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First countability, ω-well-filtered spaces and reflections ☆

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ABSTRACT

We first introduce and study two new classes of subsets in T_0 spaces — ω-Rudin sets and ω-well-filtered determined sets lying between the class of all closures of countable directed subsets and that of irreducible closed subsets, and two new types of spaces — ω-d-spaces and ω-well-filtered spaces. We prove that an ω-well-filtered T_0 space is locally compact iff it is core compact. One immediate corollary is that every core compact well-filtered space is sober, answering Jia-Jung problem with a new method. We also prove that all irreducible closed subsets in a first countable ω-well-filtered T_0 space are directed. Therefore, a first countable T_0 space X is sober iff X is well-filtered iff X is an ω-well-filtered d-space. Using ω-well-filtered determined sets, we present a direct construction of the ω-well-filtered reflections of T_0 spaces, and show that products of ω-well-filtered spaces are ω-well-filtered.

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1. Introduction

In domain theory and non-Hausdorff topology, the d-spaces, well-filtered spaces and sober spaces form three of the most important classes of spaces (see [2–26]). In this paper, based on the topological version

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of Rudin's Lemma by Heckmann and Keimel [6], we introduce and study two new classes of subsets in T_0 spaces — ω-Rudin sets and ω-well-filtered determined sets (WD_ω sets for short) lying between the class of all closures of countable directed subsets and that of irreducible closed subsets. We also introduce and investigate two new types of spaces — ω-d-spaces and ω-well-filtered spaces. It will be proved that an ω-well-filtered T_0 space is locally compact iff it is core compact. One immediate corollary is that every core compact well-filtered space is sober, giving a positive answer to Jia-Jung problem [16], which has been first answered by Lawson and Xi [19] using a different method. We also prove that every irreducible closed subset of a first countable ω-well-filtered T_0 space is directed. Therefore, a first countable T_0 space X is sober iff X is well-filtered iff X is an ω-well-filtered d-space.

It is well-known that the category of all sober spaces and that of d-spaces are reflective in the category of all T_0 spaces (see [7,12,23]). Recently, following Ershov's method of constructing the d-completion of T_0 spaces, Shen, Xi, Xu and Zhao [22] presented a construction of the well-filtered reflection of T_0 spaces. In the current paper, using WD_ω sets, we present a direct construction of the ω-well-filtered reflections of T_0 spaces, and show that products of ω-well-filtered spaces are ω-well-filtered. Some major properties of ω-well-filtered reflections of T_0 spaces are also investigated.

2. Preliminary

In this section, we briefly recall some basic concepts and notations to be used in this paper. Some known properties of irreducible sets and compact saturated sets are presented.

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). The set of all directed sets of P is denoted by $\mathcal{D}(P)$. P is called a *directed complete poset*, or *dcpo* for short, if for any $D \in \mathcal{D}(P)$, $\bigvee D$ exists in P. The set of all natural numbers with the usual ordering is denoted by $\mathbb{N}$. Let ω denote the ordinal (also the cardinal number) of $\mathbb{N}$ and ω_1 the first uncountable ordinal.

The upper sets of a poset Q form the (*upper*) *Alexandroff topology* $\alpha(Q)$ on Q. As in [7], the *lower topology* on Q, generated by the complements of the principal filters of Q, is denoted by $\omega(Q)$. A subset U of Q is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of Q form a topology, and we call this topology the *Scott topology* on Q and denote it by $\sigma(P)$. The space $\Sigma\, Q = (Q, \sigma(Q))$ is called the *Scott space* of Q. The common refinement $\sigma(Q)$ and $\omega(Q)$ is called the *Lawson topology* and is denoted by $\lambda(Q)$. The space $\Lambda(Q) = (Q, \lambda(Q))$ is called the *Lawson space* of Q.

The category of all T_0 spaces with continuous mappings is denoted by $\mathbf{Top}_0$. For any $X \in \mathbf{Top}_0$, $\leq_X$ denotes the *specialization order* on X: $x \leq_X y$ iff $x \in \overline{\{y\}}$. In the following, when a T_0 space X is considered as a poset, the order always refers to the specialization order if no other explanation. Let $\mathcal{O}(X)$ (resp., $\mathcal{C}(X)$) be the set of all open subsets (resp., closed subsets) of X. Define $\mathcal{S}_c(X) = \{\overline{\{x\}} : x \in X\}$ and $\mathcal{D}_c(X) = \{\overline{D} : D \in \mathcal{D}(X)\}$. A space X is called a *d-space* (or *monotone convergence space*) if X (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. [7,24]).

As in [4], a space X is *locally hypercompact* if for each $x \in X$ and each open neighborhood U of x, there is a nonempty finite subset of X such that $x \in \operatorname{int}\uparrow F \subseteq \uparrow F \subseteq U$. A space X is called a *C-space* if for each $x \in X$ and each open neighborhood U of x, there is $u \in X$ such that $x \in \operatorname{int}\uparrow u \subseteq \uparrow u \subseteq U$. A set $K \subseteq X$ is called *supercompact* if for any family $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, $K \subseteq \bigcup_{i\in I} U_i$ implies $K \subseteq U_i$ for some $i \in I$. It is easy to check that the supercompact saturated sets of X are exactly the sets $\uparrow x$ with $x \in X$ (see [13, Fact 2.2]). It is well-known that X is a C-space iff $\mathcal{O}(X)$ is a *completely distributive* lattice (cf. [2]). A space X is called *core compact* if $\mathcal{O}(X)$ is a *continuous lattice* (cf. [7]).

For a T_0 space X and a nonempty subset A of X, A is *irreducible* if for any $\{F_1, F_2\} \subseteq \mathcal{C}(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X. Clearly, every subset of X that is directed under $\leq_X$ is irreducible. X is called *sober*, if for any $F \in \mathsf{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \overline{\{a\}}$.

The following two lemmas on irreducible sets are well-known.

Lemma 2.1. *Let X be a space and Y a subspace of X. Then the following conditions are equivalent for a subset $A \subseteq Y$:*

(1) *A is an irreducible subset of Y.*
(2) *A is an irreducible subset of X.*
(3) *$\mathrm{cl}_X A$ is an irreducible closed subset of X.*

Lemma 2.2. *If $f : X \longrightarrow Y$ is continuous and $A \in \mathsf{Irr}(X)$, then $f(A) \in \mathsf{Irr}(Y)$.*

Remark 2.3. If Y is a subspace of a space X and $A \subseteq Y$, then by Lemma 2.1, $\mathsf{Irr}(Y) = \{B \in \mathsf{Irr}(X) : B \subseteq Y\} \subseteq \mathsf{Irr}(X)$ and $\mathsf{Irr}_c(Y) = \{B \in \mathsf{Irr}(X) : B \in \mathcal{C}(Y)\} \subseteq \mathsf{Irr}(X)$. If $Y \in \mathcal{C}(X)$, then $\mathsf{Irr}_c(Y) = \{C \in \mathsf{Irr}_c(X) : C \subseteq Y\} \subseteq \mathsf{Irr}_c(X)$.

Lemma 2.4. ([22]) *Let $X = \prod_{i\in I} X_i$ be the product space of T_0 spaces $X_i (i \in I)$. If A is an irreducible subset of X, then $\mathrm{cl}_X(A) = \prod_{i\in I} \mathrm{cl}_{X_i}(p_i(A))$, where $p_i : X \longrightarrow X_i$ is the ith projection.*

Lemma 2.5. *Let $X = \prod_{i\in I} X_i$ be the product space of T_0 spaces $X_i (i \in I)$ and $A_i \subseteq X_i (i \in I)$. Then the following two conditions are equivalent:*

(1) *$\prod_{i\in I} A_i \in \mathsf{Irr}(X)$.*
(2) *$A_i \in \mathsf{Irr}(X_i)$ $(i \in I)$.*

Proof. (1) $\Rightarrow$ (2): By Lemma 2.2.

(2) $\Rightarrow$ (1): Let $A = \prod_{i\in I} A_i$. For $U, V \in \mathcal{O}(X)$, if $A \cap U \neq \emptyset \neq A \cap V$, then there exist $I_1, I_2 \in I^{(<\omega)}$ and $(U_i, V_j) \in \mathcal{O}(X_i) \times \mathcal{O}(X_j)$ for all $(i, j) \in I_1 \times I_2$ such that $\bigcap_{i\in I_1} p_i^{-1}(U_i) \subseteq U$, $\bigcap_{j\in I_2} p_j^{-1}(V_j) \subseteq V$ and $A \cap \bigcap_{i\in I_1} p_i^{-1}(U_i) \neq \emptyset \neq A \cap \bigcap_{j\in I_2} p_j^{-1}(V_j)$. Let $I_3 = I_1 \cup I_2$. Then I_3 is finite. For $i \in I_3 \setminus I_1$ and $j \in I_3 \setminus I_2$, let $U_i = X_i$ and $V_j = X_j$. Then for each $i \in I_3$, we have $A_i \cap U_i \neq \emptyset \neq A_i \cap V_i$, and whence $A_i \cap U_i \cap V_i \neq \emptyset$ by $A_i \in \mathsf{Irr}(X_i)$. It follows that $A \cap \bigcap_{i\in I_1} p_i^{-1}(U_i) \cap \bigcap_{j\in I_2} p_i^{-1}(V_j) \neq \emptyset$, and consequently, $A \cap U \cap V \neq \emptyset$. Thus $A \in \mathsf{Irr}(X)$. $\square$

By Lemma 2.4 and Lemma 2.5, we obtain the following corollary.

Corollary 2.6. *Let $X = \prod_{i\in I} X_i$ be the product space of T_0 spaces $X_i (i \in I)$. If $A \in \mathsf{Irr}_c(X)$, then $A = \prod_{i\in I} p_i(A)$ and $p_i(A) \in \mathsf{Irr}_c(X_i)$ for each $i \in I$.*

For any topological space X, $\mathcal{G} \subseteq 2^X$ and $A \subseteq X$, let $\Diamond_{\mathcal{G}} A = \{G \in \mathcal{G} : G \bigcap A \neq \emptyset\}$ and $\Box_{\mathcal{G}} A = \{G \in \mathcal{G} : G \subseteq A\}$. The symbols $\Diamond_{\mathcal{G}} A$ and $\Box_{\mathcal{G}} A$ will be simply written as $\Diamond A$ and $\Box A$ respectively if there is no confusion. The *lower Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a subbase, and the resulting space is denoted by $P_H(\mathcal{G})$. If $\mathcal{G} \subseteq \mathsf{Irr}(X)$, then $\{\Diamond_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ is a topology on $\mathcal{G}$. The space $P_H(\mathcal{C}(X) \setminus \{\emptyset\})$ is called the *Hoare power space* or *lower space* of X and is denoted by $P_H(X)$ for short (cf. [21]). The *upper Vietoris topology* on $\mathcal{G}$ is the topology that has $\{\Box_{\mathcal{G}} U : U \in \mathcal{O}(X)\}$ as a base, and the resulting space is denoted by $P_S(\mathcal{G})$.

Remark 2.7. Let X be a T_0 space.

(1) If $\mathcal{S}_c(X) \subseteq \mathcal{G}$, then the specialization order on $P_H(\mathcal{G})$ is the order of set inclusion, and the *canonical mapping* $\eta_X : X \longrightarrow P_H(\mathcal{G})$, given by $\eta_X(x) = \overline{\{x\}}$, is an order and topological embedding (cf. [7,8,21]).
(2) The space $X^s = P_H(\mathsf{Irr}_c(X))$ with the canonical mapping $\eta_X : X \longrightarrow X^s$ is the *sobrification* of X (cf. [7,8]).

