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AN ANSWER TO AN OPEN PROBLEM OF HECKMANN AND $\mathscr{U}_s$-ADMITTING SPACES

CHONG SHEN, QINGYU HE, XIAOYONG XI AND DONGSHENG ZHAO

Heckmann first introduced and studied the $\mathscr{U}_s$-admitting spaces, which are defined similarly to the extensively studied well-filtered spaces. He posed this problem (1992): Is the upper space of a $\mathscr{U}_s$-admitting space still $\mathscr{U}_s$-admitting? We prove that a T_0 space is $\mathscr{U}_s$-admitting if and only if its upper space is $\mathscr{U}_s$-admitting; thus giving a positive answer to Heckmann's problem. We then carry out a systematic investigation on $\mathscr{U}_s$-admitting spaces, using the new machinery employed in studying the well-filtered spaces in the past few years. The main results include (i) a T_0 space is $\mathscr{U}_s$-admitting if and only if each closed $\mathscr{U}_s$-subset is the closure of a singleton; (ii) the locally finitary compact $\mathscr{U}_s$-admitting spaces are exactly the quasicontinuous directed complete posets (dcpos) with the Scott topology; (iii) the category of all $\mathscr{U}_s$-admitting spaces is reflective in the category of all T_0 spaces; (iv) a T_1 space X is well-filtered if and only if its Xi–Zhao dcpo model $D(X)$ is $\mathscr{U}_s$-admitting, which strengthens an existing result of the third and fourth authors. These results reveal that the $\mathscr{U}_s$-admitting spaces form a well-behaved class, lying between the classes of well-filtered spaces and d-spaces.

1. Introduction

The primary motivation for the study of domains, which was initiated by Dana Scott in the late 1960s, was to search for a denotational semantics of lambda calculus. Domain theory also provides a platform to study the interlinks between topology and order. The $\mathscr{U}_s$-admitting spaces were introduced by Heckmann [6], to provide an alternative novel type of upper power domains. The new power domain is defined by strongly compact sets, which contain fewer elements than the classical ones known as compact saturated sets. The $\mathscr{U}_s$-admitting space is defined in a very similar manner as the well-filtered spaces. The well-filtered spaces form a class of non-Hausdorff spaces, lying between those of sober spaces and d-spaces. In the past few years, the research on well-filtered spaces has made significant progress. The following are some of the newly obtained results on well-filtered spaces:

(i) Well-filtered spaces can be completely determined by KF-sets [12; 19].

(ii) Every core-compact well-filtered space is sober [10].

(iii) The category of well-filtered spaces is reflective in T_0 spaces [15].

(iv) A space is well-filtered if and only if its upper space is well-filtered [17; 21].

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Our main goal is to carry out a systematic study of $\mathscr{U}_s$-admitting spaces using the machinery invented in the study of well-filtered spaces.

Here is the outline of our paper. In Section 3, we introduce the notion of $\mathscr{U}_s$-set, and use it to give a characterization of $\mathscr{U}_s$-admitting spaces: a T_0 space X is $\mathscr{U}_s$-admitting if and only if every $\mathscr{U}_s$-set in X is the closure of a singleton. We show that the locally finitary compact $\mathscr{U}_s$-admitting spaces are exactly the quasicontinuous directed complete posets (dcpos) with the Scott topology, which strengthens an existing result [4, Exercise 8.3.39]. Lastly, by verifying Keimel and Lawson's K-conditions [8], we prove that the category of $\mathscr{U}_s$-admitting spaces is a full reflective subcategory of the category of T_0 spaces with continuous mappings.

Heckmann [6] asked this question:

> Is the upper power space of a $\mathscr{U}_s$-admitting space again $\mathscr{U}_s$-admitting (see [6, Section 4.6])?

In Section 4, by employing $\mathscr{U}_s$-sets (introduced in Section 3), we give a positive answer to Heckmann's question. More precisely, we will prove that a T_0 space is $\mathscr{U}_s$-admitting if and only if its upper space $P_s(X)$ is $\mathscr{U}_s$-admitting.

In domain theory, one of the most important topologies on posets is the Scott topology. The Scott topology is merely T_0 in general (it is T_1 if and only if it is discrete). However, for each poset P, the subspace $\mathrm{Max}(P)$ of the Scott space of P, consisting of all maximal points of P, is always T_1. A poset model of a topological space X is a poset P such that $\mathrm{Max}(P)$ is homeomorphic to X [9]. In [23], it was proved that for every T_1 space X, one can construct a dcpo model $D(X)$ (it is called the Xi–Zhao model by some authors). Recently, Shen, Wu and Zhao [11] proved that $D(X)$ is a weak domain. Several studies showed that $D(X)$ preserves many properties of X, such as sobriety, well-filteredness, Baire property, Choquet completeness and weak sobriety (see [1; 5; 17; 23]). Chen and Li [1] constructed a T_1 space X whose dcpo model $D(X)$ is not $\mathscr{U}_s$-admitting. Therefore, one naturally has the following question:

> Under what conditions is $D(X)$ $\mathscr{U}_s$-admitting?

In Section 5, we present a necessary and sufficient condition for $D(X)$ to be $\mathscr{U}_s$-admitting, answering the above question:

> A T_1 space X is well-filtered if and only if $D(X)$ is $\mathscr{U}_s$-admitting,

which improves an existing result in [17].

Our results demonstrate that $\mathscr{U}_s$-admitting spaces behave well and enjoy rich properties similar to those of sober spaces and well-filtered spaces.

2. Preliminaries

We introduce some basic concepts and results that will be used later. For more details, we refer the readers to [3; 4].

Let P be a poset. For any subset A of a poset P, we use the standard notation

$$\uparrow A = \{y \in P : \exists x \in A,\, x \leq y\}; \quad \downarrow A = \{y \in P : \exists x \in A,\, y \leq x\}.$$

For each $x \in X$, we write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. A subset A of P is called a *lower* (resp., *upper*) *set* if $A = \downarrow A$ (resp., $A = \uparrow A$).

A nonempty subset D of P is *directed* if every two elements in D have an upper bound in D. P is called a *directed complete poset*, or a *dcpo* for short, if for any directed subset $D \subseteq P$, the supremum $\bigvee D$ of D exists.

For $x, y \in P$, x is *way below* y, denoted by $x \ll y$, if for any directed subset D of P with $\bigvee D$ existing, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. Define $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$. A continuous dcpo is also called a *domain*. An element x is *compact* in P if $x \ll x$. We use $K(P)$ to denote the set of all compact elements of P. We call P *algebraic* if for each $x \in P$, the set $\downarrow x \cap K(P)$ is directed whose supremum is x.

A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \varnothing$. All Scott open subsets of P form a topology, called the *Scott topology* on P, denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

Let X be a T_0 space. A subset A of X is called *saturated* if A equals the intersection of all open sets containing it. The specialization order $\leq$ on X is defined by $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator. It is important to note that a subset A of X is saturated if and only if $A = \uparrow A$ with respect to the specialization order.

