Characterizations of Pointwise Pseudometrics via Pointwise Closed-Ball Systems

Chong Shen, Yi Shi, Fu-Gui Shi, and Hadrian Andradi

***Abstract*—Pointwise pseudoquasi-metrics play an important role in the theory of lattice-valued topology (L-topology). Bearing in mind that closed balls and their relations with pseudoquasi-metrics have historically attracted the attention of mathematicians, it is very surprising that no attention has been paid to the relations between pointwise pseudoquasi-metrics and closed balls. In this article, we first introduce the concept of pointwise closed-ball systems and prove that the resulting category is isomorphic to that of pointwise pseudoquasi-metrics. Subsequently, we study the topological properties of pointwise pseudoquasi-metrics via pointwise closed-ball systems. Interestingly, the L-topologies defined by open sets and complements of closed sets coincide for any pointwise pseudometric. Finally, we expose an important theoretical application of the pointwise closed-systems in providing a different and relatively simpler proof of the celebrated metrization theorem of the L-fuzzy real line.**

***Index Terms*—L-fuzzy number, L-fuzzy real line, L-topology, pointwise closed-ball system, pointwise pseudoquasi-metric.**

I. Introduction

METRIC, uniformity, and general topology are three closely related mathematical structures. The extension of general topology to the fuzzy setting (i.e., the well-known fuzzy topology) originates from Chang's work [1] with the unit interval as the truth value table. The notion of Chang's fuzzy topology was generalized to the L-fuzzy setting by Goguen [2], [3], which is now called L-topology. The work of finding an appropriate notion of uniformities in the framework of fuzzy topology goes back to Hutton [4]. He managed to establish the theory of uniformities on completely distributive lattices via a family of enlarging join-preserving maps. Later, Erceg [5] constructed the theory of fuzzy pseudoquasi-metrics by considering the Hausdorff distance function between two fuzzy sets (see [6], [7] for more results).

Notably, the aforementioned works cannot reflect the characteristics of L-topology clearly. To be specific, their works only considered the relations between L-fuzzy points and closed L-subsets, but ignore the relations between L-fuzzy points and their remote neighborhoods [8] (or quasi-neighborhoods [9]). Besides that, the L-topology induced by an Erceg's metric even fails to be first countable. In order to overcome these shortcomings, Shi introduced the concept of pointwise pseudometrics and showed that every pointwise pseudometric is uniquely determined by a family of remote neighborhood maps [10], [11]. The theory of pointwise pseudometrics successfully reveals the relations between L-fuzzy points and their remote neighborhoods: the set of the remote neighborhood maps of an L-fuzzy point forms one of its locally remote neighborhood base.

As a special type of fuzzy sets, the concept of fuzzy numbers was introduced by Hutton [4] and Gantner *et al.* [12] successively. Since then, plenty of scientific papers showed that fuzzy numbers have numerous practical applications, and play their important role in data analysis, decision making, and the orienteering problem [13]–[15]. Aside from their practical aspects, fuzzy numbers also have their theoretical aspects (e.g., the space of fuzzy numbers, fuzzy differentiation and integration, the construction of operators for fuzzy arithmetic, representations of fuzzy numbers, etc.). Among theoretical aspects of fuzzy numbers, the metrization of the L-fuzzy real line (the collection of fuzzy numbers) is one of the most intensively developed research directions in this field.

It is well known that the notions of quasi-neighborhood systems and remote-neighborhood systems have taken their main positions in L-topology. In fact, the notion of fuzzy neighborhood systems also have its advantages in L-topology, which is more natural and straightforward to characterize L-topologies [16]. Although the theory of pointwise pseudometrics is well established by making use of remote neighborhood maps in [10] and [11], the results therein are heavily dependent on the order-reversing involution of the truth value table, which cannot directly reflect the relations between pointwise pseudometrics and fuzzy neighborhood systems. To fill this gap, we study the fuzzy neighborhoods of L-fuzzy points in pointwise pseudoquasi-metric spaces. For this purpose, we introduce the notion of pointwise closed-ball systems, which is an important tool in this article. By making use of pointwise closed-ball systems, we successfully characterize the fuzzy neighborhoods of L-fuzzy points, without requiring the order-reversing involution

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Chong Shen is with the School of Mathematical Sciences, Nanjing Normal University, Nanjing 210046, China (e-mail: shenchong0520@163.com).

Yi Shi is with the Department of Mathematics, Shantou University, Shantou 515063, China (e-mail: shiyi7353@126.com).

Fu-Gui Shi is with the School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China (e-mail: fuguishi@bit.edu.cn).

Hadrian Andradi is with the Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta 55281, Indonesia (e-mail: hadrian.andradi@mail.ugm.ac.id).

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any more. In particular, by using the concept of pointwise closed-ball systems, we manage to establish a new approach to the metrization of the L-fuzzy real line. The methods in this article are simpler and relatively more straightforward comparing to those in the literature [10].

As a summary, we have three important aims in this article, which are as follows.

A1) To establish the categorical isomorphism between pointwise closed-ball systems and pointwise pseudoquasi-metrics.

A2) To study the topological properties of pointwise (pseudo)metrics in terms of pointwise closed-ball systems.

A3) To provide a new approach to metrization theorem of the L-fuzzy real line.

This article is structured as follows. We recall some preliminaries in Section II. In Section III, the notion of pointwise closed-ball systems is introduced and their links with pointwise pseudoquasi-metrics are revealed. Section IV provides some insights into the relations among L-topologies induced by pointwise pseudoquasi-metrics. We show that the L-fuzzy real line is metrizable by making use of pointwise closed-ball systems in Section V and Section VI concludes this article.

II. Preliminaries

In this section, we present the terminology and basic notions concerning completely distributive lattices, fuzzy sets, fuzzy topological spaces, and pointwise pseudoquasi-metric spaces. For the category theory, we will not list here, and the readers can refer to the literature [17], [18].

A. *Completely Distributive Lattices and Fuzzy Sets*

Let L be a complete lattice. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. A map $(-)': L \longrightarrow L$ is called an *order-reversing involution* if it satisfies: $(a')' = a$ for all $a \in L$; and $a \leqslant b$ implies $b' \leqslant a'$ for all $a, b \in L$. An element a in L is called *coprime* [19] if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$. The set of all nonzero coprime elements in L is denoted by $J(L)$. For $a, b \in L$, we say that a is wedge below b [20], in symbols $a \lhd b$, if for any subset $S \subseteq L$, the relation $b \leqslant \bigvee S$ always implies the existence of $c \in S$ with $a \leqslant c$. A complete lattice L is called *completely distributive* [21] if for any $a \in L$, it holds that $a = \bigvee\{b \in L \mid b \lhd a\}$.

Proposition II.1 (see [19]): Let L be a completely distributive lattice. Then, the following statements hold.

1) $a \lhd b$ implies $a \leqslant b$ for all $a, b \in L$.
2) $a \leqslant b \lhd c \leqslant d$ implies $a \lhd d$ for all $a, b, c, d \in L$.
3) $a \lhd b$ implies that there exists $c \in L$ such that $a \lhd c \lhd b$.
4) $a \lhd \bigvee\{b_j \in L \mid j \in J\}$ implies $a \lhd b_j$ for some $j \in J$.

Throughout the remainder of this article, unless otherwise stated, X always denotes a nonempty set, and L always denotes a completely distributive lattice.

The notation L^X denotes the set of all maps from X to L and every element in L^X is called an *L-subset* of X. The smallest element and the largest element in L^X are denoted by $\perp_X$ and $\top_X$, respectively. The partial order on L can be translated onto L^X in a pointwise way. In this case, L^X is also a completely distributive lattice. The set of all nonzero coprime elements in L^X is denoted by $J(L^X)$. Each element in $J(L^X)$ is called an *L-fuzzy point* or a *fuzzy point* for abbreviation.

We should note that a fuzzy point a is of the form x_λ, where $x \in X$, $\lambda \in J(L)$, and x_λ is defined by

$$\forall y \in X,\ x_\lambda(y) = \begin{cases} \lambda, & y = x \\ \perp, & y \neq x \end{cases}.$$

Let X and Y be two nonempty sets. Given a map $f: X \longrightarrow Y$, define $f_L^{\rightarrow}: L^X \longrightarrow L^Y$ and $f_L^{\leftarrow}: L^Y \longrightarrow L^X$ as follows:

$$f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) \mid f(x) = y\}$$

$$f_L^{\leftarrow}(B)(x) = B(f(x))$$

for any $x \in X, y \in Y, A \in L^X$, and $B \in L^Y$. For convenience, we write $f(a)$ for $f_L^{\rightarrow}(a) \in J(L^Y)$ whenever $a \in J(L^X)$. One can easily check that $a \in J(L^X)$ implies $f(a) \in J(L^Y)$.

B. *Fuzzy Topological Spaces*

An *L-topological space* is a pair (X, τ), where X is a set and τ is a subset of L^X, which contains $\perp_X, \top_X$ and is stable under arbitrary suprema and finite infima. Each member of τ is called an *open L-subset*. Suppose that L is a completely distributive lattice with an order-reversing involution $'$. For an L-topological space (X, τ), we say that an L-subset U is closed if $U' \in \tau$. For $A \in L^X$, $\mathrm{int}(A)$ or A° denotes the interior of A, and $\mathrm{cl}(A)$ or A^- denotes the closure of A. An *L-closure operator* on a set X is a map $\mathrm{cl}: L^X \longrightarrow L^X$ such that the following conditions hold.

LC1) $\mathrm{cl}(\perp_X) = \perp_X$.
LC2) $A \leqslant \mathrm{cl}(A)$ for all $A \in L^X$.
LC3) $\mathrm{cl}(A \vee B) = \mathrm{cl}(A) \vee \mathrm{cl}(B)$ for all $A, B \in L^X$.
LC4) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

The L-interior operator int has the dual form of (LC1)–(LC4).

Definition II.2 (see [16]): An *L-neighborhood system* on a set X is a set

$$\mathcal{N} = \left\{\mathcal{N}_a \subseteq L^X \middle| a \in J\left(L^X\right)\right\}$$

satisfying the following conditions.

CBS1) $\top_X \in \mathcal{N}_a, \perp_X \notin \mathcal{N}_a$.
CBS2) $U \notin \mathcal{N}_a$ for all $a \nleqslant U$.
CBS3) if $U \in \mathcal{N}_a$ and $U \leqslant V$, then $V \in \mathcal{N}_a$.
CBS4) if $U, V \in \mathcal{N}_a$, then $U \wedge V \in \mathcal{N}_a$.
CBS5) for any $U \in \mathcal{N}_a$, there exists $V \in L^X$ such that $a \leqslant V \leqslant U$ and $V \in \mathcal{N}_b$ for any $b \lhd V$.

The relations among L-topologies, L-closure operators, L-interior operators, and L-neighborhood systems are bijective. This is shown as follows.

Theorem II.3 (see [16]): Let (X, τ) be an L-topological space. For any $a \in J(L^X)$, define a subset $\mathcal{N}_a^\tau$ of L^X by

$$\mathcal{N}_a^\tau = \left\{A \in L^X \mid \exists U \in \tau, a \leqslant U \leqslant A\right\}.$$

Then, $\{\mathcal{N}_a^\tau \subseteq L^X \mid a \in J(L^X)\}$ is an L-neighborhood system on X.

Theorem II.4 (see [16]): Let $\mathcal{N} = \{\mathcal{N}_a \subseteq L^X \mid a \in J(L^X)\}$ be an L-neighborhood system on X. Define a subset $\tau^{\mathcal{N}}$ of L^X by

$$\tau^{\mathcal{N}} = \left\{A \in L^X \mid \quad \forall a \lhd A, A \in \mathcal{N}_a\right\}.$$

Then, $\tau^{\mathcal{N}}$ is an L-topology on X and $\mathcal{N}^{(\tau^{\mathcal{N}})} = \mathcal{N}$.

