The reflectivity of some categories of T_0 spaces

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Overview

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4. Summary

In the paper, J.F. Kennison[1] gives three types of full reflective subcategories of all topological spaces **Top** called simple, identifying, and embedding, as follows:

Definition (Kennison 1965)

Let **P** be a full subcategory of TOP, and $F : \mathbf{Top} \longrightarrow \mathbf{P}$ be a reflector from the category of topology. F is called

(1) simple: if $e_X : X \longrightarrow F(X)$ is bijective for all X;

(2) identifying: if $e_X(X) = F(X)$ for all X;

(3) embedding: if each object of P is a Hausdorff space and if $e_X(X)$ is a dense subset of $F(X)$ for all X.

The full category **P** is simple (resp., identifying or embedding) if there exists a simple (resp., identifying or embedding) reflector $F : \mathbf{Top} \longrightarrow \mathbf{P}$.

[1] J.F. Kennison, Reflective functors in general topology and elsewhere, Trans. Amer. Math. Soc. 118 (1965), 303–315.

In the paper[2], Kennison gave the characterizations of the three reflectors:

Theorem (A)

A topological property **P** *is simple iff* **P** *is hereditary, productive and contains every indiscrete space.*

Theorem (B)

A topological property **P** *is identifying iff* **P** *is hereditary and productive.*

Theorem (C)

A topological property **P** *is embedding iff* **P** *is closed-hereditary, productive and contains only Hausdorff spaces.*

[2] J.F. Kennison, Reflective functors in general topology and elsewhere, Trans. Amer. Math. Soc. 118 (1965), 303–315.

In the paper, J.F. Kennison[3] gives three types of full reflective subcategories of all topological spaces, but

- he doesn't know whether these three types include all the full reflective subcategories of all topological spaces.

In the paper[4], L. Skula gave an Negative answer, and proposed another type called *b*-embedding NOT mentioned by Kennison. Then he show that

Theorem (Skula 1969)

If **P** *is a full reflective subcategory of the category of* **Top** *containing at least one non-T_1 space, then* **P** *is a subcategory of one of the above-mentioned 4 types.*

[3] J.F. Kennison, Reflective functors in general topology and elsewhere, Trans. Amer. Math. Soc. 118 (1965), 303–315.

[4] L. Skula, On a reflective subcategory of the category of all topological spaces, Trans. Amer. Math. Soc. 142 (1969) 37–41.

Why Skula topology?

Let $A \subseteq X \in \mathbf{Top}$, $Y \in \mathbf{Top}_0$, and $f\colon A \longrightarrow Y$ a continuous map.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

It requires that there is **at most one** continuous extension g of f.

Otherwise, $\exists$ conti. map $g_1 \neq g_2$ s.t. $g_1|_A = g_2|_A = f$. Then $\exists x_0 \in X$ s.t.

- $g_1(x_0) \neq g_2(x_0)$ in Y.
- $\exists V_0 \in \mathcal{O}(Y)$, $g_1(x_0) \in V_0$ and $g_2(x_0) \notin V_0$ (without loss of generality).
- $\exists U_1, U_2 \in \mathcal{O}(X)$, $x_0 \in U_1 - U_2$ and $U_1 \cap A = U_2 \cap A$. (take $U_i = g_i^{-1}(V_0)$).

As a consequence, we obtain

Proposition (Skula 1969)

The extension is at most one for all $Y \in \mathbf{Top}_0$ iff $\forall x \in X$, $\nexists U_1, U_2 \in \mathcal{O}(X)$ s.t. $x \in U_1 - U_2$ and $U_1 \cap A = U_2 \cap A$.

Let $A \subseteq X \in \mathbf{Top}$, and define

- $x \in \widehat{A} \ \Leftrightarrow \nexists U_1, U_2 \in \mathcal{O}(X)$ s.t. $x \in U_1 - U_2$ and $U_1 \cap A = U_2 \cap A$.

Proposition (Skula 1969)

The following assertion holds:

1. $A \subseteq \widehat{A} \subseteq \overline{A}$;
2. $A \subseteq B \ \Rightarrow \ \widehat{A} \subseteq \widehat{B}$;
3. $\widehat{A} = \widehat{\widehat{A}}$;
4. $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$.

This new topology is the so called b-topology, denoted by bX.

