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Derived operators on M-fuzzifying convex spaces

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Abstract. The main objective of this paper is to provide some characterizations of M-fuzzifying convex structures in terms of derived operators. We first introduce the notion of M-fuzzifying convexly derived operators, and then establish the one-to-one correspondence between M-fuzzifying convexly derived operators and M-fuzzifying convex structures. Additionally, it is proved that M-fuzzifying CP mappings can be expressed as formulas of derived sets. Finally, we introduce the notion of M-fuzzifying restricted convexly derived operators and prove that it is isomorphic to M-fuzzifying convex structures.

Keywords: M-fuzzifying convex structure, M-fuzzifying derived operator, M-fuzzifying convexly derived operator, M-fuzzifying restricted convexly derived operator, M-fuzzifying restricted hull operator

1. Introduction

Convexity theory, which is inspired by the shape of some convex figures, has been accepted to be of increasing importance in recent years in many areas of applied mathematics. In 1993, M.L.J. van de Vel collected the theory of convexity systematically in the famous book [30]. In 1994, Rosa [17] firstly generalized convex spaces to fuzzy case. In 2009, Y. Maruyama generalized it to L-fuzzy setting in [10], where L is a completely distributive lattice. In 2014, F.G. Shi and Z.Y. Xiu gave a new approach to fuzzification of convexity in a completely different way and proposed the concept of M-fuzzifying convexity [27]. Subsequently, many properties of convexity theory are generalized to M-fuzzifying convexity [9, 28, 33, 34, 38–40] or L-convexity [5, 7, 8, 12–16, 18–20]. Afterwards, abstract convexity was extended to (L, M)-fuzzy convexity in [29].

The concept of derived sets was first introduced by Georg Cantor in 1872, and he developed set theory in large part to study derived sets on the real line. In mathematics, it is well known that derived sets and derived operators are very important in topology, modal logic [6, 21], etc.

In 2010, Xin and Shi [35] firstly applied derived operators to matroids in the sense of Whitney (see [32]). They proposed the notion of m-derived operators and used it to characterize to matroids. As noted in [30], there is a one-to-one correspondence between Whitney's matroids (called independent structures in [30]) and convex structures satisfying the Exchange Law (see Chapter I, Section 2 in [30] for detail). Based on this conclusion, Chen and Shen recently provide a characterization of convex structures by c-derived operators [2]. Also, the relationship between m-derived operators and c-operators are investigated, as shown as follows.

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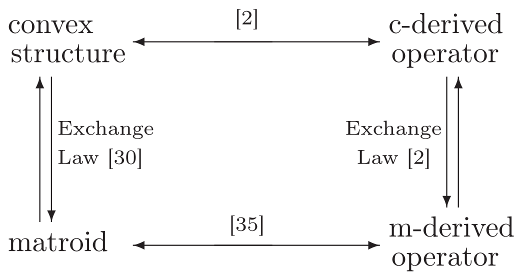
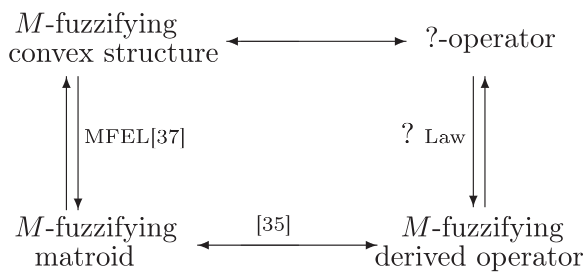


A natural idea is that does the relations in the above diagram hold in the fuzzy case? In 2009, Shi proposed the concept of M-fuzzifying matroids in [26], which is a generalization of matroids. Later in [35], Xin and Shi introduced the concept of M-fuzzifying derived operators and showed its isomorphism to M-fuzzifying matroids. In a completely different way, Zhong and Shi also proposed a new approach to M-fuzzifying derived operators on M-fuzzifying matroids, which is parallel to M-fuzzifying derived operators on topologies [41]. Recently, Yang do some research on M-fuzzifying convex structures satisfying the M-fuzzifying Exchange Law (MFEL for short) [37]. From his work, one can deduce that Shi's M-fuzzifying matroids are isomorphic to M-fuzzifying convex structures satisfying the MFEL. According to the existing work, we now obtain the following diagram:



So it will be interesting and meaningful to accomplished the above diagram. The layout of the paper is as follows. In Section 2, we recall some basic notions and results that will be used in the subsequent sections. In Sections 3 and 4, we introduce the notions of M-fuzzifying convexly derived operators and M-fuzzifying restricted convexly derived operators, which are isomorphic to M-fuzzifying convex structures.

2. Preliminaries

Throughout this paper, $(M, \vee, \wedge, ')$ denotes a completely distributive lattice with an order-reversing involution $'$. The smallest element and the largest element in M are denoted by $\perp$ and $\top$, respectively. Also, we adopt the convention that $\wedge\emptyset = \top$ and $\vee\emptyset = \perp$. For $S \subseteq M$, write $\bigvee S$ for the supremum of S and $\bigwedge S$ for the infimum of S, respectively.

An element a in M is called a co-prime element if for any $b, c \in M$, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. An element a in M is called a prime element if for any $b, c \in M$, $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. The set of non-zero co-prime elements in M is denoted by $J(M)$. For $a, b \in M$, we say that a is wedge below b, in symbols $a \prec b$, if for any subset $D \subseteq M$, $b \leq \bigvee D$ implies the existence of $d \in D$ with $a \leq d$ [3]. We denote $\beta(a) = \{x \in M \mid x \prec a\}$ and $\beta^*(a) = \beta(a) \cap J(M)$. Moreover, we define $\prec^{op}$ as follows: $a \prec^{op} b$ if and only if for every subset $D \subseteq M$, $\bigwedge D \leq a$ always implies $d \leq b$ for some $d \in D$. Therefore $a \prec b$ if and only if $b' \prec^{op} a'$. We denote $\alpha(a) = \{x \in M \mid a \prec^{op} x\}$. Two classic results on M due to Wang [31] is that

(1) $a = \bigvee \beta(a) = \bigwedge \alpha(a)$.
(2) α is a $\bigwedge$-$\bigcup$ mapping, that is, $\alpha\left(\bigwedge_{i\in\Omega} a_i\right) = \bigcup_{i\in\Omega} \alpha(a_i)$, where $\{a_i \mid i \in \Omega\} \subseteq M$.

