Topology and its Applications 367 (2025) 109340

Contents lists available at ScienceDirect

Topology and its Applications

journal homepage: www.elsevier.com/locate/topol

The answers to two problems on maximal point spaces of domains ☆

Xiaoyong Xi [a], Chong Shen [b,c,*], Dongsheng Zhao [d]

[a] School of Mathematics and Statistics, Yancheng Teachers University, Yancheng 224002, Jiangsu, China
[b] School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China
[c] Key Laboratory of Mathematics and Information Networks, Beijing University of Posts and Telecommunications, Ministry of Education, China
[d] Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, 637616, Singapore

ARTICLE INFO

Article history:
Received 26 December 2024
Received in revised form 6 March 2025
Accepted 7 March 2025
Available online 12 March 2025

MSC:
06B35
06B30
54A05

Keywords:
Continuous dcpo
Domain
Domain-representable space
Maximal point space
Scott topology

ABSTRACT

A topological space is domain-representable (or, has a domain model) if it is homeomorphic to the maximal point space $\mathrm{Max}(P)$ of a domain P (with the relative Scott topology). We first construct an example to show that the set of maximal points of an ideal domain P need not be a G_δ-set in the Scott space ΣP, thereby answering an open problem from Martin (2003). In addition, Bennett and Lutzer (2009) asked whether X and Y are domain-representable if their product space $X \times Y$ is domain-representable. This problem was first solved by Önal and Vural (2015). In this paper, we provide a new approach to Bennett and Lutzer's problem.

1. Introduction

The Scott topology of posets, introduced by Dana Scott [15,16], is one of the core structures in domain theory. Spaces that are homeomorphic to the space of maximal elements of a domain (i.e., a continuous dcpo) with the relative Scott topology are referred to as domain-representable by Bennett and Lutzer in [2], whereas Martin, in several of his papers (e.g., [12]), refers to such spaces as having a domain model.

☆ This work was supported by the National Natural Science Foundation of China (No. 12071188, 12101313, 12401607), and the Natural Science Foundation of Jiangsu Province (BK20241086, BK20241087).

* Corresponding author at: School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China.
E-mail address: shenchong0520@163.com (C. Shen).

https://doi.org/10.1016/j.topol.2025.109340
0166-8641/

The concept of representing a topological space as the set of maximal elements of a specific type of poset originates from theoretical computer science [10], which also proves to be highly valuable in the study of completeness properties of Baire spaces. Identifying domain-representable spaces has become one of the active areas in domain theory. Initially, these spaces were named by various terms such as "P-complete" or "FY-complete" by different authors, but the term "directed complete" was eventually introduced in [5]. Fleissner and Yengulalp provided a simplified definition of domain-representability in [6], proving its equivalence to the traditional definition.

One classical example is the poset $\mathrm{IR}(\mathbb{R})$ of all closed intervals of reals, with the inverse set inclusion order, is a domain model of the Euclidean space $\mathbb{R}$. Lawson confirmed that a space is Polish if and only if it has an ω-domain model satisfying the Lawson condition [9]. K. Martin [13] proved that the space $\mathrm{Max}(P)$ of an ω-continuous dcpo P is regular if and only if it is Polish. The formal balls of a metric space, initially introduced by Weihrauch and Schreiber as a domain-theoretic representation of metric spaces [17], were further employed by Edalat and Heckmann to prove that every complete metric space is domain-representable [4]. The Sorgenfrey line, which is not metrizable, is also domain-representable [1]. A more general result is that every T_1 space is homeomorphic to the set of maximal points of some dcpo with the relative Scott topology [18].

As pointed out by Martin [11], knowing that the set of maximal points is a G_δ-subset is often useful in proofs. For instance, Edalat studied the connection between measure theory and the probabilistic powerdomain [4] assuming a separable metric space that embedded as a G_δ-subset of a countably based domain.

The following are some results on G_δ-subsets $E \subseteq \mathrm{Max}(P)$ obtained by Martin:

(i) If P is an ω-domain and $\mathrm{Max}(P)$ is regular, then $\mathrm{Max}(P)$ is a G_δ-set [13].
(ii) For any ω-ideal domain P (see Section 1 for the definition of ω-ideal domain), $\mathrm{Max}(P)$ is a G_δ-set [11].

Furthermore, Brecht, Goubault-Larrecq, Jia, and Lyu [3] showed that continuous valuations on G_δ-subsets of locally compact sober spaces can be extended to measures. They also demonstrated that each G_δ-subset of the maximal point space of an ω-ideal domain is a quasi-Polish space.

