Available online at www.sciencedirect.com

ScienceDirect

Information Sciences 178 (2008) 1141–1151

INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL

www.elsevier.com/locate/ins

L-fuzzy numbers and their properties

Han-Liang Huang *, Fu-Gui Shi

Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China

Received 25 March 2006; received in revised form 27 September 2007; accepted 1 October 2007

Abstract

In this paper, the notions of L-fuzzy convex sets and L-fuzzy numbers are introduced where L is a completely distributive lattice. The notions of $[0,1]$-fuzzy convex sets and $[0,1]$-fuzzy numbers are generalized. Furthermore their properties and characterizations are presented in terms of cut sets of L-fuzzy sets.

Keywords: L-fuzzy set; L-fuzzy convex set; L-fuzzy number; Cut set

1. Introduction

Fuzzy numbers are important in the theory of fuzzy sets [29]. They have been applied in statistics, computer programming, engineering (especially communications), experimental science, and other disciplines (for more details see [1–4,6,7,10,11,13–16,21,22,24,27,28]). The notion takes into account the fact that all phenomena in the physical universe have a degree of inherent uncertainty [30]. However, because previous research on fuzzy convex sets and fuzzy numbers have been based on the $[0,1]$-fuzzy set theory, their applications are limited. Therefore, many concepts in the $[0,1]$-fuzzy set theory have been extended into the L-fuzzy set theory [9]. It is known that the fuzzy distance in the $[0,1]$-fuzzy set theory is a $[0,1]$-fuzzy number. In [12], the notion of a metric was generalized to the $[0,1]$-fuzzy set theory by setting the distance between two fuzzy points to be a non-negative fuzzy number. In the matroid theory, a matroid can be characterized by its rank function. Thus, some questions naturally arise. In the L-fuzzy set theory, how do we define the distance between two L-fuzzy points? In the L-fuzzy matroid theory, how is the rank function of an L-fuzzy matroid defined? For an L-fuzzy set, what is its cardinality? In order to answer these questions, we introduce and research L-fuzzy numbers when L is a more general lattice.

In this paper, we generalize the notions of fuzzy convex sets and fuzzy numbers to the L-fuzzy set theory when L is a completely distributive lattice. We present some of their equivalent characterizations in terms of

* Corresponding author.
E-mail addresses: hl_huang1980.student@sina.com, h_l_huang1980@eyou.com (H.-L. Huang), fuguishi@bit.edu.cn (F.-G. Shi).

0020-0255/$ - see front matter
doi:10.1016/j.ins.2007.10.001

cut sets of L-fuzzy sets similar to [17]. Moreover, we discuss their properties and introduce and discuss the cardinality of L-fuzzy sets as an application.

The paper is organized as follows: In Section 3, we discuss L-fuzzy convex sets and their characterizations. In Section 4, we present some characterizations and properties of L-fuzzy numbers. In Section 5, we concentrate on the cardinality of L-fuzzy sets.

2. Preliminaries

Throughout this paper, L denotes a completely distributive lattice. X is a non-empty set. L^X is the set of all L-fuzzy sets (or L-sets for short) over X. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X respectively. We often do not distinguish a crisp subset A of X from its characteristic function χ_A.

An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. a in L is called co-prime if a' is prime [8]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [5]. $\{a \in L | a \prec b\}$ denoted by $\beta(b)$ is called the greatest minimal family of b in the sense of [23]. Let $\beta^*(b) = \beta(b) \cap M(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L | a \prec^{op} b\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$. In a completely distributive lattice L, there exists $\alpha(b)$ and $\beta(b)$ for each $b \in L$, and $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ (see [23]).

In [23], Wang noted that $\beta(0) = \{0\}$ and $\alpha(1) = \{1\}$. In fact, it should be $\beta(0) = \emptyset$ and $\alpha(1) = \emptyset$.

Theorem 2.1 [23]. *Let L be a completely distributive lattice and $\{a_i | i \in \Omega\} \subseteq L$. Then*

(1) $\alpha(\bigvee_{i\in\Omega} a_i) = \bigcup_{i\in\Omega} \alpha(a_i)$, i.e., α is an $\bigwedge$–$\bigcup$ *mapping.*
(2) $\beta(\bigvee_{i\in\Omega} a_i) = \bigcup_{i\in\Omega} \beta(a_i)$, *i.e., β is an union-preserving mapping.*

Definition 2.2 [17–20,26]. Let $A \in L^X$ and $a \in L$, we define

$$A_{[a]} = \{x \in X | A(x) \geqslant a\}, \qquad A_{(a)} = \{x \in X | a \in \beta(A(x))\},$$
$$A^{[a]} = \{x \in X | a \notin \alpha(A(x))\}, \quad A^{(a)} = \{x \in X | A(x) \nleqslant a\}.$$

It is obvious that $a \in \beta(b)$ implies $A_{[b]} \subset A_{(a)} \subset A_{[a]}$ and, $a \in \alpha(b)$ implies $A^{[a]} \subset A^{(b)} \subset A^{[b]}$. In particular, when $L = [0, 1]$, it is easy to verify that $A_{[a]} = A^{[a]}$ and $A_{(a)} = A^{(a)}$.

For $a \in L$ and $D \subset X$, two L-fuzzy sets $a \wedge D$ and $a \vee D$ are defined as

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D \end{cases} \quad \text{and} \quad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases}$$

The proof of the following theorem was given in [17]. Here we briefly recall this proof.

