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On quasi-metrizable d-spaces ☆

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In this paper we develop some connections between the quasi-metric spaces and the d-spaces of domain theory. We do this by introducing the notion of S-quasi-metrics on dcpos, which is a quasi-metric compatible with the ordering. Results show that (i) the quasi-metrizable d-spaces are exactly the S-quasi-metric spaces; (ii) the open ball topology is generally coarser than the Scott topology in an S-quasi-metric space, and a condition is provided to make them coincide; (iii) the Scott space of formal balls of a complete metric space is S-quasi-metrizable; (iv) the quasi-metrizability, as a topological property, is generally not preserved by the D-completion, the well-filtered reflection, or the sobrification of a T_0 space.

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Domain theory, initiated by Dana Scott in the late 1960s [11,12], plays a central role in computer science. The sober spaces, well-filtered spaces and d-spaces form three of the most important and heavily studied classes of topological spaces in domain theory. It is noteworthy that all these three kind of topologies more often than that are T_0 spaces, which remove any possibility that the theory of metric spaces (which are all Hausdorff). In the spirit of *All Topologies Come from Generalised Metrics* [9], various researchers try to establish links between (generalized) metrics and non-Hausdorff topologies. A well-known kind of generalized metric structures, called partial metrics, is due to Matthews [10], which has an important position in domain theory (see [6,17–19] for example). Another important generalized metric is called quasi-metric, which is defined by rejecting the symmetry axiom of metrics, and usually used to characterize T_0 spaces.

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The original link between domain theory and topology is via the Scott topology on continuous lattices, and then the lattice structure is transferred to a more appropriate ordering, the directed complete poset (dcpo). The Scott topology on a dcpo is usually just T_0, but not T_1. As an extension of the Scott topology of dcpos to the setting of T_0 spaces, Wyler [21] introduced the notion of d-spaces (also called *monotone convergence spaces* in [4]), where one requires that directed sets in the specialization ordering have suprema, and that each open set is Scott open with respect to the specialization ordering.

In this paper, we focus on the class of d-spaces that are quasi-metrizable. In Section 2, we introduce the notion of S-quasi-metrics on dcpos, which is a quasi-metric that is compatible with the ordering of the dcpo. We end this section by showing that the quasi-metrizable d-spaces are exactly the S-quasi-metric spaces.

In Section 3, some examples of S-quasi-metric spaces are given. In particular, we observe that the dcpo of formal balls of a complete metric, introduced by Edalat and Heckmann [2], is S-quasi-metrizable. As a consequence, every complete metric space is homeomorphic to the maximal points space of an S-quasi-metric space. These examples also show that the open ball topology is generally coarser than the Scott topology, not vice-versa.

In Section 4, we provide a condition for that the Scott topology and the open ball topology coincide in an S-quasi-metric space.

As far as we know, Wyler [21] was the first to show that the category of d-spaces is reflective in the category of T_0 spaces. Later, Ershov [3] investigated d-spaces in his own right, and he proved that for any subspace X_0 of a d-space X, there exists a smallest d-subspace of X containing X_0, which is exactly the D-completion of X_0. Keimel and Lawson [8] developed the theory of d-spaces by introducing the d-topology of a T_0 space (see also in [26]), and prove that the D-completion of a subspace X_0 of a d-space X is exactly the closure of X_0 in the d-topology of X. From the viewpoint of ordering theory, Zhao and Fan [26] constructed the dcpo-completion of a poset. Zhang and Li [25] then showed that Zhao's dcpo-completion of a poset P is exactly the D-completion of the Scott space of P. The existence of the well-filtered reflection is due to Wu, Xi, Xu and Zhao [22], who proved this by using the K-fication introduced by Keimel and Lawson [8]. After that, guided by Ershov's method [3] of constructing the D-completion of a T_0 space, Shen, Xi, Xu and Zhao [13] constructed the well-filtered reflections of T_0 spaces.

A *hemi-metric space* is a set X equipped with a *hemi-metric* d, i.e., with a nonnegative function d defined on $X \times X$ satisfying the following two conditions: $\forall x, y, z \in X$,

(M1) $d(x, x) = 0$;
(M2) $d(x, z) \leq d(x, y) + d(y, z)$.

Every hemi-metric (X, d) can be considered as a topological space, with the open ball topology (we denote it by $\mathcal{O}_d$), introduced by taking the collection $\{B_r(x) : x \in X, r > 0\}$ as a base, where $B_r(x) = \{y \in X : d(x, y) < r\}$. In fact, the open ball topology of a hemi-metric d is T_0 if and only if d is a quasi-metric (see [5, Lemma 6.1.9]), i.e., it satisfies the following condition:

(M3) $\forall x, y \in X$, $d(x, y) = d(y, x) = 0$ implies $x = y$.

By Wilson's Theorem [20] (or see [5, Theorem 6.3.13]), the quasi-metrizability, as topological property, sits in between T_0 second-countability and T_0 first-countability:

$$\text{second-countable } T_0 \;\Rightarrow\; \text{quasi-metrizable} \;\Rightarrow\; \text{first-countable } T_0.$$

As far as we know: a T_0 space X is second-countable if and only if its D-completion is; but this result is generally not true for the first-countable property (see [23, Example 5.12]). Then a natural question is (refer to the table):

- For a quasi-metrizable space X, whether or not its D-completion is quasi-metrizable?

X	second-countable	quasi-metrizable	first-countable
$D(X)$	$\surd$	?	$\times$

We will give a negative answer for the above question in Section 4. Specifically, we provide a counterexample of the Alexandroff topology (which is always quasi-metrizable) on a continuous poset P whose D-completion is not quasi-metrizable.

1. Introduction

In this section, we recall some basic concepts and notations to be used in the paper (more details can be found in [4,5]).

The set of all natural numbers (resp., positive numbers) with the usual ordering is denoted by $\mathbb{N}$ (resp., $\mathbb{N}^+$). Let ω_0 denote the ordinal (also the cardinal number) of $\mathbb{N}$ and ω_1 denote the first uncountable ordinal.