Proposition

A is b-dense in X iff $U_1 \cap A = U_2 \cap A$ implies $U_1 = U_2$ for all $U_1, U_2 \in \mathcal{O}(X)$. Hence, $\mathcal{O}(A) \cong \mathcal{O}(X)$.

In the paper, Hoffmann[5] showed some properties on b-topology.

Theorem (Hoffmann)

For $x \in X$, $\mathcal{U}_b^o(x) = \{\downarrow x \cap U : x \in U \in \mathcal{B}\}$ is a nbd of x in bX.

Proposition (Hoffmann)

The topology of bX is generated by

- $\mathcal{O}(X) \cup \mathcal{C}(X)$ *(consequently, the b-topology is* **finer** *than the origin);*
- *or equivalently, by $\mathcal{O}(X) \cup \{A \subseteq X : A = \downarrow A\}$.*

Lemma (Hoffmann)

For each $x \in U \in \mathcal{O}(X)$, $\downarrow x \cap U$ is both b-closed and b-open, so bX is zero-dimensional, hence is completely regular.

Theorem (Hoffmann)

For each $X \in \mathbf{Top}_0$, bX is Hausdorff, hence is a Tychonoff space.

[5] R.E Hoffmann, On the sobrification remainder $X^s - X$, Pacific J. Math. 83(1), 1979

Using the b-topology, we see that

- Let $A \subseteq X \in \mathbf{Top}$, $Y \in \mathbf{Top}_0$, $f\colon A \longrightarrow Y$:

$$\begin{tikzcd} A \arrow[r, "e_A"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & Y \end{tikzcd}$$

Proposition (Skula 1969)

The extension is at most one for all $Y \in \mathbf{Top}_0$ iff A is b-dense in X.

Proposition (Skula 1969)

If $A \subseteq X \in \mathbf{Top}_0$ s.t. A is a b-dense retract of X, then $A = X$.

Proof:

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, dashed, "fe=id_A"] \arrow[ddr, "e"'] & & X \arrow[dl, dashed, "f"'] \arrow[ddl, "ef=id_X"] \\ & A \arrow[d, dashed, "e"'] & \\ & X & \end{tikzcd}$$

b-embedding reflector

Definition (Skula 1969)

- Let $F : \mathbf{Top} \longrightarrow \mathbf{P}$ be a reflector. F is called a b-embedding reflector iff $\mathbf{P} \subseteq \mathbf{Top}_0$ and if $e_X(X)$ is a b-dense subset of $F(X)$ for all $X \in \mathbf{Top}$.
- A topological property $\mathbf{P}$ is called a b-embedding iff there exists a b-embedding reflector $F : \mathbf{Top} \longrightarrow \mathbf{P}$.
- A topological property $\mathbf{P}$ is b-closed-hereditary if $Y \in \mathbf{P}$ whenever Y is a b-closed subspace of some $X \in \mathbf{P}$.

Theorem (Skula 1969)

A topological property $\mathbf{P}$ *is b-embedding iff* $\mathbf{P}$ *is productive, b-closed-hereditary and* $\mathbf{P} \subseteq \mathbf{Top}_0$.

Theorem (Skula 1969)

Let $\mathbf{P}$ *be a topological property, which is a reflective subcategory of* $\mathbf{Top}$. *If* $\mathbf{P} \subseteq \mathbf{Top}_0$, $\mathbf{P} \nsubseteq \mathbf{Top}_1$, *then* $\mathbf{P}$ *is b-embedding.*

Skula's example

The Sierpinski space $\mathbb{S} = \Sigma 2$, where $2 = \{0, 1\}$ with open sets $\emptyset, 2, \{1\}$. Then the power $\prod_{i \in I} \mathbb{S} = \Sigma(2^I, \subseteq)$, since $\uparrow\chi_F = \bigcap_{i \in F} p^{-1}(\{1\})$.

- **P′** the class of all spaces of the type $\prod_{i \in I} \mathbb{S}$ $(I \neq \emptyset)$.
- **P** the class of all b-closed subspaces of a space from **P′** of all spaces homeomorphic to these spaces.

Example (Skula 1969)

Let $X = \prod_{i=1}^{+\infty} \mathbb{S} = \Sigma 2^N$ and $A = X - \{N\}$. Note that each open set in X contains N, so $\widehat{A} = X$, which implies that X is the reflection of A in **P**. Also, **P** is a topological property such that

- **P** is b-embedding.
- e_A is not surjective, so **P** not simple.
- $e_A(A) \neq X$ so not identifying.
- $\mathbf{P} \nsubseteq \mathbf{Top}_2$, so not embedding.

