Topology and its Applications 282 (2020) 107316

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Sober Scott spaces are not always co-sober

Chong Shen [a,*], Guohua Wu [b], Xiaoyong Xi [b], Dongsheng Zhao [c]

[a] *School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu, 210046, PR China*
[b] *Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore*
[c] *Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, 637616, Singapore*

ARTICLE INFO

Article history:
Received 19 August 2019
Received in revised form 18 June 2020
Accepted 23 June 2020
Available online 30 June 2020

MSC:
06B35
06F30
54B99
54D30

Keywords:
Sober space
Co-sober space
Poset model
Dcpo model

ABSTRACT

A nonempty compact saturated subset F of a topological space is called k-irreducible if it cannot be written as a union of two compact saturated proper subsets. A topological space is said to be co-sober if each of its k-irreducible compact saturated sets is the saturation of a point. Wen and Xu (2018) proved that Isbell's non-sober complete lattice equipped with the lower topology is sober but not co-sober. So far, it is still unknown whether every sober Scott space is co-sober. In this paper, we construct a dcpo whose Scott space is sober but not co-sober, which strengthens Wen and Xu's result.

1. Introduction and basic notions

In [1], in order to study the dual of the Hofmann-Mislove Theorem, Escardó, Lawson and Simpson introduced the k-irreducible sets and co-sober spaces, aiming to provide an alternative approach to the theory of compactly generated spaces. A nonempty compact saturated subset K of a topological space X is *k-irreducible* if for any nonempty compact saturated subsets K_1 and K_2, $K = K_1 \cup K_2$ implies $K = K_1$ or $K = K_2$. A topological space is called *co-sober* if every k-irreducible compact saturated set is the saturation of a point. It is easy to see that a co-sober space need not be sober (for example, the Scott space of the dcpo given by Johnstone [5]). Then Escardó, Lawson and Simpson asked whether every sober space is co-sober

* Corresponding author.
E-mail addresses: shenchong0520@163.com (C. Shen), guohua.wu@ntu.edu.sg (G. Wu), xiaoyong.xi@ntu.edu.sg (X. Xi), dongsheng.zhao@nie.edu.sg (D. Zhao).

https://doi.org/10.1016/j.topol.2020.107316
0166-8641/

(Problem 9.7 in [1]). Recently, Wen and Xu [7] gave a negative answer to this problem by proving the following result:

- For a complete lattice L, the Scott topology on L is sober if and only if the lower topology on L is co-sober. Since the lower topology on L is always sober, the complete lattice constructed by Isbell [4] is sober but not co-sober with respect to the lower topology.

Note that Wen and Xu's counterexample is based on the lower topology on a complete lattice. It is natural to ask the following question:

- Is every sober Scott space of a dcpo co-sober?

In this paper, we give a negative answer to the above question. In order to do this, we first generalize the Xi-Zhao dcpo models [8,9] of T_1 spaces to T_0 spaces. Precisely, we prove that (i) every T_0 space can be topologically embedded into the Scott space of a bounded complete algebraic poset; (ii) every d-space X can be topologically embedded into the Scott space of a dcpo P_X, which has the property that X is sober if and only if the Scott space of P_X is sober.

First, we recall some basic definitions and results which will be used in this paper. We refer readers to [2,3] for more details.

Let P be a poset. For $A \subseteq P$, let $\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\}$ and $\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. A subset A is called a *lower set* (*upper set*, resp.) if $A = \downarrow A$ ($A = \uparrow A$, resp.).

A nonempty subset D of a poset P is *directed* if every two elements in D have an upper bound in D. P is called a *directed complete poset*, or *dcpo* for short, if for any directed subset $D \subseteq P$, $\bigvee D$ exists. We call a subset A of P a *subdcpo*, if for any directed subset D of A, $\bigvee D$ exists and $\bigvee D \in A$.

We say that a poset P is *bounded complete* if for any $A \subseteq P$, $\bigvee A$ exists whenever A has an upper bound in P.

A subset U of a poset P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology, called the *Scott topology* on P and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

For two elements x and y in a poset P, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P for which $\bigvee D$ exists, $y \leq \bigvee D$ implies $D \cap \uparrow x \neq \emptyset$. Let $0\!\uparrow\! x = \{y \in P : x \ll y\}$ and $\Downarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$. A continuous dcpo is also called a *domain*.

