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PAPER

# The set of maximal points of an $\omega$-domain need not be a $G_\delta$-set

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**Abstract**

A topological space has a domain model if it is homeomorphic to the maximal point space Max($P$) of a domain $P$. Lawson proved that every Polish space $X$ has an $\omega$-domain model $P$ and for such a model $P$, Max($P$) is a $G_\delta$-set of the Scott space of $P$. Martin (2003) then asked whether it is true that for every $\omega$-domain $Q$, Max($Q$) is $G_\delta$-set of the Scott space of $Q$. In this paper, we give a negative answer to Martin's long-standing open problem by constructing a counterexample. The counterexample here actually shows that the answer is no even for $\omega$-algebraic domains. In addition, we also construct an $\omega$-ideal domain $\widetilde{Q}$ for the constructed $Q$ such that their maximal point spaces are homeomorphic. Therefore, Max($Q$) is a $G_\delta$-set of the Scott space of the new model $\widetilde{Q}$ .

**Keywords:** maximal point space; Scott topology; ω-continuous dcpo; $G_\delta$-subset

## 1. Introduction

The Scott topology is the most useful topology in domain theory. In general, this topology is just $T_0$. However, by taking the sets Max($P$) of all maximal points of posets $P$, with the relative Scott topology, one can obtain all $T_1$ spaces (every $T_1$ space is homeomorphic to the subspace Max($P$) of some poset $P$) (Zhao and Xi, 2018). In general, if a space $X$ is homeomorphic to Max($P$) of a poset $P$, then we say that $P$ is a poset model of $X$. The spaces with a domain model have been studied by many authors. Lawson proved that a space $X$ is a Polish space if and only if it has an $\omega$-domain model satisfying the Lawson condition (Lawson, 1997). Moreover, if $P$ is an $\omega$-domain that satisfies the Lawson condition, then the set Max($P$) is a $G_\delta$-set of the Scott space $\Sigma P$ of $P$. As pointed out by Martin, knowing that Max($P$) is a $G_\delta$-subset of $\Sigma P$ is often useful in proofs: for instance, Edalat obtained the well-known connection between measure theory and the probabilistic powerdomain (Edalat, 1997) assuming a separable metric space that embedded as a $G_\delta$-subset of an $\omega$-domain. Martin has established several results on $G_\delta$-subsets $E \subseteq \mathrm{Max}(P)$, including (i) if $P$ is an $\omega$-domain such that Max($P$) is regular, then Max($P$) is a $G_\delta$-set (Martin, 2003b); (ii) if $P$ is an $\omega$-domain that has a countable set $C \subseteq \mathrm{Max}(P)$ such that $\uparrow x \cap C \neq \emptyset$ for each $x \in P \backslash \mathrm{Max}(P)$, then Max($P$) is a $G_\delta$-set (Martin, 2003a); and (iii) for any $\omega$-ideal domain $P$, Max($P$) is a $G_\delta$-set (Martin, 2003a).





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However, as stated by Martin (2003a) [Section 8(1)], the answer to the following problem is not known:

- Is Max($P$) a $G_\delta$-set of $\Sigma P$ for every $\omega$-domain $P$

In this paper, we construct an $\omega$-algebraic domain $P$ such that Max($P$) is not a $G_\delta$-set, which gives a negative answer to the above problem. In addition, we also construct an $\omega$-ideal domain $\widetilde{P}$ for our constructed $P$ such that their maximal points spaces are homoemorphic. Therefore, Max($P$) is a $G_\delta$-set of the Scott space of the new model $\widetilde{P}$ .

## 2. Preliminaries

This section is devoted to a brief review of some basic concepts and notations that will be used later. For more details, we refer the readers to Gierz et al. (2003) and Goubault-Larrecq (2013).

Let $P$ be a poset. For a subset $A$ of $P$, we shall adopt the following standard notations:

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\};\ \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

For each $x \in X$, we simply write $\uparrow x$ and $\downarrow x$ for $\uparrow\{x\}$ and $\downarrow\{x\}$, respectively. A subset $A$ of $P$ is called a *lower* (resp., an *upper*) *set* if $A = \downarrow A$ (resp., $A = \uparrow A$). An element $x$ is *maximal* in $A \subseteq P$, if $A \cap \uparrow x = \{x\}$. The set of all maximal elements of $A$ is denoted by Max$A$. The set of minimal elements of $A$, denoted by min$A$, is defined dually.

A nonempty subset $D$ of $P$ is *directed* if every two elements in $D$ have an upper bound in $D$. For $x, y \in P$, $x$ is *way-below* $y$, denoted by $x \ll y$, if for each directed subset $D$ of $P$ with $\bigvee D$ existing, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset $P$ is *continuous*, if for each $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$.

An element $a \in P$ is called *compact*, if $a \ll a$. The set of all compact elements of $P$ will be denoted by $K(P)$. Then, $P$ is called *algebraic* if for each $x \in P$, the set $\{a \in K(P) : a \leq x\}$ is directed and $x = \bigvee\{a \in K(P) : a \leq x\}$. A continuous (resp., algebraic) dcpo is also called a *domain* (resp., an *algebraic domain*). A subset $B \subseteq P$ is a *base* of $P$ if for each $x \in P$, $B \cap \twoheaddownarrow x$ is directed and $\bigvee (B \cap \twoheaddownarrow x) = x$. If $P$ has a countable base, then $P$ is called an *$\omega$-continuous dcpo* or *$\omega$-domain*.

**Remark 1.1.**

(1) A dcpo is a domain if and only if it has a base.

(2) For each base $B$ of a domain $P$, $K(P) \subseteq B$. As a consequence, every algebraic domain $P$ has the smallest base, namely $K(P)$.

A subset $U$ of $P$ is *Scott open* if (i) $U = \uparrow U$ and (ii) for each directed subset $D$ of $P$ for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of $P$ form a topology on $P$, called the *Scott topology* and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of $P$.

Let $X$ be a topological space. A subset $G \subseteq X$ is called a *$G_\delta$-set*, if there exists a countable family $\{U_n : n \in \mathbb{N}\}$ of open sets such that $G = \bigcap_{n\in\mathbb{N}} U_n$.

