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PAPER

Hofmann-Mislove type definitions of non-Hausdorff spaces†

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Abstract
One of the most important results in domain theory is the Hofmann-Mislove Theorem, which reveals a very distinct characterization for the sober spaces via open filters. In this paper, we extend this result to the d-spaces and well-filtered spaces. We do this by introducing the notions of Hofmann-Mislove-system (HM-system for short) and Ψ-well-filtered space, which provide a new unified approach to sober spaces, well-filtered spaces, and d-spaces. In addition, a characterization for Ψ-well-filtered spaces is provided via Ψ-sets. We also discuss the relationship between Ψ-well-filtered spaces and H-sober spaces considered by Xu. We show that the category of complete Ψ-well-filtered spaces is a full reflective subcategory of the category of T_0 spaces with continuous mappings. For each HM-system Ψ that has a designated property, we show that a T_0 space X is Ψ-well-filtered if and only if its Smyth power space $P_s(X)$ is Ψ-well-filtered.

Keywords: d-space; H-sober; Hofmann-Mislove Theorem; Ψ-well-filtered space; well-filtered space; sober space

1. Introduction

Traditionally, topologists were interested in Hausdorff spaces much more than non-Hausdorff spaces. The development of domain theory has inspired the heavy interests in non-Hausdorff spaces. Sober spaces, well-filtered spaces, and d-spaces are three of the mostly well-studied non-Hausdorff spaces in domain theory. Recent researches revealed that these three classes of spaces share quite a number of common properties: (i) their categories are reflective in the category of T_0 spaces (Eršhov 1999; Hoffmann 1981; Liu et al. 2020; Wu et al. 2020; Wyler 1981; Xu et al. 2020*a*); (ii) they can be defined in terms of special subsets (i.e., irreducible sets, KF-sets, directed sets) (Shen et al. 2019); (iii) they are preserved under Cartesian product (Hoffmann 1979; Keimel and Lawson 2009); (iv) their open sets are Scott open in the specialization order (Gierz et al. 2003; Goubault-Larrecq 2003). Recently, Li, Yuan, and Zhao (2021) introduced the notion of Θ-fine space, which provided another unified approach to such properties. Also, using a generalized notion of Rudin set (KF-set), directed set, and irreducible set in T_0 spaces, Xu (2021)

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introduced the notion of H-sober space and particularly proved that the category of H-sober spaces is a full reflective subcategory of the category of T_0 spaces.

Sober spaces have very rich properties. The single most important result about such spaces is the Hofmann-Mislove Theorem, which states that sober spaces are exactly the spaces such that there is a natural correspondence between the open filters of the lattice of open subsets and the compact saturated subsets.

In this paper, we shall extend the Hofmann-Mislove Theorem to a general class of spaces, including sober spaces, well-filtered spaces, and d-spaces. One byproduct of this approach is the finding of some new classes of T_0 spaces.

Here is the outline of the paper. In Section 3, we introduce the Ψ-well-filtered spaces, where Ψ is a property on open filters. We then show that sober spaces, well-filtered spaces, and d-spaces are all special types of Ψ-well-filtered spaces. In Section 4, we introduce the notion of Ψ-set and use it to characterize Ψ-well-filtered spaces. In Section 5, the interlink between H-sober spaces and Ψ-well-filtered spaces is discussed. Especially, it is shown that the complete Ψ-well-filtered spaces are exactly the H_Ψ-sober spaces, where H_Ψ is an R-subset system induced by Ψ. As an immediate result, the category of complete Ψ-well-filtered spaces is reflective in the category of T_0 spaces. In the last section, for an HM-system Ψ and a T_0 space X, we show that the Smyth power space $P_s(X)$ is Ψ-well-filtered if and only if X is Ψ-well-filtered and Ψ has property Q for X. As a corollary, it is deduced that a T_0 space X is sober (resp., well-filtered) if and only if $P_s(X)$ is sober (resp., well-filtered).

2. Preliminary

Next, we introduce some basic concepts and notations that will be used in the paper. For more details, see Engelking (1989), Gierz et al. (2003), Goubault-Larrecq (2003).

Let P be a poset. A nonempty subset D of P is *directed* (resp., *filtered*) if every two elements in D have an upper (resp., lower) bound in D. P is called a *directed complete poset* or a *dcpo* for short, if for any directed subset $D \subseteq P$, the supremum of D, denoted by $\bigvee D$, exists.

For any subset A of a poset P, we use the following standard notations:

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\}; \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

In particular, for each $x \in X$, we write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. A subset A of P is called a *lower* (resp., *upper*) *set* if $A = \downarrow A$ (resp., $A = \uparrow A$).

A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, called the *Scott topology* on P, denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

Let X be a T_0 space. A subset A of X is called *saturated* if A equals the intersection of all open sets containing it. The specialization order $\leq$ on X is defined by $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator. It is important to note that a subset A of X is saturated if and only if $A = \uparrow A$ in the specialization order.

Remark 2.1. Let X be a T_0 space.

(1) Every open (resp., closed) set is an upper (resp., lower) set. In particular, $\mathrm{cl}(\{x\}) = \downarrow x$ (Gierz et al. 2003; Goubault-Larrecq 2003).
(2) For each subset K of X, K is compact if and only if $\uparrow K$ is compact (Gierz et al. 2003; Goubault-Larrecq 2003).
(3) It is well-known that a subset K of X is compact saturated if and only if $\min K$ is compact and $K = \uparrow \min K$, where $\min K$ is the set of all minimal elements of K in the specialization order (see, e.g., Erné 2009, pp. 2068).

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Definition 2.2 (Gierz et al. 2003; Goubault-Larrecq 2003). *(1) A nonempty subset A of a topological space X is called* irreducible *if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$.*

(2) A T_0 space X is called sober *if every irreducible closed subset of X is the closure of a (unique) point.*

For a T_0 space X, $A \subseteq X$, and $\mathcal{F} \subseteq 2^X$, we use the following notations:

$\mathcal{O}(X)$, the family of all open subsets of X;

$\mathcal{C}(X)$, the family of all closed subsets of X;

$\mathcal{Q}(X)$, the family of all compact saturated subsets of X;

$\mathcal{N}(A)$, the family $\{U \in \mathcal{O}(X) : A \subseteq U\}$;

$\mathcal{M}(A)$, the family $\{U \in \mathcal{O}(X) : A \cap U \neq \emptyset\}$;

$\mathfrak{M}(\mathcal{F})$, the family $\{C \in \mathcal{C}(X) : \forall F \in \mathcal{F}, C \cap F \neq \emptyset\}$;

$m(\mathcal{F})$, the family of all minimal members in $(\mathfrak{M}(\mathcal{F}), \subseteq)$.

Definition 2.3 (Gierz et al. 2003; Goubault-Larrecq 2003). *A T_0 space X is called* well-filtered *if for any filtered family $\mathcal{F}$ of $\mathcal{Q}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{F}$.*

Definition 2.4 (Shen et al. 2019; Xu et al. 2020*a*). *Let X be a T_0 space. A nonempty subset A of X is called a* KF-set *(or a* Rudin set*), if there exists a filtered family $\mathcal{F}$ of $\mathcal{Q}(X)$ such that $\mathrm{cl}(A) \in m(\mathcal{F})$.*

Theorem 2.5 (Shen et al. 2019; Xu et al. 2020*a*). *A T_0 space X is well-filtered if and only if for each KF-set A, there exists $x \in X$ such that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$.*

Lemma 2.6 (Shen et al. 2019). *Let X and Y be two T_0 spaces, and $f : X \longrightarrow Y$ be a continuous mapping. If A is a KF-set in X, then $f(A)$ is a KF-set in Y.*

Definition 2.7 (Gierz et al. 2003; Goubault-Larrecq 2003). *A T_0 space X is called a d-space, if X is a dcpo and every open subset of X is Scott open in the specialization order.*

Proposition 2.8 (Xu et al. 2020*a*, Proposition 3.3). *A T_0 space X is a d-space if and only if for each directed subset D of X, there is $x \in X$ such that $\mathrm{cl}(D) = \mathrm{cl}(\{x\})$.*

Remark 2.9. Every sober space is well-filtered, and every well-filtered space is a d-space (Gierz et al. 2003; Goubault-Larrecq 2003).