A subset A of a space X is called *saturated* if A equals the intersection of all open sets containing it (equivalently, A is an upper set in the specialization order). We shall use $\mathsf{K}(X)$ to denote the set of all nonempty compact saturated subsets of X and endow it with the *Smyth preorder*, that is, for $K_1, K_2 \in \mathsf{K}(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. Let $\mathcal{S}^u(X) = \{\uparrow x : x \in X\}$. X is called *well-filtered* if it is T_0, and for any open set U and filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, $\bigcap \mathcal{K} \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

For the well-filteredness of Scott topologies on dcpos, Xi and Lawson [25] gave the following interesting results.

Lemma 2.8. ([25]) *Let P be a dcpo. If $(P, \lambda(P))$ is compact, then $(P, \sigma(P))$ is well-filtered.*

Corollary 2.9. ([25]) *For any complete lattice L, $(L, \sigma(L))$ is well-filtered.*

For any T_0 space X, the space $P_S(\mathsf{K}(X))$, denoted shortly by $P_S(X)$, is called the *Smyth power space* or *upper space* of X (cf. [11,21]). It is easy to see that the specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \sqsubseteq$). The *canonical mapping* $\xi_X : X \longrightarrow P_S(X)$, $x \mapsto \uparrow x$, is an order and topological embedding (cf. [11,13,21]). Clearly, $P_S(\mathcal{S}^u(X))$ is a subspace of $P_S(X)$ and X is homeomorphic to $P_S(\mathcal{S}^u(X))$.

Lemma 2.10. *Let X be a T_0 space and $A \subseteq X$. Then the following conditions are equivalent:*

(1) *$A \in \mathsf{Irr}(X)$.*
(2) *$\xi_X(A) \in \mathsf{Irr}(P_S(X))$.*

Proof. (1) $\Rightarrow$ (2): By Lemma 2.2.

(2) $\Rightarrow$ (1): Suppose that $B, C \in \mathcal{C}(X)$ such that $A \subseteq B \cup C$. Then $\Diamond B, \Diamond C \in \mathcal{C}(P_S(X))$ and $\xi_X(A) \subseteq \Diamond B \cup \Diamond C$, and hence $\xi_X(A) \subseteq \Diamond B$ or $\xi_X(A) \subseteq \Diamond C$ by $\xi_X(A) \in \mathsf{Irr}(P_S(X))$. It follows that $A \subseteq B$ or $A \subseteq C$. Thus $A \in \mathsf{Irr}(X)$. $\square$

Remark 2.11. Let X be a T_0 space and $\mathcal{A} \subseteq \mathsf{K}(X)$. Then $\bigcap \mathcal{A} = \bigcap \overline{\mathcal{A}}$, here the closure of $\mathcal{A}$ is taken in $P_S(X)$. Clearly, $\bigcap \overline{\mathcal{A}} \subseteq \bigcap \mathcal{A}$. On the other hand, for $K \in \overline{\mathcal{A}}$ and $U \in \mathcal{O}(X)$ with $K \subseteq U$ (that is, $K \in \Box U$), we have $\mathcal{A} \bigcap \Box U \neq \emptyset$, and hence there is a $K_U \in \mathcal{A} \bigcap \Box U$. Therefore, $K = \bigcap \{U \in \mathcal{O}(X) : K \subseteq U\} \supseteq \bigcap \{K_U : U \in \mathcal{O}(X) \text{ and } K \subseteq U\} \supseteq \bigcap \mathcal{A}$. It follows that $\bigcap \overline{\mathcal{A}} \supseteq \bigcap \mathcal{A}$. Thus $\bigcap \mathcal{A} = \bigcap \overline{\mathcal{A}}$.

Lemma 2.12. ([7]) *For a nonempty family $\{K_i : i \in I\} \subseteq \mathsf{K}(X)$, $\bigvee_{i \in I} K_i$ exists in $\mathsf{K}(X)$ iff $\bigcap_{i \in I} K_i \in \mathsf{K}(X)$. In this case, $\bigvee_{i \in I} K_i = \bigcap_{i \in I} K_i$.*

Lemma 2.13. ([17,21]) *Let X be a T_0 space. If $\mathcal{K} \in \mathsf{K}(P_S(X))$, then $\bigcup \mathcal{K} \in \mathsf{K}(X)$.*

Corollary 2.14. ([17,21]) *For any T_0 space X, the mapping $\bigcup : P_S(P_S(X)) \longrightarrow P_S(X)$, $\mathcal{K} \mapsto \bigcup \mathcal{K}$, is continuous.*

Proof. For $\mathcal{K} \in \mathsf{K}(P_S(X))$, $\bigcup \mathcal{K} \in \mathsf{K}(X)$ by Lemma 2.13. For $U \in \mathcal{O}(X)$, we have $\bigcup^{-1}(\Box U) = \{\mathcal{K} \in \mathsf{K}(P_S(X)) : \bigcup \mathcal{K} \in \Box U\} = \{\mathcal{K} \in \mathsf{K}(P_S(X)) : \mathcal{K} \subseteq \Box U\} = \eta_{P_S(X)}^{-1}(\Box(\Box U)) \in \mathcal{O}(P_S(P_S(X)))$. Thus $\bigcup : P_S(P_S(X)) \longrightarrow P_S(X)$ is continuous. $\Box$

3. ω-d-Spaces and ω-well-filtered spaces

For a T_0 space X, let $\mathcal{D}^\omega(X) = \{D \subseteq X : D \text{ is countable and directed}\}$ and $\mathcal{D}_c^\omega(X) = \{\overline{D} : D \in \mathcal{D}^\omega(X)\}$.

Definition 3.1. A poset P is called an *ω-dcpo*, if for any $D \in \mathcal{D}^\omega(X)$, $\bigvee D$ exists.

Lemma 3.2. *Let P be a poset and D a countable directed subset of P. Then there exists a countable chain $C \subseteq D$ such that $D = \downarrow C$. Hence, $\bigvee C$ exists and $\bigvee C = \bigvee D$ whenever $\bigvee D$ exists.*

Proof. If $|D| < \omega$, then D contains a largest element d, so let $C = \{d\}$, which satisfies the requirement.

Now assume $|D| = \omega$ and let $D = \{d_n : n < \omega\}$. We use induction on $n \in \omega$ to define $C = \{c_n : n < \omega\}$. More precisely, let $c_0 = d_0$ and let c_{n+1} be an upper bound of $\{d_{n+1}, c_0, c_1, c_2 \ldots, c_n\}$ in D. It is clear that C is a chain and $D = \downarrow C$. $\Box$

Corollary 3.3. *A poset P is an ω-dcpo iff for any countable chain C of P, $\bigvee C$ exists.*

Definition 3.4. Let P be a poset. A subset U of P is called *ω-Scott open* if (i) $U = \uparrow U$, and (ii) for any countable directed set D, $\bigvee D \in U$ implies that $D \cap U \neq \emptyset$. All ω-Scott open sets form a topology on P, denoted by $\sigma_\omega(P)$ and called the *ω-Scott topology*. The space $\Sigma_\omega P = (P, \sigma_\omega(P))$ is called the *ω-Scott space* of P.

Remark 3.5. (1) By Lemma 3.2, $U \in \sigma_\omega(P)$ iff $U = \uparrow U$ and for any countable chain C, $\bigvee C \in U$ implies $C \cap U \neq \emptyset$.

(2) Clearly, $\sigma(P) \subseteq \sigma_\omega(P)$. The converse need not be true, see Example 4.3.

Definition 3.6. A T_0 space X is called an ω-d-space (or an *ω-monotone convergence space*) if for any $D \in \mathcal{D}^\omega(X)$, the closure of D has a generic point, equivalently, if $\mathcal{D}_c^\omega(X) = \mathcal{S}_c(X)$.

Proposition 3.7. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is an ω-d-space.*
(2) *X (with the specialization order $\leq_X$) is an ω-dcpo and $\mathcal{O}(X) \subseteq \sigma_\omega(X)$.*
(3) *For any $D \in \mathcal{D}^\omega(X)$ and $U \in \mathcal{O}(X)$, $\bigcap_{d \in D} \uparrow d \subseteq U$ implies $\uparrow d \subseteq U$ (i.e., $d \in U$) for some $d \in D$.*
(4) *For any $D \in \mathcal{D}^\omega(X)$ and $A \in \mathcal{C}(X)$, if $D \subseteq A$, then $A \cap \bigcap_{d \in D} \uparrow d \neq \emptyset$.*
(5) *For any $D \in \mathcal{D}^\omega(X)$ and $A \in \mathsf{Irr}_c(X)$, if $D \subseteq A$, then $A \cap \bigcap_{d \in D} \uparrow d \neq \emptyset$.*
(6) *For any $D \in \mathcal{D}^\omega(X)$, $\overline{D} \cap \bigcap_{d \in D} \uparrow d \neq \emptyset$.*

Proof. (1) $\Leftrightarrow$ (2): Clearly, (2) $\Rightarrow$ (1). Conversely, if condition (1) holds, then for each $D \in \mathcal{D}^\omega(X)$ and $A \in \mathcal{C}(X)$ with $D \subseteq A$, there is $x \in X$ such that $\overline{D} = \overline{\{x\}}$, and consequently, $\bigvee D = x$ and $\bigvee D \in A$ since $\overline{D} \subseteq A$. Thus X is an ω-dcpo and $\mathcal{O}(X) \subseteq \sigma_\omega(X)$.

(2) $\Rightarrow$ (3): By condition (2), $\uparrow \bigvee D = \bigcap_{d \in D} \uparrow d \subseteq U \in \sigma_\omega(X)$. Therefore, $\bigvee D \in U$, and whence $d \in U$ for some $d \in D$.

(3) ⇒ (4): If $A \cap \bigcap_{d\in D} \uparrow d = \emptyset$, then $\bigcap_{d\in D} \uparrow d \subseteq X \setminus A$. By condition (3), $\uparrow d \subseteq X \setminus A$ for some $d \in D$, which is in contradiction with $D \subseteq A$.

(4) ⇒ (5) ⇒ (6): Trivial.

(6) ⇒ (1): For each $D \in \mathcal{D}^\omega(X)$ and $A \in \mathcal{C}(X)$ with $D \subseteq A$, by condition (6), $\overline{D} \cap \bigcap_{d\in D} \uparrow d \neq \emptyset$. Select an $x \in \overline{D} \cap \bigcap_{d\in D} \uparrow d$. Then $D \subseteq \downarrow x \subseteq \overline{D}$, and hence $\overline{D} = \downarrow x$. Thus X is an ω-d-space. □

A poset P is said to be *Noetherian* if it satisfies the *ascending chain condition* (ACC for short): every ascending chain has a greatest member (cf. [28]). Clearly, P is Noetherian iff every directed set of P has a largest element. The poset P is said to be *ω-Noetherian* if it satisfies the *ω-ascending chain condition* (ω-ACC for short): every countable ascending chain has a greatest member. By Lemma 3.2, P is ω-Noetherian iff every countable directed set of P has a largest element.

Proposition 3.8. *For a poset P, the following conditions are equivalent:*

(1) *$(P, \alpha(P))$ is an ω-d-space.*
(2) *P is an ω-Noetherian poset.*
(3) *P is a Noetherian poset.*
(4) *P is an ω-dcpo such that every element of P is ω-compact (i.e., $x \ll_\omega x$ for all $x \in P$).*
(5) *P is an ω-dcpo such that $\alpha(P) = \sigma_\omega(P)$.*

Proof. (1) ⇒ (2): Suppose that D is a countable directed set of P. Then $\downarrow D$ is a closed subset in $(P, \alpha(P))$. Since $(P, \alpha(P))$ is an ω-d-space, by Proposition 3.7, $\bigcap_{d\in D} \uparrow d \cap \downarrow D \neq \emptyset$. Clearly, $x \in \bigcap_{d\in D} \uparrow d \cap \downarrow D$ iff x is the greatest element of P.

