Fuzzy Sets and Systems 492 (2024) 109070

Contents lists available at ScienceDirect

Fuzzy Sets and Systems

journal homepage: www.elsevier.com/locate/fss

Scott quasi-metric and Scott quasi-uniformity based on pointwise quasi-metrics

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ARTICLE INFO

Keywords:
Dcpo
d-space
Fuzzy topology
Pointwise quasi-metric
Quasi-uniformity
Scott topology
Scott quasi-metric

ABSTRACT

In this paper, we develop some connections between pointwise quasi-metric spaces and Scott spaces in domain theory. The main results include (i) the category of Scott quasi-metrics with S-morphisms is equivalent to that of pointwise quasi-metrics in the sense of Shi; (ii) a topological space $(X,\mathcal{T})$ is quasi-metrizable if and only if the topologically generated space $(I^X,\omega_I(\mathcal{T}))$ (where $\omega_I(\mathcal{T})$ denotes the family of all lower semi-continuous mappings from X to the unit interval I) can be induced by a pointwise quasi-metric with a property M; (iii) the notion of Scott quasi-uniformity is presented, and it is shown that d-spaces of domain theory are exactly the Scott quasi-uniformizable spaces; (iv) the relationship between Scott quasi-metrics (introduced by the first and second authors) and Scott quasi-uniformities is established. In specific, the Scott quasi-metrics are exactly the Scott quasi-uniformities that has a countable base.

1. Introduction

Topology and order are two of the three fundamental mathematical structures. The interlinks and interactions between topology and order have drawn a lot of attentions and interests by mathematicians. Domain theory, which arose from logic and computer science, is initiated by Dana Scott in the late 1960s [17,18]. Domain theory is the study of mathematical structures and applications of a special class of partially ordered sets (called domains), which establishes some new bridges between topology and order structures. Dcpos are one of the most well-studied and widely used order structures in domain theory. The Scott topology defined on dcpos plays a crucial role in the theory of non-Hausdorff topological spaces. As an extension of the Scott topology of dcpos to the setting of T_0 spaces, Wyler [26] introduced the notion of d-spaces (also called monotone convergence spaces in [4,7]), where one requires that directed sets in the specialization order have suprema, and that each open set is Scott open with respect to the specialization order.

Another overlapping and fusion between topology and order is the theory of fuzzy topology, or L-topology. Fuzzy topology originates from Chang's work [1] in the case that the truth value is the unit interval. The notion of Chang's fuzzy topology was generalized to lattice-valued topology by Goguen [5,6], which is now called L-topology. The work of finding an appropriate notion of uniformities in the framework of fuzzy topology comes back to Hutton [8]. He managed to establish the theory of uniformities on completely distributive lattices via a family of maps which are extensive and preserve joins. Later, Erceg [3] constructed the theory

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https://doi.org/10.1016/j.fss.2024.109070
Received 7 June 2022; Received in revised form 15 June 2024; Accepted 27 June 2024

Available online 6 July 2024

of fuzzy pseudo-quasi metrics by considering the Hausdorff distance function between two fuzzy sets. As for this, some scholars had made useful researches [11,15].

Nevertheless, Erceg's metric is failed to be first countable, and cannot reflect the relationship between fuzzy points and their remote neighborhoods (or quasi-neighborhoods). To overcome these shortcomings, Shi introduced the theory of pointwise pseudo-metric, which successfully revealed that the set of the r-balls of an L-fuzzy point is exactly its neighborhood base [22,23]. Later, in a direct way, Shi [24] presented the concept of neighborhood system in L-fuzzy topology, and proving that every L-fuzzy topology is completely determined by its neighborhood systems. Motivated by Shi's neighborhood systems on L-topology, Shen et al. [20] presented the notion of closed ball systems in pointwise pseudo-quasi-metric spaces, and showed that it can be used to characterize the fuzzy topology induced by a pointwise pseudo-quasi-metric. Recently, motivated by the pointwise metric theory, Shen and Shi [19] introduced the notion of Scott quasi-metric, and showed that the Scott quasi-metric spaces are exactly the quasi-metrizable d-spaces of domain theory.

In this paper, our aim is to show more connections between pointwise quasi-metric spaces and domain theory. Here is the outline of the paper.

In Section 3, we study the relationship between pointwise quasi-metrics and Scott quasi-metrics. Particularly, it is shown that the category of Scott quasi-metric spaces with S-morphisms is categorically equivalent to that of pointwise quasi-metric spaces with contractive GOHs.

In Section 4, we study the pointwise quasi-metrization of the topologically generated space of a quasi-metric space. It is shown that with a designated property M, a topological space X is quasi-metrizable if and only if the topologically generated space $\omega_I(X)$ (whose open sets are the lower semi-continuous mappings) is pointwise quasi-metrizable.

In Section 5, we introduce the notion of Scott quasi-uniformity on dcpos, and proved that the Scott quasi-uniform spaces are exactly the d-spaces of domain theory. In addition, the relationship between Scott quasi-metrics and Scott quasi-uniformities is studied. In specific, the Scott quasi-metrics are exactly the Scott quasi-uniformities that has a countable base. Their relations are summarized as follows:

$$\begin{tikzcd}
\text{Scott quasi-uniform spaces} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, "\text{(countable base)}"'] & d\text{-spaces} \arrow[d, Rightarrow, "\text{(quasi-metrizable)}"] \\
\text{Scott quasi-metric spaces} \arrow[r, Leftrightarrow] & \text{quasi-metrizable } d\text{-spaces.}
\end{tikzcd}$$

2. Preliminaries

In this section, we review some basic concepts and notations. For more details, the reader can refer to references [4,7] for domain theory, and [2,10] for topology theory.

Let P be a poset. A nonempty subset D of P is *directed* if every two elements of D have an upper bound in D. A poset P is a *directed complete poset*, or *dcpo* for short, if for any directed subset $D \subseteq P$, the supremum of D, denoted by $\bigvee D$, exists.

For any subset A of a poset P, we use the following standard notations:

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\};\ \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

In particular, for any $x \in X$, we write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. We call A a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$).

For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P for which $\bigvee D$ exists, $y \leqslant \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$. A continuous dcpo is also called a *domain*.

A subset U of a poset P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, and we call this topology the *Scott topology* on P and denote it by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P. The family of all Scott closed subset of P is denoted by $\sigma^{op}(P)$.

For a T_0 space X, we use the notation $\mathcal{T}_X$ to denote the topology of X. The *specialization order* $\leq$ of X is defined by $x \leq y$ if and only if for each $U \in \mathcal{T}_X$, $y \in U$ whenever $x \in U$. It is easy to check that $\mathrm{cl}(\{y\}) = \downarrow y$, where cl denotes the closure operator of X. In the following, when we consider a T_0 space X as a poset, it is always equipped with the specialization order.

Remark 2.1 *([4,7]).*

(1) For a poset $(P, \leq_P)$, the specialization order of ΣP is exactly $\leq_P$.
(2) If P is a continuous poset, then $\{\twoheaduparrow x : x \in P\}$ forms a base for $\sigma(P)$.
(3) In a T_0 space X, every open set is upper, and every closed set is lower. In addition, $x \leq y$ if and only if every open neighborhood of x contains y.

Proposition 2.2 *([4,7]). Let $f : P \longrightarrow Q$ be a mapping between dcpos P and Q. Then f is continuous with respect to the Scott topology (konwn as Scott continuous) if and only if f is monotone and for each directed subset D of P, $f(\bigvee D) = \bigvee f(D)$.*

Definition 2.3 *([7]).* A *quasi-metric* d on a nonempty set X is a mapping $d : X \times X \to [0, +\infty)$ satisfying the following conditions: $\forall x, y, z \in X$,

(QM1) $d(x,x)=0$;
(QM2) $d(x,z)\le d(x,y)+d(y,z)$;
(QM3) $d(x,y)=d(y,x)=0$ implies $x=y$.

