Fuzzy Sets and Systems 27 (1988) 351–372
North-Holland

ON FUZZY PSEUDO-NORMED VECTOR SPACES

Nehad N. MORSI
Department of Mathematics, Military Technical College, Cairo, Egypt

Received July 1986
Revised November 1986

We provide a method for introducing fuzzy pseudo-metric topologies on sets, and fuzzy pseudo-normed topologies on vector spaces over $\mathbb{R}$ or $\mathbb{C}$ which will be fuzzy linear topologies. We define fuzzy pseudo-metrics for pairs of crisp points, and fuzzy pseudo-norms for crisp points, as fuzzy real numbers $\geqslant \tilde{0}$ (as defined by Hutton). We define the associated fuzzy open balls, with crisp points for their centres and fuzzy real numbers $> \tilde{0}$ for their radii. These form a basis for the associated fuzzy pseudo-metric topology. The axioms governing fuzzy pseudo-metric and fuzzy pseudo-norm are straightforward extensions for the corresponding axioms in the crisp case. The formulation conforms with Zadeh's Extension Principle. We show that Katsaras' concept of fuzzy seminorm (Fuzzy Sets and Systems 12 (1984) 143–154) is equivalent to ours, in the sense that both concepts will result in same fuzzy linear topologies.

AMS Subject Classification: 54A40.

Keywords: Fuzzy real numbers, Fuzzy pseudo-metric, Fuzzy open balls, Fuzzy pseudo-norm.

We shall not distinguish in notation between a fuzzy subset U of a universe X and its membership function $U: X \to I = [0, 1]$, nor between a constant fuzzy subset of X with value p and the real number $p \in I$. We shall abbreviate *fuzzy topological space* to *fts*. We follow Chang's definition of fuzzy topology [2]. However, all fuzzy pseudo-metric topologies will turn out to be fully stratified. (A fully stratified fts is one in which all constant fuzzy subsets are open.) We denote by int_τ the fuzzy interior operator associated with a fuzzy topology τ, and by $X - U$ the fuzzy complement of a fuzzy subset U in a universe X.

1. The fuzzy real numbers

Let η be a nonascending function $\mathbb{R} \to I$. Then for all $b \in \mathbb{R}$, both $\eta(b-)$ = limit of η from the left at b, and $\eta(b+)$ = limit of η from the right at b, exist, and $\eta(b-) \geqslant \eta(b+)$. An equivalence relation $\sim$ is defined on the collection of nonascending functions $\mathbb{R} \to I$ as follows: For two such functions η and ζ, $\eta \sim \zeta$ iff for all $b \in \mathbb{R}$, $\eta(b-) = \zeta(b-)$ and $\eta(b+) = \zeta(b+)$.

Definition 1.1 [6, 8]. A *fuzzy real number* is an equivalence class, under the above equivalence relation $\sim$, of nonascending functions $\eta: \mathbb{R} \to I$ with $\eta(-\infty+) = 1$ and $\eta(+\infty-) = 0$. The set of all fuzzy real numbers is denoted by $\mathbb{R}(I)$.

Fuzzy Sets and Systems 27 (1988) 351–372
North-Holland

ON FUZZY PSEUDO-NORMED VECTOR SPACES

Nehad N. MORSI
Department of Mathematics, Military Technical College, Cairo, Egypt

Received July 1986
Revised November 1986

We provide a method for introducing fuzzy pseudo-metric topologies on sets, and fuzzy pseudo-normed topologies on vector spaces over $\mathbb{R}$ or $\mathbb{C}$ which will be fuzzy linear topologies. We define fuzzy pseudo-metrics for pairs of crisp points, and fuzzy pseudo-norms for crisp points, as fuzzy real numbers $\geq \tilde{0}$ (as defined by Hutton). We define the associated fuzzy open balls, with crisp points for their centres and fuzzy real numbers $>\tilde{0}$ for their radii. These form a basis for the associated fuzzy pseudo-metric topology. The axioms governing fuzzy pseudo-metric and fuzzy pseudo-norm are straightforward extensions for the corresponding axioms in the crisp case. The formulation conforms with Zadeh's Extension Principle. We show that Katsaras' concept of fuzzy seminorm (Fuzzy Sets and Systems 12 (1984) 143–154) is equivalent to ours, in the sense that both concepts will result in same fuzzy linear topologies.

AMS Subject Classification: 54A40.

Keywords: Fuzzy real numbers, Fuzzy pseudo-metric, Fuzzy open balls, Fuzzy pseudo-norm.

We shall not distinguish in notation between a fuzzy subset U of a universe X and its membership function $U: X \to I = [0, 1]$, nor between a constant fuzzy subset of X with value p and the real number $p \in I$. We shall abbreviate *fuzzy topological space* to *fts*. We follow Chang's definition of fuzzy topology [2]. However, all fuzzy pseudo-metric topologies will turn out to be fully stratified. (A fully stratified fts is one in which all constant fuzzy subsets are open.) We denote by int_τ the fuzzy interior operator associated with a fuzzy topology τ, and by $X - U$ the fuzzy complement of a fuzzy subset U in a universe X.

1. The fuzzy real numbers

Let η be a nonascending function $\mathbb{R} \to I$. Then for all $b \in \mathbb{R}$, both $\eta(b-)$ = limit of η from the left at b, and $\eta(b+)$ = limit of η from the right at b, exist, and $\eta(b-) \geq \eta(b+)$. An equivalence relation $\sim$ is defined on the collection of nonascending functions $\mathbb{R} \to I$ as follows: For two such functions η and ζ, $\eta \sim \zeta$ iff for all $b \in \mathbb{R}$, $\eta(b-) = \zeta(b-)$ and $\eta(b+) = \zeta(b+)$.

Definition 1.1 [6, 8]. A *fuzzy real number* is an equivalence class, under the above equivalence relation $\sim$, of nonascending functions $\eta: \mathbb{R} \to I$ with $\eta(-\infty+) = 1$ and $\eta(+\infty-) = 0$. The set of all fuzzy real numbers is denoted by $\mathbb{R}(I)$.

We shall not distinguish between a fuzzy real number and any of its representative functions $\mathbb{R} \to I$. If the supremum, as a function $\mathbb{R} \to I$, of a collection of fuzzy real numbers exists, and if 0 is the limit of this function at $+\infty$, then this supremum is a well defined fuzzy real number. It will be called the *supremum fuzzy real number* of the given collection. The *infimum fuzzy real number* of a given collection is similarly understood. We use the symbols $\vee$ and $\wedge$ for supremum and infimum, respectively.

Definition 1.2. The relation *smaller than* $<$ is defined on $\mathbb{R}(I)$ by $\eta < \zeta$ iff $\eta \neq \zeta$ and $\eta(t-) \leq \zeta(t-)$ for all $t \in \mathbb{R}$. Hence, the relation *smaller than or equal to* $\leq$ is an antisymmetric partial ordering on $\mathbb{R}(I)$. ($\eta \leq \zeta$ iff $\eta' \subseteq \zeta'$, as fuzzy subsets of $\mathbb{R}$, for some representative functions η' of η and ζ' of ζ.)

Definition 1.3. (i) A fuzzy real number η is said to be *positive* if $\eta(r) = 1$, for some real $r > 0$.

(ii) $\mathbb{R}^+(I)$ is the collection of all positive fuzzy real numbers.

(iii) $\mathbb{R}^*(I)$ is the collection of all fuzzy real numbers η with $\eta(0-) = 1$.

(iv) $\mathbb{R}^+, \mathbb{R}^* \subseteq \mathbb{R}$ are the collections of positive, respectively nonnegative, real numbers.

For every $r \in \mathbb{R}$, the fuzzy real number $\bar{r} : \mathbb{R} \to I$ is given by: For $s \in \mathbb{R}$,

$$\bar{r}(s)\begin{cases} =1 & \text{if } s < r, \\ =0 & \text{if } s \geq r. \end{cases}$$

Up to this canonical injection $r \mapsto \bar{r}$, $\mathbb{R}$ is considered a subset of $\mathbb{R}(I)$. This inclusion preserves the usual ordering of $\mathbb{R}$.

Hence, $\mathbb{R}^* \subset \mathbb{R}^*(I)$ and $\mathbb{R}^+ \subset \mathbb{R}^+(I) \subset \mathbb{R}^*(I)$. Notice that $\mathbb{R}^+(I) \cup \{\bar{0}\} \neq \mathbb{R}^*(I)$, and $\mathbb{R}^*(I) = \{\eta \in \mathbb{R}(I) : \eta \geq \bar{0}\}$.

Definition 1.4 [17]. Let U be a fuzzy subset of a universe X, and let $\alpha \in I$.

(i) The α-cut (also called the strong α-cut) of U is the crisp subset $U^\alpha = \{x \in X : U(x) > \alpha\}$ of X.

(ii) The α^*-cut (also called the weak α-cut, or the α-level set) of U is the crisp subset $U_{\alpha^*} = \{x \in X : U(x) \geq \alpha\}$ of X.

(iii) U^0 is called the support of U.

Addition of fuzzy real numbers is well defined through the addition of the fuzzy subsets of the additive group $\mathbb{R}$. Namely:

Definition 1.5 [14]. Let $\eta, \zeta \in \mathbb{R}(I)$. $\eta + \zeta$ is the fuzzy real number given by: For every $s \in \mathbb{R}$, $(\eta + \zeta)(s) = \sup\{\eta(a) \wedge \zeta(b) : a + b = s\}$.

The addition of fuzzy subsets η, ζ of an additive group is obtainable in terms of their α-cuts as follows: For $\alpha \in I$, $(\eta + \zeta)^\alpha = \eta^\alpha + \zeta^\alpha$, where addition in the right hand side is the usual addition of ordinary subsets of the given additive group. Hence, $\eta + \zeta$ is easily seen to be indeed a fuzzy real number if so are η and ζ, and we get:

Proposition 1.1. (i) *The canonical inclusion* $\mathbb{R} \subset \mathbb{R}(I)$ *preserves addition.*

(ii) *Under addition,* $\mathbb{R}(I)$ *is an Abelian cancellation monoid, with identity element* $\bar{0}$.

(iii) *The partial ordering of* $\mathbb{R}(I)$ *is translation invariant.*

(iv) *For* $\eta, \zeta \in \mathbb{R}(I)$, $\zeta > \bar{0} \Rightarrow \eta < \eta + \zeta$.