This paper focuses on two problems related to maximal point spaces of domains. The first problem was posed by Martin ([11, Section 8, Ideas (iv)]):

- Is there an ideal domain whose maximal points do not form a G_δ-set?

In Section 3, we construct an ideal domain P such that $\mathrm{Max}(P)$ is not G_δ-set, thereby providing a positive answer.

The second problem was posed by Bennett and Lutzer in [1]:

- Suppose the product space $X \times Y$ is domain-representable, where $Y \neq \emptyset$. Must X be domain-representable?

In [14], Önal and Vural proved that domain-representability is hereditary with respect to retracts, meaning that retracts of a domain-representable space are also domain-representable. Consequently, if the product of two spaces is domain-representable, the factor spaces must also be domain-representable; thus gives a positive answer to Bennett and Lutzer's problem. In Section 4, we provide a new proof for the above problem, which is distinct from that of [14] in that it relies directly on the original definition of domain-representable spaces proposed by Martin in [12]. Our method may help one solve the corresponding product problem of Scott-domain-representable spaces.

2. Preliminaries

We first review some basic concepts and notations that will be used later. For more details, we refer the reader to [7,8].

Let P be a poset. For a subset A of P, we shall adopt the following standard notations:

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\},\ \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

For each $x \in X$, we simply write $\uparrow x$ and $\downarrow x$ for $\uparrow\{x\}$ and $\downarrow\{x\}$, respectively. A subset A of P is called a *lower* (*upper*, resp.) *set* if $A = \downarrow A$ ($A = \uparrow A$, resp.). An element x is *maximal* in P, if $P \cap \uparrow x = \{x\}$. The set of all maximal elements of P is denoted by $\mathrm{Max}(P)$.

A nonempty subset D of P is *directed* if every two elements in D have an upper bound in D. For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for each directed subset D of P with $\bigvee D$ existing, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for each $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$.

An element $a \in P$ is called *compact*, if $a \ll a$. The set of all compact elements of P will be denoted by $K(P)$. Then, P is called *algebraic* if for each $x \in P$, the set $\{a \in K(P) : a \leq x\}$ is directed and $x = \bigvee\{a \in K(P) : a \leq x\}$. A continuous (algebraic, resp.) dcpo is also called a *domain* (an *algebraic domain*, resp.). A subset $B \subseteq P$ is a *base* of P if for each $x \in P$, $B \cap \twoheaddownarrow x$ is directed and $\bigvee(B \cap \twoheaddownarrow x)=x$. If P has a countable base, then P is called an *ω-continuous dcpo* or *ω-domain*.

A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for each directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology on P, called the *Scott topology* and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

Let X be a topological space. A subset $G \subseteq X$ is called a *G_δ-set*, if there exists a countable family $\{U_n : n \in \mathbb{N}\}$ of open sets such that $G = \bigcap_{n\in\mathbb{N}} U_n$.

Definition 2.1 *([11])*. A domain is called *ideal* if every element is either compact or maximal.

A poset model of a space X is a poset P such that X is homeomorphic to $\mathrm{Max}(P)$ with the relative Scott topology on P. If P is a poset model of X such that P is an (ideal) domain, then we say that X has an (ideal) domain model, or equivalently, X is (ideal) domain-representable.

3. The set of maximal points of an ideal domain need not be a G_δ-set

In [11, Section 8, Ideas (iv)], Martin asks:

- Is there an ideal domain whose maximal elements do not form a G_δ-set?

The following example shows that the set of maximal points of an ideal domain need not be a G_δ-set.

Example 3.1. Let $\mathbb{N}$ be the set of all natural numbers equipped with the usual order. Let $E = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$, and $F = \prod_{i\in\mathbb{N}} C_i$ be the Cartesian product of all C_j, where $C_i = \{i\} \times \mathbb{N}$ for each $i \in \mathbb{N}$. For each $\varphi \in F$, $\varphi(i) \in C_i$ denotes the i-th coordinate of φ. It is clear that F is uncountable and $\bigcup_{i\in\mathbb{N}} C_i \subseteq E$.

Let $L = E \cup (F \times \{0, 1\})$ and define a binary relation $\leq$ on L as follows:

(i) $\forall i \in \mathbb{N}$, $(i, m) \leq (i, n) \leq (i, \infty)$ whenever $m \leq n$ in $\mathbb{N}$;
(ii) $\forall \varphi \in F$, $\forall i \in \mathbb{N}$, $\varphi(i) \leq (\varphi, 0) \leq (\varphi, 1)$.

Then, one can easily verify that $(L, \leq)$ is an ideal domain, as shown in Fig. 1.