## 3. A Counterexample for Martin's Problem

We shall use the following notations:

$\mathbb{N}$ := the set of all nonzero natural numbers;

$\omega$ := the ordinal (also the cardinal) of natural numbers;

$\omega_1$ := the ordinal (also the cardinal) of real numbers;

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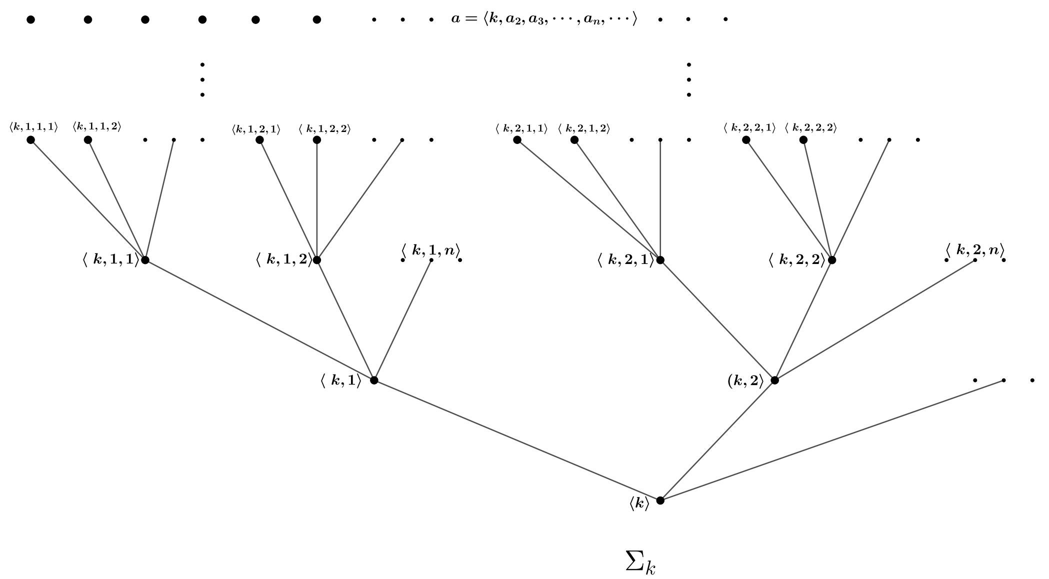


**Figure 1.** The subdcpo $\Sigma_k$.

$$\Sigma := \bigcup\{\mathbb{N}^k : k \in \mathbb{N} \cup \{\omega\}\}. \text{ In other words, } \Sigma \text{ is the set of all nonempty finite or countably infinite sequences of } \mathbb{N};$$

$$\Sigma^* := \{a^* : a \in \Sigma\}, \text{ which is a copy of } \Sigma.$$

Each element $a \in \Sigma$ can be represented as $\langle a_1, a_2, a_3, \cdots, a_k\rangle$ if it is a finite sequence, or $\langle a_n\rangle_{n\in\mathbb{N}}$ if it is an infinite sequence. We use the notation $\ell(a)$ to denote the length of $a$, which is defined as follows:

$$\ell(a) = \begin{cases} k, & a = \langle a_1, a_2, a_3, \cdots, a_k\rangle; \\ \omega, & a = \langle a_n\rangle_{n\in\mathbb{N}}. \end{cases}$$

We can also define $\ell(a^*) := \ell(a)$, as the lengths of $a$ and $a^*$ are clearly equal.

For any $a \in \Sigma$, we use $a_k$ (resp., $(a^*)_k$), if it exists, to denote the $k$-th element in the sequence $a$ (resp., $a^*$). For example, if $a := \langle 1, 2, 3, 5, \cdots, n, n+1, \cdots\rangle$, it follows that $\ell(a) = \omega$ and $a = \langle a_n\rangle_{n\in\mathbb{N}}$, where $a_n = n$ for all $n \in \mathbb{N}$.

**Definition 2.1.** *For each $a, b \in \Sigma$, $a$ is said to be a* substring *of $b$, denoted by $a \sqsubseteq b$, if any of the following cases holds:*

*(s1) $\ell(a) \in \mathbb{N}$, and $a_k = b_k$ for all $1 \le k \le \ell(a)$, that is $a$ is a prefix of $b$;*
*(s2) $\ell(a) = \omega$, and $a = b$ (in other words, $a_k = b_k$ for all $k \in \mathbb{N}$).*

It is straightforward to verify that the relation $\sqsubseteq$ defined on $\Sigma$ is a partial order, and some of its properties needed for understanding the main example are listed below.

**Remark 2.2.**

(1) $(\Sigma, \sqsubseteq)$ is an algebraic domain. In fact, for each $k \in \mathbb{N}$, the set $\Sigma_k := \{a \in \Sigma : a_1 = k\}$ together with the substring order $\sqsubseteq$ form a tree-like dcpo, as shown in Figure 1. In addition, the poset

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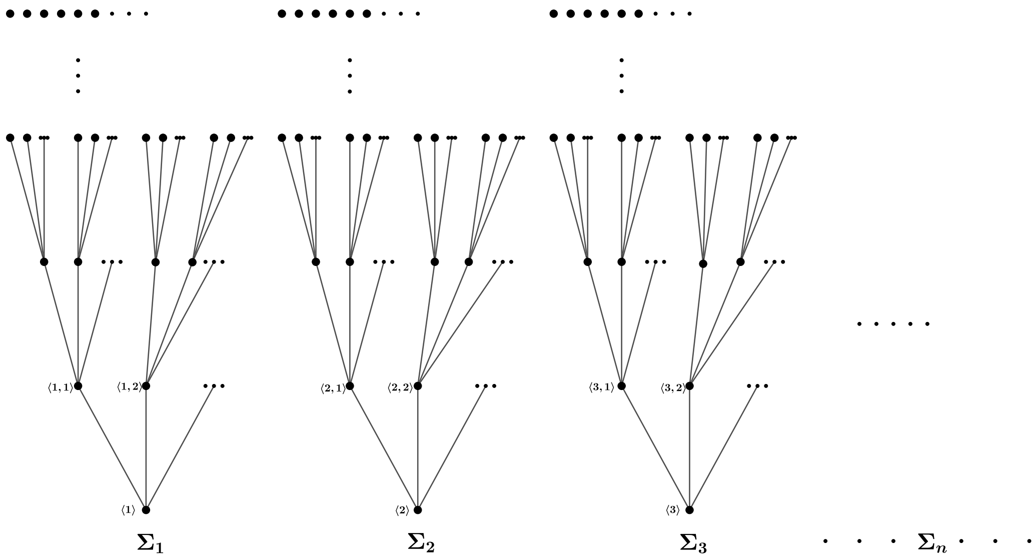


**Figure 2.** The algebraic domain $(\Sigma, \sqsubseteq)$.

$(\Sigma, \sqsubseteq)$ is the countable disjoint union of these $\Sigma_k$, where $k \in \mathbb{N}$. That is, for any $x, y \in \Sigma$, $x \sqsubseteq y$ in $\Sigma$ iff there exists some $k_0 \in \mathbb{N}$ such that $x, y \in \Sigma_{k_0}$ and $x \sqsubseteq y$ in $\Sigma_{k_0}$, as shown in Figure 2.