(2) ⇔ (3): Clearly, (3) ⇒ (2). Conversely, suppose that P is ω-Noetherian. For any ascending chain C of P, if C has no greatest member, then using induction on $n \in \omega$, we can get a strictly ascending countable chain $c_0 < c_1 < c_2 ... < c_n < c_{n+1} < ...$ in C, and hence the countable ascending chain $\{c_0, c_1, c_2, ..., c_n, ...\}$ has no greatest member, a contradiction.

(3) ⇒ (4) and (5) ⇒ (1): Trivial.

(4) ⇒ (5): For any $x \in X$, since $x \ll_\omega x$, we have $\uparrow x \in \sigma_\omega(P)$. Therefore, $\alpha(P) = \sigma_\omega(P)$. □

Definition 3.9. A T_0 space X is called *ω-well-filtered*, if for any countable filtered family $\{K_i : i < \omega\} \subseteq \mathsf{K}(X)$ and $U \in \mathcal{O}(X)$, it satisfies

$$\bigcap_{i<\omega} K_i \subseteq U \;\Rightarrow\; \exists i_0 < \omega, K_{i_0} \subseteq U.$$

By Lemma 3.2, we have the following result.

Proposition 3.10. *A T_0 space X is ω-well-filtered iff for any countable descending chain $K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$ of compact saturated subsets of X and $U \in \mathcal{O}(X)$, the following implication holds:*

$$\bigcap_{i<\omega} K_i \subseteq U \;\Rightarrow\; \exists i_0 < \omega,\; K_{i_0} \subseteq U.$$

By Proposition 3.7, we get the following result.

Proposition 3.11. *Every ω-well-filtered space is an ω-d-space.*

The following result is well-known.

Theorem 3.12. ([7,8,18]) *For a T_0 space X, the following conditions are equivalent:*

(1) *X is locally compact and sober.*
(2) *X is locally compact and well-filtered.*
(3) *X is core compact and sober.*

The above theorem will be strengthened (see Theorem 3.19).

Definition 3.13. For a T_0 space and $A, B \subseteq X$, we say A is *way below* B, or A is *compact relative to* B, written as $A \ll B$, if for each $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, $B \subseteq \bigcup_{i\in I} U_i$ implies $A \subseteq \bigcup_{i\in J} U_i$ for some finite subset J of I.

Clearly, we have $A \ll B \Rightarrow \uparrow A \subseteq \uparrow B$, and if $A, B, G, H \in \alpha(X)$ (i.e., A, B, G and H are upper sets in X), then $G \subseteq A \ll B \subseteq H \Rightarrow G \ll H$.

Lemma 3.14. *Let X be a T_0 space and $\mathcal{F}$ a filtered family of nonempty subsets of X satisfying the following property: for any $F \in \mathcal{F}$, there is $G \in \mathcal{A}$ with $G \ll F$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{F}$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{F}$.*

Proof. Let $\mathcal{C} = \{B \in \mathcal{C}(X) : B \subseteq C$ and C meets all members of $\mathcal{F}\}$.

Claim 1: $C \in \mathcal{C}$.

Claim 2: For any filtered family $\mathcal{E} \subseteq \mathcal{C}$, $\bigcap \mathcal{E} \in \mathcal{C}$.

Let $E = \bigcap \mathcal{E}$. Then $E \in \mathcal{C}(X)$ and $E \subseteq C$. Assume $E \notin \mathcal{C}$. Then there exists $F \in \mathcal{F}$ such that $E \cap F = \emptyset$. By hypothesis, there is $G \in \mathcal{A}$ with $G \ll F$. For each $H \in \mathcal{E}$, let $U_H = X \setminus H$. Then $\{U_H : H \in \mathcal{E}\} \subseteq \mathcal{O}(X)$ is directed and $G \ll F \subseteq \bigcup\limits_{H\in\mathcal{E}} U_H$, and whence $G \subseteq U_{H_0}$ for some $H_0 \in \mathcal{E}$, which is in contradiction with $H_0 \in \mathcal{E} \subseteq \mathcal{C}$. Therefore, $\bigcap \mathcal{E} \in \mathcal{C}$.

By Zorn's Lemma, there exists a minimal element A in $\mathcal{C}$. Finally, we show that $A \in \mathsf{Irr}_c(X)$. Suppose that $A_1, A_2 \in \mathcal{C}(X)$ and $A = A_1 \bigcup A_2$. If $A \neq A_1$ and $A \neq A_2$, then by the minimality of A, there are $F_1, F_2 \in \mathcal{F}$ such that $A_1 \bigcap F_1 = \emptyset = A_2 \bigcap F_2$. By the filteredness of $\mathcal{F}$, there is $F_3 \in \mathcal{F}$ with $F_3 \subseteq F_1 \bigcap F_2$, and consequently, $A \bigcap F_3 = (A_1 \bigcap F_3) \bigcup (A_2 \bigcap F_3) \subseteq (A_1 \bigcap F_1) \bigcup (A_2 \bigcap F_2) = \emptyset$, a contradiction. Thus $A \in \mathsf{Irr}_c(X)$. $\square$

Theorem 3.15. *For an ω-well-filtered T_0 space X, X is locally compact iff X is core compact.*

Proof. Suppose that X is core compact. For $x \in X$ and $U \in \mathcal{O}(X)$ with $x \in U$, since X is core compact, there is a sequence $\{U_\infty, ..., U_n, ..., U_2, U_1, U_0\} \subseteq \mathcal{O}(X)$ such that

$$x \in U_\infty \ll ... \ll U_n \ll ... \ll U_2 \ll U_1 \ll U_0 = U.$$

We show that $K = \bigcap\limits_{i\in\mathbb{N}} U_i$ is compact. Suppose that $\{W_i : i \in I\} \subseteq \mathcal{O}(X)$ is an open cover of K. Let $W = \bigcup_{i\in I} W_i$. If $U_n \nsubseteq W$ for all $n \in \mathbb{N}$, then by Lemma 3.14, there is a minimal irreducible closed subset $C \subseteq X \setminus W$ such that $U_n \cap C \neq \emptyset$. For each $n \in \mathbb{N}$, select an $x_n \in U_n \cap C$ and let $H_n = \{x_m : n \leq m\}$. Since $U_1 \supseteq U_2 \supseteq ... \supseteq U_m \supseteq ...$, $\uparrow H_n \subseteq U_n$ for each $n \in \mathbb{N}$. Now we prove that $\uparrow H_n \in \mathsf{K}(X)$ for all $n \in \mathbb{N}$. Suppose that $\{V_d : d \in D\} \subseteq \mathcal{O}(X)$ is a directed open cover of $\uparrow H_n$.

(c1) If for some $d_1 \in D$, $H_n \cap (X \setminus V_{d_1}) = H_n \setminus V_{d_1}$ is finite, then $H_n \setminus V_{d_1} \subseteq V_{d_2}$ for some $d_2 \in D$ because $H_n \subseteq \bigcup\limits_{d\in D} V_d$ and $\{V_d : d \in D\} \subseteq \mathcal{O}(X)$ is directed. By the directness of $\{V_d : d \in D\}$ again, $V_{d_1} \cup V_{d_2} \subseteq V_{d_3}$ for some $d_3 \in D$. Then $H_n \subseteq V_{d_3}$.

(c2) If for all $d \in D$, $H_n \cap (X \setminus V_d)$ is infinite, then $U_n \cap C \cap (X \setminus V_d) \neq \emptyset$ since $U_1 \supseteq U_2 \supseteq ... \supseteq U_n \supseteq ...$, and whence $C \cap (X \setminus V_d) = C$ for all $d \in D$ by the minimality of C. Therefore, $H_n \subseteq C \subseteq \bigcap_{d \in D}(X \setminus V_d) = X \setminus \bigcup_{d \in D} V_d$, which is a contradiction with $H_n \subseteq \bigcup_{d \in D} V_d$.

By (c1) and (c2), $\uparrow H_n \in \mathsf{K}(X)$. Clearly, $\{\uparrow H_n : n \in N\} \subseteq \mathsf{K}(X)$ is countable filtered and $H \subseteq \bigcap_{n \in \mathbb{N}} U_n \subseteq W$. By the ω-well-filteredness of X, $\uparrow H_m \subseteq W$ for some $m \in \mathbb{N}$, which is a contradiction with $H_m \subseteq C \subseteq X \setminus W$.

Therefore, $U_{n_0} \subseteq W = \bigcup_{i \in I} W_i$ for some $n_0 \in \mathbb{N}$. By $U_{n_0+1} \ll U_{n_0}$, $K \subseteq U_{n_0+1} \subseteq \bigcup_{i \in J} W_i$ for some $J \in I^{<\omega}$. It follows that $K \in \mathsf{K}(X)$ and $x \in U_\infty \subseteq K \subseteq U$. Thus X is locally compact. $\square$

Corollary 3.16. *A well-filtered T_0 space is locally compact iff it is core compact.*

By Theorem 3.12 and Theorem 3.15, we reobtain the following result, which was first proved by Lawson and Xi [19] using a different method.

Corollary 3.17. *Every well-filtered core compact T_0 space is sober.*

Corollary 3.17 gives a positive answer to Jia-Jung problem [16] (see [16, Question 2.5.19]).

Remark 3.18. Based on [23], Goubault independently obtained Theorem 3.15 (and hence Corollary 3.16 and Corollary 3.17), please see his blog (https://projects.lsv.ens-cachan.fr/topology/?page_id=2067).

The Theorem 3.12 can be strengthened into the following one.

Theorem 3.19. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is locally compact and sober.*
(2) *X is locally compact and well-filtered.*
(3) *X is core compact and sober.*
(4) *X is core compact and well-filtered.*

Rudin's Lemma [20] is a very useful tool in domain theory and non-Hausdorff topology (see [2–10,13,22]). In [13], Heckmann and Keimel presented the following topological variant of Rudin's Lemma.

Lemma 3.20. (Topological Rudin's Lemma) *Let X be a topological space and $\mathcal{A}$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains a minimal irreducible closed subset A that still meets all members of $\mathcal{A}$.*

In the following, using the topological Rudin's Lemma, we prove that a T_0 space X is ω-well-filtered iff the Smyth power space of X is ω-well-filtered. The corresponding results for well-filteredness were given in [26,27].

Theorem 3.21. *For a T_0 space, the following conditions are equivalent:*

(1) *X is ω-well-filtered.*
(2) *$P_S(X)$ is an ω-d-space.*
(3) *$P_S(X)$ is ω-well-filtered.*

Proof. (1) $\Rightarrow$ (2): Suppose that X is an ω-well-filtered space. For any countable filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$, by the ω-well-filteredness of X, $\bigcap \mathcal{K} \in \mathsf{K}(X)$. Therefore, by Lemma 2.12, $\mathsf{K}(X)$ is an ω-dcpo. Clearly, by the ω-well-filteredness of X, $\Box U \in \sigma_\omega(\mathsf{K}(X))$ for any $U \in \mathcal{O}(X)$. Thus $P_S(X)$ is an ω-d-space.