Let P be a poset. A nonempty subset D of P is *directed* if every two elements of D have an upper bound in D. A poset P is a *directed complete poset*, or a *dcpo* for short, if for any directed subset $D \subseteq P$, $\bigvee D$ exists. A poset P is a *chain* if for every pair of elements $x, y \in P$, $x \leq y$ or $y \leq x$. We call P *bounded-complete* if every nonempty subset of P that has an upper bound has a supremum.

For any subset A of a poset P, we use the following standard notations:

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\}, \text{ and } \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

In particular, for any $x \in X$, we write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$.

For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P such that $\bigvee D$ exists, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$. A continuous dcpo is also called a *domain*.

A subset U of a poset P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P such that $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, and we call this topology the *Scott topology* on P and denote it by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

For a T_0 space X, the specialization ordering $\leq$ is defined by $x \leq y$ iff $x \in \mathrm{cl}_X(\{y\})$, where cl_X is the closure operator of X. Clearly, $\mathrm{cl}_X(\{y\}) = \downarrow y$. In the following, when we consider a T_0 space X as a poset, it always means to be equipped with the specialization ordering.

Remark 1.1 *([4,5]).*

(1) For each poset $(P, \leq_P)$, the specialization ordering of ΣP is exactly $\leq_P$.
(2) If P is continuous, then $\{\twoheaduparrow x : x \in P\}$ forms a base for $\sigma(P)$.

For a T_0 space X, we use the notation $\mathcal{O}(X)$ to denote the topology of X. For any subset A of X, the *saturation* of A, denoted by $Sat_X(A)$, is defined to be

$$Sat_X(A) = \bigcap\{U \in \mathcal{O}(X) : A \subseteq U\}.$$

A subset A of a T_0 space X is called *saturated* if $A = Sat_X(A)$.

Remark 1.2 *([4,5]).* Let X be a T_0 space.

(1) For any subset A of X, $Sat_X(A) = \uparrow A$.
(2) For any $x \in X$, $x \in Sat_X(A)$ if and only if $\mathrm{cl}_X(\{x\}) \cap A \neq \emptyset$.
(3) For any open subset U of X, $U = Sat_X(U) = \uparrow U$, and for any closed subset F of X, $F = \downarrow F$.

Definition 1.3 *([4,5,21]).* A T_0 space X is called a *d-space*, or *a monotone convergence space*, if X under the specialization ordering $\leq$ is a dcpo, and $\mathcal{O}(X)$ is coarser than the Scott topology of $(X, \leq)$.

Definition 1.4 *([4,5]).* A T_0 space X is called *well-filtered* if for any filtered family $\{Q_i : i \in I\}$ of compact saturated subsets of X and any open set $U \subseteq X$, $\bigcap_{i\in I} Q_i \subseteq U$ implies $Q_{i_0} \subseteq U$ for some $i_0 \in I$.

Definition 1.5 *([4,5]).* A nonempty subset A of a topological space X is called *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called *sober*, if for any irreducible closed set F of X there is a unique point $x \in X$ such that $F = \mathrm{cl}_X(\{x\})$.

Remark 1.6 *([4,5]).* Every sober space is well-filtered, and every well-filtered space is a d-space.

Definition 1.7 *([5]).* Let (X, d) be a hemi-metric space. The *open ball* $B_r(x)$ with center x and radius $r > 0$, is defined by

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

The *open ball topology* of d, denoted by $\mathcal{O}_d(X)$ (or simply, $\mathcal{O}_d$), is the topology generated by the open balls.

Remark 1.8.

(1) [5, Lemma 6.1.9] Let (X, d) be a hemi-metric space. Then $(X, \mathcal{O}_d)$ is T_0 if and only if d is a quasi-metric.
(2) [5, Proposition 6.1.8] Let (X, d) be a quasi-metric space. Then $\mathcal{O}_d$ is a T_0 space, and its specialization ordering $\leq_d$ is given by

$$x \leq_d y \;\Leftrightarrow\; d(x, y) = 0.$$

A space X is called *hemi-metrizable* (resp., *quasi-metrizable*), if there exists a hemi-metric (resp., a quasi-metric) d on X such that the topology of X is precisely the open ball topology $\mathcal{O}_d$ induced by d.

Remark 1.9. By Remark 1.8(1), a T_0 space is hemi-metrizable if and only if it is quasi-metrizable.

The following property will be used in the sequel.

Lemma 1.10. *Let (X, d) be a quasi-metric space, and $x, y, z \in X$. If $y \leq_d z$, then $d(x, z) \leq_d d(x, y)$ and $d(y, x) \leq_d d(z, x)$.*

Proof. Since $y \leq_d z$, by Remark 1.8(2), $d(y, z) = 0$. Then $d(x, z) \leq_d d(x, y) + d(y, z) = d(x, y)$ and $d(y, x) \leq_d d(y, z) + d(z, x) = d(z, x)$, as desired. $\square$

2. S-quasi-metric spaces

To study quasi-metrizable d-spaces, we first give the following definition.

Definition 2.1. Let P be a poset, and $d : P \times P \longrightarrow [0, +\infty)$ be a mapping. We call d a *Scott hemi-metric*, or an *S-hemi-metric* for short, on P, if d is a hemi-metric on P satisfying the following condition:

(SM1) for any $x \in P$ and any directed subset D of P such that $\bigvee D$ exists, $d(x, \bigvee D) = \bigwedge_{y \in D} d(x, y)$.

An S-hemi-metric d on P is called an *S-quasi-metric* if it satisfies the following condition:

(SM2) $d(x, y) = 0$ implies $x \leq y$.

We call the pair (P, d) a *weak S-hemi-metric space* (resp., a *weak S-quasi-metric space*) if d is an S-hemi-metric (resp., an S-quasi-metric) on P.

An *S-hemi-metric space* (resp., an *S-quasi-metric space*) is a weak S-hemi-metric space (resp., a weak S-quasi-metric space) (P, d) such that P is a dcpo.

Remark 2.2.

(1) If (X, d) is a metric space, then d is an S-quasi-metric space on $(X, =)$.
(2) It is easy to observe that every weak S-quasi-metric space is a quasi-metric space.