The pair (X,d) is called a *quasi-metric space* if d is a quasi-metric on X.

Every quasi-metric can induce a topology naturally.

Definition 2.4 *([7]).* Let (X,d) be a hemi-metric space. The *open ball* $B_r(x)$ with center x and radius $r>0$, is defined by

$$B_r(x)=\{y\in X : d(x,y)<r\}.$$

The *open ball topology* of d, denoted by $\mathcal{T}_d(X)$ (or simply, $\mathcal{T}_d$), is the topology generated by the open balls.

Remark 2.5. [7, Lemma 6.1.9] Let (X,d) be a quasi-metric space. Then $(X,\mathcal{T}_d)$ is a T_0 space, and its specialization order $\le_d$ is given by

$$x\le_d y \Leftrightarrow d(x,y)=0.$$

The following notion is motivated by the pointwise quasi-metrics for fuzzy topology [22,23], which provides a characterization for quasi-metrizable d-spaces.

Definition 2.6 *([19]).* Let P be a poset. A mapping $d : P\times P\longrightarrow[0,+\infty)$ is called a *Scott quasi-metric* on P if it satisfies the following conditions: $\forall a,b,c\in P$,

(SM1) $d(a,a)=0$;
(SM2) $d(a,b)\le d(a,c)+d(c,b)$;
(SM3) for any directed subset D of P in which $\bigvee D$ exists,

$$d(a,\bigvee D)=\bigwedge_{c\in D} d(a,c);$$

(SM4) $d(a,b)=0$ implies $a\le b$.

The pair (P,d) is called a *Scott quasi-metric space* if P is a dcpo and d is a *Scott quasi-metric* on P.

Remark 2.7 *([19]).*

(1) Every Scott quasi-metric is a quasi-metric.
(2) If d is a Scott quasi-metric on P, then for any $a,b\in P$, $d(a,b)=0$ if and only if $a\le b$.

Theorem 2.8 *([19]). Let d be a Scott quasi-metric on $(P,\le)$. Then $\mathcal{T}_d\subseteq\sigma(P)$ and the specialization order of $\mathcal{T}_d$ is $\le$.*

Definition 2.9 *([4,7,26]).* A T_0 space X is called a *d-space*, if X is a dcpo and every open subset of X is Scott open in the specialization order.

A topological space X is *quasi-metrizable* (resp., *Scott quasi-metrizable*) if there is a quasi-metric (resp., Scott quasi-metric) d on X such that $\mathcal{T}_X=\mathcal{T}_d$.

Theorem 2.10 *([19]). The quasi-metrizable d-spaces are exactly the Scott quasi-metrizable spaces.*

3. The categorical relationship between Scott quasi-metrics and pointwise quasi-metrics

To capture the characteristics of pointwise topology, Shi establishes a theory of pointwise metric on a completely distributive lattice [21]. In this section, we demonstrate that the categories of Scott quasi-metric spaces and pointwise quasi-metric spaces are equivalent.

An element p of a poset P is called a *co-prime element* if $a\le x\vee y$ implies either $p\le x$ or $p\le y$ whenever $x\vee y$ exists. We use $J(P)$ to denote the set of all co-prime elements of P excluding the bottom element of P (if it exists). A complete lattice L is called *completely distributive* if for any $x\in L$, it holds that $x=\bigvee\Downarrow x\cap J(L)$ [4,16].

Remark 3.1 *([4,16]).* Let L be a completely distributive lattice.

(1) For any $p \in J(L)$ and $x \in L$, $p \ll x$ if and only if for each $S \subseteq L$ with $x \leq \bigvee S$, we have that $p \leq y$ for some $y \in S$.
(2) For any $p, q \in J(L)$, $p \ll q$ in $J(L)$ with the inherited order if and only if $p \ll q$ in L. For this reason, there is no need to distinguish that $p \ll q$ in $J(L)$ or in L.
(3) For any $p \in J(L)$ and $x \in L$, if $p \ll x$, then there exists $q \in J(L)$ such that $p \ll q \leq x$.
(4) $J(L)$ is a continuous dcpo such that $(\sigma^{op}(J(L)), \subseteq)$ is order-isomorphic to L. Hence, $\twoheaddownarrow p \cap J(L)$ is directed whenever $p \in J(L)$.
(5) If P is a continuous dcpo, then $(\sigma^{op}(P), \subseteq)$ is a completely distributive lattice such that $J(\sigma^{op}(P), \subseteq) = \{\downarrow x : x \in P\}$, which is order-isomorphic to P.

Definition 3.2 *([21]).* Let L be a completely distributive lattice, and $d : J(L) \times J(L) \longrightarrow [0, +\infty)$ be a mapping. Then d is called a *pointwise quasi-metric on L* if d is a Scott quasi-metric on $J(L)$. In this case, the pair (L, d) is called a *pointwise quasi-metric space*.

In this section, when we say the pair (P, d) an S-metric space, it means that P is a "continuous" dcpo and d is a Scott quasi-metric on P.

By Remark 3.1 and Definition 3.2, the following result is immediate.

Lemma 3.3. *The pair (L, d) is a pointwise quasi-metric space if and only if $(J(L), d)$ is a Scott quasi-metric space.*

Definition 3.4 *([4,7]).* Let P and Q be two posets. A pair (f, g) of mappings $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ is a *Galois adjunction* between P and Q if f and g are order-preserving and satisfies the following condition: $\forall x \in P$, $\forall y \in Q$,

$$f(x) \leq y \Leftrightarrow x \leq g(y).$$

In a Galois adjunction (f, g), the mapping f is called the *left adjoint* and g the *right adjoint*.

Remark 3.5 *([4]).* Let $f : L \longrightarrow M$ be a mapping between complete lattices L and M. Then, f has a right adjoint if and only if f is sup-preserving, i.e., $f(\bigvee S) = \bigvee f(S)$ for each $S \subseteq L$.

We consider the following type of mappings, which is introduced in [25].

Definition 3.6 *([25]).* Let L and M be two completely distributive lattices, and $f : L \longrightarrow M$ be a mapping. Then f is called a *generalized order-homomorphism* (*GOH* for short), if f has a right adjoint that is sup-preserving, i.e., it satisfies the following conditions:

(G1) f is sup-preserving;
(G2) g is sup-preserving, where g is the right ajoint of f defined by $g(b) = \bigvee \{a \in L : f(a) \leq b\}$ for any $b \in M$.

Lemma 3.7. *Let L and M be two completely distributive lattices, and $f : L \longrightarrow M$ be a GOH. Then $f(a) \in J(M)$ for each $a \in J(L)$.*

Proof. Let $a \in J(L)$. Then for any $b, c \in M$, if $f(a) \leq b \vee c$, then $a \leq g(b \vee c) = g(b) \vee g(c)$, which implies that $a \leq g(b)$ or $a \leq g(c)$, so $f(a) \leq b$ or $f(a) \leq c$. Therefore, $f(a) \in J(L)$. $\square$

Definition 3.8. Let (L, d) and (M, e) be pointwise quasi-metric spaces. A GOH $f : L \longrightarrow M$ is called *contractive* if

$$e(f(a), f(b)) \leq d(a, b)$$

for any $a, b \in J(L)$.

By Lemma 3.7, Definition 3.8 is well-defined. We use

P-QMet

to denote the category of pointwise quasi-metric spaces with contractive GOHs. To further study the relationship between pointwise quasi-metrics and Scott quasi-metrics, we consider the following type of mappings.