Definition 2.10 (Gierz et al. 2003; Goubault-Larrecq 2003). *Let X be a T_0 space, and $\mathcal{F} \subseteq \mathcal{O}(X)$. Then, $\mathcal{F}$ is called an* open filter*, if it is filtered and Scott open in $(\mathcal{O}(X), \subseteq)$. We denote by $\mathsf{OF}(X)$ the family of all open filters of $\mathcal{O}(X)$.*

Remark 2.11. For a T_0 space X, the following results can be checked easily.

(1) For each compact saturated subset K of X, $\mathcal{N}(K) \in \mathsf{OF}(X)$.
(2) For each irreducible subset A of X, $\mathcal{M}(A) \in \mathsf{OF}(X)$.
(3) For each continuous mapping $f : X \longrightarrow Y$ between T_0 spaces X and Y, if $\mathcal{F} \in \mathsf{OF}(X)$, then $f_*(\mathcal{F}) = \{V \in \mathcal{O}(Y) : f^{-1}(V) \in \mathcal{F}\} \in \mathsf{OF}(Y)$.

The following is a similar result to the Topological Rudin Lemma in Heckmann and Keimel (2013).

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Lemma 2.12. *Let X be a T_0 space, $A \in \mathcal{C}(X)$ and $\mathcal{F} \in \mathsf{OF}(X)$. Then, the following conditions hold:*

(1) *every element of $m(\mathcal{F})$ is irreducible;*
(2) *if $A \in \mathfrak{M}(\mathcal{F})$, then there is a closed subset C of A such that $C \in m(\mathcal{F})$.*

Proof. (1) Assume on the contrary there is $C \in m(\mathcal{F})$ that is not irreducible. Then, there exist closed sets C_1, C_2 such that $C = C_1 \cup C_2$ but $C \neq C_1$ and $C \neq C_2$. Since C_1, C_2 are proper subsets of C, by the minimality of C there exist $U_1, U_2 \in \mathcal{F}$ such that $C_1 \cap U_1 = \emptyset$ and $C_2 \cap U_2 = \emptyset$. Since $\mathcal{F}$ is a filter, there exists $U_3 \in \mathcal{F}$ such that $U_3 \subseteq U_1 \cap U_2$. It follows that $C_1 \cap U_3 = C_2 \cap U_3 = \emptyset$. Thus, $C \cap U_3 = (C_1 \cup C_2) \cap U_3 = (C_1 \cap U_3) \cup (C_2 \cap U_3) = \emptyset$, contradicting the fact that $C \in m(\mathcal{F})$. Therefore, C is irreducible.

(2) Note that an open set is not in $\mathcal{F}$ if and only if its complement is in $\mathfrak{M}(\mathcal{F})$. Then, using Proof (i) of Lemma II-1.19 in Gierz et al. (2003), condition (2) holds dually. □

3. Ψ-Well-Filtered Spaces

For each $K \in \mathcal{Q}(X)$, it is trivial that $\mathcal{N}(K) \in \mathsf{OF}(X)$ and $K = \bigcap \mathcal{N}(K)$. Then, the mapping $\mathcal{N} : (\mathcal{Q}(X), \supseteq) \longrightarrow (\mathsf{OF}(X), \subseteq)$, $K \mapsto \mathcal{N}(K)$ is well-defined and clearly is an order-embedding. In domain theory, the single most important result about sober spaces is the Hofmann-Mislove Theorem (see Gierz et al. 2003, Theorems II-1.20, II-1.21).

Theorem 3.1 (Hofmann-Mislove Theorem)**.** *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober;*
(2) *$\forall \mathcal{F} \in \mathsf{OF}(X)$, there is a $K \in \mathcal{Q}(X)$ such that $\mathcal{F} = \mathcal{N}(K)$;*
(3) *$\forall \mathcal{F} \in \mathsf{OF}(X)$, $\mathcal{F} = \mathcal{N}(\bigcap \mathcal{F})$.*

As a corollary of the Hofmann-Mislove Theorem, the following result is clear.

Corollary 3.2. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober;*
(2) $\mathsf{OF}(X) = \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}$;
(3) $\forall \mathcal{F} \in \mathsf{OF}(X)$, $\forall U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ *implies* $U \in \mathcal{F}$.

In the following, we would like to provide a unified characterization of Hofmann-Mislove Theorem type for the classes of d-spaces and well-filtered spaces via open filters.

Definition 3.3. *A covariant functor $\Psi : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ is called a* Hofmann-Mislove system *(an* HM-system *for short) on $\mathbf{Top_0}$ if the following two conditions are satisfied:*

(HM1) for each T_0 space X, $\{\mathcal{N}(K) : K \in \mathcal{Q}(X)\} \subseteq \Psi(X) \subseteq \mathsf{OF}(X)$;
(HM2) for each continuous mapping $f : X \longrightarrow Y$ in $\mathbf{Top_0}$, $\Psi(f)(\mathcal{F}) = f_*(\mathcal{F}) = \{V \in \mathcal{O}(Y) : f^{-1}(V) \in \mathcal{F}\} \in \Psi(Y)$ for each $\mathcal{F} \in \Psi(X)$.

Definition 3.4. *Let Ψ be an HM-system and X be a T_0 space. We call X a Ψ-*well-filtered space*, if $\forall \mathcal{F} \in \Psi(X)$, $\forall U \in \mathcal{O}(X)$,*

$$\bigcap \mathcal{F} \subseteq U \text{ implies } U \in \mathcal{F}.$$

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Lemma 3.5. *Let Ψ be an HM-system, X be a Ψ-well-filtered space, and $\mathcal{F} \in \Psi(X)$. Then $\bigcap \mathcal{F} \in \mathcal{Q}(X)$, and $\emptyset \notin \mathcal{F}$ implies $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. Let $\{U_i : i \in I\}$ be a directed family of $\mathcal{O}(X)$ such that $\bigcap \mathcal{F} \subseteq \bigcup_{i\in I} U_i$. Since X is Ψ-well-filtered, $\bigcup_{i\in I} U_i \in \mathcal{F}$, and since $\mathcal{F}$ is Scott open, there exists $i_0 \in I$ such that $U_{i_0} \in \mathcal{F}$, which implies $\bigcap \mathcal{F} \subseteq U_{i_0}$. Hence, $\bigcap \mathcal{F}$ is compact and clearly is saturated. In addition, if $\bigcap \mathcal{F} = \emptyset$, then $\emptyset \in \mathcal{F}$ because X is Ψ-well-filtered, completing the proof. □

Theorem 3.6. *Let Ψ be an HM-system and X be a T_0 space. Then, the following conditions are equivalent:*

(1) *X is Ψ-well-filtered;*
(2) *$\forall \mathcal{F} \in \Psi(X)$, $\mathcal{F} = \mathcal{N}(\bigcap \mathcal{F})$;*
(3) *$\Psi(X) = \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{F} \in \Psi(X)$. It is clear that $\mathcal{F} \subseteq \mathcal{N}(\bigcap \mathcal{F})$. Conversely, if $U \in \mathcal{N}(\bigcap \mathcal{F})$, then $\bigcap \mathcal{F} \subseteq U$ and $U \in \mathcal{O}(X)$, and since X is Ψ-well-filtered, we have that $U \in \mathcal{F}$. This shows that $\mathcal{N}(\bigcap \mathcal{F}) \subseteq \mathcal{F}$. Hence, $\mathcal{F} = \mathcal{N}(\bigcap \mathcal{F})$.

(2) $\Rightarrow$ (3). By the definition of Ψ, it is clear that $\{\mathcal{N}(K) : K \in \mathcal{Q}(X)\} \subseteq \Psi(X)$. Now suppose $\mathcal{F} \in \Psi(X)$. By Lemma 3.5, $K_0 = \bigcap \mathcal{F} \in \mathcal{Q}(X)$, and by condition (2) $\mathcal{F} = \mathcal{N}(K_0)$. This shows that $\Psi(X) \subseteq \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}$. Thus, condition (3) holds.