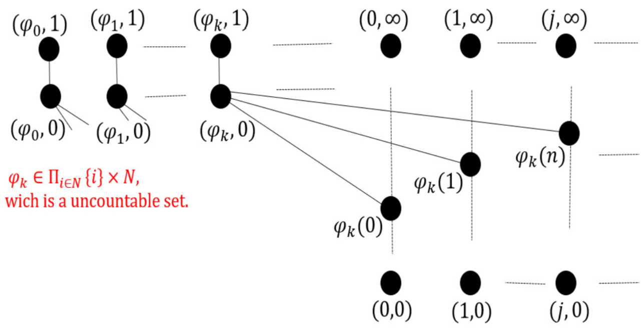

Fig. 1. The ideal domain L.

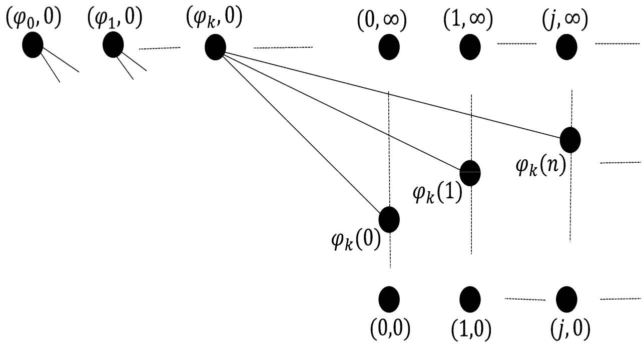

Fig. 2. The ideal domain $\widehat{L}$.

Claim: $\mathrm{Max}(L)$ is not a G_δ-set.

Suppose that $\mathrm{Max}(L)$ is a G_δ-set. Then there is a countable family $\{U_i : i \in \mathbb{N}\}$ of Scott open subsets of L such that $\mathrm{Max}(L) = \bigcap_{i\in\mathbb{N}} U_i$. For each $j \in \mathbb{N}$, since

$$\bigvee\{(j, n) : n \in \mathbb{N}\} = (j, \infty) \in \bigcap_{i=0}^{\infty} U_i \subseteq U_j,$$

there exists $n_j \in \mathbb{N}$ such that $(j, n_j) \in U_j$ (clearly $(j, n_j) \in C_j$). Define $\varphi \in F$ by $\varphi(j) = (j, n_j) \in U_j$ for each $j \in \mathbb{N}$. Then, since $(\varphi, 0) \geq \varphi(j) \in U_j$ for any $j \in \mathbb{N}$, we have that $(\varphi, 0) \in \bigcap_{j\in\mathbb{N}} U_j = \mathrm{Max}(L)$, but $(\varphi, 0) \notin \mathrm{Max}(L)$, which is a contradiction. Therefore, $\mathrm{Max}(L)$ is not a G_δ-set.

It is worth noting that though the above maximal point space $\mathrm{Max}(L)$ is not a G_δ-set in ΣL, but it is homeomorphic to a G_δ-subspace of another ideal domain $\widehat{L}$, where $\widehat{L}$ is a slight modification of L. This is demonstrated in the following example.

Example 3.2. Let $\widehat{L} = L \setminus \{(\varphi, 1) : \varphi \in F\}$ with the inherited order from L, where F and L are sets defined in Example 3.1. Then, $\widehat{L}$ is an ideal domain, as can be seen in Fig. 2.

Claim: $\mathrm{Max}(\widehat{L})$ is a G_δ-set.

For each $k \in \mathbb{N}$, define

$$U_k := \widehat{L} \setminus \bigcup\{\downarrow(i, k) : i \leq k\}.$$

Then, it is clear that $\{U_k : k \in \mathbb{N}\}$ is a collection of Scott open sets and $\bigcap_{k\in\mathbb{N}} U_k = \mathrm{Max}(\widehat{L})$. Therefore, $\mathrm{Max}(\widehat{L})$ is a G_δ-set.

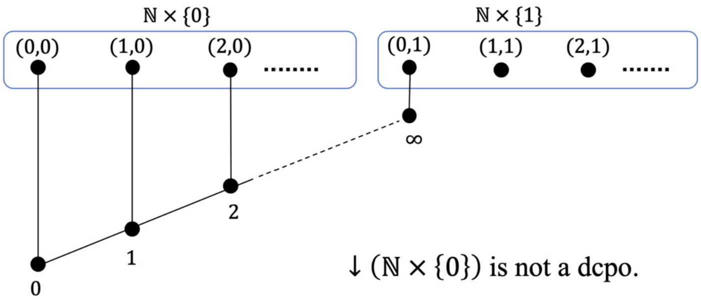

Fig. 3. The model of $\mathbb{N} \times \{0, 1\}$ with the discrete topology.

