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Lattice-equivalence of convex spaces

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Abstract. The links between order and topology have been extensively studied. The objective of this paper is to study the similar links between convex spaces and order structures via the lattices of convex sets. In particular, we prove a characterization of convex spaces uniquely determined by means of their lattices of convex sets, they are precisely sober and S_D. One adjunction between the category of convex spaces and that of continuous lattices is constructed, revealing the connection between convex structures and domain theory.

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1. Introduction

A convex structure $\mathcal{C}$ on a set X is a family of subsets of X containing the empty set, and closed under arbitrary intersections and directed unions, which can be regarded as the axiomatization of the properties of usual convex sets in Euclidean space. The pair $(X, \mathcal{C})$ is then called a convex space, and each member of $\mathcal{C}$ is called a convex subset of X. Convex structures exist in various different mathematics areas, such as lattices [6,16], algebras [11,13], metric spaces [12], graphs [4,5,10] and topological spaces [9,17]. For more about convex spaces, we refer the reader to the book [18].

It has got a long history to study the relation between order and topology. The every first order structure constructed from a topological space X might be the lattice $\Gamma(X)$ (with the inclusion order) of all closed subsets of X. Following Thron [15], we say two topological spaces X and Y are lattice-equivalent if the

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lattices $(\Gamma(X), \subseteq)$ and $(\Gamma(Y), \subseteq)$ are isomorphic. Three classic results due to Thron and Drake are as follows:

(R1) A complete lattice L is isomorphic to the lattice $\Gamma(X)$ of a topological space X if and only if the coprimes of L are join-dense in L [15]. Such lattices are said to be spatial.

(R2) For any sober spaces X and Y, $\Gamma(X)$ is isomorphic to $\Gamma(Y)$ if and only if X is homeomorphic to Y [3].

(R3) A topological space X has the property that for any T_0-space Y, $\Gamma(X)$ isomorphic to $\Gamma(Y)$ implies X is homeomorphic to Y if and only if X is sober and T_D [3].

In analogy to the above results in topology, this paper mainly studies the relation between order structures and convex spaces by means of the lattices of convex sets. For convex structures, a fundamental result analogy to the above (R1), is that a complete lattice L is isomorphic to $\mathcal{C}(X)$ of some convex space X, if and only if the compact elements of L are join-dense, or equivalently, L is algebraic (see [18, Further Topics 1.30]). However, on (R2) and (R3), there is no any responding result for convex spaces.

In this paper, we introduce the sober convex spaces, S_0 convex spaces and S_D convex spaces, and prove the following results:

(1) For any sober convex spaces X and Y, $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(Y)$ if and only if X is isomorphic to Y.

(2) A convex space X has the property that for any S_0 convex space Y, $\mathcal{C}(X)$ isomorphic to $\mathcal{C}(Y)$ implies X is isomorphic to Y if and only if X is sober and S_D.

The paper is organized as follows. In Section 2, we review some preliminaries on orders and convex spaces. In Section 3, an adjunction between the category of continuous lattices and that of convex spaces is constructed. In the last section, we introduce sober convex spaces and prove that the sobrification exists for every convex space. Then we prove that a convex space X has the property that $\mathcal{C}(X)$ isomorphic to $\mathcal{C}(Y)$ for any S_0 convex space Y implies X is isomorphic to Y if and only if X is sober and S_D.

2. Preliminaries

In this section, we will review some concepts and results on order and convex structures. For other undefined notions in this paper, the reader can refer to [1] for category theory, and [2,7] for order structures.

Let L be a complete lattice. The greatest (or the top) element and the least (or the bottom) element of L are denoted by $\top_L$ and $\bot_L$, respectively. For any $S \subseteq L$, write $\bigvee_L S$ and $\bigwedge_L S$ for the least upper bound (or the supremum) and the greatest lower bound (or the infimum) of S, respectively. In particular, $\bigvee_L \emptyset = \bot_L$ and $\bigwedge_L \emptyset = \top_L$. For any $A \subseteq L$, define $\uparrow A = \{b \in L \mid \exists a \in A, a \leq b\}$, and dually $\downarrow A = \{b \in L \mid \exists a \in A, b \leq a\}$.

For any $a, b \in L$, we denote by $a \ll b$, if for any directed subset D of L, the relation $b \leq \bigvee D$ implies $a \in \downarrow D$. A *continuous lattice* L is a complete lattice such that $x = \bigvee \downarrow\!\!x$ for all $x \in L$, where $\downarrow\!\!x = \{y \in L \mid y \ll x\}$. An element $x \in L$ is called *compact* if $x \ll x$. Note that $x \ll x$ if and only if for any subset S of L, $x \le \bigvee S$ implies $a \le \bigvee F$ for some finite subset F of S. Denote by $K(L)$ the set of all compact elements of L. A complete lattice L is called an *algebraic lattice* if $x = \bigvee\{y \in K(L) \mid y \le x\}$ for all $x \in L$.

Continuous lattices can be characterized by an infinite distributive law, which is called the *directed distributive law.*

Theorem 2.1 [7]. *For a complete lattice L, the following statements are equivalent.*

1. *L is a continuous lattice.*
2. *Let $\{x_{i,j} \mid i \in I, j \in J(i)\}$ be a non-empty family of elements in L such that $\{x_{i,j} \mid j \in J(i)\}$ is directed for all $i \in I$. Then it satisfies the* directed distributive law*:*

$$\bigwedge_{i\in I} \bigvee_{j\in J(i)} x_{i,j} = \bigvee_{f\in M} \bigwedge_{i\in I} x_{i,f(i)},$$

where M is the set of all choice functions $f : I \longrightarrow \bigcup_{i\in I} J(i)$ with $f(i) \in J(i)$ for all $i \in I$.

Definition 2.2 [2,7]. A (*monotone*) *Galois connection* between two complete lattices L and M consists of two monotone maps $f : L \longrightarrow M$ and $g : M \longrightarrow L$ such that

$$f(a) \le b \text{ if and only if } a \le g(b)$$

for all $a, b \in L$. In this case, f is called *the left adjoint* of g and g is called *the left adjoint* of f.

Theorem 2.3 [2,7]. *Let $g : M \longrightarrow L$ be a monotone map between complete lattices. Then g has a left adjoint if and only if g preserves arbitrary infima.*

Definition 2.4 [2,7]. A map $f : P \longrightarrow Q$ between two posets is called an *isomorphism* provided it is bijective and for any $x, y \in P$,

$$x \le y \text{ if and only if } f(x) \le f(y).$$

Two posets are *isomorphic* if there is an isomorphism between them.

Definition 2.5 [18]. Let X be a set. A subfamily $\mathcal{C}$ of the power set of X is called a *convex structure* on X, if it satisfies the following conditions:

(CS1) $\emptyset, X \in \mathcal{C}$;
(CS2) for any family $\{A_i \mid i \in I\} \subseteq \mathcal{C}$, $\bigcap_{i\in I} A_i \in \mathcal{C}$;
(CS3) for any directed family $\{D_i \mid i \in I\} \subseteq \mathcal{C}$, $\bigcup_{i\in I} D_i \in \mathcal{C}$.

In this case, we call the pair $(X, \mathcal{C})$, or simply X, a *convex space*, and every set A in $\mathcal{C}$ a *convex set.*

Definition 2.6 [18]. A convex space X is called S_1 if all singletons in X are convex.