(3) $\Rightarrow$ (1). Let $\mathcal{F} \in \Psi(X)$ and $U \in \mathcal{O}(X)$ such that $\bigcap \mathcal{F} \subseteq U$. By (3), there exists $K \in \mathcal{Q}(X)$ such that $\mathcal{F} = \mathcal{N}(K)$. Then, $K = \bigcap \mathcal{N}(K) = \bigcap \mathcal{F} \subseteq U$, which implies that $U \in \mathcal{N}(K) = \mathcal{F}$. Therefore, X is Ψ-well-filtered. □

The following result shows that the Ψ-well-filteredness is a topological property for each HM-system Ψ.

Proposition 3.7. *Let Ψ be an HM-system, X be a Ψ-well-filtered space, and Y be a T_0 space. If Y is homeomorphic to X, then Y is a Ψ-well-filtered space.*

Proof. Suppose $h : Y \longrightarrow X$ is a homeomorphism. Let $\mathcal{F} \in \Psi(Y)$ and $W \in \mathcal{O}(Y)$ such that $\bigcap \mathcal{F} \subseteq W$. Since h is a homeomorphism, one can easily obtain that $h_*(\mathcal{F}) = \{h(U) : U \in \mathcal{F}\}$, which implies that

$$\bigcap h_*(\mathcal{F}) = \bigcap \{h(U) : U \in \mathcal{F}\} = h\left(\bigcap \mathcal{F}\right) \subseteq h(W) \in \mathcal{O}(X).$$

By (HM2), $h_*(\mathcal{F}) \in \Psi(X)$, and since X is Ψ-well-filtered, we have that $h(W) \in h_*(\mathcal{F})$, so $W \in \mathcal{F}$. Therefore, X is Ψ-well-filtered. □

In the following, we will show that all the classes of sober spaces, well-filtered spaces, and d-spaces can be characterized via HM-systems.

Definition 3.8. *Define $\Psi_{\mathsf{sob}}, \Psi_{\mathsf{wf}}, \Psi_{\mathsf{d}} : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ as follows: for each T_0 space X,*

$$\Psi_{\mathsf{sob}}(X) = \mathsf{OF}(X),$$

$$\Psi_{\mathsf{wf}}(X) = \left\{ \bigcup_{K\in\mathcal{G}} \mathcal{N}(K) : \mathcal{G} \textit{ is a filtered family of } \mathcal{Q}(X) \right\},$$

$$\Psi_{\mathsf{d}}(X) = \left\{ \bigcup_{x\in D} \mathcal{N}(\uparrow x) : D \textit{ is a directed subset of } X \right\} \cup \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}.$$

Then, it is trivial to check that $\Psi_{\mathsf{sob}}, \Psi_{\mathsf{wf}}, \Psi_{\mathsf{d}}$ are all HM-systems.

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Theorem 3.9. *Let X be a T_0 space. The following conditions are equivalent:*

(1) *X is sober;*
(2) *X is Ψ_{sob}-well-filtered;*
(3) *$\forall \mathcal{F} \in \Psi_{\mathsf{sob}}(X)$, $\mathcal{F} = \mathcal{N}(\bigcap \mathcal{F})$;*
(4) *$\Psi_{\mathsf{sob}}(X) = \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}$.*

Proof. It is straightforward by the Hofmann-Mislove Theorem. □

Remark 3.10. Let $\Psi_0(X) = \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}$ for each T_0 space X. Then, it is easy to check that Ψ_0 is an HM-system, and it is clear that each T_0 space X is Ψ_0-well-filtered. From Theorem 3.9, for each HM-system Ψ, we have the following relations:

$$\text{sober } (\Psi_{\mathsf{sob}}\text{-well-filtered}) \text{ space} \Longrightarrow \Psi\text{-well-filtered space} \Longrightarrow T_0 \ (\Psi_0\text{-well-filtered}) \text{ space.}$$

In another words, the sober space is the strongest Ψ-well-filtered space, and the T_0 space is the weakest one.

Theorem 3.11. *Let X be a T_0 space. The following conditions are equivalent:*

(1) *X is well-filtered;*
(2) *X is Ψ_{wf}-well-filtered;*
(3) *$\forall \mathcal{F} \in \Psi_{\mathsf{wf}}(X)$, $\mathcal{F} = \mathcal{N}(\bigcap \mathcal{F})$;*
(4) *$\Psi_{\mathsf{wf}}(X) = \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}$.*

Proof. That (2) ⇔ (3) ⇔ (4) follows immediately from Theorem 3.6.

(1) ⇒ (2). Suppose X is well-filtered. Let $\mathcal{F} \in \Psi_{\mathsf{wf}}(X)$ and $U \in \mathcal{O}(X)$ such that $\bigcap \mathcal{F} \subseteq U$. Then, there exists a filtered family $\{K_i : i \in I\} \subseteq \mathcal{Q}(X)$ such that $\mathcal{F} = \bigcup_{i \in I} \mathcal{N}(K_i)$. Note that $\bigcap_{i \in I} K_i = \bigcap \mathcal{F} \subseteq U$, which implies that $K_{i_0} \subseteq U$ for some $i_0 \in I$ because X is well-filtered. It follows that $U \in \mathcal{N}(K_{i_0}) \subseteq \mathcal{F}$. Hence, X is Ψ_{wf}-well-filtered.

(2) ⇒ (1). Let $\{K_i : i \in I\}$ be a filtered family of $\mathcal{Q}(X)$ and $U \in \mathcal{O}(X)$ such that $\bigcap_{i \in I} K_i \subseteq U$. Then, $\mathcal{F} = \bigcup_{i \in I} \mathcal{N}(K_i) \in \Psi_{\mathsf{wf}}(X)$. Note that $\bigcap \mathcal{F} = \bigcap_{i \in I} K_i \subseteq U$, and since X is Ψ-well-filtered, it follows that $U \in \mathcal{F}$. By the definition of $\mathcal{F}$, there exists $i_0 \in I$ such that $U \in \mathcal{N}(K_{i_0})$, that is, $K_{i_0} \subseteq U$. Hence, X is well-filtered. □

Theorem 3.12. *Let X be a T_0 space. The following conditions are equivalent:*

(1) *X is a d-space;*
(2) *X is Ψ_{d}-well-filtered;*
(3) *$\forall \mathcal{F} \in \Psi_{\mathsf{d}}(X)$, $\mathcal{F} = \mathcal{N}(\bigcap \mathcal{F})$;*
(4) *$\Psi_{\mathsf{d}}(X) = \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}$.*

Proof. That (2) ⇔ (3) ⇔ (4) follows immediately from Theorem 3.6.

(1) ⇒ (2). Let $\mathcal{F} \in \Psi_{\mathsf{d}}(X)$ and $U \in \mathcal{O}(X)$. If $\mathcal{F} = \mathcal{N}(K)$ for some $K \in \mathcal{Q}(X)$, then it is clear that $K = \bigcap \mathcal{N}(K) \subseteq U$ implies $U \in \mathcal{N}(K)$. Now assume there is a directed subset D of X such that $\mathcal{F} = \bigcup_{x \in D} \mathcal{N}(\uparrow x)$ and $\bigcap \mathcal{F} \subseteq U$. Since X is a d-space, $\bigvee D$ exists. We have that $\bigcap \mathcal{F} = \bigcap_{d \in D} \uparrow d = \uparrow \bigvee D \subseteq U$, which implies $\bigvee D \in U$. Since every open set in a d-space is Scott open, $D \cap U \neq \emptyset$, and take $x_0 \in D \cap U$. It follows that $U \in \mathcal{N}(\uparrow x_0) \subseteq \mathcal{F}$. Hence, X is Ψ_{d}-well-filtered.