(2) ⇒ (3): Suppose that $\{\mathcal{K}_n : n < \omega\} \subseteq \mathsf{K}(P_S(X))$ is countable filtered, $\mathcal{U} \in \mathcal{O}(P_S(X))$, and $\bigcap_{n<\omega} \mathcal{K}_n \subseteq \mathcal{U}$. If $\mathcal{K}_n \nsubseteq \mathcal{U}$ for all $n < \omega$, then by Lemma 3.20, $\mathsf{K}(X) \setminus \mathcal{U}$ contains an irreducible closed subset $\mathcal{A}$ that still meets all $\mathcal{K}_n$ $(n < \omega)$. For each $n < \omega$, let $K_n = \bigcup \uparrow_{\mathsf{K}(X)}(\mathcal{A} \bigcap \mathcal{K}_n)$ $(= \bigcup(\mathcal{A} \bigcap \mathcal{K}_n))$. Then by Lemma 2.13, $\{K_n : n < \omega\} \subseteq \mathsf{K}(X)$ is countable filtered, and $K_n \in \mathcal{A}$ for all $n < \omega$ since $\mathcal{A} = \downarrow_{\mathsf{K}(X)}\mathcal{A}$. Let $K = \bigcap_{n<\omega} K_n$. Then $K \in \mathsf{K}(X)$ and $K = \bigvee_{\mathsf{K}(X)}\{K_n : n < \omega\} \in \mathcal{A}$ by Lemma 2.12 and condition (2). We claim that $K \in \bigcap_{n<\omega} \mathcal{K}_n$. Suppose, on the contrary, that $K \notin \bigcap_{n<\omega} \mathcal{K}_n$. Then there is a $n_0 < \omega$ such that $K \notin \mathcal{K}_{n_0}$. Select a $G \in \mathcal{A} \bigcap \mathcal{K}_{n_0}$. Then $K \nsubseteq G$ (otherwise, $K \in \uparrow_{\mathsf{K}(X)}\mathcal{K}_{n_0} = \mathcal{K}_{n_0}$, being a contradiction with $K \notin \mathcal{K}_{n_0}$), and hence there is a $g \in K \setminus G$. It follows that $g \in K_n = \bigcup(\mathcal{A} \bigcap \mathcal{K}_n)$ for all $n < \omega$ and $G \notin \Diamond_{\mathsf{K}(K)}\overline{\{g\}}$. For each $n < \omega$, by $g \in K_n = \bigcup(\mathcal{A} \bigcap \mathcal{K}_n)$, there is a $K_n^g \in \mathcal{A} \bigcap \mathcal{K}_n$ such that $g \in K_n^g$, and hence $K_n^g \in \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \bigcap \mathcal{A} \bigcap \mathcal{K}_n$. Thus $\Diamond_{\mathsf{K}(K)}\overline{\{g\}} \bigcap \mathcal{A} \bigcap \mathcal{K}_n \neq \emptyset$ for all $n < \omega$. By the minimality of $\mathcal{A}$, we have $\mathcal{A} = \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \bigcap \mathcal{A}$, and consequently, $G \in \mathcal{A} \bigcap \mathcal{K}_{n_0} = \Diamond_{\mathsf{K}(K)}\overline{\{g\}} \bigcap \mathcal{A} \bigcap \mathcal{K}_{n_0}$, which is a contradiction with $G \notin \Diamond_{\mathsf{K}(K)}\overline{\{g\}}$. Thus $K \in \bigcap_{n<\omega} \mathcal{K}_n \subseteq \mathcal{U} \subseteq \mathsf{K}(X) \setminus \mathcal{A}$, being a contradiction with $K \in \mathcal{A}$. Therefore, $P_S(X)$ is ω-well-filtered.

(3) ⇒ (1): Suppose that $\mathcal{K} \subseteq \mathsf{K}(X)$ is countable filtered, $U \in \mathcal{O}(X)$, and $\bigcap \mathcal{K} \subseteq U$. Let $\widetilde{\mathcal{K}} = \{\uparrow_{\mathsf{K}(X)}K : K \in \mathcal{K}\}$. Then $\widetilde{\mathcal{K}} \subseteq \mathsf{K}(P_S(X))$ is countable filtered and $\bigcap \widetilde{\mathcal{K}} \subseteq \Box U$. By the ω-well-filteredness of $P_S(X)$, there is a $K \in \mathcal{K}$ such that $\uparrow_{\mathsf{K}(X)}K \subseteq \Box U$, and whence $K \subseteq U$, proving that X is ω-well-filtered. □

4. First countable ω-well-filtered spaces

In this section, we show that in a first countable ω-well-filtered T_0 space X, all irreducible closed subsets of X are directed. Therefore, every first countable ω-well-filtered d-space (in particular, every first countable well-filtered T_0 space) is sober.

Theorem 4.1. *Let X be a first countable ω-well-filtered T_0 space and $A \in \mathsf{Irr}(X)$. Then $\overline{A}$ is directed.*

Proof. For each $x \in X$, since X is first countable, there is an open neighborhood base $\{U_n(x) : n \in \mathbb{N}\}$ of x such that

$$U_1(x) \supseteq U_2(x) \supseteq \ldots \supseteq U_k(x) \supseteq \ldots,$$

that is, $\{U_n(x) : n \in \mathbb{N}\}$ is a decreasing sequence of open subsets.

Suppose that $A \in \mathsf{Irr}(X)$. We show that $\overline{A}$ is directed. Let $a_1, a_2 \in \overline{A}$. It needs to show $\uparrow a_1 \cap \uparrow a_2 \cap \overline{A} \neq \emptyset$. Since $a_1, a_2 \in \overline{A}$, $A \cap U_1(a_1) \neq \emptyset \neq A \cap U_1(a_2)$, and hence $A \cap U_1(a_1) \cap U_1(a_2) \neq \emptyset$ by the irreducibility of A. Choose $c_1 \in U_1(a_1) \cap U_1(a_2) \cap A$. Now suppose we already have a set $\{c_1, c_2, \ldots, c_n\}$ such that for each $2 \leq i \leq n$,

$$c_i \in U_i(c_1) \cap U_i(c_2) \cap \ldots \cap U_i(c_i) \cap U_i(a_1) \cap U_i(a_2) \cap A.$$

Note that above condition implies that for any positive integer k,

$$U_k(c_1) \cap U_k(c_2) \cap \ldots \cap U_k(c_n) \cap U_k(a_1) \cap U_k(a_2) \cap A \neq \emptyset.$$

So we can choose $c_{n+1} \in U_{n+1}(c_1) \cap U_{n+1}(c_2) \cap \ldots \cap U_{n+1}(c_n) \cap U_{n+1}(a_1) \cap U_{n+1}(a_2) \cap A \neq \emptyset$. By induction, we can obtain a set $\{c_n : n \in \mathbb{N}\}$.

Let $K_n = \uparrow\{c_k : k \geq n\}$ for each $n \in \mathbb{N}$.

Claim 1: $\forall n \in \mathbb{N}$, K_n is compact.

Suppose $\{V_i : i \in I\}$ is an open cover of K_n, i.e., $K_n \subseteq \bigcup_{i\in I} V_i$. Then there is $i_0 \in I$ such that $c_n \in V_{i_0}$, and thus there is $m \geq n$ such that $c_n \in U_m(c_n) \subseteq V_{i_0}$. It follows that $c_k \in U_m(c_n) \subseteq V_{i_0}$ for all $k \geq m$. Thus $\{c_k : k \geq m\} \subseteq V_{i_0}$. For each c_k, where $n \leq k \leq m$, choose a V_{i_k} such that $c_k \in V_{i_k}$. Then the finite family $\{V_{i_k} : n \leq k \leq m\} \cup \{V_{i_0}\}$ covers K_n. So K_n is compact.

Claim 2: $\bigcap_{n\in\mathbb{N}} K_n \cap \overline{A} \neq \emptyset$.

Assume $\bigcap_{n\in\mathbb{N}} K_n \cap \overline{A} = \emptyset$. Then $\bigcap_{n\in\mathbb{N}} K_n \subseteq X \setminus \overline{A}$. Since $\{K_n : n \in \mathbb{N}\}$ is a countable filtered family of compact saturated set in X and X is ω-well-filtered, there exists $n_0 \in \mathbb{N}$ such that $K_{n_0} \subseteq X \setminus \overline{A}$, a contradiction.

Claim 3: $\uparrow a_1 \cap \uparrow a_2 \cap \overline{A} \neq \emptyset$.

Note that for each $n \in \mathbb{N}$, $K_n = \uparrow\{c_k : k \geq n\} \subseteq U_n(a_1) \cap U_n(a_2)$. This implies that

$$\emptyset \neq \bigcap_{n\in\mathbb{N}} K_n \cap \overline{A} \subseteq \bigcap_{n\in\mathbb{N}} U_n(a_1) \cap \bigcap_{n\in\mathbb{N}} U_n(a_2) \cap \overline{A} = \uparrow a_1 \cap \uparrow a_2 \cap \overline{A}.$$

Thus $\uparrow a_1 \cap \uparrow a_2 \cap \overline{A} \neq \emptyset$, and whence there is $a_3 \in \uparrow a_1 \cap \uparrow a_2 \cap \overline{A}$, that is, $a_3 \in \overline{A}$ such that $a_1 \leq a_3$ and $a_2 \leq a_3$. Therefore, $\overline{A}$ is directed. $\square$

Theorem 4.2. *Let X be a first countable T_0 space. Then the following conditions are equivalent:*

(1) *X is a sober space.*
(2) *X is a well-filtered space.*
(3) *X is an ω-well-filtered d-space.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Let $A \in \mathsf{Irr}_c(X)$. Then by Theorem 4.1, $A \in \mathcal{D}(X)$. Since X is a d-space, $\bigvee A \in A$, and hence $A = \overline{\{\bigvee A\}}$. Thus X is sober. $\square$

Example 4.3. Let L be the complete chain $[0, \omega_1]$. Consider the space $\Sigma_\omega L = (L, \sigma_\omega(L))$.

(1) It is a first countable ω-well-filtered space.
(2) It is not a d-space. In fact, we have that $\{\omega_1\} \in \sigma_\omega(L)$ but $\{\omega_1\} \notin \sigma(L)$.

Therefore, $\Sigma_\omega L$ is not well-filtered, and hence non-sober. So in Theorem 4.2, condition (3) cannot be weakened to the condition that X is only an ω-well-filtered space.

Example 4.4. Let L be the complete lattice constructed by Isbell [15]. Then ΣL is non-sober. By Corollary 2.9 and Theorem 4.2, ΣL is well-filtered but not first countable.

Recently, Jung[1] asked whether there is a countable complete lattice whose Scott space is non-sober. If there is such a countable complete lattice L, then $(L, \sigma(L))$ cannot be first countable (see Corollary 4.7).

By Lemma 2.8, Corollary 2.9 and Theorem 4.2, we get the following results.

Corollary 4.5. *For a dcpo P, if $(P, \sigma(P))$ is first countable and $(P, \lambda(P))$ is compact, then $(P, \sigma(P))$ is sober.*

Corollary 4.6. *For a dcpo P, if $(P, \sigma(P))$ is a first countable ω-well-filtered space, then it is sober.*

Corollary 4.7. *For a complete lattice L, if $(L, \sigma(L))$ is first countable, then it is sober.*

[1] A. Jung, Four dcpos, a theorem, and an open problem, an academic report at National Institute of Education, Singapore, 1 February, 2019.

The reader may wonder whether we can answer Jung's question positively by showing that the Scott topology on every countable complete lattice is first countable. Unfortunately, the following example crashes this hope.

Example 4.8. Let $L = \{\bot\} \cup (\mathbb{N} \times \mathbb{N}) \cup \{\top\}$ and define a partial order $\leq$ on L as follows:

(i) $\forall (n, m) \in \mathbb{N} \times \mathbb{N}$, $\bot \leq (n, m) \leq \top$;
(ii) $\forall (n_1, m_1), (n_2, m_2) \in \mathbb{N} \times \mathbb{N}$, $(n_1, m_1) \leq (n_2, m_2)$ iff $n_1 = n_2$ and $m_1 \leq m_2$.