One should note that in the definition of convex space, the underlying set could be empty. In this case, one can easily check that the power set $\{\emptyset\}$ of $\emptyset$ satisfies (CS1)–(CS3) and hence, $(\emptyset, \{\emptyset\})$ is a convex space.

In the sequel, we always use the notation $\mathcal{C}(X)$ to denote the family of all convex sets in a convex space X. The set $\mathcal{C}(X)$ is a partially ordered set when ordered by inclusion.

Definition 2.7 [18]. Let X be a convex space. For any subset A of X, the *hull* $co_X(A)$ of A is defined as

$$co_X(A) = \bigcap\{B \in \mathcal{C}(X) \mid A \subseteq B\}.$$

The operator co_X is called the *hull* on X. A convex set C is called a *polytope* if it is the hull of a finite subset of X. For convenience, we write $co_X(x)$ for $co_X(\{x\})$ when $x \in X$.

Definition 2.8 [18]. Let X be a space, and $Y \subseteq X$. Then the family

$$\mathcal{C}(Y) = \{C \cap Y \mid C \in \mathcal{C}(X)\}$$

is a convex structure on Y, and $(Y, \mathcal{C}(Y))$ is called a *subspace* of X.

Proposition 2.9 [18]. *The hull operator co_Y of a subspace Y of X satisfies*

$$\forall A \subseteq Y,\ co_Y(A) = co_X(A) \cap Y.$$

Proposition 2.10 [18]. *The hull operator co_X on a convex space X is a closure operator satisfying the following two equivalent conditions.*

(A1) $co_X(A) = \bigcup\{co_X(F) \mid F$ *is a finite subset of* $A\}$, *for every subset A of X.*
(A2) *If $\{D_i \mid i \in I\}$ is directed family of subsets of X, then* $co_X\left(\bigcup_{i\in I} D_i\right) = \bigcup_{i\in I} co_X(D_i)$.

Definition 2.11 [18]. Let $f : X \longrightarrow Y$ be a map between two convex spaces. Then f is called

(1) *convexity-preserving* (*CP* for short) if for any $B \in \mathcal{C}(Y)$, $f^{-1}(B) \in \mathcal{C}(X)$;
(2) *convex-to-convex* (*CC* for short) if for any $A \in \mathcal{C}(X)$, $f(A) \in \mathcal{C}(Y)$;
(3) a *isomorphism* if it is bijective, CC and CP.

We say that a convex space X is *isomorphic* to Y if there exists an isomorphism between X and Y.

Let **CS** be the category of convex spaces and CP maps.

Theorem 2.12 [18]. *Let $f : X \longrightarrow Y$ be a map between two convex spaces. Then the following statements are equivalent.*

(1) *f is a CP map.*
(2) *For every finite subset F of X, $f(co_X(F)) \subseteq co_Y(f(F))$.*
(3) *For every subset A of X, $f(co_X(A)) \subseteq co_Y(f(A))$.*

Theorem 2.13 [18]. *Let $f : X \longrightarrow Y$ be a map between two convex spaces. Then the following statements are equivalent.*

(1) *f is a CC map.*
(2) *For every finite subset F of X, $co_Y(f(F)) \subseteq f(co_X(F))$.*
(3) *For every subset A of X, $co_Y(f(A)) \subseteq f(co_X(A))$.*

3. An adjunction between continuous lattices and convex spaces

In this section, we study convex spaces by means of their convex set lattices. We show that there is an adjunction between the category **CS** and the category **CLat** of continuous lattices. This result will be used in Section 4.

A map $f : L \longrightarrow M$ between two continuous lattices is called a *continuous homomorphism* if it is pointed ($f(\bot_L) = \bot_M$) and preserves arbitrary infima and directed suprema. We denote by **CLat** the category of continuous lattices and continuous homomorphisms.

For any convex space X, $L = (\mathcal{C}(X), \subseteq)$ is a continuous lattice because it satisfies the directed distributive law (see Theorem 2.1). If $f : X \longrightarrow Y$ is a CP map between two convex spaces X and Y, then f^{-1} restricts to a map $\mathcal{C}(f) : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$, which is clearly pointed (i.e., $\mathcal{C}(f)(\emptyset) = \emptyset$), and preserves arbitrary intersections and directed unions. Then $\mathcal{C}$ then defines a contravariant functor from **CS** to **CLat**.

Let $\mathbf{2} = \{0, 1\}$ be the two element chain. For a complete lattice L, let $[L \to \mathbf{2}]$ be the set of all continuous homomorphisms from L to $\mathbf{2}$. Define

$$\mathcal{C}([L \to \mathbf{2}]) = \{\lambda(a) \mid a \in L\},$$

where $\lambda(a) = \{\phi \in [L \to \mathbf{2}] \mid \phi(a) = 1\}$. Note that for any $\phi \in [L \to \mathbf{2}]$, $\phi(\top_L) = \phi(\bigwedge_L \emptyset) = \bigwedge_{\mathbf{2}} \phi(\emptyset) = 1$.

The result below can be verified easily.

Lemma 3.1. *Let L be a complete lattice. Then λ satisfies the following statements.*

(1) $\lambda(\bot_L) = \emptyset$ *and* $\lambda(\top_L) = [L \to \mathbf{2}]$.
(2) *For any subset* $S \subseteq L$, $\lambda\left(\bigwedge_L S\right) = \bigcap_{a \in S} \lambda(a)$.
(3) *For any directed subset* $D \subseteq L$, $\lambda\left(\bigvee_L D\right) = \bigcup_{d \in D} \lambda(d)$.

Corollary 3.2. *For a complete lattice L, the family $\mathcal{C}([L \to \mathbf{2}]) = \{\lambda(a) \mid a \in L\}$ forms a convex structure on set $[L \to \mathbf{2}]$.*

Remark 3.3. For a complete lattice L, the set $[L \to \mathbf{2}]$ might be empty (e.g., $[I \to \mathbf{2}]$ where I denotes the unit interval). In this case, the induced convex space is $(\emptyset, \{\emptyset\})$.

In [18, Further Topics 1.30], it provides another approach to inducing convex spaces from complete lattices by using non-bottom compact elements. Precisely, for a complete lattice L, let $X = \mathsf{cpt}(L)$, the set of all non-bottom compact elements of L, and the convex subsets of X are the forms $K_a = {\downarrow} a \cap \mathsf{cpt}(L)$, $a \in L$. The requirement "$\bot_L \notin \mathsf{cpt}(L)$" is needed, because only in this way can we guarantee that $\emptyset = K_{\bot_L}$ is convex in $\mathsf{cpt}(L)$.

In the following, we shall show that, for any complete lattice L, the space $[L \to \mathbf{2}]$ is isomorphic to $\mathsf{cpt}(L)$.

Theorem 3.4. *Let L be a complete lattice. Then the map $\Theta : \mathsf{cpt}(L) \longrightarrow [L \to \mathbf{2}]$, $u \longmapsto \chi_{\uparrow u}$, is a isomorphism, where $\chi_{\uparrow u}$ is the characterize function of ${\uparrow} u$, that is,*

$$\forall a \in L,\ \chi_{\uparrow u}(a) = \begin{cases} 1, & u \leq a; \\ 0, & u \nleq a. \end{cases}$$

Proof. We prove this result in three steps.