Let X be a non-empty set. We denote the set of all subsets (resp., all finite subsets) of X by 2^X (resp., 2^X_{fin}). Denote by M^X the set of all M-sets on X. Usually, we do not distinguish a crisp subset A of X and its characteristic function χ_A.

Definition 2.1 ([25]). For each $a \in M$ and each $A \in M^X$, we define

$$A_{[a]} = \{x \in X \mid A(x) \geq a\},$$

$$A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}.$$

More properties and applications on above two cut sets can be found in [4, 11, 22–24].

Next, we will review some basic concepts and results on the convexity theory. For notions related to convex structures used in the paper, the reader can refer to [30].

Definition 2.2 ([30]). Let X be a non-empty set. A *convex structure* on X is a subfamily $\mathscr{C}$ of 2^X satisfying the following conditions:

(C1) $\emptyset, X \in \mathscr{C}$;
(C2) for any non-empty family $\{A_i \mid i \in \Omega\} \subseteq \mathscr{C}$, $\bigcap_{i\in\Omega} A_i \in \mathscr{C}$;
(C3) for any directed family $\{A_i \mid i \in \Omega\} \subseteq \mathscr{C}$, $\bigcup_{i\in\Omega} A_i \in \mathscr{C}$.

The pair $(X, \mathscr{C})$ is called a *convex space* if $\mathscr{C}$ is a convex structure on X.

Definition 2.3. Let $(X, \mathscr{C})$ be a convex space. For any $A \in 2^X$, the *hull* of A is defined by

$$co(A) = \bigcap \{C \in \mathscr{C} \mid A \subseteq C\}.$$

The operator co is called the *hull operator* in $(X, \mathscr{C})$.

Recently, Chen and Shen [2] prove that convex structures, like topologies, can be completely described by derived operators.

Definition 2.4 ([2]). Let X be a non-empty set. A *convexly derived operator* (*c-derived operator* for short) is a mapping $d : 2^X \longrightarrow 2^X$ satisfying the following axioms:

(CD1) Normalization : $d(\emptyset) = \emptyset$;
(CD2) Representation: $x \in d(A)$ implies $x \in d(A - \{x\})$;
(CD3) Idempotency: $d(d(A) \cup A) \subseteq d(A) \cup A$;
(CD4) Domain-finiteness: $d(A) = \bigcup_{F \in 2^A_{fin}} d(F)$.

The pair (X, d) is called a *convexly derived space* (*c-derived space*) if d is a convexly derived operator on X.

Theorem 2.5 ([2]). *Let X be a non-empty set. Each convex structure $\mathscr{C}$ on X can induce a c-derived operator $d_{\mathscr{C}} : 2^X \longrightarrow 2^X$ by*

$$d_{\mathscr{C}}(A) = \{x \in X \mid x \in co(A - \{x\})\}.$$

Conversely, every c-derived operator d on X can induce a convex structure $\mathscr{C}_d \subseteq 2^X$ by

$$\mathscr{C}_d = \{C \subseteq X \mid d(C) \subseteq C\}.$$

Also, it holds that $d_{\mathscr{C}_d} = d$ and $\mathscr{C}_{d_{\mathscr{C}}} = \mathscr{C}$.

Definition 2.6 ([2]). A mapping $f : (X, d_X) \longrightarrow (Y, d_Y)$ between c-derived spaces is called a *derived homomorphism* provided that

$$\forall A \subseteq X, \quad f(d_X(A)) \subseteq f(A) \cup d_Y(f(A)).$$

In [27], Shi and Xiu extend the notion of convex structures to the lattice-valued case, which are called M-fuzzifying convex structures.

Definition 2.7 ([27]). Let X be a non-empty set. An *M-fuzzifying convex structure* on X is a mapping $\mathscr{C} : 2^X \longrightarrow M$ satisfying the following conditions:

(MFC1) $\mathscr{C}(\emptyset) = \mathscr{C}(X) = \top$;
(MFC2) for any non-empty family $\{A_i \mid i \in \Omega\} \subseteq 2^X$, $\bigwedge_{i\in\Omega} \mathscr{C}(A_i) \leq \mathscr{C}\big(\bigcap_{i\in\Omega} A_i\big)$;
(MFC3) for any directed family $\{A_i \mid i \in \Omega\} \subseteq 2^X$, $\bigwedge_{i\in\Omega} \mathscr{C}(A_i) \leq \mathscr{C}\big(\bigcup_{i\in\Omega} A_i\big)$.

The pair $(X, \mathscr{C})$ is called an *M-fuzzifying convex space* if $\mathscr{C}$ is an M-fuzzifying convex structure on X.

Definition 2.8 ([27, 28]). Let X be a non-empty set. An *M-fuzzifying hull operator* on X is a mapping $co : 2^X \longrightarrow M^X$ satisfying the following conditions:

(MFC1) $co(\emptyset)(x) = \perp$;
(MFC2) for any $x \in A$, $co(A)(x) = \top$;
(MFC3) for any $x \notin A$,

$$co(A)(x) = \bigwedge_{x\notin B\supseteq A} \Big[\bigvee_{y\notin B} co(B)(y)\Big];$$

(MDF) $co(A)(x) = \bigvee\{co(F)(x) \mid F \in 2^A_{fin}\}$.