We show that $(L, \sigma(L))$ does not have any countable base at $\top$. Assume, on the contrary, there exists a countable base $\{U_n : n \in \mathbb{N}\}$ at $\top$. Then for each $n \in \mathbb{N}$, as

$$\bigvee(\{n\} \times \mathbb{N}) = \top \in U_n,$$

there exists $m_n \in \mathbb{N}$ such that $(n, m_n) \in U_n$. Let $U = \bigcup_{n \in \mathbb{N}} \uparrow(n, m_n + 1)$. Then $U \in \sigma(L)$. But for each $n \in N$, $(n, m_n) \in U_n \setminus U$, which contradicts that $\{U_n : n \in \mathbb{N}\}$ is a base at $\top$. Therefore, $(L, \sigma(L))$ is not first countable. One can easily check that $(L, \sigma(L))$ is sober.

5. ω-Rudin-sets and ω-well-filtered determined sets

In this section, based on the topological Rudin's Lemma, we introduce and study two new classes of closed subsets in T_0 spaces - ω-Rudin sets and ω-well-filtered determined closed sets lying between the class of all closures of countable directed subsets and that of irreducible closed subsets.

For a T_0 space X and $\mathcal{K} \subseteq \mathsf{K}(X)$, let $M(\mathcal{K}) = \{A \in \mathcal{C}(X) : K \bigcap A \neq \emptyset \text{ for all } K \in \mathcal{K}\}$ (that is, $\mathcal{K} \subseteq \Diamond A$) and $m(\mathcal{K}) = \{A \in \mathcal{C}(X) : A \text{ is a minimal member of } M(\mathcal{K})\}$.

Definition 5.1. Let X be a T_0 space. A nonempty subset A of X is said to have the *ω-Rudin property*, if there exists a countable filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\mathsf{RD}_\omega(X) = \{A \in \mathcal{C}(X) : A \text{ has } \omega\text{-Rudin property}\}$. The sets in $\mathsf{RD}_\omega(X)$ will also be called *ω-Rudin sets*.

Lemma 5.2. *Let X, Y be two T_0 spaces and $f : X \longrightarrow Y$ a continuous mapping. If $A \in \mathsf{RD}_\omega(X)$, then $\overline{f(A)} \in \mathsf{RD}_\omega(Y)$.*

Proof. Since $A \in \mathsf{RD}_\omega(X)$, there exists a countable filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $A \in m(\mathcal{K})$. Let $\mathcal{K}_f = \{\uparrow f(K \cap A) : K \in \mathcal{K}\}$. Then $\mathcal{K}_f \subseteq \mathsf{K}(Y)$ is countable filtered. For each $K \in \mathcal{K}$, since $K \cap A \neq \emptyset$, we have $\emptyset \neq f(K \cap A) \subseteq \uparrow f(K \cap A) \cap \overline{f(A)}$. So $\overline{f(A)} \in M(\mathcal{K}_f)$. If B is a closed subset of $\overline{f(A)}$ with $B \in M(\mathcal{K}_f)$, then $B \cap \uparrow f(K \cap A) \neq \emptyset$ for every $K \in \mathcal{K}$. So $K \cap A \cap f^{-1}(B) \neq \emptyset$ for all $K \in \mathcal{K}$. It follows that $A = A \cap f^{-1}(B)$ by the minimality of A, and consequently, $\overline{f(A)} \subseteq B$. Therefore, $\overline{f(A)} = B$. Thus $\overline{f(A)} \in \mathsf{RD}_\omega(Y)$. $\square$

Proposition 5.3. *Let X be a T_0 space and Y an ω-well-filtered space. If $f : X \longrightarrow Y$ is continuous and $A \in \mathsf{RD}_\omega(X)$, then there exists a unique $y_A \in X$ such that $\overline{f(A)} = \overline{\{y_A\}}$.*

Proof. Since $A \in \mathsf{RD}_\omega(X)$, there exists a countable filtered family $\mathcal{K} \subseteq \mathsf{K}(X)$ such that $\overline{A} \in m(\mathcal{K})$. Let $\mathcal{K}_f = \{\uparrow f(K \cap \overline{A}) : K \in \mathcal{K}\}$. Then $\mathcal{K}_f \subseteq \mathsf{K}(Y)$ is countable filtered. By the proof of Lemma 5.2, $\overline{f(A)} \in m(\mathcal{K}_f)$. Since Y is ω-well-filtered, we have $\bigcap_{K \in \mathcal{K}} \uparrow f(K \cap \overline{A}) \cap \overline{f(A)} \neq \emptyset$. Select a $y_A \in \bigcap_{K \in \mathcal{K}} \uparrow f(K \cap \overline{A}) \cap \overline{f(A)}$. Then $\overline{\{y_A\}} \subseteq \overline{f(A)}$ and $K \cap \overline{A} \cap f^{-1}(\overline{\{y_A\}}) \neq \emptyset$ for all $K \in \mathcal{K}$. It follows that $\overline{A} = \overline{A} \cap f^{-1}(\overline{\{y_A\}})$ by the minimality of $\overline{A}$, and consequently, $\overline{f(A)} \subseteq \overline{\{y_A\}}$. Therefore, $\overline{f(A)} = \overline{\{y_A\}}$. The uniqueness of y_A follows from the T_0 separation of Y. $\square$

Motivated by Proposition 5.3, we introduce the following concept.

Definition 5.4. A subset A of a T_0 space X is called an *ω-well-filtered determined set*, WD_ω *set* for short, if for any continuous mapping $f : X \longrightarrow Y$ to an ω-well-filtered space Y, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Denote by $\mathsf{WD}_\omega(X)$ the set of all closed ω-well-filtered determined subsets of X.

Obviously, a subset A of a space X is ω-well-filtered determined iff $\overline{A}$ is ω-well-filtered determined.

Proposition 5.5. *Let X be a T_0 space. Then $\mathcal{S}_c(X) \subseteq \mathcal{D}_c^\omega(X) \subseteq \mathsf{RD}_\omega(X) \subseteq \mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$.*

Proof. Obviously, $\mathcal{S}_c(X) \subseteq \mathcal{D}_c^\omega(X)$. Now we prove that the closure of a countable directed subset D of X is an ω-Rudin set. Let $\mathcal{K}_D = \{\uparrow d : d \in D\}$. Then $\mathcal{K}_D \subseteq \mathsf{K}(X)$ is countable filtered and $\overline{D} \in M(\mathcal{K}_D)$. If $A \in M(\mathcal{K}_D)$, then $d \in A$ for every $d \in D$, and hence $\overline{D} \subseteq A$. So $\overline{D} \in m(\mathcal{K}_D)$. Therefore $\overline{D} \in \mathsf{RD}_\omega(X)$. By Proposition 5.3, $\mathsf{RD}_\omega(X) \subseteq \mathsf{WD}_\omega(X)$. Finally we show $\mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$. Let $A \in \mathsf{WD}_\omega(X)$. Since $\eta_X : X \longrightarrow X^s$, $x \mapsto \downarrow x$, is a continuous mapping to an ω-well-filtered space (X^s is sober), there exists $C \in \mathsf{Irr}_c(X)$ such that $\overline{\eta_X(A)} = \overline{\{C\}}$. Let $U \in \mathcal{O}(X)$. Note that

$$\begin{aligned} A \cap U \neq \emptyset &\Leftrightarrow \eta_X(A) \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow \overline{\{C\}} \cap \Diamond U \neq \emptyset \\ &\Leftrightarrow C \in \Diamond U \\ &\Leftrightarrow C \cap U \neq \emptyset. \end{aligned}$$

This implies that $A = C$, and hence $A \in \mathsf{Irr}_c(X)$. $\square$

Lemma 5.6. *Let X, Y be two T_0 spaces. If $f : X \longrightarrow Y$ is a continuous mapping and $A \in \mathsf{WD}_\omega(X)$, then $\overline{f(A)} \in \mathsf{WD}_\omega(Y)$.*

Proof. Let Z be an ω-well-filtered space and $g : Y \longrightarrow Z$ is a continuous mapping. Since $g \circ f : X \longrightarrow Z$ is continuous and $A \in \mathsf{WD}_\omega(X)$, there is $z \in Z$ such that $\overline{g(\overline{f(A)})} = \overline{g \circ f(A)} = \overline{\{z\}}$. Thus $\overline{f(A)} \in \mathsf{WD}_\omega(Y)$. $\square$

Lemma 5.7. *Let $X = \prod_{i<\omega} X_i$ be the product of a countable family $\{X_i : i < \omega\}$ of T_0 spaces and $A \in \mathsf{Irr}_c(X)$. Then the following conditions are equivalent:*

(1) *A is an ω-Rudin set.*
(2) *$p_i(A)$ is an ω-Rudin set for each $i \in I$.*

Proof. (1) $\Rightarrow$ (2): By Corollary 2.6 and Lemma 5.2.

(2) $\Rightarrow$ (1): For each $i < \omega$, by Corollary 2.6, $p_i(A) \in \mathsf{Irr}_c(X_i)$, and hence by condition (2), there is a countable filtered family $\mathcal{K}_i \in \mathsf{K}(X_i)$ such that $p_i(A) \in m(\mathcal{K}_i)$. Let $\mathcal{K} = \{K_\varphi = \prod_{i<\omega} \varphi(i) : \varphi \in \prod_{i<\omega} \mathcal{K}_i\}$. Then by Tychonoff's Theorem (see [1, pp. 184, 3.2.4]), $\mathcal{K} \subseteq \mathsf{K}(\prod_{i<\omega} X_i)$ and $\mathcal{K}$ is countable filtered because all $\mathcal{K}_i$ are countable filtered. $\forall \varphi \in \prod_{i<\omega} \mathcal{K}_i, i < \omega$, $\varphi(i) \cap p_i(A) \neq \emptyset$ since $p_i(A) \in m(\mathcal{K}_i)$. So by Lemma 2.4 and Corollary 2.6 we have

$$K_\varphi \cap A = \left(\prod_{i<\omega} \varphi(i)\right) \bigcap \left(\prod_{i<\omega} p_i(A)\right) = \prod_{i<\omega} (\varphi(i) \cap p_i(A)) \neq \emptyset.$$

It follows that $A \in M(\mathcal{K})$. Now we show that $A \in m(\mathcal{K})$. If B is a closed subset of A that meets all members of $\mathcal{K}$, then by Lemma 3.20, B contains a minimal irreducible closed subset C that still meets all members of $\mathcal{K}$. Then for each $i < \omega$, $p_i(C) \in M(\mathcal{K}_i)$, so $p_i(C) = p_i(A)$ by $p_i(A) \in m(\mathcal{K}_i)$ and $p_i(C) \subseteq p_i(A)$. By Corollary 2.6 again, we have $A = \prod_{i<\omega} p_i(A) = \prod_{i<\omega} p_i(C) = C$, and hence $A = C$. Thus $A \in \mathsf{RD}_\omega(\prod_{i<\omega} X_i)$. $\square$

Lemma 5.8. *Let $\{X_i : 1 \leq i \leq n\}$ be a finite family of T_0 spaces and $X = \prod\limits_{i=1}^{n} X_i$ the product space. For $A \in \mathsf{Irr}_c(X)$, the following conditions are equivalent:*

(1) $A \in \mathsf{WD}_\omega(X)$.
(2) $p_i(A) \in \mathsf{WD}_\omega(X_i)$ *for each* $1 \leq i \leq n$.

Proof. (1) $\Rightarrow$ (2): By Corollary 2.6 and Lemma 5.6.