Proposition 2.3. *Let (P, d) be a weak S-hemi-metric space. Then for any $x, y, z \in P$ and any directed subset D of P such that $\bigvee D$ exists, it holds that*

(1) $y \leq z$ implies $d(x, z) \leq d(x, y)$;
(2) $x \leq y$ implies $d(x, y) = 0$;
(3) $x \leq y$ implies $d(x, z) \leq d(y, z)$;
(4) $d(\bigvee D, y) = \bigvee_{x \in D} d(x, y)$.

Proof. (1) Suppose $y \leq z$. From (SM1), it follows that

$$d(x, z) = d(x, \bigvee \downarrow z) = \bigwedge_{u \leq z} d(x, u) \leq d(x, y),$$

as desired.

(2) Suppose $x \leq y$. Using (1), it holds that $d(x, y) \leq d(x, x) = 0$, as desired.

(3) Suppose $x \leq y$. By (2), $d(x, y) = 0$. It then follows that

$$d(x, z) \leq d(x, y) + d(y, z) = 0 + d(y, z) = d(y, z),$$

as desired.

(4) From (3), it follows that $\bigvee_{x \in D} d(x, y) \leq d(\bigvee D, y)$. Now suppose that $\bigvee_{x \in D} d(x, y) < d(\bigvee D, y)$. Then there exist $s, t > 0$ such that

$$\bigvee_{x \in D} d(x, y) < s < t < d(\bigvee D, y).$$

For each $x \in D$, we have that $d(x, y) < s$, and hence

$$t < d(\bigvee D, y) \leq d(\bigvee D, x) + d(x, y) < d(\bigvee D, x) + s,$$

which implies that $d(\bigvee D, x) > t - s > 0$. By (M1) and (SM1), we have that

$$0 = d(\bigvee D, \bigvee D) = \bigwedge_{x\in D} d(\bigvee D, x) \geq t - s > 0,$$

which is a contradiction. Therefore, $d(\bigvee D, y) = \bigvee_{x\in D} d(x, y)$. □

The reader may wonder whether (SM1) in Definition 2.1 is equivalent to statement (4) in Proposition 2.3 for a hemi-metric space (P, d), where P is a poset. The answer is negative, as shown in the following counterexample.

Example 2.4. Let $P = \mathbb{N} \cup \{a, \top\}$, where $a, \top \notin \mathbb{N}$ and $a \neq \top$. The ordering $\leq$ on P is defined by $x \leq y$ if and only if $y = \top$, or $x = a$ and $y \in \{a, \top\}$, or $x, y \in \mathbb{N}$ and x is smaller than y with the usual ordering. We define $d : P \times P \longrightarrow [0, +\infty)$ as follows:

$$d(x, y) = \begin{cases} 0 & x \leq y, \\ 1 & x \not\leq y. \end{cases}$$

It is easy to check that d is a hemi-metric on P. In fact, d does not satisfy (SM1) in Definition 2.1, since $d(a, \bigvee \mathbb{N}) = d(a, \top) = 0$, but $\bigwedge_{n\in\mathbb{N}} d(a, n) = 1$. In addition, for any $y \in P$ and any directed subset D of P, note that $\bigvee D \leq y$ if and only if $x \leq y$ for every $x \in D$, which implies that $d(\bigvee D, y) = \bigvee_{x\in D} d(x, y)$ by the definition of d. Thus statement (4) in Proposition 2.3 is satisfied.

The following result follows immediately from Proposition 2.3.

Proposition 2.5. *Let (P, d) be a weak S-hemi-metric space. Then d satisfies (SM2) in Definition 2.1 if and only if it satisfies the following condition:*

(SM2)* $\forall x, y \in P,\ d(x, y) = 0 \Leftrightarrow x \leq y$.

Proposition 2.6. *Let (P, d) be a weak S-hemi-metric space. If P is continuous, then d satisfies (SM1) in Definition 2.1 if and only if it satisfies the following condition:*

(SM1)* $\forall x, y \in P,\ d(x, y) = \bigwedge_{z\ll y} d(x, z)$.

Proof. (SM1) ⇒ (SM1*). Since P is continuous, we have that $\Downarrow y$ is directed and $y = \bigvee \Downarrow y$. From (SM1), it follows that

$$d(x, y) = d(x, \bigvee \Downarrow y) = \bigwedge_{z\ll y} d(x, z),$$

which gives (SM1*).

(SM1*) ⇒ (SM1). Let D be a directed subset of P such that $\bigvee D$ exists. Since P is continuous, it is clear that $y \ll \bigvee D$ iff $y \ll z$ for some $z \in D$. Using (SM1*), we have that

$$d(x, \bigvee D) = \bigwedge_{y\ll\bigvee D} d(x, y) = \bigwedge_{z\in D} \bigwedge_{y\ll z} d(x, y) = \bigwedge_{z\in D} d(x, z),$$

which gives (SM1). □

Proposition 2.7. *Let $(P, \leq_P)$ be a poset, and d be an S-hemi-metric on P. Then the following statements are equivalent:*

(1) d is an S-quasi-metric;
(2) $(P, \mathcal{O}_d)$ is a T_0 space whose specialization ordering is $\leq_P$.

Proof. It is straightforward by Remarks 1.8, 2.2(2) and Proposition 2.5. □

Lemma 2.8. *Let (P, d) be a weak S-hemi-metric space. Then for any $r > 0$ and $x \in X$, the open ball $B_r(x)$ is a Scott open subset of P.*

Proof. We first show that $B_r(x)$ is an upper subset of P. If $z \geq y \in B_r(x)$, then $d(x, y) < r$, and by Proposition 2.3(1), it follows that $d(x, z) \leq d(x, y) < r$, which implies that $z \in B_r(x)$. Hence, $B_r(x)$ is an upper set.

Let D be a directed subset of P such that $\bigvee D$ exists. Suppose $\bigvee D \in B_r(x)$. Then $d(x, \bigvee D) < r$, and from (SM1) it follows that $\bigwedge_{y \in D} d(x, y) = d(x, \bigvee D) < r$. Then there exists $y_0 \in D$ such that $d(x, y_0) < r$, which implies that $y_0 \in B_r(x) \cap D$.