Step 1. Θ is well-defined. It suffices to prove that $\chi_{\uparrow u}$ is a continuous homomorphism from L to $\mathbf{2}$ for any $u \in \mathsf{cpt}(L)$.

(i) Since $u \neq \perp_L$, it follows that $\chi_{\uparrow u}(\perp_L) = 0$, i.e., $\chi_{\uparrow u}$ is pointed.

(ii) Take any subset $S \subseteq L$. It holds that

$$\begin{aligned} \chi_{\uparrow u}(\bigwedge_L S) = 1 &\Leftrightarrow u \leq \bigwedge_L S \\ &\Leftrightarrow u \leq a \text{ for all } a \in S \\ &\Leftrightarrow \chi_{\uparrow u}(a) = 1 \text{ for all } a \in S \\ &\Leftrightarrow \bigwedge_{\mathbf{2}} \{\chi_{\uparrow u}(a) \mid a \in S\} = 1. \end{aligned}$$

Thus $\chi_{\uparrow u}(\bigwedge_L S) = \bigwedge_{\mathbf{2}} \{\chi_{\uparrow u}(a) \mid a \in S\}$.

(iii) Take any directed subset $D \subseteq L$. Then we have

$$\begin{aligned} \chi_{\uparrow u}(\bigvee_L D) = 1 &\Leftrightarrow u \leq \bigvee_L D \\ &\Leftrightarrow u \leq d_0 \text{ for some } d_0 \in D \\ &\Leftrightarrow \chi_{\uparrow u}(d_0) = 1 \text{ for some } d_0 \in D \\ &\Leftrightarrow \bigvee_{\mathbf{2}} \{\chi_{\uparrow u}(d) \mid d \in D\} = 1. \end{aligned}$$

The second equivalence holds because u is a compact element of L. Hence $\chi_{\uparrow u}(\bigvee_L D) = \bigvee_{\mathbf{2}} \{\chi_{\uparrow u}(d) \mid d \in D\}$.

Step 2. Θ is bijective. On one hand, if $u \nleq v$ in $\mathsf{cpt}(L)$, then $\chi_{\uparrow u}(v) = 0 \neq 1 = \chi_{\uparrow v}(v)$. Thus $\chi_{\uparrow u} \neq \chi_{\uparrow v}$. This means that Θ is injective. On the other hand, for any $\phi \in [L \to \mathbf{2}]$, let $u = \bigwedge_L \phi^{-1}(\{1\})$.

(i) u is compact. Suppose $D \subseteq L$ is directed such that $u \leq \bigvee_L D$. We have

$$\begin{aligned} \phi(u) &= \phi(\bigwedge_L \phi^{-1}(\{1\})) \\ &= \bigwedge_{\mathbf{2}} \{\phi(x) \mid x \in \phi^{-1}(\{1\})\} \\ &= 1. \end{aligned}$$

Because ϕ is monotone, it follows $\phi(u) \leq \phi(\bigvee_L D)$. Thus

$$\phi(\bigvee_L D) = \bigvee_{\mathbf{2}} \{\phi(d) \mid d \in D\} = 1.$$

Then there exists $d_0 \in D$ such that $\phi(d_0) = 1$, i.e., $d_0 \in \phi^{-1}(\{1\})$. Therefore $u = \bigwedge_L \phi^{-1}(\{1\}) \leq d_0$.

(ii) $\Theta(u) = \chi_{\uparrow u} = \phi$. This can be obtained by the following:

$$\begin{aligned}\chi_{\uparrow u}(a) = 0 &\Leftrightarrow u = \bigwedge_L \phi^{-1}\{1\} \nleq a\\ &\Leftrightarrow a \notin \phi^{-1}(\{1\})\\ &\Leftrightarrow \phi(a) = 0.\end{aligned}$$

It follows that $\chi_{\uparrow u} = \phi$.

The above (i) and (ii) together imply Θ is surjective.

Step 3. Θ is CP and CC. Take any $u \in L$. It is easy to check that $\Theta^{-1}(\lambda(u)) = K_u$, which means Θ is CP. Furthermore, take any $\phi \in \lambda(u)$, and let $v = \bigwedge_L \phi^{-1}(\{1\})$. By the proof of Step 2 (ii), we obtain $\chi_{\uparrow v} = \phi$ and $v \leq u$. It thus follows $\phi \in \Theta(K_u)$, showing that $\lambda(u) \subseteq \Theta(K_u)$. The inclusion $\Theta(K_u) = \{\chi_{\uparrow v} \mid v \leq u\} \subseteq \lambda(u)$ trivially holds. Thus $\Theta(K_u) = \lambda(u)$. This shows that Θ is a CC map. □

Theorem 3.5. *There is a contravariant functor* cpt *from* **CLat** *to* **CS***, defined by mapping every continuous lattice L to* $\mathsf{cpt}(L)$*, and for any continuous morphism $g : M \longrightarrow L$ in* **CLat***, by letting* $\mathsf{cpt}(g)$ *be restriction of the left adjoint of g to* $\mathsf{cpt}(L) \longrightarrow \mathsf{cpt}(M)$*.*

Proof. We prove this result in two steps.

Step 1. $\mathsf{cpt}(g)$ is well-defined.

Since g preserves arbitrary infima, the left adjoint of g exists. It suffices to show $\mathsf{cpt}(g)(u) \in \mathsf{cpt}(M)$ for all $u \in \mathsf{cpt}(L)$. Firstly, $\mathsf{cpt}(g)(u) \neq \perp_M$ (otherwise $u \leq g(\perp_M) \leq \perp_L$). If D is a directed subset of M such that $\mathsf{cpt}(g)(u) \leq \bigvee D$, then $u \leq g(\bigvee D) = \bigvee g(D)$. By the compactness of u, there exists $d \in D$ such that $u \leq g(d)$, which means $\mathsf{cpt}(g)(u) \leq d$. Thus $\mathsf{cpt}(g)(u)$ is compact.

Step 2. $\mathsf{cpt}(g)$ is a CP map.

Recall that the convex sets in $\mathsf{cpt}(L)$ are the form of K_a, $a \in L$. Now for any $b \in M$, we have

$$\begin{aligned}(\mathsf{cpt}(g))^{-1}(K_b) &= \{u \in \mathsf{cpt}(L) \mid \mathsf{cpt}(g)(u) \in K_b\}\\ &= \{u \in \mathsf{cpt}(L) \mid \mathsf{cpt}(g)(u) \leq b\}\\ &= \{u \in \mathsf{cpt}(L) \mid u \leq g(b)\}\\ &= \mathsf{cpt}(L) \cap {\downarrow} g(b) = K_{g(b)}.\end{aligned}$$

The third equation above holds because $\mathsf{cpt}(g)$ is the restriction of the left adjoint of g. Thus $\mathsf{cpt}(g)$ is a CP map from $\mathsf{cpt}(L)$ to $\mathsf{cpt}(M)$.