4. The productivity of domain-representable spaces

In this section, we provide a direct construction of domain models for the factor spaces X and Y, assuming that the product space $X \times Y$ is domain-representable. This offers another proof to the problem posed by Bennett and Lutzer in [1], as mentioned in the Introduction. It should be noted that this construction is not straightforward and cannot be derived directly, as the following example demonstrates:

Example 4.1. Let $P = (\mathbb{N} \times \{0, 1\}) \cup \mathbb{N} \cup \{\infty\}$ equipped with the partial order given by

(i) $\infty \leq (0, 1)$;
(ii) $\forall n \in \mathbb{N}$, $n \leq n + 1 \leq \infty$.

It is clear that P is a domain model of $\mathbb{N} \times \{0, 1\}$ with the discrete topology. Now, if we want to construct a domain model for $\mathbb{N} \times \{0\}$, which is homeomorphic to the factor space $\mathbb{N}$ (which is also the discrete space), a natural candidate would be $Q = \downarrow(\mathbb{N} \times \{0\})$ with the inherited order, which indeed is its poset model in this example. However, this is not a dcpo. Therefore, Q is not a domain model of $\mathbb{N}$ with the discrete order. (See Fig. 3.)

In Example 4.1, one can easily observe that $\downarrow(\mathbb{N} \times \{1\})$ is a domain model of $\mathbb{N}$ with the discrete topology. As a supplementary conclusion to the example, we add the following propositions.

Proposition 4.2. *Let P be a (Scott-) domain model of the product $X \times Y$ of two spaces X and Y. If there exists $y \in Y$ such that $\downarrow(X \times \{y\})$ is a Scott closed subset (or equivalently, a subdcpo) of P, then, with the inherited order, it is a (Scott-) domain model of X.*

Proof. The proof is clear by using the following facts:

(i) The Scott topology on $\downarrow(X \times \{y\})$ agrees with the relative Scott topology from P (see Exercise II-1.26 on page 151 in [7]).
(ii) Every Scott subset of a domain is a closure system, hence is a domain (see Corollary I-2.5 in [7]).
(iii) It is easy to check that every lower subset of a bounded complete dcpo is also bounded complete. □

Lemma 4.3. *Let P be an ideal domain. If $X \subseteq Max(P)$ is closed in the maximal point space, then $\downarrow X$ is a Scott closed subset of P, and hence is an ideal domain.*

Proof. We first show that $\downarrow X$ is a Scott closed subset of P. Suppose D is a directed subset of $\downarrow X$. Then, since P is an ideal domain, there are two cases:

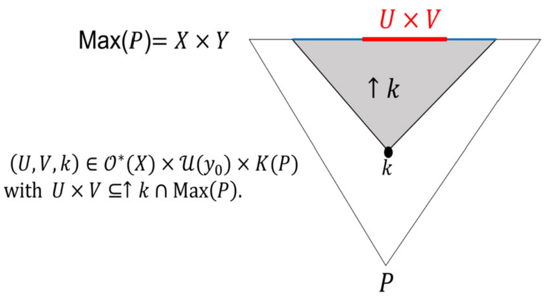

Fig. 4. The elements in Q.

(c1) $\bigvee D$ is compact. This means $\bigvee D \in D \subseteq \downarrow X$.

(c2) $\bigvee D$ is a maximal point of P. Since X is closed in $\mathrm{Max}(P)$, there exists a Scott closed subset C of P such that $X = C \cap \mathrm{Max}(P)$, which means $\downarrow X = \downarrow(C \cap \mathrm{Max}(P))$. Then, $D \subseteq \downarrow X = \downarrow(C \cap \mathrm{Max}(P)) \subseteq \downarrow C = C$, which follows that $\bigvee D \in C \cap \mathrm{Max}(P) = X$.

Thus, $\downarrow X$ is Scott closed. It is trivial to check that every Scott closed subset of an ideal domain is an ideal domain. □

From Lemma 4.3, one can deduce the following corollary.

Corollary 4.4. *Let P be an ideal domain model of the product $X \times Y$ of two spaces X and Y. Then, for each $y \in Y$, $\downarrow(X \times \{y\})$ is a domain model of X.*

Note that the above corollary does not hold for Scott domains. For example, consider the P in Example 4.1, which serves as a Scott domain model for $\mathbb{N} \times \{0, 1\}$ with the discrete topology, but $\downarrow(\mathbb{N} \times \{0\})$ is not a Scott domain model of $\mathbb{N}$ with the discrete topology.

Next, we provide a new construction method for the domain model of factor spaces. Before doing so, we state a lemma, which is well-known, will be used in our proof.