There is a one-to-one correspondence between M-fuzzifying convex structures and M-fuzzifying hull operators, shown as follows:

Theorem 2.9 ([27]). *Let X be a non-empty set. Every M-fuzzifying convex structure $\mathscr{C}$ on X can determine an M-fuzzifying hull operator $co_{\mathscr{C}} : 2^X \longrightarrow M^X$ by*

$$co_{\mathscr{C}}(A)(x) = \bigwedge_{x\notin B\supseteq A} \mathscr{C}(B)'.$$

Conversely, every M-fuzzifying hull operator $co : 2^X \longrightarrow M^X$ can induce an M-fuzzifying convex structure $\mathscr{C}_{co} : 2^X \longrightarrow M$ by

$$\mathscr{C}_{co}(A) = \bigwedge_{x\notin A} [co(A)(x)]'.$$

Furthermore, $\mathscr{C}_{co_{\mathscr{C}}} = \mathscr{C}$ and $co_{\mathscr{C}_{co}} = co$.

Definition 2.10 ([37]). An M-fuzzifying convex structure $\mathscr{C}$ on X is called an *M-fuzzifying matriod* if it satisfies the following condition:

(MFEL) $\forall x, y \in X,\ \forall A \in 2^X,\ co(A \cup \{y\})(x) \leq co(A \cup \{x\})(y) \vee co(A)(x)$.

Definition 2.11 ([27]). A mapping $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ between two M-fuzzifying convex spaces is called an *M-fuzzifying convexity preserving mapping* (*MCP* for short) provided that

$$\forall B \subseteq Y, \mathscr{C}_X(f^{-1}(B)) \geq \mathscr{C}_Y(B).$$

Theorem 2.12 ([36]). *Let $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ be a mapping between M-fuzzifying convex spaces. Then the following statements are equivalent.*

(1) *f is MCP.*
(2) $\forall A \in 2^X$, $co_{\mathscr{C}_X}(A)(x) \leq co_{\mathscr{C}_Y} f(A)(f(x))$.
(3) $\forall F \in 2^X_{fin}$, $co_{\mathscr{C}_X}(F)(x) \leq co_{\mathscr{C}_Y} f(F)(f(x))$.

3. M-fuzzifying convexly derived operators

In this section, the notion of M-fuzzifying convexly derived operators is introduced and its isomorphic relationship to M-fuzzifying convex structures is constructed.

Definition 3.1. Let X be a non-empty set. An *M-fuzzifying convexly derived operator* on X is a mapping $d : 2^X \longrightarrow M^X$ satisfying the following conditions:

(MCD1) $\forall x \in X$, $d(\emptyset)(x) = \perp$;
(MCD2) $\forall x \in X$ and $\forall A \in 2^X$, $d(A)(x) = d(A - \{x\})(x)$;
(MCD3) $\forall x \in X$ and $\forall A \in 2^X$, $d(A)(x) = \bigwedge_{x\notin B\supseteq A-\{x\}} \left[\bigvee_{y\notin B} d(B)(y)\right]$;
(MCD4) $\forall x \in X$ and $\forall A \in 2^X$, $d(A)(x) = \bigvee_{F\in 2^A_{fin}} d(F)(x)$.

The pair (X, d) is called an *M-fuzzifying convexly derived space* if d is an M-fuzzifying convexly derived operator on X.

The notation $d(A)(x)$ can be interpreted as the degree that the point x belongs to the derived set of A. In the crisp situation, that is, $M = \{\perp, \top\}$, (MCD1)–(MCD4) can reduce to (CD1)–(CD4) in Definition 2 , respectively. In this sense, the notion of M-fuzzifying convexly derived operators is a reasonable generalization of the c-derived operators.

Lemma 3.2. *Let (X, d) be an M-fuzzifying convexly derived space.*

(1) *For any $A, B \in 2^X$, $A \subseteq B$ implies $d(A) \leq d(B)$.*
(2) *Axiom* (MCD3) *is equivalent to each of the following statements:*
(MCD3′) $\forall A \in 2^X$ *and* $\forall x \notin A$, $d(A)(x) = \bigwedge_{x\notin B\supseteq A} \left[\bigvee_{y\notin B} d(B)(y)\right]$;
(MCD3″) $\forall A \in 2^X$ *and* $\forall a \in J(M)$, $d\big(A \cup d(A)_{[a]}\big)_{[a]} \subseteq A \cup d(A)_{[a]}$;
(MCD3‴) $\forall A \in 2^X$ *and* $\forall a \in \alpha(\perp)$, $d\big(A \cup d(A)^{[a]}\big)^{[a]} \subseteq A \cup d(A)^{[a]}$.

Proof. (1) It is trivial by (MCD4).

(2) It is straightforward that (MCD3) $\Rightarrow$ (MCD3′).

(MCD3′) $\Rightarrow$ (MCD3). Let $A \in 2^X$ and $x \in X$. It is trivial if $x \notin A$. Now suppose $x \in A$. Since $x \notin A - \{x\}$, by assumption and (MCD2), we have

$$\begin{aligned} d(A)(x) &= d(A - \{x\})(x) \\ &= \bigwedge\nolimits_{x\notin B\supseteq A-\{x\}} \left[\bigvee\nolimits_{y\notin B} d(B)(y)\right]. \end{aligned}$$

Therefore, (MCD3) holds.

(MCD3′) $\Rightarrow$ (MCD3″). Suppose $a \in J(M)$ and $x \notin A \cup d(A)_{[a]}$. Then $x \notin A$ and $x \notin d(A)_{[a]}$. By (MCD3′), we have

$$d(A)(x) = \bigwedge_{x\notin B\supseteq A} \left[\bigvee_{y\notin B} d(B)(y)\right] \not\geq a.$$

Then there exists $B_* \in 2^X$ such that $x \notin B_* \supseteq A$ and

$$\begin{aligned} &\bigvee\nolimits_{y\notin B_*} \left[d(B_*)(y)\right] \not\geq a \\ \Rightarrow\ &\forall y \notin B_*, d(B_*)(y) \not\geq a \\ \Rightarrow\ &\forall y \notin B_*, y \notin d(B_*)_{[a]} \\ \Rightarrow\ &d(B_*)_{[a]} \subseteq B_*. \end{aligned}$$

It follows that $d(A)_{[a]} \subseteq d(B_*)_{[a]} \subseteq B_*$. Additionally, we have

$$d\left(d(A)_{[a]} \cup A\right)_{[a]} \subseteq d(B_* \cup A)_{[a]} = d(B_*)_{[a]} \subseteq B_*.$$

Since $x \notin B_*$, we have $x \notin d(B_*)_{[a]}$, which implies $x \notin d(d(A)_{[a]} \cup A)_{[a]}$.