(2) $\mathrm{Max}\Sigma = \{a \in \Sigma : \ell(a) = \omega\}$, $|\Sigma| = |\mathrm{Max}\Sigma| = \omega_1$ and $|\Sigma \setminus \mathrm{Max}\Sigma| = \omega$.

(3) The compact elements of $\Sigma$ are the nonempty finite sequences of $\mathbb{N}$. In other words,

$$K(\Sigma) = \{a \in \Sigma : \ell(a) \in \mathbb{N}\}.$$

(4) For any two elements $a, b \in \Sigma$, $\{a, b\}$ has upper bounds iff $a$ and $b$ are comparable (i.e., either $a \leq b$ or $b \leq a$).

We can also define a partial order $\sqsubseteq^*$ on $\Sigma^*$ as follows: $\forall a, b \in \Sigma$,

$$a^* \sqsubseteq^* b^* \text{ iff } a \sqsubseteq b.$$

It is clear that the posets $(\Sigma, \sqsubseteq)$ and $(\Sigma^*, \sqsubseteq^*)$ are order-isomorphic. Furthermore, the properties of $(\Sigma, \sqsubseteq)$ listed in Remark 2.2 also hold for $(\Sigma^*, \sqsubseteq^*)$.

**Example 2.3.** Let $X := \{x_{m,n} : m \in \mathbb{N}, n \in \mathbb{N} \cup \{\omega\}\}$ with the partial order $\leq_X$ defined as follows: $\forall m_1, m_2 \in \mathbb{N}$ and $n_1, n_2 \in \mathbb{N} \cup \{\omega\}$,

$$x_{m_1,n_1} \leq_X x_{m_2,n_2} \Leftrightarrow m_1 = m_2 \text{ and } n_1 \leq n_2.$$

Then, $X$ is an $\omega$-algebraic domain, as shown in Figure 3. In fact, one easily sees that $\bigvee_{n\in\mathbb{N}} x_{k,n} = x_{k,\omega}$ for all $n \in \mathbb{N}$. Moreover, we have that

$$\mathrm{Max}X = \{x_{m,\omega} : m \in \mathbb{N}\} \text{ and } K(X) = X \setminus \mathrm{Max}X = \{x_{m,n} : m, n \in \mathbb{N}\}.$$

It then follows that $|K(X)| = |\mathrm{Max}X| = |X| = \omega$.

Before presenting the counterexample, let us briefly discuss the order structure defined by combining two ordered structures, which will be helpful to understand the order relationship of our main counterexample.

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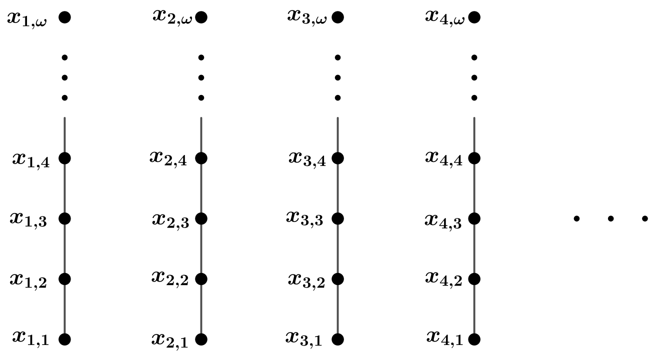


**Figure 3.** The algebraic domain $(X, \leq_X)$.

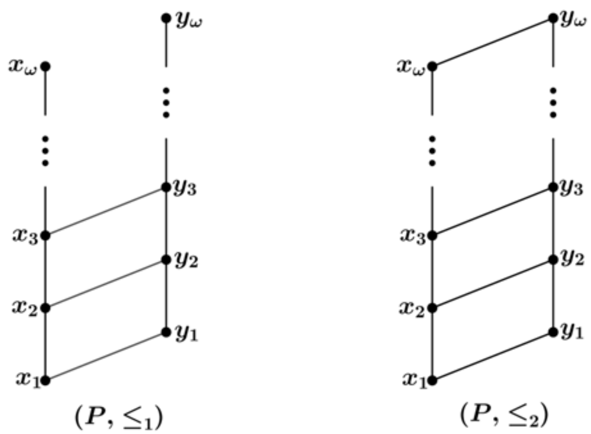


**Figure 4.** The posets $(P, \leq_1)$ and $(P, \leq_2)$.

**Remark 2.4.** Let $P = \{x_n : n \in \mathbb{N} \cup \{\omega\}\} \cup \{y_n : n \in \mathbb{N} \cup \{\omega\}\}$. We define two relations $\leq_1$ and $\leq_2$ on $P$.

(1) The relation $\leq_1$ is given as follows:

(P1) $x_1 \leq_1 x_2 \leq_1 x_3 \leq_1 \cdots \leq_1 x_n \leq_1 x_{n+1} \leq_1 \cdots \leq_1 x_\omega$;
(P2) $y_1 \leq_1 y_2 \leq_1 y_3 \leq_1 \cdots \leq_1 y_n \leq_1 y_{n+1} \leq_1 \cdots \leq_1 y_\omega$;
(P3) $\forall m, n \in \mathbb{N}, x_m \leq_1 y_n \Leftrightarrow m \leq n$.

Then, we define $\leq_2 := \leq_1 \cup \{(x_\omega, y_\omega)\}$. It is easy to verify that both $\leq_1$ and $\leq_2$ are partial orders on $P$, as shown in Figure 4.

(2) In the poset $(P, \leq_1)$, the chain $C := \{x_n : n \in \mathbb{N}\}$ has two incomparable upper bounds, namely $x_\omega$ and $y_\omega$. Hence, the supremum of $C$ in $(P, \leq_1)$ does not exist, and consequently $\bigvee_{(P, \leq_1)} C \neq x_\omega$. However, since $x_\omega \leq_2 y_\omega$, it is clear that the supremum of $C$ in $(P, \leq_2)$ exists and equals $x_\omega$, that is, $\bigvee_{(P, \leq_2)} C = x_\omega$.