Remark 2.1 [3; 4]. (1) For a poset P, the specialization order of $(P, \sigma(P))$ coincides with the order of P. If P is continuous, then $\{\twoheaduparrow x : x \in P\}$ forms a base for $\sigma(P)$.

(2) Every open (resp., closed) set is an upper (resp., lower) set. In particular, $\mathrm{cl}(\{x\}) = \downarrow x$ holds for every $x \in X$ in the specialization order.

(3) For each subset K of X, K is compact if and only if $\uparrow K$ is compact.

Definition 2.2 [6]. A subset K of a T_0 space X is *strongly compact* if for each open set O with $K \subseteq O$, there is a finite set $F \subseteq X$ such that $K \subseteq \uparrow F \subseteq O$.

Proposition 2.3 [6]. *Let X be a T_0 space, and $A, B \subseteq X$.*

(1) *Every finite set is strongly compact, and every strongly compact set is compact.*

(2) *The set A is strongly compact if and only if $\uparrow A$ is so.*

(3) *If A and B are strongly compact, then so is $A \cup B$.*

(4) *The continuous image of a strongly compact set is also strongly compact.*

For a T_0 space X, $A \subseteq X$, and $\mathscr{F} \subseteq 2^X$, we shall use the notation

- $\mathscr{O}(X)$, the family of all open subsets of X;
- $\mathscr{C}(X)$, the family of all closed subsets of X;
- $\mathscr{Q}_s(X)$, the family of all strongly compact saturated subsets of X;
- $\mathscr{Q}_s^*(X)$, $\mathscr{Q}_s(X) \setminus \{\varnothing\}$;
- $\mathscr{M}(\mathscr{F})$, the family $\{C \in \mathscr{C}(X) : \forall F \in \mathscr{F}, C \cap F \neq \varnothing\}$;
- $m(\mathscr{F})$, the family of all minimal members in $(\mathscr{M}(\mathscr{F}), \subseteq)$.

Definition 2.4 [3; 4]. Let X be a T_0 space. A nonempty subset A of X is called *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. The space X is called *sober* if every irreducible closed subset of X is the closure of a (unique) point.

The following results on irreducible sets are well-known and will be used in the sequel.

Lemma 2.5. *Let X be a T_0 space and Y a subspace of X. The following statements are equivalent for a subset $A \subseteq Y$:*

(1) *A is an irreducible subset of Y.*

(2) *A is an irreducible subset of X.*

(3) *$\mathrm{cl}_X(A)$ is an irreducible subset of X.*

Definition 2.6 [3; 4]. A T_0 space X is called *well-filtered* if for any filtered family $\mathscr{F}$ of compact saturated sets in X and $U \in \mathscr{O}(X)$, $\bigcap \mathscr{F} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathscr{F}$.

Definition 2.7 [16]. A T_0 space X is called a *d-space* if X is a dcpo and every open subset of X is Scott open in the specialization order.

Remark 2.8. (1) Every sober space is well-filtered, and every well-filtered space is a d-space [3; 4].

(2) A T_0 space is a d-space if and only if for each directed set D in X with respect to the specialization order, there is $x \in X$ such that $\mathrm{cl}(D) = \downarrow x$ [19, Proposition 3.3].

We will use the following result, derived from the topological Rudin lemma (see Lemma 3.1 in [7]).

Lemma 2.9 [7]. *Let X be a T_0 space, C a closed subset of X and $\mathscr{F}$ a filtered family of compact saturated sets in X. Every closed set $C \in \mathscr{M}(\mathscr{F})$ contains a subset $A \in m(\mathscr{F})$. Every member in $m(\mathscr{F})$ is irreducible.*

3. The reflectivity of $\mathscr{U}_s$-admitting spaces

The notion of $\mathscr{U}_s$-admitting space introduced by Heckmann [6] is originally defined for Scott spaces of dcpos. It can be generalized to T_0 spaces naturally.

Definition 3.1 [6]. A T_0 space is called *$\mathscr{U}_s$-admitting* if for any filtered family $\mathscr{F}$ of strongly compact saturated sets and any open set U in X, $\bigcap \mathscr{F} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathscr{F}$.

It is an immediate conclusion that every well-filtered space is $\mathscr{U}_s$-admitting.

Proposition 3.2. *If X is a $\mathscr{U}_s$-admitting space, then $(\mathscr{Q}_s^*(X), \supseteq)$ is a dcpo.*

Proof. Suppose $\mathscr{F}$ is a filtered family of $\mathscr{Q}_s^*(X)$. It suffices to prove $\bigcap \mathscr{F} \in \mathscr{Q}_s^*(X)$. If $\bigcap \mathscr{F} = \varnothing$, then since X is $\mathscr{U}_s$-admitting, we have that $K = \varnothing$ for some $K \in \mathscr{F}$, contradicting the fact that $\mathscr{F} \subseteq \mathscr{Q}_s(X) \backslash \{\varnothing\}$. Hence, $\bigcap \mathscr{F}$ is nonempty. In addition, if W is an open neighborhood of $\bigcap \mathscr{F}$, then since X is $\mathscr{U}_s$-admitting, there is $K_0 \in \mathscr{F}$ such that $K_0 \subseteq W$, and since K_0 is strongly compact, there is a finite subset F of X such that $K_0 \subseteq \uparrow F \subseteq W$. Hence, $\bigcap \mathscr{F} \subseteq K_0 \subseteq \uparrow F \subseteq W$. Thus, $\bigcap \mathscr{F}$ is strongly compact, and it is saturated since all members of $\mathscr{F}$ are saturated. Therefore, $\bigcap \mathscr{F} \in \mathscr{Q}_s^*(X)$. $\square$

It is easy to verify that in a T_1 space, the strongly compact sets are exactly the finite sets. Thus, we have the following.

Proposition 3.3. *Every T_1 space is $\mathscr{U}_s$-admitting.*

It is known that every locally compact well-filtered space is sober, as proved in [4, Proposition 8.3.8]; every first-countable well-filtered space is sober, as proved in [18, Theorem 4.2] or [20, Theorem 6.7]. However, it should be noted that these two conclusions on well-filtered spaces cannot be generalized to $\mathscr{U}_s$-admitting spaces, as demonstrated by the following example.

Example 3.4. Let $\mathbb{N}_{\mathrm{cof}}$ be the set of natural numbers equipped with the cofinite topology (the open sets are empty set and the complement of finite sets). Then, $\mathbb{N}_{\mathrm{cof}}$ is a locally compact and first-countable T_1 space, and hence is $\mathscr{U}_s$-admitting by Proposition 3.3. But $\mathbb{N}_{\mathrm{cof}}$ is not sober since $\mathbb{N}$ itself is irreducible closed but does not equal to the closure of some singleton.

Definition 3.5 [4, Exercise 5.1.42]. A topological space X is *locally finitary compact* (also called *multi-continuous* in [6]) if for every $x \in X$ and every open neighborhood U of x, there is a finite subset F of X such that $x \in (\uparrow F)^o \subseteq \uparrow F \subseteq U$.