An element x in a poset P is called *compact* if $x \ll x$, and we use $\mathsf{K}(P)$ to denote the set of all compact elements of P. Note that if $x \in \mathsf{K}(P)$, then $\uparrow x \in \sigma(P)$. A poset P is *algebraic*, if for any $x \in P$, the set $\mathsf{K}(P) \cap \downarrow x$ is directed and $x = \bigvee(\mathsf{K}(P) \cap \downarrow x)$. In an algebraic domain P, the family $\{\uparrow x : x \in \mathsf{K}(P)\}$ forms a base for the Scott topology on P.

Given a topological space X, $\mathcal{O}(X)$ always denotes the topology of X, and $\mathcal{O}^*(X) = \mathcal{O}(X) \setminus \{\emptyset\}$.

A nonempty subset A of a topological space X is *irreducible* if for any closed sets F_1, F_2 of X, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. A T_0 space X is *sober*, if for any irreducible closed set F of X there is a unique point $x \in X$ such that $F = \mathrm{cl}(x)$.

For any T_0 space X, we use $\sqsubseteq_X$ to denote the *specialization order* on X: $x \sqsubseteq_X y$ if and only if $x \in \mathrm{cl}(y)$, where cl is the closure operator. A T_0 space is called a *d-space* if $(X, \sqsubseteq_X)$ is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X, \sqsubseteq_X)$.

A subset A of a T_0 space X is *saturated* if $A = \uparrow_{\sqsubseteq_X} A$, where $\uparrow_{\sqsubseteq_X} A = \{x \in X : a \sqsubseteq_X x \text{ for some } a \in A\}$ (equivalently, if A equals the intersection of all open sets containing A).

2. Main results

A poset model of a topological space X is a poset P such that $\mathrm{Max}(P)$, the set of maximal elements of P with the relative Scott topology is homeomorphic to X [6]. Every space having a poset model is T_1.

For a topological space X, let $\mathrm{Zh}(X)$ be the set of all filters of nonempty open sets of X that have nonempty intersections. For $A \subseteq X$, let $\mathcal{N}(A)$ be the set of all open neighborhoods of A, that is, $\mathcal{N}(A) := \{U \in \mathcal{O}(X) : A \subseteq U\}$. For $x \in X$, we write $\mathcal{N}(x)$ for $\mathcal{N}(\{x\})$. One can easily check that $\mathrm{Zh}(X)$ is a bounded complete algebraic poset under the inclusion order, and the compact elements of $\mathrm{Zh}(X)$ are $\mathcal{N}(U)$ $(U \in \mathcal{O}^*(X))$. Thus the family $\{\uparrow_{\mathrm{Zh}(X)}\mathcal{N}(U) : U \in \mathcal{O}^*(X)\}$ forms a base for the Scott topology on $\mathrm{Zh}(X)$. When X is a T_1 space, Zhao proves that $\mathrm{Zh}(X)$ under the inclusion order is a bounded complete algebraic poset model of X [8]. We now prove that this result can be generalized to T_0 spaces.

Proposition 2.1. *Let X be a T_0 space. Then the mapping $\phi : X \longrightarrow \mathrm{Zh}(X)$ defined by*

$$\phi(x) = \mathcal{N}(x), \ \forall x \in X,$$

is a topological embedding satisfying the following properties:

(i) $\mathrm{Zh}(X) = \downarrow_{\mathrm{Zh}(X)}\phi(X)$;
(ii) *if X is a d-space, then $\phi(X)$ is a subdcpo of* $\mathrm{Zh}(X)$.

Proof. The mapping ϕ is injective because X is T_0. For any nonempty open set U in X, we have

$$\phi(U) = \{\mathcal{N}(x) : x \in U\} = \phi(X) \cap \uparrow_{\mathrm{Zh}(X)}\mathcal{N}(U),$$

where $\uparrow_{\mathrm{Zh}(X)}\mathcal{N}(U) = \{\mathcal{F} \in \mathrm{Zh}(X) : \mathcal{N}(U) \subseteq \mathcal{F}\} = \{\mathcal{F} \in \mathrm{Zh}(X) : U \in \mathcal{F}\}$. As $\mathcal{N}(U)$ is a compact element of $\mathrm{Zh}(X)$, we have that $\uparrow_{\mathrm{Zh}(X)}\mathcal{N}(U)$ is Scott open in $\mathrm{Zh}(X)$. Thus $\phi(U)$ is open in the subspace $\phi(X)$ of $\mathrm{Zh}(X)$ equipped with the Scott topology. The same is true for $\phi(\emptyset) = \emptyset$. Furthermore, since for any nonempty open set U in X, we have

$$\begin{aligned}&\phi^{-1}\left(\left(\uparrow_{\mathrm{Zh}(X)}\mathcal{N}(U)\right) \cap \phi(X)\right)\\ &= \{x \in X : \mathcal{N}(U) \subseteq \mathcal{N}(x)\}\\ &= \{x \in X : x \in U\} = U,\end{aligned}$$

which shows that ϕ is continuous. Therefore, ϕ is a topological embedding.