(v) $\mathbb{R}^+(I)$ *and* $\mathbb{R}^*(I)$ *are closed under addition.* $\mathbb{R}^*(I)$ *is also closed under taking infima.* □

Scalar multiplication of fuzzy real numbers by nonnegative reals is well defined through the scalar multiplication of fuzzy subsets of the vector space $\mathbb{R}$ (over itself) [10]. Namely:

Definition 1.6. Let $\eta \in \mathbb{R}(I)$. 0η is the fuzzy real number $\bar{0}$, while for a positive real r, $r\eta$ is the fuzzy real number given by: For $s \in \mathbb{R}$, $(r\eta)(s) = \eta(s/r)$.

For all $\alpha \in I$, $r > 0$, and $\eta \in \mathbb{R}(I)$, $(r\eta)^\alpha = r\eta^\alpha$, where multiplication in the right hand side is the usual scalar multiplication of ordinary subsets of the vector space $\mathbb{R}$. Hence, $r\eta$ is easily seen to be indeed a fuzzy real number, and we get:

Proposition 1.2. (i) *The canonical inclusion* $\mathbb{R} \subset \mathbb{R}(I)$ *preserves scalar multiplication by nonnegative reals.*

(ii) *Scalar multiplication by positive reals preserves the relation* $<$ *on* $\mathbb{R}(I)$.

(iii) $\mathbb{R}^+(I)$ ($\mathbb{R}^*(I)$) *is closed under scalar multiplication by positive (nonnegative) reals.*

(iv) *For* $\eta, \zeta \in \mathbb{R}(I)$ *and* $r, s \geq 0$,

$$r(s\eta) = (rs)\eta, \qquad (r+s)\eta = r\eta + s\eta,$$
$$r(\eta + \zeta) = r\eta + r\zeta, \qquad 1\eta = \eta.$$

(v) *For* $\eta > \bar{0}$ *and* $s > 1 > r \geq 0$, $s\eta > \eta > r\eta$. □

The above two operations on fuzzy real numbers become more tenable if we notice that the α-cuts of fuzzy real numbers are themselves fuzzy real numbers in $\mathbb{R} \subset \mathbb{R}(I)$, and that the induced operations on those α-cuts then coincide with the usual operations on $\mathbb{R}$. We can also combine Proposition 1.1(iii) and Proposition 1.2(ii) in one sentence, and say that the partial ordering $\geq$ of $\mathbb{R}(I)$ is a *vector ordering*. This ordering can be extended to a vector ordering of a real vector space which includes $\mathbb{R}(I)$.

Definition 1.7 [8]. For every $b \in \mathbb{R}$, the fuzzy subsets R_b and L_b of $\mathbb{R}(I)$ are defined as follows: For all $\eta \in \mathbb{R}(I)$,

$$R_b(\eta) = \eta(b+) \quad \text{and} \quad L_b(\eta) = 1 - \eta(b-).$$

The collection $\{R_b : b \in \mathbb{R}\} \cup \{L_b : b \in \mathbb{R}\}$ is a subbase for Hutton's fuzzy topology on $\mathbb{R}(I)$.

The following properties of the fuzzy subsets R_b and L_b are put to use in the sequel:

Theorem 1.1. *Let $a, b, s \in \mathbb{R}$ and $\eta, \zeta \in \mathbb{R}(I)$.*

(i) *R_a is descending in a; i.e., $R_a \cup R_b = R_{a \wedge b}$ and $R_a \cap R_b = R_{a \vee b}$.*

(ii) *L_a is ascending in a; i.e., $L_a \cup L_b = L_{a \vee b}$ and $L_a \cap L_b = L_{a \wedge b}$.*

(iii) *$\eta < \zeta \Rightarrow R_b(\eta) \leqslant R_b(\zeta)$ and $L_b(\eta) \geqslant L_b(\zeta)$.*

(iv) *$R_a \cap \mathbb{R} = (a, \infty)$ and $L_a \cap \mathbb{R} = (-\infty, a)$.*

(v) *$R_b = \bigcup_{r>b} R_r$ and $L_b = \bigcup_{r<b} L_r$.*

(vi) $$R_b = \mathbb{R}(I) - \left[\bigcap_{r>b} L_r\right] \subset \mathbb{R}(I) - L_b,$$

$$L_b = \mathbb{R}(I) - \left[\bigcap_{r<b} R_r\right] \subset \mathbb{R}(I) - R_b.$$

Let $\sigma = \{\eta_m : m \in M\}$ be a nonempty collection of fuzzy real numbers.

(vii) *If $\bigvee \sigma$ exists in $\mathbb{R}(I)$, then $R_b(\bigvee \sigma) = \bigvee \{R_b(\eta_m) : m \in M\}$, and $L_b(\bigvee \sigma) \leqslant \bigwedge \{L_b(\eta_m) : m \in M\}$.*

(viii) *If $\bigwedge \sigma$ exists in $\mathbb{R}(I)$, then $R_b(\bigwedge \sigma) \leqslant \bigwedge \{R_b(\eta_m) : m \in M\}$, and $L_b(\bigwedge \sigma) = \bigvee \{L_b(\eta_m) : m \in M\}$.*

(ix) $$R_s(\eta + \zeta) = \bigvee \{R_r(\eta) \wedge R_t(\zeta) : r + t = s\},$$
$$L_s(\eta + \zeta) = \bigwedge \{L_r(\eta) \vee L_t(\zeta) : r + t = s\}.$$

(x) $$R_{a+b}(\eta + \zeta) \leqslant R_a(\eta) \vee R_b(\zeta), \qquad L_{a+b}(\eta + \zeta) \geqslant L_a(\eta) \wedge L_b(\zeta).$$

(xi) $[\eta + (-b)^{\sim}](s) = \eta(s + b)$.

(xii) *Addition on $\mathbb{R}(I)$ defines addition on $I^{\mathbb{R}(I)}$, with respect to which $\tilde{b} + R_a = R_{a+b}$ and $\tilde{b} + L_a = L_{a+b}$.*

(xiii) *$s > 0 \Rightarrow R_b(\eta) = R_{sb}(s\eta)$ and $L_b(\eta) = L_{sb}(s\eta)$.*

(xiv) *Scalar multiplication on $\mathbb{R}(I)$ defines multiplication by positive scalars on $I^{\mathbb{R}(I)}$, with respect to which $R_b = (1/s)R_{sb}$ and $L_b = (1/s)L_{sb}$, for all $s > 0$.*

Proof. (i)–(iv). These are well known immediate consequences of Definition 1.7.

(v) We have

$$\left[\bigcup_{r>b} R_r\right](\eta) = \sup_{r>b} \eta(r+) = \eta(b+) = R_b(\eta)$$

and

$$\left[\bigcup_{r<b} L_r\right](\eta) = \sup_{r<b} [1 - \eta(r-)]$$
$$= 1 - \inf_{r<b} \eta(r-) = 1 - \eta(b-) = L_b(\eta),$$

because η is nonascending.

(vi) For the first assertion,

$$\left[\mathbb{R}(I) - \left(\bigcap_{r>b} L_r\right)\right](\eta) = \sup_{r>b}\,[1 - L_r(\eta)]$$
$$= \sup_{r>b} \eta(r-) = \eta(b+) = R_b(\eta),$$

and from (ii), $\bigcap_{r>b} L_r \supset L_b$ but equality does not hold, as can be easily verified using (iv). The second assertion is similarly proved.

(vii) This follows from

$$R_b(\bigvee \sigma) = (\bigvee \sigma)(b+) = \sup\{\eta_m(b+) : m \in M\} = \sup\{R_b(\eta_m) : m \in M\},$$

and

$$L_b(\bigvee \sigma) = 1 - (\bigvee \sigma)(b-) \leqslant 1 - \sup\{\eta_m(b-) : m \in M\}$$
$$= \inf\{1 - \eta_m(b-) : m \in M\} = \inf\{L_b(\eta_m) : m \in M\}.$$

(viii) The proof is similar to that of (vii).

(ix) We have

$$R_s(\eta + \zeta) = (\eta + \zeta)(s+) = \sup_{r>s}\,[(\eta + \zeta)(r)]$$
$$= \sup_{r>s}\,[\sup\{\eta(c) \wedge \zeta(d) : c + d = r\}]$$
$$= \sup\{\eta(a + \delta) \wedge \zeta(b + \delta) : \delta > 0 \text{ and } a + b = s\}$$
$$= \sup\{\eta(a+) \wedge \zeta(b+) : a + b = s\}$$
$$= \sup\{R_a(\eta) \wedge R_b(\zeta) : a + b = s\}.$$

Hence from (vi),

$$L_s(\eta + \zeta) = 1 - \inf_{r<s} R_r(\eta + \zeta)$$
$$= 1 - \inf_{r<s}\,[\sup\{\eta(c+) \wedge \zeta(d+) : c + d = r\}]$$
$$= 1 - \inf_{\delta>0}\,[\sup\{\eta(a - \delta+) \wedge \zeta(b - \delta+) : a + b = s\}]$$
$$= 1 - \sup\left\{\inf_{\delta>0}\,[\eta(a - \delta+) \wedge \zeta(b - \delta+)] : a + b = s\right\}$$
$$= 1 - \sup\{\eta(a-) \wedge \zeta(b-) : a + b = s\}$$
$$= \inf\{[1 - \eta(a-)] \vee [1 - \zeta(b-)] : a + b = s\}$$
$$= \inf\{L_a(\eta) \vee L_b(\zeta) : a + b = s\}.$$

(x) For $c, d \in \mathbb{R}\cdot$ $d \geqslant b \Rightarrow \zeta(d) \leqslant \zeta(b) \Rightarrow \eta(c) \wedge \zeta(d) \leqslant \zeta(d) \leqslant \zeta(b) \leqslant \eta(a) \vee \zeta(b)$, and $c \geqslant a \Rightarrow \eta(c) \leqslant \eta(a) \Rightarrow \eta(c) \wedge \zeta(d) \leqslant \eta(c) \leqslant \eta(a) \leqslant \eta(a) \vee \zeta(b)$. Therefore, $c + d = a + b \Rightarrow \eta(c) \wedge \zeta(d) \leqslant \eta(a) \vee \zeta(b)$, and

$$[1 - \eta(c)] \vee [1 - \zeta(d)] = 1 - [\eta(c) \wedge \zeta(d)]$$
$$\geqslant 1 - [\eta(a) \vee \zeta(b)] = [1 - \eta(a)] \wedge [1 - \zeta(b)].$$