Theorem 2.3. *For each L-fuzzy set A in L^X, we have*:

(1) $A = \bigvee_{a\in L}(a \wedge A_{[a]}) = \bigvee_{a\in M(L)} (a \wedge A_{[a]}) = \bigvee_{a\in L}(a \wedge A_{(a)}) = \bigvee_{a\in M(L)}(a \wedge A_{(a)})$.
(2) $A = \bigwedge_{a\in L}(a \vee A^{[a]}) = \bigwedge_{a\in P(L)} (a \vee A^{[a]}) = \bigwedge_{a\in L}(a \vee A^{(a)}) = \bigwedge_{a\in P(L)}(a \vee A^{(a)})$.
(3) $\forall a \in L$, $A_{[a]} = \bigcap_{b\in\beta(a)} A_{[b]} = \bigcap_{b\in\beta(a)} A_{(b)}$.
(4) $\forall a \in L$, $A_{(a)} = \bigcup_{a\in\beta(b)} A_{[b]} = \bigcup_{a\in\beta(b)} A_{(b)}$.
(5) $\forall a \in L$, $A^{[a]} = \cap_{a\in\alpha(b)} A^{[b]} = \cap_{a\in\alpha(b)} A^{(b)}$.
(6) $\forall a \in L$, $A^{(a)} = \bigcup_{b\in\alpha(a)} A^{[b]} = \bigcup_{b\in\alpha(a)} A^{(b)}$.

Proof. We only give the proof of (1)–(3). The others are analogous.

(1) $$A(x) = \bigvee\{a \in M(L) | a \leqslant A(x)\}, \\ = \bigvee\{a \in M(L) | A_{[a]}(x) = 1\}, \\ = \bigvee_{a \in M(L)} (a \wedge A_{[a]}(x)),$$

$$A(x) = \bigvee\{a \in M(L) | a \in \beta^*(A(x))\}, \\ = \bigvee\{a \in M(L) | A_{(a)}(x) = 1\}, \\ = \bigvee_{a \in M(L)} (a \wedge A_{(a)}(x)).$$

The other equalities of (1) can be proven analogously.

(2) $$A(x) = \bigwedge\{a \in P(L) | a \in \alpha^*(A(x))\}, \\ = \bigwedge\{a \in P(L) | A^{[a]}(x) = 0\}, \\ = \bigwedge_{a \in P(L)} (a \vee A^{[a]}(x)),$$

$$A(x) = \bigwedge\{a \in P(L) | A(x) \leqslant a\}, \\ = \bigwedge\{a \in P(L) | A^{(a)}(x) = 0\}, \\ = \bigwedge_{a \in P(L)} (a \vee A^{(a)}(x)).$$

Analogously, we can prove the other equalities of (2).

(3) For each $b \in \beta(a)$, by noting that $A_{[a]} \subset A_{(b)} \subset A_{[b]}$ we can obtain

$$A_{[a]} \subset \bigcap_{b \in \beta(a)} A_{(b)} \subset \bigcap_{b \in \beta(a)} A_{[b]}.$$

On the other hand, we have

$$x \in \bigcap_{b \in \beta(a)} A_{[b]} \Rightarrow \forall b \in \beta(a), A(x) \geqslant b \\ \Rightarrow A(x) \geqslant \bigvee\{b \in L | b \in \beta(a)\} = a \\ \Rightarrow x \in A_{[a]}.$$

Thus

$$A_{[a]} = \bigcap_{b \in \beta(a)} A_{(b)} = \bigcap_{b \in \beta(a)} A_{[b]}. \qquad \square$$

3. *L*-fuzzy convex set and its characterizations

The collection of all L-fuzzy sets over the set $\mathbb{R}$ of all real numbers is denoted by $L^{\mathbb{R}}$.

Definition 3.1. $A \in L^{\mathbb{R}}$ is called an L-fuzzy convex set if $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0, 1]$,

$$A(\lambda x + (1 - \lambda)y) \geqslant A(x) \wedge A(y).$$

Theorem 3.2. *If $A \in L^{\mathbb{R}}$, then the following conditions are equivalent*:

(1) *A is an L-fuzzy convex set*;
(2) $\forall a \in L, A_{[a]}$ *is a convex set*;
(3) $\forall a \in M(L), A_{[a]}$ *is a convex set*;
(4) $\forall a \in L, A^{[a]}$ *is a convex set*;
(5) $\forall a \in P(L), A^{[a]}$ *is a convex set*;
(6) $\forall a \in P(L), A^{(a)}$ *is a convex set.*

Proof. We only give the proof of (1) $\Longleftrightarrow$ (3), (1) $\Longleftrightarrow$ (4), and (1) $\Longleftrightarrow$ (6) because the others are analogous.

(1) $\Rightarrow$ (3) Suppose A is an L-fuzzy convex set. $\forall a \in M(L)$, let $x, y \in A_{[a]}$, then

$$A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y) \geqslant a.$$

This implies $\lambda x + (1-\lambda)y \in A_{[a]}$. Therefore $A_{[a]}$ is a convex set.

(3) $\Rightarrow$ (1) Assume that $\forall a \in M(L)$, $A_{[a]}$ is a convex set. For any $x, y \in \mathbb{R}$, and any $a \in M(L)$ with $a \leqslant A(x) \wedge A(y)$, we have $x \in A_{[a]}$ and $y \in A_{[a]}$. By convexity of $A_{[a]}$ we know that $\forall \lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in A_{[a]}$ which shows $A(\lambda x + (1-\lambda)y) \geqslant a$. Further we obtain

$$A(\lambda x + (1-\lambda)y) \geqslant \bigvee\{a \in M(L) | a \leqslant A(x) \wedge A(y)\} = A(x) \wedge A(y).$$

Therefore A is an L-fuzzy convex set.