To prove (i), let $\mathcal{F} \in \mathrm{Zh}(X)$ and $x \in \bigcap \mathcal{F} (\neq \emptyset)$. This implies that $\mathcal{F} \subseteq \mathcal{N}(x) \in \phi(X)$. Thus $\mathcal{F} \in \downarrow_{\mathrm{Zh}(X)}\phi(X)$. So $\mathrm{Zh}(X) = \downarrow_{\mathrm{Zh}(X)}\phi(X)$.

Since X is T_0, $x \sqsubseteq_X y$ if and only if $\mathcal{N}(x) \subseteq \mathcal{N}(y)$ for all $x, y \in X$. Hence, ϕ is an order embedding from $(X, \sqsubseteq_X)$ into $(\mathrm{Zh}(X), \sqsubseteq_{\mathrm{Zh}(X)})$. Let $\{x_i : i \in I\}$ be a directed subset of $(X, \sqsubseteq_X)$. Since X is a d-space, $\bigvee_{i\in I} x_i$ exists, denoted by x_0. To show $\bigvee\{\mathcal{N}(x_i) : i \in I\} \in \phi(X)$, it suffices to verify that $\bigcup\{\mathcal{N}(x_i) : i \in I\} = \mathcal{N}(x_0)$. Let $U \in \mathcal{N}(x_0)$. Then $x_0 = \bigvee_{i\in I} x_i \in U$, and since U is Scott open, there exists $i_0 \in I$ such that $x_{i_0} \in U$, which implies that $U \in \mathcal{N}(x_{i_0}) \subseteq \bigcup\{\mathcal{N}(x_i) : i \in I\}$. Thus $\mathcal{N}(x_0) \subseteq \bigcup\{\mathcal{N}(x_i) : i \in I\}$. The reverse inclusion is trivial because $U = \uparrow U$. Thus $\bigcup\{\mathcal{N}(x_i) : i \in I\} = \mathcal{N}(x_0)$. Therefore, $\phi(X)$ is a subdcpo of $\mathrm{Zh}(X)$. □

Note that $\downarrow_{\mathrm{Zh}(X)}\phi(X) = \mathrm{Zh}(X)$ shows that the closure of $\phi(X)$ in the Scott space $\Sigma\mathrm{Zh}(X)$ equals $\mathrm{Zh}(X)$. Thus by Proposition 2.1, we obtain the following corollary.

Corollary 2.2. *Every T_0 space is homeomorphic to a dense subspace of the Scott space of a bounded complete algebraic poset.*

Based on Zhao's bounded complete algebraic poset model [8], Xi and Zhao [9] further constructed a dcpo model, denoted by $\widehat{\mathrm{Zh}}(X)$, for every T_1 space X, and showed that X is sober if and only if $\widehat{\mathrm{Zh}}(X)$ is sober. This result can be extended to d-spaces.

Proposition 2.3. *Let $(P, \leq_P)$ be a bounded complete algebraic poset and $A \subseteq P$ satisfy the following conditions:*

(P1) $P = \downarrow A$;
(P2) *A is a subdcpo of P.*

Then there exists a dcpo $\widehat{P}$ such that the subspace A of ΣP is homeomorphic to a subspace of $\Sigma\widehat{P}$.

Proof. Let $\widehat{P} = \{(x, a) : x \in P, a \in A \text{ and } x \leq_P a\}$. Define a binary relation $\leq$ on $\widehat{P}$ as follows:

$$(x, a) \leq (y, b) \text{ iff } x \leq_P y \text{ and in addition } a = b \text{ or } y = b.$$

Note that for any $a, b \in A$, $a \leq_P b$ in P iff $(a, a) \leq (b, b)$ in $\widehat{P}$.

It is not difficult to show that $\widehat{P}$ is a poset. We now prove that $\widehat{P}$ is a dcpo. Let $\mathcal{D}$ be a directed subset of $\widehat{P}$. There are two cases:

Case 1. There are $(x_1, a_1), (x_2, a_2) \in \mathcal{D}$ with $a_1 \neq a_2$.