Lemma 4.5 *([1]). If X has a domain model, then it has an algebraic domain model.*

Theorem 4.6. *If the product $X \times Y$ of two spaces X and Y is domain-representable, then X and Y are domain-representable.*

Proof. According to Lemma 4.5, we can assume that $(P, \leq)$ is an algebraic domain model of $X \times Y$. Without loss of generality, we may assume that the maximal point space $\mathrm{Max}(P)$ equals $X \times Y$, that is,

$$\mathrm{Max}(P) = X \times Y \text{ and } \mathcal{O}(X \times Y) = \{U \cap \mathrm{Max}(P) : U \in \sigma(P)\}.$$

Let y_0 be a fixed element of Y. To construct a domain model of X, it suffices to construct a domain model of the subspace $X \times \{y_0\}$ of $X \times Y$. We do this in three steps.

Step 1. Construct a poset Q.
Let

$$Q = \{(U, V, k) \in \mathcal{O}^*(X) \times \mathcal{U}(y_0) \times K(P) : U \times V \subseteq \uparrow k \cap \mathrm{Max}(P)\},$$

where $\mathcal{U}(y_0)$ is the set of all open neighborhoods of y_0. The elements of Q are illustrated by Fig. 4. Define a binary relation $\sqsubseteq$ on Q by $(U_1, V_1, k_1) \sqsubseteq (U_2, V_2, k_2)$ iff it satisfies the following two conditions:

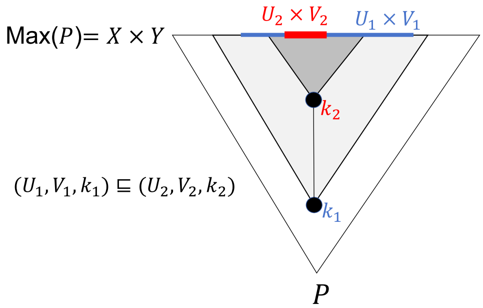

Fig. 5. The poset Q.

(i) $k_1 \leq k_2$;
(ii) $\uparrow k_2 \cap \mathrm{Max}(P) \subseteq U_1 \times V_1$,

as illustrated in Fig. 5.

Claim 1. $\sqsubseteq$ is a partial order on Q.

Reflexivity is clear. To show the antisymmetry, suppose $(U_1, V_1, k_1) \sqsubseteq (U_2, V_2, k_2)$ and $(U_2, V_2, k_2) \sqsubseteq (U_1, V_1, k_1)$. Then, on the one hand, it follows that $k_1 \leq k_2$ and $k_2 \leq k_1$, so $k_1 = k_2$. On the other hand, by the definition of Q and the relation $\sqsubseteq$, we have that

$$\uparrow k_2 \cap \mathrm{Max}(P) \subseteq U_1 \times V_1 \subseteq \uparrow k_1 \cap \mathrm{Max}(P) \subseteq U_2 \times V_2 \subseteq \uparrow k_2 \cap \mathrm{Max}(P),$$

which follows that $U_1 \times V_1 = U_2 \times V_2$, that is, $U_1 = U_2$ and $V_1 = V_2$. Therefore, $(U_1, V_1, k_1) = (U_2, V_2, k_2)$.

Now we verify the transitivity. Suppose $(U_1, V_1, k_1) \sqsubseteq (U_2, V_2, k_2) \sqsubseteq (U_3, V_3, k_3)$. On the one hand, we have that $k_1 \leq k_2 \leq k_3$, so $k_1 \leq k_3$. On the other hand, it follows that

$$\uparrow k_3 \cap \mathrm{Max}(P) \subseteq U_2 \times V_2 \subseteq \uparrow k_2 \cap \mathrm{Max}(P) \subseteq U_1 \times V_1,$$

and consequently, $(U_1, V_1, k_1) \sqsubseteq (U_3, V_3, k_3)$. Therefore, $\sqsubseteq$ is a partial order.

Step 2. Let $P_X = (Idl(Q), \subseteq)$, where $Idl(Q)$ denotes the set of all ideals of Q. Then, P_X is an algebraic domain whose compact elements are the principal ideals (see [7, Proposition I-4.10]). Next, we show that $\mathrm{Max}(P_X) = \{\mathcal{J}(x) : x \in X\}$, where

$$\mathcal{J}(x) \triangleq \{(U, V, k) \in Q : x \in U\}.$$

Claim 2. For any $x \in X$, $\mathcal{J}(x) \in P_X$.