(2) ⇒(1). Let D be a directed subset of X, and $\mathcal{F} = \bigcup_{x \in D} \mathcal{N}(\uparrow x)$. We claim that $\mathrm{cl}(D) \cap \bigcap \mathcal{F} \neq \emptyset$. Otherwise, $\bigcap \mathcal{F} \subseteq X \setminus \mathrm{cl}(D)$, and since $\mathcal{F} \in \Psi_{\mathsf{d}}(X)$ and X is Ψ_{d}-well-filtered, $X \setminus \mathrm{cl}(D) \in \mathcal{F} = \bigcup_{x \in D} \mathcal{N}(\uparrow x)$. Then, there exists $x_0 \in D$ such that $X \setminus \mathrm{cl}(D) \in \mathcal{N}(\uparrow x_0)$, which follows that $x_0 \in X \setminus$

$\mathrm{cl}(D)$, contradicting the fact that $x_0 \in D$. Hence, there is $y \in \mathrm{cl}(D) \cap \bigcap \mathcal{F} \neq \emptyset$. Then, $y \in \bigcap \mathcal{F} = \bigcap_{x \in D} \uparrow x$, so y is an upper bound of D. It follows that $D \subseteq \downarrow y = \mathrm{cl}(\{y\})$, and since $y \in \mathrm{cl}(D)$, it follows that $\mathrm{cl}(D) = \mathrm{cl}(\{y\})$. Hence, by Proposition 2.8 X is a d-space. □

Remark 3.13. Note that $\Psi_{\mathsf{d}}(X) \subseteq \Psi_{\mathsf{wf}}(X) \subseteq \Psi_{\mathsf{sob}}(X)$ for each T_0 space X. Then by Theorems 3.9, 3.11 and 3.12, the following relations are clear:

$$\text{sober space} \Longrightarrow \text{well-filtered space} \Longrightarrow d\text{-space}.$$

4. A Characterization for Ψ-Well-Filtered Spaces

In this section, we will show that a Ψ-well-filtered space X is determined by a class of subsets of X, called Ψ-sets.

Definition 4.1. *Let Ψ be an HM-system and X be a T_0 space. A nonempty subset A of X is called a* Ψ-set *(relative to $\mathcal{F}$), if there exists $\mathcal{F} \in \Psi(X)$ such that $\mathrm{cl}(A) \in m(\mathcal{F})$.*

Remark 4.2. Let Ψ be an HM-system, X be a T_0 space, and $A \subseteq X$.

(1) It is clear that A is a Ψ-set if and only if $\mathrm{cl}(A)$ is a Ψ-set.
(2) Every Ψ-set is irreducible by Lemma 2.12.

Lemma 4.3. *Let X be a T_0 space, $K \in \mathcal{Q}(X)$, and $A \subseteq X$. The following two conditions are equivalent:*

(1) $\mathrm{cl}(A) \cap K \neq \emptyset$;
(2) $\forall U \in \mathcal{N}(K)$, $A \cap U \neq \emptyset$.

Proof. That (1) $\Rightarrow$ (2) is trivial. Conversely, if $\mathrm{cl}(A) \cap K = \emptyset$, then $X \setminus \mathrm{cl}(A) \in \mathcal{N}(K)$, but $A \cap (X \setminus \mathrm{cl}(A)) = \emptyset$, contradicting the assumption (2). This shows that (2) implies (1). □

Lemma 4.4. *Let X be a T_0 space and $K \in \mathcal{Q}(X)$. Then,*

$$m(\mathcal{N}(K)) = \{\mathrm{cl}(\{x\}) : x \in \min K\},$$

where $\min K$ *is the set of all minimal elements of K in the specialization order of X.*

Proof. First, it is well-known that $K = \uparrow \min K$ (see, e.g., Erné 2009, pp. 2068). Suppose $x \in \min K$. It is clear that $\mathrm{cl}(\{x\}) \in \mathfrak{M}(\mathcal{N}(K))$. Now assume C is a closed subset of $\mathrm{cl}(\{x\})$ such that $C \in \mathfrak{M}(\mathcal{N}(K))$. By Lemma 4.3, there is $a \in C \cap K \neq \emptyset$. Then, $a \in C \subseteq \mathrm{cl}(\{x\})$, so $a \leq x$. Since x is minimal in K, we have that $a = x \in C$, so $\mathrm{cl}(\{x\}) \subseteq C$. Thus, $C = \mathrm{cl}(\{x\})$. All this shows that $\mathrm{cl}(\{x\}) \in m(\mathcal{N}(K))$.

Now assume $A \in m(\mathcal{N}(K))$. By Lemma 4.3, there is $a \in A \cap K \neq \emptyset$. Since $a \in K = \uparrow \min K$, there exists $x \in \min K$ such that $x \leq a$, which follows that $x \in \mathrm{cl}(\{a\}) \subseteq A$. Then, $\mathrm{cl}(\{x\})$ is a closed subset of A such that $\mathrm{cl}(\{x\}) \in \mathfrak{M}(\mathcal{N}(K))$. By the minimality of A, we have that $A = \mathrm{cl}(\{x\})$. This shows that $m(\mathcal{N}(K)) \subseteq \{\mathrm{cl}(\{x\}) : x \in \min K\}$, completing the proof. □

Using Lemma 4.4, we deduce that $m(\mathcal{N}(\uparrow x)) = \{\mathrm{cl}(\{x\})\}$ for each point x of a T_0 space X, and since $\mathcal{N}(\uparrow x) \in \Psi(X)$ for each HM-system Ψ, we have the following result.

Proposition 4.5. *Let Ψ be an HM-system and X be a T_0 space. Every singleton of X is a Ψ-set.*

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Theorem 4.6. *Let Ψ be an HM-system and X be a T_0 space. The following conditions are equivalent:*

(1) *X is Ψ-well-filtered;*
(2) *for each Ψ-set $A \subseteq X$, there exists $x \in X$ such that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$;*
(3) *for each Ψ-set $A \subseteq X$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{M}(A) \subseteq U$ implies $U \in \mathcal{M}(A)$;*
(4) *for each Ψ-set $A \subseteq X$ and $U \in \mathcal{O}(X)$, $\bigcap_{a \in A} \uparrow a \subseteq U$ implies $\uparrow a \subseteq U$ for some $a \in A$.*

Proof. (1) $\Rightarrow$ (2). Let A be a Ψ-set in X. Then, there exists $\mathcal{F} \in \Psi(X)$ such that $\mathrm{cl}(A) \in m(\mathcal{F})$. Since X is Ψ-well-filtered, it follows that $(\bigcap \mathcal{F}) \cap \mathrm{cl}(A) \neq \emptyset$. Take $x \in (\bigcap \mathcal{F}) \cap \mathrm{cl}(A) \neq \emptyset$. Then, $\mathrm{cl}(\{x\})$ is a subset of $\mathrm{cl}(A)$ such that $\mathrm{cl}(\{x\}) \in \mathfrak{M}(\mathcal{F})$. By the minimality of $\mathrm{cl}(A)$, we deduce that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$.

(2) $\Rightarrow$ (3). It is trivial since $\mathcal{M}(\mathrm{cl}(\{x\})) = \mathcal{N}(\{x\})$ for each $x \in X$.

(3) $\Rightarrow$ (1). Let $\mathcal{F} \in \Psi(X)$ and $O \in \mathcal{O}(X)$ such that $\bigcap \mathcal{F} \subseteq O$. We need to show that $O \in \mathcal{F}$, or equivalently, $U \subseteq O$ for some $U \in \mathcal{F}$, because $\mathcal{F}$ is an upper set. On the contrary, we assume $U \nsubseteq O$, that is, $U \cap (X \setminus O) \neq \emptyset$ for each $U \in \mathcal{F}$. By Lemma 2.12, there exists a closed subset A of $X \setminus O$ such that $A \in m(\mathcal{F})$. Then, A is a Ψ-set in X and note that $\mathcal{F} \subseteq \mathcal{M}(A)$, so $\bigcap \mathcal{M}(A) \subseteq \bigcap \mathcal{F} \subseteq O$. From condition (3), it follows that $O \in \mathcal{M}(A)$, that is, $O \cap A \neq \emptyset$, contradicting the fact that $A \subseteq X \setminus O$. This implies that $O \in \mathcal{F}$. Therefore, X is Ψ-well-filtered.