(2) $\Rightarrow$ (1): By induction, we need only to prove the implication for the case of $n = 2$. Let $A_1 = p_1(A)$ and $A_2 = p_2(A)$. Then by condition (2), $(A_1, A_2) \in \mathsf{WD}_\omega(X_1) \times \mathsf{WD}_\omega(X_2)$ (note that $(A_1, A_2) \in \mathsf{Irr}_c(X_1) \times \mathsf{Irr}_c(X_2)$ by Corollary 2.6). Now we show that the product $A_1 \times A_2 \in \mathsf{WD}_\omega(X)$. Let $f : X_1 \times X_2 \longrightarrow Y$ be a continuous mapping from $X_1 \times X_2$ to an ω-well-filtered space Y. For each $b \in X_2$, X_1 is homeomorphic to $X_1 \times \{b\}$ (as a subspace of $X_1 \times X_2$) via the homeomorphism $\mu_b : X_1 \longrightarrow X_1 \times \{b\}$ defined by $\mu_b(x) = (x, b)$. Let $i_b : X_1 \times \{b\} \longrightarrow X_1 \times X_2$ be the embedding of $X_1 \times \{b\}$ in $X_1 \times X_2$. Then $f_b = f \circ i_b \circ \mu_b : X_1 \longrightarrow Y$, $f_b(x) = f((x, b))$, is continuous. Since $A_1 \in \mathsf{WD}_\omega(X_1)$, there is a unique $y_b \in Y$ such that $\overline{f(A_1 \times \{b\})} = \overline{f_b(A_1)} = \overline{\{y_b\}}$. Define a mapping $g_A : X_2 \longrightarrow Y$ by $g_A(b) = y_b$. For each $V \in \mathcal{O}(Y)$,

$$\begin{aligned} g_A^{-1}(V) &= \{b \in X_2 : g_A(b) \in V\} \\ &= \{b \in X_2 : \overline{f_b(A_1)} \cap V \neq \emptyset\} \\ &= \{b \in X_2 : \overline{f(A_1 \times \{b\})} \cap V \neq \emptyset\} \\ &= \{b \in X_2 : f(A_1 \times \{b\}) \cap V \neq \emptyset\} \\ &= \{b \in X_2 : (A_1 \times \{b\}) \cap f^{-1}(V) \neq \emptyset\}. \end{aligned}$$

Therefore, for each $b \in g_A^{-1}(V)$, there is an $a_1 \in A_1$ such that $(a_1, b) \in f^{-1}(V) \in \mathcal{O}(X_1 \times X_2)$, and hence there is $(U_1, U_2) \in \mathcal{O}(X_1) \times \mathcal{O}(X_2)$ such that $(a_1, b) \in U_1 \times U_2 \subseteq f^{-1}(V)$. It follows that $b \in U_2 \subseteq g_A^{-1}(V)$. Thus $g_A : X_2 \longrightarrow Y$ is continuous. Since $A_2 \in \mathsf{WD}_\omega(X_2)$, there is a unique $y_A \in Y$ such that $\overline{g_A(A_2)} = \overline{\{y_A\}}$. Therefore, by Lemma 2.4 and Corollary 2.6, we have

$$\begin{aligned} \overline{f(A)} &= \overline{f(A_1 \times A_2)} \\ &= \overline{\bigcup_{a_2 \in A_2} f(A_1 \times \{a_2\})} \\ &= \overline{\bigcup_{a_2 \in A_2} \overline{f(A_1 \times \{a_2\})}} \\ &= \overline{\bigcup_{a_2 \in A_2} \overline{\{g_A(a_2)\}}} \\ &= \overline{\bigcup_{a_2 \in A_2} \{g_A(a_2)\}} \\ &= \overline{g_A(A_2)} \\ &= \overline{\{y_A\}}. \end{aligned}$$

Thus $A \in \mathsf{WD}_\omega(X)$. $\square$

By Corollary 2.6 and Lemma 5.8, we get the following result.

Corollary 5.9. *Let $X = \prod\limits_{i=1}^{n} X_i$ be the product of a finite family $\{X_i : 1 \leq i \leq n\}$ of T_0 spaces. If $A \in \mathsf{WD}_\omega(X)$, then $A = \prod\limits_{i=1}^{n} p_i(A)$ and $p_i(A) \in \mathsf{WD}_\omega(X_i)$ for all $1 \leq i \leq n$.*

Question 5.10. Let $X = \prod_{i<\omega} X_i$ be the product space of a countable family $\{X_i : i < \omega\}$ of T_0 spaces. If all $A_i \subseteq X_i$ $(i < \omega)$ are WD_ω sets, must the product set $\prod_{i<\omega} A_i$ be a WD_ω set of X?

Theorem 5.11. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is sober.*
(2) *X is an ω-d-space and $\mathsf{Irr}_c(X) = \mathsf{D}_c^\omega(X)$.*
(3) *X is ω-well-filtered and $\mathsf{Irr}_c(X) = \mathsf{D}_c^\omega(X)$.*
(4) *X is ω-well-filtered and $\mathsf{Irr}_c(X) = \mathsf{RD}_\omega(X)$.*
(5) *X is ω-well-filtered and $\mathsf{Irr}_c(X) = \mathsf{WD}_\omega(X)$.*

Proof. By Proposition 3.7, Proposition 3.11 and Proposition 5.5, we only need to check (5) $\Rightarrow$ (1). Assume X is ω-well-filtered and $\mathsf{Irr}_c(X) = \mathsf{WD}_\omega(X)$. Let $A \in \mathsf{Irr}_c(X)$. Since the identity $id_X : X \longrightarrow X$ is continuous, there is a unique $x \in X$ such that $\overline{A} = \overline{\{x\}}$. So X is sober. $\square$

Example 5.12. Let X be a countable infinite set and endow X with the cofinite topology (having the complements of the finite sets as open sets). The resulting space is denoted by X_{cof}. Then $\mathsf{K}(X_{cof}) = 2^X \setminus \{\emptyset\}$ (that is, all nonempty subsets of X), and hence X_{cof} is a locally compact and first countable T_1 space. Let $\mathcal{K} = \{X \setminus F : F \in X^{(<\omega)}\}$. It is easy to check that $\mathcal{K} \subseteq \mathsf{K}(X_{cof})$ is countable filtered and $X \in m(\mathcal{K})$. Therefore, $X \in \mathsf{RD}_\omega(X)$ but $X \notin \mathsf{D}_c(X)$, and hence $X \notin \mathsf{D}_c^\omega(X)$. Thus $\mathsf{RD}_\omega(X) \neq \mathsf{D}_c^\omega(X)$ and $\mathsf{WD}_\omega(X) \neq \mathsf{D}_c^\omega(X)$. X_{cof} is a d-space. Since $X \in \mathsf{Irr}_c(X_{cof}) \setminus \mathcal{S}_c(X_{cof})$, X_{cof} is non-sober, and hence is not ω-well-filtered by Theorem 4.2. In fact, $\mathcal{K} = \{X \setminus F : F \in X^{(<\omega)}\} \subseteq \mathsf{K}(X_{cof})$ is countable filtered and $\bigcap \mathcal{K} = X \setminus \bigcup X^{(<\omega)} = X \setminus X = \emptyset$, but $X \setminus F \neq \emptyset$ for all $F \in X^{(<\omega)}$.

Example 5.13. Let L be the complete lattice constructed by Isbell [15]. Then it is not sober, and by Corollary 2.9, ΣL is a well-filtered space, and hence an ω-well-filtered. By Theorem 5.11, $\mathsf{Irr}_c(X) \neq \mathsf{RD}_\omega(X)$ and $\mathsf{Irr}_c(X) \neq \mathsf{WD}_\omega(X)$.

Question 5.14. Does $\mathsf{RD}_\omega(X) = \mathsf{WD}_\omega(X)$ hold for every T_0 space X?

6. ω-Well-filtered reflections of T_0 spaces

In this section, we present a direct construction of the ω-well-filtered reflections of T_0 spaces. Some basic properties of ω-well-filtered reflections of T_0 spaces are investigated.

Definition 6.1. Let X be a T_0 space. An *ω-well-filtered reflection* of X is a pair $\langle \widetilde{X}, \mu \rangle$ consisting of an ω-well-filtered space $\widetilde{X}$ and a continuous mapping $\mu : X \longrightarrow \widetilde{X}$ satisfying that for any continuous mapping $f : X \longrightarrow Y$ to an ω-well-filtered space, there exists a unique continuous mapping $f^* : \widetilde{X} \longrightarrow Y$ such that $f^* \circ \mu = f$, that is, the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr, "f"'] & \widetilde{X} \arrow[d, dotted, "f^*"] \\ & Y \end{tikzcd}$$

ω-well-filtered reflections, if they exist, are unique up to homeomorphism. We shall use $X^{\omega\text{-}w}$ to denote the space of the ω-well-filtered reflection of X if it exists.

Let X be a T_0 space. Then by Proposition 5.5, $\mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$, and whence the space $P_H(\mathsf{WD}_\omega(X))$ has the topology $\{\Diamond U : U \in \mathcal{O}(X)\}$, where $\Diamond U = \{A \in \mathsf{WD}_\omega(X) : A \cap U \neq \emptyset\}$. The closed subsets of $P_H(\mathsf{WD}_\omega(X))$ are exactly the set of forms $\Box C = \downarrow_{\mathsf{WD}_\omega(X)} C$ with $C \in \mathcal{C}(X)$.

Lemma 6.2. *Let X be a T_0 space and $A \subseteq X$. Then $\overline{\eta_X(A)} = \overline{\eta_X\left(\overline{A}\right)} = \overline{\Box A} = \Box \overline{A}$ in $P_H(\mathsf{WD}_\omega(X))$.*

Proof. Clearly, $\eta_X(A) \subseteq \Box A \subseteq \Box \overline{A}$, $\eta_X\left(\overline{A}\right) \subseteq \Box \overline{A}$ and $\Box \overline{A}$ is closed in $P_H(\mathsf{WD}_\omega(X))$. It follows that

$$\overline{\eta_X(A)} \subseteq \overline{\Box A} \subseteq \Box \overline{A} \ \text{ and } \ \overline{\eta_X(A)} \subseteq \overline{\eta_X\left(\overline{A}\right)} \subseteq \Box \overline{A}.$$

To complete the proof, we need to show $\Box \overline{A} \subseteq \overline{\eta_X(A)}$. Let $F \in \Box \overline{A}$. Suppose $U \in \mathcal{O}(X)$ such that $F \in \Diamond U$, that is, $F \cap U \neq \emptyset$. Since $F \subseteq \overline{A}$, we have $A \cap U \neq \emptyset$. Let $a \in A \cap U$. Then $\downarrow a \in \Diamond U \cap \eta_X(A) \neq \emptyset$. This implies that $F \in \overline{\eta_X(A)}$. Whence $\Box \overline{A} \subseteq \overline{\eta_X(A)}$. $\Box$

Lemma 6.3. *The mapping $\eta_X : X \longrightarrow P_H(\mathsf{WD}_\omega(X))$ defined by*

$$\forall x \in X,\ \eta_X(x) = \downarrow x,$$

is a topological embedding.

Proof. For $U \in \mathcal{O}(X)$, we have

$$\eta_X^{-1}(\Diamond U) = \{x \in X : \downarrow x \in \Diamond U\} = \{x \in X : x \in U\} = U,$$

so η_X is continuous. In addition, we have

$$\eta_X(U) = \{\downarrow x : x \in U\} = \{\downarrow x : \downarrow x \in \Diamond U\} = \Diamond U \cap \eta_X(X),$$

which implies that η_X is an open mapping to $\eta_X(X)$, as a subspace of $P_H(\mathsf{WD}_\omega(X))$. As η_X is an injection, η_X is a topological embedding. $\Box$

Lemma 6.4. *Let X be a T_0 space and A a nonempty subset of X. Then the following conditions are equivalent:*

(1) *A is irreducible in X.*
(2) *$\Box A$ is irreducible in $P_H(\mathsf{WD}_\omega(X))$.*
(3) *$\Box \overline{A}$ is irreducible in $P_H(\mathsf{WD}_\omega(X))$.*

Proof. (1) $\Rightarrow$ (3): Assume A is irreducible. Then $\eta_X(A)$ is irreducible in $P_H(\mathsf{WD}_\omega(X))$ by Lemma 2.2 and Lemma 6.3. By Lemma 2.1 and Lemma 6.2, $\Box \overline{A} = \overline{\eta_X(A)}$ is irreducible in $P_H(\mathsf{WD}_\omega(X))$.