Characterize sobriety via b-topology

Definition

A nonempty subset A of a T_0 space is called irreducible if for any closed sets F_1, F_2, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is called sober, if for any irreducible closed set F of X there is a (unique) point $x \in X$ such that $F = \mathrm{cl}(\{x\})$. [a]

[a] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, D. Scott, Continuous lattices and Domains, Encyclopedia of Mathematics and Its Applications, Vol. 93, Cambridge University Press, 2003.

The b-topology is a very effective tool for studying sober spaces.

Theorem (Keimel and Lawson 2009)

*Let $A \subseteq X \in$ **Sober**. TFAE:*

(1) *A is sober iff A is b-closed.*

(2) $A^s \cong \mathrm{cl}_b(A)$. [a]

[a] K. Keimel, J.D. Lawson, *D*-completions and the *d*-topology, Ann. Pure Appl. Logic 159 (2009) 292–306.

For d-spaces (also called monotone convergence spaces), Kemimel and Lawson [6] presented the notion of d-topology (see also Zhao and Fan [7]), a natural question is that

- what about the well-filtered spaces?

More precisely,

- whether the class of all well-filtered subspaces of a well-filtered spaces forms a (co-)topology?

[6] K. Keimel, J.D. Lawson, *D*-completions and the *d*-topology, Ann. Pure Appl. Logic 159 (2009) 292–306.

[7] D. Zhao, T. Fan, Dcpo-completion of posets, Theor. Comp. Sci. 411 (2010) 2167–2173.

In domain theory, the mostly concerned topological spaces are usually just T_0. We use

Top$_0$ all T_0 spaces + continuous maps.

It is a popular topic for studying the reflectivity of subcategories of T_0, for example:

- well-filtered spaces ✓, solved by Wu, Xu, Xi, Zhao (2019)
- k-bounded sober spaces $\times$, solved by Lu, Wang, Wu, Zhao (2020)
- strong-d-spaces $\times$, solved by Jin, Miao, Li (2021)

The following problem are still open:

- open well-filtered spaces ? introduced by Shen, Xi, Xu, Zhao (2020)
- co-sober spaces ? asked by Xu, Zhao (2020)

Next, using Skula's b-topology, we give negative answers for the last two questions. Also, the non-reflectivity of k-bounded sober spaces and strong d-spaces are easily obtained [8].

[8] C. Shen, X. Xi, D. Zhao, the reflectivity of some categories of T_0 spaces in domain theory, arXiv:submit/3952036 [math.GN].

Related papers of the last page

1 G. Wu, X. Xi, X. Xu, D. Zhao, Existence of well-filtered reflections of T_0 topological spaces, Topol. Appl. 267 (2019) 107044.

2 X. Xu, D. Zhao, Some open problems on well-filtered spaces and sober spaces, Topol. Appl. (2020) 107540.

3 M. Jin, H. Miao, Q. Li, On some open problems concerning strong d-spaces and super H-sober spaces, arXiv:2109.11299 [math.GN].

4 C. Shen, X. Xi, X. Xu, D. Zhao, On open well-filtered spaces, Logic Meth. Computer Sci. 16 (4) (2020) 4–18.

Definition

A subcategory $\mathbf{K}$ of $\mathbf{Top_0}$ is called reflective, if $\forall X \in \mathbf{Top_0}$, $\exists X^k \in \mathbf{K}$ (the $\mathbf{K}$-completion) and a continuous map $\mu_X : X \longrightarrow X^k$ (the $\mathbf{K}$-reflection) s.t. for any conti. map $f\colon X \longrightarrow Z \in \mathbf{K}$, $\exists$ a unique conti. map $g : X^k \longrightarrow Z$ such that $g \circ \mu_X = f$:

$$\begin{tikzcd} X \arrow[r, "\mu_X"] \arrow[dr, "f"'] & X^k \arrow[d, dotted, "g"] \\ & Z. \end{tikzcd}$$

Definition

Let $f\colon X \longrightarrow Y$ be a map between topological spaces. We call f a *b*-dense embedding, if it is a topological embedding such that $e(X)$ is b-dense in Y.

Theorem

Let $\mathbf{K}$ be a reflective subcategory of $\mathbf{Top_0}$ such that $\mathbf{K} \nsubseteq \mathbf{Top_1}$. Then each $\mathbf{K}$-reflection is a b-dense embedding.