That cpt preserves identities and composition is straightforward. Therefore cpt is a contravariant functor from **CLat** to **CS**. □

Proposition 3.6. *Let L be a complete lattice and let $\delta_L : L \longrightarrow \mathcal{C}(\mathsf{cpt}(L))$ be the map that sends $u \in L$ to K_u.*

(1) *The map δ_L is monotone and surjective.*
(2) *The following statements are equivalent.*
   (i) *L is algebraic.*
   (ii) *δ_L is injective.*

(iii) δ_L *is isomorphic.*

Proof. (1) That δ_L is monotone is trivial, and it is surjective because

$$\mathcal{C}(\mathsf{cpt}(L)) = \{K_a \mid a \in L\}.$$

(2) (i) $\Rightarrow$ (iii). Assume L is algebraic. If $a, b \in L$ such that $a \not\leq b$, then there exists $u \in \mathsf{cpt}(L)$ such that $u \leq a$ and $u \not\leq b$, implying that $u \in K_a$ but $u \notin K_b$. Then $K_a \nsubseteq K_b$. Therefore, δ_L is injective.

(iii) $\Rightarrow$ (ii). It is trivial.

(ii) $\Rightarrow$ (i). If δ_L is injective, then it is bijective by (1). Also, it is easy to check that $\bigvee K_a = a$ for all $a \in L$. Thus L is an algebraic lattice. $\square$

Corollary 3.7 [18, Further Topics, 1.30.2]. *A complete lattice L is algebraic if and only if there is a convex space X such that L is isomorphic to $\mathcal{C}(X)$.*

Corollary 3.8. *$\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(\mathsf{cpt}(\mathcal{C}(X)))$, for every convex space X.*

Theorem 3.9. *The contravariant functor $\mathsf{cpt} : \mathbf{CLat} \longrightarrow \mathbf{CS}$ is right adjoint to the contravariant functor $\mathcal{C} : \mathbf{CS} \longrightarrow \mathbf{CLat}$.*

Proof. We prove this result in three steps.

Step 1. There is a natural transformation $\eta : I_{\mathbf{CS}} \longrightarrow \mathsf{cpt} \circ \mathcal{C}$ ($I_{\mathbf{CS}}$ is the identity functor on **CS**) defined by

$$\eta_X : X \longrightarrow \mathsf{cpt}(\mathcal{C}(X)),\ x \longmapsto co_X(x).$$

In fact, if $g : X \longrightarrow Y$ is a CP map between two convex spaces and $A \in \mathsf{cpt}(\mathcal{C}(X))$, i.e., A is a non-empty polytope in X, then there exists a non-empty finite subset F of X such that $A = co_X(F)$. Note that

$$\begin{aligned}
\mathsf{cpt}(\mathcal{C}(g))(A) &= \bigcap\{B \in \mathsf{cpt}(\mathcal{C}(Y)) \mid A \subseteq g^{-1}(B)\} \\
&= \bigcap\{B \in \mathsf{cpt}(\mathcal{C}(Y)) \mid F \subseteq g^{-1}(B)\} \\
&= \bigcap\{B \in \mathsf{cpt}(\mathcal{C}(Y)) \mid g(F) \subseteq B\} \\
&= co_{\mathsf{cpt}(\mathcal{C}(Y))}(g(F)).
\end{aligned}$$

From above, one can deduce that $\eta_Y \circ g = \mathsf{cpt}(\mathcal{C}(g)) \circ \eta_X$, i.e., the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "g"'] & \mathsf{cpt}(\mathcal{C}(X)) \arrow[d, "\mathsf{cpt}(\mathcal{C}(g))"] \\
Y \arrow[r, "\eta_Y"] & \mathsf{cpt}(\mathcal{C}(Y))
\end{tikzcd}$$

Therefore η is a natural transformation.

Step 2. Let L be a continuous lattice, and $f : X \longrightarrow \mathsf{cpt}(L)$ a morphism in **CS**. Define the map $\widehat{f} : L \longrightarrow \mathcal{C}(X)$ that sends $a \in L$ to $f^{-1}(K_a)$. Then $\widehat{f}$ is well-defined because f is CP, and $\widehat{f}$ is a continuous homomorphism by

Proposition 3.6. Thus $\widehat{f}$ is a morphism in **CLat**. In addition, for any $x \in X$, we have

$$\begin{aligned}
\mathsf{cpt}(\widehat{f})(\eta_X(x)) &= \bigwedge_{\mathsf{cpt}(L)} \{u \in \mathsf{cpt}(L) \mid co_X(x) \subseteq \widehat{f}(u)\} \\
&= \bigwedge_{\mathsf{cpt}(L)} \{u \in \mathsf{cpt}(L) \mid co_X(x) \subseteq f^{-1}(K_u)\} \\
&= \bigwedge_{\mathsf{cpt}(L)} \{u \in \mathsf{cpt}(L) \mid f(x) \in K_u\} \\
&= \bigwedge_{\mathsf{cpt}(L)} \{u \in \mathsf{cpt}(L) \mid f(x) \leq u\} \\
&= f(x).
\end{aligned}$$

Hence the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \mathsf{cpt}(\mathcal{C}(X)) \arrow[d, dotted, "\mathsf{cpt}(\widehat{f})"] & \mathcal{C}(X) \arrow[d, dotted, "\widehat{f}"] \\
& \mathsf{cpt}(L) & L
\end{tikzcd}$$

Step 3. Now we confirm the uniqueness of $\widehat{f}$. Let $g : L \longrightarrow \mathcal{C}(X)$ be a continuous homomorphism, i.e, a morphism in **CLat**, such that $\mathsf{cpt}(g) \circ \eta_X = f$ and $g \neq \widehat{f}$. Then there exists $a \in L$ such that $g(a) \neq \widehat{f}(a)$.

Case 1. $g(a) \nsubseteq \widehat{f}(a)$. There exists $x \in X$ satisfying $x \in g(a)$ but $x \notin \widehat{f}(a)$. As $\widehat{f}(a) = f^{-1}(K_a)$, we have $f(x) \nleq a$. Thus $\mathsf{cpt}(g)(co_X(x)) = f(x) \nleq a$. That $\mathsf{cpt}(g)$ is the left adjoint of g implies $co_X(x) \nsubseteq g(a)$, a contradiction.

Case 2. $\widehat{f}(a) \nsubseteq g(a)$. There exists $x \in X$ satisfying $x \in \widehat{f}(a)$ but $x \notin g(a)$. Then $f(x) \leq a$ because $\widehat{f}(a) = f^{-1}(K_a)$. Further, we have

$$f(x) = \mathsf{cpt}(g)(\eta_X(x)) = \mathsf{cpt}(g)(co_X(x)) \leq a,$$

implying $co_X(x) \nsubseteq g(a)$, a contradiction. $\square$

4. Sobriety and sobrification of convex spaces

Sobriety plays an every important role in the theory of Non-Hausdorff topological spaces. One of the key properties of sober topological spaces is that every topological space has a sobrification [8, Proposition 8.2.22]. In the current section, we introduce sober convex spaces and prove some properties similar to that of sober topological spaces. In particular, the sobrification of each convex space exists. Based on these ideas, we further characterize those convex spaces which are uniquely determined by their convex set lattices.

Recall that a topological space is T_0 if and only if for any $x, y \in X$, $x \neq y$ implies $\mathrm{cl}_X(x) \neq \mathrm{cl}_X(y)$, where $\mathrm{cl}_X(x)$ is the closure of x. In the following, we will introduce an analogous concept to T_0, called S_0.