Theorem II.5 (see [16]): Let $\mathcal{N} = \{\mathcal{N}_a \subseteq L^X \mid a \in J(L^X)\}$ be an L-neighborhood system on X. Define a map $\mathrm{int}^{\mathcal{N}} : L^X \longrightarrow L^X$ by

$$\mathrm{int}^{\mathcal{N}}(A) = \bigvee \left\{a \in J\left(L^X\right) \mid A \in \mathcal{N}_a\right\}.$$

Then, $\mathrm{int}^{\mathcal{N}}$ is an L-interior operator on X.

Theorem II.6 (see [16]): Let $\mathrm{int} : L^X \longrightarrow L^X$ be an L-interior operator on X. For any $a \in J(L^X)$, define a subset $\mathcal{N}_a^{\mathrm{int}}$ of L^X by

$$\mathcal{N}_a^{\mathrm{int}} = \left\{A \in L^X \mid a \leqslant \mathrm{int}(A)\right\}.$$

Then, $\mathcal{N}^{\mathrm{int}} = \{\mathcal{N}_a^{\mathrm{int}} \mid a \in J(L^X)\}$ is an L-neighborhood system and $\mathcal{N}^{(\mathrm{int}^{\mathcal{N}})} = \mathcal{N}$.

C. Pointwise Pseudoquasi-Metrics

Throughout this article, let

$\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = (0, +\infty)$.

Definition II.7 (see [10] and [11]): A *pointwise pseudoquasi-metric* (briefly, *pointwise p.q.-metric*) on X is a map $d : J(L^X) \times J(L^X) \longrightarrow [0, +\infty)$ such that for any $a, b, c \in J(L^X)$, the following conditions hold.

M1) $d(a, a) = 0$.

M2) $d(a, c) \leqslant d(a, b) + d(b, c)$.

M3) $d(a, b) = \bigwedge_{c \lhd b} d(a, c)$.

The pair (X, d) is called a *pointwise p.q.-metric space*.

A map $f : X \longrightarrow Y$ between fuzzy pointwise p.q.-metric spaces (X, d_1) and (Y, d_2) is called a *contraction* if

$$\forall a, b \in J\left(L^X\right), \quad d(f(a), f(b)) \leqslant d(a, b).$$

It is easy to check that all pointwise p.q.-metric spaces and their contractions form a category, denoted by *PQ-Met*.

Proposition II.8 (see [11] and [22]): Let (X, d) be a pointwise p.q.-metric space. Then, the following statements hold.

M1*) $\forall a, b \in J(L^X)$, $a \leqslant b$ implies $d(a, b) = 0$.

M3*) $\forall a, b \in J(L^X)$, $d(a, b) = \bigvee_{c \lhd a} d(c, b)$.

M0) $\forall a, b, c \in J(L^X)$, $a \leqslant b$ implies $d(a, c) \leqslant d(b, c)$.

M0*) $\forall a, b, c \in J(L^X)$, $b \geqslant c$ implies $d(a, b) \leqslant d(a, c)$.

Definition II.9 (see [10]): Let (X, d) be a pointwise p.q.-metric space. For any $r \in \mathbb{R}^+$, a *remote-neighborhood map* (briefly, *R-nbd map*) induced by d is a map $P_r^d : J(L^X) \longrightarrow L^X$ defined by

$$\forall a \in J\left(L^X\right), \; P_r^d(a) = \bigvee \left\{b \in J\left(L^X\right) \,\middle|\, d(a, b) \geqslant r\right\}.$$

The family of R-nbd maps induced by d is denoted by $\mathcal{P}^d$.

Lemma II.10 (see [10]): Let (X, d) be a pointwise p.q.-metric space. Then, for any $r \in \mathbb{R}^+$ and $a, b \in J(L^X)$

$$b \leqslant P_r^d(a) \Leftrightarrow d(a, b) \geqslant r.$$

Theorem II.11 (see [10]): Let (X, d) be a pointwise p.q.-metric space. Then, $\mathcal{P}^d = \{P_r^d(a) \mid a \in J(L^X), r \in \mathbb{R}^+\}$ is a base for an L-cotopology on X, and the L-cotopology is denoted by $\eta^{\mathcal{P}^d}$.

Theorem II.12 (see [10]): If d is a pointwise p.-metric on X, then $(\eta^{\mathcal{P}^d})' = \tau^{\mathcal{P}^d}$.

Theorem II.13 (see [10]): If d is a pointwise p.q.-metric on X, then for each $A \in L^X$, its closure with respect to $(X, \tau^{\mathcal{P}^d})$ is given as follows:

$$A^- = \bigvee \left\{a \in J(L^X) \,\middle|\, d(a, A) = \textstyle\bigwedge_{c \leqslant A} d(a, c) = 0\right\}.$$

III. Closed-Ball Systems in Pointwise Pseudoquasi-Metric Spaces

In this section, we consider the closed balls in pointwise pseudoquasi-metric spaces and show that every pointwise pseudoquasi-metric can be characterized by a family of closed balls. This result provides a new and relatively direct approach to study the topologies of a pointwise pseudoquasi-metric.

A *pseudoquasi-metric* (also called *hemi-metric*) ρ on X is a map from $X \times X$ to $[0, +\infty)$ such that the following conditions hold.

1) $\rho(x, x) = 0$ for every $x \in X$.

2) $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

The pair (X, ρ) is called a *pseudoquasi metric space.*

In the classical case, closed balls play a crucial role in pseudoquasi-metric spaces since the topology induced by a metric (even the metric itself) can be completely described by the closed balls. More precisely, U is open in a metric space (X, ρ) if and only if for each $x \in U$, there exists $r \in \mathbb{R}^+$ such that $B_r^\rho(x) \subseteq U$, where $B_r^\rho(x) = \{y \in X \mid \rho(x, y) \leqslant r\}$. In what follows, we focus more on the family of closed balls in the framework of pointwise pseudoquasi-metrics.

Definition III.1: A *pointwise closed-ball system* on X is defined to be a set $\mathcal{B} = \{B_r \mid r \in \mathbb{R}^+\}$ of maps $B_r : J(L^X) \longrightarrow L^X$ fulfilling the following conditions.

CB1) $\forall a, b \in J(L^X), \exists r \in \mathbb{R}^+, b \leqslant B_r(a)$.

CB2) $\forall a \in J(L^X) \quad \forall r \in \mathbb{R}^+, a \leqslant B_r(a)$.

CB3) $\forall r, s \in \mathbb{R}^+, \quad B_s \circledcirc B_r \leqslant B_{r+s}, \quad$ where $(B_s \circledcirc B_r)(a) = \bigvee\{B_s(b) \mid b \leqslant B_r(a)\}$ for all $a \in J(L^X)$.

CB4) $\forall a \in J(L^X) \quad \forall r \in \mathbb{R}^+, B_r(a) = \bigwedge_{s > r} B_s(a)$.

CB5) $\forall a, b \in J(L^X) \quad \forall r \in \mathbb{R}^+$

$$b \leqslant B_r(a) \Rightarrow \quad \forall s > r, \exists c \lhd a, b \leqslant B_s(c).$$

The pair $(X, \mathcal{B})$ is called a *pointwise closed-ball space*.

A map $f : X \longrightarrow Y$ between two pointwise closed-ball spaces $(X, \mathcal{B}_1)$ and $(Y, \mathcal{B}_2)$ is called *continuous* if for any $a \in J(L^X)$ and $r \in \mathbb{R}^+$, it holds that $f_L^{\rightarrow}(B_r(a)) \leqslant B_r(f(a))$. It is easy to check that all pointwise closed-ball systems and their continuous maps form a category, denoted by *F-CBS*.

Proposition III.2: Let $(X, \mathcal{B})$ be a pointwise closed-ball space. Then, for any $a, b \in J(L^X)$ and $r, s \in \mathbb{R}^+$, the following statements hold.

1) $r \leqslant s$ implies $B_r(a) \leqslant B_s(a)$.

2) $a \leqslant b$ implies $B_r(a) \leqslant B_r(b)$.

3) $B_r(a) = \bigwedge_{s > r} \bigvee_{c \lhd a} B_s(c)$.

Proof:

1) It is key to show that $r < s$ implies $B_r(a) \leqslant B_s(a)$. Since $s - r > 0$, it follows from (CB2) that $a \leqslant B_{s-r}(a)$. By (CB3), we have

$$\begin{aligned} B_r(a) &\leqslant \bigvee \{B_r(b) \mid b \leqslant B_{s-r}(a)\} \\ &= (B_r \circledcirc B_{s-r})(a) \\ &\leqslant B_s(a) \end{aligned}$$

completing the proof.

2) By (CB2), for any $s > r$, we have $b \leqslant B_{s-r}(b)$. Since $a \leqslant b$, it follows from (CB3) that

$$\begin{aligned} B_r(a) &\leqslant \bigvee \{B_r(e) \mid e \leqslant B_{s-r}(b)\} \\ &= (B_r \circledcirc B_{s-r})(b) \\ &\leqslant B_s(b). \end{aligned}$$

From (CB4), we then have $B_r(a) \leqslant \bigwedge_{s>r} B_s(b) = B_r(b)$.

3) It is immediate from Proposition III.2(1), (2) and (CB5). ■

Theorem III.3: Let d be a pointwise p.q.-metric on X. For any $r \in \mathbb{R}^+$, define a map $B_r^d : J(L^X) \longrightarrow L^X$ by

$$\forall a \in J\left(L^X\right),\ B_r^d(a) = \bigvee \left\{b \in J\left(L^X\right) \middle| d(b,a) \leqslant r\right\}.$$

Then, the family $\mathcal{B}^d = \{B_r^d \mid r \in \mathbb{R}^+\}$ is a pointwise closed-ball system on X.

Before proving Theorem III.3, we first give a useful lemma.

Lemma III.4: Let d be a pointwise p.q.-metric on X. Then, for any $r \in \mathbb{R}^+$ and $a, b \in J(L^X)$

$$b \leqslant B_r^d(a) \Leftrightarrow d(b,a) \leqslant r.$$

Proof: Let $a, b \in J(L^X)$ and $r \in \mathbb{R}^+$ such that $b \leqslant B_r^d(a)$. For any $c \in J(L^X)$, if $c \lhd b$, then

$$c \lhd B_r^d(a) = \bigvee \left\{e \in J\left(L^X\right) \middle| d(e,a) \leqslant r\right\}.$$

It follows that there exists $e \in J(L^X)$ such that $c \leqslant e$ and $d(e,a) \leqslant r$. By Proposition II.8(M0), we have $d(c,a) \leqslant d(e,a) \leqslant r$. From (M3*), we get $d(b,a) = \bigvee_{c \lhd b} d(c,a) \leqslant r$. The converse holds trivially. ■

We now prove Theorem III.3.

Proof: To check that $\mathcal{B}^d$ satisfies (CB1)–(CB5).

CB1) For any $a, b \in J(L^X)$, let $r = d(b,a) + 1$. Then, $r \in \mathbb{R}^+$ and one can easily obtain that $b \leqslant B_r^d(a)$ by using Lemma III.4.

CB2) Suppose that there exist $r \in \mathbb{R}^+$ and $a \in J(L^X)$ such that $a \not\leqslant B_r^d(a)$. By Lemma III.4, it holds that $d(a,a) > r > 0$, a contradiction.