Therefore, $d\big(A \cup d(A)_{[a]}\big)_{[a]} \subseteq A \cup d(A)_{[a]}$.

(MCD3″) $\Rightarrow$ (MCD3′). Suppose $A \in 2^X$ and $x \notin A$. If $B \in 2^X$ such that $x \notin B \supseteq A$, then by (1) we have

$$d(A)(x) \leq d(B)(x) \leq \bigvee_{y\notin B} d(B)(y).$$

Therefore $d(A)(x) \leq \bigwedge_{x\notin B\supseteq A} \left[\bigvee_{y\notin B} d(B)(y)\right]$. Conversely, suppose $a \in J(M)$ such that $a \not\leq d(A)(x)$. Then there exists $b \in \beta^*(a)$ such that $b \not\leq d(A)(x)$. Let $B_* = A \cup d(A)_{[b]}$. Then it satisfies that $x \notin B_* \supseteq A$. By (MCD3), we have $B_* = d(B_*)_{[b]}$. Thus

$$\begin{aligned} &\forall y \notin B_*, y \notin d(B_*)_{[b]} \\ \Rightarrow\ &\forall y \notin B_*, b \not\leq d(B_*)(y). \end{aligned}$$

Since $b \in \beta^*(a)$, we have

$$a \not\leq \bigvee_{y \notin B_*} d(B_*)(y) \geq \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} d(B)(y).$$

It follows that $a \not\leq \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} d(B)(y)$. By the arbitrariness of a, we obtain

$$\bigwedge_{x \notin B \supseteq A} \left[\bigvee_{y \notin B} d(B)(y) \right] \leq d(A)(x).$$

Thus $d(A)(x) = \bigwedge_{x \notin B \supseteq A} \left[\bigvee_{y \notin B} d(B)(y) \right]$.

(MCD3′) ⇒ (MCD3‴). Let $a \in \alpha(\perp)$ and $x \notin A \cup d(A)^{[a]}$. Then $x \notin d(A)^{[a]}$, i.e., $a \in \alpha(d(A)(x))$. Since $\alpha(\cdot)$ is a $\bigwedge - \bigcup$ mapping and (MCD3′), we have

$$\begin{aligned} a &\in \alpha(d(A)(x)) \\ &= \alpha\left(\bigwedge_{x \notin B \supseteq A} \left[\bigvee_{y \notin B} d(B)(y) \right]\right) \\ &= \bigcup_{x \notin B \supseteq A} \alpha\left(\bigvee_{y \notin B} d(B)(y)\right) \\ &\subseteq \bigcup_{x \notin B \supseteq A} \bigcap_{y \notin B} \alpha(d(B)(y)). \end{aligned}$$

Thus there exists $B_* \in 2^X$ such that $x \notin B_* \supseteq A$, and that

$$\begin{aligned} &\forall y \notin B_*, a \in \alpha(d(B_*)(y)) \\ \Rightarrow\ &\forall y \notin B_*, y \notin d(B_*)^{[a]} \\ \Rightarrow\ &d(B_*)^{[a]} \subseteq B_*. \end{aligned}$$

From $A \subseteq B_*$, it follows that

$$A \cup d(A)^{[a]} \subseteq A \cup d(B_*)^{[a]} \subseteq A \cup B_* = B_*.$$

By (MCD3′), we have

$$\begin{aligned} a &\in \bigcap_{y \notin B_*} \alpha(d(B_*)(y) \\ &= \alpha\left(\bigvee_{y \notin B_*} d(B_*)(y)\right) \\ &\subseteq \bigcup_{x \notin B \supseteq A \cup d(A)^{[a]}} \alpha\left(\bigvee_{y \notin B} d(B)(y)\right) \\ &= \alpha\left(\bigwedge_{x \notin B \supseteq A \cup d(A)^{[a]}} \bigvee_{y \notin B} d(B)(y)\right) \\ &= \alpha\left(d\left(A \cup d(A)^{[a]}\right)(x)\right). \end{aligned}$$

This implies $a \in \alpha\left(d\left(A \cup d(A)^{[a]}\right)(x)\right)$, i.e., $x \notin d\left(d(A)^{[a]} \cup A\right)^{[a]}$.

Therefore, $d\left(A \cup d(A)^{[a]}\right)^{[a]} \subseteq A \cup d(A)^{[a]}$.

(MCD3‴) ⇒ (MCD3′). Let $x \in X$ and $A \in 2^X$ with $x \notin A$. Suppose $a \in \alpha(\perp)$ such that $a \in \alpha(d(A)(x))$. Then there exists $b \in J(M)$ satisfying $a \in \alpha(b)$ and $b \in \alpha(d(A)(x))$. It follows that $x \notin d(A)^{[b]} \cup A$. Let $B_* = A \cup d(A)^{[b]}$. Then $x \notin B_* \supseteq A$. By (MCD3‴), $B_* = d(B_*)^{[b]}$, which implies that

$$\begin{aligned} &\forall y \notin B_*, y \notin d(B_*)^{[b]} \\ \Rightarrow\ &\forall y \notin B_*, b \in \alpha(d(B_*)(y)). \end{aligned}$$

In addition, since $a \in \alpha(b)$, we have

$$\begin{aligned} a &\in \alpha\left(\bigvee_{y \notin B_*} d(B_*)(y)\right) \\ &\subseteq \alpha\left(\bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} d(B)(y)\right). \end{aligned}$$

Hence, $d(A)(x) \geq \bigwedge_{x \notin B \supseteq A} \left[\bigvee_{y \notin B} d(B)(y) \right]$. The converse can be shown using the same method of (MCD3″) ⇒ (MCD3′). Thus (MCD3′) holds. □

Definition 3.3. A mapping $f : (X, d_X) \longrightarrow (Y, d_Y)$ between M-fuzzifying convexly derived spaces is called an *M-fuzzfying convexly derived homomorphism* if for any $x \in X$ and $A \in 2^X$,
$d_X(A)(x) \leq f(A)(f(x)) \vee d_Y(f(A))(f(x))$.

