping $\widetilde{g} : X^k \longrightarrow Y$ such that $\widetilde{g} \circ \mu_X = \widetilde{f}$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "\widetilde{f}"'] & X^k \arrow[d, dotted, "\widetilde{g}"] \\ & Y \end{tikzcd}$$

Claim 1: $\mu_X(V) = \widetilde{g}^{-1}(Y \setminus \downarrow y_1) \cap \mu_X(X)$.

If $x \in V$, then $\widetilde{g}(\mu_X(x)) = \widetilde{f}(x) = y_2 \in Y \setminus \downarrow y_1$, which implies that $\mu_X(x) \in \widetilde{g}^{-1}(Y \setminus \downarrow y_1) \cap \mu_X(X)$, and this proves $\mu_X(V) \subseteq \widetilde{g}^{-1}(Y \setminus \downarrow y_1) \cap \mu_X(X)$. Conversely, if $\mu_X(x) \in \widetilde{g}^{-1}(Y \setminus \downarrow y_1) \cap \mu_X(X)$, then $\widetilde{f}(x) = \widetilde{g}(\mu_X(x)) \in Y \setminus \downarrow y_1$. It follows that $\widetilde{f}(x) = y_2$, which is equivalent to $x \in V$, so $\mu_X(x) \in \mu_X(V)$. This implies that $\widetilde{g}^{-1}(Y \setminus \downarrow y_1) \cap \mu_X(X) \subseteq \mu_X(V)$. Therefore, $\mu_X(V) = \widetilde{g}^{-1}(Y \setminus \downarrow y_1) \cap \mu_X(X)$.

Note that $Y \setminus \downarrow y \in \mathcal{O}(Y)$ and g is continuous, so $g^{-1}(Y \setminus \downarrow y_1) \in \mathcal{O}(X^k)$. By Claim 1, it follows that $\mu_X(V) = \widetilde{g}^{-1}(Y \setminus \downarrow y_1) \cap \mu_X(X)$ is open in the subspace $\mu_X(X)$ of X^k. Thus, together with Step 1, we have that μ_X is an embedding, proving Step 2.

Step 3: $\mu_X(X)$ is b-dense in X^k.

Suppose, on the contrary, $\mu_X(X)$ is not b-dense in X^k. Then there exist a point $z_0 \in X^k$ and an open subset W of X^k such that $z_0 \in W$, and $\downarrow z_0 \cap W \cap \mu_X(X) = \emptyset$. Define two mappings $g_1, g_2 : X^k \longrightarrow Y$ by

$$g_1(z) = \begin{cases} y_1 & z \notin W \\ y_2 & z \in W, \end{cases} \qquad g_2(z) = \begin{cases} y_1 & z \notin W \setminus \downarrow z_0 \\ y_2 & z \in W \setminus \downarrow z_0. \end{cases}$$

Then by Lemma 3.1, g_1, g_2 are both continuous mappings. It should note that $g_1 \neq g_2$ as $g_1(z_0) = y_2$ and $g_2(z_0) = y_1$.

Claim 2: $g_1 \circ \mu_X = g_2 \circ \mu_X$.

Let $x \in X$. If $g_1(\mu_X(x)) = y_2$, then $\mu_X(x) \in W$, and since $\downarrow z_0 \cap W \cap \mu_X(X) = \emptyset$, it follows that $\mu_X(x) \notin \downarrow z_0$, and hence $\mu_X(x) \in W \setminus \downarrow z_0$, which implies

$g_2(\mu_X(x)) = y_2$. If $g_1(\mu_X(x)) = y_1$, then $\mu_X(x) \notin W$, and hence $\mu_X(x) \notin W \setminus {\downarrow} z_0$, which implies $g_2(\mu_X(x)) = y_1$. Then we have that $g_1 \circ \mu_X = g_2 \circ \mu_X$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "g_1\circ\mu_X = g_2\circ\mu_X"'] & X^k \arrow[d, "g_1, g_2"] \\ & Z \end{tikzcd}$$

Since μ_X is a $\mathbf{K}$-reflection for X, we have that $g_1 = g_2$, which contradicts that $g_1 \neq g_2$. Therefore, ${\downarrow} z_0 \cap W \cap \mu_X(X) \neq \emptyset$, which implies that $\mu_X(X)$ is b-dense in X^k.

All these show that the $\mathbf{K}$-reflection μ_X is a b-dense embedding. $\square$

Remark 3.3. It is noteworthy that there is a difference between b-dense embedding of Definition 2.7 and Skula's b-embedding, where the "embedding" condition is not required in the definition of Skula's b-embedding (see [11, Definition 3.1]). As a consequence, Theorem 3.2 above cannot be deduced directly from the result in [11, Theorem 3.5]. Undoubtedly, Skula's method plays an important role in the proof of Theorem 3.2, while for the sake of completeness, some parts of the proof overlap slightly.