CB3) For any $a, c \in J(L^X)$, if $c \lhd (B_s^d \circledcirc B_r^d)(a)$, i.e., $c \lhd \bigvee\{B_s^d(b) \mid b \leqslant B_r^d(a)\}$, then there exists $b \leqslant B_r^d(a)$ such that $c \leqslant B_s^d(b)$. Using Lemma III.4, we have $d(b,a) \leqslant r$ and $d(c,b) \leqslant s$. It follows from (M2) that

$$d(c,a) \leqslant d(c,b) + d(b,a) \leqslant s + r$$

and thus, $c \leqslant B_{r+s}^d(a)$ by using Lemma III.4 again. Since both a and c are arbitrary, we have $B_s^d \circledcirc B_r^d \leqslant B_{r+s}^d$.

CB4) For any $a, b \in J(L^X)$ and $r \in \mathbb{R}^+$, it follows from Lemma III.4 that:

$$\begin{aligned} b \leqslant \bigwedge_{s>r} B_s^d(a) &\Leftrightarrow \quad \forall s > r,\ b \leqslant B_s^d(a) \\ &\Leftrightarrow \quad \forall s > r,\ d(b,a) \leqslant s \\ &\Leftrightarrow d(b,a) \leqslant r \\ &\Leftrightarrow b \leqslant B_r^d(a). \end{aligned}$$

By the arbitrariness of b, we have $B_r^d(a) = \bigwedge_{s>r} B_s^d(a)$.

CB5) Let $a, b \in J(L^X)$ and $r \in \mathbb{R}^+$ such that $b \leqslant B_r^d(a)$. It follows from Lemma III.4 that $d(b,a) \leqslant r$. Then, for any $s > r$, $d(b,a) < s$. From (M3), we get $d(b,a) = \bigwedge_{c \lhd a} d(b,c) < s$, which implies that there exists $c \lhd a$ such that $d(b,c) < s$, and thus, $d(b,c) \leqslant s$. Therefore, it holds that $b \leqslant B_s(c)$. ■

Proposition III.5: If $f : X \longrightarrow Y$ is a contraction between pointwise p.q.-metric spaces (X, d_1) and (Y, d_2), then f is a continuous map between the pointwise closed-ball spaces $(X, \mathcal{B}^{d_1})$ and $(Y, \mathcal{B}^{d_2})$.

Proof: For any $a, b \in J(L^X)$ and $r \in \mathbb{R}^+$, it follows from the contraction of f that $d_2(f(b), f(a)) \leqslant d_1(b,a)$. It holds that

$$\begin{aligned} & f_L^{\rightarrow}((B^{d_1})_r(a)) \\ = \ & f_L^{\rightarrow}\left(\bigvee\left\{b \in J\left(L^X\right) \mid d_1(b,a) \leqslant r\right\}\right) \\ = \ & \bigvee\{f(b) \mid d_1(b,a) \leqslant r\} \\ \leqslant \ & \bigvee\left\{f(b) \mid b \in J\left(L^X\right), d_2\left(f(b), f(a)\right) \leqslant r\right\} \\ \leqslant \ & \bigvee\left\{e \in J\left(L^Y\right) \mid d_2\left(e, f(a)\right) \leqslant r\right\} \\ = \ & (B^{d_2})_r(f(a)). \end{aligned}$$

Hence, f is continuous. ■

Theorem III.6: Let $(X, \mathcal{B})$ be a pointwise closed-ball space. Define a map $d^{\mathcal{B}} : J(L^X) \times J(L^X) \longrightarrow [0, +\infty)$ by

$$\forall a, b \in J\left(L^X\right),\ d^{\mathcal{B}}(a,b) = \bigwedge\{r \in \mathbb{R}^+ \mid a \leqslant B_r(b)\}.$$

Then, $d^{\mathcal{B}}$ is a pointwise p.q.-metric on X.

Proof: We prove it in following three steps.

Step 1: To show $\forall a, b \in J(L^X)\quad \forall s \in \mathbb{R}^+$

$$d^{\mathcal{B}}(a,b) \leqslant s \Leftrightarrow a \leqslant B_s(b).$$

Suppose that $d^{\mathcal{B}}(a,b) \leqslant s$. For any $t > s$, it holds that

$$d^{\mathcal{B}}(a,b) = \bigwedge\{r \in \mathbb{R}^+ \mid a \leqslant B_r(b)\} < t$$

which implies that there exists $r \in \mathbb{R}^+$ such that $r < t$ and $a \leqslant B_r(b)$. It follows from Proposition III.2(1) that $a \leqslant B_t(b)$. From (CB4), we get $a \leqslant \bigwedge_{t>s} B_t(b) = B_s(b)$. The reverse implication is obvious.

Step 2: To show $\forall a, b \in J(L^X)$, $d^{\mathcal{B}}(a,b) \in [0, +\infty)$.

By (CB2), we first have

$$\begin{aligned} d^{\mathcal{B}}(a,a) &= \bigwedge\{r \in \mathbb{R}^+ \mid a \leqslant B_r(a)\} \\ &= \bigwedge \mathbb{R}^+ \\ &= 0 \\ &\leqslant d^{\mathcal{B}}(a,b). \end{aligned}$$

Moreover, it follows from (CB1) that there exists $r \in \mathbb{R}^+$ such that $b \leqslant B_r(a)$. Thus, $d^{\mathcal{B}}(a,b) \leqslant r$, and then $d^{\mathcal{B}}(a,b) \in [0,+\infty)$.

Step 3: To show $d^{\mathcal{B}}$ is a pointwise p.q.-metric on X.

M1) For any $a \in J(L^X)$, it follows from (CB2) that

$$d^{\mathcal{B}}(a,a) = \bigwedge\{r \in \mathbb{R}^+ \mid a \leqslant B_r(a)\} = \bigwedge \mathbb{R}^+ = 0.$$

M2) Suppose that $d^{\mathcal{B}}(a,c) = s$ and $d^{\mathcal{B}}(c,b) = t$. From Step 1, we obtain that $a \leqslant B_s(c)$ and $c \leqslant B_t(b)$. Hence, it holds that $a \leqslant \bigvee\{B_s(e) \mid e \leqslant B_t(b)\} = (B_s \circledcirc B_t)(b)$. By (CB3), we have $a \leqslant B_{t+s}(b)$, implying that $d^{\mathcal{B}}(a,b) \leqslant t+s$. Therefore, $d^{\mathcal{B}}(a,b) \leqslant d^{\mathcal{B}}(a,c) + d^{\mathcal{B}}(c,b)$.

M3) For any $a,b \in J(L^X)$ and $r \in \mathbb{R}^+$, if $c \lhd b$, then $B_r(c) \leqslant B_r(b)$ by Proposition III.2(2). It holds that

$$\begin{aligned} d^{\mathcal{B}}(a,b) &= \bigwedge\{r \in \mathbb{R}^+ \mid a \leqslant B_r(b)\} \\ &\leqslant \bigwedge\{r \in \mathbb{R}^+ \mid a \leqslant B_r(c)\} \\ &= d^{\mathcal{B}}(a,c). \end{aligned}$$

Hence, $d^{\mathcal{B}}(a,b) \leqslant \bigwedge_{c \lhd b} d^{\mathcal{B}}(a,c)$. Next, we need to show that $\bigwedge_{c\lhd b} d^{\mathcal{B}}(a,c) \leqslant d^{\mathcal{B}}(a,b)$. Let $d^{\mathcal{B}}(a,b) = t$. Then, $a \leqslant B_t(b)$. By (CB5), we can obtain that for any $s > t$, there exists $c_s \lhd b$ such that $a \leqslant B_s(c_s)$, and then $d^{\mathcal{B}}(a,c_s) \leqslant s$. It holds that

$$\bigwedge_{c\lhd b} d^{\mathcal{B}}(a,c) \leqslant \bigwedge_{s>t} d^{\mathcal{B}}(a,c_s) \leqslant \bigwedge_{s>t} s = t = d^{\mathcal{B}}(a,b)$$

completing the proof. ∎

Proposition III.7: If $f : X \longrightarrow Y$ is a continuous map between pointwise closed-ball spaces $(X,\mathcal{B}_1)$ and $(Y,\mathcal{B}_2)$, then f is a contraction between the pointwise p.q.-metric spaces $(X, d^{\mathcal{B}_1})$ and $(Y, d^{\mathcal{B}_2})$.

Proof: Let $d^{\mathcal{B}_1}(a,b) = t$. Then, for any $s > t$, $d^{\mathcal{B}_1}(a,b) \leqslant s$, which implies that $a \leqslant (B_1)_s(b)$ by Step 1 in the proof of Theorem III.6. Since f is continuous, it follows that:

$$f(a) \leqslant f_L^{\rightarrow}((B_1)_s(b)) \leqslant (B_2)_s(f(b)).$$

Hence, $d^{\mathcal{B}_2}(f(a), f(b) \leqslant s$. It holds that

$$d^{\mathcal{B}_2}(f(a), f(b)) \leqslant \bigwedge_{s>t} s = t = d^{\mathcal{B}_1}(a,b).$$

Therefore, f is a contraction map. ∎

By Lemma III.4 and Step 1 in the proof of Theorem III.6, the following proposition is straightforward.

Proposition III.8:

1) If $\mathcal{B}$ is a pointwise closed-ball system on X, then $\mathcal{B}^{d^{\mathcal{B}}} = \mathcal{B}$.
2) If d is a pointwise p.q.-metric on X, then $d = d^{\mathcal{B}^d}$.

By Propositions III.5, III.7, and III.8, we further obtain the following result.

Theorem III.9: The category F-CBS is isomorphic to PQ-Met.

IV. Topological Properties of Pointwise Pseudoquasi-Metrics

In this section, we mainly study the topological properties of pointwise pseudoquasi-metrics by making use of pointwise closed-ball systems.

Theorem IV.1: Let d be a pointwise p.q.-metric on X. For every $a \in J(L^X)$, define a subset $\mathcal{N}_a^d$ of L^X by

$$\mathcal{N}_a^d = \left\{A \in L^X \mid \exists r \in \mathbb{R}^+, B_r^d(a) \leqslant A\right\}.$$

The family $\mathcal{N}^d = \{\mathcal{N}_a^d \mid a \in J(L^X)\}$ forms an L-neighborhood system on X. Hence, it can induce an L-topology on X, denoted by $\tau^{\mathcal{N}^d}$.

Proof: Verify that $\mathcal{N}^d$ satisfies (CBS1)–(CBS5).

(CBS1)–(CBS3) They are trivial.

(CBS4) If $U,V \in \mathcal{N}_a^d$, then there exist $r,s \in \mathbb{R}^+$ such that $B_r^d(a) \leqslant U$ and $B_s^d(a) \leqslant V$. Let $t = \min\{r,s\}$. By Proposition III.2(1), we have $B_t^d(a) \leqslant B_r^d(a) \wedge B_s^d(a) \leqslant U \wedge V$. Thus, $U \wedge V \in \mathcal{N}_a^d$.

(CBS5) If $U \in \mathcal{N}_a^d$, then there exists $r \in \mathbb{R}^+$ such that $B_r^d(a) \leqslant U$. Let

$$V = \bigvee\left\{B^d_{\frac{s}{2}}(e) \,\middle|\, B_s^d(e) \leqslant B_r^d(a)\right\}.$$

Obviously, $a \leqslant V \leqslant U$. Now, let $b \in J(L^X)$ with $b \lhd V$. Then, there exist $l \in \mathbb{R}^+$ and $c \in J(L^X)$ such that

$$b \leqslant B^d_{l/2}(c) \leqslant B_l^d(c) \leqslant B_r^d(a).$$

It holds that

$$\begin{aligned} B^d_{\frac{l}{2}}(b) &\leqslant \bigvee\left\{B^d_{\frac{l}{2}}(u) \mid u \leqslant B^d_{\frac{l}{2}}(c)\right\} \\ &= \left(B^d_{\frac{l}{2}} \circledcirc B^d_{\frac{l}{2}}\right)(c) \\ &\leqslant B_l^d(c) \qquad \text{[by (CB3)]} \\ &\leqslant B_r^d(a). \end{aligned}$$

Hence, $B^d_{l/4}(b) \leqslant V$ and thus, $V \in \mathcal{N}_b^d$. ∎

From Theorems II.4 and IV.1, we have

$$\begin{aligned} \tau^{\mathcal{N}^d} &= \left\{A \in L^X \mid \quad \forall a \lhd A, A \in \mathcal{N}_a^d\right\} \\ &= \left\{A \in L^X \mid \quad \forall a \lhd A, \exists r \in \mathbb{R}^+,\ B_r^d(a) \leqslant A\right\}. \end{aligned}$$

By Theorems II.4, II.5, and II.6, we deduce the following corollaries.