(2) $\Rightarrow$ (4). Suppose condition (2) is satisfied. Then, there exists $x \in X$ such that $\mathrm{cl}(A) = \mathrm{cl}(\{x\}) = \downarrow x$, which follows that

$$\uparrow x = \bigcap_{a \in \downarrow x} \uparrow a = \bigcap_{a \in \mathrm{cl}(A)} \uparrow a \subseteq \bigcap_{a \in A} \uparrow a \subseteq U.$$

This shows that U is an open neighborhood of x, and since $x \in \mathrm{cl}(A)$, there is $a_0 \in U \cap A \neq \emptyset$, so $\uparrow a_0 \subseteq U$. This gives (4).

(4) $\Rightarrow$ (2). Suppose A is a Ψ-set in X. By Remark 4.2, $\mathrm{cl}(A)$ is also a Ψ-set. Since $\uparrow a \nsubseteq X \setminus \mathrm{cl}(A)$ for each $a \in \mathrm{cl}(A)$, by condition (4) there exists $x \in \mathrm{cl}(A) \cap \bigcap_{a \in \mathrm{cl}(A)} \uparrow a \neq \emptyset$. Then, we can easily obtain that $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$. □

Lemma 4.7. *Let X and Y be two T_0 spaces and A be an irreducible subset of X.*

(1) *$\mathcal{M}(A) \in \mathsf{OF}(X)$ and $\mathrm{cl}(A) \in m(\mathcal{M}(A))$.*
(2) *For each $B \in m(\mathcal{M}(A))$, $\mathrm{cl}(A) = B$. Hence, $\mathcal{M}(B) = \mathcal{M}(A)$.*
(3) *If $f : X \longrightarrow Y$ is a continuous mapping, then $f_*(\mathcal{M}(A)) = \mathcal{M}(f(A))$, where*

$$f_*(\mathcal{M}(A)) = \{V \in \mathcal{O}(Y) : f^{-1}(V) \in \mathcal{M}(A)\}.$$

Hence, $\mathrm{cl}_Y(f(A)) \in m(f_*(\mathcal{M}(A)))$.

Proof. (1) It is trivial that $\mathcal{M}(A) \in \mathsf{OF}(X)$ and $\mathrm{cl}(A) \in \mathfrak{M}(\mathcal{M}(A))$. To show $\mathrm{cl}(A) \in m(\mathcal{M}(A))$, assume C is a closed subset of $\mathrm{cl}(A)$ such that $C \in \mathfrak{M}(\mathcal{M}(A))$. We need to show that $A \subseteq C$. Otherwise, $A \cap (X \setminus C) \neq \emptyset$. Then, $X \setminus C \in \mathcal{M}(A)$. Since $C \in \mathfrak{M}(\mathcal{M}(A))$, it follows that $C \cap (X \setminus C) \neq \emptyset$, a contradiction. Thus, $C = \mathrm{cl}(A)$. This shows that $\mathrm{cl}(A) \in m(\mathcal{M}(A))$.

(2) Suppose $B \in m(\mathcal{M}(A))$. Let $x \in A$. For each open neighborhood U of x, $U \in \mathcal{M}(A)$, hence $B \cap U \neq \emptyset$. This implies $x \in \mathrm{cl}(B) = B$. We then deduce that $A \subseteq B$, so $\mathrm{cl}(A) \subseteq B$. From the minimality of B, it follows that $\mathrm{cl}(A) = B$.

(3) For each $V \in \mathcal{O}(Y)$, we have that $V \in f_*(\mathcal{M}(A))$ iff $f^{-1}(V) \cap A \neq \emptyset$ iff $V \cap f(A) \neq \emptyset$ iff $V \in \mathcal{M}(f(A))$, which implies that $f_*(\mathcal{M}(A)) = \mathcal{M}(f(A))$. Since f is continuous, it follows that $f(A)$ is an irreducible set in Y. Hence, by (1), we have $\mathrm{cl}_Y(f(A)) \in m(\mathcal{M}(f(A))) = m(f_*(\mathcal{M}(A)))$. □

Proposition 4.8. *Let X be a T_0 space.*

(1) *The Ψ_{sob}-sets are exactly the irreducible sets in X.*

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(2) *The* $\Psi_{\mathsf{wf}-}$ *sets are exactly the KF-sets in* X.
(3) *The closed* $\Psi_{\mathsf{d}-}$*sets are exactly the closure of directed sets in* X.

Proof. (1) Suppose A is an irreducible subset of X. By Lemma 4.7, $\mathcal{M}(A) \in \Psi_{\mathsf{sob}}(X) = \mathsf{OF}(X)$ and $\mathrm{cl}(A) \in m(\mathcal{M}(A))$, so A is a Ψ_{sob}-set in X. The converse is trivial by Remark 4.2.

(2) Suppose $\mathcal{F} \in \Psi_{\mathsf{wf}}(X)$. Then, there exists a filtered family $\{K_i : i \in I\} \subseteq \mathcal{Q}(X)$ such that $\mathcal{F} = \bigcup_{i\in I} \mathcal{N}(K_i)$. By Lemma 4.3, for each subset $A \subseteq X$, $\mathrm{cl}(A) \in \mathfrak{M}(\mathcal{F})$ iff $\mathrm{cl}(A) \cap K_i \neq \emptyset$ for each $i \in I$. Then, we deduce that the Ψ_{wf}-sets are exactly the KF-sets.

(3) Suppose A is a closed Ψ_{d}-set in X. Then, there exists a directed subset D of X such that $A \in m(\mathcal{M}(D))$. By Lemma 4.7, we have $\mathrm{cl}(D) = A$. Now suppose E is a directed subset of X. Then, by Lemma 4.7, $\mathrm{cl}(E) \in m(\mathcal{M}(E))$. This means that $\mathrm{cl}(E)$ is a closed Ψ_{d}-set in X. □

5. Relations between Ψ-Well-Filtered Spaces and *H*-Sober Spaces

In the paper Xu (2021), Xu introduces the notions of R-subset system and H-sober space, which provides a uniform approach to d-spaces, well-filtered spaces, and sober spaces. In this section, we study the relations between H-sober spaces and Ψ-well-filtered spaces.

Definition 5.1 (Xu 2021). *A covariant functor* $H : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ *is called an* R-subset system *on* $\mathbf{Top_0}$ *if it satisfies the following two conditions:*

(H1) for each T_0 *space* X, $\{\{x\} : x \in X\} \subseteq H(X) \subseteq Irr(X)$;
(H2) For each continuous mapping $f : X \longrightarrow Y$ *in* $\mathbf{Top_0}$ *and each* $A \in H(X)$, $H(f)(A) = f(A) \in H(Y)$.

For an R-subset system H and a T_0 *space X, we call* $A \subseteq X$ *an* H-set *if* $A \in H(X)$.

Definition 5.2 (Xu 2021). *Let H be an R-subset system. A* T_0 *space X is called* H-sober *if for each* $A \in H(X)$, *there is a (unique) point* $x \in X$ *such that* $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$.

Next, we study the relationship between Ψ-well-filtered spaces and H-sober spaces, the following concept is needed.

Definition 5.3. *An HM-system* Ψ *is called* complete *if for each continuous mapping* $f : X \longrightarrow Y$ *between* T_0 *spaces X and Y, and each* Ψ*-set A in X,* $f(A)$ *is a* Ψ*-set in Y.*

Lemma 5.4. *An HM-system* Ψ *is complete if and only if for each continuous mapping* $f : X \longrightarrow Y$ *between* T_0 *spaces X and Y, and each closed* Ψ*-set A in X,* $f(A)$ *is a* Ψ*-set in Y.*

Proof. It suffices to prove the sufficiency. Suppose A is a Ψ-set in X. By assumption, $f(\mathrm{cl}_X(A))$ is a Ψ-set in Y. By Remark 4.2, $\mathrm{cl}_Y(f(\mathrm{cl}_X(A))) = \mathrm{cl}_Y(f(A))$ is a Ψ-set, so is $f(A)$, completing the proof. □

Proposition 5.5. *The HM-systems* Ψ_{sob}, Ψ_{wf}, *and* Ψ_{d} *are all complete.*

Proof. By Proposition 4.8 and Lemma 2.6, Ψ_{wf} is complete. The completeness of Ψ_{sob} and Ψ_{d} is trivial (see Gierz et al. 2003) by Proposition 4.8. □

Lemma 5.6. *Let X and Y be two* T_0 *spaces, K be a compact saturated subset of X, and* $f : X \longrightarrow Y$ *be a continuous mapping.*

(1) $f_*(\mathcal{N}(K)) = \mathcal{N}(f(K))$.
(2) *If* $A \in m(\mathcal{N}(K))$, *then* $\mathrm{cl}_Y(f(A)) \in m(\mathcal{N}(\uparrow f(K \cap A)))$.