Proposition 3.4. *Let $f : (X, d_X) \longrightarrow (Y, d_Y)$ be a mapping between M-fuzzifying convexly derived spaces. Then the following statements are equivalent.*

(1) *f is an M-fuzzifying convexly derived homomorphism.*
(2) *$\forall x \in X$ and $\forall F \in 2^X_{fin}$, $d_X(F)(x) \leq d_Y(f(F))(f(x)) \vee f(F)(f(x))$.*

Proof. (1) ⇒ (2). It is straightforward.
(2) ⇒ (1). Let $x \in X$ and $A \in 2^X$. We have

$$\begin{aligned} &d_X(A)(x) \\ &= \bigvee_{F \in 2^A_{fin}} d_X(F)(x) \\ &\leq \bigvee_{F \in 2^A_{fin}} f(F)(f(x)) \vee d_Y(f(F))(f(x)) \\ &\leq f(A)(f(x)) \vee d_Y(f(A))(f(x)). \end{aligned}$$

Thus d is an M-fuzzifying convexly derived homomorphism. □

Next, we will study the relationship between M-fuzzifying convexly derived operators and M-fuzzifying convex structures.

Proposition 3.5. *Let $d : 2^X \longrightarrow M^X$ be an M-fuzzifying convexly derived operator. Define a mapping $\mathscr{C}_d : 2^X \longrightarrow M$ as follows:*

$$\forall A \in 2^X, \mathscr{C}_d(A) = \bigwedge_{x \notin A} [d(A)(x)]'.$$

Then $\mathscr{C}_d$ is an M-fuzzifying convex structure on X.

Proof. We need to check (MFC1)–(MFC3).

(MFC1) Since $\{x \mid x \notin X\} = \emptyset$, we conclude that $\mathscr{C}_d(X) = \bigwedge \emptyset = \top$. By (MCD1), we have

$$\mathscr{C}_d(\emptyset) = \bigwedge_{x \notin \emptyset} \left[d(\emptyset)(x)\right]' = \bigwedge_{x \in X} \bot' = \top.$$

(MFC2) Suppose $\{A_i \mid i \in \Omega\} \subseteq 2^X$ is non-empty. If $x \notin \bigcap_{i\in\Omega} A_i$, then there exists $i_x \in \Omega$ such that $x \notin A_{i_x}$. By Lemma 3, we have

$$\begin{aligned}[d(\textstyle\bigcap_{i\in\Omega} A_i)(x)]' &\geq [d(A_{i_x})(x)]' \\ &\geq \textstyle\bigwedge_{y \notin A_{i_x}} [d(A_{i_x})(y)]' \\ &= \mathscr{C}_d(A_{i_x}) \\ &\geq \textstyle\bigwedge_{i\in\Omega} \mathscr{C}_d(A_i).\end{aligned}$$

Then $\mathscr{C}_d\big(\bigcap_{i\in\Omega} A_i\big) \geq \bigwedge_{i\in\Omega} \mathscr{C}_d(A_i)$.

(MFC3) Suppose $\{A_i \mid i \in \Omega\} \subseteq 2^X$ be is directed by inclusion. By Definition (2), we get

$$\begin{aligned}&\mathscr{C}_d\big(\textstyle\bigcup_{i\in\Omega} A_i\big) = \textstyle\bigwedge_{x \notin \bigcup_{i\in\Omega} A_i} [d(\bigcup_{i\in\Omega} A_i)(x)]' \\ &= \textstyle\bigwedge_{x \notin \bigcup_{i\in\Omega} A_i} [\bigvee_{F \in 2_{fin}^{\bigcup_{i\in\Omega} A_i}} d(F)(x)]' \\ &= \textstyle\bigwedge_{x \notin \bigcup_{i\in\Omega} A_i} [\bigvee_{i\in\Omega} \bigvee_{F \in 2_{fin}^{A_i}} d(F)(x)]' \\ &= \textstyle\bigwedge_{x \notin \bigcup_{i\in\Omega} A_i} \bigwedge_{i\in\Omega} \bigwedge_{F \in 2_{fin}^{A_i}} [d(F)(x)]' \\ &\geq \textstyle\bigwedge_{i\in\Omega} \bigwedge_{x \notin A_i} \bigwedge_{F \in 2_{fin}^{A_i}} [d(F)(x)]' \\ &\geq \textstyle\bigwedge_{i\in\Omega} \bigwedge_{x \notin A_i} d(A_i)(x)' = \bigwedge_{i\in\Omega} \mathscr{C}_d(A_i).\end{aligned}$$

The proof is completed. □

Proposition 3.6. *Let $d : 2^X \longrightarrow M^X$ be an M-fuzzifying convexly derived operator. Then for any $x \in X$ and $A \in 2^X$,*

$$co_d(A)(x) = d(A)(x) \vee A(x),$$

where co_d is the M-fuzzifying hull operator on $\mathscr{C}_d$.