Definition 3.9. Let (P, d) and (Q, e) be two Scott quasi-metric spaces. A mapping $f : P \longrightarrow Q$ is called

(1) 1-*Lipschitz* if for any $x, y \in P$, $e(f(x), f(y)) \leq d(x, y)$.
(2) *way-below preserving* (*WP* for short) if $f(x) \ll f(y)$ whenever $x \ll y$.

We call $f : (P, d) \longrightarrow (Q, e)$ an *S-morphism* if (P, d) and (Q, e) are Scott quasi-metric spaces, and $f : P \longrightarrow Q$ is a 1-Lipschitz and WP mapping.

Remark 3.10. It is clear that every S-morphism is continuous with respect to the open ball topology.

In the following, we use

S-QMet

to denote the category of Scott quasi-metric spaces with S-morphisms.

Proposition 3.11. *Let (P,d) and (Q,e) be two Scott quasi-metric spaces. If $f:(P,\mathcal{T}_d)\longrightarrow(Q,\mathcal{T}_e)$ is a continuous mapping, then f is Scott continuous.*

Proof. Note that every continuous mapping is monotone with respect to the specialization order (see [7, Proposition 4.3.9]). Then by Theorem 2.8, it suffices to prove $f(\bigvee D)\leq\bigvee f(D)$ for any directed subset D. Since $f:(P,\mathcal{T}_d)\longrightarrow(Q,\mathcal{T}_e)$ is continuous, we have $\bigvee D\in f^{-1}(B_r^e(f(\bigvee D)))\in\mathcal{T}_d$ for any $r>0$, and since $\mathcal{T}_d\subseteq\sigma(P)$ by Theorem 2.8, we have $\bigvee D\in f^{-1}(B_r^e(f(\bigvee D)))\in\sigma(P)$. Then there exists $x_r\in D\cap f^{-1}(B_r^e(f(\bigvee D)))\neq\emptyset$, which follows that $e(f(\bigvee D),f(x_r))<r$. It follows that

$$e(f(\bigvee D),\bigvee f(D))=\bigwedge_{x\in D}(f(\bigvee D),f(x))\leq\bigwedge_{r>0}(f(\bigvee D),f(x_r))\leq\bigwedge_{r>0}r=0.$$

This implies $f(\bigvee D)\leq\bigvee f(D)$. $\square$

As a corollary of Remark 3.10 and Proposition 3.11, the following result is clear.

Corollary 3.12. *Every S-morphism is Scott continuous.*

Lemma 3.13. *If $f:(L,d)\longrightarrow(M,e)$ is a contractive GOH, then the restriction $f^*:(J(L),d)\longrightarrow(J(M),e)$ of f is an S-morphism.*

Proof. First, we have that $(J(L),d)$ and $(J(M),e)$ are Scott quasi-metric spaces by Lemma 3.3, and the restriction $f^*:J(L)\longrightarrow J(M)$ of f is well-defined by Lemma 3.7. In addition, since f is a GOH, its right adjoint g exists, and it is trivial that f^* is 1-Lipschitz and Scott continuous, so it remains to prove that f^* is WP. Suppose $a,b\in J(L)$ such that $a\ll b$. It suffices to prove $f(a)\ll f(b)$ by Remark 3.1. Suppose D is a directed subset of $J(M)$ such that $f(b)\leq\bigvee D$. Then $b\leq g(\bigvee D)=\bigvee\{g(c):c\in D\}$. Since $a\ll b$, there is $c_0\in D$ such that $a\leq g(c_0)$, which follows that $f(a)\leq c_0$. This implies that $f(a)\ll f(b)$. Hence, f^* is WP. Therefore, f^* is an S-morphism. $\square$

By Lemmas 3.3 and 3.13, we obtain a functor F: **P-QMet** $\longrightarrow$ **S-QMet** as follows:

$$\begin{cases}(L,d)\longmapsto(J(L),d)\\ f\qquad\longmapsto f^*.\end{cases}$$

Let (P,d) be a Scott quasi-metric space. Then, by Remark 3.1, $\sigma^{op}(P)$ with the inclusion order is a completely distributive lattice such that

$$J(\sigma^{op}(P))=\{\downarrow x:x\in P\}.$$

Then by Remark 3.1, $J(\sigma^{op}(P))$ is a continuous dcpo that is order-isomorphic to P via the isomorphism $\downarrow x\mapsto x$ for each $x\in P$. Now define $\widehat{d}:J(\sigma^{op}(P))\times J(\sigma^{op}(P))\longrightarrow[0,+\infty)$ as follows: $\forall x,y\in P$,

$$\widehat{d}(\downarrow x,\downarrow y)=d(x,y).$$

Then the following lemma is clear.

Lemma 3.14. *The pair $(\sigma^{op}(P),\widehat{d})$ defined above is a pointwise quasi-metric space.*

Lemma 3.15. *Let L and M be two completely distributive lattices, and $f:J(L)\longrightarrow J(M)$ be a WP mapping. Define $\widehat{f}:L\longrightarrow M$ as follows:*

$$\widehat{f}(x)=\bigvee\{f(y):y\in J(L)\cap\Downarrow x\}$$

for each $x\in L$. Then the following conditions are equivalent:

(1) *f is Scott continuous;*
(2) *$\widehat{f}$ is a GOH that extends f.*

Proof. That (2) ⇒ (1) is trivial. We prove that (1) ⇒ (2) in two steps.

Step 1: $\widehat{f}$ is an extension of f.

For each $x \in J(L)$, $\Downarrow x \cap J(L)$ is directed, and since f is Scott continuous, we have that

$$f(x) = f\left(\bigvee \Downarrow x \cap J(L)\right) = \bigvee\{f(y) : y \in \Downarrow x \cap J(L)\} = \widehat{f}(x).$$

Hence, $\widehat{f}$ is an extension of f.

Step 2: $\widehat{f}$ is a GOH.

First, note that $\widehat{f}$ is an order-preserving mapping since f is. It suffices to check (G1) and (G2):

(G1) For each $A \subseteq L$, we have that

$$\begin{aligned} \widehat{f}(\bigvee A) &= \bigvee\{f(x) : x \in J(L) \cap \Downarrow \bigvee A\} \\ &= \bigvee_{y\in A} \bigvee\{f(x) : x \in J(L) \cap \Downarrow y\} \\ &= \bigvee_{y\in A} \widehat{f}(y) \\ &= \bigvee \widehat{f}(A), \end{aligned}$$

which gives (G1).

(G2) Since $\widehat{f}$ is sup-preserving by (G1), the right adjoint g of $\widehat{f}$ exists. We need to prove that g is sup-preserving. Let $B \subseteq M$. Since g is order-preserving, it follows that $\bigvee g(B) \le g(\bigvee B)$. To prove $g(\bigvee B) \le \bigvee g(B)$, take an arbitrary $x \in J(L)$ such that $x \ll g(\bigvee B)$. By Remark 3.1 there is $y \in J(L)$ such that $x \ll y \le g(\bigvee B)$. Since f is WP, we have that

$$f(x) \ll f(y) = \widehat{f}(y) \le \widehat{f}(g(\bigvee B)) \le \bigvee B.$$

Since $f(x) \in J(M)$, by Remark 3.1 there is $y_0 \in B$ such that $f(x) = \widehat{f}(x) \le y_0$, which follows that $x \le g(y_0) \le \bigvee g(B)$. This implies that $g(\bigvee B) \le \bigvee g(B)$. Therefore, g is sup-preserving. □

Let (P,d) and (Q,e) be Scott quasi-metric spaces. Then, $(\sigma^{op}(P),\widehat{d})$ and $(\sigma^{op}(Q),\widehat{e})$ are pointwise quasi-metric spaces (see Lemma 3.14). For each mapping $f : P \longrightarrow Q$, define $\widehat{f} : \sigma^{op}(P) \longrightarrow \sigma^{op}(Q)$ as follows: $\forall a \in \sigma^{op}(P)$,

$$\widehat{f}(a) = \bigvee\{f(b) : b \in J(\sigma^{op}(P)) \cap \Downarrow a\}.$$

Then by Lemma 3.15, the following result is trivial.