Corollary IV.2: Let d be a pointwise p.q.-metric on X. For any $A \in L^X$, define a map $\mathrm{int}^d : L^X \longrightarrow L^X$ by

$$\mathrm{int}^d(A) = \bigvee\left\{a \in J\left(L^X\right) \mid \exists r \in \mathbb{R}^+, B_r^d(a) \leqslant A\right\}.$$

Then, int^d is an L-interior operator on X. Hence, it can induce an L-topology on X, denoted by τ^{int^d}.

Corollary IV.3: Let d be a pointwise p.q.-metric on X and define a subset τ^d of L^X by

$$\tau^d = \left\{A \in L^X \mid \quad \forall a \lhd A, \exists r \in \mathbb{R}^+, B_r^d(a) \leqslant A\right\}.$$

Then, τ^d is an L-topology on X.

Corollary IV.4: If (X,d) is a pointwise p.q.-metric space, then $\tau^d = \tau^{\mathcal{N}^d} = \tau^{\mathrm{int}^d}$.

The following theorem provides a direct way to construct an open set in (X,τ^d) via a certain function.

Theorem IV.5: Let d be a pointwise p.q.-metric on X. For any $r\in\mathbb{R}^+$, define a map $U_r^d : J(L^X)\longrightarrow L^X$ by

$$\forall a\in J\left(L^X\right),\ U_r^d(a)=\bigvee\left\{b\in J\left(L^X\right)\,\middle|\, d(b,a)<r\right\}.$$

Then, the following statements hold.

1) $c\lhd U_r^d(a)$ implies $d(c,a)<r$.
2) $U_r^d(a)$ is an open L-subset in (X,τ^d).

Proof: (1) If $c\lhd U_r^d(a)$, then there exists $b\geqslant c$ such that $d(b,a)<r$. It follows from (M0) that $d(c,a)\leqslant d(b,a)<r$.

(2) By Theorem IV.1 and Corollary IV.3, it suffices to show that the following statement holds:

$$\forall c\lhd U_r^d(a),\ \exists s\in\mathbb{R}^+,\ B_s^d(c)\leqslant U_r^d(a).$$

Let $c\in J(L^X)$ such that $c\lhd U_r^d(a)$. Then, $d(c,a)<r$. Let $s=(r-d(c,a))/2$ and $e\leqslant B_s^d(c)$. It holds that $d(e,c)\leqslant(r-d(c,a))/2$. Since

$$\begin{aligned} d(e,a)&\leqslant d(e,c)+d(c,a)\\ &\leqslant \frac{r-d(c,a)}{2}+d(c,a)\\ &=\frac{r+d(c,a)}{2}\\ &<r \end{aligned}$$

it follows that $e\leqslant U_r^d(a)$. By the arbitrariness of e, we get $B_s^d(c)\leqslant U_r^d(a)$. ■

Definition IV.6: Let d be a pointwise p.q.-metric on X and define a map $\mathrm{cl}^d : L^X\longrightarrow L^X$ by

$$\mathrm{cl}^d(A)=\bigvee\left\{a\in J\left(L^X\right)\,\middle|\, d(a,A)=\bigwedge_{c\leqslant A} d(a,c)=0\right\}.$$

Let us give a lemma first.

Lemma IV.7: Let d be a pointwise p.q.-metric on X. For any $a\in J(L^X)$ and $A\in L^X$, consider the followings.

1) $a\leqslant \mathrm{cl}^d(A)$.
2) $d(a,A)=\bigwedge_{c\leqslant A} d(a,c)=0$.
3) $a\leqslant \bigwedge_{r>0}\bigvee_{e\leqslant A} B_r^d(e)$.
4) $a\lhd \bigwedge_{r>0}\bigvee_{e\leqslant A} B_r^d(e)$.

Then, (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (2).

Proof: (1) $\Rightarrow$ (2) Suppose $a\leqslant \mathrm{cl}^d(A)$. Then, for any $e\lhd a$, there exists $b_x\geqslant x$ such that $d(b_x,A)=\bigwedge_{c\leqslant A} d(b_x,c)=0$. Then, for any $s\in\mathbb{R}^+$, there always exists $c_s\leqslant A$ such that $d(b_x,c_s)<s$. Since

$$\begin{aligned} d(a,c_s)&=d(a,b_x)+d(b_x,c_s) && \text{[by (M2)]}\\ &<d(a,b_x)+s\\ &\leqslant d(a,x)+s && \text{[by (M0*)]} \end{aligned}$$

it follows that

$$d(a,c_s)\leqslant\bigwedge_{x\lhd a} d(a,x)+s=d(a,a)+s=s.$$

Hence, $\bigwedge_{c\leqslant A} d(a,c)\leqslant d(a,c_s)\leqslant s$. Therefore, $d(a,A)=\bigwedge_{c\leqslant A} d(a,c)=0$.

(2) $\Rightarrow$ (1) It is trivial.

(2) $\Rightarrow$ (3) Since $\bigwedge_{c\leqslant A} d(a,c)=0$, it follows that for any $r\in\mathbb{R}^+$, there exists $c\leqslant A$ such that $d(a,c)\leqslant r$, implying that $a\leqslant B_r^d(c)$. Hence, $a\leqslant\bigwedge_{r>0}\bigvee_{e\leqslant A} B_r^d(e)$.

(4) $\Rightarrow$ (2) If $a\lhd\bigwedge_{r>0}\bigvee_{e\leqslant A} B_r^d(e)$, then for any $r>0$, there exists $e_r\leqslant A$ such that $a\leqslant B_r^d(e_r)$, implying that $d(a,e_r)\leqslant r$. Hence, $\bigwedge_{r>0} d(a,e_r)=0$. Therefore

$$\bigwedge_{e\leqslant A} d(a,e)\leqslant\bigwedge_{r>0} d(a,e_r)=0$$

completing the proof. ■

From Lemma IV.7, we can obtain the following corollary. This result will bring great convenience to discuss the relations between the L-topology τ^{int^d} and the L-cotopology η^{cl^d}.

Corollary IV.8: Let d be a pointwise p.q.-metric on X. Then

$$\mathrm{cl}^d(A)=\bigwedge_{r>0}\bigvee_{e\leqslant A} B_r^d(e).$$

Theorem IV.9: Let d be a pointwise p.q.-metric on X. Then, cl^d is an L-closure operator on X. Hence, it can induce an L-cotopology on X, denoted by η^{cl^d}.

Proof: It suffices to show that cl^d satisfies (LC1)–(LC4).

LC1) It is valid since $\bigvee\emptyset=\bot_X$.

LC2) It is valid since $d(a,a)=0$.

LC3) We first note that cl^d is order-preserving, hence, $\mathrm{cl}^d(A\vee B)\geqslant \mathrm{cl}^d(A)\vee\mathrm{cl}^d(B)$. It remains to show that $\mathrm{cl}^d(A\vee B)\leqslant \mathrm{cl}^d(A)\vee\mathrm{cl}^d(B)$. Let $b\in J(L^X)$ with $b\lhd\mathrm{cl}^d(A\vee B)$. It follows that there exists $a\geqslant b$ such that

$$d(a,A\vee B)=\bigwedge_{c\leqslant A\vee B} d(a,c)=0$$

which implies that for any $r\in\mathbb{R}^+$, there exists $c_r\leqslant A\vee B$ such that $d(a,c_r)<r$. Thus, we further obtain that there is $c_r\leqslant A$ or $c_r\leqslant B$ such that $d(a,c_r)<r$. From (M0), we get $d(b,c_r)<r$. Hence, it holds that

$$\bigwedge_{c\leqslant A} d(b,c)\leqslant\bigwedge_{r>0} d(b,c_r)=0$$

or

$$\bigwedge_{c\leqslant B} d(b,c)\leqslant\bigwedge_{r>0} d(b,c_r)=0.$$

Thus, $b\leqslant\mathrm{cl}^d(A)$ or $b\leqslant\mathrm{cl}^d(B)$. Therefore, $b\leqslant\mathrm{cl}^d(A)\vee\mathrm{cl}^d(B)$.

LC4). It is key to show that $\mathrm{cl}^d(\mathrm{cl}^d(A))\leqslant\mathrm{cl}^d(A)$. Let $a\in J(L^X)$ with $a\leqslant\mathrm{cl}^d(\mathrm{cl}^d(A))$. It follows from Lemma IV.7 that

$$d\left(a,\mathrm{cl}^d(A)\right)=\bigwedge_{c\leqslant\mathrm{cl}^d(A)} d(a,c)=0.$$

Thus, for any $r\in\mathbb{R}^+$, there exists $c_r\leqslant\mathrm{cl}^d(A)$ such that $d(a,c_r)<r/2$. Since $c_r\leqslant\mathrm{cl}^d(A)$, we have

$$d(c_r,A)=\bigwedge_{e\leqslant A} d(c_r,e)=0$$

implying that there exists $e_r \leqslant A$ such that $d(c_r, e_r) < r/2$. From (M2), we get

$$d(a, e_r) \leqslant d(a, c_r) + d(c_r, e_r) < \frac{r}{2} + \frac{r}{2} = r.$$

Hence, it holds that

$$\bigwedge_{e \leqslant A} d(a, e) \leqslant \bigwedge_{r>0} d(a, e_r) = 0.$$

Therefore, $a \leqslant \mathrm{cl}^d(A)$ by using Lemma IV.7. ■

Theorem IV.10: Let d be a pointwise p.q.-metric on X. For any $a \in J(L^X)$ and $r \in \mathbb{R}^+$, the L-subset $B_r^d(a)$ is closed in $(X, \eta^{\mathrm{cl}^d})$.

Proof: It suffices to verify that $\mathrm{cl}^d(B_r^d(a)) \leqslant B_r^d(a)$. Let $e \in J(L^X)$ with $e \leqslant \mathrm{cl}^d(B_r^d(a))$. It follows from Lemma IV.7 that

$$d(e, B_r^d(a)) = \bigwedge_{c \leqslant B_r^d(a)} d(e, c) = 0$$

and thus for any $s \in \mathbb{R}^+$, there exists $c_s \leqslant B_r^d(a)$ such that $d(e, c_s) < s$. Since

$$d(e, a) \leqslant d(e, c_s) + d(c_s, a) < s + r$$

we have $d(e, a) \leqslant \bigwedge_{s>0} s + r = r$. Hence, $e \leqslant B_r(a)$. By the arbitrariness of e, it holds that $\mathrm{cl}^d(B_r^d(a)) \leqslant B_r^d(a)$. ■

Next, we study the relations between the L-topology τ^{int^d} and the L-cotopology η^{cl^d}. First, we recall some basic notions related to pointwise pseudometrics.

In what follows, L always denotes a completely distributive lattice with an order-reversing involution$'$.

Definition IV.11 (see [10]): A pointwise p.q.-metric d on X is called a *pointwise pseudometric* (briefly, *pointwise p.-metric*) if it is symmetric, i.e., it holds that

M4) $\forall u, v \in J(L^X)$, $\bigwedge_{a \not\leqslant u'} d(a, v) = \bigwedge_{b \not\leqslant v'} d(b, u)$.