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Proof. (1) It is trivial.

(2) First, by Lemma 4.4 there exists $x \in \min K$ such that $A = \mathrm{cl}_X(\{x\})$. It follows that $A \cap K = \{x\}$, so $\uparrow f(K \cap A) = \uparrow f(x)$. In addition, using Lemma 4.3, one can easily obtain that $\mathrm{cl}_Y(f(A)) \in \mathfrak{M}(\mathcal{N}(\uparrow f(x)))$. Suppose C is a closed subset of $\mathrm{cl}_Y(f(A))$ such that $C \in \mathfrak{M}(\mathcal{N}(\uparrow f(x)))$. By Lemma 4.3, $C \cap \uparrow f(x) \neq \emptyset$, and since C is a lower set, we have that $f(x) \in C$, so $A = \mathrm{cl}_X(\{x\}) \subseteq f^{-1}(C)$. It follows that $\mathrm{cl}_Y(f(A)) \subseteq C$. Thus, $C = \mathrm{cl}_Y(f(A))$. Therefore, $\mathrm{cl}_Y(f(A)) \in m(\mathcal{N}(\uparrow f(K \cap A)))$. □

Theorem 5.7. *Let $H : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ be an R-subset system. Define $\Psi_H : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ by*

$$\Psi_H(X) = \{\mathcal{M}(A) : A \in H(X)\} \cup \{\mathcal{N}(K) : K \in \mathcal{Q}(X)\}.$$

for each T_0 space X, and for each continuous mapping $f : X \longrightarrow Y$ between T_0 spaces X and Y, we define $\Psi_H(f) : \Psi_H(X) \longrightarrow \Psi_H(Y)$ by

$$\Psi_H(f)(\mathcal{F}) = f_*(\mathcal{F}) = \{V \in \mathcal{O}(Y) : f^{-1}(V) \in \mathcal{F}\}$$

for each $\mathcal{F} \in \Psi_H(X)$.

(1) *Ψ_H is a complete HM-system.*
(2) *A T_0 space X is H-sober if and only if it is Ψ_H-well-filtered.*

Proof. (1) We first prove that Ψ_H is an HM-system.

Note that each member of $H(X)$ is irreducible in X, so it is clear that Ψ_H satisfies (H1). To show (HM2), let $f : X \longrightarrow Y$ be a continuous mapping between T_0 spaces X and Y, and $A \in H(X)$, that is, $\mathcal{M}(A) \in \Psi_H(X)$. By (H2), $f(A) \in H(Y)$, and it follows from Lemma 4.7 that $f_*(\mathcal{M}(A)) = \mathcal{M}(f(A)) \in \Psi_H(Y)$, and since $f_*(\mathcal{N}(K)) = \mathcal{N}(f(K)) \in \Psi_H(Y)$, (H2) holds. In addition, it is trivial to check that Ψ_H is a covariant functor. Hence, Ψ_H is an HM-system.

Now we prove that Ψ_H is complete. Suppose A is a closed Ψ_H-set in a T_0 space X and $f : X \longrightarrow Y$ is a continuous mapping to a T_0 space Y. We need to prove that $f(A)$ is a Ψ_H-set in Y. We consider the following cases:

(c1) there exists $K \in \mathcal{Q}(X)$ such that $A \in m(\mathcal{N}(K))$. Note that the intersection $K \cap A$ is compact and since f is continuous, $f(K \cap A)$ is compact in Y, so $\uparrow f(K \cap A) \in \mathcal{Q}(Y)$. By Lemma 5.6, $\mathrm{cl}_Y(f(A)) \in m(\mathcal{N}(\uparrow f(K \cap A)))$. This shows that $f(A)$ is a Ψ_H-set in Y.

(c2) there exists a $B \in H(X)$ such that $A \in m(\mathcal{M}(B))$. By Lemma 4.7, $A = \mathrm{cl}_X(B)$ and $\mathrm{cl}_Y(f(A)) \in m(\mathcal{M}(f(A)))$. Since $f(A) \in H(Y)$ by (H2), it follows that $\mathcal{M}(f(A)) = \mathcal{M}(f(\mathrm{cl}_X(B))) = \mathcal{M}(f(B)) \in \Psi_H(Y)$. Thus $f(A)$ is a Ψ_H-set in Y.

By Lemma 5.4, Ψ is a complete HM-system.

(2) ($\Rightarrow$). Assume X is H-sober. Let A be a Ψ_H-set in X. There are two cases:

(c1) $A \in m(\mathcal{M}(B))$ for some $B \in H(X)$. By Lemma 4.7, $\mathrm{cl}(B) = \mathrm{cl}(A)$, and since X is H-sober, $\mathrm{cl}(B) = \mathrm{cl}(A) = \mathrm{cl}(\{x\})$ for some $x \in X$.

(c2) $A \in m(\mathcal{N}(K))$ for some $K \in \mathcal{Q}(X)$. By Lemma 4.4, $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$ for some $x \in \min K$.

Then by Theorem 4.6, X is Ψ_H-well-filtered.

($\Leftarrow$). Assume X is a Ψ_H-well-filtered space. Let $A \in H(X)$. Then by Lemma 4.7, $\mathrm{cl}(A) \in m(\mathcal{M}(A)) \in \Psi_H(X)$, so A is a Ψ_H-set in X. Since X is Ψ_H-well-filtered, by Theorem 4.6 $\mathrm{cl}(A) = \mathrm{cl}(\{x\})$ for some $x \in X$. Therefore, X is H-sober. □

Theorem 5.8. *Let $\Psi : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ be a complete HM-system. Define $H_\Psi : \mathbf{Top_0} \longrightarrow \mathbf{Set}$ by*

$$H_\Psi(X) = \{A \subseteq X : A \textit{ is a } \Psi\textit{-set in } X\}$$

for each T_0 space X, and for each continuous mapping $f : X \longrightarrow Y$ between T_0 spaces X and Y, define $H_\Psi(f) : H_\Psi(X) \longrightarrow H_\Psi(Y)$ by

$$H_\Psi(f)(A) = f(A)$$

for each $A \in H_\Psi(X)$.

(1) H_Ψ *is an R-system.*
(2) *A* T_0 *space X is* Ψ*-well-filtered if and only if it is* H_Ψ*-sober.*

Proof. (1) We first prove that H_Ψ satisfies (H1) and (H2). For (H1), it follows from Proposition 4.5 that $\{\{x\} : x \in X\} \subseteq H_\Psi(X)$. By Lemma 2.12, every Ψ-set is irreducible; hence, $H_\Psi(X) \subseteq Irr(X)$. Thus, (H1) holds. Condition (H2) holds immediately since Ψ is complete. It is trivial to check that H is a covariant functor. Therefore, H_Ψ is an R-system.

(2) It is straightforward by Theorem 4.6. □

For an HM-system Ψ, we use Ψ-**WF** to denote the category of all Ψ-well-filtered spaces with continuous mappings. In Xu (2021), Xu proved that the category of all H-sober spaces with continuous mappings is reflective in the category $\mathbf{Top_0}$ of T_0 spaces. Then by Theorem 5.8, we have the following corollary.

Corollary 5.9. *For a complete HM-system* Ψ*, the category* Ψ-**WF** *is a reflective subcategory of* $\mathbf{Top_0}$.