Proof. If $x \in A$, the proof is completed. Otherwise, by Theorem 2 and (MCD3) of Definition 2, we have

$$\begin{aligned}co_d(A)(x) &= \textstyle\bigwedge_{x \notin B \supseteq A} \left[\mathscr{C}_d(B)\right]' \\ &= \textstyle\bigwedge_{x \notin B \supseteq A} \left[\bigvee_{y \notin B} d(B)(y)\right] \\ &= d(A)(x) = d(A)(x) \vee A(x),\end{aligned}$$

completing the proof. □

Proposition 3.7. *Let $f : (X, d_X) \longrightarrow (Y, d_Y)$ be an M-fuzzifying convexly derived homomorphism between M-fuzzifying convexly derived spaces. Then $f : (X, \mathscr{C}_{d_X}) \longrightarrow (Y, \mathscr{C}_{d_Y})$ is an MCP mapping.*

Proof. Denote by co_X and co_Y the M-fuzzifying hull operators on $\mathscr{C}_{d_X}$ and $\mathscr{C}_{d_Y}$, respectively. Let $x \in X$ and $A \in 2^X$. If $f(x) \in f(A)$, then it is trivial by Theorem 2. Now suppose $f(x) \notin f(A)$. Then $x \notin A$. By Proposition 3, we have

$$\begin{aligned}co_X(A)(x) &= d_X(A)(x) \vee A(x) \\ &= d_X(A)(x) \\ &\leq d_Y(f(A))(f(x)) \vee f(A)(f(x)) \\ &= co_Y(f(A))(f(x)).\end{aligned}$$

Therefore, f is an MCP mapping. □

Theorem 3.8. *Let $\mathscr{C}$ be an M-fuzzifying convex structure on X. Define a mapping $d_{\mathscr{C}} : 2^X \longrightarrow M^X$ as follows:*

$$d_{\mathscr{C}}(A)(x) = co_{\mathscr{C}}(A - \{x\})(x)$$

for all $x \in X$ and $A \in 2^X$. Then $d_{\mathscr{C}}$ is an M-fuzzifying convexly derived operator on X.

Proof. We need to check (MCD1)–(MCD4).

That (MCD1) and (MCD2) are trivial.

(MCD3) It can be proved by the following equations:

$$\begin{aligned}&d_{\mathscr{C}}(A)(x) \\ &= co_{\mathscr{C}}(A - \{x\})(x) \\ &= \textstyle\bigwedge_{x \notin B \supseteq A - \{x\}} \left[\bigvee_{y \notin B} co(B)(y)\right] \\ &= \textstyle\bigwedge_{x \notin B \supseteq A - \{x\}} \left[\bigvee_{y \notin B} co(B - \{y\})(y)\right] \\ &= \textstyle\bigwedge_{x \notin B \supseteq A - \{x\}} \left[\bigvee_{y \notin B} d(B)(y)\right].\end{aligned}$$

(MCD4) First, we have

$$\begin{aligned}d_{\mathscr{C}}(A)(x) &= co_{\mathscr{C}}(A - \{x\})(x) \\ &= \textstyle\bigvee_{F \in 2_{fin}^{A - \{x\}}} co(F)(x) \\ &= \textstyle\bigvee_{F \in 2_{fin}^{A - \{x\}}} co(F - \{x\})(x) \\ &= \textstyle\bigvee_{F \in 2_{fin}^{A - \{x\}}} d_{\mathscr{C}}(F)(x) \\ &\leq \textstyle\bigvee_{F \in 2_{fin}^{A}} d_{\mathscr{C}}(F)(x).\end{aligned}$$

Table 1
I-fuzzifying convex structure $(X, \mathscr{C})$ and its I-fuzzifying convexly derived operator

A	$\mathscr{C}(A)$	$d_{\mathscr{C}}(A)(x)$	$d_{\mathscr{C}}(A)(y)$	$d_{\mathscr{C}}(A)(z)$
X	1	0.29	0.5	0.7
$\{x, y\}$	0.3	0.29	0.39	0.7
$\{x, z\}$	0.5	0.2	0.5	0.39
$\{y, z\}$	0.71	0.29	0.2	0.6
$\{x\}$	0.61	0	0.39	0.39
$\{y\}$	0.4	0.29	0	0.6
$\{z\}$	0.8	0.2	0.2	0
$\emptyset$	1	0	0	0

The inverse inequality holds obviously. □

Example 3.9 Let $X = \{x, y, z\}$ and $I = [0, 1]$. Define $\mathscr{C} : 2^X \to I$ by

$$\mathscr{C}(\emptyset) = \mathscr{C}(X) = 1;$$

$$\mathscr{C}(\{x\}) = 0.61, \mathscr{C}(\{y\}) = 0.4, \mathscr{C}(\{z\}) = 0.8;$$

$$\mathscr{C}(\{x, y\}) = 0.3, \mathscr{C}(\{x, z\}) = 0.5, \mathscr{C}(\{y, z\}) = 0.71.$$

Then $\mathscr{C}$ satisfies (MFC1)–(MFC3), and hence $\mathscr{C}$ is an I-fuzzifying convex structure. By Theorem 3, we obtain the I-fuzzifying convex structure and its I-fuzzifying convexly derived operator, as illustrated in Table 1.

The following theorem shows that M-fuzzifying convex structures can be completely determined by derived operators.

Theorem 3.10. *M-fuzzifying convex structures are isomorphic to M-fuzzifying convex derived operators.*

Proof. Suppose $\mathscr{C}$ is an M-fuzzifying convex structure on X and d is an M-fuzzifying convexly derived operator on X. It suffices to prove $\mathscr{C}_{d_{\mathscr{C}}} = \mathscr{C}$ and $d_{\mathscr{C}_d} = d$.