The pair (X, d) is called a *pointwise p.-metric space*.

We first give a characterization of the symmetry of a pointwise pseudometric in terms of its pointwise closed-ball system.

Proposition IV.12: Let $\mathcal{B}^d$ be the pointwise closed-ball system of a pointwise p.q.-metric d on X. Then, d is symmetric if and only if $\mathcal{B}^d$ satisfies

CB6) $\forall a, b \in J(L^X) \quad \forall r \in \mathbb{R}^+$

$$\bigvee_{s<r} B_s^d(a) \not\leqslant b' \Leftrightarrow \bigvee_{s<r} B_s^d(b) \not\leqslant a'.$$

Proof: Necessity. Let $a, b \in J(L^X)$ and $r \in \mathbb{R}^+$ such that $\bigvee_{s<r} B_s^d(a) \not\leqslant b'$. Then, there exists $l < r$ such that $B_l^d(a) = \bigvee\{e \in J(L^X) \mid d(e, a) \leqslant l\} \not\leqslant b'$. It follows that there exists $c \in J(L^X)$ such that $d(c, a) \leqslant l < r$ and $c \not\leqslant b'$. By the symmetry of d, we get

$$\bigwedge_{v \not\leqslant a'} d(v, b) = \bigwedge_{u \not\leqslant b'} d(u, a) < d(c, a) < r$$

implying that there exists $w \not\leqslant a'$ such that $d(w, b) < r$. Suppose $d(w, b) = t < r$. Then, $w \leqslant B_t^d(b) \leqslant \bigvee_{s<r} B_s^d(b)$. Since $w \not\leqslant a'$, we have $\bigvee_{s<r} B_s^d(b) \not\leqslant a'$. The reverse implication holds dually.

Sufficiency. Let $a, b \in J(L^X)$ with $\bigwedge_{u \not\leqslant a'} d(u, b) = t$. It follows that for any $r > t$, there exists $e \not\leqslant a'$ such that $d(e, b) < r$. Suppose $d(e, b) = k < t$. Then, $e \leqslant B_k^d(b)$. Since $e \not\leqslant a'$, we have $B_k^d(b) \not\leqslant a'$. Thus, $\bigvee_{s<r} B_s^d(b) \not\leqslant a'$. By (CB6), we get $\bigvee_{s<r} B_s^d(a) \not\leqslant b'$. It follows that there exists $l < r$ such that $B_l^d(a) \not\leqslant b'$, and thus there exists w such that $w \leqslant B_l^d(a)$ (i.e., $d(w, a) \leqslant l$) and $w \not\leqslant b'$. Hence

$$\bigwedge_{v \not\leqslant b'} d(v, a) \leqslant d(w, a) \leqslant l < r.$$

By the arbitrariness of r, it holds that

$$\bigwedge_{v \not\leqslant b'} d(v, a) \leqslant \bigwedge_{r>t} r = t = \bigwedge_{u \not\leqslant a'} d(u, b).$$

The reverse inequality can be analogously proved. ■

Theorem IV.13: If d is a pointwise p.-metric on X, then

$$\left(\eta^{\mathrm{cl}^d}\right)' = \tau^{\mathrm{int}^d}.$$

Proof: For convenience, we write int^d for the interior operator of $(X, \tau^{\mathrm{int}^d})$ and cl^d for the closure operator of $(X, \eta^{\mathrm{cl}^d})$. We have to show that $U \in \tau^{\mathrm{int}^d}$ if and only if $U' \in \eta^{\mathrm{cl}^d}$. It suffices to show that $\mathrm{cl}^d(A) = A$ if and only if $\mathrm{int}^d(A') = A'$, or equivalently, $\mathrm{cl}^d(A)' = \mathrm{int}^d(A')$. Concretely, we prove the following equality:

$$\bigvee_{s>0} \bigwedge_{e \leqslant A} B_s^d(e)' = \bigvee_{r>0} \bigvee \left\{a \in J\left(L^X\right) \mid B_r^d(a) \leqslant A'\right\}.$$

Claim 1: $\forall e \leqslant A$, $B_r^d(a) \leqslant A' \Rightarrow a \leqslant B_{r/2}^d(e)'$.

Suppose, on the contrary, that there exists $e \leqslant A$ such that $a \not\leqslant B_{r/2}^d(e)'$. Then, $\bigvee_{s<r} B_s^d(e) \not\leqslant a'$. It follows from (CB6) in Proposition IV.12 that $\bigvee_{s<r} B_s^d(a) \not\leqslant e'$. By Proposition III.2(1), we get $B_r^d(a) \geqslant \bigvee_{s<r} B_s^d(a)$. Since $e' \geqslant A'$, we have $B_r^d(a) \not\leqslant A'$. Therefore, Claim 1 holds.

From Claim 1, one can deduce that

$$\bigvee_{r>0} \bigvee \left\{a \in J\left(L^X\right) \mid B_r^d(a) \leqslant A'\right\} \leqslant \bigvee_{s>0} \bigwedge_{e \leqslant A} B_s^d(e)'.$$

Claim 2: $\forall s > 0$, $c \leqslant \bigwedge_{e \leqslant A} B_s^d(e)' \Rightarrow B_{s/2}^d(c) \leqslant A'$.

Suppose, on the contrary, that $B_{s/2}^d(c) \not\leqslant A'$. Since $A' = \bigwedge\{e' \mid e \lhd A\}$, it follows that there exists $u \lhd A$ such that $B_{s/2}^d(c) \not\leqslant u'$, implying that $\bigvee_{r<s} B_r^d(c) \not\leqslant u'$. It follows from (CB6) in Proposition IV.12 that $\bigvee_{r<s} B_r^d(u) \not\leqslant c'$. From Proposition III.2(1), we get $B_s^d(u) \geqslant \bigvee_{r<s} B_r^d(u)$ and hence, $B_s^d(u) \not\leqslant c'$, i.e., $c \not\leqslant B_s^d(u)'$. Since $u \lhd A$, we have $B_s^d(u) \leqslant \bigvee_{e \leqslant A} B_s^d(e)$, implying that $\bigwedge_{e \leqslant A} B_s^d(e)' \leqslant B_s^d(u)'$. Thus, $c \not\leqslant \bigwedge_{e \leqslant A} B_s^d(e)'$. Therefore, Claim 2 holds.

From Claim 2, one can deduce that

$$\bigvee_{s>0} \bigwedge_{e \leqslant A} B_s^d(e)' \leqslant \bigvee_{r>0} \bigvee \left\{a \in J\left(L^X\right) \mid B_r^d(a) \leqslant A'\right\}.$$

■

In a pseudoquasi-metric space (X, ρ), for every subset A of X, define $\rho(x, A) = \bigwedge_{y \in A} \rho(x, y)$. In particular, we take this to be $+\infty$ if A is empty.

Interestingly, in (X, ρ), the set of points x such that $\rho(x, A) = 0$ is exactly the closure of A in the open ball topology. However, this result is generally not true in the framework of pointwise pseudoquasi-metrics. This can be seen from the following example, which means that the requirement in Theorem IV.13 that d is a pointwise pseudometric is necessary.

Example IV.14: Let $\mathbb{I} = [0,1]$ and $\mathbb{R}^* = [0, +\infty)$. For any $x \in \mathbb{R}^*$ and $\alpha \in \mathbb{I}$, define $x_\alpha : \mathbb{R}^* \longrightarrow \mathbb{I}$ as follows:

$$\forall y \in \mathbb{R}^*,\ x_\alpha(y) = \begin{cases} \alpha, & y = x \\ 0, & y \neq x \end{cases}.$$

Note that $J(\mathbb{I}^{\mathbb{R}^*}) = \{x_\alpha \mid x \in \mathbb{R}^*, \alpha \in (0,1]\}$.

Define a map $d^{\mathbb{R}^*} : \mathbb{R}^* \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ as follows:

$$d_{\mathbb{R}^*}(x_\alpha, y_\beta) = \max\{x - y, 0\} + \max\{\alpha - \beta, 0\}.$$

We prove that $(\mathbb{I}, d_{\mathbb{R}^*})$ is a pointwise p.q.-metric space, but not a pointwise p.-metric space.

M1) For any $x_\alpha \in J(\mathbb{I}^{\mathbb{R}^*})$, we have $d_{\mathbb{R}^*}(x_\alpha, x_\alpha) = \max\{x - x, 0\} + \max\{\alpha - \alpha, 0\} = 0$.

M2) Let $x_\alpha, y_\beta, z_\gamma \in J(\mathbb{I}^{\mathbb{R}^*})$. Then, we have

$$\begin{aligned} & d_{\mathbb{R}^*}(x_\alpha, y_\beta) \\ =\ & \max\{x - y, 0\} + \max\{\alpha - \beta, 0\} \\ \leqslant\ & \max\{(x - z) + (z - y), 0\} \\ & + \max\{(\alpha - \gamma) + (\gamma - \beta), 0\} \\ \leqslant\ & \max\{x - z, 0\} + \max\{z - y, 0\} \\ & + \max\{\alpha - \gamma, 0\} + \max\{\gamma - \beta, 0\} \\ =\ & d_{\mathbb{R}^*}(x_\alpha, z_\gamma) + d_{\mathbb{R}^*}(z_\gamma, y_\beta). \end{aligned}$$

M3) Let $x_\alpha, y_\beta \in J(\mathbb{I}^{\mathbb{R}^*})$. Note that $z_\gamma \lhd y_\beta$ if and only if $z = y$ and $\gamma < \beta$ for any $z_\gamma \in J(\mathbb{I}^{\mathbb{R}^*})$. Then, we have

$$\begin{aligned} & \textstyle\bigwedge_{z_\gamma \lhd y_\beta} d_{\mathbb{R}^*}(x_\alpha, z_\gamma) \\ =\ & \textstyle\bigwedge_{\gamma < \beta} d_{\mathbb{R}^*}(x_\alpha, y_\gamma) \\ =\ & \textstyle\bigwedge_{\gamma < \beta} \max\{x - y, 0\} + \max\{\alpha - \gamma, 0\} \\ =\ & \max\{x - y, 0\} + \textstyle\bigwedge_{\gamma < \beta} \max\{\alpha - \gamma, 0\}. \end{aligned}$$

There are two cases, which are as follows.

Case 1: $\alpha \geqslant \beta$. Then, $\bigwedge_{\gamma < \beta} \max\{\alpha - \gamma, 0\} = \alpha - \beta$.

Case 2: $\alpha < \beta$. Then, $\bigwedge_{\gamma < \beta} \max\{\alpha - \gamma, 0\} = 0$.

These cases show $\bigwedge_{\gamma < \beta} \max\{\alpha - \gamma, 0\} = \max\{\alpha - \beta, 0\}$. It follows that $\bigwedge_{z_\gamma \lhd y_\beta} d_{\mathbb{R}^*}(x_\alpha, z_\gamma) = \max\{x - y, 0\} + \max\{\alpha - \beta, 0\} = d_{\mathbb{R}^*}(x_\alpha, y_\beta)$.