Theorem

Let $\mathbf{K}$ *be a reflective subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{Top_1}$. *Then the following statements hold.*

(1) $\mathbf{K}$ *is b-closed-hereditary.*

(2) *The Sierpiński space* $\Sigma 2 \in \mathbf{K}$. *Hence, for any set* M, *the product* $(\Sigma 2)^M \in \mathbf{K}$.

(3) $\mathbf{Sob} \subseteq \mathbf{K}$.

Corollary

Let $\mathbf{K}$ *be a reflective subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{Top_1}$. *Suppose* $A \subseteq X \in \mathbf{K}$. *If* $A = \uparrow A$, *then* A *as a subspace of* X *is in* $\mathbf{K}$.

Let **Sier** be the full subcategory of $\mathbf{Top_0}$ consisting of all spaces $X \cong \Sigma 2$.

Corollary (Nel, Wilson 1972)

The reflective hull of **Sier** *in* $\mathbf{Top_0}$ *is* **Sob**[a].

[a] L.D. Nel, R.G. Wilson, Epireflections in the category of T_0-spaces, Fund. Math. 75 (1972) 69–74.

Theorem ((A))

Let $\mathbf{K}$ *be a subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{Top_1}$. *Then* $\mathbf{K}$ *is reflective if and only if it is productive and b-closed-hereditary.*

Definition

A category $\mathbf{K}$ has equalizers if for any morphisms $f, g : X \longrightarrow Y$ in $\mathbf{K}$, the equalizer $E_{f,g} = \{x \in X : f(x) = g(x)\}$ of f and g belongs to $\mathbf{K}$.

Lemma

Let $X \in \mathbf{Top_0}$ *and* $E \subseteq X$. *TFAE:*

(1) E *is b-closed in* X;

(2) *there exist continuous maps* $f, g : X \longrightarrow (\Sigma 2)^M$ *for some set* M *such that* $E = \{x \in X : f(x) = g(x)\}$;

(3) *there exist continuous maps* $f, g : X \longrightarrow Y$ *for some* $Y \in \mathbf{Top_0}$ *such that* $E = \{x \in X : f(x) = g(x)\}$.

Theorem

Let $\mathbf{K}$ *be a subcategory of* $\mathbf{Top_0}$ *s.t.* $\{(\Sigma 2)^M : M \text{ is a set}\} \subseteq \mathbf{K}$. *Then* $\mathbf{K}$ *has equalizers iff* $\mathbf{K}$ *is b-closed-hereditary.*

As an immediate result, we deduce the following.

Theorem (B)

Let $\mathbf{K}$ *be a reflective subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{Top_1}$. *Then* $\mathbf{K}$ *is reflective if and only if it is productive and has equalizers.*

The above theorem was given by Nel and Wilson [9].

[9] L.D. Nel, R.G. Wilson, Epireflections in the category of T_0-spaces, Fund. Math. 75 (1972) 69–74.

In 2009, Keimel and Lawson[10] showed that a subcategory $\mathbf{K}$ of T_0 spaces is reflective in the category $\mathbf{Top_0}$ of all T_0 spaces if it satisfies the following **four conditions**:

(K1) $\mathbf{K}$ contains all sober spaces;

(K2) If $X \in \mathbf{K}$ and Y is homeomorphic to X, then $Y \in \mathbf{K}$;

(K3) If $\{X_i : i \in I\} \subseteq \mathbf{K}$ is a family of subspaces of a sober space, then the subspace $\bigcap_{i \in I} X_i \in \mathbf{K}$.

(K4) If $f \colon X \longrightarrow Y$ is a continuous map from a sober space X to a sober space Y, then for any subspace Y_1 of Y, $Y_1 \in \mathbf{K}$ implies that $f^{-1}(Y_1) \in \mathbf{K}$.

Using the above conditions, Wu, Xi, Xu, Zhao[11] firstly proved that the category of well-filtered spaces are reflective in $\mathbf{Top_0}$.

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[10] K. Keimel, J.D. Lawson, *D*-completions and the *d*-topology, Ann. Pure Appl. Logic 159 (2009) 292–306.

[11] G. Wu, X. Xi, X. Xu and D. Zhao, Existence of well-filterification, Topol. Appl. 267 (2019) 107044.