For this case, let (x, a_3) be an upper bound of $\{(x_1, a_1), (x_2, a_2)\}$ in $\mathcal{D}$, then $x = a_3$, as otherwise, $a_3 = a_1$ and $a_3 = a_2$, which is not true. Thus $\mathcal{D}_0 := \mathcal{D} \cap \{(a, a) : a \in A\} \neq \emptyset$. Then $\mathcal{D} \subseteq \downarrow\mathcal{D}_0$, since for any $(x, a) \in \mathcal{D}$, each upper bound of $\{(x, a), (x_1, a_1), (x_2, a_2)\}$ in $\mathcal{D}$ must be of the form (a, a), which belongs to $\mathcal{D}_0$. We have that $\downarrow\mathcal{D} = \downarrow\mathcal{D}_0$ and therefore $\mathcal{D}_0$ is directed since $\mathcal{D}$ is directed. As a consequence, $D := \{a \in A : (a, a) \in \mathcal{D}_0\}$ is a directed subset of A. By (P2), $\bigvee D$ exists and $\bigvee D \in A$. It then follows that $\bigvee \mathcal{D} = \bigvee \mathcal{D}_0 = (\bigvee D, \bigvee D)$ holds in $(\widehat{P}, \leq)$.

Case 2. There exists $a \in A$ such that $\mathcal{D} = \{(x_i, a) : i \in I\}$.

In this case, $\{x_i : i \in I\}$ is a directed subset of P with an upper bound a. As P is bounded complete, $\bigvee\{x_i : i \in I\}$ exists and clearly $\bigvee\{x_i : i \in I\} \leq a$, which gives that $\bigvee \mathcal{D} = (\bigvee\{x_i : i \in I\}, a)$.

Now define a mapping $\varphi : A \longrightarrow \Sigma\widehat{P}$ by

$$\varphi(a) = (a, a), \ \forall a \in A.$$

Obviously, φ is injective. For any Scott open subset U of P, we have

$$\varphi(U \cap A) = \{(a, a) : a \in U \cap A\} = \widehat{U} \cap \varphi(A),$$

where $\widehat{U} = \{(x, a) \in \widehat{P} : x \in U, a \in A\}$. We now show that $\widehat{U}$ is a Scott open set in $\widehat{P}$. Clearly, $\widehat{U}$ is an upper set because U is an upper set. Let $\mathcal{D}$ be a directed subset of $\widehat{P}$ such that $\bigvee \mathcal{D} \in \widehat{U}$. It suffices to show that $\mathcal{D} \cap \widehat{U} \neq \emptyset$. We consider two cases:

(i) There are $(x_1, a_1), (x_2, a_2) \in \mathcal{D}$ with $a_1 \neq a_2$.

By the above conclusion, $\mathcal{D}_0 = \mathcal{D} \cap \{(a, a) : a \in A\} \neq \emptyset$ and $\bigvee \mathcal{D} = (\bigvee D, \bigvee D)$, where $D := \{a \in A : (a, a) \in \mathcal{D}_0\}$. It follows that $\bigvee D \in U$, and $D \cap U \neq \emptyset$ because $U \subseteq P$ is Scott open. Let $b \in D \cap U$. Then $(b, b) \in \widehat{U} \cap \mathcal{D}$.

(ii) $\mathcal{D} = \{(x_i, a) : i \in I\}$ for some $a \in A$.

In this case, $\bigvee \mathcal{D} = (\bigvee\{x_i : i \in I\}, a)$. The set $\{x_i : i \in I\} \subseteq P$ is directed and $\bigvee\{x_i : i \in I\} \in U$. As U is Scott open, there exists $i_0 \in I$ such that $x_{i_0} \in U$ and hence $(x_{i_0}, a) \in \widehat{U} \cap \mathcal{D}$.

Hence, $\widehat{U}$ is a Scott open set in $\widehat{P}$. Therefore, $\varphi(U \cap A) = \widehat{U} \cap \varphi(A)$ is an open set in the subspace $\varphi(A)$ of $\Sigma\widehat{P}$, so φ is an open mapping from A to $\varphi(A)$.