The aforementioned results show that d is a pointwise p.q.-metric. Next, we show that d is not symmetric. Note that $x_\alpha \nleqslant (y_\beta)'$ if and only if $x = y$ and $\alpha > 1 - \beta$ for all $x_\alpha, y_\beta \in J(\mathbb{I}^{\mathbb{R}^*})$. Then

$$\begin{aligned} & \textstyle\bigwedge_{x_\alpha \nleqslant (1_{0.5})'} d_{\mathbb{R}^*}(x_\alpha, 2_{0.5}) \\ =\ & \textstyle\bigwedge_{\alpha > 0.5} d_{\mathbb{R}^*}(1_\alpha, 2_{0.5}) \\ =\ & \textstyle\bigwedge_{\alpha > 0.5} \max\{1 - 2, 0\} + \max\{\alpha - 0.5, 0\} \\ =\ & 0.5 \end{aligned}$$

and

$$\begin{aligned} & \textstyle\bigwedge_{y_\beta \nleqslant (2_{0.5})'} d_{\mathbb{R}^*}(y_\beta, 1_{0.5}) \\ =\ & \textstyle\bigwedge_{\beta > 0.5} d_{\mathbb{R}^*}(2_\beta, 1_{0.5}) \\ =\ & \textstyle\bigwedge_{\beta > 0.5} \max\{2 - 1, 0\} + \max\{\beta - 0.5, 0\} \\ =\ & 1.5 \end{aligned}$$

that is

$$\bigwedge_{x_\alpha \nleqslant (1_{0.5})'} d_{\mathbb{R}^*}(x_\alpha, 2_{0.5}) \neq \bigwedge_{y_\beta \nleqslant (2_{0.5})'} d_{\mathbb{R}^*}(y_\beta, 1_{0.5}).$$

Therefore, d is not a pointwise p.-metric

$$\begin{aligned} & B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5}) \\ =\ & \textstyle\bigvee \{x_\alpha \mid d_{\mathbb{R}^*}(x_\alpha, 0_{0.5}) \leqslant 0.1\} \\ =\ & \textstyle\bigvee \{x_\alpha \mid \max\{x, 0\} + \max\{\alpha - 0.5, 0\} \leqslant 0.1\}. \end{aligned}$$

Thus, for any $x \in \mathbb{R}^*$, we have

$$B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5})(x) = \begin{cases} 0.6, & x \leqslant 0 \\ 0.6 - x, & 0 < x \leqslant 0.1 \\ 0, & x > 0.1 \end{cases}.$$

By Theorem IV.10, it holds that $B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5})$ is closed in $(X, \eta^{\mathrm{cl}^{d_{\mathbb{R}^*}}})$. Next, we show that $(B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5}))'$ is not open in $\tau^{\mathrm{int}^{d_{\mathbb{R}^*}}}$. First, for any $x \in \mathbb{R}^*$, it holds that

$$\left(B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5})\right)'(x) = \begin{cases} 0.4, & x \leqslant 0 \\ 0.4 + x, & 0 < x \leqslant 0.1 \\ 1, & x > 0.1 \end{cases}.$$

Since $0.4 < (B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5}))'(0.1) = 0.5$, we have $0.1_{0.4} \lhd (B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5}))'$. Take an arbitrary r in $\mathbb{R}^+$. Then

$$\begin{aligned} & B_r^{d_{\mathbb{R}^*}}(0.1_{0.4}) \\ =\ & \textstyle\bigvee \{x_\alpha \mid d_{\mathbb{R}^*}(x_\alpha, 0.1_{0.4}) \leqslant r\} \\ =\ & \textstyle\bigvee \{x_\alpha \mid \max\{x - 0.1, 0\} + \max\{\alpha - 0.4, 0\} \leqslant r\}. \end{aligned}$$

Note that

$$0.4 + r = B_r^{d_{\mathbb{R}^*}}(0.1_{0.4})(0)$$

and

$$0.4 + r \nleqslant (B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5}))'(0) = 0.4.$$

Hence

$$0_{0.4+r} \leqslant B_r^{d_{\mathbb{R}^*}}(0.1_{0.4}) \text{ and } 0_{0.4+r} \nleqslant \left(B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5})\right)'.$$

Thus, $B_r(0.1_{0.4}) \nleqslant (B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5}))'$. It follows from the arbitrariness of r that $(B_{0.1}^{d_{\mathbb{R}^*}}(0_{0.5}))'$ is not open in $\tau^{\mathrm{int}^{d_{\mathbb{R}^*}}}$. Therefore, $\tau^{\mathrm{int}^{d_{\mathbb{R}^*}}} \neq (\eta^{\mathrm{cl}^{d_{\mathbb{R}^*}}})'$.

Combining Theorems II.12, IV.13 and Corollary IV.4, the following corollary is straightforward.

Corollary IV.15: If d is a pointwise p.-metric on X, then $\eta^{\mathcal{P}^d} = \eta^{\mathcal{B}^d}$.

The following theorem reveals the link between closed-balls and remote neighborhoods in pointwise p.-metric spaces.

Theorem IV.16: Let d be a pointwise p.-metric on X. For any $a \in J(L^X)$ and $r \in \mathbb{R}^+$, it holds that

$$B_r(a) = \bigwedge_{s > r} \bigvee_{b \nleqslant a'} P_s(b)'.$$

Proof: Claim 1: $\bigvee_{b \nleqslant a'} P_s(b)' \leqslant B_s(a)$.

Suppose, on the contrary, that there exist $a, b \in J(L^X)$ such that $b \nleqslant a'$ and $P_s(b)' \nleqslant B_s(a)$. Since $B_s(a)' \nleqslant P_s(b)$, it follows that there exists $u \lhd B_s(a)'$ such that $u \nleqslant P_s(b)$, implying that $d(b, u) < s$ by Lemma II.10. Let $d(b, u) = t$. Then, $b \leqslant B_t(u) \leqslant \bigvee_{l<s} B_l(u)$. Since $b \nleqslant a'$, we have $\bigvee_{l<s} B_l(u) \nleqslant a'$. It follows from (CB6) that $\bigvee_{l<s} B_l(a) \nleqslant u'$. By Proposition III.2(1), we get $B_s(a) \geqslant \bigvee_{l<s} B_l(a)$, which implies that $B_s(a) \nleqslant u'$. Hence, $u \nleqslant B_s(a)'$, a contradiction. Therefore, Claim 1 holds.

By (CB4), we get $B_r(a) = \bigwedge_{s>r} B_s(a)$. Using Claim 1, one can deduce that $B_r(a) \geqslant \bigwedge_{s>r} \bigvee_{b \nleqslant a'} P_s(b)'$.

Claim 2: $\forall s > r$, $B_r(a) \leqslant \bigvee_{b \nleqslant a'} P_s(b)'$.

Suppose, on the contrary, that there exists $s > r$ such that $B_r(a) \nleqslant \bigvee_{b \nleqslant a'} P_s(b)'$. Then, $\bigwedge_{b \nleqslant a'} P_s(b) \nleqslant B_r(a)'$, and so there exists $c \in J(L^X)$ such that $c \leqslant \bigwedge_{b \nleqslant a'} P_s(b)$ and $c \nleqslant B_r(a)'$, and so $\bigvee_{l<s} B_l(a) \geqslant B_r(a) \nleqslant c'$, i.e., $\bigvee_{l<s} B_l(a) \nleqslant c'$. By (CB6), we have $\bigvee_{l<s} B_l(c) \nleqslant a'$. Then, there exists $l < s$ such that $B_l(c) \nleqslant a'$. Thus, there exists $e \in J(L^X)$ such that $e \leqslant B_l(c)$ and $e \nleqslant a'$. Since $c \leqslant \bigwedge_{b \nleqslant a'} P_s(b)$, we have $c \leqslant P_s(e)$. It follows from Lemma II.10 that $d(e, c) \geqslant s > l$. Then, $d(e, c) \nleqslant l$, implying that $e \nleqslant B_l(c)$, a contradiction. Therefore, Claim 2 holds.

From Claim 2, we get $B_r(a) \leqslant \bigwedge_{s>r} \bigvee_{b \nleqslant a'} P_s(b)'$.

Claims 1 and 2 lead us to the fact that $B_r(a) = \bigwedge_{s>r} \bigvee_{b \nleqslant a'} P_s(b)'$. ■

The following contents concerning pointwise metrics will be used in the next section.

Definition IV.17 (see [11] and [23]): A pointwise p.-metric d on X is called a *pointwise metric* if it satisfies:

M5) $\forall x, y \in X$, $\bigvee_{\lambda \in J(L)} d(x_\lambda, y_\lambda) = 0$ implies $x = y$.

The pair (X, d) is called a *pointwise metric space* if d is a pointwise metric.

Proposition IV.18: Let $\mathcal{B}^d$ be the pointwise closed-ball system of a pointwise p.-metric d on X. Then, d is a pointwise metric if and only if $\mathcal{B}^d$ satisfies

CB7) $\forall x_\lambda, y_\lambda \in J(L^X), x_\lambda \leqslant \bigwedge_{r>0} B^d_r(y_\lambda) \Rightarrow x = y$.

Proof: It is trivial. ■

At the end of this section, we make a conclusion regarding the relations between pointwise closed-ball systems and pointwise p.q.-metrics as follows:

$$\begin{tikzcd}
\text{p.c.s} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, "+\text{(CB6)}"] & \text{pointwise p.q.-metric} \arrow[d, Rightarrow, "+\text{(M4)}"] \\
\text{p.c.s + (CB6)} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, "+\text{(CB7)}"] & \text{pointwise p.-metric} \arrow[d, Rightarrow, "+\text{(M5)}"] \\
\text{p.c.s + (CB6)+(CB7)} \arrow[r, Leftrightarrow] & \text{pointwise metric}
\end{tikzcd}$$

where p.c.s is the abbreviation of a pointwise closed-ball system.

V. Metrization Theorem of the L-Fuzzy Real Line

The space of the L-fuzzy real line is of utmost importance in fuzzy topology. The related concept of fuzzy numbers was successively defined by Hutton [4] and Gantner *et al.* [12]. Many researchers have shown that fuzzy numbers have many practical and theoretical applications. In this section, we will provide a direct approach to the metrization of the L-fuzzy real line (the collection of all fuzzy numbers) via the closed-ball systems. The result shows that the topology induced by this way is exactly the standard topology when L is reduced to $\{\bot, \top\}$.

Definition V.1 (see [4] and [12]): The *L-fuzzy real line* $\mathbb{R}(L)$ is defined as the set of all left-continuous decreasing maps $u : \mathbb{R} \longrightarrow L$ satisfying

$$\bigvee_{r\in\mathbb{R}} u(r) = \top \text{ and } \bigwedge_{r\in\mathbb{R}} u(r) = \bot.$$

Every element in $\mathbb{R}(L)$ is called an *L-fuzzy number*. For any $r \in \mathbb{R}$, define two L-sets $\mathcal{L}_r, \mathcal{R}_r \in L^{\mathbb{R}(L)}$ by

$$\mathcal{L}_r(u) = u(r),' \quad \mathcal{R}_r(u) = u(r+).$$

The *standard L-topology* on $\mathbb{R}(L)$ is generated from the subbase $\{\mathcal{L}_r, \mathcal{R}_r \mid r \in \mathbb{R}\}$.

Remark V.2: By the definitions of $\mathcal{L}_r$ and $\mathcal{R}_r$, one can easily observe that for any $r, s \in \mathbb{R}$, $r \leqslant s$ implies $\mathcal{L}_r \leqslant \mathcal{L}_s$ (i.e., $\mathcal{L}_r(u) \leqslant \mathcal{L}_s(u) \quad \forall u \in \mathbb{R}(L)$), and $\mathcal{R}_r \geqslant \mathcal{R}_s$.

Define two maps $\alpha, \beta : J(L^{\mathbb{R}(L)}) \longrightarrow \mathbb{R}$ as follows:

$$\alpha(a) = \bigvee \{t \in \mathbb{R} \mid a \leqslant \mathcal{L}'_t\}, \; \beta(a) = \bigwedge \{t \in \mathbb{R} \mid a \leqslant \mathcal{R}'_t\}.$$

The following results will be used later.

Lemma V.3 (see [24]): The following statements are valid.