6. The Smyth Power Space of a Ψ-Well-Filtered Space

For a topological space X, the *upper Vietoris topology* on $\mathcal{Q}^*(X) = \mathcal{Q}(X) \setminus \{\emptyset\}$ is generated by the following family (as a base)

$$\Box U = \{K \in \mathcal{Q}^*(X) : K \subseteq U\},$$

where U ranges over the open subsets of X. The resulting space, denoted by $P_s(X)$, is called the *Smyth power space* or the *upper space*.

Remark 6.1 (Goubault-Larrecq 2003; Jia and Jung 2016; Schalk 1993)**.** Let X be a T_0 space.

(1) The specialization order of $P_s(X)$ is $\supseteq$. Hence, for each $\mathcal{A} \subseteq \mathcal{Q}^*(X)$,

$$\uparrow_{P_s(X)}\mathcal{A} = \{K \in \mathcal{Q}^*(X) : K \subseteq G \text{ for some } G \in \mathcal{A}\}$$

in the specialization order of $P_s(X)$.
(2) Define $\xi : X \longrightarrow P_s(X)$, $x \mapsto \uparrow x$. Then $\xi^{-1}(\Box U) = U$ for each $U \in \mathcal{O}(X)$, and hence ξ is continuous.
(3) If $\mathcal{K}$ is compact in $P_s(X)$, then $\bigcup \mathcal{K}$ is compact in X.

Definition 6.2 (Xu 2021)**.** *Let X be a* T_0 *space. An R-subset system H is said to satisfy* property Q for *X if for each* $\mathcal{A} \in H(P_s(X))$ *and each* $C \in \mathfrak{M}(\mathcal{A})$*, there is an H-subset F of C such that* $\mathrm{cl}(F) \in \mathfrak{M}(\mathcal{A})$.

Lemma 6.3. *Let X be a* T_0 *space,* $\mathcal{A} \subseteq \mathcal{Q}(X)$*. Then,* $\mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathrm{cl}_{P_s(X)}(\mathcal{A}))$*. Hence,* $m(\mathcal{A}) = m(\mathrm{cl}_{P_s(X)}(\mathcal{A}))$.

Proof. We only need to prove that $\mathfrak{M}(\mathcal{A}) \subseteq \mathfrak{M}(\mathrm{cl}_{P_s(X)}(\mathcal{A}))$. Assume on the contrary there exists $C \in \mathfrak{M}(\mathcal{A})$ such that $C \notin \mathfrak{M}(\mathrm{cl}_{P_s(X)}(\mathcal{A}))$. Then, there exists $K \in \mathrm{cl}_{P_s(X)}(\mathcal{A})$ such that $C \cap K = \emptyset$. This implies that $\Box(X \setminus C)$ is an open neighborhood of K in $P_s(X)$, and since $K \in \mathrm{cl}_{P_s(X)}(\mathcal{A})$, there is $G \in \mathcal{A} \cap \Box(X \setminus C) \neq \emptyset$. Thus, G is a member of $\mathcal{A}$ such that $G \cap C = \emptyset$, contradicting the fact that $C \in \mathfrak{M}(\mathcal{A})$. Hence, $\mathfrak{M}(\mathcal{A}) \subseteq \mathfrak{M}(\mathrm{cl}_{P_s(X)}(\mathcal{A}))$. □

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The following theorem strengthens a result in Xu (2021).

Theorem 6.4. *Let H be an R-subset system and X be a T_0 space. The following two conditions are equivalent:*

(1) *$P_s(X)$ is H-sober;*
(2) *X is H-sober and H has property Q for X.*

Proof. By Xu (2021, Theorem 5.12), we only need to check that condition (1) implies that H has property Q for X. Suppose $\mathcal{A} \in H(P_s(X))$ and $C \in \mathfrak{M}(\mathcal{A})$. Since $P_s(X)$ is H-sober, there exists $K \in \mathcal{Q}^*(X)$ such that $\mathrm{cl}_{P_s(X)}(\mathcal{A}) = \mathrm{cl}_{P_s(X)}(\{K\})$. By Lemma 6.3, $C \in \mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathrm{cl}_{P_s(X)}(\mathcal{A})) = \mathfrak{M}(\mathrm{cl}_{P_s(X)}(\{K\})) = \mathfrak{M}(\{K\})$, which follows that $C \cap K \neq \emptyset$. Take $x \in C \cap K$. Then, $\{x\}$ is an H-set, and it is clear that $\mathrm{cl}(\{x\}) \in \mathfrak{M}(\{K\}) = \mathfrak{M}(\mathcal{A})$. □

Definition 6.5. *Let X be a T_0 space. An HM-system Ψ is said to satisfy* property Q for X *if for each Ψ-set $\mathcal{A}$ in $P_s(X)$ and each $C \in \mathfrak{M}(\mathcal{A})$, there is a Ψ-subset F of C such that* $\mathrm{cl}(F) \in \mathfrak{M}(\mathcal{A})$.

From Theorems 5.7 and 5.8, it turns out that there is a one to one correspondence between H-sober spaces and Ψ-well-filtered spaces when Ψ is complete. In other words, the class of Ψ-well-filtered spaces are more general than H-sober spaces. As a generalized result of Theorem 6.4, we have the following result.

Theorem 6.6. *Let Ψ be an HM-system and X be a T_0 space. Then, the following conditions are equivalent:*

(1) *$P_s(X)$ is Ψ-well-filtered;*
(2) *X is Ψ-well-filtered and Ψ has property Q for X;*
(3) *for each Ψ-set $\mathcal{A}$ in $P_s(X)$ and each $U \in \mathcal{O}(X)$, $\bigcap \mathcal{A} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{A}$.*

Proof. (1) $\Rightarrow$ (2). We first prove that X is Ψ-well-filtered. Let $\mathcal{F} \in \Psi(X)$ and $U \in \mathcal{O}(X)$ with $\bigcap \mathcal{F} \subseteq U$. Consider the continuous mapping $\xi : X \longrightarrow P_s(X)$, $x \mapsto \uparrow x$. Then, $\xi_*(\mathcal{F}) = \{\mathcal{U} \in \mathcal{O}(P_s(X)) : \xi^{-1}(\mathcal{U}) \in \mathcal{F}\} \in \Psi(P_s(X))$ by the definition of Ψ. Let

$$\mathfrak{F} = \xi_*(\mathcal{F}) \cap \{\Box V : V \in \mathcal{O}(X)\}.$$

For each $V \in \mathcal{O}(X)$, observe that $\Box V \in \mathfrak{F}$ iff $\xi^{-1}(\Box V) = V \in \mathcal{F}$, which implies that $\mathfrak{F} = \{\Box V : V \in \mathcal{F}\}$. If $K \in \bigcap \mathfrak{F}$, then $K \in \Box V$ (i.e., $K \subseteq V$) for all $V \in \mathcal{F}$, so $K \subseteq \bigcap \mathcal{F} \subseteq U$, which implies $K \in \Box U$. Thus, $\bigcap \mathfrak{F} \subseteq \Box U$. Since $P_s(X)$ is Ψ-well-filtered and $\bigcap \xi_*(\mathcal{F}) \subseteq \bigcap \mathfrak{F} \subseteq \Box U$, we have that $\Box U \in \xi_*(\mathcal{F})$, so $\xi^{-1}(\Box U) = U \in \mathcal{F}$. Hence, X is Ψ-well-filtered. Using a similar method of Theorem 6.4, one can prove that Ψ has property Q for X.

(2) $\Rightarrow$ (3). Assume on the contrary that $K \cap (X \setminus U) \neq \emptyset$ for each $K \in \mathcal{A}$. It follows that $X \setminus U \in \mathfrak{M}(\mathcal{A})$. Since Ψ satisfies property Q for X, there exists a Ψ-set $F \subseteq X \setminus U$ such that $\mathrm{cl}(F) \in \mathfrak{M}(\mathcal{A})$. Since X is Ψ-well-filtered, by Theorem 4.6 there exists $x \in X$ such that $\mathrm{cl}(F) = \mathrm{cl}(\{x\})$. For each $K \in \mathcal{A}$, since $K = \uparrow K$ and $\mathrm{cl}(F) \cap K = \mathrm{cl}(\{x\}) \cap K \neq \emptyset$, we deduce that $x \in K$. Thus, $x \in \bigcap \mathcal{A} \subseteq U$, contradicting the fact that $x \in \mathrm{cl}(F) \subseteq X \setminus U$. This shows that $K \subseteq U$ for some $K \in \mathcal{A}$, which gives (3).