We first show that $\mathcal{J}(x)$ is a lower set. Suppose $(U, V, k) \sqsubseteq (U_0, V_0, k_0) \in \mathcal{J}(x)$. Then, we have that

$$(x, y_0) \in U_0 \times V_0 \subseteq \uparrow k_0 \cap \mathrm{Max}(P) \subseteq U \times V,$$

which follows that $x \in U$. This means that $(U, V, k) \in \mathcal{J}(x)$.

Now we show that $\mathcal{J}(x)$ is directed. That $\mathcal{J}(x) \neq \emptyset$ is clear. Suppose $(U_1, V_1, k_1), (U_2, V_2, k_2) \in \mathcal{J}(x)$. By the definition of $\mathcal{J}(x)$ and Q, we have that

$$(x, y_0) \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{O}(X \times Y) = \mathcal{O}(\mathrm{Max}(P)),$$

so there exists $k_3 \in K(P)$ such that

$$(x, y_0) \in \uparrow k_3 \cap \mathrm{Max}(P) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2) \subseteq \uparrow k_1 \cap \uparrow k_2 \cap \mathrm{Max}(P).$$

It follows that $k_1, k_2, k_3 \leq (x, y_0)$, and since P is algebraic, there exists $k_0 \in K(P)$ such that $k_1, k_2, k_3 \leq k_0 \leq (x, y_0)$. We have that

$$(x, y_0) \in \uparrow k_0 \cap \mathrm{Max}(P) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2) \subseteq \uparrow k_1 \cap \uparrow k_2 \cap \mathrm{Max}(P).$$

On the other hand, since $(x, y_0) \in \uparrow k_0 \cap \mathrm{Max}(P) \in \mathcal{O}(X \times Y)$ and $\mathcal{O}(X) \times \mathcal{O}(Y)$ forms a basis of $\mathcal{O}(X \times Y)$, there exist $U_0 \in (X)$ and $V_0 \in \mathcal{O}(Y)$ (we may assume $U_0 \subseteq U_1 \cap U_2$ and $V_0 \subseteq V_1 \cap V_2$) such that

$$(x, y_0) \in U_0 \times V_0 \subseteq \uparrow k_0 \cap \mathrm{Max}(P).$$

It follows that $(U_1, V_1, k_1), (U_2, V_2, k_2) \sqsubseteq (U_0, V_0, k_0) \in \mathcal{J}(x)$. Therefore, $\mathcal{J}(x)$ is an ideal of Q, i.e., $\mathcal{J}(x) \in P_X$.

Claim 3. $\mathrm{Max}(P_X) \subseteq \{\mathcal{J}(x) : x \in X\}$.

Suppose $\mathcal{I} \in \mathrm{Max}(P_X)$. Let

$$W_1 \triangleq \bigcap\{U \times V : (U, V, k) \in \mathcal{I}\} \text{ and } W_2 \triangleq \bigcap\{\uparrow k \cap \mathrm{Max}(P) : (U, V, k) \in \mathcal{I}\}.$$

We show that $W_1 = W_2$. First, by the definition of Q, it is clear that $W_1 \subseteq W_2$. Conversely, suppose $(U, V, k) \in \mathcal{I}$, i.e., $U \times V \in W_1$. Since $\mathcal{I}$ is a lower set, there is $(U_0, V_0, k_0) \in \mathcal{I}$ such that $(U, V, k) \sqsubseteq (U_0, V_0, k_0)$, which follows that $U \times V \supseteq \uparrow k_0 \cap \mathrm{Max}(P) \in W_2$. From this fact it follows that $W_2 \subseteq W_1$. Therefore, $W_1 = W_2$.

Now let $D = \{k : (U, V, k) \in \mathcal{I}\}$. By using the directedness of $\mathcal{I}$, one can easily deduce that D is a directed subset of P. Let $k_0 = \bigvee D$ as P is a domain. We have that

$$\begin{aligned} W_1 = W_2 &= \bigcap\{\uparrow k \cap \mathrm{Max}(P) : (U, V, k) \in \mathcal{I}\} \\ &= \bigcap\{\uparrow k : (U, V, k) \in \mathcal{I}\} \cap \mathrm{Max}(P) \\ &= \uparrow k_0 \cap \mathrm{Max}(P) \neq \emptyset. \end{aligned}$$

Taking $(x, y) \in W_1$. Then, for any $(U, V, k) \in \mathcal{I}$, we have that $x \in U$, which implies that $(U, V, k) \in \mathcal{J}(x)$. We then have that $\mathcal{I} \subseteq \mathcal{J}(x)$. By the maximality of $\mathcal{I}$, we obtain that $\mathcal{I} = \mathcal{J}(x)$. Therefore, $\mathrm{Max}(P) \subseteq \{\mathcal{J}(x) : x \in X\}$.