1) $\alpha(b) = \max\{t \in \mathbb{R} \mid b \leqslant \mathcal{L}'_t\}$ and $\beta(b) = \min\{t \in \mathbb{R} \mid b \leqslant \mathcal{R}'_t\}$.
2) $\forall a, b \in J(\mathbb{R}(L))$, $a \leqslant b$ implies $\alpha(a) \geqslant \alpha(b)$ and $\beta(a) \leqslant \beta(b)$.
3) $\forall a \in J(\mathbb{R}(L))$, $\alpha(a) = \bigwedge_{b \lhd a} \alpha(b)$, and $\beta(a) = \bigvee_{b \lhd a} \beta(b)$.
4) $\forall a, b \in J(\mathbb{R}(L))$, there exists $c \nleqslant a'$ such that $\alpha(b) < \alpha(c) + r$ iff there exists $e \nleqslant b'$ such that $\beta(a) > \beta(e) - r$.

Lemma V.4 (see [24]): For any $a, b \in J(L^{\mathbb{R}(L)})$, define

$$d(a, b) = \max\{\alpha(a) - \alpha(b), \beta(b) - \beta(a), 0\}.$$

Then, d is a pointwise p.-metric and the standard L-topology on $\mathbb{R}(L)$ is exactly the topology induced by d.

Remark V.5: It is easy to check that for each $u_\lambda \in J(\mathbb{R}(L))$, $\alpha(u_\lambda) = \max\{t \in \mathbb{R} \mid \lambda \leqslant u(r)\}$ and $\beta(u_\lambda) = \min\{t \in \mathbb{R} \mid \lambda \leqslant u(r+)'\}$.

Proposition V.6: Let $a \in J(L^{\mathbb{R}(L)})$. Then, the following conditions hold.

1) $\alpha(a) = \bigvee\{s \in \mathbb{R}^+ \mid a \leqslant \mathcal{R}_s\}$.
2) $\beta(a) = \bigwedge\{s \in \mathbb{R}^+ \mid a \leqslant \mathcal{L}_s\}$.

Proof: For convenience, let $a = u_\lambda$ ($u \in \mathbb{R}(L)$ and $\lambda \in J(L)$). Denote

$$\alpha^*(u_\lambda) := \bigvee\{s \mid u_\lambda \leqslant \mathcal{R}_s\}, \; \beta^*(u_\lambda) := \bigwedge\{s \mid u_\lambda \leqslant \mathcal{L}_s\}.$$

1) Let $u_\lambda \leqslant \mathcal{L}'_t$, i.e., $\lambda \leqslant u(t)$. For any $r < t$, it holds that $\lambda \leqslant u(t) \leqslant u(r+)$, implying that $u_\lambda \leqslant \mathcal{R}_r$. Hence, $r \leqslant \alpha^*(u_\lambda)$ and thus, $t \leqslant \alpha^*(u_\lambda)$. Therefore, $\alpha(u_\lambda) \leqslant \alpha^*(u_\lambda)$. For the converse, let $\lambda \leqslant u(s+)$. Then, $\lambda \leqslant u(s+) \leqslant u(s)$, showing that $u_\lambda \leqslant \mathcal{L}'_s$, so $s \leqslant \alpha(u_\lambda)$. Hence, $\alpha^*(u_\lambda) \leqslant \alpha(u_\lambda)$.

2) Let $\lambda \leqslant u(t+)'$. Then, $\lambda \leqslant u(t+)' = \bigwedge_{s>t} u(s)'$. It follows that $\lambda \leqslant u(s)'$, i.e., $u_\lambda \leqslant \mathcal{L}_s$ and thus, $s \geqslant \beta^*(u_\lambda)$, for all $s > t$. Hence, $t \geqslant \beta^*(u_\lambda)$. Therefore, $\beta^*(u_\lambda) \leqslant \beta(u_\lambda)$. For the converse, if $u_\lambda \leqslant \mathcal{L}_s$, then $\lambda \leqslant u(s)' \leqslant u(s+)'$, which implies that $u_\lambda \leqslant \mathcal{R}'_s$. Thus, $s \geqslant \beta(u_\lambda)$. Therefore, $\beta^*(u_\lambda) \geqslant \beta(u_\lambda)$. ■

Lemma V.7: Let $a, b \in J(L^{\mathbb{R}(L)})$ and $r \in \mathbb{R}$. Then, the following conditions hold.

1) $r \leqslant \alpha(a)$ if and only if $a \leqslant \mathcal{L}'_r$.
2) $\beta(b) \leqslant r$ if and only if $b \leqslant \mathcal{R}'_r$.

Proof: 1) Suppose $r \leqslant \alpha(a)$. By Lemma V.3(1), we have that $\alpha(a) = \max\{t \in \mathbb{R} : a \leqslant \mathcal{L}'_t\}$, which implies that $a \leqslant \mathcal{L}'_{\alpha(a)}$. Since $r \leqslant \alpha(a)$, it follows that $\mathcal{L}_r \leqslant \mathcal{L}_{\alpha(a)}$ by Remark V.2, and thus

$$a \leqslant \mathcal{L}'_{\alpha(a)} \leqslant \mathcal{L}'_r.$$

The converse is trivial since $\alpha(a) = \max\{t \in \mathbb{R} : a \leqslant \mathcal{L}'_t\}$ by Lemma V.3(1).

2) Suppose $r \geqslant \beta(b)$. By Lemma V.3(1), we have that $\beta(b) = \min\{t \in \mathbb{R} \mid b \leqslant \mathcal{R}'_t\}$, which implies that $b \leqslant \mathcal{R}'_{\beta(b)}$. Since $r \geqslant \beta(b)$, it follows that $\mathcal{R}_r \leqslant \mathcal{R}_{\beta(b)}$ by Remark V.2. Thus:

$$b \leqslant \mathcal{R}'_{\beta(b)} \leqslant \mathcal{R}'_r.$$

The converse is trivial since $\beta(b) = \min\{t \in \mathbb{R} \mid b \leqslant \mathcal{R}'_t\}$ by Lemma V.3(1). ■

Lemma V.8: Let $u, v \in \mathbb{R}(L)$. If $\alpha(u_\lambda) \leqslant \alpha(v_\lambda)$ and $\beta(v_\lambda) \leqslant \beta(u_\lambda)$ for all $\lambda \in J(L)$, then $u = v$.

Proof: Suppose $u \neq v$. Then, there exists $s \in \mathbb{R}^+$ such that $u(s) \neq v(s)$, i.e., $u(s) \nleqslant v(s)$ or $v(s) \nleqslant u(s)$.

Case 1. $u(s) \nleqslant v(s)$.

Then, there exists $\lambda \in J(L)$ such that $\lambda \leqslant u(s)$ and $\lambda \nleqslant v(s)$. It follows that $u_\lambda \leqslant \mathcal{L}'_s$ and $v_\lambda \nleqslant \mathcal{L}'_s$. Thus, $s \leqslant \alpha(u_\lambda)$ and $s \nleqslant \alpha(v_\lambda)$, contradicting that $\alpha(u_\lambda) \leqslant \alpha(v_\lambda)$.

Case 2. $v(s) \nleqslant u(s)$.

In this case, we first claim that there exists $r < s$ such that $v(r+) \nleqslant u(r+)$. Otherwise, $v(t+) \leqslant u(t+)$ for all $t < s$, implying that

$$v(s) = \bigwedge_{t<s} v(t+) \leqslant \bigwedge_{t<s} u(t+) = u(s)$$

a contradiction. Thus, $u(r+)' \nleqslant v(r+)'$. Then, there exists $\lambda \in J(L)$ such that $\lambda \leqslant u(r+)'$ and $\lambda \nleqslant v(r+)'$, i.e., $u_\lambda \leqslant \mathcal{R}'_r$ and $v_\lambda \nleqslant \mathcal{R}'_r$. It follows from Lemma V.7 that $r \geqslant \beta(u_\lambda)$ and $r \ngeqslant \beta(v_\lambda)$, contradicting $\beta(v_\lambda) \leqslant \beta(u_\lambda)$.

All these show that $u = v$. ■

The following theorem provides a new approach to the metrization theorem of the L-fuzzy real line.

Theorem V.9: The L-fuzzy real line $\mathbb{R}(L)$ is metrizable, i.e., the standard L-topology on $\mathbb{R}(L)$ is that of a pointwise metric.

Proof: For $r \in \mathbb{R}^+$, define a map $B_r : J(L^{\mathbb{R}(L)}) \longrightarrow L^X$ by

$$B_r(a) = \mathcal{L}'_{\alpha(a)-r} \wedge \mathcal{R}'_{\beta(a)+r}.$$

First, one should note that for any $b \in J(L^{\mathbb{R}(L)})$

$$\begin{aligned} b \leqslant B_r(a) &\Leftrightarrow b \leqslant \mathcal{L}'_{\alpha(a)-r} \text{ and } b \leqslant \mathcal{R}'_{\beta(a)+r} \\ &\Leftrightarrow r \geqslant \max\{\alpha(a)-\alpha(b), \beta(b)-\beta(a)\}. \end{aligned}$$

Next, we prove that $\{B_r \mid r \in \mathbb{R}^+\}$ satisfies (CB1)–(CB6).

(CB1) For any $a, b \in J(L^{\mathbb{R}(L)})$, let

$$r = \max\{\alpha(a)-\alpha(b), \beta(b)-\beta(a), 1\}.$$

Then, $b \leqslant B_r(a)$.

(CB2) Let $r \in \mathbb{R}^+$ and $a \in J(L^{\mathbb{R}(L)})$. Since $r \geqslant 0 = \alpha(a) - \alpha(a)$ and $r \geqslant 0 = \beta(a) - \beta(a)$, it follows from Lemma V.7 that $a \leqslant \mathcal{L}'_{\alpha(a)-r}$ and $a \leqslant \mathcal{R}'_{\beta(a)+r}$. Hence, $a \leqslant B_r(a)$.

(CB3) Let $r, s \in \mathbb{R}^+$ and $a \in J(L^{\mathbb{R}(L)})$. It suffices to show that $B_s \circledcirc B_r(a) \leqslant B_{s+r}(a)$, where

$$B_s \circledcirc B_r(a) = \bigvee\{B_s(b) \mid b \leqslant B_r(a)\}.$$

Suppose $b \leqslant B_r(a)$ and $c \leqslant B_s(b)$. Then

$$r \geqslant \max\{\alpha(a)-\alpha(b), \beta(b)-\beta(a)\}$$

and

$$s \geqslant \max\{\alpha(b)-\alpha(c), \beta(c)-\beta(b)\}.$$

It follows that:

$$r+s \geqslant \max\{\alpha(a)-\alpha(c), \beta(c)-\beta(a)\} \Leftrightarrow c \leqslant B_{s+r}(a).$$

Hence, $B_s \circledcirc B_r(a) \leqslant B_{s+r}(a)$.

(CB4) Let $a \in J(L^{\mathbb{R}(L)})$ and $r \in \mathbb{R}^+$. Then

$$\begin{aligned} & b \leqslant \bigwedge_{s>r} B_s(a) \\ &\Leftrightarrow \forall s > r,\ b \leqslant B_s(a) \\ &\Leftrightarrow \forall s > r,\ s \geqslant \max\{\alpha(a)-\alpha(b), \beta(b)-\beta(a)\} \\ &\Leftrightarrow r \geqslant \max\{\alpha(a)-\alpha(b), \beta(b)-\beta(a)\} \\ &\Leftrightarrow b \leqslant B_r(a) \end{aligned}$$

which implies that $B_r(a) = \bigwedge_{s>r} B_s(a)$.