All this shows that $B_r(x)$ is a Scott open subset of P. □

Theorem 2.9. *Let P be a poset, and d be a hemi-metric on P. Then d is an S-hemi-metric if and only if $\mathcal{O}_d \subseteq \sigma(P)$.*

Proof. $(\Rightarrow)$. It is trivial by Lemma 2.8.

$(\Leftarrow)$. It suffices to verify condition (SM1). Suppose $x \in P$ and D is a directed subset of P such that $\bigvee D$ exists. Then for any $r > 0$, since $B_r(x)$ is Scott open, it follows that

$$\bigvee D \in B_r(x) \Leftrightarrow D \cap B_r(x) \neq \emptyset.$$

Then we have that

$$\begin{aligned} d(x, \bigvee D) < r \quad &\Leftrightarrow \quad \bigvee D \in B_r(x) \\ &\Leftrightarrow \quad D \cap B_r(x) \neq \emptyset \\ &\Leftrightarrow \quad \exists y \in D, d(x, y) < r \\ &\Leftrightarrow \quad \bigwedge_{y \in D} d(x, y) < r, \end{aligned}$$

which implies that $d(x, \bigvee D) = \bigwedge_{y \in D} d(x, y)$, completing the proof. □

Definition 2.10. We call a T_0 space X *weak d-space*, if $\mathcal{O}(X)$ is coarser than the Scott topology of X under the specialization ordering.

It is clear that every d-space is a weak d-space, and every Scott space of a poset is a weak d-space. By Proposition 2.5, Proposition 2.7 and Theorem 2.9, we have the following result.

Theorem 2.11. *Let $(P, \leq)$ be a poset, and d be a hemi-metric on P. Then the following statements are equivalent:*

(1) (P, d) is an (resp., a weak) S-quasi-metric space;
(2) $(P, \mathcal{O}_d)$ is a (resp., weak) d-space whose specialization ordering is $\leq$.

Corollary 2.12. *Let (X, d) be a hemi-metric space. Then $(X, \mathcal{O}_d)$ is a (resp., weak) d-space if and only if (X, d) is an (resp., a weak) S-quasi-metric space with respect to the specialization ordering.*

We call a T_0 space X (resp., *weak*) *S-quasi-metrizable* if there exists a hemi-metric d on X such that $\mathcal{O}(X) = \mathcal{O}_d$ and (X, d) under the specialization ordering is an (resp., a weak) S-quasi-metric space.

Corollary 2.13. *Let X be a T_0 space. Then X is a quasi-metrizable (resp., weak) d-space if and only if it is (resp., weak) S-quasi-metrizable.*

From the above result, we obtain the following:

$$\text{quasi-metrizable } d\text{-spaces} = \text{S-quasi-metric spaces.}$$

Corollary 2.14. *The Scott topology of a dcpo (resp., a poset) is quasi-metrizable if and only if it is (resp., weak) S-quasi-metrizable.*

Since the Scott space of an ω-continuous dcpo is a second-countable d-space (see [4, Theorem III-4.5]), we have the following corollary.

Corollary 2.15. *Every Scott space of an ω-continuous dcpo is S-quasi-metrizable.*

3. Some examples of S-quasi-metric spaces

We have introduced the notion of S-quasi-metric spaces in the last section, and notably every open set in the open ball topology of d is a Scott open subset of P in any S-quasi-metric space (P, d). In this part, we will show that the Scott topology and the open ball topology may not coincide in an S-quasi-metric space (see Example 3.4 below). Furthermore, it is shown that many common examples (including the space of formal balls, see Example 3.8) in domain theory are S-quasi-metrizable.

For posets $(P, \leq_P)$ and $(Q, \leq_Q)$, the *pointwise ordering* on the product $P \times Q$ is defined by $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq_P x_2$ and $y_1 \leq_Q y_2$. One should note that $P \times Q$ with the pointwise ordering is a dcpo whenever P and Q are dcpos.

Lemma 3.1. *If (P, d_1) and (Q, d_2) are S-quasi-metric spaces, then both $d_1 \vee d_2$ (the maximum of d_1 and d_2) and $d_1 + d_2$ are S-quasi-metrics on $P \times Q$ with the pointwise ordering, where*

$$(d_1 \vee d_2)(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}, \text{ and } (d_1 + d_2)(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$$

for any $x = (x_1, x_2), y = (y_1, y_2) \in P \times Q$.

Proof. We only check that $d_1 \vee d_2$ satisfies condition (SM1).

Suppose $x = (x_1, x_2) \in P \times Q$ and D is a directed subset of $P \times Q$. Let $D_1 = \{y_1 \in P : \exists y_2 \in Q, (y_1, y_2) \in D\}$ and $D_2 = \{y_2 \in Q : \exists y_1 \in P, (y_1, y_2) \in D\}$. Then D_1 and D_2 are directed sets such that $D \subseteq D_1 \times D_2$ and $\bigvee D = (\bigvee D_1, \bigvee D_2)$.

On the one hand, we have

$$\begin{aligned}
(d_1 \vee d_2)(x, \textstyle\bigvee D) &= \max\{d_1(x_1, \textstyle\bigvee D_1), d_2(x_2, \textstyle\bigvee D_2)\} \\
&= \max\{\textstyle\bigwedge_{y_1 \in D_1} d_1(x_1, y_1), \textstyle\bigwedge_{y_2 \in D_2} d_2(x_2, y_2)\} \\
&= \textstyle\bigwedge_{(y_1, y_2) \in D_1 \times D_2} \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\
&\leq \textstyle\bigwedge_{(y_1, y_2) \in D} \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\
&= \textstyle\bigwedge_{y \in D} (d_1 \vee d_2)(x, y).
\end{aligned}$$