Let $x \in X$ and $A \in 2^X$. On one hand, we have

$$\begin{aligned}\mathscr{C}_{d_{\mathscr{C}}}(A) &= \bigwedge\nolimits_{x \notin A} \left[d_{\mathscr{C}}(A)(x)\right]' \\ &= \bigwedge\nolimits_{x \notin A} \left[co_{\mathscr{C}}(A - \{x\})(x)\right]' \\ &= \bigwedge\nolimits_{x \notin A} \left[co_{\mathscr{C}}(A)(x)\right]' = \mathscr{C}(A).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}d_{\mathscr{C}_d}(A)(x) &= co_{\mathscr{C}_d}(A - \{x\})(x) \\ &= d(A - \{x\})(x) \vee (A - \{x\})(x) \\ &= d(A - \{x\})(x) = d(A)(x).\end{aligned}$$

The proof is completed. □

From Definition 2 and Theorem 3, we obtain the following diagram.

$$\begin{tikzcd}[column sep=large, row sep=large]
\text{M-fuzzifying convex structure} \arrow[r, leftrightarrow] \arrow[d, shift right] & \text{M-fuzzifying convexly derived operator} \arrow[d, shift right, "\text{MFEL}"'] \\
\text{M-fuzzifying matroid} \arrow[u, shift right, "\text{MFEL [37]}"'] \arrow[r, leftrightarrow, "{[35]}"] & \text{M-fuzzifying derived operator} \arrow[u, shift right]
\end{tikzcd}$$

Theorem 3.11. *Let $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ be an MCP mapping between M-fuzzifying convex spaces. Then $f : (X, d_{\mathscr{C}_X}) \longrightarrow (Y, d_{\mathscr{C}_Y})$ is an M-fuzzifying convexly derived homomorphism.*

Proof. It suffices to check

$$d_{\mathscr{C}_X}(A)(x) \le f(A)(f(x)) \vee d_{\mathscr{C}_Y}(f(A))(f(x)).$$

It is trivial when $f(x) \in f(A)$. Now suppose $f(x) \notin f(A)$, which implies $x \notin A$. Since $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ is an MCP mapping, we have

$$\begin{aligned}d_{\mathscr{C}_X}(A)(x) &= co_{\mathscr{C}_X}(A - \{x\})(x) \\ &= co_{\mathscr{C}_X}(A)(x) \le co_{\mathscr{C}_Y}(f(A))(f(x)) \\ &= co_{\mathscr{C}_Y}(f(A) - \{f(x)\})(f(x)) \\ &= d_{\mathscr{C}_Y}(f(A))(f(x)) \\ &\le d_{\mathscr{C}_Y}(f(A))(f(x)) \bigvee f(A)(f(x)).\end{aligned}$$

The proof is completed. □

4. M-fuzzifying restricted convexly derived operators

In this section, we will introduce the notion of M-fuzzifying restricted convexly derived operators which is isomorphic to M-fuzzifying restricted hull operators.

Definition 4.1 ([28]). A mapping $h : 2^X_{fin} \longrightarrow M^X$ is called an *M-fuzzifying restricted hull operator* if it satisfies the following conditions:

(MH1) $\forall x \in X$, $h(\emptyset)(x) = \bot$;
(MH2) $\forall F \in 2^X_{fin}$ and $\forall x \in F$, $h(F)(x) = \top$;
(MH3) $\forall G, F \in 2^X_{fin}$ and $\forall x \in X$,

$$h(G)(x) \wedge \Big[\bigwedge_{y \in G} h(F)(y)\Big] \le h(F)(x).$$

Remark 4.2. If a mapping $h : 2^X_{fin} \longrightarrow M^X$ satisfies (MH1) and (MH2), then (MH3) is equivalent to the following condition:

(MH3′) $\forall x \in X$, and $\forall F, G \in 2^X_{fin}$, $h(G)(x) \wedge \left[\bigwedge_{y\in G-F} h(F)(y)\right] \leq h(F)(x)$.

Definition 4.3. ([2]) A mapping $\overline{d} : 2^X_{fin} \longrightarrow 2^X$ is called a *restricted convexly derived operator* provided for any $F, G \in 2^X_{fin}$, it satisfies the following conditions:

(RD1) $\overline{d}(\emptyset) = \emptyset$;
(RD2) $F \subseteq G$ implies $\overline{d}(F) \subseteq \overline{d}(G)$;
(RD3) $\forall x \in \overline{d}(F), x \in \overline{d}(F - \{x\})$;
(RD4) $G \subseteq \overline{d}(F)$ implies $\overline{d}(G \cup F) \subseteq \overline{d}(F) \cup F$.

Remark 4.4. It can be easily proved that (RD4) can be replaced by the following condition:

1. $G \subseteq \overline{d}(F) \cup F$ implies $\overline{d}(G) \subseteq \overline{d}(F) \cup F$.

Next, we will extend the notion of restricted convexly derived operator to the fuzzy case.

Definition 4.5 A mapping $\overline{d} : 2^X_{fin} \longrightarrow M^X$ is called an *M-fuzzifying restricted convexly derived operator* provided for any $x \in X$ and $F, G \in 2^X_{fin}$, it satisfies the following conditions:

(MRD1) $\overline{d}(\emptyset)(x) = \perp$;
(MRD2) $F \subseteq G$ implies $\overline{d}(F) \leq \overline{d}(G)$;
(MRD3) $\overline{d}(F)(x) = \overline{d}(F - \{x\})(x)$;
(MRD4) $\forall x \notin F$, $\overline{d}(G)(x) \wedge \left[\bigwedge_{y\in G-F} \overline{d}(F)(y)\right] \leq \overline{d}(F)(x)$.

If $M = \{\perp, \top\}$, then (MRD1)–(MRD3) can be simplified to (RD1)–(RD3). By Remark 4, we prove the equivalence of (MRD4) and (RD4′).

(MRD4) $\Rightarrow$ (RD4′). Let $F, G \in 2^X$ with $G \subseteq \overline{d}(F) \cup F$. We needs to show $\overline{d}(G) \subseteq \overline{d}(F) \cup F$. Suppose $x \in \overline{d}(G)$ and $x \notin F$. It follows that $\overline{d}(G)(x) = \top$ and $G - F \subseteq (\overline{d}(F) \cup F) - F \subseteq \overline{d}(F)$, implying

$$\overline{d}(G)(x) \wedge \Big[\bigwedge_{y\in G-F} \overline{d}(F)(y)\Big] = \top.$$

By (MRD4), $\overline{d}(F)(x) = \top$. Hence $x \in \overline{d}(F)$.
The converse implication is trivial.