Definition 4.1. A convex space X is called S_0 if for any $x, y \in X$, $x \neq y$ implies $co_X(x) \neq co_X(y)$.

Recall that a convex space X is called S_1 (see [18]) if all singletons in X are convex. Then clearly every S_1 convex space is S_0.

The following results is straightforward.

Proposition 4.2. *For any convex space X, define $\eta_X : X \longrightarrow \mathsf{cpt}(\mathcal{C}(X))$, $x \mapsto co_X(x)$.*

(1) *η_X is a CP map.*
(2) *The following statements are equivalent.*
   (a) *η_X is injective.*
   (b) *X is an S_0 convex space.*
   (c) *η_X is an embedding, i.e., X and $\eta_X(X)$ are isomorphism.*

Remark 4.3. Note that the lattice $\mathcal{C}(X)$ of convex sets in a convex space X is algebraic, and the non-bottom compact elements $\mathsf{cpt}(\mathcal{C}(X))$ are exactly the non-empty polytopes, i.e., the hull of non-empty finite subsets of X. Then η_X is surjective if and only if every non-empty polytope in X is the hull of some point $x \in X$.

Recall that a non-empty subset A of a topological space X is *irreducible* if $A \subseteq B \cup C$ for closed subsets B and C implies $A \subseteq B$ or $A \subseteq C$. A topological space is *sober* if every irreducible closed set is the closure of a unique singleton. In the following, the concept of sober in convex spaces is introduced, which will be used later.

Definition 4.4. A convex space X is called *sober* if every non-empty polytope is the hull of a unique singleton.

Let **Sob** be the category of sober convex spaces and CP maps.

Remark 4.5. Every sober space is S_0.

Theorem 4.6. *Let X and Y be two sober convex spaces. Then $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(Y)$ if and only if X is isomorphic to Y.*

Proof. It suffices to prove the only if. Clearly, $\mathcal{C}(X)$ isomorphic to $\mathcal{C}(Y)$ implies that $\mathsf{cpt}(\mathcal{C}(X))$ is isomorphic to $\mathsf{cpt}(\mathcal{C}(Y))$. As X and Y are sober, there is an isomorphism $\mu : \{co_X(x) \mid x \in X\} \longrightarrow \{co_Y(y) \mid y \in Y\}$.

Now define a map $g : X \longrightarrow Y$ by $g(x) = y$ such that $\mu(co_X(x)) = co_Y(y)$. Since μ is an isomorphism, it follows that g is a bijective map. It remains to prove that g is CC and CP. For any $x' \in co_X(x)$, we have $co_X(x') \subseteq co_X(x)$, which implies $co_Y(g(x')) = \mu(co_X(x')) \subseteq \mu(co_X(x)) = co_Y(g(x))$. It then holds that

$$g(co_X(x)) = \bigcup\{g(x') \mid x' \in co_X(x)\} \subseteq co_Y(g(x)).$$

Conversely, if $y' \in co_Y(g(x))$, then $co_Y(y') \subseteq co_Y(g(x))$. Then

$$\mu(co_X(g^{-1}(y'))) = co_Y(y') \subseteq co_Y(g(x)) = \mu(co_X(x)).$$

It follows that $co_X(g^{-1}(y')) \subseteq co_X(x)$, implying that $g^{-1}(y') \in co_X(x)$, i.e., $y' \in g(co_X(x))$. Thus $co_Y(g(x)) \subseteq g(co_X(x))$. Hence $co_Y(g(x)) = g(co_X(x))$. Now let F be any non-empty finite subset of X. There exits $x^* \in X$ such that $co_X(x^*) = co_X(F)$ as X is sober. Then

$$\begin{aligned} co_Y(g(F)) &= \bigvee_{\mathcal{C}(Y)} \{co_Y(g(x)) \mid x \in F\} \\ &= \bigvee_{\mathcal{C}(Y)} \{\mu(co_X(x)) \mid x \in F\} \\ &= \mu(co_X(F)). \end{aligned}$$

Note that

$$\mu(co_X(F)) = \mu(co_X(x^*)) = co_Y(g(x^*)) = g(co_X(x^*)) = g(co_X(F)).$$

So $co_Y(g(F)) = g(co_X(F))$. Hence g is CC and CP by Theorems 2.12 and 2.13. $\square$

Proposition 4.7. *For any algebraic lattice L, $\mathsf{cpt}(L)$ is sober.*

Proof. First, note that the non-empty polytopes in $\mathsf{cpt}(L)$ are exactly K_u, $u \in \mathsf{cpt}(L)$. Furthermore, for any K_u where $u \in \mathsf{cpt}(L)$,

$$co_{\mathsf{cpt}(L)}(u) = \bigcap \{K_v \mid v \in L, u \in K_v\} = \bigcap \{K_v \mid v \in L, u \leq v\} = K_u,$$

which shows that K_u is the hull of singleton $\{u\}$ in $\mathsf{cpt}(L)$. The uniqueness of u is determined by the bijective map $u \longmapsto K_u$ (see Proposition 3.6). Hence $\mathsf{cpt}(L)$ is sober. $\square$

Corollary 4.8. *Let X be a convex space. Then $\mathsf{cpt}(\mathcal{C}(X))$ equipped with the convex structure $\{K_C \mid C \in \mathcal{C}(X)\}$ is a sober space.*

Definition 4.9. A *sobrification* of a convex space X is a sober convex space Y together with a CP map $\eta_X : X \longrightarrow Y$, such that for any CP map $f : X \longrightarrow Z$ into a sober convex space Z, there exists a unique CP map $\widehat{f} : Y \longrightarrow Z$ satisfying $f = \widehat{f} \circ \eta_X$.

Lemma 4.10. *The sobrification of a convex space is unique up to isomorphism.*

Proof. Suppose both (Y, η_X) and (Z, γ_X) are sobrifications of the convex space X. Then there exist CP maps $\widehat{f} : Y \longrightarrow Z$ and $\widehat{g} : Z \longrightarrow Y$ such that $\gamma_X = \widehat{f} \circ \eta_X$ and $\eta_X = \widehat{g} \circ \gamma_X$, that is, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\gamma_X"'] & Y \arrow[d, dotted, "\widehat{f}"] \\ & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\gamma_X"'] & Y \\ & Z \arrow[u, dotted, "\widehat{g}"'] \end{tikzcd}$$

It follows that $\eta_X = \widehat{g} \circ \widehat{f} \circ \eta_X$ and $\gamma_X = \widehat{f} \circ \widehat{g} \circ \gamma_X$. Since the identities id_X and id_Y are uniquely such that the following diagram commutes,

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\eta_X"'] & Y \arrow[d, dotted, "\mathrm{id}_Y"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\gamma_X"] \arrow[dr, "\gamma_X"'] & Z \arrow[d, dotted, "\mathrm{id}_Z"] \\ & Z \end{tikzcd}$$

we have that $\mathrm{id}_X = \widehat{g} \circ \widehat{f}$ and $\mathrm{id}_Y = \widehat{f} \circ \widehat{g}$, meaning that $\widehat{f}$ and $\widehat{g}$ are mutually inverse. Thus $\widehat{f}$ is a isomorphism from Y to Z. $\square$

Lemma 4.11. *Let $f : X \longrightarrow Y$ be a CP map between two convex spaces. Then $co_Y(f(A)) \in \mathsf{cpt}(\mathcal{C}(Y))$ for all $A \in \mathsf{cpt}(\mathcal{C}(X))$.*

Proof. If $A \in \mathsf{cpt}(\mathcal{C}(X))$, then there exists a non-empty finite subset F of X such that $A = co_X(F)$. As f is a CP map, it follows $f(co_X(F)) \subseteq co_Y(f(F))$. Then we have $co_Y(f(A)) = co_Y(f(co_X(F))) \subseteq co_Y(co_Y(f(F))) = co_Y(f(F))$, showing that $co_Y(f(A)) = co_Y(f(F))$. Note that $f(F)$ is finite. Hence $co_Y(f(A))$ is a non-empty polytope in Y. $\square$

Theorem 4.12. *For any convex space X, the space $\mathsf{cpt}(\mathcal{C}(X))$ with map $x \longmapsto co_X(x)$ is a sobrification of X.*

Proof. Let Y be a sober convex space and let $f : X \longrightarrow Y$ be a CP map.