On the other hand, for any $y_1 \in D_1$ and $z_2 \in D_2$, there exist $y_2 \in Q$ and $z_1 \in P$ such that $y = (y_1, y_2) \in D$ and $z = (z_1, z_2) \in D$. Since D is directed, there exists $u = (u_1, u_2) \in D$ such that $y, z \leq u$, which implies that $y_1 \leq u_1$ and $z_2 \leq u_2$. From Proposition 2.3, it follows that

$$d_1(x_1, u_1) \leq d_1(x_1, y_1) \text{ and } d_2(x_2, u_2) \leq d_2(x_2, z_2),$$

which implies that $(d_1 \vee d_2)(x, u) \leq \max\{d_1(x_1, y_1), d_2(x_2, z_2)\}$. By the arbitrariness of y, z, we deduce that

$$\begin{aligned}\bigwedge_{u \in D}(d_1 \vee d_2)(x, u) &\leq \bigwedge_{y_1 \in D_1, z_2 \in D_2} \max\{d_1(x_1, y_1), d_2(x_2, z_2)\} \\ &= \max\{\bigwedge_{y_1 \in D_1} d_1(x_1, y_1), \bigwedge_{y_2 \in D_2} d_2(x_2, y_2)\} \\ &= \max\{d_1(x_1, \bigvee D_1), d_2(x_2, \bigvee D_2)\} \\ &= (d_1 \vee d_2)(x, \bigvee D).\end{aligned}$$

Therefore, $d_1 \vee d_2$ satisfies (SM1).

Similarly, one can prove that $d_1 + d_2$ also satisfies (SM1). □

Example 3.2 *([5, Example 6.1.2]).* Let $\mathbb{R}$ be the set of real numbers with the usual ordering. There is a simple example of a hemi-metric on $\mathbb{R}$, defined by

$$d_{\mathbb{R}}(x, y) = \max\{x - y, 0\}.$$

In fact, one can easily check that $d_{\mathbb{R}}$ is an S-quasi-metric on $\mathbb{R}$. If we restrict $\mathbb{R}$ to the dcpo $\mathbb{I} = [0, 1]$, then we obtain an S-quasi-metric $d_{\mathbb{I}}$ on $\mathbb{I}$, that is, $(\mathbb{I}, d_{\mathbb{I}})$ is an S-quasi-metric space. In addition, we have that $\sigma(\mathbb{R}) = \mathcal{O}_{d_{\mathbb{R}}}$ and $\sigma(\mathbb{I}) = \mathcal{O}_{d_{\mathbb{I}}}$, since $B_r(x) = (x - r, +\infty)$ and $B_r(y) = (y - r, 1] \cap \mathbb{I}$ for any $x \in \mathbb{R}$, $y \in \mathbb{I}$ and $r > 0$. Therefore, $\Sigma\mathbb{R}$ is weak S-quasi-metrizable, and $\Sigma\mathbb{I}$ is S-quasi-metrizable.

Example 3.3. Let $\mathbb{I}^2 = [0, 1] \times [0, 1]$ with the pointwise ordering, and define a mapping $d_{\mathbb{I}^2}$ from $\mathbb{I}^2 \times \mathbb{I}^2$ to $[0, +\infty)$ by

$$d_{\mathbb{I}^2}((x_1, y_1), (x_2, y_2)) = \max\{x_1 - y_1, x_2 - y_2, 0\}.$$

Note that $d_{\mathbb{I}^2} = d_{\mathbb{I}} \vee d_{\mathbb{I}}$. By Example 3.2 and Lemma 3.1, $d_{\mathbb{I}^2}$ is an S-quasi-metric on the dcpo $\mathbb{I}^2$. Since $B_r((x, y)) = (x - r, 1] \times (y - r, 1] \cap \mathbb{I}^2$ for any $(x, y) \in \mathbb{I}^2$ and $r > 0$, we have that $\mathcal{O}_{d_{\mathbb{I}^2}} = \sigma(\mathbb{I}^2)$. Therefore, the Scott topology of $\mathbb{I}^2$ is S-quasi-metrizable.

The following example shows that the Scott topology and the open ball topology may not coincide in an S-quasi-metric space.

Example 3.4. Let $P = [0, 1] \times [0, 1]$ with the ordering $\leq$ defined by

$$(x_1, y_1) \leq (x_2, y_2) \;\Leftrightarrow\; x_1 = x_2 \text{ and } y_1 \leq y_2.$$

Then $(P, \leq)$ is a continuous dcpo. Define $d : P \times P \longrightarrow [0, \infty)$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{y_1 - y_2, 0\} + |x_1 - x_2|.$$

Note that $(x_1, y_1) \ll (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 \ll y_2$ in $[0, 1]$ with the usual ordering, where $y_1 \ll y_2$ if and only if $y_1 = 0$ or $y_1 < y_2$. One can easily check that d is an S-quasi-metric on P by using Proposition 2.6. From Lemma 2.8, it follows that $\mathcal{O}_d \subseteq \sigma(P)$. We claim that $\sigma(P) \nsubseteq \mathcal{O}_d$. Consider

$U = [\frac{1}{2}, 1] \times (\frac{1}{2}, 1]$. Then $U = \bigcup_{\frac{1}{2} \leq x \leq 1} \uparrow(x, \frac{1}{2})$, hence is Scott open. In addition, we have that $(\frac{1}{2}, 1) \in U$, and for each $r > 0$ (we may assume $r < 1$), there is a point $(\frac{1-r}{2}, 1) \in B_r((\frac{1}{2}, 1)) \setminus U$, which means that $B_r((\frac{1}{2}, 1)) \nsubseteq U$. Then we deduce that U is not open in the open ball topology $\mathcal{O}_d$. Therefore, $\mathcal{O}_d \neq \sigma(P)$.