Lemma 3.16. *If f is an S-morphism, then $\widehat{f}$ defined above is a contractive GOH.*

By Lemmas 3.14 and 3.16, we obtain a functor G : **S-QMet** $\longrightarrow$ **P-QMet**

$$\begin{cases} (P,d) \longmapsto (\sigma^{op}(P),\widehat{d}) \\ f \qquad\ \longmapsto \widehat{f}. \end{cases}$$

From the above arguments, the following result is trivial.

Corollary 3.17. *The categories **P-QMet** and **S-QMet** are equivalent.*

4. Embedding quasi-metric spaces to pointwise quasi-metric spaces

The theory of pointwise quasi-metric, introduced by Shi [22,23], has an important position in fuzzy topology. Let X be a nonempty set, and L be a completely distributive lattice. Then the set L^X of all mappings from X to L is also a completely distributive lattice with the pointwise order. Every member of $J(L^X)$ is of the form x_a (usually called fuzzy points), defined by

$$\forall y \in X,\ x_a(y) = \begin{cases} a, & y = x, \\ \bot, & y \neq x, \end{cases}$$

where $x \in X$, $a \in J(L)$, and $\bot$ denotes the least point in L.

Definition 4.1 *([22,23]).* A function $d : J(L^X) \times J(L^X) \longrightarrow [0,+\infty)$ is called a *pointwise quasi-metric* on L^X if d is a quasi-metric on $J(L^X)$ satisfying the following conditions: $\forall x_a, y_b \in J(L^X)$,

(PM1) $d(x_a, y_b) = \bigwedge_{c\in J(L)\cap\Downarrow b} d(x_a, y_c)$;
(PM2) $d(x_a, y_b) = 0$ implies $x = y$ and $a \le b$.

The pair (L^X, d) is called a *pointwise quasi-metric space* if d is a pointwise quasi-metric on L^X.

Remark 4.2. In fact, the set L^X (with the pointwise order) in Definition 4.1 also forms a completely distributive lattice. Therefore, the pointwise quasi-metric defined above is consistent with the one in Definition 3.2.

In the following, we consider the fuzzy topology induced by pointwise quasi-metric spaces. For each $\epsilon > 0$ and $y_b \in J(L^X)$, define

$$\overline{B}^d_\epsilon(y_b) = \bigvee \{x_a \in J(L^X) : d(x_a, y_b) \leq \epsilon\}.$$

Proposition 4.3 *([20]).*

(1) *For any $x_a \in J(L^X)$, $x_a \leq \overline{B}^d_\epsilon(y_b)$ if and only if $d(x_a, y_b) \leq \epsilon$.*
(2) *The L-topology τ_d is induced as follows:*

$$\tau_d = \{\varphi \in L^X : \forall x_a \in J(L^X), x_a \ll \varphi \Rightarrow \exists \epsilon > 0, \overline{B}^d_\epsilon(x_a) \leq \varphi\}.$$

In fuzzy topology theory, every classic topological space can naturally generate a fuzzy topological space, the topologically generated space, which is proposed by Lowen [13]. In this section, we study the relationship between quasi-metrics and pointwise quasi-metrics from the viewpoint of topologically generated space. Results show that every quasi-metric d can induce a pointwise quasi-metric d_I such that the fuzzy topology τ induced by d_I is exactly the fuzzy topology that is topologically generated by $\mathcal{T}_d$. Therefore, we deduce that if a topological space is quasi-metrizable, then its topologically generated space is pointwise quasi-metrizable, as shown in the following:

$$\begin{tikzcd} d \arrow[r, "\text{defined in Theorem 4.7}"] \arrow[d, dashed, "\text{inducing}"'] & d_I \arrow[d, dashed, "\text{inducing}"] \\ (X, \mathcal{T}_d) \arrow[r, "\text{topologically generated}"] & (I^X, \omega_I(\mathcal{T}_d)) \end{tikzcd}.$$

In the following, we always assume that $I = [0, 1]$. Then, it holds that

$$J(I^X) = \{x_r : x \in X, r \in (0, 1]\}.$$

For any $x_r \in J(I^X)$ and $\varphi \in I^X$, we should note that

$$x_r \ll \varphi \Leftrightarrow r < \varphi(x).$$

To further explore the connection between the (quasi-)metrizations of topological spaces and fuzzy topological spaces, we consider the following condition:

(QM3*) $d(x, y) = 0$ implies $x = y$.

In the subsequent discussion, when we refer to d as a *quasi-metric* (called T_1 *hemi-metric* in [7], as the name suggests, a hemi-metric whose open ball topology is T_1) on X, it always means that d satisfies the conditions (QM1) and (QM2) of Definition 2.3, as well as the additional condition (QM3*).

Proposition 4.4. *Let (X, d) be a quasi-metric space. Define $d_I : J(I^X) \times J(I^X) \longrightarrow [0, \infty)$ as follows: $\forall x_r, y_s \in J(I^X)$,*

$$d_I(x_r, y_s) = d(y, x) + \max\{r - s, 0\}.$$

Then, d_I is a pointwise quasi-metric on I^X that satisfies the following property:

(M) $\forall x, y \in X$, $\bigwedge_{r>0} \bigvee_{s>0} d_I(y_r, x_s) = 0 \Leftrightarrow x = y$.

Proof. We prove this conclusion in three steps.
Step 1: we prove that d_I is a pointwise quasi-metric.
(QM1): For each $x_r \in J(I^X)$, it is clear that $d_I(x_r, x_r) = 0$.
(QM2): Let $x_r, y_s, z_t \in J(I^X)$,

$$\begin{aligned} & d_I(x_r, y_s) + d_I(y_s, z_t) \\ = \; & d(y, x) + \max\{r - s, 0\} + d(z, y) + \max\{s - t, 0\} \\ \geqslant \; & d(z, x) + \max\{r - t, 0\} \\ = \; & d_I(x_r, z_t), \end{aligned}$$

which shows (QM2).

(PM1): Let $x_r, y_s \in J(I^X)$. We have that

$$\begin{aligned}\bigwedge_{0<t<s} d_I(x_r, y_t) &= \bigwedge_{0<t<s}(d(y,x)+\max\{r-t,0\}) \\ &= d(y,x)+\bigwedge_{0<t<s}\max\{r-t,0\} \\ &= d(y,x)+\max\left\{\bigwedge_{0<t<s} r-t,0\right\} \\ &= d(y,x)+\max\{r-s,0\} \\ &= d_I(x_r,y_s),\end{aligned}$$

which shows (PM1).

(PM2): If $d_I(x_r, y_s)=0$, then $d(y,x)=0$ and $\max\{r-s,0\}=0$, which implies $x=y$ and $r\le s$. It follows that $x_r \le y_s$, and thus (PM2) is satisfied.

From (PM2) one can easily deduce condition (QM3). Hence, d_I is a pointwise quasi-metric.

Step 2: we prove that d_I satisfies property M.