(1) $\Rightarrow$ (4) Assume that A is an L-fuzzy convex set. $\forall a \in L$, if $A^{[a]} = \emptyset$, then it is a convex set. Suppose $A^{[a]} \neq \emptyset$. Let $x, y \in A^{[a]}$, then $a \notin \alpha(A(x))$ and $a \notin \alpha(A(y))$. Since α is an $\bigwedge$–$\bigcup$ mapping, we obtain

$$a \notin \alpha(A(x)) \cup \alpha(A(y)) = \alpha(A(x) \wedge A(y)).$$

By

$$A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y),$$

we know that $a \notin \alpha(A(\lambda x + (1-\lambda)y))$ which implies $\lambda x + (1-\lambda)y \in A^{[a]}$. Therefore $A^{[a]}$ is a convex set.

(4) $\Rightarrow$ (1) Assume that $\forall a \in L$, $A^{[a]}$ is a convex set. $\forall a \in L$, if $a \notin \alpha(A(x) \wedge A(y))$, then by the following equality

$$\alpha(A(x) \wedge A(y)) = \alpha(A(x)) \cup \alpha(A(y)),$$

we obtain $a \notin \alpha(A(x)) \cup \alpha(A(y))$. This implies that $a \notin \alpha(A(x))$ and $a \notin \alpha(A(y))$, that is, $x, y \in A^{[a]}$. By convexity of $A^{[a]}$ we know that $\lambda x + (1-\lambda)y \in A^{[a]}$ which means $a \notin \alpha(A(\lambda x + (1-\lambda)y))$. Therefore

$$A(\lambda x + (1-\lambda)y) = \bigwedge \alpha(A(\lambda x + (1-\lambda)y)) \geqslant \bigwedge \alpha(A(x) \wedge A(y)) = A(x) \wedge A(y).$$

This shows that A is an L-fuzzy convex set.

(1) $\Rightarrow$ (6) Assume that A is an L-fuzzy convex set. $\forall a \in L$, if $A^{(a)} = \emptyset$, then it is a convex set. Now we suppose $A^{(a)} \neq \emptyset$. Take $x, y \in A^{(a)}$, then $A(x) \nleqslant a$ and $A(y) \nleqslant a$. Since a is prime, we have $A(x) \wedge A(y) \nleqslant a$. By

$$A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y),$$

we know $A(\lambda x + (1-\lambda)y) \nleqslant a$. This shows that $\lambda x + (1-\lambda)y \in A^{(a)}$. Therefore $A^{(a)}$ is a convex set.

(6) $\Rightarrow$ (1) Assume that $\forall a \in P(L)$, $A^{(a)}$ is a convex set, that is, $\forall x, y \in A^{(a)}$, $\lambda x + (1-\lambda)y \in A^{(a)}$. Suppose $A(x) \wedge A(y) \nleqslant a$. Then $A(x) \nleqslant a$ and $A(y) \nleqslant a$, that is, $x, y \in A^{(a)}$. Hence $\lambda x + (1-\lambda)y \in A^{(a)}$, that is, $A(\lambda x + (1-\lambda)y) \nleqslant a$. This shows that $A(x) \wedge A(y) \nleqslant a$ implies $A(\lambda x + (1-\lambda)y) \nleqslant a$. So we obtain

$$A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y).$$

This proves that A is an L-fuzzy convex set. $\square$

Theorem 3.3. *Let $A \in L^{\mathbb{R}}$. If for any $a,b \in L$, $\beta(a \wedge b) = \beta(a) \cap \beta(b)$, then the following conditions are equivalent*:

(1) *A is an L-fuzzy convex set*;
(2) $\forall a \in L$, $A_{(a)}$ *is a convex set*;
(3) $\forall a \in M(L)$, $A_{(a)}$ *is a convex set.*

Proof. (1) $\Rightarrow$ (2) Assume that A is an L-fuzzy convex set. $\forall a \in L$, if $A_{(a)} = \emptyset$, then it is a convex set. Suppose $A_{(a)} \neq \emptyset$. Take $x, y \in A_{(a)}$. Then $a \in \beta(A(x))$ and $a \in \beta(A(y))$, that is, $a \in \beta(A(x)) \cap \beta(A(y))$. By $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ we have $a \in \beta(A(x) \wedge A(y))$. From Theorem 2.1(2) and

$$A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y),$$

we can obtain $a \in \beta(A(\lambda x + (1-\lambda)y))$, that is, $\lambda x + (1-\lambda)y \in A_{(a)}$. Therefore $A_{(a)}$ is a convex set.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Assume that $\forall a \in M(L)$, $A_{(a)}$ is a convex set. If $A(x) \wedge A(y) = 0$, then obviously $A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y)$. Suppose $A(x) \wedge A(y) \neq 0$, then $\beta(A(x) \wedge A(y)) \neq \emptyset$. Take $a \in \beta(A(x) \wedge A(y))$. By $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ we have $a \in \beta(A(x))$ and $a \in \beta(A(y))$, that is, $x, y \in A_{(a)}$. Hence $\lambda x + (1-\lambda)y \in A_{(a)}$ which implies $a \in \beta(A(\lambda x + (1-\lambda)y))$. Therefore

$$A(\lambda x + (1-\lambda)y) = \bigvee \beta(A(\lambda x + (1-\lambda)y)) \geqslant \bigvee \beta(A(x) \wedge A(y)) = A(x) \wedge A(y).$$

This proves that A is an L-fuzzy convex set. $\square$

Remark 3.4. In general, if $\beta(a \wedge b) \neq \beta(a) \cap \beta(b)$, then (1) and (2) in Theorem 3.3 are not equivalent. This can be seen from the following example.