**Example 2.5.** Let $L := X \cup \Sigma \cup \Sigma^*$, where $X$ is the poset defined in Example 2.3. For any $u, v \in L$, define $u \leq v$ if any of the following conditions holds:

(i) $u \sqsubseteq v$ in $\Sigma$;
(ii) $u \sqsubseteq^* v$ in $\Sigma^*$;
(iii) $u \leq_X v$ in $X$;

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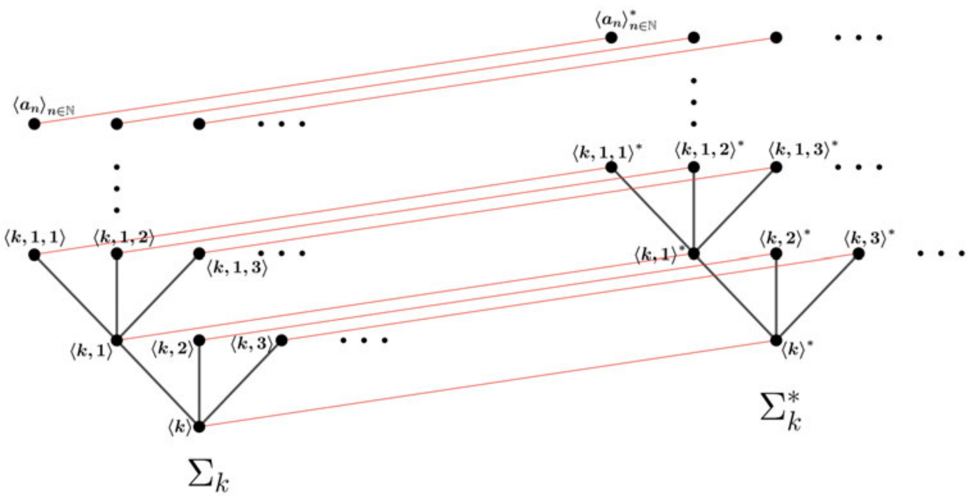


**Figure 5.** The relations between $\Sigma$ and $\Sigma^*$.

(iv) $u \in \Sigma$, $v = b^* \in \Sigma^*$ and $u \sqsubseteq b$. The order is illustrated in Figure 5.

(v) $\exists k, m \in \mathbb{N}$, $u = x_{k,m}$, $\ell(v) \geq k$ and $v_k \geq m$ (refer to Figure 6, where the black lines in the figure represent the relationship between the points in $X$ and $\Sigma$, while the dotted red lines represent the relations between the points in $\Sigma$).

In fact, Condition (v) describes the relations between $X$ and $\Sigma \cup \Sigma^*$, which can be characterized as follows: for any $k, m \in \mathbb{N}$, $v \in \Sigma \cup \Sigma^*$,

$$x_{k,m} \leq v \text{ iff } \exists n_1, n_2, \cdots, n_k \in \mathbb{N} \text{ such that } n_k \geq m \text{ and } \langle n_1, n_2, \cdots, n_k \rangle \leq v.$$

For example, we have the following facts:

$$x_{4,11} \leq \langle 1,5,7,11 \rangle \leq \langle 1,5,7,11,11 \rangle^* \leq \langle 1,5,7,11,11,22,22 \rangle^*;$$

$$x_{4,11} \leq \langle 1,5,7,111 \rangle \leq \langle 1,5,7,111 \rangle \leq \langle 1,5,7,111,11 \rangle^* \leq \langle 1,5,7,111,11,22,22 \rangle^*;$$

$$x_{3,3} \leq \langle 1,2,3 \rangle \leq \langle 1,2,3,\cdots,n,n+1,\cdots \rangle \leq \langle 1,2,3,\cdots,n,n+1,\cdots \rangle^*;$$

$$x_{3,3} \leq \langle 2,3,4 \rangle \leq \langle 2,3,4,\cdots,n,n+1,\cdots \rangle \leq \langle 2,3,4,\cdots,n,n+1,\cdots \rangle^*.$$

Next, we prove that $L$ is an $\omega$-algebraic dcpo such that Max$L$ is not a $G_\delta$-set in the Scott space of $L$. We achieve this in a few steps.

(1) The relation $\leq$ is a partial order on $L$.

The reflexivity and antisymmetry are almost trivial. For showing the transitivity, we need the following:

**Claim 1:** $\forall k, m \in \mathbb{N}, \forall a \in \Sigma, x_{k,m} \leq a \Leftrightarrow x_{k,m} \leq a^*$.

By the definition of $a^*$, it holds that $\ell(a^*) = \ell(a)$ and $(a^*)_k = a_k$. Therefore, Claim 1 follows from the following facts:

$$x_{k,m} \leq a \text{ iff } \ell(a^*) = \ell(a) \geq k \text{ and } (a^*)_k = a_k \geq m \text{ iff } x_{k,m} \leq a^*.$$

**Claim 2:** $\forall a, b \in \Sigma, a \leq b \Leftrightarrow a \leq b^* \Leftrightarrow a^* \leq b^* \Leftrightarrow a \sqsubseteq b$.

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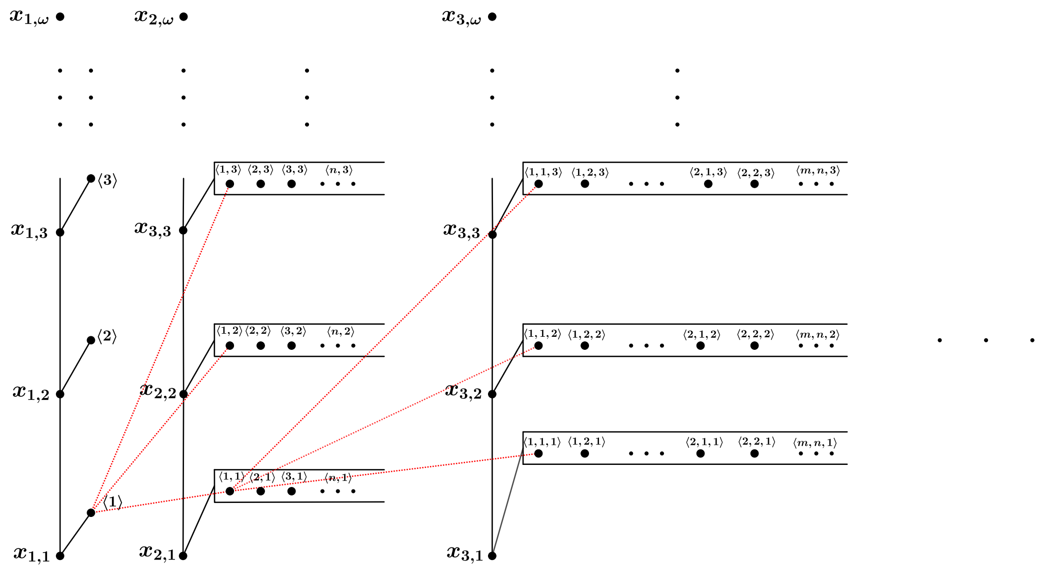