Lemma 3.4 ([1]). *Let $f : X \longrightarrow Y$ be a continuous mapping between T_0 spaces. Then f is an epimorphism if and only if $f(X)$ is b-dense.*

By Theorem 3.2 and Lemma 3.4, we have the following result.

Corollary 3.5. *If $\mathbf{K}$ is a reflective subcategory of $\mathbf{Top}_0$ such that $\mathbf{K} \nsubseteq \mathbf{Top_1}$, then it is epireflective.*

The following lemma follows directly from Lemma 2.4.

Lemma 3.6. *Let X be a T_0 space and Y be a subset of X. If Y is b-dense in X, then $X = {\uparrow} Y$.*

Theorem 3.7. *Let $\mathbf{K}$ be a reflective subcategory of $\mathbf{Top}_0$ such that $\mathbf{K} \nsubseteq \mathbf{Top_1}$. If $X \in \mathbf{K}$ and Y is a saturated subspace of X, then $Y \in \mathbf{K}$.*

Proof. Let $\mu_Y : Y \longrightarrow Y^k$ be the $\mathbf{K}$-reflection for Y. By Theorem 3.2, $\mu_Y(Y)$ is b-dense in Y^k, and by Lemma 3.6, we have $Y^k = {\uparrow} \mu_Y(Y)$. Let $e : Y \longrightarrow X$ be the inclusion mapping. Then there exists a unique continuous mapping $f : Y^k \longrightarrow X$

such that $f \circ \mu_Y = e$:

$$\begin{tikzcd} Y \arrow[r, "\mu_Y"] \arrow[dr, "e"'] & Y^k \arrow[d, dotted, "f"] \\ & X \end{tikzcd}$$

Claim: $f(Y^k) = Y$. On the one hand, $Y = e(Y) = f(\mu_Y(Y)) \subseteq f(Y^k)$. On the other hand, note that each continuous mapping is order-preserving and $Y^k = \uparrow\mu_Y(Y)$, so $f(Y^k) = f(\uparrow\mu_Y(Y)) \subseteq \uparrow f(\mu_Y(Y)) = \uparrow e(Y) = \uparrow Y = Y$. Hence, $f(Y^k) = Y$.

Now consider the co-restriction $\widehat{f} : Y^k \longrightarrow Y$ (i.e., $\widehat{f}(x) = f(x)$) of f. Then it is clear that $\widehat{f} \circ \mu_Y = \mathrm{id}_Y$, the identity mapping on Y:

$$\begin{tikzcd} Y \arrow[r, "\mu_Y"] \arrow[dr, "\widehat{f}\circ\mu_Y = \mathrm{id}_Y"'] & Y^k \arrow[d, "\widehat{f}"] \\ & Y \end{tikzcd}$$

We then deduce that Y is a b-retract of Y^k, hence by Proposition 2.11, Y is homeomorphic to Y^k. As $\mathbf{K}$ is required to be replete, we obtain that $Y \in \mathbf{K}$. $\square$

4. The non-reflectivity of open well-filtered spaces

Definition 4.1. [12] Let X be a T_0 space.

(1) For any $U, V \in \mathcal{O}(X)$, we define $U \ll V$ if and only if each open cover of V has a finite subfamily that covers U.

(2) A subfamily $\mathcal{F} \subseteq \mathcal{O}(X)$ is called a *$\ll$-filtered family* if for any $U_1, U_2 \in \mathcal{F}$, there exists $U_3 \in \mathcal{F}$ such that $U_3 \ll U_1$ and $U_3 \ll U_2$.

(3) X is called *open well-filtered* if for each $\ll$-filtered family $\mathcal{F} \subseteq \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$,

$$\bigcap \mathcal{F} \subseteq U \ \Rightarrow \ V \subseteq U \text{ for some } V \in \mathcal{F}.$$

The category of all open well-filtered spaces with continuous mappings is denoted by $\mathbf{Owf}$. Then $\mathbf{Owf}$ is a full subcategory of $\mathbf{Top}_0$.

On open well-filtered spaces, a natural question is

- whether the category $\mathbf{Owf}$ is reflective in $\mathbf{Top}_0$?

In the following, we will give a negative answer to the above question.

Recall that a subset U of a poset P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, and we call this topology the *Scott* *topology* on P and denote it by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

Let $\mathbb{N}_{\mathrm{cof}}$ be the set $\mathbb{N}$ of natural numbers with the cofinite topology (the open sets are $\emptyset$, $\mathbb{N}$, and all the complements of finite subsets of $\mathbb{N}$). Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ be the Johnstone's dcpo [4, 7], which is ordered by $(m, n) \leq (m', n')$ iff either $m = m'$ and $n \leq n'$ or $n' = \infty$ and $n \leq m'$ (see Figure 1).