Now we show that φ is continuous. Let V be a Scott open set in $\widehat{P}$. Then $\varphi^{-1}(V) = \{a \in A : (a,a) \in V\}$. Let $\underline{V} = \uparrow\{x \in \mathsf{K}(P) : \exists a \in A, (x,a) \in V\}$. We show that

$$\varphi^{-1}(V) = \underline{V} \cap A.$$

Let $b \in \underline{V} \cap A$. There exists $x \in \mathsf{K}(P)$ and $a \in A$ such that $(x,a) \in V$ and $x \leq_P b$. Note that $(x,a) \leq (b,b)$ and V is an upper set, so $\varphi(b) = (b,b) \in V$. Thus $\underline{V} \cap A \subseteq \varphi^{-1}(V)$. Conversely, let $a \in \varphi^{-1}(V)$. Then $\varphi(a) = (a,a) \in V$. Since P is algebraic, $\mathsf{K}(P) \cap {\downarrow}a$ is a directed subset of P and $a = \bigvee(\mathsf{K}(P) \cap {\downarrow}a)$. Thus $\{(x,a) : x \in \mathsf{K}(P) \cap {\downarrow}a\}$ is a directed subset of $\widehat{P}$ and $(a,a) = \bigvee\{(x,a) : x \in \mathsf{K}(P) \cap {\downarrow}a\}$. Since V is Scott open and $(a,a) \in V$, there exists $x_0 \in \mathsf{K}(P) \cap {\downarrow}a$ such that $(x_0,a) \in V$. We have that $a \in \underline{V}$ since $x_0 \leq_P a$. Thus $\varphi^{-1}(V) \subseteq \underline{V}$.

Clearly, $\underline{V}$ is a Scott open subset of P. It then follows that $\varphi^{-1}(V) = \underline{V} \cap A$ is an open set in the subspace A of ΣP, and hence φ is continuous.

All this shows that φ is a topological embedding. $\square$

Remark 2.4. Let $\mathcal{D}$ be a directed subset of $\widehat{P}$. The following results on $\mathcal{D}$ obtained in the proof of Proposition 2.3 will be used later.

(1) If $\mathcal{D} \cap \{(a,a) : a \in A\} \neq \emptyset$, then $D = \{a \in A : (a,a) \in \mathcal{D}\}$ is directed and $\bigvee \mathcal{D} = (\bigvee D, \bigvee D)$.
(2) If $\mathcal{D} \cap \{(a,a) : a \in A\} = \emptyset$, then there is a directed subset $\{x_i : i \in I\}$ of P and $a_0 \in A$ such that $\mathcal{D} = \{(x_i, a_0) : i \in I\}$, and in this case, $\bigvee \mathcal{D} = (\bigvee\{x_i : i \in I\}, a_0)$.

Proposition 2.5. *Let P be a bounded complete algebraic poset, A a subset of P satisfying the conditions (P1) and (P2) in Proposition 2.3, and $\widehat{P}$ the dcpo constructed from P in Proposition 2.3. Then the subspace A of ΣP is sober if and only if $\Sigma\widehat{P}$ is sober.*

Proof. Assume that $\Sigma\widehat{P}$ is sober. Recall that the mapping $\varphi : A \longrightarrow \widehat{P}$, $a \mapsto (a,a)$, is a topological embedding by Proposition 2.3. By the fact that a saturated subspace of a sober space is sober [2, Exercise O-5.16] and Proposition 2.3, it suffices to show that $\varphi(A)$ is an upper subset of $\widehat{P}$ (hence is saturated in $\Sigma\widehat{P}$). Let $(a,a) \in \varphi(A)$ and $(x,b) \in \widehat{P}$ with $(a,a) \leq (x,b)$. There are two cases:

(i) $a \leq_P x$ and $a = b$. Since $x \leq_P b$, we have $x = b = a$. Therefore, $(x,b) = (a,a) \in \varphi(A)$.
(ii) $a \leq_P x$ and $x = b$. In this case, we have $(x,b) = (b,b) = \varphi(b) \in \varphi(A)$.

Therefore, $\varphi(A)$ is an upper set. Thus $\varphi(A)$ is sober, and so is A.

Conversely, assume that A, as a subspace of ΣP, is sober. By Proposition 2.3, $\varphi(A)$ is sober as a subspace of $\Sigma\widehat{P}$. We need to show that $\Sigma\widehat{P}$ is sober. The following facts are clear now:

(P1′) $\widehat{P} = {\downarrow}\varphi(A)$;
(P2′) $\varphi(A)$ is a subdcpo of $\widehat{P}$.