Step 1. Take any $A \in \mathsf{cpt}(\mathcal{C}(X))$, i.e., A is a non-empty polytope in X. Then by Lemma 4.11, $co_Y(f(A)) \in \mathsf{cpt}(\mathcal{C}(Y))$. Since Y is sober, there exists a unique $y \in Y$ such that $co_Y(f(A)) = co_Y(y)$. Define $\widehat{f} : K(\mathcal{C}(X)) \longrightarrow Y$ by $A \longmapsto y$, i.e., $co_Y(f(A))$ is the hull of y. Then $\widehat{f}$ is a CP map because $\widehat{f}^{-1}(D) = K_{f^{-1}(D)}$.

Step 2. Take any $x \in X$. Suppose $\widehat{f}(\eta(x)) = \widehat{f}(co_X(x)) = y$, which means $co_Y(y) = co_Y(f(co_X(x))) = co_Y(f(x))$. Since

$$co_Y(f(co_X(x))) \in \mathsf{cpt}(\mathcal{C}(Y))$$

and Y is sober, we have $f(x) = y = \widehat{f}(\eta(x))$. So $f = \widehat{f} \circ \eta$, i.e., the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & \mathsf{cpt}(\mathcal{C}(X)) \arrow[d, dotted, "\widehat{f}"] \\ & Y \end{tikzcd}$$

Step 3. Suppose that $g : \mathsf{cpt}(\mathcal{C}(X)) \longrightarrow Y$ is a CP map from $\mathsf{cpt}(\mathcal{C}(X))$ to Y such that $f = g \circ \eta_X$. Take any $A \in \mathsf{cpt}(\mathcal{C}(X))$, and let $\widehat{f}(A) = y$. By the definition of $\widehat{f}$, we have $co_Y(f(A)) = co_Y(y)$, and

$$f(A) = \bigcup_{x \in A} f(x) = \bigcup_{x \in A} g(co_X(x)) \subseteq co_Y(y),$$

it follows that $co_X(x) \in g^{-1}(co_Y(y))$ for all $x \in A$. Since $g^{-1}(co_Y(y))$ is a convex set in $\mathsf{cpt}(\mathcal{C}(X))$, we have $A \in co_{\mathsf{cpt}(\mathcal{C}(X))}(\{co_X(x) \mid x \in A\}) = K_A \subseteq g^{-1}(co_Y(y))$. Hence $g(A) \in co_Y(y)$, i.e., $g(A) \subseteq co_Y(y)$. Conversely, as

$f(x) = g(co_X(x)) \in co_Y(g(A))$ for all $x \in A$, it follows that $y \in co_Y(f(A)) \subseteq co_Y(g(A))$. Thus $co_Y(g(A)) = co_Y(y)$. Since Y is S_0, $g(A) = y = \widehat{f}(A)$. Therefore $g = \widehat{f}$. $\square$

The following is a straightforward consequence of Theorem 4.12.

Corollary 4.13. *The full subcategory* **Sob** *of* **CS** *is reflective in* **CS**.

Proposition 4.14. *If Y is the sobrification of a convex space X, then $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(Y)$.*

Proof. This is straightforward by Corollary 3.8, Theorem 4.12 and Lemma 4.10. $\square$

Let **AgLat** be the category of all algebraic lattices with continuous homomorphisms as morphisms.

According to Theorem 3.9, there is an adjunction $\mathcal{C} \dashv \mathsf{cpt}$ between **CLat** and **CS**. In generally, they are not inverse to each other. However, when restricting **CLat** to its full subcategory **AgLat** and **CS** to its full subcategory **Sob**, the functors $\mathcal{C}$ and cpt are categorically equivalent.

Theorem 4.15. *The adjunction $\mathcal{C} \dashv \mathsf{cpt}$ restricts to an equivalence between* **Sob** *and* **AgLat**.

Proof. It is trivial by Proposition 3.6, Proposition 4.2 and Remark 4.3. $\square$

A convex space X is called S_D if for every $x \in X$, $co_X(x) - \{x\}$ is convex. An element x of a complete lattice L is called *strongly irreducible* if for any $S \subseteq L$, $x = \bigvee S$ implies $x = y$ for some $y \in S$.

Remark 4.16 [14]. A topological space X is called T_D if for every $x \in X$, $\mathrm{cl}_X(x) - \{x\}$ is closed.

Lemma 4.17. *Every S_1 convex space is S_D, and every S_D convex space is S_0.*

Proof. That S_1 implies S_D is easy since the empty set is always convex. Let X be an S_D convex space. We need to show X is S_0. Suppose there exist $x, y \in X$ such that $co_X(x) = co_X(y)$ and $x \neq y$. Since X is S_D, $co_X(x) - \{y\}$ is convex. That $x \in co_X(x) - \{y\}$ implies $co_X(x) \subseteq co_X(x) - \{y\}$. It follows that $y \notin co_X(x)$, a contradiction. Thus X is S_0. $\square$

Example 4.18. (1) Let $X = \mathbb{R}$ be the real line with the usual order. A subset C of X is called order convex provided for any $x, y \in X$,

$$x \leq z \leq y \text{ implies } z \in C.$$

Denote by $\mathcal{C}(X)$ the set of all the order convex sets in X. Clearly, $(X, \mathcal{C}(X))$ is an S_1 convex space. Note that $co_X(\{1,2\}) \neq co_X(x)$ for any $x \in X$. Thus X is not sober.

(2) Let $\mathbb{N}$ be the set of all nonnegative integers, and $Y = \mathbb{N} \cup \{a, \omega\}$. Define a partial order $\leq$ on Y as follows (see Figure 1):

(i) for any $x \in Y$, $0 \leq x \leq \omega$;

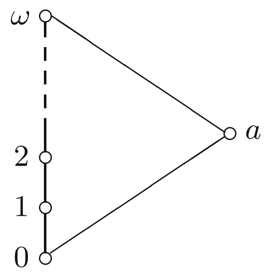

Figure 1. The poset $(Y, \leq)$

$$\begin{tikzcd}
S_1 \arrow[r, "\times" marking] \arrow[d] & \text{Sober} \arrow[d] \arrow[dl, "\times" marking] \\
S_D \arrow[r] & S_0
\end{tikzcd}$$

Figure 2. An illustration of the relationships

(ii) for any $x \in \mathbb{N} - \{0\}$, x and a are incomparable.