Hence from (ix),

$$R_{a+b}(\eta+\zeta)=\sup\{\eta(c+)\wedge\zeta(d+):c+d=a+b\}$$
$$\leq \eta(a+)\vee\zeta(b+)=R_a(\eta)\vee R_b(\zeta),$$

and

$$L_{a+b}(\eta+\zeta)=\bigwedge\{[1-\eta(c-)]\vee[1-\zeta(d-)]:c+d=a+b\}$$
$$\geq [1-\eta(a-)]\wedge[1-\zeta(b-)]=L_a(\eta)\wedge L_b(\zeta).$$

(xi) We have

$$[\eta+(-b)^{\sim}](s)=\bigvee\{\eta(s-a)\wedge(-b)^{\sim}(a):a\in\mathbb{R}\}$$
$$=\eta(s-(-b))=\eta(s+b).$$

(xii) From (xi),

$$(\tilde{b}+R_a)(\eta)=R_a(\eta+(-b)^{\sim})=[\eta+(-b)^{\sim}](a+)$$
$$=\eta(a+b+)=R_{a+b}(\eta)$$

and

$$(\tilde{b}+L_a)(\eta)=L_a(\eta+(-b)^{\sim})=1-[\eta+(-b)^{\sim}](a-)$$
$$=1-\eta(a+b-)=L_{a+b}(\eta).$$

(xiii) $R_{sb}(s\eta)=(s\eta)(sb+)=\eta(b+)=R_b(\eta)$, and $L_{sb}(s\eta)=1-(s\eta)(sb-)=1-\eta(b-)=L_b(\eta)$.

(xiv) From (xiii), $[(1/s)R_{sb}](\eta)=R_{sb}(s\eta)=R_b(\eta)$, and $[(1/s)L_{sb}](\eta)=L_{sb}(s\eta)=L_b(\eta)$. □

Proposition 1.3. *Let U and V be fuzzy subsets of a group $(X,+)$. Then for all $\alpha\in I$, $U_{\alpha^*}+V_{\alpha^*}\subseteq(U+V)_{\alpha^*}$.*

Proof. For all $x\in X$, $x\in[U_{\alpha^*}+V_{\alpha^*}]\Leftrightarrow$ there are $y,z\in X$ with $x=y+z$ and $y\in U_{\alpha^*}$, $z\in V_{\alpha^*}\Rightarrow(U+V)(x)\geq U(y)\wedge V(z)\geq\alpha\Rightarrow x\in(U+V)_{\alpha^*}$. Hence, $U_{\alpha^*}+V_{\alpha^*}\subseteq(U+V)_{\alpha^*}$. □

Theorem 1.2. *Let $\eta,\zeta\in\mathbb{R}(I)$ and $r\in\mathbb{R}$. Then,*

$$(\eta+\zeta)(r)=\inf\{\eta(a)\vee\zeta(b):a+b=r\}.$$

Proof. Choose and fix representative functions $\mathbb{R}\to I$ for η and ζ. Those functions are also considered fuzzy subsets of $\mathbb{R}$. It is easy to check that for all $\beta\in I$, $\mathbb{R}-(\eta_{\beta^*}+\zeta_{\beta^*})=(\mathbb{R}-\eta_{\beta^*})+(\mathbb{R}-\zeta_{\beta^*})$, as ordinary subsets of $\mathbb{R}$. Hence for all $\alpha\in I$,

$$[\mathbb{R}-(\eta+\zeta)]^{\alpha}=\mathbb{R}-(\eta+\zeta)_{(1-\alpha)^*}$$
$$\subseteq\mathbb{R}-[\eta_{(1-\alpha)^*}+\zeta_{(1-\alpha)^*}]\quad\text{(from Proposition 1.3)}$$
$$=[\mathbb{R}-\eta_{(1-\alpha)^*}]+[\mathbb{R}-\zeta_{(1-\alpha)^*}]$$
$$=(\mathbb{R}-\eta)^{\alpha}+(\mathbb{R}-\zeta)^{\alpha}=[(\mathbb{R}-\eta)+(\mathbb{R}-\zeta)]^{\alpha}.$$

Hence, $\mathbb{R}-(\eta+\zeta)\subseteq(\mathbb{R}-\eta)+(\mathbb{R}-\zeta)$. Consequently, $\eta+\zeta\supseteq\mathbb{R}-[(\mathbb{R}-\eta)+$

$(\mathbb{R} - \zeta)]$, and

$$\begin{aligned}(\eta+\zeta)(r) &\geqslant 1 - [(\mathbb{R}-\eta) + (\mathbb{R}-\zeta)](r) \\ &= 1 - \sup\{[1-\eta(a)] \wedge [1-\zeta(b)] : a+b=r\} \\ &= \inf\{\eta(a) \vee \zeta(b) : a+b=r\}.\end{aligned}$$

The inverse inequality follows from Theorem 1.1(x). □

Corollary 1.1. *For* $\eta, \zeta \in \mathbb{R}(I)$, $\eta + \zeta = \mathbb{R} - [(\mathbb{R}-\eta) + (\mathbb{R}-\zeta)]$, *as fuzzy subsets of* $\mathbb{R}$.

Proof. This follows from the above proof. □

Corollary 1.2. (Compare with Theorem 1.1(x).) *Let* $\eta, \zeta \in \mathbb{R}(I)$ *and* $s \in \mathbb{R}$. *Then*:
(i) $R_s(\eta+\zeta) = \bigwedge \{R_a(\eta) \vee R_b(\zeta) : a+b=s\}$.
(ii) $L_s(\eta+\zeta) = \bigvee \{L_a(\eta) \wedge L_b(\zeta) : a+b=s\}$.

Proof. (i) Follows from Theorem 1.2 by taking limits from the right.
(ii) By taking limits from the left in Theorem 1.2, we get

$$\begin{aligned}L_s(\eta+\zeta) = 1 - (\eta+\zeta)(s-) &= 1 - \wedge\{\eta(a-) \vee \zeta(b-) : a+b=s\} \\ &= \bigvee \{[1-\eta(a-)] \wedge [1-\zeta(b-)] : a+b=s\} \\ &= \bigvee \{L_a(\eta) \wedge L_b(\zeta) : a+b=s\}. \quad \square\end{aligned}$$

2. Fuzzy pseudo-metric

Definition 2.1. A *fuzzy pseudo-metric* on a nonempty set X is a mapping $d: X \times X \to \mathbb{R}^*(I)$ which satisfies: For $x, y, z \in X$,
(i) $d(x,x) = \tilde{0}$,
(ii) $d(x,y) = d(y,x)$ (symmetry),
(iii) $d(x,y) + d(y,z) \geqslant d(x,z)$ (triangle inequality).
(X, d) is called a *fuzzy pseudo-metric space* (abbreviated fpms). If, in addition, d satisfies:
(iv) $x \neq y \Rightarrow d(x,y)$ is a positive fuzzy real number (i.e., $d(x,y) \in \mathbb{R}^+(I)$),
then the fuzzy pseudo-metric d is called a *fuzzy metric*, and (X, d) is called a *fuzzy metric space*.

Definition 2.2. Let (X, d) be a fpms, $x \in X$, and $\eta \in \mathbb{R}^*(I) - \{\tilde{0}\}$. The *fuzzy open ball* in (X, d) with centre x and radius η is the fuzzy subset $B(x; \eta) \in I^X$ defined as follows: For $y \in X$,

$$B(x;\eta)(y) = \sup\{R_s(\eta) \wedge L_s[d(x,y)] : s \in \mathbb{R}\}.$$

The notation $B(\, ; \,)$ for fuzzy open balls will be maintained in the sequel.

Definition 2.3. Let (X, d) be a fpms. The *fuzzy (pseudo-metric) topology* on (X, d) is the fuzzy topology on X with subbase the collection of all fuzzy open balls in (X, d).

This fuzzy topology is also called the fuzzy (pseudo-metric) topology associated with d. When d is a fuzzy metric, its associated fuzzy topology will be called a *fuzzy metric topology*.

Notation 2.1. Let $\eta \in \mathbb{R}^*(I)$ and $0 \leqslant q \leqslant 1$. As a fuzzy subset of $\mathbb{R}$, $\bar{0} \cup (\eta \cap q)$ is a fuzzy real number in $\mathbb{R}^*(I)$. We shall denote this fuzzy real number by $\eta \pitchfork q$. (Hence, $\bar{0} = \eta \pitchfork 0 \leqslant \eta \pitchfork q \leqslant \eta \pitchfork 1 = \eta$.)

Theorem 2.1. *Let τ be the fuzzy pseudo-metric topology on a fpms (X, d). Let $x, y \in X$, $\eta \in \mathbb{R}^*(I) - \{\bar{0}\}$, and let r, q be positive real numbers with $q \leqslant 1$. Then*:

(i) $B(x; \eta)(y) = B(y; \eta)(x) = \sup\{R_s(\eta) \wedge L_s[d(x, y)] : s > 0\}$.

(ii) $B(x; \bar{r})(y) = L_r[d(x, y)] = 1 - d(x, y)(r-)$.

(iii) *If $d(x, y) = \bar{b}$ for some nonnegative real number b, then* $B(x; \eta)(y) = R_b(\eta) = \eta(b+)$, *and*

$$B(x; \bar{r})(y) = \begin{cases} 1 & \text{if } d(x, y) = \bar{b} < \bar{r}, \\ 0 & \text{if } d(x, y) = \bar{b} \geqslant \bar{r}. \end{cases}$$

(Hence, the definition of fuzzy open balls reduces in the crisp case to the definition of open balls.)

(iv) $B(x; \bar{r})(y) = 0 \Leftrightarrow d(x, y) \geqslant \bar{r}$.

(v) $B(x; \bar{r})(y) = 1 \Leftrightarrow d(x, y) < \bar{r}$.

(vi) $B(x; \eta \pitchfork q) = B(x; \eta) \cap q$.

(vii) $B(x; \eta \pitchfork q)(x) = \eta(0+) \wedge q$. *This means that, if η is positive, we get* $B(x; \eta \pitchfork q)(x) = q$.