Theorem (C)

Let $\mathbf{K}$ *be a subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{Top_1}$. *Then* $\mathbf{K}$ *is reflective in* $\mathbf{Top_0}$ *if and only if* $\mathbf{K}$ *satisfies the conditions (K1)–(K4).*

From the above results, the characterizations for the reflectivity of $\mathbf{K}$ can be summarized as follows:

Theorem

Let $\mathbf{K}$ *be a subcategory of* $\mathbf{Top_0}$ *such that* $\mathbf{K} \not\subseteq \mathbf{Top_1}$. *Then the following statements are equivalent:*

(1) $\mathbf{K}$ *is reflective in* $\mathbf{Top_0}$;

(2) $\mathbf{K}$ *satisfies conditions (K1)–(K4);*

(3) $\mathbf{K}$ *is productive and b-closed-hereditary;*

(4) $\mathbf{K}$ *is productive and has equalizers.*

The category of open well-filtered spaces

In 2020, Shen, Xi, Xu and Zhao[12] introducing the notion of open well-filtered spaces for providing a more natural proof of Jia-Jung problem: core-compact + well-filtered ⇒ sober, which was firstly solved by Lawson, Wu and Xi [13].

Definition (Shen-Xi-Xu-Zhao 2020)

Let X be a T_0 space.

(1) $\forall U, V \in \mathcal{O}(X)$, define $U \ll V$ iff each open cover of V has a finite subfamily that covers U.

(2) A subfamily $\mathcal{F} \subseteq \mathcal{O}(X)$ is called a $\ll$-filtered family if $\forall U_1, U_2 \in \mathcal{F}$, $\exists U_3 \in \mathcal{F}$ s.t $U_3 \ll U_1, U_2$ in $(\mathcal{O}(X), \subseteq)$.

(3) X is called open well-filtered if for each $\ll$-filtered family $\mathcal{F} \subseteq \mathcal{O}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U \;\Rightarrow\; V \subseteq U$ for some $V \in \mathcal{F}$.

[12] C. Shen, X. Xi, X. Xu, D. Zhao, On open well-filtered spaces, Logic Meth. Computer Sci. 16 (4) (2020) 4–18.

[13] J. Lawson, G. Wu and X. Xi, Well-filtered spaces, compactness, and the lower topology, Houston J. Math. 46 (2020) 283–294.

Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ be the Johnstone's dcpo [14], which is ordered by $(m, n) \leq (m', n')$ iff either $m = m'$ and $n \leq n' \leq \infty$ or $n' = \infty$ and $n \leq m'$, shown in Figure 1:

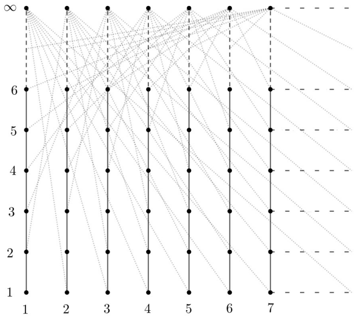

Figure: The Johnstone's dcpo $\mathbb{J}$

[14] P.T. Johnstone, Scott is not always sober, In: Banaschewski B., Hoffmann RE. (eds) Continuous Lattices. Lecture Notes in Mathematics, vol. 871 (1981) 282-283. Springer, Berlin, Heidelberg, https://doi.org/10.1007/BFb0089911.

Proposition

The Scott space $\Sigma\mathbb{J}$ is open well-filtered, and clearly not T_1.

Proposition

The maximal points space $\mathrm{Max}_\sigma\mathbb{J}$ (homeomorphic to $\mathbb{N}_{\mathrm{cof}}$) is not open well-filtered.

Corollary

The category of open well-filtered spaces is not reflective in $\mathbf{Top_0}$.

The category of k-bounded sober spaces

In 2015, Zhao and Ho[15] introduced another weaker notion of sobriety:

Definition (Zhao-Ho 2015)

A T_0 space X is k-bounded sober if for any irreducible closed subset F of X whose $\bigvee F$ exists, there is a unique point $x \in X$ such that $F = \downarrow x$.

The category of all k-bounded sober spaces with continuous maps is denoted by **KSob**. Then

$$\mathbf{Sob} \subseteq \mathbf{KSob} \subseteq \mathbf{Top_0},$$

and since $\mathbf{Sob} \not\subseteq \mathbf{Top_1}$, it follows

$$\mathbf{KSob} \not\subseteq \mathbf{Top_1}.$$

[15] D. Zhao, W. Ho, On topologies defined by irreducible sets, J. Log. Algebr. Methods Program 84 (1) (2015) 185–195.