Proposition 3.6. *Every locally finitary compact $\mathscr{U}_s$-admitting space is sober.*

Proof. Assume that X is a locally finitary compact and $\mathscr{U}_s$-admitting space. Let A be an irreducible closed subset of X and

$$\mathscr{F}_A := \{\uparrow F : F \text{ is a finite subset of } X,\ (\uparrow F)^o \cap A \neq \varnothing\}.$$

Claim: $\mathscr{F}_A$ is a filtered family.

Since $A \neq \varnothing$, there is $x \in A$, and since X is locally finitary compact, there is a finite subset F of X such that $x \in (\uparrow F)^o$, so $x \in (\uparrow F)^o \cap A \neq \varnothing$. It follows that $\uparrow F \in \mathscr{F}_A$. Thus, $\mathscr{F}_A \neq \varnothing$.

Let F_1, F_2 be two finite subsets of X such that $(\uparrow F_1)^o \cap A \neq \varnothing$ and $(\uparrow F_2)^o \cap A \neq \varnothing$. Since A is irreducible, there is $x \in (\uparrow F_2)^o \cap (\uparrow F_2)^o \cap A \neq \varnothing$. Since X is locally finitary compact, there exists a finite subset F_3 of X such that $x \in (\uparrow F_3)^o \subseteq \uparrow F_3 \subseteq (\uparrow F_2)^o \cap (\uparrow F_2)^o$. Note that $x \in (\uparrow F_3)^o \cap A \neq \varnothing$, so $\uparrow F_3 \in \mathscr{F}_A$. Hence, $\mathscr{F}_A$ is a filtered family.

Since X is $\mathscr{U}_s$-admitting, there is $x_0 \in A \cap \bigcap \mathscr{F}_A \neq \varnothing$. We show that $A = \mathrm{cl}(\{x_0\})$. It is clear that $\mathrm{cl}(\{x_0\}) \subseteq A$. Now suppose $x \in A$. For each open neighborhood U of x, since X is locally finitary, there is a finite subset F of X such that $x \in (\uparrow F)^o \subseteq \uparrow F \subseteq U$, which implies that $x \in (\uparrow F)^o \cap A \neq \varnothing$, so $\uparrow F \in \mathscr{F}_A$. It follows that $x_0 \in \bigcap \mathscr{F}_A \subseteq \uparrow F \subseteq U$. This implies that $x \in \mathrm{cl}(\{x_0\})$. Then, $A \subseteq \mathrm{cl}(\{x_0\})$. Thus, $A = \mathrm{cl}(\{x_0\})$. Therefore, X is sober. □

A remarkable result in domain theory is that locally finitary compact sober spaces are exactly the Scott space of quasicontinuous dcpos [4, Exercise 8.3.39]. Thus, by Proposition 3.6 we have the following corollary.

Corollary 3.7. *The locally finitary compact $\mathscr{U}_s$-admitting spaces are exactly the quasicontinuous dcpos with their Scott topology.*

Definition 3.8. A nonempty subset A of a T_0 space X is called a *$\mathscr{U}_s$-set* if there is a filtered subfamily $\mathscr{F}$ of $\mathscr{Q}_s(X)$ such that $\mathrm{cl}(A) \in m(\mathscr{F})$.

The $\mathscr{U}_s$-sets defined above will play a crucial role in all the proofs in rest of this section.

Remark 3.9. Let X be a T_0 space, and $A \subseteq X$.

(1) It is clear that A is a $\mathscr{U}_s$-set if and only if $\mathrm{cl}(A)$ is a $\mathscr{U}_s$-set.

(2) Every $\mathscr{U}_s$-set is irreducible by Lemma 2.9.

(3) If A is directed with respect to the specialization order of X, it is easy to verify that $\mathscr{F} = \{\uparrow x : x \in A\}$ is a filtered family of strongly compact saturated sets in X such that $\mathrm{cl}(A) \in m(\mathscr{F})$, so A is a $\mathscr{U}_s$-set. Hence, every (closure of a) directed set is a $\mathscr{U}_s$-set. In particular, every (closure of a) singleton is a $\mathscr{U}_s$-set.

Lemma 3.10. *Let X be a T_0 space, and $A \subseteq X$. Then $\bigcap_{x\in A} \uparrow x = \bigcap_{x\in \mathrm{cl}(A)} \uparrow x$.*

Proof. For each $y \in X$, we have

$$\begin{aligned} y \in \bigcap_{x\in A} \uparrow x &\iff \forall x \in A, x \leq y \\ &\iff A \subseteq \downarrow y \\ &\iff \mathrm{cl}(A) \subseteq \downarrow y \\ &\iff \forall x \in \mathrm{cl}(A), x \leq y \\ &\iff y \in \bigcap_{x\in \mathrm{cl}(A)} \uparrow x, \end{aligned}$$

which implies that $\bigcap_{x\in A} \uparrow x = \bigcap_{x\in \mathrm{cl}(A)} \uparrow x$. $\square$

Theorem 3.11. *For every T_0 space X, the following statements are equivalent:*

(1) *X is $\mathscr{U}_s$-admitting.*
(2) *For each $\mathscr{U}_s$-set $A \subseteq X$, there exists $x \in X$ such that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$.*
(3) *For each closed $\mathscr{U}_s$-set $A \subseteq X$, there exists $x \in X$ such that $A = \mathrm{cl}(\{x\})$.*
(4) *For each $\mathscr{U}_s$-set $A \subseteq X$ and $U \in \mathscr{O}(X)$, $\bigcap_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$.*
(5) *For each closed $\mathscr{U}_s$-set $A \subseteq X$ and $U \in \mathscr{O}(X)$, $\bigcap_{a\in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$.*

Proof. (1) $\Longrightarrow$ (2): Let A be a $\mathscr{U}_s$-set in X. Then, there exists a filtered family $\mathscr{F}$ of $\mathscr{Q}_s(X)$ such that $\mathrm{cl}(A) \in m(\mathscr{F})$. Since X is $\mathscr{U}_s$-admitting, it follows that $\left(\bigcap \mathscr{F}\right) \cap \mathrm{cl}(A) \neq \varnothing$. Take $x \in \left(\bigcap \mathscr{F}\right) \cap \mathrm{cl}(A)$. Then, $\mathrm{cl}(\{x\})$ is a subset of $\mathrm{cl}(A)$ such that $\mathrm{cl}(\{x\}) \in \mathscr{M}(\mathscr{F})$. Since $\mathrm{cl}(A) \in m(\mathscr{F})$, we deduce that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$.

(2) $\iff$ (3): It is trivial by Remark 3.9(1).