(3) $\Rightarrow$ (1): Assume $\Box \overline{A}$ is irreducible. Let $A \subseteq B \cup C$ with $B, C \in \mathcal{C}(X)$. By Proposition 5.5, $\mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$, and consequently, we have $\Box \overline{A} \subseteq \Box B \cup \Box C$. Since $\Box \overline{A}$ is irreducible, $\Box \overline{A} \subseteq \Box B$ or $\Box \overline{A} \subseteq C$, showing that $\overline{A} \subseteq B$ or $\overline{A} \subseteq C$, and consequently, $A \subseteq B$ or $A \subseteq C$, proving A is irreducible.

(2) $\Leftrightarrow$ (3): By Lemma 2.1 and Lemma 6.2. $\Box$

Lemma 6.5. *Let X be a T_0 space and $f : X \longrightarrow Y$ a continuous mapping from X to an ω-well-filtered space Y. Then there exists a unique continuous mapping $f^* : P_H(\mathsf{WD}_\omega(X)) \longrightarrow Y$ such that $f^* \circ \eta_X = f$, that is, the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & P_H(\mathsf{WD}_\omega(X)) \arrow[d, dotted, "f^*"] \\ & Y \end{tikzcd}$$

Proof. For each $A \in \mathsf{WD}_\omega(X)$, there exists a unique $y_A \in Y$ such that $\overline{f(A)} = \overline{\{y_A\}}$. Then we can define a mapping $f^* : P_H(\mathsf{WD}_\omega(X)) \longrightarrow Y$ by

$$\forall A \in \mathsf{WD}_\omega(X), \quad f^*(A) = y_A.$$

Claim 1: $f^* \circ \eta_X = f$.

Let $x \in X$. Since f is continuous, we have $\overline{f\left(\overline{\{x\}}\right)} = \overline{f(\{x\})} = \overline{\{f(x)\}}$, so $f^*\left(\overline{\{x\}}\right) = f(x)$. Thus $f^* \circ \eta_X = f$.

Claim 2: f^* is continuous.

Let $V \in \mathcal{O}(Y)$. Then

$$\begin{aligned} (f^*)^{-1}(V) &= \{A \in \mathsf{WD}_\omega(X) : f^*(A) \in V\} \\ &= \{A \in \mathsf{WD}_\omega(X) : \overline{\{f^*(A)\}} \cap V \neq \emptyset\} \\ &= \{A \in \mathsf{WD}_\omega(X) : \overline{f(A)} \cap V \neq \emptyset\} \\ &= \{A \in \mathsf{WD}_\omega(X) : f(A) \cap V \neq \emptyset\} \\ &= \{A \in \mathsf{WD}_\omega(X) : A \cap f^{-1}(V) \neq \emptyset\} \\ &= \Diamond f^{-1}(V), \end{aligned}$$

which shows that $(f^*)^{-1}(V)$ is open in $P_H(\mathsf{WD}_\omega(X))$. Thus f^* is continuous.

Claim 3: The mapping f^* is unique such that $f^* \circ \eta_X = f$.

Assume $g : P_H(\mathsf{WD}_\omega(X)) \longrightarrow Y$ is a continuous mapping such that $g \circ \eta_X = f$. Let $A \in \mathsf{WD}_\omega(X)$. We need to show $g(A) = f^*(A)$. Let $a \in A$. Then $\overline{\{a\}} \subseteq A$, implying that $g(\overline{\{a\}}) \leq_Y g(A)$, that is, $g\left(\overline{\{a\}}\right) = f(a) \in \overline{\{g(A)\}}$. Thus $\overline{\{f^*(A)\}} = \overline{f(A)} \subseteq \overline{\{g(A)\}}$. In addition, since $A \in \overline{\eta_X(A)}$ and g is continuous, $g(A) \in g\left(\overline{\eta_X(A)}\right) \subseteq \overline{g(\eta_X(A))} = \overline{f(A)} = \overline{\{f^*(A)\}}$, which implies that $\overline{\{g(A)\}} \subseteq \overline{\{f^*(A)\}}$. So $\overline{\{g(A)\}} = \overline{\{f^*(A)\}}$. Since Y is T_0, $g(A) = f^*(A)$. Thus $g = f^*$. $\square$

Lemma 6.6. *Let X be a T_0 space and $C \in \mathcal{C}(X)$. Then the following conditions are equivalent:*

(1) *C is ω-well-filtered determined in X.*
(2) *$\Box C$ is ω-well-filtered determined in $P_H(\mathsf{WD}_\omega(X))$.*

Proof. (1) $\Rightarrow$ (2): By Propositions 5.6, Lemma 6.2 and Lemma 6.3.

(2) $\Rightarrow$ (1). Let Y be an ω-well-filtered space and $f : X \longrightarrow Y$ a continuous mapping. By Lemma 6.5, there exists a continuous mapping $f^* : P_H(\mathsf{WD}_\omega(X)) \longrightarrow Y$ such that $f^* \circ \eta_X = f$. Since $\Box C = \overline{\eta_X(C)}$ is ω-well-filtered determined and f^* is continuous, there exists a unique $y_C \in Y$ such that $\overline{f^*\left(\overline{\eta_X(C)}\right)} = \overline{\{y_C\}}$. Furthermore, we have

$$\overline{\{y_C\}} = \overline{f^*\left(\overline{\eta_X(C)}\right)} = \overline{f^*(\eta_X(C))} = \overline{f(C)}.$$

So C is ω-well-filtered determined. $\square$

Theorem 6.7. *Let X be a T_0 space. Then $P_H(\mathsf{WD}_\omega(X))$ is an ω-well-filtered space.*

Proof. Since X is T_0, one can deduce that $P_H(\mathsf{WD}_\omega(X))$ is T_0. Let $\{\mathcal{K}_i : i \in I\} \subseteq \mathsf{K}(P_H(\mathsf{WD}_\omega(X)))$ be a countable filtered family and $U \in \mathcal{O}(X)$ such that $\bigcap_{i\in I} \mathcal{K}_i \subseteq \Diamond U$. We need to show $\mathcal{K}_i \subseteq \Diamond U$ for some $i \in I$. Assume, on the contrary, $\mathcal{K}_i \nsubseteq \Diamond U$, i.e., $\mathcal{K}_i \cap \Box(X \setminus U) \neq \emptyset$, for any $i \in I$.

Let $\mathcal{A} = \{C \in \mathcal{C}(X) : C \subseteq X \setminus U$ and $\mathcal{K}_i \cap \Box C \neq \emptyset$ for all $i \in I\}$. Then we have the following two facts.

(a1) $\mathcal{A} \neq \emptyset$ because $X \setminus U \in \mathcal{A}$.

(a2) For any filtered family $\mathcal{F} \subseteq \mathcal{A}$, $\bigcap \mathcal{F} \in \mathcal{A}$.

Let $F = \bigcap \mathcal{F}$. Then $F \in \mathcal{C}(X)$ and $F \subseteq X \setminus U$. Assume, on the contrary, $F \notin \mathcal{A}$. Then there exists $i_0 \in I$ such that $\mathcal{K}_{i_0} \cap \Box F = \emptyset$. Note that $\Box F = \bigcap_{C\in\mathcal{F}} \Box C$, implying that $\mathcal{K}_{i_0} \subseteq \bigcup_{C\in\mathcal{F}} \Diamond(X \setminus C)$ and $\{\Diamond(X \setminus C) : C \in \mathcal{F}\}$ is a directed family since $\mathcal{F}$ is filtered. Then there is $C_0 \in \mathcal{F}$ such that $\mathcal{K}_{i_0} \subseteq \Diamond(X \setminus C_0)$, i.e., $\mathcal{K}_{i_0} \cap \Box C_0 = \emptyset$, contradicting $C_0 \in \mathcal{A}$. Hence $F \in \mathcal{A}$.

By Zorn's Lemma, there exists a minimal element C_m in $\mathcal{A}$ such that $\Box C_m$ intersects all $\mathcal{K}_i$ $(i \in I)$. Clearly, $\Box C_m$ is also a minimal closed set that intersects all $\mathcal{K}_i$ $(i \in I)$, hence is an ω-Rudin set in $P_H(\mathsf{WD}_\omega(X))$. By Proposition 5.5 and Lemma 6.6, C_m is ω-well-filtered determined. So $C_m \in \Box C_m \cap \bigcap_{i\in I} \mathcal{K}_i \neq \emptyset$. It follows that $\bigcap_{i\in I} \mathcal{K}_i \nsubseteq \Diamond(X \setminus C_m) \supseteq \Diamond U$, which implies that $\bigcap_{i\in I} \mathcal{K}_i \nsubseteq \Diamond U$, a contradiction. $\Box$

By Lemma 6.5 and Theorem 6.7, we have the following result.

Theorem 6.8. *Let X be a T_0 space and $X^{\omega\text{-}w} = P_H(\mathsf{WD}_\omega(X))$. Then the pair $\langle X^{\omega\text{-}w}, \eta_X\rangle$, where $\eta_X : X \longrightarrow X^{\omega\text{-}w}$, $x \mapsto \overline{\{x\}}$, is the ω-well-filtered reflection of X.*

Corollary 6.9. *The category $\mathbf{Top}_{\omega\text{-}w}$ of all ω-well-filtered spaces is a reflective full subcategory of $\mathbf{Top}_0$.*

Corollary 6.10. *Let X, Y be two T_0 spaces and $f : X \longrightarrow Y$ a continuous mapping. Then there exists a unique continuous mapping $f^{\omega\text{-}w} : X^{\omega\text{-}w} \longrightarrow Y^{\omega\text{-}w}$ such that $f^{\omega\text{-}w} \circ \eta_X = \eta_Y \circ f$, that is, the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & X^{\omega\text{-}w} \arrow[d, "f^{\omega\text{-}w}"] \\ Y \arrow[r, "\eta_Y"] & Y^{\omega\text{-}w} \end{tikzcd}$$

For each $A \in \mathsf{WD}_\omega(X)$, $f^{\omega\text{-}w}(A) = \overline{f(A)}$.

Corollary 6.10 defines a functor $W_\omega : \mathbf{Top}_0 \longrightarrow \mathbf{Top}_{\omega\text{-}w}$, which is the left adjoint to the inclusion functor $I : \mathbf{Top}_{\omega\text{-}w} \longrightarrow \mathbf{Top}_0$.

Corollary 6.11. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is ω-well-filtered.*
(2) *$\mathsf{RD}_\omega(X) = \mathcal{S}_c(X)$.*
(3) *$\mathsf{WD}_\omega(X) = \mathcal{S}_c(X)$, that is, for each $A \in \mathsf{WD}_\omega(X)$, there exists a unique $x \in X$ such that $A = \overline{\{x\}}$.*
(4) *$X \cong X^{\omega\text{-}w}$.*

Proof. (1) $\Rightarrow$ (2): Applying Proposition 5.3 to the identity $id_X : X \longrightarrow X$.

(2) $\Rightarrow$ (3): By Proposition 5.5.