Example

Let $X = \Sigma[0,3]$ (i.e., the open sets are $\emptyset$, $[0,3]$ and all sets of the form $(x,3]$, $x \in [0,3]$). For each $n \geq 2$, let

$$X_n = [0,1) \cup (2 - \frac{1}{n}, 2 + \frac{1}{n}).$$

(1) X is sober, hence is k-bounded sober.

(2) Each X_n is a k-bounded sober subspace of X.

(3) The intersection $\bigcap_{n\geq 2} X_n = [0,1) \cup \{2\}$ is not k-bounded sober.

Corollary (Lu-Wang-Wu-Zhao 2020)

The category of k-bounded sober spaces is not reflective in $\mathbf{Top_0}$. [a]

[a] J. Lu, K. Wang, G. Wu, B. Zhao, Nonexistence of k-bounded sobrification, Topol. Appl. (2020), arXiv:2011.11606 [math.GN].

To study the dual Hofmann-Mislove Theorem, Escardó, Lawson and Simpson introduced the co-sober spaces[16], which are defined below.

Definition (Escardó-Lawson-Simpson 2004)

Let X be a T_0 space. and Q be a compact saturated subset of X.

(1) Q is called *k-irreducible* if for any compact saturated subsets Q_1, Q_2 of X, $Q = Q_1 \cup Q_2$ implies $Q = Q_1$ or $Q = Q_2$.

(2) X is called *co-sober* if for each k-irreducible set Q, there exists a unique $x \in X$ such that $Q = \uparrow x$.

The category of all co-sober spaces with continuous maps is denoted by **Co-Sob**. Note that **Co-Sob** is a subcategory of $\mathbf{Top_0}$.

[16] M. Escardó, J. Lawson, A. Simpson, Comparing Cartesian closed categories of (core) compactly generated spaces, Topol. Appl. 143 (2004) 105–145.

Let $\mathbb{N}_\alpha$ be the set $\mathbb{N}$ of all natural numbers with the Alexandorff topology (the open sets are $\emptyset$, $\mathbb{N}$ and all the sets of form $\uparrow n$, $n \in \mathbb{N}$).

Lemma

The space $\mathbb{N}_\alpha$ is co-sober, and not T_1.

Theorem (Wen-Xu 2018)

The Isbell's complete lattice equipped with the lower topology is sober but not co-sober. [a]

[a] X.P. Wen and X.Q. Xu, Sober is not always co-sober, Topol. Appl. 250 (2018) 48–52.

From the above results, we have that

$$\mathbf{Sob} \not\subseteq \mathbf{Co\text{-}Sob} \not\subseteq \mathbf{Top_1}.$$

Corollary

The category **Co-Sob** *is not reflective in* $\mathbf{Top_0}$.

The category of strong d-spaces

The strong d-spaces were introduced by Xu and Zhao [17], which lie between the classes of T_1 spaces and that of d-spaces.

Definition (Xu-Zhao 2020)

A T_0 space X is called a strong d-space if for any $x \in X$, directed subset D of X and open subset U of X, $\bigcap_{d \in D} \uparrow d \cap \uparrow x \subseteq U$ implies $\uparrow d_0 \cap \uparrow x \subseteq U$ for some $d_0 \in D$.
The category of strong d-spaces with continuous maps is denoted by **StrongD**.

[17] X. Xu, D. Zhao, On topological Rudin's lemma, well-filtered spaces and sober spaces, Topol. Appl. 272 (2020) 107080.

The following results can be found in Xu and Zhao's paper [18].

Lemma (Xu-Zhao 2020)

(1) *There exists a continuous dcpo P whose Scott topology is not strong d-space (Example 3.34).*

(2) *The Scott topology on every continuous lattice is a strong d-space (Remark 3.21).*

From the above results, we have that

$$\mathbf{Sob} \not\subseteq \mathbf{StrongD} \not\subseteq \mathbf{Top_1}.$$

Corollary

The category of strong d-spaces is not reflective in $\mathbf{Top_0}$.

[18] X. Xu, D. Zhao, On topological Rudin's lemma, well-filtered spaces and sober spaces, Topol. Appl. 272 (2020) 107080.

Summary

The reflectivity of some T_0 spaces:

well-filter	✓
k-bounded sober	×
strong d-space	×
open well-filter	×
co-sober	×

Thanks