(3) $\Rightarrow$ (1). Suppose $\mathcal{A}$ is a Ψ-set in $P_s(X)$ and $\mathcal{U}$ is an open set in $P_s(X)$ such that $\bigcap_{K \in \mathcal{A}} \uparrow_{P_s(X)} K \subseteq \mathcal{U}$. Then, there exists a family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$ such that $\mathcal{U} = \bigcup_{i \in I} \Box U_i$. Using a similar proof of Lemma 3.5, one can obtain that $K_0 = \bigcap \mathcal{A} \in \mathcal{Q}^*(X)$. Note that $\uparrow_{P_s(X)} K = \{G \in \mathcal{Q}^*(X) : G \subseteq K\}$ for each $K \in \mathcal{A}$, so $K_0 \in \bigcap_{K \in \mathcal{A}} \uparrow_{P_s(X)} K \subseteq \bigcup_{i \in I} \Box U_i$. Then, there exists $i_0 \in I$ such that $K_0 = \bigcap \mathcal{A} \in \Box U_{i_0}$, so $\bigcap \mathcal{A} \subseteq U_{i_0}$. By condition (3), there exists $K_1 \in \mathcal{A}$ such that $K_1 \subseteq U_{i_0}$. This implies that $\uparrow_{P_s(X)} K_1 \subseteq \Box U_{i_0} \subseteq \mathcal{U}$. By Theorem 4.6, we deduce that $P_s(X)$ is Ψ-well-filtered. □

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Definition 6.7. *Let X be a T_0 space. An HM-system Ψ is said to satisfy* property T for X *if for any Ψ-set $\mathcal{A}$ in $P_s(X)$, $\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)\in\Psi(X)$.*

Remark 6.8. Using Lemma 4.3, one can easily deduce that $\mathfrak{M}(\mathcal{A})=\mathfrak{M}(\mathcal{F}_{\mathcal{A}})$ in Definition 6.7, where $\mathcal{F}_{\mathcal{A}}=\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)$. Then by Lemma 2.12, if an HM-system Ψ satisfies property T, then it must satisfy Q.

Lemma 6.9. *Both Ψ_{sob} and Ψ_{wf} satisfy property T for each T_0 space X.*

Proof. Let X be a T_0 space. It is trivial that Ψ_{sob} satisfies property T for X. Next, we verify that Ψ_{wf} satisfies property T for X. To show this, let $\mathfrak{F}\in\Psi_{\mathsf{wf}}(P_s(X))$ and $\mathcal{A}\in m(\mathfrak{F})$. Then, there exists a filtered family $\{\mathcal{K}_i : i\in I\}$ of compact saturated subsets of $P_s(X)$ such that $\mathfrak{F}=\{\mathcal{U}\in\mathcal{O}(P_s(X)) : \exists i\in I, \mathcal{K}_i\subseteq\mathcal{U}\}$. It suffices to prove $\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)\in\Psi_{\mathsf{wf}}(X)$.

Claim: $\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)=\bigcup_{i\in I}\mathcal{N}(\bigcup(\mathcal{A}\cap\mathcal{K}_i))$.

On the one hand, suppose $U\in\bigcup_{i\in I}\mathcal{N}(\bigcup(\mathcal{A}\cap\mathcal{K}_i))$. Then, there exists $i_0\in I$ such that $\bigcup(\mathcal{A}\cap\mathcal{K}_{i_0})\subseteq U$. Choose one $K_{i_0}\in\mathcal{A}\cap\mathcal{K}_{i_0}\neq\emptyset$. Then, $K_{i_0}\subseteq\bigcup(\mathcal{A}\cap\mathcal{K}_{i_0})\subseteq U$, which implies that $U\in\mathcal{N}(K_{i_0})\subseteq\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)$. This shows that $\bigcup_{i\in I}\mathcal{N}(\bigcup(\mathcal{A}\cap\mathcal{K}_i))\subseteq\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)$. On the other hand, suppose $U\notin\bigcup_{i\in I}\mathcal{N}(\bigcup(\mathcal{A}\cap\mathcal{K}_i))$. For each $i\in I$, $\bigcup(\mathcal{A}\cap\mathcal{K}_i)\nsubseteq U$, so there exists $G_i\in\mathcal{A}\cap\mathcal{K}_i$ such that $G_i\nsubseteq U$; that is, $G_i\notin\Box U$. Then, $G_i\in(P_s(X)\setminus\Box U)\cap\mathcal{A}\cap\mathcal{K}_i\neq\emptyset$ for all $i\in I$. Thus, $(P_s(X)\setminus\Box U)\cap\mathcal{A}$ is a closed subset of $\mathcal{A}$ such that $(P_s(X)\setminus\Box U)\cap\mathcal{A}\in\mathfrak{M}(\mathfrak{F})$. Since $\mathcal{A}\in m(\mathfrak{F})$, $\mathcal{A}\cap(P_s(X)\setminus\Box U)=\mathcal{A}$, that is, $\mathcal{A}\subseteq P_s(X)\setminus\Box U$. It follows that $K\nsubseteq U$ for all $K\in\mathcal{A}$; hence, $U\notin\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)$. Therefore, $\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)\subseteq\bigcup_{i\in I}\mathcal{N}(\bigcup(\mathcal{A}\cap\mathcal{K}_i))$.

Recall that the intersection of a closed set and a compact set is always compact, so $\mathcal{A}\cap\mathcal{K}_i$ is compact in $P_s(X)$ for each $i\in I$. Since $\{\mathcal{K}_i : i\in I\}$ is a filtered family, by Remark 6.1 $\{\uparrow\bigcup(\mathcal{A}\cap\mathcal{K}_i) : i\in I\}$ is a filtered family of $\mathcal{Q}^*(X)$. Note that for each open subset U of X, $\bigcup(\mathcal{A}\cap\mathcal{K}_i)\subseteq U$ if and only if $\uparrow\bigcup(\mathcal{A}\cap\mathcal{K}_i)\subseteq U$, we have that $\bigcup_{K\in\mathcal{A}}\mathcal{N}(K)=\bigcup_{i\in I}\mathcal{N}(\bigcup(\mathcal{A}\cap\mathcal{K}_i))=\bigcup_{i\in I}\mathcal{N}(\uparrow\bigcup\mathcal{A}\cap\mathcal{K}_i)\in\Psi_{\mathsf{wf}}(X)$. Therefore, Ψ_{wf} satisfies property T. $\square$

As a direct consequence of Theorem 6.6, Remark 6.8 and Lemma 6.9, we have the following corollaries.

Corollary 6.10 (Heckmann and Keimel 2013)**.** *Let X be a T_0 space. The following conditions are equivalent:*

(1) *X is sober;*
(2) *$P_s(X)$ is sober.*

Corollary 6.11 (Xu et al. 2021)**.** *Let X be a T_0 space. The following conditions are equivalent:*

(1) *X is well-filtered;*
(2) *$P_s(X)$ is well-filtered.*

7. Conclusion

Motivated by the Hofmann-Mislove Theorem on sober spaces, we introduce the Ψ-well-filtered spaces, where $\Psi:\mathbf{Top}_0\longrightarrow\mathbf{Set}$ is a covariant functor that assigns each T_0 space X a family of open filters of $\mathcal{O}(X)$. The classes of d-spaces, well-filtered spaces, and sober spaces are all special types of Ψ-well-filtered spaces. The $\mathcal{U}_s$-admitting spaces (Heckmann 1991), and the recently introduced open well-filtered spaces (Shen et al. 2020) and the ω-well-filtered spaces (Xu et al. 2020*b*) can also be viewed as special types of Ψ-well-filtered spaces. The results in this paper reveal some features of such spaces similar to that of sober space as shown by the Hofmann-Mislove Theorem. We

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hope that based on this general notion, some new classes of interesting spaces can be identified in the future.

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