(2) $\Longrightarrow$ (4): Since A is a $\mathscr{U}_s$-set, there exists $x \in X$ such that $\mathrm{cl}(A) = \downarrow x$, which implies that

$$\uparrow x = \bigcap_{a\in \downarrow x} \uparrow a = \bigcap_{a\in \mathrm{cl}(A)} \uparrow a \subseteq \bigcap_{a\in A} \uparrow a \subseteq U.$$

This shows that U is an open neighborhood of x, and since $x \in \mathrm{cl}(A)$, $U \cap A \neq \varnothing$. Take $a_0 \in U \cap A$, so $\uparrow a_0 \subseteq U$. This gives (4).

(4) $\iff$ (5): It is straightforward by Lemma 3.10.

(5) $\Longrightarrow$ (1): Suppose $\mathscr{F}$ is a filtered family of $\mathscr{Q}_s(X)$ and $U \in \mathscr{O}(X)$ such that $\bigcap \mathscr{F} \subseteq U$. We need to show that $K \subseteq U$ for some $K \in \mathscr{F}$. Assume, on the contrary, that $K \nsubseteq U$. Then $K \cap (X \setminus U) \neq \varnothing$, for all $K \in \mathscr{F}$. By Lemma 2.9, there exists a closed subset $A \subseteq X \setminus U$ such that $A \in m(\mathscr{F})$. It follows that A is a $\mathscr{U}_s$-set in X, and since $A \cap U = \varnothing$, it holds that $\uparrow a \nsubseteq U$ for all $a \in A$. Thus by (5), $\bigcap_{a\in A} \uparrow a \nsubseteq U$, so there is $y \in \bigcap_{a\in A} \uparrow a$ such that $y \notin U$. In addition, for each $K \in \mathscr{F}$, there is $a \in A \cap K \neq \varnothing$, and since K is saturated, we deduce that $y \in \uparrow a \subseteq K$, so $y \in \bigcap \mathscr{F} \subseteq U$, which contradicts the fact that $y \notin U$. This implies that $K \subseteq U$ for some $K \in \mathscr{F}$. Hence, X is $\mathscr{U}_s$-admitting. $\square$

By Remarks 2.8, 3.9 and Theorem 3.11, we immediately deduce the following.

Proposition 3.12. *Every $\mathscr{U}_s$-admitting space is a d-space.*

As a summary of Remark 2.8, Propositions 3.3 and 3.12, we have the relations

$$\begin{tikzcd} T_2 \arrow[rrr, Rightarrow] \arrow[d, Rightarrow] & & & T_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & T_0 \\ \text{sober} \arrow[r, Rightarrow] & \text{well-filtered} \arrow[rr, Rightarrow] & & \mathscr{U}_s\text{-admitting} \arrow[r, Rightarrow] & d\text{-space} \arrow[u, Rightarrow] \end{tikzcd}$$

Proposition 3.13. *Let X be a T_0 space. Then, for each closed set C and each strongly compact set K in X, $C \cap K$ is strongly compact.*

Proof. Let $U \in \mathscr{O}(X)$ such that $C \cap K \subseteq U$. Then $K \subseteq U \cup (X \setminus C) \in \mathscr{O}(X)$. Since K is strongly compact, there is a finite subset F of X such that $K \subseteq \uparrow F \subseteq U \cup (X \setminus C)$. Let $G = F \cap C$. We have the following two claims:

(i) $C \cap K \subseteq \uparrow G$. In fact, let $x \in C \cap K$. Since $K \subseteq \uparrow F$, there is $a \in F$ such that $a \leq x$. Then, $a \in \downarrow x \subseteq \downarrow C = C$, so $a \in F \cap C = G$. Thus, $x \in \uparrow a \subseteq \uparrow G$.

(ii) $\uparrow G \subseteq U$. To verify this, it suffices to prove that $G \subseteq U$ because U is an upper set. Let $x \in G$. As $G = F \cap C$, $x \in F \subseteq \uparrow F \subseteq U \cup (X \setminus C)$, and from $x \in C$, we deduce that $x \in U$. Thus, $G \subseteq U$.

Thus, G is a finite set satisfying that $C \cap K \subseteq \uparrow G \subseteq U$. Therefore, $C \cap K$ is strongly compact. □

Theorem 3.14. *Let $f : X \to Y$ be a continuous mapping between T_0 spaces X and Y. Then, for each $\mathscr{U}_s$-set A in X, $f(A)$ is a $\mathscr{U}_s$-set in Y.*

Proof. Since A is a $\mathscr{U}_s$-set in X, there is a filtered family $\mathscr{F}$ of $\mathscr{Q}_s(X)$ such that $\mathrm{cl}_X(A) \in m(\mathscr{F})$. For each $K \in \mathscr{F}$, let $\widehat{K} = \uparrow f(K \cap \mathrm{cl}_X(A))$. By Proposition 3.13 and the fact that $\mathrm{cl}_X(A) \in m(\mathscr{F})$, the intersection $K \cap \mathrm{cl}_X(A)$ is a nonempty strongly compact set. Since f is continuous, by Proposition 2.3, the family $\widehat{\mathscr{F}} = \{\widehat{K} : K \in \mathscr{F}\}$ is a filtered family of $\mathscr{Q}_s(Y)$. To prove $f(A)$ is a $\mathscr{U}_s$-set in Y, it suffices to prove $\mathrm{cl}_Y(f(A)) \in m(\widehat{\mathscr{F}})$. In fact, we have the following:

(i) For each $K \in \mathscr{F}$, since $f(K \cap \mathrm{cl}_X(A)) \subseteq f(\mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A))$, it follows that

$$\varnothing \neq f(K \cap \mathrm{cl}_X(A)) \subseteq \mathrm{cl}_Y(f(A)) \cap \uparrow f(K \cap \mathrm{cl}_X(A)) = \mathrm{cl}_Y(f(A)) \cap \widehat{K},$$

so $\mathrm{cl}_Y(f(A)) \in \mathscr{M}(\widehat{\mathscr{F}})$.

(ii) Now assume C is a closed subset of $\mathrm{cl}_Y(f(A))$ such that $C \in \mathscr{M}(\widehat{\mathscr{F}})$. Then for each $K \in \mathscr{F}$, it holds that $C \cap \uparrow f(K \cap \mathrm{cl}_X(A)) \neq \varnothing$, and since C is a lower set, we deduce that $C \cap f(K \cap \mathrm{cl}_X(A)) \neq \varnothing$, which implies that $(f^{-1}(C) \cap \mathrm{cl}_X(A)) \cap K \neq \varnothing$. Since $\mathrm{cl}_X(A) \in m(\mathscr{F})$, it follows that $f^{-1}(C) \cap \mathrm{cl}_X(A) = \mathrm{cl}_X(A)$, so $\mathrm{cl}_X(A) \subseteq f^{-1}(C)$. Then we have that $f(\mathrm{cl}_X(A)) \subseteq C$, so $\mathrm{cl}_Y(f(A)) = \mathrm{cl}_Y(f(\mathrm{cl}_X(A))) \subseteq C$. This implies that $\mathrm{cl}_Y(f(A)) \in m(\widehat{\mathscr{F}})$.