Example 3.5. Let $P = \mathbb{N}^+ \cup \{\omega_0\}$ with the ordering $\leq$ defined by

$$1 \leq 2 \leq 3 \leq 4 \leq \ \cdots \ \leq \omega_0.$$

From Corollary 2.15, it follows that the Scott space ΣP (i.e., the open sets are $\emptyset$, P and all sets of the form $\uparrow n$, $n \in \mathbb{N}^+$) is S-quasi-metrizable. In addition, the mapping $d : P \times P \longrightarrow [0, +\infty)$ defined as follows:

(1) $d(x, y) = 0$, if $x \leq y$;
(2) $d(m, n) = \max\{\frac{1}{n} - \frac{1}{m}, 0\}$ for $m, n \in \mathbb{N}^+$;
(3) $d(\omega_0, n) = \frac{1}{n}$ for $n \in \mathbb{N}^+$,

is an S-quasi-metric on P that exactly induces $\sigma(P)$, that is, the open ball topology of d and the Scott topology on P coincide. By Lemma 2.8, it suffices to show that $\sigma(P) \subseteq \mathcal{O}_d$. For each $n \in \mathbb{N}^+$, if $n = 1$, then $\uparrow n = P$, which is clearly open in $\mathcal{O}_d$. Now assume $n \geq 2$. Let $r_n = \frac{1}{2(n-1)}$ and $x_n = 2(n-1)$. Then $r_n > 0$ and $x_n \in P$, and we have the following:

$$\begin{aligned} k \in B_{r_n}(x_n) \Leftrightarrow d(x_n, k) < r_n \ &\Leftrightarrow \ \max\{\tfrac{1}{k} - \tfrac{1}{2(n-1)}, 0\} < \tfrac{1}{2(n-1)} \\ &\Leftrightarrow \tfrac{1}{k} - \tfrac{1}{2(n-1)} < \tfrac{1}{2(n-1)} \ \Leftrightarrow \ k > n-1 \ \Leftrightarrow \ k \in \uparrow n, \end{aligned}$$

which implies that $B_{r_n}(x_n) = \uparrow n \in \mathcal{O}_d$. This shows that $\sigma(P) \subseteq \mathcal{O}_d$. Therefore, $\sigma(P) = \mathcal{O}_d$.

An element a of a poset P is called a *co-prime element* if $a \leq x \vee y$ implies either $a \leq x$ or $a \leq y$ whenever $x \vee y$ exists. We use $J(P)$ to denote the set of all co-prime elements of P excluding the bottom element of P (if it exists).

Remark 3.6. For any complete lattice [26, Lemma 1] (resp., completely distributive lattice [4, Theorem V-1.7]) L, $J(L)$ is a dcpo (resp., a continuous dcpo) under the inherited ordering from L.

Example 3.7. The theory of pointwise quasi-metric spaces, introduced by Shi [14,15], has an important position in fuzzy topology. Let X be a nonempty set, and L be a completely distributive lattice. Note that the set L^X of all mappings from X to L with the pointwise ordering (i.e., $\phi \leq \varphi$ in L^X iff $\phi(x) \leq \varphi(x)$ in L for all $x \in X$) is also a completely distributive lattice. A function $d : J(L^X) \times J(L^X) \longrightarrow [0, +\infty)$ is called a *pointwise quasi-metric* on L^X if d is a hemi-metric on $J(L^X)$ and satisfies the following conditions: $\forall \phi, \varphi \in J(L^X)$,

(i) $d(\phi, \varphi) = \bigwedge_{\psi \ll \varphi} d(\phi, \psi)$;
(ii) $d(\phi, \varphi) = 0$ implies $\phi \leq \varphi$.

By Proposition 2.6, d is an S-quasi-metric on $J(L^X)$. Therefore, every pointwise quasi-metric in the sense of Shi is an S-quasi-metric.

The following example shows that the poset of formal balls of a complete metric space, with its Scott topology is always S-quasi-metrizable.

Example 3.8. A formal ball in a metric space (X, d) is a pair (x, r) with $x \in X$ and $r \in [0, +\infty)$. The ordering $\sqsubseteq$ on the set $\mathbf{B}X$ of formal balls is defined by

$$(x, r) \sqsubseteq (y, s) \;\Leftrightarrow\; d(x, y) \leq r - s.$$

Note that $\mathbf{B}X$ is a continuous poset, and it is a continuous dcpo iff X is complete (see [2,5] for more details). A mapping $d^+ : \mathbf{B}X \times \mathbf{B}X \longrightarrow [0, +\infty)$ is defined by

$$d^+((x, r), (y, s)) = \max\{d(x, y) - r + s, 0\}.$$

Then d^+ is an S-quasi-metric on $\mathbf{B}X$ such that $\sigma(\mathbf{B}X) = \mathcal{O}_{d^+}$ (refer to [5, Definition 7.3.1, Exercise 7.3.13]). Therefore, the space of formal balls of any metric space (resp., complete metric space) is weak S-quasi-metrizable (resp., S-quasi-metrizable).

Note that every metric space (X, d) is homeomorphic to the maximal points space of $\mathbf{B}X$ [2]. By Example 3.8, we obtain the following corollary.

Corollary 3.9. *Every (resp., complete) metric space is homeomorphic to the maximal points space of a weak S-quasi-metric (resp., an S-quasi-metric) space.*

4. Coincidence of the open ball topology and the Scott topology

In the previous sections, we have seen that the Scott topology is generally coarser than the open ball topology in an S-quasi-metric space, not vice versa (see Lemma 2.8 and Example 3.4). In this part, we give a condition for that the open ball topology and the Scott topology coincide in an S-quasi-metric space.

Definition 4.1. Let P be a poset, and d be an S-quasi-metric on P.

(1) A sequence $\{x_n\}_{n \in \mathbb{N}^+}$ of P *converges* to x if $\lim\limits_{n \to +\infty} d(x, x_n) = 0$.
(2) The space (P, d) is said to satisfy *property M* if for each sequence $\{x_n\}_{n \in \mathbb{N}^+}$ that converges to x, there exists a directed subset D of $\downarrow\{x_n : n \in \mathbb{N}^+\}$ such that $\bigvee D$ exists and $x \leq \bigvee D$.