**Figure 6.** The subset $X \cup \Sigma$ of $L$.

Claim 2 follows from the following facts:

$$a \leq b \overset{(i)}{\Leftrightarrow} a \sqsubseteq b \text{ in } \Sigma \overset{(iv)}{\Leftrightarrow} a \leq b^* \overset{(iv)}{\Leftrightarrow} a \sqsubseteq b \text{ in } \Sigma \Leftrightarrow a^* \sqsubseteq^* b^* \text{ in } \Sigma^* \overset{(ii)}{\Leftrightarrow} a^* \leq b^*.$$

We now verify the transitivity only by considering the following cases: for any $k, m, k', m' \in \mathbb{N}$ and $a, b, c \in \Sigma$,

- (c1) $x_{k,m} \leq x_{k',m'} \leq a \leq b$. Then, as $x_{k,m} \leq x_{k',m'}$, we have that $k = k'$ and $m \leq m'$. Also, since $x_{k',m'} \leq a \leq b$, we have that $\ell(b) \geq \ell(a) \geq k' = k$ and $b_k = b_{k'} = a_k = a_{k'} \geq m' \geq m$, and by (v), it follows that $x_{k,m} \leq b$;
- (c2) $x_{k,m} \leq a \leq b$. Then, $\ell(b) \geq \ell(a) \geq k$ and $b_k = a_k \geq m$, which implies that $x_{k,m} \leq b$;
- (c3) By Claims 1 and 2, both cases $x_{k,m} \leq a \leq b^*$ and $x_{k,m} \leq a^* \leq b^*$ are equivalent to $x_{k,m} \leq a \leq b$. Then by (c2) , one can imply that $x_{k,m} \leq b$, and this is equivalent to $x_{k,m} \leq b^*$ by Claim 1;
- (c4) By Claim 2, both cases $a \leq b^* \leq c^*$ and $a \leq b \leq c^*$ are equivalent to $a \leq b \leq c$, which means $a \sqsubseteq b \sqsubseteq c$ in $\Sigma$. This implies $a \sqsubseteq c$, which is equivalent to $a \leq c^*$.

(2) $\mathrm{Max}L = \mathrm{Max}\Sigma^* \cup \mathrm{Max}X$ and $|\mathrm{Max}L| = \omega_1$.

First, one easily observes that

$$\begin{aligned} \mathrm{Max}\Sigma &= \mathbb{N}^\omega; \\ \mathrm{Max}\Sigma^* &= \{a^* : a \in \mathbb{N}^\omega\}; \\ \mathrm{Max}X &= \{x_{m,\omega} : m \in \mathbb{N}\}, \end{aligned}$$

and

$$|\mathrm{Max}\Sigma| = |\mathrm{Max}\Sigma^*| = \omega_1, \ \ |\mathrm{Max}X| = \omega.$$

In addition, by the definition of the order $\leq$ on $L$, we have that $\uparrow x_{m,\omega} \cap L = x_{m,\omega}$, and therefore, $\mathrm{Max}X \subseteq \mathrm{Max}L$. Then, we conclude that $\mathrm{Max}L = \mathrm{Max}\Sigma^* \cup \mathrm{Max}X$, and $|\mathrm{Max}L| = \omega_1$.

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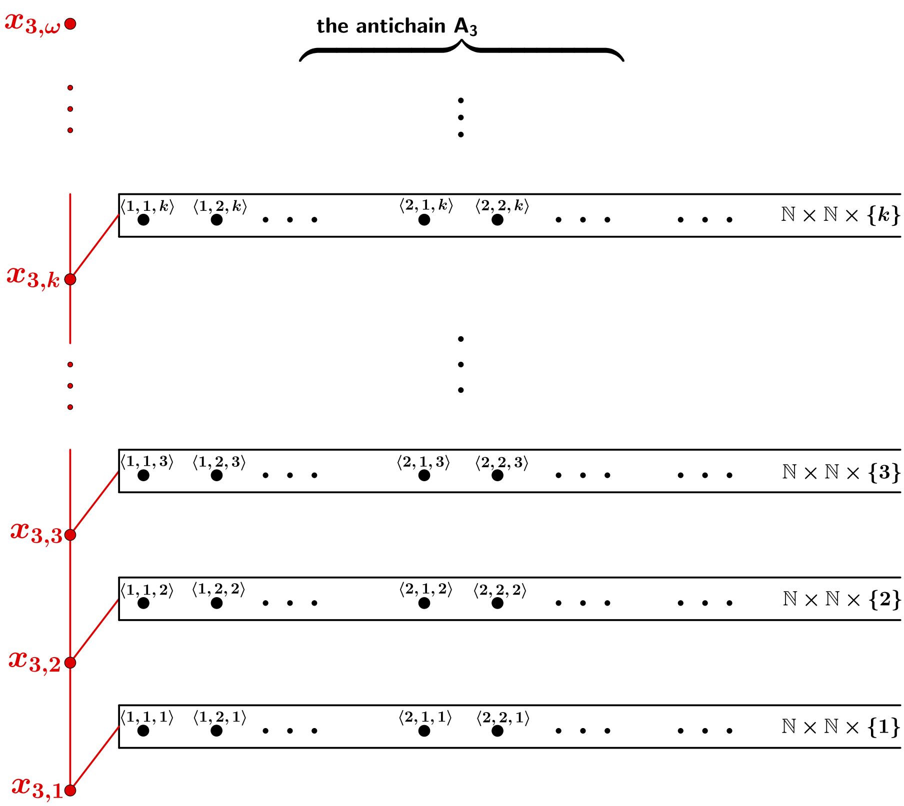


**Figure 7.** The set $\{x_{3,k} : k \in \mathbb{N} \cup \{\omega\}\} \cup A_3$.

In view of Remark 2.4, we need to check the following fact.

(3) $L$ is a dcpo.

Before proving this, we need the following remark about the characteristics of the relations of points $X \cup \Sigma$.

**Remark.** For any $m \in \mathbb{N}$, the set $A_m$ of all elements $a \in \Sigma$ that satisfy $x_{m,k} \le a$ and $\ell(a) = m$ for some $k \in \mathbb{N}$ is an antichain. That is, $A_m = \mathbb{N}^m = \bigcup_{k\in\mathbb{N}} \min\,(\uparrow x_{m,k} \cap \Sigma)$ is an antichain. For example, $A_3$ is the antichain of all black points (on the right side) in Figure 7.