Therefore, $f(A)$ is a $\mathscr{U}_s$-set in Y. □

Proposition 3.15. *Let X be a T_0 space, and $A \subseteq Y \subseteq X$. If A is a $\mathscr{U}_s$-set in the subspace Y, then A is also a $\mathscr{U}_s$-set in X.*

Proof. Suppose A is a $\mathscr{U}_s$-set in Y. Consider the embedding $e : Y \to X$, $x \mapsto x$. Since e is continuous, by Theorem 3.14, $e(A) = A$ is a $\mathscr{U}_s$-set in X. □

Proposition 3.16. *Let X be a sober space, Y be a $\mathscr{U}_s$-admitting subspace of X, and $A \subseteq Y$. If A is a $\mathscr{U}_s$-set in Y, then there exists a unique $x \in Y$ such that* $\mathrm{cl}_X(A) = \mathrm{cl}_X(\{x\})$.

Proof. Since A is a $\mathscr{U}_s$-set in Y and Y is a $\mathscr{U}_s$-admitting space, by Theorem 3.11, there exists a unique $y \in Y$ such that $\mathrm{cl}_Y(A) = \mathrm{cl}_Y(\{y\})$. By Remark 3.9, A is an irreducible set in Y, and hence is also irreducible in X. Since X is sober, there exists $x \in X$ such that $\mathrm{cl}_X(A) = \mathrm{cl}_X(\{x\})$. It remains to prove $x = y$. First, we have that $y \in \mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y \subseteq \mathrm{cl}_X(\{x\})$, so $\mathrm{cl}_X(\{y\}) \subseteq \mathrm{cl}_X(\{x\})$. In addition, since $A \subseteq \mathrm{cl}_Y(\{y\}) = \mathrm{cl}_X(\{y\}) \cap Y \subseteq \mathrm{cl}_X(\{y\})$, we have that $\mathrm{cl}_X(\{x\}) = \mathrm{cl}_X(A) \subseteq \mathrm{cl}_X(\{y\})$. Thus, $\mathrm{cl}_X(\{x\}) = \mathrm{cl}_X(\{y\})$. Since X is T_0, we have that $x = y \in Y$. □

Assume **K** is a full subcategory of the category $\mathbf{Top_0}$ of T_0 spaces with continuous mappings. Recall that **K** is *reflective* in $\mathbf{Top_0}$ if for each $X \in \mathbf{Top_0}$, there exists a $X^k \in \mathbf{K}$ and a continuous mapping $\eta_X : X \to X^k$ such that for any continuous mapping $f : X \to Y$ to a space $Y \in \mathbf{K}$, there exists a unique continuous mapping $g : X^k \to Y$ such that $f = g \circ \eta_X$:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & X^k \arrow[d, dotted, "g"] \\ & Y \end{tikzcd}$$

Let **K** be a full subcategory of $\mathbf{Top_0}$. Keimel and Lawson [8] proved that if **K** satisfies the following four conditions, then **K** is a reflective subcategory of $\mathbf{Top_0}$:

(K1) **K** contains all sober spaces.

(K2) If $X \in \mathbf{K}$ and the space Y is homeomorphic to X, then $Y \in \mathbf{K}$.

(K3) If $\{X_i : i \in I\}$ is a family of subspaces of a sober space such that each $X_i \in \mathbf{K}$, then the subspace $\bigcap_{i \in I} X_i$ is contained in **K**.

(K4) If $X \to Y$ is a continuous mapping between sober spaces X and Y, then for any subspace Z of Y, $Z \in \mathbf{K}$ implies $f^{-1}(Z) \in \mathbf{K}$.

It is well-known that the categories of sober spaces and d-spaces are reflective in $\mathbf{Top_0}$. Recently, Wu, Xi, Xu and Zhao [15] firstly proved that the category of well-filtered spaces satisfies (K1)–(K4); therefore, it is reflective in $\mathbf{Top_0}$. Keimel and Lawson's K-conditions are also proved to be necessary under some condition; see [2; 13; 14] for more details. Next, by verifying conditions (K1)–(K4), we will show that the category of $\mathscr{U}_s$-admitting spaces is also reflective in $\mathbf{Top_0}$.

First, by Remark 2.8, every sober space is $\mathscr{U}_s$-admitting; hence, the category of $\mathscr{U}_s$-admitting spaces satisfies (K1). One can easily check that (K2) is also satisfied.

Lemma 3.17 (K3). *Let X be a sober space, and $\{X_i : i \in I\}$ be a family of $\mathscr{U}_s$-admitting subspaces of X. Then, $\bigcap_{i \in I} X_i$ is a $\mathscr{U}_s$-admitting subspace of X.*

Proof. Let $X_0 = \bigcap_{i \in I} X_i \neq \varnothing$, and A be a $\mathscr{U}_s$-set in X_0. By Remark 3.9, A is irreducible in X_0, and hence is irreducible in X, and since X is sober, there exists $x \in X$ such that $\mathrm{cl}_X(A) = \mathrm{cl}_X(\{x\})$. For each $i \in I$, by Proposition 3.15, A is a $\mathscr{U}_s$-set in X_i, and by Proposition 3.16, there exists $x_i \in X_i$ such that $\mathrm{cl}_X(A) = \mathrm{cl}_X(\{x_i\})$. Since X is T_0, all x_i equals x. We then deduce that $x \in X_0$ and $\mathrm{cl}_{X_0}(A) = \mathrm{cl}_{X_0}(\{x\})$. By Theorem 3.11, $\bigcap_{i \in I} X_i$ is $\mathscr{U}_s$-admitting. □

Lemma 3.18 (K4). *Let $f : X \to Y$ be a continuous mapping between sober spaces X and Y. Then, for each $\mathscr{U}_s$-admitting subspace Z of Y, $f^{-1}(Z)$ is a $\mathscr{U}_s$-admitting subspace of X.*

Proof. Suppose A is a $\mathscr{U}_s$-set in $f^{-1}(Z)$. Then A is an irreducible closed set in $f^{-1}(Z)$, and hence is irreducible in X. Since X is sober, there exists $x \in X$ such that $\mathrm{cl}_X(A) = \mathrm{cl}_X(\{x\})$. We need to show that $x \in f^{-1}(Z)$. By the continuity of f, we have that

$$\mathrm{cl}_Y(f(A)) = \mathrm{cl}_Y(f(\mathrm{cl}_X(A))) = \mathrm{cl}_Y\big(f(\mathrm{cl}_X(\{x\}))\big) = \mathrm{cl}_Y(\{f(x)\}).$$

The restriction $\hat{f} : f^{-1}(Z) \to Z$ ($\forall x \in f^{-1}(Z)$, $\hat{f}(x) = f(x)$) of f is continuous. Since A is a $\mathscr{U}_s$-set in $f^{-1}(Z)$, by Theorem 3.14, $\hat{f}(A) = f(A)$ is a $\mathscr{U}_s$-set in Z, and since Z is a $\mathscr{U}_s$-admitting subspace of Y, by Proposition 3.16, there is $y \in Z$ such that $\mathrm{cl}_Y(f(A)) = \mathrm{cl}_Y(\{y\})$. Recall that $\mathrm{cl}_Y(f(A)) = \mathrm{cl}_Y(\{f(x)\})$, and since Y is T_0, $y = f(x) \in Z$, so $x \in f^{-1}(Z)$. Then we have that $\mathrm{cl}_{f^{-1}(Z)}(A) = \mathrm{cl}_X(A) \cap f^{-1}(Z) = \mathrm{cl}_X(\{x\}) \cap f^{-1}(Z) = \mathrm{cl}_{f^{-1}(Z)}(\{x\})$. By Theorem 3.11, $f^{-1}(Z)$ is a $\mathscr{U}_s$-admitting space. □

Now we have proved that the category of $\mathscr{U}_s$-admitting spaces satisfies (K1)–(K4); thus, we have the following result.