Theorem 4.2. *Let P be a poset, and d be an S-quasi-metric on P. If it satisfies property M, then $\sigma(P) = \mathcal{O}_d(P)$.*

Proof. By Theorem 2.9, it suffices to prove $\sigma(P) \subseteq \mathcal{O}_d(P)$. Let $U \in \sigma(P)$ and $x \in U$. We need to find a $r > 0$ satisfying that $B_r(x) \subseteq U$. Assume on the contrary that $B_r(x) \nsubseteq U$ for all $r > 0$. Then for each $n \in \mathbb{N}^+$, there exists $x_n \in B_{\frac{1}{n}}(x)$ such that $x_n \notin U$. Then we have that $\lim\limits_{n \to +\infty} d(x, x_n) = 0$. Since (P, d) satisfies property M, there exists a directed subset D of $\downarrow\{x_n : n \in \mathbb{N}^+\}$ such that $x \leq \bigvee D$. Since U is Scott open and $x \in U$, it follows that $\bigvee D \in U$, and there exists $y \in D \cap U$. Recall that $D \subseteq \downarrow\{x_n : n \in \mathbb{N}^+\}$, so there exists $n_0 \in \mathbb{N}^+$ such that $y \leq x_{n_0}$. Since U is an upper set and $y \in U$, we have that $x_{n_0} \in U$, contradicting the fact that $x_{n_0} \notin U$. Hence, there exists $r_0 > 0$ such that $B_{r_0}(x) \subseteq U$. This implies that $U \in \mathcal{O}_d(P)$. Therefore, $\sigma(P) \subseteq \mathcal{O}_d(P)$. $\square$

Theorem 4.3. *For each bounded-complete chain P and each S-quasi-metric d on P, $\sigma(P) = \mathcal{O}_d(P)$.*

Proof. By Theorem 4.2, it suffices to prove that (P, d) satisfies property M. Suppose $\{x_n\}_{n \in \mathbb{N}^+}$ is a sequence in (P, d) that converges to x. Let $D = \{x \wedge x_n : n \in \mathbb{N}^+\}$. Clearly, D is a (nonempty) directed subset of

P that has an upper bound x. Since P is bounded-complete, $\bigvee D$ exists. Note that $d(x, x \wedge x_n) = d(x, x_n)$ if $x \leq x_n$, and $d(x, x \wedge x_n) = d(x, x) = 0$ otherwise. We then have that $d(x, x \wedge x_n) \leq d(x, x_n)$ for each $n \in \mathbb{N}^+$, and since d is an S-quasi-metric, we have that

$$d(x, \bigvee D) = \bigwedge_{n \in \mathbb{N}^+} d(x, x \wedge x_n) \leq \bigwedge_{n \in \mathbb{N}^+} d(x, x_n) \leq \lim_{n \to +\infty} d(x, x_n) = 0.$$

By Proposition 2.3, $x \leq \bigvee D$. Thus property M is satisfied. $\square$

As an immediate result of Theorem 4.3, we have the following corollary.

Corollary 4.4. *For each S-quasi-metric d on $\mathbb{R}$ with the usual ordering, $\sigma(\mathbb{R}) = \mathcal{O}_d(\mathbb{R})$.*

5. The quasi-metrizability of *D*-completion

An essential tool for the investigation of sober spaces is the b-topology, which is introduced by L. Skula [16] (see also [1]).

Definition 5.1 *([1,16]).* Let X be a T_0 space. The *b-topology* associated with X is the topology which has the family $\{U \cap \mathrm{cl}_X(\{x\}) : x \in U \in \mathcal{O}(X)\}$ as an base, and the resulting space is denoted by bX. We say a subset B of X *b-dense* in X, if it is dense in bX.

Lemma 5.2. *Let X be a T_0 space, and Y be a b-dense subspace of X.*

(1) For each $V \in \mathcal{O}(X)$, $V = Sat_X(V \cap Y)$.
(2) For each $U \in \mathcal{O}(Y)$, $Sat_X(U) \in \mathcal{O}(X)$.

Proof. (1) Since every open set is saturated, it follows that $Sat_X(V \cap Y) \subseteq V$. Let $x \in V$. Then $\mathrm{cl}_X(\{x\}) \cap V$ is b-open in X, and since Y is b-dense in X, it follows that $\mathrm{cl}_X(\{x\}) \cap V \cap Y \neq \emptyset$, which implies that $x \in Sat_X(V \cap Y)$. Hence, $V \subseteq Sat_X(V \cap Y)$.

(2) Since U is open in Y, there exists an open set V in X such that $U = V \cap Y$. By (1), we have that $V = Sat_X(V \cap Y) = Sat_X(U)$. Hence, $Sat_X(U)$ is open in X. $\square$

Theorem 5.3. *Let X be a T_0 space, and Y be a b-dense subspace of X. Define $\varphi : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ and $\psi : \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ by*

$$\varphi(U) = Sat_X(U) \text{ and } \psi(V) = V \cap Y,$$

for any $U \in \mathcal{O}(Y)$ and any $V \in \mathcal{O}(X)$. Then φ and ψ are order-isomorphisms under the inclusion ordering, and they are inverse to each other.

Proof. First, φ is well-defined by Lemma 5.2, and it is clear that both φ and ψ are order-preserving. In addition, we have that $\psi \circ \varphi(U) = Sat_X(U) \cap Y = Sat_Y(U) = U$ for any $U \in \mathcal{O}(Y)$, and from Lemma 5.2 it follows that $\varphi \circ \psi(V) = Sat_X(V \cap Y) = V$ for any $V \in \mathcal{O}(X)$. Therefore, φ and ψ are order-isomorphisms that are inverse to each other. $\square$

Definition 5.4 *([3,21]).* Let X be a T_0 space. The *D-completion* (resp., *well-filtered reflection*, *sobrification*) of X is a d-space (resp., well-filtered space, sober space) Y together with a continuous mapping $\eta : X \longrightarrow Y$, such that for any continuous mapping $f : X \longrightarrow Z$ to any d-space (resp., well-filtered space, sober space)

Z, there exists a unique continuous mapping $g : Y \longrightarrow Z$ satisfying $f = g \circ \eta$, that is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & Y \arrow[d, dotted, "g"] \\ & Z. \end{tikzcd}$$

Remark 5.5. In the following, we only use Y to denote the D-completion (well-filtered reflection or sobrification) of X instead of the pair (Y, η), if it exists.