In fact, for any $x, y\in X$, we have that

$$\begin{aligned}&\bigwedge_{r>0}\bigvee_{s>0} d_I(y_r, x_s) \\ =\ &\bigwedge_{r>0}\bigvee_{s>0} d(x,y)+\max\{r-s,0\} \\ =\ &d(x,y)+\bigwedge_{r>0}\bigvee_{s>0}\max\{r-s,0\} \\ =\ &d(x,y)+\bigwedge_{r>0} r \\ =\ &d(x,y)+0 \\ =\ &d(x,y),\end{aligned}$$

and this implies that

$$\bigwedge_{r>0}\bigvee_{s>0} d_I(y_r, x_s)=0 \ \Leftrightarrow\ d(x,y)=0 \ \Leftrightarrow\ x=y,$$

completing the proof. □

Definition 4.5 *([12]).* Let $(X,\mathcal{T})$ be a topological space. Let $\omega_I(\mathcal{T})$ denote the family of all lower semi-continuous mappings from $(X,\mathcal{T})$ to I, i.e.,

$$\omega_I(\mathcal{T})=\{\varphi\in I^X : \forall r\in I, \varphi^{-1}((r,1])\in\mathcal{T}\}.$$

Then, $\omega_I(\mathcal{T})$ is a fuzzy topology, and $\omega_I(X)=(X,\omega_I(\mathcal{T}))$ is called *the topologically generated space* of $(X,\mathcal{T})$.

Remark 4.6.

(1) $U\in\mathcal{T}$ if and only if $\chi_U\in\omega_I(\mathcal{T})$, where χ_U maps the elements of U to 1, and others to 0.
(2) Recall that $I=[0,1]$ and the Scott topology $\sigma(I)=\{\emptyset, I\}\cup\{(r,1] : r\ge 0\}$, and hence $\varphi\in\omega_I(\mathcal{T})$ if and only if the mapping $\varphi : (X,\mathcal{T})\longrightarrow \Sigma I$ is continuous.

Now we are in a position of establishing our main result of this section.

Theorem 4.7. *Let $(X,\mathcal{T})$ be a topological space. Then, the following conditions are equivalent:*

(1) *$(X,\mathcal{T})$ is quasi-metrizable;*
(2) *$(I^X,\omega_I(\mathcal{T}))$ is pointwise quasi-metrizable that has property M.*

Proof. (1) ⇒ (2): Suppose d is a quasi-metric on X inducing $\mathcal{T}$. Define $d_I : J(I^X)\times J(I^X)\longrightarrow[0,\infty)$ as follows:

$$d_I(x_r, y_s)=d(y,x)+\max\{r-s,0\}.$$

Then, by Proposition 4.4, d_I is a pointwise quasi-metric on I^X that satisfies property M. Denote by τ the fuzzy topology induced by d_I. Recall that

$$\tau=\{\varphi\in I^X : \forall x_r\in J(I^X), x_r\ll\varphi\Rightarrow \exists\epsilon>0, \overline{B}^{d_I}_{\epsilon}(x_r)\le\varphi\}.$$

We need to prove $\omega_I(\mathcal{T})=\tau$.

(i) We prove $\tau\subseteq\omega_I(\mathcal{T})$.
Let $\varphi\in\tau$. To prove $\varphi\in\omega_I(\mathcal{T})$, it suffices to prove $\varphi^{-1}((r,1])\in\mathcal{T}$ for each $r\in(0,1)$.
Suppose $x\in\varphi^{-1}((r,1])$, i.e., $r<\varphi(x)$. This implies that $x_r\ll\varphi$ in I^X. Since $\varphi\in\tau$, there is $\epsilon>0$ (we may assume $\epsilon<\min\{r,1-r\}$) such that $\overline{B}^{d_I}_{\epsilon}(x_r)\le\varphi$.

Claim 1. $B^d_{\frac{\epsilon}{2}}(x) \subseteq \varphi^{-1}((r,1])$.

Suppose $z \in B^d_{\frac{\epsilon}{2}}(x)$. Then $d(x,z) < \frac{\epsilon}{2}$. Let $t = r + \frac{\epsilon}{2}$. We have that

$$d_I(z_t, x_r) = d(x,z) + \max\{t-r, 0\} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

By Proposition 4.3, it follows that $z_t \leq \overline{B}^{d_I}_{\epsilon}(x_r) \leq \varphi$, so $r < t \leq \varphi(z)$, that is, $z \in \varphi^{-1}((r,1])$. Hence, $B^d_{\epsilon}(x) \subseteq \varphi^{-1}((r,1])$.

From Claim 1, it follows that $\varphi^{-1}((r,1]) \in \mathcal{T}$. Therefore, $\tau \subseteq \omega_I(\mathcal{T})$.

(ii) We prove $\omega_I(\mathcal{T}) \subseteq \tau$.

Let $\varphi \in \omega_I(\mathcal{T})$. Suppose $x_r \in J(I^X)$ such that $x_r \ll \varphi$ (i.e., $0 < r < \varphi(x)$). Then there exists $s > 0$ such that $r < s < \varphi(x)$, which implies that $x \in \varphi^{-1}((s,1]) \in \mathcal{T}$. Since $\mathcal{T}$ is induced by d, there exists $\epsilon > 0$ (we may assume $\epsilon < s - r$) such that $B^d_{\epsilon}(x) \subseteq \varphi^{-1}((s,1])$.

Claim 2. $\overline{B}^{d_I}_{\frac{\epsilon}{2}}(x_r) \leq \varphi$.

Let $z_t \leq \overline{B}^{d_I}_{\frac{\epsilon}{2}}(x_r)$. By Proposition 4.3, we have that

$$d_I(z_t, x_r) = d(x,z) + \max\{t-r, 0\} \leq \frac{\epsilon}{2},$$

which implies that $t - r \leq \frac{\epsilon}{2} < \epsilon < s - r$, so $t < s$, and that $d(x,z) \leq \frac{\epsilon}{2} < \epsilon$. It follows that $z \in B^d_{\epsilon}(x) \subseteq \varphi^{-1}((s,1])$, which implies that $t < s < \varphi(z)$, so $z_t \leq \varphi$. Therefore, $\overline{B}^{d_I}_{\frac{\epsilon}{2}}(x_r) \leq \varphi$.

From Claim 2, it follows that $\varphi \in \tau$. Hence, $\omega_I(\mathcal{T}) \subseteq \tau$.

From above (i) and (ii), we obtain that $\tau = \omega_I(\mathcal{T})$.

(2) ⇒ (1): Suppose d_I is a pointwise quasi-metric inducing $\omega_I(\mathcal{T})$. Define $d : X \times X \longrightarrow [0, +\infty)$ by

$$d(x,y) = \bigwedge_{r>0} \bigvee_{s>0} d_I(y_r, x_s).$$

We first show that d is a quasi-metric on X. Since d_I satisfies property M, it suffices to check that d satisfies the Transitivity. Let $k, l > 0$ such that $d(x,y) < k$ and $d(y,z) < l$. Then, there are $r_1, r_2 > 0$ such that

$$\bigvee_{s>0} d_I(y_{r_1}, x_s) < k \text{ and } \bigvee_{s>0} d_I(z_{r_2}, y_s) < l,$$

and this implies that

$$\begin{aligned} d(x,z) &= \textstyle\bigwedge_{r>0} \bigvee_{s>0} d_I(z_r, x_s) \\ &\leq \textstyle\bigvee_{s>0} d_I(z_{r_2}, x_s) \\ &\leq \textstyle\bigvee_{s>0} d_I(z_{r_2}, y_{r_1}) + d_I(y_{r_1}, x_s) \\ &= \textstyle d_I(z_{r_2}, y_{r_1}) + \bigvee_{s>0} d_I(y_{r_1}, x_s) \\ &\leq \textstyle\bigvee_{s>0} d_I(z_{r_2}, y_s) + \bigvee_{s>0} d_I(y_{r_1}, x_s) \\ &< l + k. \end{aligned}$$

This shows that $d(x,z) \leq d(x,y) + d(y,z)$. Hence, d is a quasi-metric on X.

Next, we prove that the topology $\mathcal{T}_d$ induced by d is exactly $\mathcal{T}$.

(i) We prove $\mathcal{T} \subseteq \mathcal{T}_d$.