(viii) *The fuzzy topology τ is fully stratified.*

Proof. (i) Since $d(x, y) \geqslant \bar{0}$, it follows that $s \leqslant 0 \Rightarrow L_s[d(x, y)] = 0$. Hence (i) follows from Definition 2.2.

(ii) Since

$$R_s(\bar{r}) = \begin{cases} 1 & \text{if } s < r, \\ 0 & \text{if } s \geqslant r, \end{cases}$$

it follows that

$$B(x; \bar{r})(y) = \sup\{L_s[d(x, y)] : 0 < s < r\} = L_r[d(x, y)],$$

using Theorem 1.1(v).

(iii) Suppose $d(x, y) = \bar{b} > \bar{0}$. Since for $s > 0$,

$$L_s(\bar{b}) = \begin{cases} 1 & \text{if } s > b, \\ 0 & \text{if } s \leqslant b, \end{cases}$$

it follows that

$$B(x; \eta)(y) = \sup\{R_s(\eta) \wedge L_s(\bar{b}) : s > 0\} = \sup\{R_s(\eta) : s > b\} = R_b(\eta),$$

by Theorem 1.1(v). Hence also, $B(x; \bar{r})(y) = R_b(\bar{r}) = 1$ if $b < r$, and $=0$ if $b \geqslant r$.

(iv) From (ii),

$$0 = B(x; \bar{r})(y) = L_r[d(x, y)] = 1 - d(x, y)(r-) \Leftrightarrow d(x, y)(r-) = 1 \Leftrightarrow d(x, y) \geq \bar{r}.$$

(v) From (ii), we have $1 = B(x; \bar{r})(y) = 1 - d(x, y)(r-) \Leftrightarrow d(x, y)(r-) = 0 \Leftrightarrow d(x, y) < \bar{r}$.

(vi) For all $z \in X$,

$$B(x; \eta \pitchfork q)(z) = \sup\{R_s(\eta \pitchfork q) \wedge L_s[d(x, z)] : s > 0\} = \sup\{q \wedge R_s(\eta) \wedge L_s[d(x, z)] : s > 0\} = q \wedge B(x; \eta)(z).$$

Hence, $B(x; \eta \pitchfork q) = B(x; \eta) \cap q$.

(vii) By (iii) and (vi), $B(x; \eta \pitchfork q)(x) = q \wedge B(x; \eta)(x) = q \wedge R_0(\eta)$. Hence, if η is positive, then $B(x; \eta \pitchfork q)(x) = q$.

(viii) For every $0 < q \leq 1$, let $U_q = \bigcup\{B(x; \bar{1} \pitchfork q) : x \in X\} \in I^X$. From (vi), $U_q \subseteq q$. From (vii), $U_q \supseteq q$. Hence, $q = U_q \in \tau$. This proves that τ is fully stratified. □

Theorem 2.2. *Let* $\{\eta_m : m \in M\}$ *and* $\{\zeta_h : h \in H\}$ *be nonempty subcollections of* $\mathbb{R}^*(I) - \{\bar{0}\}$, *which satisfy*

$$\sup\{\eta_m : m \in M\} = \eta \in \mathbb{R}^*(I) \quad \textit{and} \quad \inf\{\zeta_h : h \in H\} = \zeta > \bar{0}.$$

Let (X, d) *be a fpms, and* $x \in X$. *Then*:

(i) $B(x; \eta) = \bigcup\{B(x; \eta_m) : m \in M\}$.

(ii) $B(x; \zeta) \subseteq \bigcap\{B(x; \zeta_h) : h \in H\}$, *and equality holds when H is finite and all* ζ_h *belong to* $\mathbb{R}^+ \subset \mathbb{R}^*(I) - \{\bar{0}\}$.

Proof. (i) For $y \in X$ we have from Theorem 1.1(vii),

$$\begin{aligned} B(x; \eta)(y) &= \sup\{R_s[\textstyle\bigvee\{\eta_m : m \in M\}] \wedge L_s[d(x, y)] : s > 0\} \\ &= \sup\{[\sup\{R_s(\eta_m) : m \in M\}] \wedge L_s[d(x, y)] : s > 0\} \\ &= \sup\{\sup\{R_s(\eta_m) \wedge L_s[d(x, y)] : s > 0\} : m \in M\} \\ &= \sup\{B(x; \eta_m)(y) : m \in M\}. \end{aligned}$$

This proves (i).

(ii) From (i), $B(x; \zeta)$ is isotone in ζ. Hence, $B(x; \zeta) \subseteq \bigcap\{B(x; \zeta_h) : h \in H\}$. The second assertion follows from Theorem 2.1(ii) and Theorem 1.1(ii). □

Definition 2.4 [4, 18]. Let X be a nonempty set, $x \in X$, and $0 < q \leq 1$. The *fuzzy singleton* q_x with value q and support x is the fuzzy subset of X given by: For $y \in X$,

$$q_x(y) = \begin{cases} q & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

When $0 < q < 1$, q_x is also called a *fuzzy point*, while $1_x = \{x\}$ is called a *crisp singleton*.

Let $U \in I^X$. A fuzzy point q_x is said to be *in* U, written $q_x \mathrel{\bar{\in}} U$, if $q < U(x)$.

Theorem 2.3. *Let (X, d) be a fpms, $x, y \in X$, $0<q<1$, and $\eta > \bar{0}$ in $\mathbb{R}(I)$. If $q_y \,\bar{\in}\, B(x; \eta)$, then there is $\zeta > \bar{0}$ in $\mathbb{R}(I)$ such that $q_y \,\bar{\in}\, B(y; \zeta) \subseteq B(x; \eta)$.*

Proof. Since $q_y \,\bar{\in}\, B(x; \eta)$, there is $p \in I$ with

$$q < p < B(x; \eta)(y) = \sup\{R_s(\eta) \wedge L_s[d(x, y)] : s > 0\}.$$

Hence, there is $r > 0$ with

$$p < R_r(\eta) \wedge L_r[d(x, y)]. \tag{1}$$

Since $\eta(r+) = R_r(\eta) > p > 0$, it follows that $\eta \not\leqslant \bar{r}$. Hence, $(\eta \vee \bar{r}) + (-r)^{\sim} \geqslant \eta + (-r)^{\sim} \not\leqslant \bar{0}$. Also, $(\eta \vee \bar{r}) + (-r)^{\sim} \geqslant \bar{r} + (-r)^{\sim} = \bar{0}$. Hence, the fuzzy real number $\zeta = [(\eta \vee \bar{r}) + (-r)^{\sim}] \pitchfork p$ is $> \bar{0}$. From Theorem 1.1(xi) we get for all $s > 0$,

$$\begin{aligned}\zeta(s) &= p \wedge [(\eta \vee \bar{r}) + (-r)^{\sim}](s) \\ &= p \wedge [(\eta \vee \bar{r})(s + r)] = p \wedge [\eta(s + r) \vee \bar{r}(s + r)].\end{aligned}$$

Hence,

$$\zeta(s) = p \wedge \eta(s + r). \tag{2}$$

Hence from Theorem 2.1(vii) and (1) above, $B(y; \zeta)(y) = R_0(\zeta) = \zeta(0+) = p \wedge \eta(r+) = p \wedge R_r(\eta) = p > q$. Hence, $q_y \,\bar{\in}\, B(y; \zeta)$. On the other hand, for every $z \in X$,

$$\begin{aligned}B(y; \zeta)(z) &= \sup\{R_s(\zeta) \wedge L_s[d(y, z)] : s > 0\} \\ &= \sup\{R_{s+r}(\eta) \wedge p \wedge L_s[d(y, z)] : s > 0\} \quad \text{(from (2) above)} \\ &\leqslant \sup\{R_{s+r}(\eta) \wedge L_r[d(x, y)] \wedge L_s[d(y, z)] : s > 0\} \quad \text{(from (1) above).}\end{aligned}$$

Hence from Theorem 1.1(x), the triangle inequality, and Theorem 1.1(iii),

$$\begin{aligned}B(y; \zeta)(z) &\leqslant \sup\{R_{s+r}(\eta) \wedge L_{s+r}[d(x, y) + d(y, z)] : s > 0\} \\ &\leqslant \sup\{R_{s+r}(\eta) \wedge L_{s+r}[d(x, z)] : s > 0\} \\ &\leqslant \sup\{R_s(\eta) \wedge L_s[d(x, z)] : s > 0\} = B(x; \eta)(z).\end{aligned}$$

This completes the proof that $q_y \,\bar{\in}\, B(y; \zeta) \subseteq B(x; \eta)$. □

Theorem 2.4. *The collection of all fuzzy open balls in a fpms is a base for its associated fuzzy topology.*

Proof. Let τ be the fuzzy pseudo-metric topology on a fpms (X, d). Suppose that for $x_1, x_2, y \in X$, $\eta_1, \eta_2 > \bar{0}$, and $0 < q < 1$,

$$q_y \,\bar{\in}\, B(x_1; \eta_1) \cap B(x_2; \eta_2).$$

From Theorem 2.3, there are $\zeta_1, \zeta_2 > \bar{0}$ such that for $i = 1, 2$, $q_y \,\bar{\in}\, B(y; \zeta_i) \subseteq B(x_i; \eta_i)$. Hence from Theorem 2.2(ii),

$$q_y \,\bar{\in}\, B(y; \zeta_1 \wedge \zeta_2) \subseteq \bigcap_{i=1,2} B(y; \zeta_i) \subseteq \bigcap_{i=1,2} B(x_i; \eta_i).$$

This proves that the subbase of τ which consists of the fuzzy open balls in (X, d) is a base of τ. □

Definition 2.5 [18]. A fuzzy topological space (X, τ) is said to be *first countable* if the neighbourhood (abbreviated nhd) system of every fuzzy point q_x in X, $\{V \in I^X : \text{there is } U \in \tau \text{ with } q_x \,\bar{\in}\, U \subseteq V\}$, has a countable base. (Cf. [4].)