(3) $\Rightarrow$ (4): By assumption, $\mathsf{WD}_\omega(X) = \left\{\overline{\{x\}} : x \in X\right\}$, so $X^{\omega\text{-}w} = P_H(\mathsf{WD}_\omega(X)) = P_H(\{\overline{\{x\}} : x \in X\})$, and whence $X \cong X^{\omega\text{-}w}$.

(4) $\Rightarrow$ (1): By Theorem 6.7. $\Box$

Remark 6.12. By Proposition 3.7, Proposition 5.5 and Corollary 6.11, we can get Proposition 3.11.

Corollary 6.13. *A retract of an ω-well-filtered space is ω-well-filtered.*

Proof. Suppose that Y is a retract of an ω-well-filtered space X. Then there are continuous mappings $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g = id_Y$. Let $B \in \mathsf{WD}_\omega(Y)$, then by Lemma 5.6 and Corollary 6.11, there exists a unique $x_B \in X$ such that $\overline{g(B)} = \overline{\{x_B\}}$. Therefore, $B = \overline{f \circ g(B)} = \overline{f(\overline{g(B)})} = \overline{f(\overline{\{x_B\}})} = \overline{\{f(x_B)\}}$. By Corollary 6.11, Y is ω-well-filtered. $\square$

Theorem 6.14. *Let $\{X_i : 1 \leq i \leq n\}$ be a finite family of T_0 spaces. Then $(\prod_{i=1}^{n} X_i)^{\omega\text{-}w} = \prod_{i=1}^{n} X_i^{\omega\text{-}w}$ (up to homeomorphism).*

Proof. Let $X = \prod_{i=1}^{n} X_i$. By Corollary 5.9, we can define a mapping $\gamma : P_H(\mathsf{WD}_\omega(X)) \longrightarrow \prod_{i=1}^{n} P_H(\mathsf{WD}_\omega(X_i))$ by

$$\forall A \in \mathsf{WD}_\omega(X),\ \gamma(A) = (p_1(A), p_2(A), ..., p_n(A)).$$

By Lemma 5.8 and Corollary 5.9, γ is bijective. Now we show that γ is a homeomorphism. For any $(U_1, U_2, ..., U_n) \in \mathcal{O}(X_1) \times \mathcal{O}(X_2) \times ... \times \mathcal{O}(X_n)$, by Lemma 5.8 and Corollary 5.9, we have

$$\begin{aligned}
\gamma^{-1}(\Diamond U_1 \times \Diamond U_2 \times ... \times \Diamond U_n) &= \{A \in \mathsf{WD}_\omega(X) : \gamma(A) \in \Diamond U_1 \times \Diamond U_2 \times ... \times \Diamond U_n\} \\
&= \{A \in \mathsf{WD}_\omega(X) : p_1(A) \cap U_1 \neq \emptyset, p_2(A) \cap U_2 \neq \emptyset, ..., p_n(A) \cap U_n \neq \emptyset\} \\
&= \{A \in \mathsf{WD}_\omega(X) : A \cap U_1 \times U_2 \times ... \times U_n \neq \emptyset\} \\
&= \Diamond(U_1 \times U_2 \times ... \times U_n) \in \mathcal{O}(P_H(\mathsf{WD}_\omega(X)),\ \text{and}
\end{aligned}$$

$$\begin{aligned}
\gamma(\Diamond(U_1 \times U_2 \times ... \times U_n)) &= \{\gamma(A) : A \in \mathsf{WD}_\omega(X) \text{ and } A \cap U_1 \times U_2 \times ... \times U_n \neq \emptyset\} \\
&= \{\gamma(A) : A \in \mathsf{WD}_\omega(X), \text{ and } p_1(A) \cap U_1 \neq \emptyset, ..., p_n(A) \cap U_n \neq \emptyset\} \\
&= \Diamond U_1 \times \Diamond U_2 \times ... \times \Diamond U_n \in \mathcal{O}(\prod_{i=1}^{n} P_H(\mathsf{WD}_\omega(X_i))).
\end{aligned}$$

Therefore, $\gamma : P_H(\mathsf{WD}_\omega(X)) \longrightarrow \prod_{i=1}^{n} P_H(\mathsf{WD}_\omega(X_i))$ is a homeomorphism. Thus $X^{\omega\text{-}w}$ $(= P_H(\mathsf{WD}_\omega(X)))$ and $\prod_{i=1}^{n} X_i^{\omega\text{-}w}$ $(= \prod_{i=1}^{n} P_H(\mathsf{WD}_\omega(X_i)))$ are homeomorphic. $\square$

Question 6.15. Does $(\prod_{i<\omega} X_i)^{\omega\text{-}w} = \prod_{i<\omega} X_i^{\omega\text{-}w}$ (up to homeomorphism) hold for any countable family $\{X_i : i < \omega\}$ of T_0 spaces?

Using WD_ω sets and Corollary 6.11, we show that products of ω-well-filtered spaces are ω-well-filtered.

Theorem 6.16. *Let $\{X_i : i \in I\}$ be a family of T_0 spaces. Then the following two conditions are equivalent:*

(1) *The product space $\prod_{i \in I} X_i$ is ω-well-filtered.*
(2) *For each $i \in I$, X_i is ω-well-filtered.*

Proof. (1) $\Rightarrow$ (2): For each $i \in I$, X_i is a retract of $\prod_{i \in I} X_i$. By Corollary 6.13, X_i is ω-well-filtered.

(2) ⇒ (1): Let $X = \prod_{i\in I} X_i$. Suppose $A \in \mathsf{WD}_\omega(X)$. Then by Corollary 2.6, Proposition 5.5 and Lemma 5.6, $A \in \mathsf{Irr}_c(X)$ and for each $i \in I$, $p_i(A) \in \mathsf{WD}_\omega(X_i)$, and consequently, there is a $u_i \in X_i$ such that $p_i(A) = cl_{X_i}(\{u_i\})$ by condition (2) and Corollary 6.11. Let $u = (u_i)_{i\in I}$. Then by Lemma 2.4, Corollary 2.6 and [1, Proposition 2.3.3], we have $A = \prod_{i\in I} p_i(A) = \prod_{i\in I} \mathrm{cl}_{X_i}(\{u_i\}) = \mathrm{cl}_X(\{u\})$. Therefore, X is ω-well-filtered by Corollary 6.11. □

Theorem 6.17. *For a T_0 space X, the following conditions are equivalent:*

(1) *$X^{\omega\text{-}w}$ is the sobrification of X, in other words, the ω-well-filtered reflection of X and sobrification of X are the same.*
(2) *$X^{\omega\text{-}w}$ is sober.*
(3) *$\mathsf{Irr}_c(X) = \mathsf{WD}_\omega(X)$.*

Proof. (1) ⇒ (2): Trivial.

(2) ⇒ (3): Let $\eta_X^{\omega\text{-}w} : X \longrightarrow X^{\omega\text{-}w}$ be the canonical topological embedding defined by $\eta_X^{\omega\text{-}w}(x) = \overline{\{x\}}$ (see Theorem 6.8). Since the pair $\langle X^s, \eta_X\rangle$ is the sobrification of X (see Remark 2.7) and $X^{\omega\text{-}w}$ is sober, there exists a unique continuous mapping $(\eta_X^{\omega\text{-}w})^* : X^s \longrightarrow X^{\omega\text{-}w}$ such that $(\eta_X^{\omega\text{-}w})^* \circ \eta_X = \eta_X^{\omega\text{-}w}$, that is, the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\eta_X^{\omega\text{-}w}"'] & X^s \arrow[d, dotted, "(\eta_X^{\omega\text{-}w})^*"] \\
 & X^{\omega\text{-}w}
\end{tikzcd}$$

So for each $A \in \mathsf{Irr}_c(X)$, there exists a unique $B \in \mathsf{WD}_\omega(X)$ such that

$$\downarrow_{\mathsf{WD}_\omega(X)} A = \overline{\eta_X^{\omega\text{-}w}(A)} = \overline{\{B\}} = \downarrow_{\mathsf{WD}_\omega(X)} B.$$

Clearly, we have $B \subseteq A$. On the other hand, for each $a \in A$, $\overline{\{a\}} \in \downarrow_{\mathsf{WD}_\omega(X)} A = \downarrow_{\mathsf{WD}_\omega(X)} B$, and whence $\overline{\{a\}} \subseteq B$. Thus $A \subseteq B$, and consequently, $A = B$. Thus $A \in \mathsf{WD}_\omega(X)$.

(3) ⇒ (1): If $\mathsf{WD}_\omega(X) = \mathsf{Irr}_c(X)$, then $X^{\omega\text{-}w} = P_H(\mathsf{WD}_\omega(X)) = P_H(\mathsf{Irr}_c(X)) = X^s$, with $\eta_X^{\omega\text{-}w} = \eta_X : X \longrightarrow X^{\omega\text{-}w}$, is the sobrification of X. □

Proposition 6.18. *A T_0 space X is compact iff $X^{\omega\text{-}w}$ is compact.*

Proof. By Proposition 5.5, we have $\mathcal{S}_c(X) \subseteq \mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$. Suppose that X is compact. For $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, if $\mathsf{WD}_\omega(X) \subseteq \bigcup_{i\in I} \Diamond U_i$, then $X \subseteq \bigcup_{i\in I} U_i$ since $\mathcal{S}_c(X) \subseteq \mathsf{WD}_\omega(X)$, and consequently, $X \subseteq \bigcup_{i\in I_0} U_i$ for some $I_0 \in I^{(<\omega)}$. It follows that $\mathsf{WD}_\omega(X) \subseteq \bigcup_{i\in I_0} \Diamond U_i$. Thus $X^{\omega\text{-}w}$ is compact. Conversely, if $X^{\omega\text{-}w}$ is compact and $\{V_j : j \in J\}$ is an open cover of X, then $\mathsf{WD}_\omega(X) \subseteq \bigcup_{j\in J} \Diamond V_j$. By the compactness of $X^{\omega\text{-}w}$, there is a finite subset $J_0 \subseteq J$ such that $\mathsf{WD}_\omega(X) \subseteq \bigcup_{j\in J_0} \Diamond V_j$, and whence $X \subseteq \bigcup_{j\in J_0} V_j$, proving the compactness of X. □

Since $\mathcal{S}_c(X) \subseteq \mathsf{WD}_\omega(X) \subseteq \mathsf{Irr}_c(X)$ (see Proposition 5.5), the correspondence $U \leftrightarrow \Diamond_{\mathsf{WD}_\omega(X)} U$ is a lattice isomorphism between $\mathcal{O}(X)$ and $\mathcal{O}(X^{\omega\text{-}w})$, and whence we have the following proposition.

Proposition 6.19. *Let X be a T_0 space. Then*

(1) *X is locally hypercompact iff $X^{\omega\text{-}w}$ is locally hypercompact.*
(2) *X is a C-space iff X^w is a C-space.*

Proposition 6.20. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is core compact.*
(2) *$X^{\omega\text{-}w}$ is core compact.*
(3) *$X^{\omega\text{-}w}$ is locally compact.*

Proof. (1) $\Leftrightarrow$ (2): Since $\mathcal{O}(X)$ and $\mathcal{O}(X^{\omega\text{-}w})$ are lattice isomorphic.

(2) $\Rightarrow$ (3): By Theorem 6.7, $X^{\omega\text{-}w}$ is ω-well-filtered. If $X^{\omega\text{-}w}$ is core compact, then $X^{\omega\text{-}w}$ is locally compact by Theorem 3.15.

(3) $\Rightarrow$ (2): Trivial. □

Remark 6.21. In [6] (see also [7, Exercise V-5.25]) Hofmann and Lawson constructed a core compact T_0 space X that is not locally compact. By Proposition 6.20, $X^{\omega\text{-}w}$ is locally compact. So the local compactness of $X^{\omega\text{-}w}$ does not imply the local compactness of X.

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