(CB5) Let $b \leqslant B_r(a)$ and $s > r$. Then

$$\begin{aligned} & s > r \geqslant \max\{\alpha(a)-\alpha(b), \beta(b)-\beta(a)\} \\ &\Rightarrow \alpha(b)+s > \alpha(a) = \bigwedge_{c \lhd a} \alpha(c), \text{ and} \\ &\quad \beta(b)-s < \beta(a) = \bigvee_{c \lhd a} \beta(c) \\ &\Rightarrow \exists c_1, c_2 \lhd a,\ \alpha(b)+s > \alpha(c_1), \beta(b)-s < \beta(c_2). \end{aligned}$$

Choose $c \lhd a$ such that $c_1, c_2 \leqslant c$. By Lemma V.3, we have

$$\alpha(b)+s > \alpha(c_1) \geqslant \alpha(c) \text{ and } \beta(b)-s < \beta(c_2) \leqslant \beta(c)$$

which implies that $b \leqslant B_s(c)$.

(CB6) For any $a, b \in J(L^{\mathbb{R}(L)})$ and $r \in \mathbb{R}^+$, it suffices to show that

$$\bigvee_{s<r} B_s(a) \nleqslant b' \Leftrightarrow \bigvee_{s<r} B_s(b) \nleqslant a'.$$

Let $\bigvee_{s<r} B_s(a) \nleqslant b'$. Then, there exists $s < r$ such that $B_s(a) \nleqslant b'$, and thus there exists $c \leqslant B_s(a)$ such that $c \nleqslant b'$, which implies that

$$s \geqslant \max\{\alpha(a)-\alpha(c), \beta(c)-\beta(a)\}.$$

Then, $r > \alpha(a) - \alpha(c)$ and $r > \beta(c) - \beta(a)$. By Lemma V.3, there exist $e_1 \nleqslant a'$ and $e_2 \nleqslant a'$ such that $\beta(b) > \beta(e_1) - r$ and $\alpha(b) < \alpha(e_2) + r$. Since a' is a prime element, we have $e_1 \wedge e_2 \nleqslant a'$. Let $e \leqslant e_1 \wedge e_2$ such that $e \nleqslant a'$. By Lemma V.3, we have $r > \beta(e_1) - \beta(b) \geqslant \beta(e) - \beta(b)$ and $r > \alpha(b) - \alpha(e_2) \geqslant \alpha(b) - \alpha(e)$. Choose $t \geqslant 0$ such that $r > t \geqslant \max\{\alpha(b) - \alpha(e), \beta(e) - \beta(b)\}$. Then, $e \leqslant B_t(b)$. Since $e \nleqslant a'$, we have $B_t(b) \nleqslant a'$. Hence, $\bigvee_{s<r} B_s(b) \nleqslant a'$.

By Theorem III.9, we know that $\{B_r : r \in \mathbb{R}^+\}$ can induce a pointwise p.q.-metric, denoted by $d_{\mathbb{R}}$. By Theorem IV.12, we know that $d_{\mathbb{R}}$ is a pointwise p.-metric.

Finally, we show that $d_{\mathbb{R}}$ is a pointwise metric.

Let $u, v \in \mathbb{R}(L)$ such that $u_\lambda \leqslant B_r(v_\lambda)$ for all $\lambda \in J(L)$ and $r \in \mathbb{R}^+$. Then, $r \geqslant \max\{\alpha(v_\lambda) - \alpha(u_\lambda), \beta(u_\lambda) - \beta(v_\lambda)\}$ for all $r \in \mathbb{R}^+$. It follows that $\alpha(u_\lambda) \leqslant \alpha(v_\lambda)$ and $\beta(v_\lambda) \leqslant \beta(u_\lambda)$ for all $\lambda \in J(L)$. By Lemma V.8, we have $u = v$. Therefore, by Theorem IV.18, $d_{\mathbb{R}}$ is a pointwise metric. Furthermore, we have

$$\begin{aligned} d_{\mathbb{R}}(a,b) &= \bigwedge\{r > 0 \mid a \leqslant B_r(b)\} \\ &= \bigwedge\{r > 0 \mid \max\{\alpha(a) - \alpha(b), \beta(b) - \beta(a)\} \leqslant r\} \\ &= \max\{\alpha(a) - \alpha(b), \beta(b) - \beta(a), 0\}. \end{aligned}$$

By Lemma V.4, we know that the standard L-topology on L-fuzzy real line is exactly the topology induced by $d_{\mathbb{R}}$. Hence, the L-fuzzy real line is metrizable. ■

We end this section with two usual examples.

Example V.10: For a real number $u \in \mathbb{R}$, define $\underline{u} \in \mathbb{R}_F$ such that

$$\underline{u}(t) = \begin{cases} 1, & t \leqslant u \\ 0, & \text{otherwise} \end{cases}.$$

Then, $\underline{u}$ is an L-fuzzy real number. In this case

$$\mathcal{L}_s(\underline{u}) = \underline{u}(s) = 1 \Leftrightarrow\ u \in (-\infty, s)$$
$$\mathcal{R}_s(\underline{u}) = \underline{u}(s)' = 1 \Leftrightarrow\ u \in (s, +\infty).$$

Let $\underline{\mathbb{R}} = \{\underline{u} \mid u \in \mathbb{R}\}$. Then, $\underline{\mathbb{R}}$ can be regarded as a subspace of $\mathbb{R}_F$. By restricting $\mathcal{L}_s$ (resp., $\mathcal{R}_s$) to $\underline{\mathbb{R}}$, it will become $(-\infty, s)$ (resp., $(s, +\infty)$). All these show that the standard L-topology on $\mathbb{R}_F$ is an extension of the standard L-topology on $\mathbb{R}$.

Example V.11: Let $u \triangleq \langle x, \varepsilon\rangle$ be a [0,1]-fuzzy number with $x \in \mathbb{R}, \varepsilon > 0$, i.e., u satisfies

$$\forall s \in \mathbb{R},\ u(s) = \begin{cases} 1, & s \leqslant x \\ \dfrac{1}{\varepsilon}(x - s) + 1, & x < s \leqslant x + \varepsilon \\ 0, & s > x + \varepsilon \end{cases}.$$

Note that when $\varepsilon = 0$, the interval $(x, x + \varepsilon]$ is an empty set. So, $u = \langle x, 0\rangle$ equals

$$\forall s \in \mathbb{R},\ \underline{x}(s) = \begin{cases} 1, & s \leqslant x \\ 0, & s > x \end{cases}.$$

In this case

$$\mathcal{L}_s(u) = u(s)' = \begin{cases} 0, & s \leqslant x \\ \frac{1}{\varepsilon}(s - x), & x < s \leqslant x + \varepsilon \\ 1, & s > x + \varepsilon \end{cases}$$

and

$$\mathcal{R}_s(u) = u(s+) = u(s).$$

For each $\lambda \in (0, 1]$, we have

$$\begin{aligned} \alpha(u_\lambda) &= \max\{s \in \mathbb{R} \mid u_\lambda \leqslant \mathcal{L}'_s\} \\ &= \max\{s \in \mathbb{R} \mid \lambda \leqslant \mathcal{L}'_s(u) = u(s)\} \\ &= x + (1 - \lambda)\varepsilon \end{aligned}$$

and

$$\begin{aligned} \beta(u_\lambda) &= \min\{s \in \mathbb{R} \mid u_\lambda \leqslant \mathcal{R}'_s\} \\ &= \min\{s \in \mathbb{R} \mid \lambda \leqslant R'_s(u) = u(s+)'\} \\ &= x + \lambda\varepsilon. \end{aligned}$$

Let $v = \langle y, \delta\rangle$ and $\mu \in (0, 1]$. Then

$$\alpha(v_\mu) = y + (1 - \mu)\delta \text{ and } \beta(v_\mu) = y + \mu\delta.$$

Since

$$\begin{aligned} v_\mu \leqslant B_r(u_\lambda) &\Leftrightarrow v_\mu \leqslant \mathcal{L}'_{\alpha(u_\lambda)-r}, v_\mu \leqslant \mathcal{R}'_{\beta(u_\lambda)+r} \\ &\Leftrightarrow r \geqslant \max\{\alpha(u_\lambda) - \alpha(v_\mu), \beta(v_\mu) - \beta(u_\lambda)\} \\ &\Leftrightarrow r \geqslant \max\{x + (1 - \lambda)\varepsilon - y - (1 - \mu)\delta \\ &\qquad y + \mu\delta - x - \lambda\varepsilon\} \end{aligned}$$

we have

$$\begin{aligned} d(v_\mu, u_\lambda) = \ &\max\{x + (1 - \lambda)\varepsilon - y - (1 - \mu)\delta, y \\ &+ \mu\delta - x - \lambda\varepsilon, 0\}. \end{aligned}$$

Let $\mathbb{R}^T(L) = \{\langle x, \varepsilon\rangle \mid x \in \mathbb{R}, \varepsilon > 0\}$. Then, $\mathbb{R}^T_F$ is a subspace of $\mathbb{R}(L)$. The subspace topology is the restriction of the standard L-topology to $\mathbb{R}^T(L)$.

Note that when u, v are classic numbers, i.e., $\delta = \varepsilon = 0$, it follows that

$$d(v_\mu, u_\lambda) = d(v_1, u_1) = \max\{x - y, y - x, 0\} = |x - y|$$

which is the classical case.

VI. Conclusion

In this work, we explore essential connections between pointwise closed-ball systems and pointwise pseudoquasi-metrics. The main results include the following.

1) Pointwise closed-ball spaces are categorically isomorphic to pointwise pseudoquasi-metric spaces.
2) The relations among pointwise pseudoquasi-metric topologies are studied.
3) By using the concept of pointwise closed-ball systems, a direct approach to the metrization theorem of L-fuzzy real line is given.

These main results fully illustrate that pointwise closed-ball systems are an effective tool to study pointwise (pseudoquasi-) metrics and the L-fuzzy real line. It is convenient for us to study the applications of L-fuzzy real line in the future.

We close this article with some problems and tasks for further exploration, which are as follows.

1) We would like to study the completeness of pointwise metric spaces.
2) We are further interested in the metric on L-fuzzy Sorgenfrey line via pointwise closed-ball systems.

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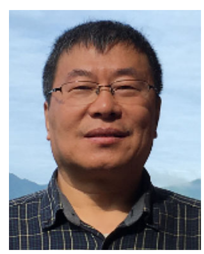
Chong Shen received the B.S. and M.S. degrees from the Hebei University of Science and Technology, Hebei, China, in 2013 and 2016, respectively, and the Ph.D. degree from the Beijing Institute of Technology, Beijing, China, in 2020.

He is currently a Teaching Staff with the School of Mathematical Sciences, Nanjing Normal University, Nanjing, China. His research interests include topology, domain theory, and fuzzy mathematics.

Yi Shi received the M.S. degree from Wuyi University, Jiangmen, China, in 2016, and the Ph.D. degree from the Beijing Institute of Technology, Beijing, China, in 2020.

He is currently a Postdoc with the Department of Mathematics, Shantou University, Shantou, China. His research interests include fuzzy topology and fuzzy convex spaces.



Fu-Gui Shi received the B.S. degree from Mudanjiang Teachers College, Heilongjiang, in 1985, and the M.S. and the Ph.D. degrees from Capital Normal University, Beijing, China, in 1988, and 2001, respectively.

He is currently a Professor with the Beijing Institute of Technology, Beijing, China. His research interests include fuzzy topology, fuzzy matroid, and fuzzy convex spaces.

Prof. Shi is one of the Editors of the *Iranian Journal of Fuzzy Systems*, *Journal of Fuzzy Systems and Mathematics*, *Asian Journal of Fuzzy and Applied Mathematics*, *Research in Applied Mathematics*, *Journal of Advanced Studies in Topology*, etc.

Hadrian Andradi received the M.Sc. degree from the Universitas Gadjah Mada, Yogyakarta, Indonesia, in 2012, and the Ph.D. degree from Nanyang Technological University, Singapore, in 2019.

He is currently a Teaching Staff with the Department of Mathematics, Universitas Gadjah Mada. His research interests include topology and domain theory.