By Remark 2.2(4), we see that any two distinct elements of $A_m$ have no upper bound in $L$. Thus, for any $a \in \Sigma$ and $k, l \in \mathbb{N}$, we have that $a \ge x_{m,k}$ iff there exists a unique element $b \in A_m$ such that $a \ge b$. This fact plays a crucial role in guaranteeing that the supremum of the chain $\{x_{m,k} : k \in \omega\}$ exists and is equal to $x_{m,\omega}$ (comparable to Remark 2.4). The convergence of all such chains in $X$ to their respective maximal points is the key to proving that $L$ is a dcpo, as shown below in Claim 1.

Now $L$ being a dcpo can be confirmed by the following four claims.

**Claim 3:** For any $m \in \mathbb{N}$ and any infinite subset $\{n_k : k \in \mathbb{N}\}$ of $\mathbb{N}$, $\bigvee_{k\in\mathbb{N}} x_{m,n_k} = x_{m,\omega}$.

Comparing to Remark 2.4, it is not obvious.

First, note that $x_{m,\omega}$ is an upper bound of $\{x_{m,n_k} : k \in \mathbb{N}\}$ in $L$. It remains to verify that $x_{m,\omega}$ is the unique upper bound. We show this by considering the two cases below:

(i) Assume $\{x_{m,n_k} : k \in \mathbb{N}\}$ has an upper bound in $\Sigma$. Then, as $\Sigma \subseteq \downarrow\mathrm{Max}\Sigma$, there must exist an element $a := \langle a_k\rangle_{k\in\mathbb{N}} \in \mathrm{Max}\Sigma$ such that $x_{m,n_k} \le a$ for all $k \in \mathbb{N}$. On the one hand, as $\{n_k : k \in \mathbb{N}\}$ is

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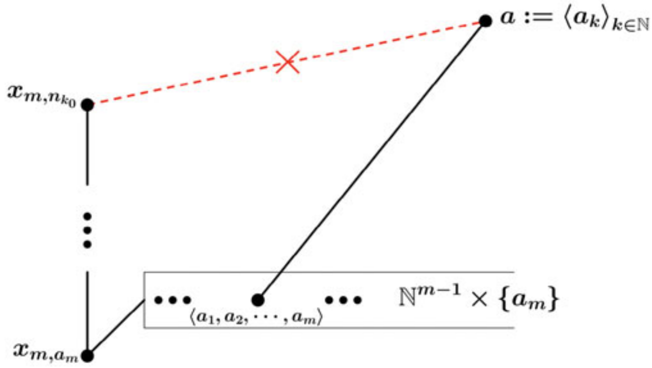


**Figure 8.** Order relations of $x_{m,a_m}$, $x_{m,n_{k_0}}$, and $a$.

infinite, it follows that $\bigvee_{k\in\mathbb{N}} n_k = \omega > a_m \in \mathbb{N}$, so there exists $k_0 \in \mathbb{N}$ such that $n_{k_0} > a_m$. On the other hand, as $x_{m,n_{k_0}} \leq a$, we have that $a_m \geq n_{k_0} > a_m$, a contradiction, as shown in Figure 8.

Therefore, $\{x_{m,n_k} : k \in \mathbb{N}\}$ has no upper bound in $\Sigma$.

(ii) Assume that there exists $a^* \in \Sigma^*$ that is an upper bound of $\{x_{m,n_k} : k \in \mathbb{N}\}$. Then, by Claim 1 in (1), $a \in \Sigma$ is also an upper bound of $\{x_{m,n_k} : k \in \mathbb{N}\}$, which contradicts (i).

Then (i) and (ii) together show that $\{x_{m,n_k} : k \in \mathbb{N}\}$ has no upper bound in $\Sigma \cup \Sigma^*$, and as it is clear that the only upper bound in $X$ is $x_{m,\omega}$, we deduce that $x_{m,\omega}$ is the unique upper bound of $\{x_{m,n_k} : k \in \mathbb{N}\}$ in $L$. Therefore, $\bigvee_{k\in\mathbb{N}} x_{m,n_k} = x_{m,\omega}$.

**Claim 4:** All sets $X$, $\Sigma$ and $\Sigma^*$ are subdcpos of $L$.

Clearly both $\Sigma$ and $\Sigma^*$ are subdcpos of $L$. We now check that $X$ is a subdcpos of $L$. Note that $X$ with the restriction order $\leq$ is exactly $\leq_X$ defined in Example 2.3. Note that all infinite directed subsets of $X$ are of the form $\{x_{m,n_k} : k \in \mathbb{N}\}$, where $m \in \mathbb{N}$ and $\{n_k : k \in \mathbb{N}\}$ is an infinite subset of $\mathbb{N}$; hence, we only need to prove that $\bigvee_{k\in\mathbb{N}} x_{m,n_k} = x_{m,\omega}$.

**Claim 5:** For any directed subset $D$ of $L$, if $\bigvee D \in \Sigma^*$, then $D \cap \Sigma^*$ is a directed set such that $\bigvee D = \bigvee (D \cap \Sigma^*)$.

It suffices to prove $D \subseteq {\downarrow}(D \cap \Sigma^*)$. Note that $X \cup \Sigma$ is both a lower set and a subdcpo of $L$, thus Scott closed. Then, $\Sigma^* = L \setminus (X \cup \Sigma)$ is a Scott open subset of $L$, so there exists $d_0 \in D \cap \Sigma^*$, it follows that ${\uparrow} d_0 \subseteq {\uparrow}\Sigma^* = \Sigma^*$. Hence, $D \subseteq {\downarrow}(D \cap {\uparrow} d_0) \subseteq {\downarrow}(D \cap \Sigma^*)$. This shows that $D \cap \Sigma^*$ is cofinal in $D$, and so $D \cap \Sigma^*$ is a directed set whose supremum equals $\bigvee D$ (see pp.61 in Goubault-Larrecq (2013)).

**Claim 6:** For any directed subset $D$ of $L$, if $\bigvee D \in \Sigma$, then $D \cap \Sigma$ is a directed set such that $\bigvee D = \bigvee (D \cap \Sigma)$.