Claim 4. $\mathrm{Max}(P_X) = \{\mathcal{J}(x) : x \in X\}$.

We need to show that every $\mathcal{J}(x)$ is maximal. Note that P_X is a dcpo, so $\mathcal{J}(x) \in P_X \subseteq \downarrow_{P_X} \mathrm{Max}(P_X)$. Then, there exists $\mathcal{I} \in \mathrm{Max}(P_X)$ such that $\mathcal{J}(x) \subseteq \mathcal{I}$. From Claim 3, there exists $x_0 \in X$ such that $\mathcal{I} = \mathcal{J}(x_0) \supseteq \mathcal{J}(x)$. From the T_1-separation of X, it follows that

$$\{x_0\} = \bigcap\{U : (U, V, k) \in \mathcal{J}(x_0)\} \subseteq \bigcap\{U : (U, V, k) \in \mathcal{J}(x)\} = \{x\},$$

which implies that $x = x_0$. Thus, $\mathcal{J}(x) = \mathcal{J}(x_0) \in \mathrm{Max}(P_X)$. This means that $\{\mathcal{J}(x) : x \in X\} \subseteq \mathrm{Max}(P_X)$. Therefore, $\mathrm{Max}(P_X) = \{\mathcal{J}(x) : x \in X\}$.

Step 4. P_X is an algebraic domain model of X.

Define a mapping $f : X \mapsto P_X$ as follows: $\forall x \in X$,

$$f(x) = \mathcal{J}(x).$$

By Step 3, f is well-defined, and one can easily check that it is bijective by using the T_1-separation of X.

Claim 5. f is continuous.

Note that $K(P_X) = \{\downarrow_Q(U, V, k) : (U, V, k) \in Q\}$, and thus $\{\uparrow_{P_X}(\downarrow_Q(U, V, k)) \cap \mathrm{Max}(P_X) : (U, V, k) \in Q\}$ forms a basis for $\mathrm{Max}(P_X)$. We have that for any $(U, V, k) \in Q$,

$$\begin{aligned} & x \in f^{-1}(\uparrow_{P_X}(\downarrow_Q(U, V, k)) \cap \mathrm{Max}(Q)) \\ \Leftrightarrow\ & f(x) = \mathcal{J}(x) \in \uparrow_{P_X}(\downarrow_Q(U, V, k)) \\ \Rightarrow\ & \downarrow_Q(U, V, k) \subseteq \mathcal{J}(x) = \downarrow\mathcal{J}(x) \\ \Leftrightarrow\ & (U, V, k) \in \mathcal{J}(x) \\ \Leftrightarrow\ & x \in U, \end{aligned}$$

which deduces that $f^{-1}(\uparrow_{P_X}(\downarrow_Q(U, V, k)) \cap \mathrm{Max}(Q)) = U \in \mathcal{O}(X)$. Therefore, f is continuous.

Claim 6. f is open.

Suppose $U \in \mathcal{O}(X)$ and $x_0 \in X$ such that $f(x_0) = \mathcal{J}(x_0) \in f(U)$. Since $(x_0, y_0) \in X \times Y = \mathrm{Max}(P)$ and P is algebraic, there exists $k_0 \in K(P)$ such that $(x_0, y_0) \in \uparrow k_0 \cap \mathrm{Max}(P) \in \mathcal{O}(X \times Y)$. Then, there exists $U_0 \in \mathcal{O}(X)$ (we may require that $U_0 \subseteq U$) and $V_0 \in \mathcal{O}(Y)$ such that

$$(x_0, y_0) \in U_0 \times V_0 \subseteq \uparrow k_0 \cap \mathrm{Max}(P).$$

Clearly, $(U_0, V_0, k_0) \in \mathcal{J}(x_0)$, and then $f(x_0) = \mathcal{J}(x_0) \in \uparrow_{P_X}(\downarrow_Q(U_0, V_0, k_0)) \cap \mathrm{Max}(P_X)$. Furthermore, for each $x \in X$, if $f(x) = \mathcal{J}(x) \in \uparrow_{P_X}(\downarrow_Q(U_0, V_0, k_0)) \cap \mathrm{Max}(P_X)$, then $(U_0, V_0, k_0) \in \mathcal{J}(x)$. This implies that $x \in U_0 \subseteq U$, and consequently $f(x) \in f(U_0) \subseteq f(U)$. Thus, we conclude that $f(U)$ is an open subset of $\mathrm{Max}(P_X)$. Therefore, f is an open mapping.