Example 3.5. Let $L = [0, 0.5] \cup \{a, b, 1\}$, $0.5 < a < 1, 0.5 < b < 1, a \nleqslant b, b \nleqslant a$. By

$$\beta(a \wedge b) = \beta(0.5) = [0, 0.5), \quad \beta(a) \cap \beta(b) = \{[0, 0.5] \cup \{a\}\} \cap \{[0, 0.5] \cup \{b\}\} = [0, 0.5],$$

we know that $\beta(a \wedge b) \neq \beta(a) \cap \beta(b)$. Define $A \in L^{\mathbb{R}}$ by setting

$$A(x) = \begin{cases} a, & x \leqslant -1, \\ 0.5, & x \in (-1, 1), \\ b, & x \geqslant 1. \end{cases}$$

It is easy to verify that A is an L-fuzzy convex set, and $-1 \in A_{(0.5)}$, $1 \in A_{(0.5)}$. But for all $x \in (-1, 1)$, $x \notin A_{(0.5)}$. Therefore $A_{(0.5)}$ is not a convex set.

4. *L*-fuzzy number and its characterizations

Definition 4.1. $A \in L^{\mathbb{R}}$ is called an L-fuzzy number if

(1) A is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $A(x_0) = 1$;
(2) $\forall a \in L$, $A_{[a]}$ is a closed interval.

An L-fuzzy number is an extension of a fuzzy number. Since many concepts in the $[0,1]$-fuzzy set theory have been extended into the L-fuzzy set theory, there are some questions that naturally have arisen. In the L-fuzzy set theory, how do we define the distance between two L-fuzzy points? In the L-fuzzy matroid theory, how is the rank function of an L-fuzzy matroid defined? For an L-fuzzy set, what is its cardinality? In order to answer these questions, we need to introduce and research L-fuzzy numbers. Therefore it is necessary to generalize the notions of fuzzy numbers in the $[0,1]$-fuzzy set theory to the L-fuzzy set theory.

To this end, we shall use $LF^*(\mathbb{R})$ to denote the set of all L-fuzzy numbers. Some operations of fuzzy numbers can be generalized as follows:

Definition 4.2. Assume that $*$ is a binary operation. For $A, B \in L^{\mathbb{R}}$, define an extension of $*$ as

$$(A * B)(z) = \bigvee_{z = x * y} (A(x) \wedge B(y)).$$

If the operation $*$ is substituted by the operations "+", "−", "×" and "÷" respectively, then we can obtain the following operations:

$$(A+B)(z) = \bigvee_{z=x+y} (A(x) \wedge B(y));$$
$$(A-B)(z) = \bigvee_{z=x-y} (A(x) \wedge B(y));$$
$$(A\times B)(z) = \bigvee_{z=x\times y} (A(x) \wedge B(y));$$
$$(A\div B)(z) = \bigvee_{z=x\div y} (A(x) \wedge B(y)).$$

Theorem 4.3. *If $A, B \in L^{\mathbb{R}}$, then*

(1) $\forall a \in L$, $(A+B)_{(a)} \subset A_{(a)} + B_{(a)} \subset A_{[a]} + B_{[a]} \subset (A+B)_{[a]}$;
(2) $\forall a \in L$, $(A+B)^{(a)} \subset A^{(a)} + B^{(a)} \subset A^{[a]} + B^{[a]} \subset (A+B)^{[a]}$;
(3) $A+B = \bigvee_{a\in L}(a \wedge (A_{[a]} + B_{[a]})) = \bigvee_{a\in L}(a \wedge (A_{(a)} + B_{(a)}))$
$\quad = \bigvee_{a\in M(L)}(a \wedge (A_{[a]} + B_{[a]})) = \bigvee_{a\in M(L)}(a \wedge (A_{(a)} + B_{(a)}))$;
(4) $A+B = \bigwedge_{a\in L}(a \vee (A^{[a]} + B^{[a]})) = \bigwedge_{a\in L}(a \vee (A^{(a)} + B^{(a)}))$
$\quad = \bigwedge_{a\in P(L)}(a \vee (A^{[a]} + B^{[a]})) = \bigwedge_{a\in P(L)}(a \vee (A^{(a)} + B^{(a)}))$;
(5) $\forall a \in L$, $(A+B)_{[a]} = \cap_{b\in\beta(a)}(A_{[b]} + B_{[b]}) = \cap_{b\in\beta(a)}(A_{(b)} + B_{(b)})$;
(6) $\forall a \in L$, $(A+B)_{(a)} = \bigcup_{a\in\beta(b)}(A_{[b]} + B_{[b]}) = \bigcup_{a\in\beta(b)}(A_{(b)} + B_{(b)})$;
(7) $\forall a \in L$, $(A+B)^{[a]} = \cap_{a\in\alpha(b)}(A^{[b]} + B^{[b]}) = \cap_{a\in\alpha(b)}(A^{(b)} + B^{(b)})$;
(8) $\forall a \in L$, $(A+B)^{(a)} = \bigcup_{b\in\alpha(a)}(A^{[b]} + B^{[b]}) = \bigcup_{b\in\alpha(a)}(A^{(b)} + B^{(b)})$.