Corollary 3.19. *The category of $\mathscr{U}_s$-admitting spaces is a reflective subcategory of* $\mathbf{Top_0}$.

4. The Smyth power space of a $\mathscr{U}_s$-admitting space

For a topological space X, the *upper topology* on $\mathscr{Q}_s^*(X) = \mathscr{Q}_s(X) \setminus \{\varnothing\}$ is generated by the family (as a base)

$$\Box U = \{K \in \mathscr{Q}_s^*(X) : K \subseteq U\},$$

where U ranges over the open subsets of X. The resulting space, denoted by $P_s(X)$, is called the *Smyth power space* or the *upper space* of X.

Remark 4.1. Let X be a T_0 space. The following results hold trivially:

(1) The specialization order of $P_s(X)$ is $\supseteq$. Hence, for each $\mathscr{A} \subseteq \mathscr{Q}_s^*(X)$,

$$\uparrow_{P_s(X)} \mathscr{A} = \{K \in \mathscr{Q}_s^*(X) : K \subseteq G \text{ for some } G \in \mathscr{A}\}$$

in the specialization order of $P_s(X)$.

(2) Define $\xi : X \to P_s(X)$, $x \mapsto \uparrow x$. Then $\xi^{-1}(\Box U) = U$ for each $U \in \mathscr{O}(X)$, and hence ξ is continuous.

(3) For each $\mathscr{A} \subseteq \mathscr{Q}_s^*(X)$, $\bigcap \mathrm{cl}_{P_s(X)}(\mathscr{A}) = \bigcap \mathscr{A}$ (using a similar proof to [19, Remark 2.12]).

Proposition 4.2. *If $\mathscr{K}$ is a strongly compact set in $P_s(X)$, then so is $\bigcup \mathscr{K}$ in X.*

Proof. Suppose U is an open set in X such that $\bigcup \mathscr{K} \subseteq U$. Then $\mathscr{K} \subseteq \Box U$. Since $\mathscr{K}$ is strongly compact, there is a finite family $\{S_1, S_2, \dots, S_n\} \subseteq \mathscr{Q}_s^*(X)$ such that $\mathscr{K} \subseteq \uparrow_{P_s(X)}\{S_1, S_2, \dots, S_n\} \subseteq \Box U$. Then, for each $1 \le k \le n$, it follows that $S_k \subseteq U$, and since S_k is strongly compact, there is a finite subset F_k of X such that $S_k \subseteq \uparrow F_k \subseteq U$. Let $F = \bigcup_{1 \le k \le n} F_k$.

Claim: $\bigcup \mathscr{K} \subseteq \uparrow F \subseteq U$.

Suppose $x \in \bigcup \mathscr{K}$. Then there is $K_0 \in \mathscr{K}$ such that $x \in K_0$. Since $\mathscr{K} \subseteq \uparrow_{P_s(X)}\{S_1, S_2, \dots, S_n\}$, there is k_0 ($1 \le k_0 \le n$) such that $K_0 \subseteq S_{k_0}$. Then, $x \in K_0 \subseteq S_{k_0} \subseteq \uparrow F_{k_0} \subseteq \uparrow F$. Since x is an arbitrary element of $\bigcup \mathscr{K}$, we have that $\bigcup \mathscr{K} \subseteq \uparrow F$. In addition, since for each k $(1 \leq k \leq n)$, $\uparrow F_k \subseteq U$, it follows that $\uparrow F = \bigcup_{1 \leq k \leq n} \uparrow F_k \subseteq U$. Therefore, the claim holds.

From the claim, we have that $\bigcup \mathscr{K}$ is a strongly compact set in X. □

Theorem 4.3. *Let X be a T_0 space. Then the following conditions are equivalent*:

(1) *$P_s(X)$ is $\mathscr{U}_s$-admitting.*

(2) *$P_s(X)$ is a d-space.*

(3) *X is $\mathscr{U}_s$-admitting.*

(4) *For each closed $\mathscr{U}_s$-set $\mathscr{A}$ in $P_s(X)$ and each $U \in \mathscr{O}(X)$, $\bigcap \mathscr{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathscr{A}$.*

Proof. (1) $\Longrightarrow$ (2): It is clear by Proposition 3.12.

(2) $\Longrightarrow$ (3): Suppose $\mathscr{F}$ is a filtered family of strongly compact sets in X and $U \in \mathscr{O}(X)$ such that $\bigcap \mathscr{F} \subseteq U$. Then, $\mathscr{F}$ is a directed set in $P_s(X)$ with the specialization order $\supseteq$, and since $P_s(X)$ is a d-space, the supremum $\bigvee_{P_s(X)} \mathscr{F}$ of $\mathscr{F}$ in $P_s(X)$ exists, denoted by K_0. It follows that $K_0 \subseteq \bigcap \mathscr{F} \subseteq U$, so $K_0 = \bigvee_{P_s(X)} \mathscr{F} \in \Box U$. Since $P_s(X)$ is a d-space, we have that $\Box U$ is Scott open, there is $G_0 \in \mathscr{F}$ such that $G_0 \in \Box U$, that is, $G_0 \subseteq U$. Thus, X is $\mathscr{U}_s$-admitting.

(3) $\Longrightarrow$ (4): Suppose $\bigcap \mathscr{A} \subseteq U$. Since $\mathscr{A}$ is a $\mathscr{U}_s$-set, there is a filtered family $\mathfrak{F} = \{\mathscr{K}_i : i \in I\}$ of strongly compact saturated subsets of $P_s(X)$ such that $\mathscr{A} \in m(\mathfrak{F})$. For each $i \in I$, let $\widehat{K}_i = \bigcup (\mathscr{A} \cap \mathscr{K}_i)$. By Proposition 3.13, $\mathscr{A} \cap \mathscr{K}_i$ is strongly compact in $P_s(X)$, and by Proposition 4.2, $\widehat{K}_i$ is strongly compact in X, and since every element in $\mathscr{K}_i \cap \mathscr{A}$ is saturated, $\widehat{K}_i$ is saturated. Thus, $\{\widehat{K}_i : i \in I\}$ is a filtered family of $\mathscr{Q}_s(X)$.

Claim: $\bigcap_{i \in I} \widehat{K}_i \subseteq \bigcap \mathscr{A}$.