Theorem 2.5. *Let (X, d) be a fpms with associated fuzzy topology τ. Then:*

(i) *The nhd system of every fuzzy point q_x in X has the countable open base $\{B(x; \bar{r} \pitchfork p)$: r and p are positive rationals, and $1 > p > q\}$.*

(ii) *(X, τ) is first countable.*

(iii) *The collection $\{B(x; \bar{r} \pitchfork p) : x \in X$, r and p are positive rationals and $1 > p > 0\} \subseteq \tau$ is a base of τ.*

Proof. (i) Suppose $q_x \,\bar{\in}\, U \in \tau$. From Theorem 2.4, $q_x \,\bar{\in}\, B(y; \eta) \subseteq U$, for some $y \in X$ and $\eta > \tilde{0}$. From Theorem 2.3, $q_x \,\bar{\in}\, B(x; \zeta) \subseteq B(y; \eta) \subseteq U$, for some $\zeta > \tilde{0}$. But from Theorem 2.1(iii), $B(x; \zeta)(x) = R_0(\zeta) = \zeta(0+)$. Hence there is a rational number p such that $q < p < \zeta(0+)$. Therefore, as a fuzzy subset of $\mathbb{R}$, ζ has p-cut $\zeta^p \supseteq (-\infty, r]$, for some positive rational number r. Hence, as fuzzy real numbers, $\zeta \geqslant \bar{r} \pitchfork p$. Hence from Theorem 2.2, $B(x; \bar{r} \pitchfork p) \subseteq B(x; \zeta) \subseteq U$. Also from Theorem 2.1(vii), $B(x; \bar{r} \pitchfork p)(x) = p > q$. Consequently, $q_x \,\bar{\in}\, B(x; \bar{r} \pitchfork p) \subseteq U$. This proves (i).

(ii) and (iii). These follow from (i). □

Theorem 2.6. *Let τ be a fuzzy pseudo-metric topology on X, and let τ_i, $i = 1, \ldots, n$, be pseudo-metric or fuzzy pseudo-metric topologies on X_i, $i = 1, \ldots, n$, respectively, with at least one τ_i a fuzzy pseudo-metric topology. Let $\prod$ denote fuzzy topological product. Then, a function*

$$f : \prod_{i=1}^{n} (X_i, \tau_i) \to (X, \tau)$$

is continuous if for all $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i$ and all $1 > p$, $r > 0$, there are $r_1, \ldots, r_n > 0$ such that

$$p \cap \prod_{i=1}^{n} B(x_i; \bar{r}_i) \subseteq f^{-1}[B(x; \bar{r})],$$

where $x = f(x_1, \ldots, x_n)$.

Proof. From Theorem 2.1(vi), the stated condition implies that for all $(x_1, \ldots, x_n)$, $r > 0$ and $1 > p > 0$, there are $r_1, \ldots, r_n > 0$ such that

$$\prod_{i=1}^{n} B(x_i; \bar{r}_i \pitchfork p) = \left[\prod_{i=1}^{n} B(x; \bar{r}_i)\right] \cap p$$
$$\subseteq f^{-1}[B(x; \bar{r})] \cap p = f^{-1}[B(x; \bar{r} \pitchfork p)].$$

Hence f is continuous (using Theorem 2.5(i)). □

Definition 2.6. Let $0 \leqslant \alpha < 1$ $(0 < \alpha \leqslant 1)$. A fts (X, τ) is said to be

(i) α-T_0 (α^*-T_0) if for every $x \neq y$ in X there is $U \in \tau$ such that $U(x) > \alpha$ and

$U(y)=0$; or $U(y)>\alpha$ and $U(x)=0$ ($U(x)\geqslant\alpha$ and $U(y)=0$; or $U(y)\geqslant\alpha$ and $U(x)=0$).

(ii) α-T_1 (α^*-T_1) if for every $x\neq y$ in X there is $U\in\tau$ such that $U(x)>\alpha$ and $U(y)=0$ ($U(x)\geqslant\alpha$ and $U(y)=0$).

(iii) [16] α-T_2 (α^*-T_2) if for every $x\neq y$ in X there are disjoint $U, V\in\tau$ such that $U(x)\wedge V(y)>\alpha$ ($U(x)\wedge V(y)\geqslant\alpha$).

The above separation axioms are easily seen to be related as follows: For $0\leqslant\alpha<1$ ($0<\alpha\leqslant 1$), α-$\mathrm{T}_2\Rightarrow\alpha$-$\mathrm{T}_1\Rightarrow\alpha$-$\mathrm{T}_0$ (α^*-$\mathrm{T}_2\Rightarrow\alpha^*$-$\mathrm{T}_1\Rightarrow\alpha^*$-$\mathrm{T}_0$). For $0<\alpha<\beta<1$ and $i=0, 1, 2$, 1^*-$\mathrm{T}_i\Rightarrow\beta$-$\mathrm{T}_i\Rightarrow\beta^*$-$\mathrm{T}_i\Rightarrow\alpha$-$\mathrm{T}_i\Rightarrow\alpha^*$-$\mathrm{T}_i\Rightarrow 0$-$\mathrm{T}_i$. Also, a fts is 1^*-T_1 iff all its crisp singletons are closed.

The concept of fuzzy metric has its merit in:

Theorem 2.7. *Let (X, d) be a fpms with associated fuzzy topology τ. Let $0\leqslant\alpha<1$, $0<\beta\leqslant 1$, and $i, j=0, 1, 2$. The following are equivalent statements:*

(i) *(X, τ) is α-T_i.*
(ii) *(X, τ) is β^*-T_j.*
(iii) *d is a fuzzy metric.*

Proof. The proof is arranged as follows: [(i) or (ii)] $\Rightarrow$ (a) $\Rightarrow$ (iii) $\Rightarrow$ (b) $\Rightarrow$ [(i) and (ii)], where, (a) (X, τ) is 0-T_0, (b) (X, τ) is 1^*-T_2.

The first and last implications above follow from the relations listed before the theorem. We now prove the middle two implications.

(a) $\Rightarrow$ (iii): Suppose (X, τ) is 0-T_0, and let $x\neq y$ in X. Then, there is $U\in\tau$ with $U(x)>0$ and $U(y)=0$, say. Hence by Theorem 2.5, there are $r>0$ and $1>p>0$, such that $B(x;\tilde{r}\Cap p)(y)=0$. Hence, $L_r[d(x, y)]=0$, so that $d(x, y)\geqslant\tilde{r}$. This proves that d is a fuzzy metric.

(iii) $\Rightarrow$ (b): Suppose d is a fuzzy metric, and let $x\neq y$ in X. Then there is $r>0$ such that $d(x, y)\geqslant(2r)^{\sim}$. Hence from Theorem 1.1(vii), (x), and from the triangle inequality for d, we have for all $z\in X$,

$$\begin{aligned}0&=L_{2r}[(2r)^{\sim}]\geqslant L_{2r}[d(x, y)]\geqslant L_{2r}[d(x, z)+d(y, z)]\\&\geqslant L_r[d(x, z)]\wedge L_r[d(y, z)]=B(x;\tilde{r})(z)\wedge B(y;\tilde{r})(z)\\&=[B(x;\tilde{r})\cap B(y;\tilde{r})](z).\end{aligned}$$

Hence, $B(x;\tilde{r})\cap B(y;\tilde{r})=\emptyset$. But, $B(x;\tilde{r})(x)=B(y;\tilde{r})(y)=1$. This proves that (X, τ) is 1^*-T_2. □

3. Fuzzy pseudo-norm

In the sequel, (K, κ) stands for the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, in their usual topologies. All vector spaces are assumed over K, and $\mathbf{0}$ always denotes the zero vector.

Familiarity with the basic notions of Katsaras' papers [10, 11, 12] is recommended for reading what follows.

Definition 3.1 [7]. A *fuzzy pseudo-norm* on a vector space X is a mapping $\|\ \|: X \to \mathbb{R}^*(I)$ which satisfies: For $x, y \in X$ and $r \in K$,

(i) $\|rx\| = |r|\,\|x\|$,

(ii) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality).

$(X, \|\ \|)$ is called a *fuzzy pseudo-normed space*. If, in addition, $\|\ \|$ satisfies:

(iii) $x \neq \mathbf{0} \Rightarrow \|x\|$ is a positive fuzzy real number, then $\|\ \|$ is called a *fuzzy norm*, and $(X, \|\ \|)$ is called a *fuzzy normed space*.

Definition 3.2. The fuzzy pseudo-metric d associated with a fuzzy pseudo-normed space $(X, \|\ \|)$ is defined on X by: For $x, y \in X$, $d(x, y) = \|y - x\|$.

The fuzzy topology on X associated with d is called a *fuzzy pseudo-normed topology*, and it is said to be associated with $\|\ \|$. This fuzzy topology is also called a *fuzzy normed topology* when $\|\ \|$ is a fuzzy norm.

The following proposition can be proved in exactly the same way as in the crisp case.

Proposition 3.1. *The mapping* $d: X \times X \to \mathbb{R}^*(I)$, *in Definition* 3.2 *above, is indeed a fuzzy pseudo-metric, and it is a fuzzy metric iff* $\|\ \|$ *is a fuzzy norm.* □

Definition 3.3 [11]. Let τ be a fully stratified fuzzy topology on a vector space X. Then, τ will be called a *fuzzy linear topology* if vector addition is a continuous mapping $(X, \tau) \times (X, \tau) \to (X, \tau)$, and scalar multiplication is a continuous mapping $(K, \omega(\kappa)) \times (X, \tau) \to (X, \tau)$, where $\omega(\kappa)$ is the fuzzy topology topologically generated by κ (induced by κ, cf. [15, §5]).

Remark 3.1. In [15, Theorem 5.1] Lowen's term *topologically generated fuzzy topology* (induced fuzzy topology) was given further substance by showing that for every topological space (Y, λ), $\omega(\lambda)$ is the smallest fully stratified fuzzy topology on Y which includes λ; specifically, $\omega(\lambda)$ has subbase $\lambda \cup I$. (Recall that the symbol I also denotes the collection of all constant fuzzy subsets of Y.) In the same theorem, it is also shown that the natural bijection $(Y, \omega(\lambda)) \to (Y, \lambda) \times (\{\cdot\}, I)$: $y \mapsto (y, \cdot)$ is a fuzzy homeomorphism, where $(\{\cdot\}, I)$ is a singleton fts with the discrete fuzzy topology.