Proof. (1) If $z \in (A+B)_{(a)}$, then $a \in \beta((A+B)(z))$. By

$$\beta((A+B)(z)) = \beta(\bigvee_{z=x+y} (A(x) \wedge B(y))) = \bigcup_{z=x+y} \beta(A(x) \wedge B(y)),$$

we know there exists $x, y \in \mathbb{R}$ such that $x+y=z$ and $a \in \beta(A(x) \wedge B(y))$. This implies $a \in \beta(A(x)) \cap \beta(B(y))$ since $\beta(A(x) \wedge B(y)) \subset \beta(A(x)) \cap \beta(B(y))$. Hence $a \in \beta(A(x))$ and $a \in \beta(B(y))$, that is, $x \in A_{(a)}$ and $y \in B_{(a)}$. This shows that $z = x+y \in A_{(a)} + B_{(a)}$. Thus $(A+B)_{(a)} \subset A_{(a)} + B_{(a)}$.

Because $A_{(a)} \subset A_{[a]}$ and $B_{(a)} \subset B_{[a]}$, we know $A_{(a)} + B_{(a)} \subset A_{[a]} + B_{[a]}$.

In order to prove $A_{[a]} + B_{[a]} \subset (A+B)_{[a]}$, let $z \in A_{[a]} + B_{[a]}$. Then there exists $x \in A_{[a]}$ and $y \in B_{[a]}$ such that $x+y=z$. Since

$$x \in A_{[a]} \Rightarrow A(x) \geqslant a \quad \text{and} \quad y \in B_{[a]} \Rightarrow B(y) \geqslant a,$$

we know $A(x) \wedge B(y) \geqslant a$. Hence

$$(A+B)(z) = \bigvee_{z=x+y} (A(x) \wedge B(y)) \geqslant a,$$

which implies $z \in (A+B)_{[a]}$. Therefore $A_{[a]} + B_{[a]} \subset (A+B)_{[a]}$. (1) is proven.

(2) If $z \in (A+B)^{(a)}$, then $(A+B)(z) \not\leqslant a$ which means $\bigvee_{z=x+y}(A(x) \wedge B(y)) \not\leqslant a$. Hence there exists x, y satisfying $x+y=z$ such that $A(x) \wedge B(y) \not\leqslant a$. Further $A(x) \not\leqslant a$ and $B(y) \not\leqslant a$, that is, $x \in A^{(a)}$ and $y \in B^{(a)}$. Thus $z = x+y \in A^{(a)} + B^{(a)}$. This shows that $(A+B)^{(a)} \subset A^{(a)} + B^{(a)}$.

Since $A^{(a)} \subset A^{[a]}$ and $B^{(a)} \subset B^{[a]}$, we can obtain $A^{(a)} + B^{(a)} \subset A^{[a]} + B^{[a]}$.

In order to prove $A^{[a]} + B^{[a]} \subset (A+B)^{[a]}$, let $z \in A^{[a]} + B^{[a]}$. Then there exists $x \in A^{[a]}$ and $y \in B^{[a]}$ such that $z = x+y$. Since

$$x \in A^{[a]} \Longleftrightarrow a \notin \alpha(A(x)) \quad \text{and} \quad y \in B^{[a]} \Longleftrightarrow a \notin \alpha(B(y)),$$

we know $a \notin \alpha(A(x)) \cup \alpha(B(y))$. Since

$$\alpha(A(x) \wedge B(y)) = \alpha(A(x)) \cup \alpha(B(y)),$$

we have $a \notin \alpha(A(x) \wedge B(y))$. Furthermore

$$a \notin \alpha\left(\bigvee_{z=x+y} A(x) \wedge B(y)\right) = \alpha((A+B)(z)),$$

which implies $z \in (A+B)^{[a]}$. Therefore $A^{[a]} + B^{[a]} \subset (A+B)^{[a]}$. (2) is proven.

The proof of (3)–(8) can be obtained from (1), (2) and Theorem 2.3. □

Theorem 4.4. *Let $A, B \in L^{\mathbb{R}}$. Then*

(1) $\forall a \in P(L)$, $(A + B)^{(a)} = A^{(a)} + B^{(a)}$;
(2) *If* $\beta(a \wedge b) = \beta(a) \cap \beta(b)$, *then* $(A + B)_{(a)} = A_{(a)} + B_{(a)}$.

Proof. (1) By (2) in Theorem 4.3 we can obtain $(A+B)^{(a)} \subset A^{(a)} + B^{(a)}$. Next we need to prove that $(A+B)^{(a)} \supset A^{(a)} + B^{(a)}$. Suppose $z \in A^{(a)} + B^{(a)}$. Then there exists $x \in A^{(a)}$ and $y \in B^{(a)}$ such that $z = x + y$. Hence $A(x) \nleqslant a$ and $B(y) \nleqslant a$. This implies that $A(x) \wedge B(y) \nleqslant a$ since a is prime. So we have

$$(A+B)(z) = \bigvee_{z=x+y} (A(x) \wedge B(y)) \nleqslant a,$$

then $z \in (A+B)^{(a)}$. This proves that $(A+B)^{(a)} \supset A^{(a)} + B^{(a)}$.

(2) By (1) in Theorem 4.3 we can obtain $(A+B)_{(a)} \subset A_{(a)} + B_{(a)}$. In order to prove $(A+B)_{(a)} \supset A_{(a)} + B_{(a)}$, suppose $z \in A_{(a)} + B_{(a)}$. Then there exists $x \in A_{(a)}$ and $y \in B_{(a)}$ such that $z = x + y$. Hence $a \in \beta(A(x))$ and $a \in \beta(B(y))$. By $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ we obtain $a \in \beta(A(x) \wedge B(y))$. So we have

$$a \in \bigcup_{z=x+y} \beta(A(x) \wedge B(y)) = \beta\left(\bigvee_{z=x+y} (A(x) \wedge B(y))\right) = \beta((A+B)(z)),$$

which implies $z \in (A+B)_{(a)}$. This shows that $(A+B)_{(a)} \supset A_{(a)} + B_{(a)}$. □

Remark 4.5. In Theorems 4.3 and 4.4, "+" can be substituted by "−", "×", "÷" respectively.