Let $x \in \bigcap_{i \in I} \widehat{K}_i$. For each $i \in I$, $x \in \widehat{K}_i = \bigcup (\mathscr{A} \cap \mathscr{K}_i)$, so there is $G_i \in \mathscr{A} \cap \mathscr{K}_i$ such that $x \in G_i$. Then, $\mathrm{cl}_{P_s(X)}(\{G_i : i \in I\})$ is a closed subset of $\mathscr{A}$ that has a nonempty intersection with all elements of $\mathfrak{F}$. Since $\mathscr{A} \in m(\mathfrak{F})$, we have that $\mathscr{A} = \mathrm{cl}_{P_s(X)}(\{G_i : i \in I\})$. By Remark 4.1(3), we have that

$$x \in \bigcap \{G_i : i \in I\} = \bigcap \mathrm{cl}_{P_s(X)}(\{G_i : i \in I\}) = \bigcap \mathscr{A}.$$

Thus, by the arbitrariness of x, we have that $\bigcap_{i \in I} \widehat{K}_i \subseteq \bigcap \mathscr{A}$.

Since $\bigcap \mathscr{A} \subseteq U$, by the above claim $\bigcap_{i \in I} \widehat{K}_i \subseteq U$. Since X is $\mathscr{U}_s$-admitting, there is $i_0 \in I$ such that $\widehat{K}_{i_0} = \bigcup (\mathscr{A} \cap \mathscr{K}_{i_0}) \subseteq U$. Since $\mathscr{A} \cap \mathscr{K}_{i_0} \neq \emptyset$, choose a $K_0 \in \mathscr{A} \cap \mathscr{K}_{i_0}$. It follows that $K_0 \subseteq U$, as desired.

(4) $\Longrightarrow$ (1): We prove this conclusion by using Theorem 3.11(5). Suppose $\mathscr{A}$ is a $\mathscr{U}_s$-set in $P_s(X)$ and $\mathscr{U}$ is an open set in $P_s(X)$ such that $\bigcap_{K \in \mathscr{A}} \uparrow_{P_s(X)} K \subseteq \mathscr{U}$. Then there exists a family $\{U_i : i \in I\} \subseteq \mathscr{O}(X)$ such that $\mathscr{U} = \bigcup_{i \in I} \Box U_i$.

Claim 1: $\bigcap \mathscr{A}$ is nonempty strongly compact saturated.

The proof of claim 1 is analogous to that of Proposition 3.2.

Claim 2: there is $K \in \mathscr{A}$ such that $\uparrow_{P_s(X)} K \subseteq \mathscr{U}$.

For each $K \in \mathscr{A}$, since $\uparrow_{P_s(X)} K = \{G \in \mathscr{Q}_s^*(X) : G \subseteq K\}$, it follows that $\bigcap \mathscr{A} \in \bigcap_{K \in \mathscr{A}} \uparrow_{P_s(X)} K \subseteq \mathscr{U} = \bigcup_{i \in I} \Box U_i$. Then, there exists $i_0 \in I$ such that $\bigcap \mathscr{A} \in \Box U_{i_0}$, that is, $\bigcap \mathscr{A} \subseteq U_{i_0}$. By condition (4), there exists $K_0 \in \mathscr{A}$ such that $K_0 \subseteq U_{i_0}$. This implies that $\uparrow_{P_s(X)} K_0 \subseteq \Box U_{i_0} \subseteq \mathscr{U}$.

By Theorem 3.11, we deduce that $P_s(X)$ is $\mathscr{U}_s$-admitting. □

5. The Xi–Zhao dcpo model of $\mathscr{U}_s$-admitting spaces

For a poset P, we use $\mathrm{Max}_\sigma(P)$ to denote the space of the maximal points set $\mathrm{Max}(P)$ with the relative Scott topology on P. A *poset model* of a topological space X is a poset P such that $\mathrm{Max}_\sigma(P)$ is homeomorphic to X.

Observe that topological spaces having a poset model must be T_1. Conversely, Xi and Zhao [22; 23] proved that one can obtain all T_1 spaces from the Scott spaces of dcpos by taking the subspaces of all maximal points, as shown below.

Theorem 5.1 [22; 23]. *Every T_1 space X has a dcpo model, denoted by $D(X)$.*

The following are the main steps of constructing $D(X)$:

(i) Given a T_1 space X, let P be the set of all filters of open sets of X with nonempty intersection. With the inclusion order $\subseteq$, P is a bounded complete algebraic poset model of X.

(ii) From the above $(P, \leq_P)$, construct a dcpo $\widehat{P}$ as follows:

$$\widehat{P} = \{(x, e) : x \in P,\ e \in \mathrm{Max}(P) \text{ and } x \leq_P e\},$$

and $(x, e) \leq (y, d)$ in $\widehat{P}$ if and only if either $e = d$ and $x \leq_P y$, or $y = d$ and $x \leq_P d$.

(iii) Then, $\mathrm{Max}(\widehat{P}) = \{(e, e) : e \in \mathrm{Max}(P)\}$ and the mapping $h : \mathrm{Max}_\sigma(P) \to \mathrm{Max}_\sigma(\widehat{P})$, where $h(e) = (e, e)$ for each $e \in \mathrm{Max}(P)$, is a homeomorphism.

(iv) Thus, $D(X) = \widehat{P}$ is a dcpo model of X, called the Xi–Zhao model of X.

Remark 5.2 [23]. For a bounded complete algebraic poset P, if $\mathscr{D}$ is a directed subset of $\widehat{P}$ and it does not have a largest element, then there is $e \in \mathrm{Max}(P)$ and a directed subset $\{x_i : i \in I\}$ of P such that $\mathscr{D} = \{(x_i, e) : i \in I\}$, and in this case, $\bigvee \mathscr{D} = (\bigvee_{i \in I} x_i, e)$.

Recently, Chen and Li [1] provided an example of a $\mathscr{U}_s$-admitting T_1 space X such that $D(X)$ is not $\mathscr{U}_s$-admitting. Since every T_1 space is $\mathscr{U}_s$-admitting (see Proposition 3.3), their result can be restated as follows:

Proposition 5.3 [1]. *The Xi–Zhao dcpo model of a T_1 space need not be $\mathscr{U}_s$-admitting.*

This leads to a natural question:

For what spaces X are $D(X)$ $\mathscr{U}_s$-admitting?

We will prove that the $\mathscr{U}_s$-admitting property and well-filteredness are equivalent for the Scott space of $D(X)$.