Let $\mathcal{C}$ be an irreducible closed set in $\Sigma\widehat{P}$, and $\mathcal{C}^* = \mathcal{C} \cap \varphi(A)$. For each $a \in A$, let $\widehat{P}_a := \{(x,a) \in \widehat{P} : x \in P\}$. We complete the proof by considering two cases.

Case 1. $\mathcal{C}^* = \emptyset$.

Note that $\mathcal{C} = \bigcup\{\mathcal{C} \cap \widehat{P}_a : a \in A\}$. For each $a \in A$, we show that $\mathcal{C} \cap \widehat{P}_a$ is Scott closed. Let $(y,b) \leq (x,a)$ and $(x,a) \in \mathcal{C} \cap \widehat{P}_a$. Since $\mathcal{C}$ is a lower set, it follows that $(y,b) \in \mathcal{C}$. Since $\mathcal{C} \cap \varphi(A) = \emptyset$, we have that $x \neq a$, which implies $y \le_P x$ and $b = a$. Thus $(y, b) = (y, a) \in \mathcal{C} \cap \widehat{P}_a$. This implies that $\mathcal{C} \cap \widehat{P}_a$ is a lower set. Let $\mathcal{D} = \{(x_i, a) : i \in I\}$ be a directed subset of $\mathcal{C} \cap \widehat{P}_a$. By Remark 2.4, $\bigvee \mathcal{D} = (\bigvee_{i \in I} x_i, a) \in \widehat{P}_a$. Since $\mathcal{D}$ is also a directed subset of $\mathcal{C}$ and $\mathcal{C}$ is Scott closed, $\bigvee \mathcal{D} \in \mathcal{C}$, and thus $\bigvee \mathcal{D} \in \mathcal{C} \cap \widehat{P}_a$. Hence, $\mathcal{C} \cap \widehat{P}_a$ is Scott closed. Since $\mathcal{C} \neq \emptyset$, there exists $a_0 \in P$ such that $\mathcal{C} \cap \widehat{P}_{a_0} \neq \emptyset$. Then $\mathcal{C} = (\mathcal{C} \cap \widehat{P}_{a_0}) \cup \bigcup\{\mathcal{C} \cap \widehat{P}_a : a \in A, a \neq a_0\}$. We claim that $\bigcup\{\mathcal{C} \cap \widehat{P}_a : a \in A, a \neq a_0\}$ is also Scott closed. To see this, let $\mathcal{D}$ be a directed subset of $\bigcup\{\mathcal{C} \cap \widehat{P}_a : a \in A, a \neq a_0\}$. Since $\mathcal{C}^* = \emptyset$, we have that $\mathcal{D} \cap \{(a, a) : a \in A\} = \emptyset$. By Remark 2.4, it holds that $\mathcal{D} \subseteq \mathcal{C} \cap \widehat{P}_{a_1}$ for some $a_1 \in A \setminus \{a_0\}$. This means $\bigvee \mathcal{D} \in \mathcal{C} \cap \widehat{P}_{a_1} \subseteq \bigcup\{\mathcal{C} \cap \widehat{P}_a : a \in A, a \neq a_0\}$ since $\mathcal{C} \cap \widehat{P}_{a_1}$ is Scott closed. Thus $\bigcup\{\mathcal{C} \cap \widehat{P}_a : a \in A, a \neq a_0\}$ is Scott closed. As $\mathcal{C}$ is irreducible, and $\mathcal{C} \cap \widehat{P}_{a_0}$ and $\bigcup\{\mathcal{C} \cap \widehat{P}_a : a \in A, a \neq a_0\}$ are disjoint Scott closed sets with $\mathcal{C} \cap \widehat{P}_{a_0} \neq \emptyset$, we have that $\mathcal{C} = \mathcal{C} \cap \widehat{P}_{a_0}$. So $\mathcal{C} \subseteq \widehat{P}_{a_0}$. Since $\mathcal{C}$ is Scott closed in $\widehat{P}$, we see easily that $\mathcal{C}$ is an irreducible Scott closed set in the subposet $\widehat{P}_{a_0}$ of $\widehat{P}$. Note that the subposet $\widehat{P}_{a_0}$ is order isomorphic to the subposet $\downarrow a_0$ of P via the mapping $(x, a_0) \mapsto x$. Thus $\mathcal{C}$ is homeomorphic to an irreducible Scott closed subset H of $\downarrow a_0$. But $\downarrow a_0$ is an algebraic dcpo as P is a bounded complete algebraic poset, its Scott topology is sober. Hence, $H = \downarrow x_0$ for some $x_0 \le a_0$, which then implies that $\mathcal{C} = \downarrow(x_0, a_0)$.