Remark 4.6. In general, if $\beta(a \wedge b) \neq \beta(a) \cap \beta(b)$, (2) in Theorem 4.4 is not true. This can be seen from the following example.

Example 4.7. Take L as mentioned in Example 3.5. Define $A, B \in LF^*(\mathbb{R})$ by setting

$$A(x) = \begin{cases} 1, & x = -1, \\ a, & x \in (-1, 0], \\ 0.5 - x, & x \in (0, 0.5], \\ 0, & x < -1 \text{ or } x > 0.5; \end{cases}$$

$$B(y) = \begin{cases} 1, & y = -1, \\ b, & y \in (-1, 0], \\ 0.5 - y, & y \in (0, 0.5], \\ 0, & y < -1 \text{ or } y > 0.5. \end{cases}$$

It is easy to check that A, B are both L-fuzzy numbers. By

$$(A+B)(z) = \bigvee_{z=x+y} (A(x) \wedge B(y)),$$

we have

$$(A+B)(0) = (A(0) \wedge B(0)) \vee \bigvee_{x \neq 0} (A(x) \wedge B(-x)) = 0.5,$$

that is, $0 \notin (A+B)_{(0.5)}$. But $0 \in A_{(0.5)}$, $0 \in B_{(0.5)}$. This shows that

$$A_{(0.5)} + B_{(0.5)} \neq (A+B)_{(0.5)}.$$

Lemma 4.8. *Let A be an L-fuzzy number. Then*

(1) *A is an L-fuzzy convex set*;
(2) *If $A(x_0) = 1$, then $A(x)$ is non-decreasing for $x \leqslant x_0$, and non-increasing for $x \geqslant x_0$.*

Proof

(1) From Definition 4.1, it can be seen that $A_{[a]}$ is a closed interval for each $a \in L$ since A is an L-fuzzy number. Obviously, $A_{[a]}$ is a convex set. From Theorem 3.2, we can obtain that A is an L-fuzzy convex set.
(2) Because $A_{[a]}$ is a closed interval for each $a \in L$, take $x_1, x_2 \in \mathbb{R}$ such that $x_1 \leqslant x_2 \leqslant x_0$. Let $a = A(x_1)$. From $A(x_0) = 1 \geqslant a$, we know $x_0 \in A_{[a]}$. Then $[x_1, x_0] \subset A_{[a]}$. Hence $x_2 \in [x_1, x_0] \subset A_{[a]}$, which implies $A(x_2) \geqslant a$, that is, $A(x_1) \leqslant A(x_2)$. Therefore $A(x)$ is non-decreasing for $x \leqslant x_0$. Analogously it can be proven that $A(x)$ is non-increasing for $x \geqslant x_0$. □

Lemma 4.9. *If $A, B \in LF^*(\mathbb{R})$, then $A + B$, $A - B$ and $A \times B$ are normal.*

Proof. Assume that $A(x_0) = 1$, $B(y_0) = 1$. Letting

$$z_1 = x_0 + y_0, \quad z_2 = x_0 - y_0 \quad \text{and} \quad z_3 = x_0 \cdot y_0,$$

we can easily prove that $(A+B)(z_1) = 1$, $(A-B)(z_2) = 1$ and $(A \times B)(z_3) = 1$. □

Lemma 4.10. *If A, $B \in LF^*(\mathbb{R})$, then $\forall a \in L$, $A_{[a]} + B_{[a]}$, $A_{[a]} - B_{[a]}$ and $A_{[a]} \cdot B_{[a]}$ are closed intervals.*

Proof. Since $A, B \in LF^*(\mathbb{R})$, both $A_{[a]}$ and $B_{[a]}$ are closed intervals, let $A_{[a]} = [x_1, y_1]$, $B_{[a]} = [x_2, y_2]$. Thus, we can easily prove that

$$A_{[a]} + B_{[a]} = [x_1 + x_2, y_1 + y_2], \quad A_{[a]} - B_{[a]} = [x_1 - y_2, y_1 - x_2] \quad \text{and} \quad A_{[a]} \cdot B_{[a]} = [x, y],$$

where $x = \min(x_1x_2, x_1y_2, y_1x_2, y_1y_2)$, $y = \max(x_1x_2, x_1y_2, y_1x_2, y_1y_2)$. □

Definition 4.11. Let $A \in LF^*(\mathbb{R})$. A is called a positive L-fuzzy number if $A(x) = 0$ while $x \leqslant 0$; A is called a negative L-fuzzy number if $A(x) = 0$ while $x \geqslant 0$.

Lemma 4.12. *Let $A, B \in LF^*(\mathbb{R})$. If B is a positive L-fuzzy number* (*or a negative L-fuzzy number*), *then $A_{[a]} \div B_{[a]}$ is a closed interval.*

Proof. Since $A, B \in LF^*(\mathbb{R})$, both $A_{[a]}$ and $B_{[a]}$ are closed intervals, let B be a positive L-fuzzy number and $A_{[a]} = [x_1, y_1]$, $B_{[a]} = [x_2, y_2]$, $x_2 > 0$. It follows that $A_{[a]} \div B_{[a]} = [x_1/y_2, y_1/x_2]$ is a closed interval. It is analogous when B is a negative L-fuzzy number. □

Theorem 4.13. *If $A, B \in LF^*(\mathbb{R})$, then $A + B$, $A - B$, $A \times B \in LF^*(\mathbb{R})$; If B is a positive L-fuzzy number* (*or a negative L-fuzzy number*), *then $A \div B \in LF^*(\mathbb{R})$ holds.*