Let $U \in \mathcal{T}$ and $x \in U$. Then we have $x_{\frac{1}{2}} \ll \chi_U \in \omega_I(\mathcal{T})$, and since $\omega_I(\mathcal{T})$ is induced by d_I, there exists $\epsilon > 0$ such that $\overline{B}^{d_I}_{\epsilon}(x_{\frac{1}{2}}) \leq \chi_U$.

Claim 3. $B^d_{\epsilon}(x) \subseteq U$.

Let $y \in B^d_{\epsilon}(x)$. Then $d(x,y) = \bigwedge_{r>0} \bigvee_{s>0} d_I(y_r, x_s) < \epsilon$, and there is $r_0 > 0$ such that $\bigvee_{s>0} d_I(y_{r_0}, x_s) < \epsilon$. Specially, $d_I(y_{r_0}, x_{\frac{1}{2}}) < \epsilon$, and by Proposition 4.3, $y_{r_0} \leq \overline{B}^{d_I}_{\epsilon}(x_{\frac{1}{2}}) \leq \chi_U$. Then we have that $0 < r_0 \leq \chi_U(y)$, that is, $\chi_U(y) = 1$, or equivalently, $y \in U$. Hence, $B^d_{\epsilon}(x) \subseteq U$.

From Claim 3, we deduce that U is open in $\mathcal{T}_d$. This shows that $\mathcal{T} \subseteq \mathcal{T}_d$.

(ii) We prove $\mathcal{T}_d \subseteq \mathcal{T}$.

Let $\epsilon > 0$ and $x \in X$. We only need to prove that $B^d_\epsilon(x)$ is open in $\mathcal{T}$. By Remark 4.6 it suffices to prove $\chi_{B^d_\epsilon(x)}$ is in $\omega_I(\mathcal{T})$. Note that $\omega_I(\mathcal{T})$ is induced by d_I. Suppose $y_t \in J(I^X)$ such that $y_t \ll \chi_{B^d_\epsilon(x)}$. It follows that $0 < t < 1$ and $y \in B^d_\epsilon(x)$, so $d(x, y) < \epsilon$. Then we can choose one $\eta > 0$ such that

$$d(x,y) = \bigwedge_{r>0} \bigvee_{s>0} d_I(y_r, x_s) < \eta < \epsilon.$$

Then, there is $r_0 > 0$ such that $\bigvee_{s>0} d_I(y_{r_0}, x_s) < \eta$.

Claim 4. $\overline{B}^{d_I}_{\epsilon-\eta}(y_{r_0}) \le \chi_{B^d_\epsilon(x)}$.

Let $z_k \in J(I^X)$ such that $z_k \le \overline{B}^{d_I}_{\epsilon-\eta}(y_{r_0})$. Then, $d_I(z_k, y_{r_0}) \le \epsilon - \eta$, and we have that

$$\begin{aligned} d(x,z) &= \textstyle\bigwedge_{r>0} \bigvee_{s>0} d_I(z_r, x_s) \\ &\le \textstyle\bigvee_{s>0} d_I(z_k, x_s) \\ &\le \textstyle\bigvee_{s>0} (d_I(z_k, y_{r_0}) + d_I(y_{r_0}, x_s)) \\ &= d_I(z_k, y_{r_0}) + \textstyle\bigvee_{s>0} d_I(y_{r_0}, x_s) \\ &< (\epsilon - \eta) + \eta \\ &= \epsilon. \end{aligned}$$

It follows that $z \in B^d_\epsilon(x)$, which implies that $z_k \le \chi_{B^d_\epsilon(x)}$.
From Claim 4, it follows that $\chi_{B^d_\epsilon(x)}$ is open in the topology induced by d_I, hence is in $\omega_I(\mathcal{T})$. Hence, $\mathcal{T}_d \subseteq \mathcal{T}$.

From above (i) and (ii), we have that $\mathcal{T}_d = \mathcal{T}$. Therefore, $(X, \mathcal{T})$ is quasi-metrizable. □

We end this section by giving an example.

Example 4.8. Let $\mathbb{R}$ be the set of real numbers. There exists a commonly used metric (automatically a quasi-metric) d on $\mathbb{R}$ defined by $\forall x, y \in \mathbb{R}$,

$$d(x,y) = |x - y|.$$

According to Theorem 4.7, the topologically generated space $(\mathbb{R}, \omega_I(\mathcal{T}_{d_\mathbb{R}}))$ can be induced by a pointwise quasi-metric d_I, where $\forall x_r, y_s \in J(I^X)$,

$$d_I(x_r, y_s) = |x - y| + \max\{r - s, 0\}.$$

5. Scott quasi-uniformities on posets

5.1. On quasi-uniformity

Uniform spaces generalize the notion of metric spaces naturally. The non-Hausdorff variant of these spaces is known as quasi-uniform spaces. In this section, we use the concept of neighbornets introduced by Junila [9].

We define a *relation* R on a set X as a mapping $R : X \longrightarrow 2^X$, where 2^X denotes the power set of X. Let R and S be two relations on X.

(1) We write $R \subseteq S$ if $R(x) \subseteq S(x)$ for each $x \in X$.
(2) We define the relations $R \cup S$ and $R \cap S$ by

$$(R \cup S)(x) = R(x) \cup S(x) \quad \text{and} \quad (R \cap S)(x) = R(x) \cap S(x) \quad \text{for each } x \in X.$$

(3) For each $A \subseteq X$, we denote $R(A) = \bigcup\{R(x) : x \in A\}$. Then, the relation $S \circ R$ on X is defined by $(S \circ R)(x) = S(R(x))$ for each $x \in X$.
(4) A relation R is called *reflexive* if $x \in R(x)$ for each $x \in X$.

Definition 5.1 *([2,10])*. A *quasi-uniformity* $\mathscr{U}$ on X is a filter of reflexive relations on X that satisfies for each $R \in \mathscr{U}$ there exists $S \in \mathscr{U}$ such that $S \circ S \subseteq R$. Explicitly, it is a family of reflexive relations on X satisfying the following conditions:

(U1) if $R \in \mathscr{U}$ and S is a relation on X such that $R \subseteq S$, then $S \in \mathscr{U}$;
(U2) if $R, S \in \mathscr{U}$, then $R \cap S \in \mathscr{U}$;
(U3) for each $R \in \mathscr{U}$, there exists $S \in \mathscr{U}$ such that $S \circ S \subseteq R$.

A subfamily $\mathcal{B} \subseteq \mathcal{U}$ is called a *base* for the uniformity $\mathcal{U}$ if for every $R \in \mathcal{U}$ there exists $S \in \mathcal{B}$ such that $S \subseteq R$. Any base $\mathcal{B}$ for a uniformity on a set X has the following properties:

(B1) for each $R \in \mathcal{B}$, $x \in R(x)$;
(B2) if $R, S \in \mathscr{B}$, then $R \cap S \in \mathscr{B}$;
(B3) for each $R \in \mathscr{B}$, there exists $S \in \mathscr{B}$ such that $S \circ S \subseteq R$.

Definition 5.2 *([2,10]).* Every quasi-uniformity $\mathscr{U}$ on X induces a topology, denoted by $\mathcal{T}_{\mathscr{U}}$, which is defined by

$$\mathcal{T}_{\mathscr{U}} = \{O \subseteq X : \forall x \in O, \exists R \in \mathscr{U} \text{ such that } R(x) \subseteq O\}.$$

Remark 5.3 *([2,10]).* Let $\mathscr{U}$ be a quasi-uniformity on X.

(1) If $\mathscr{B}$ is a base for $\mathscr{U}$, then

$$\mathcal{T}_{\mathscr{U}} = \{O \subseteq X : \forall x \in O, \exists R \in \mathscr{B} \text{ such that } R(x) \subseteq O\}.$$

(2) For $R \in \mathscr{U}$, $R(x)$ need not be an open set in $(X, \mathcal{T}_{\mathscr{U}})$.