Case 2. $\mathcal{C}^* \neq \emptyset$.

We first prove that $\mathcal{C} = \mathrm{cl}_{\widehat{P}}(\mathcal{C}^*) \cup \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$. Note that $\downarrow\mathcal{C}^* \subseteq \mathrm{cl}_{\widehat{P}}(\mathcal{C}^*)$, so

$$\mathcal{C} \subseteq \downarrow\mathcal{C}^* \cup \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\} \subseteq \mathrm{cl}_{\widehat{P}}(\mathcal{C}^*) \cup \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}.$$

For the converse, since $\mathcal{C}$ is Scott closed and $\mathcal{C}^* \subseteq \mathcal{C}$, one has $\mathrm{cl}_{\widehat{P}}(\mathcal{C}^*) \subseteq \mathcal{C}$, and thus $\mathrm{cl}_{\widehat{P}}(\mathcal{C}^*) \cup \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\} \subseteq \mathcal{C}$.

We have the following two properties:

(c1) $\varphi(A) \cap \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\} = \emptyset$. Otherwise, there exists $b \in A$ such that $\varphi(b) = (b, b) \in \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$, then $(b, b) \in \mathcal{C} \cap \widehat{P}_{a_0}$ for some $(a_0, a_0) \notin \downarrow\mathcal{C}^*$, which implies that $b = a_0$ and $(b, b) \in \mathcal{C}^*$, a contradiction.

(c2) $(x, b) \in \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$ implies that $(b, b) \notin \downarrow\mathcal{C}^*$. If $(x, b) \in \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$, then there exists $(a, a) \notin \downarrow\mathcal{C}^*$ such that $(x, b) \in \mathcal{C} \cap \widehat{P}_a$, which implies that $b = a$ by the definition of $\widehat{P}_a$. Thus $(b, b) = (a, a) \notin \downarrow\mathcal{C}^*$.

We now show that $\bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$ is Scott closed. Let $(x_1, a_1) \in \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$ and $(x_2, a_2) \le (x_1, a_1)$. Then $x_1 \neq a_1$ by (c1) and $(a_1, a_1) \notin \downarrow\mathcal{C}^*$ by (c2). It follows that $x_2 \le_P x_1$ and $a_2 = a_1$. Note that $(x_1, a_1) \in \mathcal{C} \cap \widehat{P}_{a_1}$ and $\mathcal{C}$ is Scott closed. Thus $(x_2, a_2) = (x_2, a_1) \in \mathcal{C} \cap \widehat{P}_{a_1} \subseteq \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$. Hence, $\bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$ is a lower subset of $\widehat{P}$. Let $\mathcal{D}$ be a directed subset of $\bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$. Then by Remark 2.4, there exists a directed subset $\{x_i : i \in I\}$ of P and $a_0 \in A$ such that $\mathcal{D} = \{(x_i, a_0) : i \in I\}$. We have that $\bigvee \mathcal{D} = (\bigvee_{i \in I} x_i, a_0) \in \mathcal{C} \cap \widehat{P}_{a_0} \subseteq \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$. Thus $\bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$ is Scott closed.

Since $\mathcal{C}$ is irreducible, we have that $\mathcal{C} = \mathrm{cl}_{\widehat{P}}(\mathcal{C}^*)$ or $\mathcal{C} = \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$. Since $\mathcal{C}^* = \mathcal{C} \cap \varphi(A) \neq \emptyset$ and $\bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\} \cap \varphi(A) = \emptyset$, it follows that $\mathcal{C} \neq \bigcup\{\mathcal{C} \cap \widehat{P}_a : (a, a) \notin \downarrow\mathcal{C}^*\}$. This implies $\mathcal{C} = \mathrm{cl}_{\widehat{P}}(\mathcal{C}^*)$. Thus $\mathcal{C}^*$ is irreducible in $\Sigma\widehat{P}$ (note that a subset is irreducible if and only if its closure is irreducible). By $\mathcal{C}^* = \mathcal{C} \cap \varphi(A)$, it is irreducible closed in $\varphi(A)$. Since $\varphi(A)$ is sober, there exists $(a_0, a_0) \in \varphi(A)$ such that $\mathcal{C}^* = \downarrow_{\varphi(A)}(a_0, a_0)$. It follows that $\mathcal{C} = \mathrm{cl}_{\widehat{P}}(\mathcal{C}^*) = \downarrow(a_0, a_0)$.