Hence, if (X, τ) is a fully stratified fts, and (Y, λ) is a topological space, then the product fts $(Y, \omega(\lambda)) \times (X, \tau)$ is identical with $(Y, \lambda) \times (X, \tau)$ through the natural fuzzy homeomorphisms

$$\begin{aligned}(Y, \omega(\lambda)) \times (X, \tau) &\simeq [(Y, \lambda) \times (\{\cdot\}, I)] \times (X, \tau)\\ &= (Y, \lambda) \times [(\{\cdot\}, I) \times (X, \tau)] \simeq (Y, \lambda) \times (X, \tau)\end{aligned}$$

(in the last homeomorphism we use the full stratification of τ). So, in Definition 3.3 above,

$$(K, \omega(\kappa)) \times (X, \tau) = (K, \kappa) \times (X, \tau).$$

Proposition 3.2. *In a fuzzy pseudo-normed vector space* $(X, \|\ \|)$, *the vector addition is continuous in the associated fuzzy topology.*

Proof. Let $f: X\times X\to X$ denote the vector addition. Let $y_1, y_2\in X$ and $x=y_1+y_2=f(y_1, y_2)$, and let $r>0$. Then for all $(z_1, z_2)\in X\times X$ and all $r_1, r_2>0$ with $r_1+r_2=r$,

$$\begin{aligned}\Big[\prod_{i=1,2} B(y_i;\bar{r}_i)\Big](z_1, z_2) &= \bigwedge_{i=1,2} B(y_i;\bar{r}_i)(z_i)\\ &= \bigwedge_{i=1,2} L_{r_i}(\|y_i-z_i\|) \quad \text{(from Theorem 2.1(ii))}\\ &\leqslant L_r(\|y_1-z_1\|+\|y_2-z_2\|) \quad \text{(from Theorem 1.1(x))}\\ &\leqslant L_r(\|y_1+y_2-(z_1+z_2)\|)\\ &\qquad \text{(from the triangle inequality for } \|\ \|\text{, and Theorem 1.1(vii))}\\ &= L_r(\|x-f(z_1, z_2)\|)=B(x;\bar{r})(f(z_1, z_2))\\ &= f^{-1}[B(x;\bar{r})](z_1, z_2).\end{aligned}$$

Hence, $\prod_{i=1,2} B(y_i;\bar{r}_i)\subseteq f^{-1}[B(x;\bar{r})]$. In consequence, from Theorem 2.6, the vector addition f is continuous. □

Proposition 3.3. *Let τ be the fuzzy topology associated with a fuzzy pseudo-normed vector space $(X, \|\ \|)$. Then, the scalar multiplication is a continuous function $g:(K, \omega(\kappa))\times(X, \tau)\to(X, \tau)$.*

Proof. Let $y\in X$, $t\in K$, and $x=ty=g(t, y)$, and let $1>p$, $r>0$. Since $\|y\|(+\infty-)=0$, there is $c>0$ such that $c\,\|y\|<(r/3)^{\sim}\vee(1-p)$. Now, for all $(s, z)\in K\times X$,

$$\begin{aligned}g^{-1}[B(x;\bar{r})](s, z) &= B(x;\bar{r})(sz)=L_r(\|x-sz\|)\\ &= L_r(\|ty-sz\|)=L_r(\|t(y-z)+(s-t)(y-z)-(s-t)y\|)\\ &\geqslant L_r(|t|\,\|y-z\|+|s-t|\,\|y-z\|+|s-t|\,\|y\|)\end{aligned}$$

(from same relations used in the previous proof). So, when $|s-t|<c$, we get

$$\begin{aligned}g^{-1}[B(x;\bar{r})](s, z) &\geqslant L_r([|t|+c]\,\|y-z\|+c\,\|y\|)\\ &\geqslant L_{r/2}([|t|+c]\,\|y-z\|)\wedge L_{r/2}(c\,\|y\|) \quad \text{(from Theorem 1.1(x))}\\ &\geqslant L_u(\|y-z\|)\wedge p=B(y;\bar{u})(z)\wedge p,\end{aligned}$$

where $u=r/(2|t|+2c)>0$. Since in the usual metric for (K, κ),

$$B(t;c)(s)=\begin{cases}0 & \text{when } |s-t|\geqslant c,\\ 1 & \text{when } |s-t|<c,\end{cases}$$

the above proves

$$\begin{aligned}g^{-1}[B(x;\bar{r})](s, z) &\geqslant B(t;c)(s)\wedge B(y;\bar{u})(z)\wedge p\\ &= [B(t;c)\times B(y;\bar{u})](s, z)\wedge p.\end{aligned}$$

Hence, $g^{-1}[B(x;\bar{r})]\supseteq B(t;c)\times B(y;\bar{u})\cap P$. Therefore, from Theorem 2.6, the scalar multiplication g is continuous $(K, \kappa) \times (X, \tau) \to (X, \tau)$. Hence, the assertion follows from Remark 3.1 above. □

The above two propositions are summarized in:

Theorem 3.1. *Every fuzzy pseudo-normed topology is a fuzzy linear topology.* □

The basic behaviour of the fuzzy open balls in a fuzzy pseudo-normed vector space are exposed in the next two theorems.

Theorem 3.2. *Let $(X, \| \ \|)$ be a fuzzy pseudo-normed vector space with associated fuzzy topology τ. Then for all $x, y \in X$, $r \in \mathbb{R}^+$, and $\eta \in \mathbb{R}^*(I) - \{\bar{0}\}$:*

(i) $B(x; \bar{r})(y) = 1 - \|y - x\|\,(r-) = L_r(\|y - x\|)$.

(ii) $B(\mathbb{0}; \bar{r}) = rB(\mathbb{0}; \bar{1})$.

(iii) $\|y\|\,(r-) = 1 - [rB(\mathbb{0}; \bar{1})](y)$.

(iv) $B(x + y; \eta) = x + B(y; \eta)$.

(v) *The collection $\{sB(\mathbb{0}; \bar{r}) \cap p : s > 0$ and $1 > p > 0\}$ is a local base at $\mathbb{0}$, and its translations form a base of τ. Hence, $B(\mathbb{0}; \bar{r})$ generates τ by linearity and full stratification.*

(vi) $B(rx; r\eta)(ry) = B(x; \eta)(y)$.

(vii) $B(rx; r\eta) = rB(x; \eta)$.

Proof. (i) From Theorem 2.1(ii).

(ii) for every $z \in X$,

$$B(\mathbb{0}; \bar{r})(z) = L_r(\|z\|) = L_1\left(\frac{1}{r}\|z\|\right) \quad \text{(from Theorem 1.1(xiii))}$$

$$= B(\mathbb{0}; \bar{1})\left(\frac{1}{r} z\right) = [rB(\mathbb{0}; \bar{1})](z).$$

Hence, $B(\mathbb{0}; \bar{r}) = rB(\mathbb{0}; \bar{1})$.

(iii) From (i) and (ii), $\|y\|\,(r-) = 1 - B(\mathbb{0}; \bar{r})(y) = 1 - rB(\mathbb{0}; \bar{1})(y)$.

(iv) For every $z \in X$,

$$B(x + y; \eta)(z) = \sup\{R_s(\eta) \wedge L_s[\|z - x - y\|] : s > 0\}$$
$$= B(y; \eta)(z - x) = [x + B(y; \eta)](z).$$

Hence, $B(x + y; \eta) = x + B(y; \eta)$.

(v) This follows from Theorem 2.5, Theorem 2.1(vii), and parts (ii) and (iv).

(vi) We have

$$B(rx; r\eta)(ry) = \sup\{R_s(r\eta) \wedge L_s[\|rx - ry\|] : s > 0\}$$
$$= \sup\{R_{s/r}(\eta) \wedge L_{s/r}[\|x - y\|] : s > 0\}$$
$$= \sup\{R_s(\eta) \wedge L_s[\|x - y\|] : s > 0\} = B(x; \eta)(y).$$

(vii) Follows directly from (vi). □

Definition 3.4. Let U and V be fuzzy subsets of a fuzzy pseudo-normed vector space $(X, \|\ \|)$.

(i) We say that U *1*-absorbs* V if there is $r > 0$ such that $V \subseteq rU$.

(ii) We say that U *is 1*-absorbing* if U 1*-absorbs every crisp singleton in X, equivalently if the crisp subset U_{1^*} (=the 1*-cut of U, cf. Definition 1.4) is absorbing in X. (Hence, the nomenclature.)

Definition 3.5 [10]. A fuzzy set U in a vector space X is said to be

(a) *convex* if $tU + (1-t)U \subseteq U$, for all $t \in I$;

(b) *balanced* if $kU \subseteq U$, for all $k \in K$ with $|k| \leq 1$;

(c) *absorbing* if $\bigcup_{r>0} rU = X$.

Remark 3.2. (i) Absorbency is a weaker condition than 1*-absorbency. Because the fuzzy subset U of $\mathbb{R}$ given by $U(x) = (1-|x|) \vee 0$, $x \in \mathbb{R}$, is absorbing. However, $U_{1^*} = \{0\}$, which is not absorbing. Hence, U is not 1*-absorbing.

(ii) If $U \in I^X$ is absorbing, then $U(\mathbb{0}) = 1$ [11]. Hence, if U is also convex, then $tU \subseteq U$ for all $t \in I$. Consequently, a convex absorbing $U \in I^X$ is balanced iff $kU = U$ for all $k \in K$ with $|k| = 1$; iff $kU = |k|\, U$ for all $k \in K$.