Lemma 5.4. *For a bounded complete algebraic poset $(P, \leq_P)$, if $(x, e) \in \widehat{P}$ and $x \in K(P)$, then $\uparrow(x, e) \cap \mathrm{Max}(\widehat{P})$ is open in $\mathrm{Max}_\sigma(\widehat{P})$.*

Proof. Define $U = \{(y, d) \in \widehat{P} : x \leq_P y\}$. We first prove that U is Scott open in $\widehat{P}$. Suppose $(y_2, d_2) \geq (y_1, d_1) \in U$. Then, $x \leq_P y_1$, which implies that $(x, d_1) \leq (y_1, d_1) \leq (y_2, d_2)$. Then, from $(x, d_1) \leq (y_2, d_2)$ we deduce that $x \leq_P y_2$. This means $(y_2, d_2) \in U$. Hence, U is an upper set. Suppose $\mathscr{D}$ is a directed subset of $\widehat{P}$ such that $\bigvee \mathscr{D} \in U$. If $\mathscr{D}$ has a largest element, then trivially $\bigvee \mathscr{D} \in \mathscr{D} \cap U \neq \varnothing$. Otherwise, $\mathscr{D} = \{(x_i, e) : i \in I\}$, where $e \in \mathrm{Max}(P)$ and $\{x_i : i \in I\}$ is a directed subset of P. Then, $\bigvee \mathscr{D} = \left(\bigvee_{i \in I} x_i, e\right) \in U$, which implies that $x \leq_P \bigvee_{i \in I} x_i$. Since x is a compact element in P, there is

$i_0 \in I$ such that $x \leq_P x_{i_0}$. This means that $(x_{i_0}, e) \in \mathscr{D} \cap U \neq \varnothing$. All these show that U is a Scott open subset of $\widehat{P}$.

Now we claim that $\uparrow(x, e) \cap \mathrm{Max}(\widehat{P}) = U \cap \mathrm{Max}(\widehat{P})$. This is trivial since for each $(d, d) \in \mathrm{Max}(\widehat{P})$, $(d, d) \in U \iff x \leq_P d \iff (x, e) \leq (d, d)$.

All the above shows that $\uparrow(x, e) \cap \mathrm{Max}(\widehat{P})$ is open in $\mathrm{Max}_\sigma(\widehat{P})$. □

Lemma 5.5. *Let P be a bounded complete algebraic poset. Then, every compact set in $\mathrm{Max}_\sigma(\widehat{P})$ is strongly compact in $\Sigma\widehat{P}$.*

Proof. Suppose K is a compact set in $\mathrm{Max}_\sigma(\widehat{P})$, and U is an open set in $\Sigma\widehat{P}$ with $K \subseteq U$. Then, for each $(e, e) \in U \cap \mathrm{Max}(\widehat{P})$, since P is algebraic, $\downarrow e \cap K(P)$ is a directed subset of P whose supremum equals e. Thus, $\{(x, e) : x \in \downarrow e \cap K(P)\}$ is a directed subset of $\widehat{P}$ such that $\bigvee\{(x, e) : x \in \downarrow e \cap K(P)\} = (e, e) \in U$, so there is $x_e \in \downarrow e \cap K(P)$ such that $(x_e, e) \in U$. Thus, by Lemma 5.4,

$$\{\uparrow(x_e, e) \cap \mathrm{Max}(\widehat{P}) : (e, e) \in U \cap \mathrm{Max}(\widehat{P})\}$$

is a family of open sets in $\mathrm{Max}_\sigma(\widehat{P})$ that covers $U \cap \mathrm{Max}(\widehat{P})$, and hence covers K. Since K is compact in $\mathrm{Max}_\sigma(\widehat{P})$, there is a finite subset $\{(e_k, e_k) : 1 \leq k \leq n\} \subseteq U \cap \mathrm{Max}(\widehat{P})$ such that

$$K \subseteq \bigcup_{1 \leq k \leq n} \uparrow(x_{e_k}, e_k) \cap \mathrm{Max}(\widehat{P}),$$

which implies that

$$K \subseteq \uparrow\{(e_k, e_k) : 1 \leq k \leq n\} \subseteq U.$$

Thus, K is strongly compact in $\Sigma\widehat{P}$. □

Lemma 5.6. *For a bounded complete algebraic poset P, if $\widehat{P}$ is $\mathscr{U}_s$-admitting, then $\mathrm{Max}_\sigma(\widehat{P})$ is well-filtered.*

Proof. Suppose $\{K_i : i \in I\}$ is a filtered family of compact sets in $\mathrm{Max}_\sigma(\widehat{P})$ and U is an open set in $\mathrm{Max}_\sigma(\widehat{P})$ such that $\bigcap_{i \in I} K_i \subseteq U$. Then, there is a Scott open subset W of $\widehat{P}$ such that $U = W \cap \mathrm{Max}(\widehat{P})$. By Lemma 5.5, $\{K_i : i \in I\}$ is a filtered family of strongly compact saturated sets in $\Sigma\widehat{P}$ whose intersection $\bigcap_{i \in I} K_i$ is contained in $U \subseteq W$. Since $\Sigma\widehat{P}$ is $\mathscr{U}_s$-admitting, there is $i_0 \in I$ such that $K_{i_0} \subseteq W$. Note that $K_{i_0} \subseteq \mathrm{Max}(\widehat{P})$, so $K_{i_0} \subseteq W \cap \mathrm{Max}(\widehat{P}) = U$. Therefore, $\mathrm{Max}_\sigma(\widehat{P})$ is well-filtered. □

For a T_1 space X, applying Lemma 5.6 to the dcpo $D(X)$ we obtain the following result.

Theorem 5.7. *For a T_1 space X, if $D(X)$ is $\mathscr{U}_s$-admitting, then X is well-filtered.*

For a T_1 space X, it was shown by Xi and Zhao [17] that X is well-filtered if and only if $D(X)$ is well-filtered. Thus, by Theorem 5.7 we have the following.

Theorem 5.8. *For a T_1 space X, the following statements are equivalent*:

(1) *X is well-filtered.*

(2) *$D(X)$ is well-filtered.*

(3) *$D(X)$ is $\mathscr{U}_s$-admitting.*

6. Conclusion

Our results demonstrate that the class of $\mathscr{U}_s$-admitting spaces, whose definition makes use of more special compact saturated sets, enjoys many pleasant properties that are similar to those of well-filtered spaces and sober spaces. In particular, we proved that the category of all $\mathscr{U}_s$-spaces is reflective in the category of T_0 spaces. A T_0 space is $\mathscr{U}_s$-admitting if and only if its Smyth space is $\mathscr{U}_s$-admitting, providing a positive solution to Heckmann's problem. In summary, the $\mathscr{U}_s$-spaces form a well-behaved class that lies between the class of well-filtered spaces and that of d-spaces, and they might be used in the study of other non-Hausdorff topological properties.

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CHONG SHEN: shenchong0520@163.com
School of Science, Beijing University of Posts and Telecommunications, Beijing, China

QINGYU HE: qyhe@yzu.edu.cn
School of Mathematics Science, Yangzhou University, Yangzhou, China

XIAOYONG XI: xixy@yctu.edu.cn
School of Mathematics and Statistics, Yancheng Teachers University, Yancheng, China

DONGSHENG ZHAO: dongsheng.zhao@nie.edu.sg
Mathematics and Mathematics Education Department, Nanyang Technological University, Singapore, Singapore

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