Proof. First, we know that $A+B$, $A-B$, and $A\times B$ are normal from Lemma 4.9. $A\div B$ is normal when B is a positive L-fuzzy number or a negative L-fuzzy number. $A,B\in LF^*(\mathbb{R})$ implies that both $A_{[a]}$ and $B_{[a]}$ are closed intervals. From Lemma 4.10 and $(A+B)_{[a]}=\cap_{b\in\beta(a)}(A_{[b]}+B_{[b]})$ in Theorem 4.3, we know $(A+B)_{[a]}$ is also a closed interval. Analogously, both $(A-B)_{[a]}$ and $(A\cdot B)_{[a]}$ are closed intervals. When B is a positive L-fuzzy number or a negative L-fuzzy number, $(A\div B)_{[a]}$ is also a closed interval. This result is obtained from Definition 4.1. □

By means of Theorem 4.4(1), we can obtain the following result:

Theorem 4.14. *The set $LF^*_+(\mathbb{R})$ of all positive L-fuzzy numbers is a semi-ring, i.e.,*

(1) $A,B\in LF^*_+(\mathbb{R})\Rightarrow A+B, A\times B\in LF^*_+(\mathbb{R})$;
(2) $A+B=B+A$, $A\times B=B\times A$;
(3) $(A+B)+C=A+(B+C)$, $(A\times B)\times C=A\times(B\times C)$;
(4) $A\times(B+C)=A\times B+A\times C$;
(5) $A+0=A$, $A\times 1=A$.

5. The cardinality of *L*-fuzzy set

From the viewpoint of pure set theory, the most basic question about a set is: how many elements does it have? In order to answer this question, the notion of cardinality of sets has been presented. Analogously, one can also ask, what is the cardinality of an L-fuzzy set? We find that an L-fuzzy number can be used to denote the cardinality of an L-fuzzy set.

The following theorem is, therefore, clear.

Theorem 5.1. *For any $r\in\mathbb{R}$, define $\underline{r}$ by setting*

$$\underline{r}(t)=\begin{cases}0, & \text{if } t>r;\\ 1, & \text{if } t\leqslant r.\end{cases}$$

Then $\underline{r}\in LF^*(\mathbb{R})$.

Remark 5.2. If we do not distinguish r from $\underline{r}$, then $\mathbb{R}$ can be regarded as a subset of $LF^*(\mathbb{R})$. In the sequel, we shall not distinguish r from $\underline{r}$.

Theorem 5.3. *Let A be an L-fuzzy set. Define a map $|A|:\mathbb{R}\to L$ by setting*

$$|A|(r)=\begin{cases}\bigvee\left\{a\in L\,|\,|A_{[a]}|\geqslant r\right\}, & r\geqslant 0,\\ 0, & r<0,\end{cases}$$

where $r\in\mathbb{R}$. Then $|A|\in LF^*(\mathbb{R})$, $|A|$ is called the cardinality of A.

Proof. Obviously $|A|(0)=1$, i.e., $|A|$ is normal. It is easy to prove that $|A|(r)$ is non-increasing for $r\geqslant 0$. Suppose $b\in L$ and $x\in|A|_{[b]}$. Then for any $c\in\beta(b)$, there exists $a\in L$ such that $c\leqslant a$ and $|A_{[a]}|\geqslant x$. Hence $|A_{[c]}|\geqslant|A_{[a]}|\geqslant x$. This shows that for any $c\in\beta(b)$, $|A_{[c]}|\geqslant\sup\{x|x\in|A|_{[b]}\}=\sup|A|_{[b]}$. Thus we obtain $|A|(\sup|A|_{[b]})\geqslant b$. This proves that $|A|_{[b]}$ is a closed interval and $|A|_{[b]}=[0,\sup|A|_{[b]}]$. Therefore $|A|\in LF^*(\mathbb{R})$. □

Example 5.4. Let $X=\{1,2,3,4,5,6\}$ and $L=\{0,1,a,b,c,d\}$, where $0<d<b<a<1$, $0<d<c<a<1$, and b and c are incomparable. Define an L-fuzzy set A as

$$A(x)=\begin{cases}1, & x=1;\\ a, & x=2;\\ b, & x=3;\\ c, & x=4;\\ d & x=5;\\ 0, & x=6.\end{cases}$$

Then

$$|A|(r) = \begin{cases} 0, & r < 0; \\ 1, & 0 \leqslant r \leqslant 1; \\ a, & 1 < r \leqslant 3; \\ d, & 3 < r \leqslant 5; \\ 0, & r > 5. \end{cases}$$

Theorem 5.5. *For any L-fuzzy set $A \in L^X$ and any $r \in \mathbb{R}$, we have*

$$|A|(r) = \bigvee \left\{a \in L \| A_{(a)}| \geqslant r\right\}.$$

Proof. For any $r \in \mathbb{R}$, let $\lambda = \bigvee \{a \in L \| A_{(a)}| \geqslant r\}$. It is obvious that $\lambda \leqslant |A|(r)$. Next we will prove $\lambda \geqslant |A|(r)$. Suppose $b \in L$ and $b \in \beta(|A|(r))$. Then there exists $a \in L$ such that $b \in \beta(a)$ and $|A_{[a]}| \geqslant r$. In this case, $|A_{(b)}| \geqslant |A_{[a]}| \geqslant r$ which implies $\lambda = \bigvee \{a \in L \| A_{(a)}| \geqslant r\} \geqslant b$. Thus we have