Proposition 3.4. *In a fuzzy pseudo-normed vector space $(X, \|\ \|)$, the fuzzy unit ball at $\mathbb{0}$, $B(\mathbb{0}; \tilde{1})$, is absorbing.*

Proof. Let $y \in X$ and $1 > \alpha > 0$. Since $\|y\| \in \mathbb{R}(I)$, then there is $r > 0$ such that $\|y\|\,(r-) < 1 - \alpha$. So, from Theorem 3.2(iii), $[rB(\mathbb{0}; \tilde{1})](y) > \alpha$. This proves that $B(\mathbb{0}; \tilde{1})$ is absorbing. □

Theorem 3.3. *In a fuzzy pseudo-normed vector space $(X, \|\ \|)$, $B(\mathbb{0}; \tilde{r})$ is convex, balanced, and absorbing, for every $r > 0$.*

Proof. Since $B(\mathbb{0}; \tilde{r}) = rB(\mathbb{0}; \tilde{1})$, then it suffices to consider the case $r = 1$. For all $1 > t > 0$,

$$\begin{aligned} tB(\mathbb{0}; \tilde{1}) + (1-t)B(\mathbb{0}; \tilde{1}) &= B(\mathbb{0}; \tilde{t}) + B(\mathbb{0}; (1-t)^{\sim}) \\ &\subseteq B(\mathbb{0} + \mathbb{0}; (t + (1-t))^{\sim}) \quad \text{(cf. proof of Proposition 3.2)} \\ &= \mathrm{B}(\mathbb{0}; \tilde{1}). \end{aligned}$$

Hence, $B(\mathbb{0}; \tilde{1})$ is convex. Since $B(\mathbb{0}; \tilde{1})(\mathbb{0}) = 1$, it follows that $0B(\mathbb{0}; \tilde{1}) = \{\mathbb{0}\} \subseteq B(\mathbb{0}; \tilde{1})$. So let $k \in K$ with $1 \geq |k| > 0$. Then for every $y \in X$,

$$\begin{aligned} [kB(\mathbb{0}; \tilde{1})](y) &= B(\mathbb{0}; \tilde{1})\left(\frac{1}{k} y\right) = L_1\left(\left\|\frac{1}{k} y\right\|\right) \\ &= L_1\left(\frac{1}{|k|} \|y\|\right) = L_{|k|}(\|y\|) = B(\mathbb{0}, |k|^{\sim})(y). \end{aligned}$$

Hence, $kB(\mathbb{0}; \tilde{1}) = B(\mathbb{0}; |k|^{\sim})$. Since $|k|^{\sim} \leq \tilde{1}$, from Theorem 2.2, $kB(\mathbb{0}; \tilde{1}) \subseteq B(\mathbb{0}; \tilde{1})$. This completes the proof that $B(\mathbb{0}; \tilde{1})$ is also balanced. That $B(\mathbb{0}; \tilde{1})$ is absorbing is a restatement of Proposition 3.4 above. □

Definition 3.6. Let (X, d) be a fuzzy pseudo-metric space. For $x \in X$ and $r \geq 0$, we put $C(x; \tilde{r}) = \bigcap_{s>r} B(x; \tilde{s})$.

Proposition 3.5. *Let X be a fuzzy pseudo-normed vector space, $x, y \in X$, and $r, s > 0$. Then*:

(i) $B(x; \tilde{r}) \subseteq C(x; \tilde{r})$.
(ii) $C(\mathbb{0}; \tilde{r})$ *is convex, balanced, and absorbing.*
(iii) $sC(\mathbb{0}; \tilde{r}) = C(\mathbb{0}; (sr)^{\sim})$.
(iv) $\bigcup_{0<t<r} C(x; \tilde{t}) = B(x; \tilde{r})$.
(v) *If* $0 < s < 1$, *then* $sC(\mathbb{0}; \tilde{r}) \subseteq B(\mathbb{0}; \tilde{r})$ *and* $sB(\mathbb{0}; \tilde{r}) \subseteq C(\mathbb{0}; \tilde{r})$.
(vi) $x + C(y; \tilde{r}) = C(x + y; \tilde{r})$.
(vii) $\|y\|(r+) = 1 - [rC(\mathbb{0}; \tilde{1})](y)$.

Proof. (i) Follows from Theorem 2.2.

(ii) From Theorem 3.3 and [10, Proposition 4.6], $C(\mathbb{0}; \tilde{r})$ is convex and balanced. It is also absorbing since, from (i), it includes the absorbing fuzzy subset $B(\mathbb{0}; \tilde{r})$.

(iii) From Theorem 3.2(ii),

$$sC(\mathbb{0}; \tilde{r}) = \bigcap_{t>r} sB(\mathbb{0}; \tilde{t}) = \bigcap_{t>r} B(\mathbb{0}; (st)^{\sim}) = \bigcap_{t>sr} B(\mathbb{0}; \tilde{t}) = C(\mathbb{0}; (sr)^{\sim}).$$

(iv) From Theorem 2.2 and from (i) above,

$$B(x; \tilde{r}) = \bigcup_{0<t<r} B(x; \tilde{t}) \subseteq \bigcup_{0<t<r} C(x; \tilde{t}) \subseteq B(x; \tilde{r}),$$

by the definition of $C(x; \tilde{t})$. Hence, equality holds.

(v) The first assertion follows from (iii) and (iv). The second follows from $sB(\mathbb{0}; \tilde{r}) \subseteq B(\mathbb{0}; \tilde{r})$ and (i).

(vi) Follows from Theorem 3.2(iv).

(vii) From (iii) and Theorem 3.2(iii),

$$\begin{aligned} 1 - [rC(\mathbb{0}; \tilde{1})](y) &= 1 - \bigwedge_{s>r} B(\mathbb{0}; \tilde{s})(y) \\ &= 1 - \bigwedge_{s>r} [1 - \|y\|(s-)] \\ &= \bigvee_{s>r} \|y\|(s-) = \|y\|(r+). \qquad \square \end{aligned}$$

Proposition 3.6. *In a fuzzy pseudo-normed vector space* $(X, \| \ \|)$, $B(x; \tilde{r}) = \operatorname{int}_\tau[C(x; \tilde{r})]$, *where τ is the associated fuzzy topology, for all $x \in X$ and $r > 0$.*

Proof. From Proposition 3.5(i), $B(x; \tilde{r}) \subseteq \operatorname{int}_\tau[C(x; \tilde{r})]$. On the other hand, whenever a fuzzy point q_y in X is in $\operatorname{int}_\tau[C(x; \tilde{r})]$, there are, by Theorem 2.5, positive reals b, p with $1>p>q$ and $p\cap B(y;\bar{b})\subseteq C(x;\bar{r})\subseteq B(x;\bar{s})$, for all $s>r$ (using Proposition 3.5(v)). Hence for all $t>0$,

$$\begin{aligned} B(x;\bar{r})(y) &= L_r[\|y-x\|] \\ &= L_{r(1+t)}[(1+t)\,\|y-x\|] \quad \text{(from Theorem 1.1(xiii))} \\ &= L_{r(1+t)}[\|y+t(y-x)-x\|] \\ &= B(x;(r(1+t))^{\sim})(y+t(y-x)) \\ &\geqslant [p\cap B(y;\bar{b})](y+t(y-x)) \\ &= p\wedge L_b[t\,\|y-x\|] = p\wedge L_{b/t}[\|y-x\|]. \end{aligned}$$

Since $L_{b/t}[\|y-x\|]\to 1$ for t small enough, $B(x;\bar{r})(y)\geqslant p>q$. Hence, $q_y \tilde{\in} B(x;\bar{r})$. This completes the proof that $B(x;\bar{r})=\operatorname{int}_\tau[C(x;\bar{r})]$. □

Proposition 3.7. *Let D be a fuzzy subset of a fuzzy pseudo-normed vector space $(X,\|\ \|)$, such that $B(\mathbb{0};\bar{s})\subseteq D\subseteq C(\mathbb{0};\bar{s})$ for some $s>0$. Then in the associated fuzzy topology τ,*

$$\operatorname{int}_\tau(D)=B(\mathbb{0};\bar{s})=\bigcup_{0<r<1} rD.$$

Proof. From the preceding proposition, $B(\mathbb{0};\bar{s})=\operatorname{int}_\tau(D)$. From Theorem 2.2 and Proposition 3.5,

$$B(\mathbb{0};\bar{s})=\bigcup_{0<r<1} rB(\mathbb{0};\bar{s})\subseteq\bigcup_{0<r<1} rD\subseteq\bigcup_{0<r<1} rC(\mathbb{0};\bar{s})=B(\mathbb{0};\bar{s}).$$

Hence, equality holds. □

Proposition 3.8. *Let D be a fuzzy subset of a fuzzy pseudo-normed vector space $(X,\|\ \|)$, such that $B(\mathbb{0};\bar{s})\subseteq D\subseteq C(\mathbb{0};\bar{s})$ for some $s>0$. The following two statements are equivalent*:

(i) $\|\ \|$ *is a fuzzy norm.*

(ii) $\bigcap_{r>0}[r\ \text{support}(D)]=\{\mathbb{0}\}$.

Proof. Notice that for $r>0$, $r\,\text{support}(D)=\text{support}(rD)$.

(i)$\Rightarrow$(ii). Suppose $\|\ \|$ is a fuzzy norm. Then for $y\neq\mathbb{0}$ in X, there is $r>0$ with $\|y\|\geqslant 2\bar{r}$. Hence,

$$\left(\frac{r}{s}D\right)(y)\leqslant\left[\frac{r}{s}C(\mathbb{0};\bar{s})\right](y)\leqslant\left[\frac{r}{s}B(\mathbb{0};2\bar{s})\right](y)=L_{2r}(\|y\|)\leqslant L_{2r}(2\bar{r})=0.$$

Since also $(rD)(\mathbb{0})=1$ for all $r>0$, then the above proves (ii).

(ii)$\Rightarrow$(i). Suppose (ii), and let $y\neq\mathbb{0}$ in X. Then, there is $r>0$ with $L_r(\|y\|)=B(\mathbb{0};\bar{r})(y)\leqslant((r/s)D)(y)=0$. Hence, $\|y\|\geqslant\bar{r}$. This proves that $\|\ \|$ is a fuzzy norm. □

4. The fuzzy Minkowski functionals

In this section, we show that the concept of fuzzy pseudo-norm is equivalent to Katsaras' concept of fuzzy seminorm, in the sense that both concepts result in the same class of fuzzy linear topologies. The tools linking fuzzy pseudo-norms and F-seminorms are fuzzy versions of the Minkowski functionals, which we call the fuzzy Minkowski functionals.

Definition 4.1 [12]. A *fuzzy seminorm* on a vector space X is a fuzzy set D in X which is convex, balanced, and absorbing. If in addition $\bigcap_{t>0} tD = \{\mathbb{0}\}$, then D is called a *fuzzy norm* (not to be confused with the fuzzy norm of Definition 3.1 above). A vector space X equipped with a fuzzy seminorm (resp. fuzzy norm) D is called a fuzzy seminormed (resp. fuzzy normed) space.