For this, we only need to prove $D \subseteq {\downarrow}(D \cap \Sigma)$. By the definition of $\leq$ on $L$, we have that $D \subseteq {\downarrow} \bigvee D \subseteq {\downarrow}\Sigma = \Sigma \cup (X \setminus \mathrm{Max}X)$, and hence $D = (D \cap \Sigma) \cup (D \cap (X \setminus \mathrm{Max}X))$. Thus, to show $D \subseteq {\downarrow}(D \cap \Sigma)$, it suffices to prove $D \cap (X \setminus \mathrm{Max}X) \subseteq {\downarrow}(D \cap \Sigma)$. Let $x_{m,n} \in D \cap (X \setminus \mathrm{Max}X)$. Note that $D \cap \Sigma \neq \emptyset$ (if $D \cap \Sigma = \emptyset$, then $D \subseteq X$, and since $X$ is a subdcpo of $L$, $\bigvee D \in X$, a contradiction), so we choose an arbitrary point $b \in D \cap \Sigma$. Since $D$ is directed, there exists $e \in D$ such that $b \leq e$ and $x_{m,n} \leq e$. Recall that $D \subseteq {\downarrow}\Sigma$ and ${\uparrow} b \cap {\downarrow}\Sigma \subseteq \Sigma$ (as $b \in \Sigma$), so $e \in {\uparrow} b \cap D \subseteq ({\uparrow} b \cap {\downarrow}\Sigma) \cap D \subseteq \Sigma \cap D$, which implies that $x_{m,n} \in {\downarrow} e \subseteq {\downarrow}(\Sigma \cap D)$. Thus, $D \cap (X \setminus \mathrm{Max}X) \subseteq {\downarrow}(D \cap \Sigma)$. Therefore, $D \subseteq {\downarrow}(D \cap \Sigma)$. This shows that $D \cap \Sigma$ is cofinal in $D$, so $D \cap \Sigma$ is a directed set such that $\bigvee D = \bigvee (D \cap \Sigma)$ (see pp.61 in Goubault-Larrecq (2013)).

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(5) $L$ is an $\omega$-algebraic domain and $K(L) = L \setminus (\mathrm{Max}(\Sigma) \cup \mathrm{Max}(\Sigma^*) \cup \mathrm{Max}X)$.

Let $B := L \setminus (\mathrm{Max}(\Sigma) \cup \mathrm{Max}(\Sigma^*) \cup \mathrm{Max}X)$. Then, $B = \{x_{m,n} : m, n \in \mathbb{N}\} \cup \{a \in \Sigma : \ell(a) \in \mathbb{N}\} \cup \{a^* : a \in \Sigma, \ell(a) \in \mathbb{N}\}$. Suppose $m \in \mathbb{N}$ and $a = \langle a_1, a_2, \cdots, a_m\rangle \in \Sigma$.

*Step 1:* We prove $a^* \in K(L)$.

Suppose $D$ is a directed subset of $L$ such that $a^* \leq \bigvee D$. Then, $\bigvee D \in \uparrow a^* \subseteq \Sigma^*$, so by Claim 3 of (3), $D \cap \Sigma^*$ is a directed subset of $L$ and $a^* \leq \bigvee D = \bigvee (D \cap \Sigma^*)$. It follows that $D \cap \Sigma^*$ is a directed subset of $\Sigma^*$ and $a^* \sqsubseteq \bigvee D = \bigvee (D \cap \Sigma^*)$. Note that $a^*$ is a finite sequence of $\Sigma^*$, so it is compact in $(\Sigma^*, \sqsubseteq^*)$, and then there exists $b^* \in D \cap \Sigma^*$ such that $a^* \sqsubseteq^* b^*$. This implies that $a \leq b^*$ in $L$. Therefore, $a^* \in K(L)$.

*Step 2:* We prove $a \in K(L)$.

Suppose $D$ is a directed subset of $L$ such that $a \leq \bigvee D$. Then, $\bigvee D \in \uparrow a \subseteq \Sigma \cup \Sigma^*$. We consider the following two cases:

(e1) $\bigvee D \in \Sigma^*$.

Then, by Claim 2 of (1), $a^* \leq \bigvee D$, and as we have proved in Step 1 that $a^* \in K(L)$, there exists $e \in D$ such that $a \leq a^* \leq e$, so $a \leq e$.

(e2) $\bigvee D \in \Sigma$.

Then, by Claim 2 of (3), $D \cap \Sigma$ is a directed subset of $L$ and $a \leq \bigvee D = \bigvee (D \cap \Sigma)$. It follows that $D \cap \Sigma$ is a directed subset of $\Sigma$ and $a \sqsubseteq \bigvee D = \bigvee (D \cap \Sigma)$. Note that $a$ is a finite sequence of $\Sigma$, so it is compact in $(\Sigma, \sqsubseteq)$, and then there exists $e \in D \cap \Sigma$ such that $a \sqsubseteq e$. This implies that $a \leq e$ in $L$.

By (e1) and (e2), we conclude that $a \in K(L)$.

*Step 3:* $\forall k, m \in \mathbb{N}$, $x_{k,m} \in K(L)$.

As a matter of fact, assume that $D$ is a directed subset of $L$ such that $x_{k,m} \leq \bigvee D$.

(i) Let $\bigvee D \in X$. Then, $D \subseteq \downarrow X = X$. Note that $x_{k,m} \leq \bigvee D$ in $L$ implies $x_{k,m} \leq_X \bigvee D$ in $X$, and it is clear that $x_{k,m} \in K(X)$. Then, there exists $e \in D$ such that $x_{k,m} \leq_X e$, and hence $x \leq e$ in $L$.

(ii) Let $\bigvee D \in \Sigma \cup \Sigma^*$. Then, as $x_{k,m} \leq \bigvee D$, there exist $n_1, n_2, \cdots, n_k \in \mathbb{N}$ such that $n_k \geq m$ and $x_{k,m} \leq \langle n_1, n_2, \cdots, n_k\rangle \leq \bigvee D$. As we have proved in Step 2, $\langle n_1, n_2, \cdots, n_k\rangle \in K(L)$, there exists $e \in D$ such that $\langle n_1, n_2, \cdots, n_k\rangle \leq e$. Thus, $x_{k,m} \leq e$.

Therefore, by above (i)–(ii), $x_{k,m}$ is compact.

From Steps 1–3, we deuce that $B \subseteq K(L)$. Clearly, every element of $L$ can be represented as the supremum of some directed subset of $B$, which means that $B$ is a base of $L$. Note that $K(L)$ is included in any base of $L$, so $K(L) \subseteq B$. Therefore $B = K(L)$. Also, one easily see that $|B| = |K(L)| = \omega$. Hence, $L$ is an $\omega$-algebraic domain.

(6) $\mathrm{Max}L$ is not a $G_\delta$-set in $\Sigma L$.