Define $\mathcal{C}(Y) \subseteq \mathbf{2}^Y$ by $\mathcal{C}(Y) = \{\downarrow y \mid y \in Y\} \cup \{\mathbb{N}, \emptyset\}$. Then we have the following easy facts regarding $(Y, \mathcal{C}(Y))$.

(i) $\mathcal{C}(Y)$ is a convex structure on Y.

(ii) For any non-empty finite set $F \subseteq Y$, we have

$$co_Y(F) = co_Y\left(\bigvee F\right) = \downarrow \bigvee F.$$

Thus Y is a sober space.

(iii) Y is not S_D because the set $co_Y(\omega) - \{\omega\} = Y - \{\omega\}$ is not convex.

The example in Figure 1 shows that S_1 (not to speak of S_0) does not imply sober, and sober does not imply S_D (not to speak of S_1). The relationships among sober, S_1, S_D and S_0 are depicted in Figure 2.

In [18], the quotient convex space is introduced. Precisely, let X be a convex space and let R be an equivalence relation on X. The quotient set X/R consists of all R-equivalence classes, and the quotient function $q : X \longrightarrow X/R$ sends $x \in X$ to its R-equivalence class ($[x]_R := \{x' \mid (x, x') \in R\}$). A convex structure $\mathcal{C}(X)/R$ on X/R is defined as follows:

$$\mathcal{C}(X)/R = \left\{\widehat{C} \subseteq X/R \mid q^{-1}(\widehat{C}) \in \mathcal{C}(X)\right\}.$$

Lemma 4.19. *Let X be a convex space and let $R_{co} = \{(x_1, x_2) \mid co_X(x_1) = co_X(x_2)\}$.*

(1) *R_{co} is an equivalence relation on X.*
(2) *$\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(X)/R_{co}$.*
(3) *The quotient space X/R_{co} is an S_0 convex space.*

Proof. The result (1) is trivial.

(2) First, we prove $\mathcal{C}(X)/R_{co} = \{C/R_{co} \mid C \in \mathcal{C}(X)\}$, where $C/R_{co} = \{[y]_{R_{co}} \mid y \in C\}$ for each $C \in \mathcal{C}(X)$. In fact, for any $C \in \mathcal{C}(X)$, it is easy to check $q^{-1}(C/R_{co}) = C \in \mathcal{C}(X)$, showing that $C/R_{co} \in \mathcal{C}(X)/R_{co}$. Conversely, if $\widehat{C} \in \mathcal{C}(X)/R_{co}$, then $q^{-1}(\widehat{C}) \in \mathcal{C}(X)$. Since q is surjective, it follows that $\widehat{C} = q(q^{-1}(\widehat{C})) = q^{-1}(\widehat{C})/R_{co}$.

Secondly, let $\varphi : \mathcal{C}(X) \longrightarrow \mathcal{C}(X)/R_{co}$ be the map that sends $C \in \mathcal{C}(X)$ to C/R_{co}. Then it is a surjective map, and it is easy to verify that

$$\forall C, C' \in \mathcal{C}(X),\ C \subseteq C' \Leftrightarrow C/R_{co} \subseteq C'/R_{co}.$$

Hence φ is an isomorphism.

(3) By using (2), one can easily obtain that

$$co_{X/R_{co}}([x]_{R_{co}}) = co_X(x)/R_{co}$$

for all $x \in X$. Now suppose $[x]_{R_{co}} \neq [x']_{R_{co}}$. This implies $co_X(x) \neq co_X(x')$. Since ϕ is an isomorphism, it follows that $co_{X/R_{co}}([x]_{R_{co}}) = co_X(x)/R_{co} \neq co_X(x')/R_{co} = co_{X/R_{co}}([x']_{R_{co}})$. Thus X/R_{co} is S_0. $\square$

Lemma 4.20. *Suppose X is a convex space, and there is a $x_0 \in X$ such that $co_X(x_0) - \{x_0\} \notin \mathcal{C}(X)$. Let $X' = X - \{x_0\}$ and $\mathcal{C}(X') = \{C - \{x_0\} \mid C \in \mathcal{C}(X)\}$.*

1. *$(X', \mathcal{C}(X'))$ is a convex space.*
2. *$\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(X')$.*
3. *If X is S_0, then X' is S_0.*

Proof. (1) It is easy to check $(X', \mathcal{C}(X'))$ is a subspace of X (see Definition 2.8), and hence by proposition 2.9, $co_{X'}(A) = co_X(A) \cap X' = co_X(A) - \{x_0\}$.

(2) Let $\psi : \mathcal{C}(X) \longrightarrow \mathcal{C}(X')$ be the map that sends $C \in \mathcal{C}(X)$ to $C - \{x_0\}$. Clearly ψ is monotone and surjective. Now we prove that for any $C, D \in \mathcal{C}(X)$, $C - \{x_0\} \subseteq D - \{x_0\}$ implies $C \subseteq D$. It is trivial if $x_0 \notin C$. Suppose $x_0 \in C$. It only needs to prove $x_0 \in D$. As $x_0 \in C$ and C is a convex set, we have $co_X(x_0) \subseteq C$, and hence $co_X(x_0) - \{x_0\} \subseteq D$. Thus $co_X(co_X(x_0) - \{x_0\}) \subseteq C - \{x_0\} \subseteq D$. Note that $co_X(x_0) - \{x_0\} \subsetneq co_X(co_X(x_0) - \{x_0\}) \subseteq co_X(x_0)$. So $co_X(x_0) = co_X(co_X(x_0) - \{x_0\}) \subseteq D$. Thus $C \subseteq D$. Therefore, $\psi(C) \subseteq \psi(D)$ implies $C \subseteq D$. Hence ψ is an isomorphism.

(3) Let $y, y' \in Y$ such that $co_Y(y) = co_Y(y')$. This means $co_X(y) - \{x_0\} = co_X(y') - \{x_0\}$. Since ψ is an isomorphism, it follows that $co_X(y) = co_X(y')$, and since X is S_0, we have $y = y'$. Hence X' is S_0. $\square$

Theorem 4.21. *Let X be a convex space. Then the following statements are equivalent.*

1. *X is sober and S_D.*
2. *X is an S_0 convex space whose every polytope is strongly irreducible elements of $\mathcal{C}(X)$.*
3. *For any S_0 convex space Y, $\mathcal{C}(X)$ isomorphic to $\mathcal{C}(Y)$ implies X is isomorphic to Y.*

Proof. First note that for any $\{A_i \mid i \in I\} \subseteq \mathcal{C}(X)$, the supremum $\bigvee_{\mathcal{C}(X)}\{A_i \mid i \in I\}$ of A_i, $i \in I$ in the lattice $\mathcal{C}(X)$ equals $co_X\left(\bigcup\{A_i \mid i \in I\}\right)$.