In this sense, the notion of M-fuzzifying restricted convexly derived operators is a reasonable generalization of restricted convexly derived operator.

Proposition 4.6. *If a mapping $\overline{d} : 2^X_{fin} \longrightarrow M^X$ satisfies* (MRD1)–(MRD3)*, then the following conditions are equivalent.*

(MRD4) $\forall x \notin F$ *and* $\forall F, G \in 2^X_{fin}$, $\overline{d}(G)(x) \wedge \left[\bigwedge_{y\in G-F} \overline{d}(F)(y)\right] \leq \overline{d}(F)(x)$.

(MRD4′) $\forall x \notin F$ *and* $\forall F, G \in 2^X_{fin}$, $\overline{d}(G)(x) \wedge \left[\bigwedge_{y\in G} \overline{d}(F)(y) \vee F(y)\right] \leq \overline{d}(F)(x)$.

(MRD4″) $\forall x \in X$ *and* $\forall F, G \in 2^X_{fin}$, $\overline{d}(G)(x) \wedge \left[\bigwedge_{y\in G} \overline{d}(F)(y) \vee F(y)\right] \leq \overline{d}(F)(x) \vee F(x)$.

Proposition 4.7. *Let $\overline{d} : 2^X_{fin} \longrightarrow M^X$ be an M-fuzzifying restricted convexly derived operator on X. Then the mapping $h_{\overline{d}} : 2^X_{fin} \longrightarrow M^X$ defined by*

$$\forall F \in 2^X_{fin}, h_{\overline{d}}(F) = F \vee \overline{d}(F)$$

is an M-fuzzifying restricted hull operators.

Proof. The verifications of (MH1) and (MH2) are trivial. We only verify (MH3).

Let $x \in X$ and $F, G \in 2^X_{fin}$. By (MRD4″), it is trivial whenever $x \in F \cup G$. Now suppose $x \notin G \cup F$, It follows from (MRD4′) that

$$\begin{aligned} & h_{\overline{d}}(G)(x) \wedge \left[\bigwedge_{y\in G} h_{\overline{d}}(F)(y)\right] \\ &= \overline{d}(G)(x) \wedge \left[\bigwedge_{y\in G} \overline{d}(F)(y) \vee F(y)\right] \\ &\leq \overline{d}(F)(x) \vee F(x) \leq h_{\overline{d}}(F)(x). \end{aligned}$$

The proof is completed. □

Proposition 4.8. *Let $h : 2^X_{fin} \longrightarrow M^X$ be an M-fuzzifying restricted hull operator on X. Then the mapping $\overline{d}_h : 2^X_{fin} \longrightarrow M^X$ defined by*

$$\forall F \in 2^X_{fin}, \overline{d}_h(F)(x) = h(F - \{x\})(x)$$

is an M-fuzzifying restricted convexly derived operator.

Proof. The verifications of (MRD1)–(MRD3) are straightforward, and (MRD4) holds by using Remark 4. □

Theorem 4.9. *M-fuzzifying restricted convexly derived operators are isomorphic to M-fuzzifying restricted hull operators.*

Proof. Let $\overline{d}$ and h be an M-fuzzifying restricted convexly derived operator and an M-fuzzifying restricted hull operator on X, respectively. It suffices to prove (1) $\overline{d}_{h_{\overline{d}}} = \overline{d}$ and (2) $h_{\overline{d}_h} = h$.

(1) For any $x \in X$ and $F \in 2^X_{fin}$, we have

$$\begin{aligned}\overline{d}_{h_{\overline{d}}}(F)(x) &= h_{\overline{d}}(F - \{x\})(x)\\ &= (F - \{x\})(x) \vee \overline{d}(F - \{x\})(x)\\ &= \overline{d}(F - \{x\})(x) = \overline{d}(F)(x).\end{aligned}$$

(2) Let $F \in 2^X_{fin}$. We have

$$\begin{aligned}h_{\overline{d}_h}(F)(x) &= F(x) \vee \overline{d}_h(F)(x)\\ &= F(x) \vee h(F - \{x\})(x).\end{aligned}$$

It is trivial when $x \in F$. Now suppose $x \notin F$. It follows that

$$h_{\overline{d}_h}(F)(x) = h(F - \{x\})(x) = h(F)(x).$$

The proof is completed. □

In [28], it is proved that there is a one-to-one correspondence between M-fuzzifying restricted hull operators and M-fuzzifying convex structures. Thus, by Theorem 4, the following result is trivial.

Theorem 4.10. *M-fuzzifying restricted convexly derived operators is isomorphic to M-fuzzifying convex structures.*

5. Conclusions and remarks for further work

In this paper, we study the derived operators on M-fuzzifying convex spaces, proposing the notions of M-fuzzifying convexly derived operators and M-fuzzifying restricted convexly derived operators. It is then shown that M-fuzzifying convexly derived operators, M-fuzzifying restricted convexly derived operators and M-fuzzifying convex structures are isomorphic. This result fully demonstrates the validity of derived operators to characterize M-fuzzifying convex structures.

The following are some related problems and tasks for further study on this topic.

(1) In [35], the notion of M-fuzzifying difference derived operators is introduced, which can also characterize M-fuzzifying matroids. Can it characterize M-fuzzifying convex structures?

(2) Recently, Shen and Shi provided some characterizations of L-convex structures via domain theory [19]. Is it possible to provide a characterization of L-convex structures in terms of derived operators?

(3) Try to apply derived operators to more set-structures, such as L-convex systems [18], topological hypergroupoids [1], topological convex structures [30], etc.

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