Suppose, on the contrary, that $\mathrm{Max}L$ is a $G_\delta$-set. Then, there exists a countable family $\{U_k : k \in \mathbb{N}\} \subseteq \sigma(L)$ such that $\mathrm{Max}L = \bigcap_{k\in\mathbb{N}} U_k$. Recall that $\mathrm{Max}L = \mathrm{Max}(\Sigma^*) \cup \mathrm{Max}X$.

*Procedure 1:* Since $x_{1,\omega} = \bigvee_{n\in\mathbb{N}} x_{1,n} \in \mathrm{Max}X \subseteq \mathrm{Max}L \subseteq U_1 \in \sigma(L)$, there exists $n_1 \in \mathbb{N}$ such that $x_{1,n_1} \in U_1$. Note that $\langle n_1\rangle \geq x_{1,n_1} \in U_1 = \uparrow U_1$, so

$$\langle n_1\rangle \in U_1.$$

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*Procedure 2:* Since $x_{2,\omega} = \bigvee_{n\in\mathbb{N}} x_{2,n} \in \mathrm{Max}X \subseteq \mathrm{Max}L \subseteq U_2 \in \sigma(L)$, *there exists* $n_2 \in \mathbb{N}$ *such that* $x_{2,n_2} \in U_2$. *Note that* $\langle n_1, n_2\rangle \geq x_{2,n_2} \in U_2 = \uparrow U_2$, *so* $\langle n_1, n_2\rangle \in U_2$. *Since* $\langle n_1, n_2\rangle \geq \langle n_1\rangle \in U_1 = \uparrow U_1$, *we have that* $\langle n_1, n_2\rangle \in U_1$, *which follows that*

$$\langle n_1, n_2\rangle \in U_1 \cap U_2.$$

*Procedure 3:* Since $x_{3,\omega} = \bigvee_{n\in\mathbb{N}} x_{3,n} \in \mathrm{Max}X \subseteq \mathrm{Max}L \subseteq U_3 \in \sigma(L)$, *there exists* $n_3 \in \mathbb{N}$ *such that* $x_{3,n_3} \in U_3$. *Note that* $\langle n_1, n_2, n_3\rangle \geq x_{3,n_3} \in U_3 = \uparrow U_3$, *so* $\langle n_1, n_2, n_3\rangle \in U_3$. *Since* $\langle n_1, n_2, n_3\rangle \geq \langle n_1, n_2\rangle \in U_1 \cap U_2 = \uparrow(U_1 \cap U_2)$, *we have* $\langle n_1, n_2, n_3\rangle \in U_1 \cap U_2$, *which follows that*

$$\langle n_1, n_2, n_3\rangle \in U_1 \cap U_2 \cap U_3.$$

*Repeating the above procedures, we obtain a countably infinite sequence* $a := \langle n_k\rangle_{k\in\mathbb{N}} \in \bigcap_{k\in\mathbb{N}} U_k$. *Note that* $a \notin \mathrm{Max}L$ *(as* $a < a^*$*), which contradicts our assumption* $\mathrm{Max}L = \bigcap_{k\in\mathbb{N}} U_k$. *Therefore,* Max$L$ *is not a* $G_\delta$*-set in* $\Sigma L$.

From the preceding example, we obtain our main result as follows, which gives a negative answer to Martin's problem.

**Corollary 2.6.** *There exists an* $\omega$*-algebraic domain* $P$ *such that* Max$(P)$ *is not a* $G_\delta$*-set of the Scott space* $\Sigma P$.

## 4. Max($L$) Has an $\omega$-Ideal Model

A poset model of a topological space $X$ is a poset $P$ such that Max$(P)$, the set of maximal elements of $P$, with the relative Scott topology, is homeomorphic to $X$. Clearly, every space having a poset model is $T_1$.

**Definition 3.1** (Martin (2003a)). *A continuous dcpo is called* ideal *if every element is either compact or maximal. It is called* $\omega$-ideal *if it is ideal and has a countable basis.*

If $P$ is a poset model of $X$ such that $P$ is an $\omega$-ideal domain, then we say that $X$ has an $\omega$-ideal domain model, or equivalently, and $X$ can be modeled by an $\omega$-ideal domain.

In this section, we construct an $\omega$-ideal domain model of Max$(L)$, which shows that Max$(L)$ can be modeled by an $\omega$-ideal domain model. Therefore, Max$(L)$ is a $G_\delta$-set in this model.

Let $\tilde{L} = X \cup \Sigma$. We project $L$ onto $\tilde{L}$ and define a mapping $f : L \to L$ as follows:

$$f(x) = \bigvee\{y \in K(\tilde{L}) : y \leq x\}, \forall x \in L.$$

It is easy to observe the follows facts.

*Fact 1:* i) $\forall x \in X \cap K(L), f(x) = x$; ii) $\forall x \in \Sigma \cap K(L), f(x) = x$; iii) $\forall x^* \in \Sigma^* \cap K(L), f(x^*) = x$.

*Fact 2:* $f$ preserves the supremum of directed subsets of $L$ and then is Scott-continuous.

*Fact 3:* The image of $f$ is $\tilde{L}$, and Max$(L)$ are Max$(\tilde{L})$ are bijective.

By the definition of $f$, we have that: i)$\forall x \in X \cap \mathrm{Max}(L), f(x) = x \in X \cap \mathrm{Max}(\tilde{L})$; ii)$\forall x \in \Sigma \cap \mathrm{Max}(L), f(x) = x \in \Sigma \cap \mathrm{Max}(\tilde{L})$; and iii)$\forall x^* \in X \cap \mathrm{Max}(L^*), f(x^*) = x \in X \cap \mathrm{Max}(\tilde{L})$.

*Fact 4:* The restriction of $f$ to the maximal point space is an open mapping.

This can be seen from the results:

(i) $\forall x \in K(L) \cap X, f(\uparrow x \cap \mathrm{Max}(L)) = \uparrow x \cap \mathrm{Max}(\tilde{L})$;
(ii) $\forall x \in K(L) \cap \Sigma, f(\uparrow x \cap \mathrm{Max}(L)) = \uparrow x \cap \mathrm{Max}(\tilde{L})$;
(iii) $\forall x^* \in K(L) \cap \Sigma^*, f(\uparrow x^* \cap \mathrm{Max}(L)) = \uparrow x \cap \mathrm{Max}(\tilde{L})$.

From these facts, we obtain the following results:

**Theorem 3.2.** Max($L$) *can be modeled by an $\omega$-ideal domain and therefore the maximal point space is a $G_\delta$-set of the Scott space of the model.*

The results here naturally lead to the following problems for further study on this topic:

**Problem 1** Is there a $T_1$ space which has an algebraic domain model and does not have any ideal model?

**Problem 2** Which spaces have an ideal domain model?

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