(1) $\Rightarrow$ (2). Let C be a non-empty polytope of X. Then $C = co_X(x)$ for some $x \in X$ because X is sober. If $C = \bigvee_{\mathcal{C}(X)}\{A_i \mid i \in I\}$, where $A_i \in \mathcal{C}(X)$ for all $i \in I$, then $\bigcup\{A_i \mid i \in I\} \subseteq C = co_X(x)$. It follows that $A_i \subseteq co_X(x)$ for all $i \in I$. We claim that $co_X(x) = A_{i_0}$ for some $i_0 \in I$. Otherwise $A_i \neq co_X(x)$, implying $x \notin A_i$ for all $i \in I$. Then $A_i \subseteq co_X(x) - \{x\}$. Therefore $\bigcup\{A_i \mid i \in I\} \subseteq co_X(x) - \{x\}$. Since X is S_D, we have $co_X(x) - \{x\}$ is convex, and $C = co_X\left(\bigcup\{A_i \mid i \in I\}\right) \subseteq co_X(x) - \{x\}$, which contradicts $C = co_X(x)$. Hence $C = A_{i_0}$ for some $i_0 \in I$, showing that C is strongly irreducible.

(2) $\Rightarrow$ (1). Let C be a non-empty polytope in X. Then $C = \bigcup\{co_X(x) \mid x \in C\} = \bigvee_{\mathcal{C}(X)}\{co_X(x) \mid x \in C\}$, implying $C = co_X(x)$ for some $x \in C$. Since X is S_0, the element x such that $C = co_X(x)$ is unique. So X is sober. Now let x be an element of X such that $co_X(x) - \{x\}$ is not convex. Then $co_X(x) - \{x\} \subsetneq co_X(co_X(x) - \{x\}) \subseteq co_X(x)$, which implies that $co_X(co_X(x) - \{x\}) = co_X(x)$. It follows that $co_X(x) = \bigvee_{\mathcal{C}(X)}\{co_X(y) \mid y \in co_X(x) - \{x\}\}$. Since $co_X(x)$ is strongly irreducible by the assumption, we have $co_X(x) = co_X(y)$ for some $y \in co_X(x) - \{x\}$, which is impossible because X is S_0. Thus $co_X(x) - \{x\}$ must be convex. Hence X is S_D.

(2) $\Rightarrow$ (3) Let Y be an S_0 convex space and let $\mu : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y)$ be an isomorphism.

Claim 1. For any $x \in X$, there is a unique $y \in Y$ with $\mu(co_X(x)) = co_Y(y)$. To see this, let $\mu(co_X(x)) = B \in \mathcal{C}(Y)$. It then holds that

$$\begin{aligned} co_X(x) = \mu^{-1}(B) &= \mu^{-1}\left(\bigvee_{\mathcal{C}(Y)}\{co_Y(y) \mid y \in B\}\right) \\ &= \bigvee_{\mathcal{C}(X)}\{\mu^{-1}(co_Y(y)) \mid y \in B\}. \end{aligned}$$

By the assumption (2), there exists $y \in B$ such that $\mu(co_X(x)) = co_Y(y)$. Further as X is an S_0 convex space, the point y is unique.

Claim 2. For any $y \in Y$, there exists a unique $x \in X$ such that $co_X(x) = \mu^{-}1(co_Y(y))$, or equivalently $\mu(co_X(x)) = co_Y(y)$. First, note $\mu^{-1}(co_Y(y))$ is a polytope in X (as μ^{-1} is an isomorphism). As X is a sober convex space, there exists a unique $x \in X$ such that $\mu^{-1}(co_Y(y)) = co_X(x)$. This means $\mu(co_X(x)) = co_Y(y)$.

Now define a map $g : X \longrightarrow Y$ by $g(x) = y$ such that $\mu(co_X(x)) = co_Y(y)$. In analogy to the proof of Theorem 4.6, we obtain g is CC and CP.

(3) $\Rightarrow$ (1) We prove this conclusion in three steps.

Step 1. X is an S_0 convex space. By Lemma 4.19 (2), $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(X)/R_{co}$, and then X is isomorphic to X/R_{co}. Thus by Lemma 4.19 (3) X is S_0.

Step 2. X is sober. Let $Y = \mathsf{cpt}(\mathcal{C}(X))$. Then by Proposition 4.14 and Theorem 4.12, Y is the sobrification of X, and $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(Y)$. Thus by assumption, X is isomorphic to Y. So X is sober.

Step 3. X is a T_D convex space. If not , then there exists $x_0 \in X$ such that $co_X(x_0) - \{x_0\} \notin \mathcal{C}(X)$. By Lemma 4.20, $X' = X - \{x_0\}$, as a subspace of X, is an S_0 convex space whose convex lattice $\mathcal{C}(X')$ is isomorphic to $\mathcal{C}(X)$. By assumption, X is isomorphic to X'. Thus X' is sober. As $co_X(x_0)$ is a non-empty polytope in X (i.e., a non-bottom compact element in $\mathcal{C}(X)$) and $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(X')$, it follows that $co_X(x_0) - \{x_0\}$ is a non-empty polytope in X'. Since X' is sober, there exists a unique $x \in X'$ such that $co_X(x_0) - \{x_0\} = co_{X'}(x) = co_X(x) - \{x_0\}$. Then by Lemma 4.20 (2), $co_X(x_0) = co_X(x)$. As X is S_0, it follows that $x_0 = x$, which contradicts $x \in X'$. □

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References

[1] Adámek, J., Herrlich, H., Strecker, G.E.: Abstract and Concrete Categories. Wiley, New York (1990)

[2] Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order. Cambridge University Press, Cambridge (2002)

[3] Drake, D., Thron, W.J.: On the representation of of an abstract lattice as the family of closed sets of a topological space. Trans. Am. Math. Soc. **120**, 57–71 (1965)

[4] Farber, M., Jamison, R.E.: Convex structure in graphs and hypergraphs. SIAM J. Algebraic Discret. Methods **7**, 433–444 (1986)

[5] Farber, M., Jamison, R.E.: On local convex structure in graphs. Discret. Math. **66**, 231–247 (1987)

[6] Franklin, S.P.: Some results on order convex structure. Am. Math. Mon. **69**, 357–359 (1962)

[7] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S.: Continuous Lattices and Domains. Cambridge University Press, New York (2003)

[8] Goubault-Larrecq, J.: Non-Hausdorff Topology and Domain Theory. Cambridge University Press, New York (2013)

[9] Komiya, H.: Convex structure on a topological space. Fundam. Math. **111**, 107–113 (1981)

[10] Harary, F., Nieminen, J.: Convex structure in graphs. J. Differ. Geom. **16**, 185–190 (1981)

[11] Marczewski, E.: Independence in abstract algebras results and problems. Colloq. Math. **14**, 169–188 (1966)

[12] Menger, K.: Untersuchungen über allgemeine Metrik. Math. Ann. **100**, 75–163 (1928)

[13] Nieminen, J.: The ideal structure of simple ternary algebras. Colloq. Math. **40**, 23–29 (1978)

[14] Picado, J., Pultr, A.: Frames and Locales: Topology Without Points. Springer, Science & Business Media, Berlin (2011)

[15] Thron, W.J.: Lattice-equivalence of topological spaces. Duke Math. J. **29**(4), 671–679 (1962)

[16] Van De Vel, M.: Binary convex structures and distributive lattices. Proc. Lond. Math. Soc. **48**, 1–33 (1984)

[17] Van De Vel, M.: On the rank of a topological convex structure. Fundam. Math. **119**, 17–48 (1984)

[18] Van De Vel, M.: Theory of Convex Structures. North-Holland, Amsterdam (1993)

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