The following well-known result answers the question that which topological spaces are induced by quasi-uniformity.

Theorem 5.4 *([14]). Every topological space can be induced by a quasi-uniformity.*

5.2. Neigbornets of Scott spaces

The notion of Neighbornets, introduced by Junnila, plays a significant role in the theory of both metric spaces and uniform spaces [9]. The specific definition is as follows:

Definition 5.5. [9] Let X be a topological space.

(1) A relation R on X is called a *neighbornet* of X if $R(x)$ is a neighborhood of x for each $x \in X$.
(2) When R is a relation on X, the relation $\mathring{R}$ is defined by $\mathring{R}(x) = \mathrm{Int}(R(x))$, the interior of $R(x)$, for each $x \in X$.
(3) A neighbornet R of X is called *open* if $R = \mathring{R}$.

Remark 5.6. Let X be a topological space and S, R be two relations on X.

(1) It is trivial to verify that $S \cup R$, $S \cap R$, and $S \circ R$ are neighbornets of X whenever S and R are.
(2) If $S \circ S \subseteq R$, then $\mathring{S} \circ \mathring{S} \subseteq \mathring{R}$.
   – Let $x \in X$. We have that $(\mathring{S} \circ \mathring{S})(x) = \bigcup\{\mathring{S}(y) : y \in \mathring{S}(x)\} \subseteq (S \circ S)(x) \subseteq R(x)$ and $(\mathring{S} \circ \mathring{S})(x)$ is an open subset of X. Thus, $(\mathring{S} \circ \mathring{S})(x) \subseteq \mathring{R}(x)$, which proves (2).

In what follows, if no otherwise specified, P denotes a poset.

Definition 5.7. A quasi-uniformity $\mathscr{U}$ on P is called a *Scott quasi-uniformity* if there exists a base $\mathscr{B}$ for $\mathscr{U}$ such that each member of $\mathscr{B}$ is an open neighbornet of ΣP, and $\mathscr{U}$ satisfies

(T0) for any $x, y \in P$, $y \in \bigcap_{R \in \mathcal{U}} R(x)$ implies $x \leq y$.

We call $(P, \mathscr{U})$ a *Scott quasi-uniform space* if P is a dcpo and $\mathscr{U}$ is a Scott quasi-uniformity on P.

Remark 5.8. It is easy to check that Condition (T0) in Definition 5.7 can be equivalently replaced by each of the following conditions:

(i) $\forall x, y \in P$, $y \in \bigcap_{R \in \mathcal{U}} R(x)$ iff $x \leq y$;
(ii) $\forall x, y \in P$, $y \in \bigcap_{R \in \mathcal{B}} R(x)$ implies $x \leq y$;
(iii) $\forall x, y \in P$, $y \in \bigcap_{R \in \mathcal{B}} R(x)$ iff $x \leq y$.

The following lemma will be used in the sequel.

Lemma 5.9. *Let $\mathscr{U}$ be a Scott quasi-uniformity on P. Then, $P \setminus \downarrow x \in \mathcal{T}_{\mathscr{U}}$ for all $x \in P$.*

Proof. Suppose $\mathscr{B}$ is a base for $\mathscr{U}$ such that each member of $\mathscr{B}$ is a Scott open neighbornet of P. Let $y \in P \setminus \downarrow x$, i.e., $y \nleq x$. By Remark 5.8, there is $R \in \mathscr{B}$ such that $x \notin R(y)$. We claim that $R(y) \subseteq P \setminus \downarrow x$ (since if there exists $z \in R(y)$ such that $z \leq x$, then $x \in \uparrow R(y) = R(y)$, noting that $R(y)$ is Scott open, a contradiction). This implies that $P \setminus \downarrow x \in \mathcal{T}_{\mathscr{U}}$. □

A sequence $\{R_n\}_{n\in\mathbb{N}}$ of neighbornets of X is a *normal sequence* if $R_{n+1} \circ R_{n+1} \subseteq R_n$ for every $n \in \mathbb{N}$. A neighbornet R of X is *normal* if R is a member of some normal sequence of neighbornets of X [9].

Remark 5.10.

(1) Every member of a uniformity is normal by (U3) in Definition 5.1.
(2) By Remark 5.6, for a relation R on X, $\mathring{R}$ is a normal neighbornet whenever R is.

Theorem 5.11 *([9]). The collection of all normal neighbornets of a topological space X forms a quasi-uniformity that contains every other quasi-uniformity of X; this quasi-uniformity is called the* fine quasi-uniformity *of X.*

Lemma 5.12. *Let $(P, \leq_P)$ be a poset, $\mathcal{U}_{Sfine}$ be the fine uniformity of ΣP, and $\mathcal{U}$ be a uniformity on P. Consider the following conditions:*

(1) *$\mathcal{U}$ is a Scott uniformity on P;*
(2) *$\mathcal{U}$ is coarser than $\mathcal{U}_{Sfine}$ and the specialization order of $\mathcal{T}_{\mathcal{U}}$ is exactly $\leq_P$.*
(3) *$\mathcal{T}_{\mathcal{U}}$ is a d-space whose specialization order is $\leq_P$.*

Then, (1) *is equivalent to* (2)*, and either* (1) *or* (2) *implies* (3).

Proof. Denote by $\sqsubseteq$ the specialization order of $\mathcal{T}_{\mathscr{U}}$.

(1) ⇒ (2): From Remark 5.10(1), it follows that $\mathcal{U} \subseteq \mathcal{U}_{\mathrm{Sfine}}$. In addition, if $x \nleq_P y$, then by Lemma 5.9, $P \setminus \downarrow y$ (where $\downarrow y = \{z \in P : z \leq_P y\}$) is an open neighborhood of x that does not contain y, so $x \not\sqsubseteq y$. On the other hand, if $x \not\sqsubseteq y$, then there is an open neighborhood $U \in \mathcal{T}_{\mathscr{U}}$ of x that does not contain y. Then, there exists $R \in \mathscr{B}$ such that $R(x) \subseteq U$, which follows that $y \notin R(x)$, so by Remark 5.8, $x \nleq_P y$. Therefore, the specialization order of $\mathcal{T}_{\mathcal{U}}$ is $\leq_P$.

(2) ⇒ (1): It suffices to show that $\mathcal{U}$ satisfies (T0) of Definition 5.7. Suppose $y \in \bigcap_{R\in\mathcal{U}} R(x)$. We need to verify that $x \leq_P y$, which is equivalent to prove that $x \sqsubseteq y$ by Condition (2). Suppose $U \in \mathcal{T}_{\mathcal{U}}$ such that $x \in U$. Then, there exists $R_0 \in \mathcal{U}$ such that $R_0(x) \subseteq U$. By our assumption, $y \in R_0(x) \subseteq U$. This shows that $x \sqsubseteq y$.

The conclusion of the second part follows straightforwardly. □

Theorem 5.13. *Let X be a T_0 space and $\sqsubseteq$ be the specialization order of X. Then, the following conditions are equivalent:*

(1) *X is a d-space;*
(2) *the fine quasi-uniformity of X is a Scott-uniformity on $(X, \sqsubseteq)$;*
(3) *the Pervin quasi-uniformity of X is a Scott-uniformity on $(X, \sqsubseteq)$;*
(4) *there exists a Scott-uniformity on $(X, \sqsubseteq)$ that induces $\mathcal{T}_X$.*

Proof. We use $\mathcal{U}_{\mathrm{Sfine}}$ to denote the fine uniformity of $\Sigma(X, \sqsubseteq)$.