All this shows that $\Sigma\widehat{P}$ is a sober space. □

Remark 2.6. From the above proof, we see that $\varphi(A) = \uparrow\varphi(A)$, thus $\varphi(A)$ is an upper set of $\widehat{P}$. Also, it is clear that $\widehat{P} = \downarrow\varphi(A)$, that is $\varphi(A)$ is a dense set in $\Sigma\widehat{P}$.

If X is a d-space, then by Proposition 2.1, there exists a bounded complete algebraic poset $P = \mathrm{Zh}(X)$ and a topological embedding $\phi : X \longrightarrow P$ such that the subset $A = \phi(X)$ of P satisfies the conditions (P1) and (P2) in Proposition 2.3, then there exists a dcpo $\widehat{P}$ and a topological embedding $\varphi : A \longrightarrow \Sigma\widehat{P}$, where A

is the subspace of ΣP. By Remark 2.6, we have that $\varphi(A) = \uparrow\varphi(A)$ and $\widehat{P} = \downarrow\varphi(A)$. By Proposition 2.5, we have that $X \cong \phi(X) \cong \varphi(A)$ is sober if and only if $\Sigma\widehat{P}$ is sober. Now let $P_X = \widehat{P}$ and define $\psi : X \longrightarrow \Sigma P_X$ by $\psi = \varphi \circ \phi$. We can deduce the following theorem.

Theorem 2.7. *Let X be a d-space. Then there is a dcpo P_X and a topological embedding $\psi : X \longrightarrow \Sigma P_X$ satisfying the following properties:*

(1) $\psi(X) = \uparrow\psi(X)$;
(2) $P_X = \downarrow\psi(X)$;
(3) *X is sober if and only if ΣP_X is sober.*

In [7, Example 2.8], Wen and Xu proved that Isbell's complete lattice [4] equipped with the lower topology is a sober space but not co-sober. Based on this result, we can now obtain a dcpo whose Scott topology is sober but not co-sober. The following lemma is useful.

Lemma 2.8. *[7, Theorem 2.9] Every saturated subspace of a co-sober space is co-sober.*

Example 2.9. Let X be a sober space that is not co-sober. By Theorem 2.7(3), the Scott space of the dcpo P_X is sober. But it is not co-sober by Lemma 2.8 and Theorem 2.7(1): $\psi(X)$ is not co-sober and $\psi(X)$ is saturated.

Acknowledgements

(1) We are very grateful for many helpful constructive suggestions from the anonymous referee. (2) This work was supported by China Scholarship Council (No. 20180630073); NSFC (11871097, 11361028, 61300153, 11671008, 11701500, 11626207); NSF Project of Jiangsu Province, China (BK20170483); Singapore Ministry of Education AcRF Tier 2 grant MOE2016-T2-1-083 (M4020333); NTU Tier 1 grants RG32/16 (M4011672) and RG111/19 (M4012245); NIE AcRF (RI 3/16 ZDS), Singapore.

References

[1] M. Escardó, J. Lawson, A. Simpson, Comparing Cartesian closed categories of (core) compactly generated spaces, Topol. Appl. 143 (2004) 105–145.
[2] G. Gierz, K. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D. Scott, Continuous Lattices and Domains, Encycl. Math. Appl., vol. 93, Cambridge University Press, 2003.
[3] J. Goubault-Larrecq, Non-Hausdorff Topology and Domain Theory: Selected Topics in Point-Set Topology, Cambridge University Press, 2013.
[4] J.R. Isbell, Completion of a construction of Johnstone, Proc. Am. Math. Soc. 85 (1982) 333–334.
[5] P.T. Johnstone, Scott is not always sober, Lect. Notes Math. 871 (1981) 282–283.
[6] J.D. Lawson, Spaces of maximal points, Math. Struct. Comput. Sci. 7 (1997) 543–555.
[7] X.P. Wen, X.Q. Xu, Sober is not always co-sober, Topol. Appl. 250 (2018) 48–52.
[8] D.S. Zhao, Poset models of topological spaces, in: Proceeding of International Conference on Quantitative Logic and Quantification of Software, Global-Link Publisher, 2009, pp. 229–238.
[9] D.S. Zhao, X.Y. Xi, Directed complete poset models of T_1 spaces, Math. Proc. Camb. Philos. Soc. 164 (2018) 125–134.