Theorem 4.1 [12]. *If D is a fuzzy seminorm on a vector space X, then the family $\mathbb{B} = \mathbb{B}_D = \{p \cap (tD) : t > 0$ and $0 < p < 1\}$ is a base at $\mathbb{0}$ (in the sense of [11]) for a fuzzy linear topology τ_D. (We here call τ_D the fuzzy (linear) topology associated with the fuzzy seminorm D.)* □

We abbreviate the term *fuzzy seminorm* to *F-seminorm*. We also have:

Definition 4.2. An *F-norm* D on a vector space X is an F-seminorm which satisfies $\bigcap_{t>0} \text{support}(tD) = \{\mathbb{0}\}$.

Remark 4.1. The condition $\bigcap_{t>0} \text{support}(tD) = \{\mathbb{0}\}$ is stronger than the condition $\bigcap_{t>0} (tD) = \{\mathbb{0}\}$. Because $\{\mathbb{0}\} \subseteq \bigcap_{t>0} (tD) \subseteq \bigcap_{t>0} \text{support } (tD)$, and on the other hand, letting $D \subseteq \mathbb{R}$ be the fuzzy set given by $D(x) = 1 \wedge 2e^{-|x|}$, for all $x \in \mathbb{R}$, then D is an F-seminorm on $\mathbb{R}$ and $\bigcap_{t>0} (tD) = \{\mathbb{0}\}$, but $\bigcap_{t>0} \text{support}(tD) = \mathbb{R}$.

Proposition 4.1. *Let D be a convex and absorbing fuzzy subset of a vector space X. Then for every $x \in X$, the function $P_x : \mathbb{R} \to I$ given for all $r \leqslant 0$ by $P_x(r) = 1$, and for all $r > 0$ by $P_x(r) = 1 - (rD)(x)$, is a fuzzy real number $\geqslant \tilde{0}$. In particular, $P_{\mathbb{0}} = \tilde{0}$.*

Proof. Since D is absorbing, $(rD)(x) \to 1$ as $r \to \infty$. Hence, $P_x(+\infty-) = 0$ and $P_x(0-) = 1$. Obviously, $P_{\mathbb{0}} = \tilde{0}$. Assume $x \neq \mathbb{0}$. Since D is convex, then for $r > s > 0$, $(s/r)D + ((r-s)/r)D \subseteq D$. Hence, $sD + (r-s)D \subseteq rD$. Consequently,

$$(rD)(x) \geqslant (sD)(x) \wedge [(r-s)D](\mathbb{0}) = (sD)(x) \wedge 1 = (sD)(x).$$

Hence,

$$P_x(r) = 1 - (rD)(x) \leqslant 1 - (sD)(x) = P_x(s) \leqslant P_x(0) = 1.$$

This proves that P_x is nonascending, which completes the proof that P_x is a fuzzy real number $\geqslant \tilde{0}$. □

From the above proposition, the next definition (suggested by Theorem 3.2(iii)) is well phrased.

Definition 4.3. Let D be an F-seminorm on a vector space X. The *fuzzy Minkowski functional* for D is a function $P : X \to \mathbb{R}^*(I)$ given for $x \in X$ and $r \in \mathbb{R}$

by

$$P(x)(r) = \begin{cases} 1 & \text{for } r \leqslant 0, \\ 1-(rD)(x) & \text{for } r > 0. \end{cases}$$

Definition 4.4. Two fuzzy pseudo-norms, a fuzzy pseudo-norm and an F-seminorm, or two F-seminorms, are said to be *equivalent* if their associated fuzzy topologies coincide.

Theorem 4.2. *Let X be a vector space.*

(i) *Suppose $\|\ \|$ is a fuzzy pseudo-norm on X. Then, the fuzzy unit ball at $\mathbf{0}$, $B(\mathbf{0};\bar{1})$, is an F-seminorm on X, and $\|\ \|$ is its fuzzy Minkowski functional. Also, the fuzzy pseudo-norm $\|\ \|$ is equivalent to the F-seminorm $B(\mathbf{0};\bar{1})$. Moreover, $\|\ \|$ is a fuzzy norm iff $B(\mathbf{0};\bar{1})$ is an F-norm.*

(ii) *Suppose $D \in I^X$ is an F-seminorm on X. Then, the fuzzy Minkowski functional for D, $P: X \to \mathbb{R}^*(I)$, is a fuzzy pseudo-norm on X, and in the associated fuzzy pseudo-metric, $B(\mathbf{0};\bar{1}) \subseteq D \subseteq C(\mathbf{0};\bar{1})$. Also, D is equivalent to P. Moreover, D is an F-norm iff P is a fuzzy norm.*

Proof. (i) Suppose $\|\ \|$ is a fuzzy pseudo-norm on X. From Theorem 3.3, $B(\mathbf{0};\bar{1})$ is an F-seminorm on X. From Theorem 3.2(iii), $\|\ \|$ is the fuzzy Minkowski functional for $B(\mathbf{0};\bar{1})$. From Theorem 2.5(iii), Theorem 3.2, and Theorem 4.1, $\|\ \|$ is equivalent to $B(\mathbf{0};\bar{1})$. From Proposition 3.8 $\|\ \|$ is a fuzzy norm iff $B(\mathbf{0};\bar{1})$ is an F-norm.

(ii) Suppose $D \in I^X$ is a F-seminorm on X, and let P be its fuzzy Minkoswki functional. Then for $x \in X$, $t \in K - \{0\}$, and $r > 0$,

$$P(tx)(r) = 1-(rD)(tx) = 1-\left(\frac{r}{t}D\right)(x) = 1-\left(\frac{r}{|t|}D\right)(x)$$

because D is balanced, convex, and absorbing, cf. Remark 3.2(ii). Thus

$$P(tx)(r) = P(x)\left(\frac{r}{|t|}\right) = [|t|\,P(x)](r).$$

Hence, $P(tx) = |t|\,P(x)$ in $\mathbb{R}(I)$. Also, for all $x, y \in X$ and $r, a, b > 0$ such that $r = a+b$, $rD \supseteq aD + bD$, because D is convex. Hence,

$$\begin{aligned} P(x+y)(r) &= 1-(rD)(x+y) \leqslant 1-[aD+bD](x+y) \\ &= 1-\sup\{(aD)(x') \wedge (bD)(y') : x'+y' = x+y\} \\ &\leqslant 1-[(aD)(x) \wedge (bD)(y)] \\ &= [1-(aD)(x)] \vee [1-(bD)(y)] = P(x)(a) \vee P(y)(b). \end{aligned}$$

Hence from Theorem 1.2,

$$\begin{aligned} [P(x)+P(y)](r) &= \inf\{P(x)(a) \vee P(y)(b) : a+b = r\} \\ &\geqslant P(x+y)(r). \end{aligned}$$

Consequently, $P(x+y) \leqslant P(x)+P(y)$ in $\mathbb{R}(I)$. This completes the proof that $P: X \to \mathbb{R}^*(I)$ is a fuzzy pseudo-norm. In the fuzzy pseudo-metric associated with P, we have from Theorem 3.2(iii) and Proposition 3.5(vii),

$$B(\mathbb{0}; \bar{1})(y) = 1 - P(y)(1-) \leqslant 1 - P(y)(1) = D(y)$$
$$\leqslant 1 - P(y)(1+) = C(\mathbb{0}; \bar{1})(y).$$

Hence, $B(\mathbb{0}; \bar{1}) \subseteq D \subseteq C(\mathbb{0}; \bar{1})$. Hence from Proposition 3.7, $B(\mathbb{0}; \bar{1}) = \bigcup_{0<r<1} rD$. Hence from [12, Theorem 4.3], the F-seminorm $B(\mathbb{0}; \bar{1})$ is equivalent to D. Hence from part (i) above, D is equivalent to P. Finally from Proposition 3.8, D is an F-norm iff P is a fuzzy norm. □

Remark 4.2. From the above proof it follows that, in the above theorem, the fuzzy Minkowski functionals for $B(\mathbb{0}; \bar{1})$ and for D coincide.

Corollary 4.1. *Let D be an F-seminorm on a vector space X. Then,* $\mathrm{int}_{\tau_D}(D) = \bigcup_{0<r<1} rD$, *and it is an F-seminorm equivalent to D.*

Proof. $\tau_D = \tau_P$, where P is the fuzzy pseudo-norm on X defined as the fuzzy Minkowski functional for D. Since from the above theorem, $B(\mathbb{0}; \bar{1}) \subseteq D \subseteq C(\mathbb{0}; \bar{1})$, the assertion follows from Proposition 3.7 and [12, Theorem 4.3]. □

Corollary 4.2. *Let D be an F-seminorm on a vector space X. Let $0 \leqslant \alpha < 1$, $0 < \beta \leqslant 1$, and $i, j = 0, 1, 2$. The following are equivalent statments:*

(i) (X, τ_D) *is* α-T_i.
(ii) (X, τ_D) *is* β^*-T_j.
(iii) *D is an F-norm.*

Proof. Let P be the fuzzy Minkowski functional for D. Then, $\tau_D = \tau_P$. From Theorem 2.7 and Proposition 3.1, (i)$\Leftrightarrow$(ii)$\Leftrightarrow P$ is a fuzzy norm. From Theorem 4.2, P is a fuzzy norm$\Leftrightarrow$(iii). □

Example 4.1. In a forthcoming paper, we shall show that, in a fuzzy pseudo-metric space (X, d), the collection of all fuzzy open balls constitutes a fuzzy neighbourhood base in the sense of [23], and that the associated fuzzy neighbourhood space $(X, t(d))$ is fuzzy uniform in the sense of [22]. Also, $t(d)$ coincides with the fuzzy pseudo-metric topology on (X, d) iff one of them is topologically generated (induced).

We shall use these facts to introduce a new fuzzy uniform topology on the fuzzy real line $\mathbb{R}(I)$ (uniform in the sense of [22]), starting from the following pseudo-metric d on $\mathbb{R}(I)$:

$$d(\eta, \zeta) = \inf\{\xi \in \mathbb{R}^*(I) : \xi + \eta \geqslant \zeta \text{ and } \xi + \zeta \geqslant \eta\}, \quad \eta, \zeta \in \mathbb{R}(I).$$

Obviously, d extends the usual metric on $\mathbb{R}$. But, $(\mathbb{R}(I), t(d))$ is not topologically generated. The proofs, properties and other details require some space, and so they cannot be given here.