(1) ⇒ (2): Since X is a d-space, we have that $\mathcal{T}_X \subseteq \sigma(X, \sqsubseteq)$, which implies that $\mathcal{U}_{\mathrm{fine}} \subseteq \mathcal{U}_{\mathrm{Sfine}}$. Moreover, it is clear that the specialization order of $\mathcal{T}_{\mathcal{U}_{\mathrm{fine}}} = \mathcal{T}_X$ is $\sqsubseteq$, and thus $\mathcal{U}_{\mathrm{fine}}$ satisfies Condition (2) of Lemma 5.12. Therefore, $\mathcal{U}_{\mathrm{fine}}$ is a Scott uniformity.

(2) ⇒ (3): Note that the Pervin quasi-uniformity $\mathcal{U}_{\mathrm{Pervin}} \subseteq \mathcal{U}_{\mathrm{fine}} \subseteq \mathcal{U}_{\mathrm{Sfine}}$ and $\mathcal{T}_{\mathcal{U}_{\mathrm{Pervin}}} = \mathcal{T}_X$. Then, by a similar argument to $\mathcal{U}_{\mathrm{fine}}$, we can obtain (3).

That (3) ⇒ (4) and (4) ⇒ (1) are trivial. □

The above result gives the following relationship:

$$d\text{-spaces} = \text{Scott quasi-uniform spaces.}$$

Next, we consider the quasi-metrization of Scott quasi-uniform spaces. One can easily prove the following.

Theorem 5.14. *Let d be a Scott quasi-metric on P, and $R_r(x) = \{y \in P : d(x, y) < r\}$ for each $x \in P$ and $r > 0$. Then $\mathscr{B}_d = \{R_r : r > 0\}$ is a base for a Scott quasi-uniformity, denoted by $\mathscr{U}_d$, such that each member of $\mathscr{B}_d$ is Scott open. We call $\mathscr{U}_d$* the Scott quasi-uniformity induced by *d.*

Theorem 5.15. *Let $\mathscr{U}$ be a Scott quasi-uniformity on P. Then $\mathscr{U}$ can be induced by a Scott quasi-metric if and only if it has a countable base.*

Proof. (⇒): If $\mathcal{U}$ can be induced by an Scott quasi-metric d, then obviously $\{R_{\frac{1}{n}} : n \in \mathbb{N}^+\}$ is a countable base for $\mathcal{U} = \mathcal{U}_d$.

(⇐): Assume $\mathcal{U}$ has a countable base. From topology theory, we can find a base $\mathscr{B} = \{R_n : n \in \mathbb{N}^+\}$ such that each member of $\mathscr{B}$ is basic, and satisfies the following condition (refer to [2, Theorem 8.1.21]): $\forall n \in \mathbb{N}^+$,

$$R_{n+1} \circ R_{n+1} \circ R_{n+1} \subseteq R_n.$$

Define a mapping $\rho : P \times P \longrightarrow [0, +\infty)$ as follows:

$$\rho(x,y) = \begin{cases} 0, & \text{if } y \in \bigcap_{n\in\mathbb{N}^+} R_n(x), \\ \frac{1}{2^n}, & \text{where } y \in R_n(x) \setminus R_{n+1}(x), \text{ otherwise.} \end{cases}$$

Now the desired $d : P \times P \longrightarrow [0, +\infty)$ is defined as follows:

$$d(x,y) = \bigwedge \left\{ \sum_{i=1}^{i=n} \rho(x_{i-1}, x_i) : x_0 = x, x_n = y, n \in \mathbb{N}^+ \right\}.$$

Then from the topology theory (see [2, Theorem8.1.10] for example), we have the following facts:

Fact 1: $\forall x, y \in P$, $\rho(x,y) \le \frac{1}{2^n}$ if and only if $y \in R_n(x)$;
Fact 2: $\forall x, y \in P$, $\frac{1}{2}\rho(x,y) \le d(x,y)$;
Fact 3: d is a pseudo-quasi-metric on P such that $\mathcal{U}_d = \mathcal{U}$.

It remains to verify that d is a Scott quasi-metric on P. By Fact 3, we only need to check (SM3) and (SM4).

(SM3): Let $x \in P$ and D be a directed subset of P in which $\bigvee D$ exists. We verify this condition in two steps.

Step 1: we prove $\rho(x, \bigvee D) = \bigwedge_{z\in D} \rho(x,z)$.

On the one hand, for each $z \in D$ and each $n \in \mathbb{N}^+$, it follows that

$$\begin{aligned} \rho(x,z) \le \frac{1}{2^n} &\Rightarrow z \in R_n(x). && \text{(by Fact 1)} \\ &\Rightarrow \bigvee D \in R_n(x) && \text{(by (B2))} \\ &\Rightarrow \rho(x, \bigvee D) \le \frac{1}{2^n}. && \text{(by Fact 1)} \end{aligned}$$

This shows that $\rho(x, \bigvee D) \le \rho(x,z)$ for all $z \in D$. Hence, $\rho(x, \bigvee D) \le \bigwedge_{z\in D} \rho(x,z)$.

On the other hand, for each $n \in \mathbb{N}^+$, we have that

$$\begin{aligned} \rho(x, \bigvee D) \le \frac{1}{2^n} &\Rightarrow \bigvee D \in R_n(x) && \text{(by Fact 1)} \\ &\Rightarrow \exists z \in D, \text{ s.t. } z \in R_n(x) && \text{(by (B1))} \\ &\Rightarrow \exists z \in D, \text{ s.t. } \rho(x,z) \le \frac{1}{2^n} && \text{(by Fact 1)} \\ &\Rightarrow \bigwedge_{z\in D} \rho(x,z) \le \frac{1}{2^n}. \end{aligned}$$

This shows that $\bigwedge_{z\in D} \rho(x,z) \le \rho(x, \bigvee D)$. Therefore, $\rho(x, \bigvee D) = \bigwedge_{z\in D} \rho(x,z)$.

Step 2: we prove $d(x, \bigvee D) = \bigwedge_{z\in D} d(x,z)$.

In fact, we have that

$$\begin{aligned} & d(x, \bigvee D) \\ = & \bigwedge \left\{ \textstyle\sum_{i=1}^{n} \rho(x_{i-1}, x_i) : x_0 = x, x_n = \bigvee D, x_i \in P \right\} \\ = & \bigwedge \left\{ \textstyle\sum_{i=1}^{n-1} \rho(x_{i-1}, x_i) + \rho(x_{n-1}, \bigvee D) : x_0 = x, x_i \in P \right\} \\ = & \bigwedge \left\{ \textstyle\sum_{i=1}^{n} \rho(x_{i-1}, x_i) + \bigwedge_{z\in D} \rho(x_{n-1}, z) : x_0 = x, x_i \in P \right\} \\ = & \textstyle\bigwedge_{z\in D} \bigwedge \left\{ \sum_{i=1}^{n} \rho(x_{i-1}, x_i) : x_0 = x, x_n = z, x_i \in P \right\} \\ = & \textstyle\bigwedge_{z\in D} d(x,z). \end{aligned}$$

Therefore, d satisfies (SM3).

(SM4): Suppose $d(x,y) = 0$. By Fact 2, it holds that $\rho(x,y) = 0$, which implies that $y \in \bigcap_{R\in\mathscr{B}} R(x) = \bigcap \mathcal{U}$. Since $\mathcal{U}$ satisfies (T0), it follows that $x \le y$. Thus (SM4) holds. □

CRediT authorship contribution statement

Chong Shen: Writing – original draft. **Fu-Gui Shi:** Writing – original draft, Supervision, Methodology. **Xinchao Zhao:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The author thanks the anonymous reviewers for their valuable comments and suggestions. In particular, we greatly appreciate the insights provided by the reviewer regarding the final section, where they introduced relevant knowledge about neighbornets and offered some specific suggestions. Their input significantly enhanced this manuscript.

This project was supported by the National Natural Science Foundation of China (12101313, 12271036).

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