$$\lambda \geqslant \bigvee \{b/b \in \beta(|A|(r))\} = |A|(r).$$

This proves that $\lambda = |A|(r)$. $\square$

Theorem 5.6. *For any L-fuzzy set $A \in L^X$ and any $a \in L$, we have*

$$|A|_{(a)} \leqslant |A_{(a)}| \leqslant |A_{[a]}| \leqslant |A|_{[a]}.$$

Proof. In order to prove $|A|_{(a)} \leqslant |A_{(a)}|$, suppose $r \leqslant |A|_{(a)}$. Then $a \in \beta(|A|(r))$. Thus there exists $b \in L$ such that $a \in \beta(b)$ and $|A_{[b]}| \geqslant r$. This shows that $|A_{(a)}| \geqslant |A_{[b]}| \geqslant r$. Therefore $|A|_{(a)} \leqslant |A_{(a)}|$.

$|A_{(a)}| \leqslant |A_{[a]}|$ is obvious. Moreover, it can be seen that $|A_{[a]}| \leqslant |A|_{[a]}$ from the definition of $|A|$. $\square$

6. Conclusion

In this paper, the notion of fuzzy convex sets and fuzzy numbers are generalized to an L-fuzzy set theory when L is a completely distributive lattice. As shown, L-fuzzy numbers could be used to characterize the cardinalities of L-fuzzy sets. Moreover, they can also be used to describe fuzzy rank functions of L-fuzzy matroids, L-fuzzy metrics, and others.

References

[1] S. Aytar, S. Pehlivan, Statistically monotonic and statistically bounded sequences of fuzzy numbers, Information Sciences 176 (2006) 734–744.
[2] S. Bodjanova, Alpha-bounds of fuzzy numbers, Information Sciences 152 (2003) 237–266.
[3] S. Bodjanova, Median value and median interval of a fuzzy number, Information Sciences 172 (2005) 73–89.
[4] C.B. Cheng, Fuzzy process control: construction of control charts with fuzzy numbers, Fuzzy Sets and Systems 154 (2005) 287–303.
[5] P. Dwinger, Characterization of the complete homomorphic images of a completely distributive complete lattice. I, Indagationes Mathematicae (Proceedings) 85 (1982) 403–414.
[6] T. Fan, G. Wang, Endographic approach on supremum and infimum of fuzzy numbers, Information Sciences 159 (2004) 221–231.
[7] D.P. Filev, R.R. Yager, Operations on fuzzy numbers via fuzzy reasoning, Fuzzy Sets and Systems 91 (1997) 137–142.
[8] G. Gierz et al., A Compendium of Continuous Lattices, Springer-Verlag, Berlin, 1980.
[9] J.A. Goguen, *L*-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[10] D.H. Hong, A convergence theorem for arrays of *L–R* fuzzy numbers, Information Sciences 88 (1996) 169–175.
[11] K. Horiuchi, N. Tamura, VSOP fuzzy numbers and their fuzzy ordering, Fuzzy Sets and Systems 93 (1998) 197–210.
[12] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984) 215–229.
[13] B. Matarazzo, G. Munda, New approaches for the comparison of *L–R* fuzzy numbers: a theoretical and operational analysis, Fuzzy Sets and Systems 118 (2001) 407–418.
[14] M. Oussalah, J.D. Schutter, Approximated fuzzy LR computation, Information Sciences 153 (2003) 155–175.

[15] J. Ramik, J. Rimanek, Inequality between fuzzy numbers and its use in fuzzy optimization, Fuzzy Sets and Systems 16 (1985) 123–138.
[16] E. Savaş, Mursaleen, On statistically convergent double sequences of fuzzy numbers, Information Sciences 162 (2004) 183–192.
[17] F.G. Shi, Theory of L_α-nest sets and L_β-nest sets and their applications, Fuzzy Systems and Mathematics 4 (1995) 65–72.
[18] F.G. Shi, Theory of molecular nest sets and its applications, Journal of Yantai Normal University 12 (1996) 33–36.
[19] F.G. Shi, *L*-fuzzy sets and prime element nest sets, Journal of Mathematical Research and Exposition 16 (1996) 398–402.
[20] F.G. Shi, *L*-fuzzy relation and *L*-fuzzy subgroup, Journal of Fuzzy Mathematics 8 (2000) 491–499.
[21] H. Tanaka, K. Asai, Fuzzy linear programming with fuzzy numbers, Fuzzy Sets and Systems 13 (1984) 1–10.
[22] W. Voxman, Some remarks on distances between fuzzy numbers, Fuzzy Sets and Systems 100 (1998) 353–365.
[23] G.J. Wang, Theory of topological molecular lattices, Fuzzy Sets and Systems 47 (1992) 351–376.
[24] W.J. Wang, C.H. Chiu, Entropy variation on the fuzzy numbers with arithmetic operations, Fuzzy Sets and Systems 103 (1999) 443–455.
[26] G.Q. Wu, P. Du, F.G. Shi, Decomposition theorems and representation theorem of *L*-fuzzy field, Journal of Fuzzy Mathematics 3 (1997) 353–359.
[27] M. Yamashiro, The median for a *L*–*R* fuzzy number, Microelectronics Reliability 35 (1995) 269–271.
[28] K.P. Yoon, A probabilistic approach to rank complex fuzzy numbers, Fuzzy Sets and Systems 80 (1996) 167–176.
[29] L.A. Zadeh, Fuzzy sets, Information Control 8 (1965) 338–353.
[30] L.A. Zadeh, Toward a generalized theory of uncertainty(GTU)—an outline, Information Sciences 172 (2005) 1–40.