A remarkable result on d-spaces, well-filtered spaces and sober spaces, is that every T_0 space X has a reflection in each of the corresponding categories [3,21,22]. We use $D(X)$, $W(X)$ and $S(X)$ to denote the D-completion, well-filtered reflection and sobrification of X, respectively. In fact, their relations up to homeomorphism are as follows:

$$X \subseteq D(X) \subseteq W(X) \subseteq S(X).$$

It is shown in [7, 3.1.2] that every T_0 space X is b-dense in $S(X)$ (see also [8, Proposition 3.2]), from which we deduce that X is also b-dense in both $D(X)$ and $W(X)$. Recently, Xu [24] introduced a more general structure, called H-sober spaces, which provides a uniform approach to d-spaces, sober spaces and well-filtered spaces. It turns out that each H-sobrification $H(X)$ of a T_0 space X exists, and since $H(X) \subseteq S(X)$, it follows that X is b-dense in $H(X)$. As a corollary, we obtain the following well-known result.

Corollary 5.6 *([3,5,24]). Let X be a T_0 space. The lattice of open sets of X and that of $H(X)$ are order-isomorphic, where $H(X)$ can be any one of $D(X), W(X)$ and $S(X)$.*

By Corollary 5.6, the following theorem is clear.

Theorem 5.7. *For a T_0 space X, the following statements are equivalent:*

(1) X is second-countable (resp., core-compact);
(2) $H(X)$ is second-countable (resp., core-compact), where $H(X)$ can be any one of $D(X), W(X)$ and $S(X)$.

The above theorem is generally not true for first-countability, as shown in the following counterexample.

Example 5.8 *([23, Example 5.12]).* Let $[0, \omega_1)$ be the set of all ordinals less than ω_1, and $X = \Sigma[0, \omega_1)$ (whose Scott open sets are $\emptyset$ and all sets of form $\uparrow\alpha$, where α is not a limit ordinal in $[0, \omega_1)$).

(a) X is not a d-space, since $[0, \omega_1)$ is a directed set whose supremum does not exist.
(b) X is first-countable, since each point $\alpha \in X$ has a countable neighborhood base

$$\{\uparrow\beta : \beta < \alpha \text{ and } \beta \text{ is not a limit ordinal}\}.$$

(c) Note that X itself is the unique closed directed set (hence is the unique KF-set in the sense of [13], and also is the unique irreducible set) which is not the closure of any point in X, i.e., $[0, \omega_1) \neq \downarrow\alpha$ for any $\alpha < \omega_1$. Then the D-completion $D(X)$, the well-filtered reflection $W(X)$, and the sobrification $S(X)$ of X coincide, each of which is homeomorphic to the space $Y = [0, \omega_1]$ whose open sets are $\emptyset$ and all sets of form $\uparrow\alpha$, $\alpha < \omega_1$ and α is not a limit ordinal (refer to Theorem 5.3):

$$D(X) = W(X) = S(X) = Y \text{ (up to homeomorphism).}$$

(d) It is a standard result that for any countable subset D of $[0, \omega_1)$, $\bigvee D < \omega_1$. For any family $\{\uparrow\alpha_n : n \in \mathbb{N}^+\}$ of countable open subsets (containing ω_1) of $Y = [0, \omega_1]$, since for any $n \in \mathbb{N}^+$, α_n is not a limit ordinal, we have that $\{\alpha_n : n \in \mathbb{N}^+\} \subseteq [0, \omega_1)$, and hence $\bigvee\{\alpha_n : n \in \mathbb{N}^+\} < \omega_1$. Then there exists an ordinal α such that $\bigvee\{\alpha_n : n \in \mathbb{N}^+\} < \alpha < \omega_1$ and α is not a limit ordinal. Thus $\uparrow\alpha$ is an open neighborhood of ω_1 in Y, but $\uparrow\alpha_n \nsubseteq \uparrow\alpha$ for any $n \in \mathbb{N}^+$. This shows that $\{\uparrow\alpha_n : n \in \mathbb{N}^+\}$ is not a neighborhood base for ω_1. Thus there is no countable base at the point ω_1 in Y, so Y it is not first-countable.

Then we conclude that X is a first-countable space, but its D-completion $D(X)$ (which is homeomorphic to its well-filtered reflection $W(X)$ and sobrification $S(X)$) is not first-countable.

It is then natural to ask the following:

- for a quasi-metrizable space X, whether or not $H(X)$ is hemi-metrizable, where $H(X)$ can be any one of $D(X), W(X)$ and $S(X)$.

We will give a negative answer to that question.

Given a poset P, the family $\alpha(P)$ of all upper subsets of P forms a topology, called the *Alexandroff topology*. We denote $\mathcal{A}P = (P, \alpha(P))$.

Remark 5.9. [5] Let $(P, \leq_P)$ be a poset, and $X = \mathcal{A}P$.

(1) The specialization ordering of X is $\leq_P$.
(2) The family $\{\uparrow x : x \in P\}$ forms a base for X.

Lemma 5.10. *For each poset P, $\mathcal{A}P$ is quasi-metrizable.*

Proof. Let us define a mapping $d : P \times P \longrightarrow [0, +\infty)$ by $d(x, y) = 0$ if $x \leq y$, and $d(x, y) = 1$ if $x \nleq y$. It is trivial to check that d is a quasi-metric on P. In addition, for each $x \in P$ and $1 \geq r > 0$, we have $B_r(x) = \uparrow x$. Note that $\{\uparrow x : x \in P\}$ forms a base for $\alpha(P)$, thus we deduce that $\mathcal{O}_d = \alpha(P)$. □

Example 5.11. Let $X = \mathcal{A}[0, \omega_1)$ be the Alexandroff topological space (the open sets are $\emptyset$, and all sets of form $\uparrow\alpha$, $\alpha < \omega_1$).

(a) By Lemma 5.10, X is quasi-metrizable.
(b) X is not a d-space, since $[0, \omega_1)$ is a directed set whose supremum does not exist.
(c) using similar arguments for proving Example 5.8 (c) and (d), we have that the D-completion $D(X)$ (which is homeomorphic to the well-filtered reflection $W(X)$ and the sobrification $S(X)$) of X is not first-countable, hence is not quasi-metrizable.

From Examples 5.8 and 5.11, for a T_0 space X we obtain the following:

X	second-countable	quasi-metrizable	first-countable
$D(X)$	$\surd$	$\times$	$\times$

Corollary 5.12. *There exists a quasi-metric space whose D-completion (well-filtered reflection, or sobrification) is not quasi-metrizable.*

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