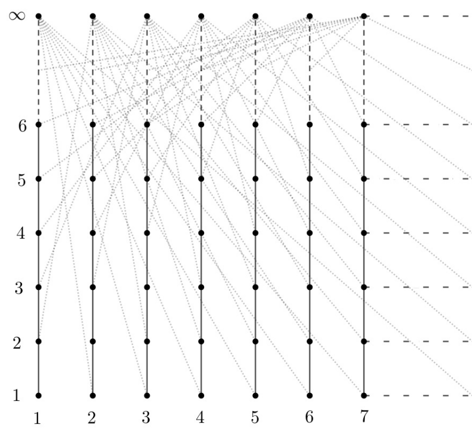

FIGURE 1. The Johnstone's dcpo $\mathbb{J}$

Remark 4.2. The following well-known results on $\mathbb{N}_{\mathrm{cof}}$ and $\Sigma\mathbb{J}$ will be used.

(1) Every set in $\mathbb{N}_{\mathrm{cof}}$ is compact.
(2) $\forall U, V \in \sigma(\mathbb{J})$, $U \ll V$ if and only if $U = \emptyset$ (see Exercise 5.2.15 in [4]).
(3) The maximal points space $\mathrm{Max}(\mathbb{J})$ of $\mathbb{J}$ with the relative topology of $\Sigma\mathbb{J}$ is homeomorphic to $\mathbb{N}_{\mathrm{cof}}$.

Theorem 4.3. *Let* $\mathbf{K}$ *be a reflective subcategory of* $\mathbf{Top}_0$. *If* $\Sigma\mathbb{J} \in \mathbf{K}$, *then* $\mathbb{N}_{\mathrm{cof}} \in \mathbf{K}$.

PROOF. If $\Sigma\mathbb{J} \in \mathbf{K}$, then by Remark 4.2(3) the maximal points space $\mathrm{Max}(\mathbb{J})$ is a saturated subspace of $\Sigma\mathbb{J}$, which is homeomorphic to $\mathbb{N}_{cof}$. Then by Theorem 3.7, $\mathbb{N}_{\mathrm{cof}} \in \mathbf{K}$. □

Proposition 4.4. *The Scott topology* $\sigma(\mathbb{J})$ *is open well-filtered.*

PROOF. Let $\mathcal{F}$ be a $\ll$-filtered family of Scott open subsets of $\mathbb{J}$, and $U \in \sigma(\mathbb{J})$ such that $\bigcap \mathcal{F} \subseteq U$. Fix a member $V_0 \in \mathcal{F}$. Since $\mathcal{F}$ is $\ll$-filtered, there exists

$V_1 \in \mathcal{F}$ such that $V_1 \ll V_0$, and from Remark 4.2(2), it follows that $V_1 = \emptyset \subseteq U$. Therefore, $\sigma(\mathbb{J})$ is open well-filtered. $\square$

Proposition 4.5. *The space* $\mathbb{N}_{\mathrm{cof}}$ *is not open well-filtered.*

PROOF. Since each set in $\mathbb{N}_{\mathrm{cof}}$ is compact, $U \ll V$ iff $U \subseteq V$ for any open subsets U, V of $\mathbb{N}_{\mathrm{cof}}$. For each $n \in \mathbb{N}$, define $U_n = \{k \in \mathbb{N} : k \geq n\}$. Then $\{U_n : n \in \mathbb{N}\}$ is a $\ll$-filtered family of open sets in $\mathbb{N}_{\mathrm{cof}}$ such that $\bigcap_{n\in\mathbb{N}} U_n = \emptyset$, but $U_n \neq \emptyset$ for any $n \in \mathbb{N}$. This implies $\mathbb{N}_{\mathrm{cof}}$ is not open well-filtered. $\square$

From Propositions 4.4 and 4.5, we have that $\mathbb{N}_{\mathrm{cof}} \notin \mathbf{Owf}$ and $\Sigma\mathbb{J} \in \mathbf{Owf}$. Hence, by Theorem 4.3, we have the following result.

Theorem 4.6. *The category* $\mathbf{Owf}$ *is not reflective in* $\mathbf{Top}_0$.

5. Conclusion

In this paper, we use Skula's b-topology to study the reflective subcategories of $\mathbf{Top}_0$, and prove the non-reflectivity of open well-filtered spaces in $\mathbf{Top}_0$. These results enrich the b-topology applications in domain theory, which fully illustrates that Skula's b-topology is an effective tool to study the reflectivity of subcategories of T_0 spaces. It also provides a possibility of solving some open problems on the reflectivity of some other T_0 spaces in domain theory (see [13] for example) by using the approach in this paper. We leave these as our future work.

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Chong Shen, School of Science, Beijing University of Posts and Telecommunications, Beijing, China

Email address: `shenchong0520@163.com`

Xiaoyong Xi, School of Mathematics and Statistics, Yancheng Teachers University, Yancheng, Jiangsu, China

Email address: `xixy@yctu.edu.cn`

Dongsheng Zhao, Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616

Email address: `dongsheng.zhao@nie.edu.sg`