From Claims 5 and 6, it follows that f is a homeomorphism. Therefore, P_X is an algebraic domain model of X. □

5. Conclusions

(1) Regarding Martin's problem, which asks whether there exists an ideal domain whose maximal elements do not form a G_δ-set, we provide a positive answer. Specifically, Example 3.1 demonstrates that $\mathrm{Max}(L)$ is not a G_δ-set for the ideal domain L. However, $\mathrm{Max}(L)$ is (up to homeomorphism) a G_δ-set in some other ideal domain $\widehat{L}$. This raises the following questions:

(i) Is there an ideal domain M such that $\mathrm{Max}(M)$ is not homeomorphic to a G_δ-subset of any ideal domain?

(ii) How can we characterize the maximal point spaces of ideal domains?

In [3], Brecht, Goubault-Larrecq, Jia, and Lyu asked whether every sober convergence Choquet-complete space is domain-complete (see Question (iii) in the Conclusion of [3]). Since we know that every ideal domain is sober convergence Choquet-complete, a negative answer to question (i) would imply that the answer to the question in [3] is also negative. We leave these problems for future study.

(2) In this paper, we provided a new approach to Bennett and Lutzer's problem: If $X \times Y$ is domain-representable, must X and Y be domain-representable? There is another similar open problem posed in [1]:

(iii) If $X \times Y$ has a Scott domain model, must X and Y have a Scott domain model?

Here a Scott domain is a bounded complete (every upper bounded subset has a supremum) continuous dcpo. Currently, we are still not able to solve this problem. However, the approach taken in Section 4, due to its direct construction feature, may help to find a method to answer this problem.

Acknowledgements

We would like to express our gratitude to the reviewer who pointed out the reference [14], which addressed the second problem in our paper. Additionally, the reviewer provided insightful historical context on the concept of domain-representability. These contributions have significantly enriched our research and offered new perspectives. Moreover, we would like to express our sincere gratitude to Dr. Hualin Miao, a postdoctoral researcher at Hunan University, for his careful review of the initial draft.

References

[1] H. Bennett, D. Lutzer, Domain-representable spaces and completeness, Topol. Proc. 34 (2009) 223–244.
[2] H. Bennett, D. Lutzer, Strong completeness properties in topology, Quest. Answ. Gen. Topol. 27 (2009) 107–124.
[3] M. de Brecht, J. Goubault-Larrecq, X. Jia, Z. Lyu, Domain-complete and LCS-complete spaces, Electron. Notes Theor. Comput. Sci. 345 (2019) 3–35.
[4] A. Edalat, When Scott is weak on the top, Math. Struct. Comput. Sci. 7 (1997) 401–417.
[5] W. Fleissner, L. Yengulalp, Baire spaces, G_δ's and directed complete spaces, Topol. Proc. 59 (2022) 111–151.
[6] W. Fleissner, L. Yengulalp, From subcompact to domain-representable, Topol. Appl. 195 (2015) 174–195.
[7] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, D. Scott, Continuous Lattices and Domains, Encyclopedia of Mathematics and Its Applications, vol. 93, Cambridge University Press, 2003.
[8] J. Goubault-Larrecq, Non-Hausdorff Topology and Domain Theory: Selected Topics in Point-Set Topology, Cambridge University Press, 2013.
[9] J. Lawson, Spaces of maximal points, Math. Struct. Comput. Sci. 7 (5) (1997) 543–555.
[10] K. Martin, M. Mislove, G.M. Reed, Topology and domain theory, in: M. Husek, J. van Mill (Eds.), Recent Progress in General Topology II, Elsevier, Amsterdam, 2002, pp. 371–391.
[11] K. Martin, Ideal models of spaces, Theor. Comput. Sci. 305 (1–3) (2003) 277–297.
[12] K. Martin, Topological games in domain theory, Topol. Appl. 129 (2003) 177–186.
[13] K. Martin, The regular spaces with countably based model, Theor. Comput. Sci. 305 (1–3) (2003) 299–310.
[14] S. Önal, C. Vural, Domain representability of retracts, Topol. Appl. 194 (2015) 1–3.
[15] D. Scott, Outline of a mathematical theory of computation, in: Proceedings of the 4th Annual Princeton Conference on Information Science and Systems, Princeton University Press, Princeton, NJ, 1970.
[16] D. Scott, Continuous Lattices, Topos, Algebraic Geometry and Logic, Lecture Notes in Mathematics, vol. 274, Springer-Verlag, Berlin, 1972.
[17] K. Weihrauch, U. Schreiber, Embedding metric spaces into cpo's, Theor. Comput. Sci. 16 (1981) 5–24.
[18] D. Zhao, X. Xi, Directed complete poset models of T_1 spaces, Math. Proc. Camb. Philos. Soc. 164 (1) (2018) 125–134.