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On a problem about strictly completely regular ordered spaces posed by Lawson

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ABSTRACT

Lawson (1991) asked whether every continuous dcpo equipped with the Lawson topology is a strictly completely regular ordered space. Xu (2001) gave a partial answer by showing that if P is a quasicontinuous dcpo and the Lawson open lower sets coincide with the open sets in the lower topology, then P with the Lawson topology is a strictly completely regular ordered space.
In this paper, we shall construct a counterexample to give a negative answer to Lawson's problem. Furthermore, we introduce the strongly completely regular ordered spaces, and then prove that Xu's condition is sufficient and necessary for a quasicontinuos poset (equipped with the Lawson topology) to be a strongly completely regular ordered space. Other relationships among pairwise normal bitopological spaces, strongly completely regular ordered spaces and strictly completely regular ordered spaces are also summarized.

1. Introduction

Kelly [4] introduced bitopological spaces in order to get a systematic approach to the study on quasi-metrics. In order to study the bitopological spaces in the context of domain theory, Jimmie Lawson [5] introduced the strictly completely regular spaces, and posed the following problem (see [5, Problem 4.1]):

- Is every continuous dcpo equipped with its Lawson topology a strictly completely regular ordered space?

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In [7,8], Xu proved that for any quasicontinuous dcpo P, if the family $\lambda(P)^{\downarrow}$ of all Lawson open lower subsets of P coincides with the lower topology $\omega(P)$ on P, then $(P, \lambda(P))$ is a strictly completely regular ordered space.

In this paper, we introduce a new class of partially ordered topological spaces, called strongly completely regular ordered spaces, which is strictly smaller than that of strictly completely regular ordered spaces. We prove that a quasicontinuous dcpo P with the Lawson topology $\lambda(P)$ is a strongly completely regular ordered space if and only if $\lambda(P)^{\downarrow} = \omega(P)$. In addition, a continuous dcpo is constructed to give a negative answer to Lawson's problem. In the last, other relationships among pairwise normal spaces, strongly completely regular ordered spaces, and strictly completely regular ordered spaces are also summarized.

2. Preliminaries

In this section, we introduce some basic notions and results needed in this paper. For more information on topology, order and domain theory, see [1–3].

In this paper, we use $\mathbb{N}$ to denote the set of all positive integers. Let P be a poset. For a subset A of P, we shall adopt the following standard notations:

$$\uparrow A = \{y \in P : \exists x \in A, x \leq y\};\ \downarrow A = \{y \in P : \exists x \in A, y \leq x\}.$$

For each $x \in X$, we write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. A subset A of P is called a *lower* (resp., an *upper*) *set* if $A = \downarrow A$ (resp., $A = \uparrow A$). An element x is an *upper bound of* A if $A \subseteq \downarrow x$. We call P *bounded complete* if each subset A of P that has an upper bound has a supremum, denoted by $\bigvee A$. Clearly, P is bounded complete if and only if each nonempty subset A of P has an infimum, denoted by $\bigwedge A$.

A nonempty subset D of P is *directed* if every two elements in D have an upper bound in D. P is called a *directed complete poset*, or a *dcpo* for short, if every directed subset of P has a supremum.

For $x, y \in P$, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P with $\bigvee D$ existing, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. Denote $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. A poset P is *continuous*, if for any $x \in P$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$.

An element $x \in P$ is called *compact*, if $x \ll x$. Denote by $K(P)$ the set of all compact elements. We say that P is *algebraic*, if for any $x \in P$, the set $K(P) \cap \downarrow x$ is directed and $x = \bigvee (K(P) \cap \downarrow x)$.

For subsets G and H of a poset P, G is way-below H, denoted by $G \ll H$, if for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in \uparrow H$ implies $D \cap \uparrow G \neq \emptyset$. In particular, if $x \in P$ and $F \ll \{x\}$, then we just write $F \ll x$. A poset P is *quasicontinuous* if for any $x \in P$, the family

$$fin(x) = \{\uparrow F \subseteq P : F \text{ is finite and } F \ll x\}$$

is a filtered family and $\uparrow x = \bigcap fin(x)$. It is well-known that every continuous poset is quasicontinuous.

A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D of P for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open subsets of P form a topology on P, called the *Scott topology* and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the *Scott space* of P.

The topology generated by the collection of sets $P \setminus \uparrow x$ (as a subbasis) is called the *lower topology* on P, denoted by $\omega(P)$. The space $\Omega P = (P, \omega(P))$ is called the *lower space* of P. For the sake of convenience, each open (resp., closed) set in the lower space will be called *ω-open* (resp., *ω-closed*).

The *Lawson topology* $\lambda(P)$ on P is the common refinement $\sigma(P) \vee \omega(P)$ of the Scott and the lower topologies. Let $\lambda(P)^{\downarrow} = \{U \in \lambda(P) : U = \downarrow U\}$ and $\lambda(P)^{\uparrow} = \{U \in \lambda(P) : U = \uparrow U\}$. We will call each $U \in \lambda(P)^{\uparrow}$ (resp., $U \in \lambda(P)^{\downarrow}$) a *Lawson open upper* (resp., *lower*) *set*. The *Lawson closed upper* (resp., *lower*) *sets* are defined dually. A poset P is called *Lawson compact* if $(P, \lambda(P))$ is a compact space.

In the following, for each set X, we use $\mathcal{P}_{fin}(X)$ to denote the family of all finite subsets of X.

Remark 2.1.

(1) For a poset P, $\sigma(P) = \lambda(P)^{\uparrow}$ (see Proof of [2, Proposition III-1.6]), and clearly $\omega(P) \subseteq \lambda(P)^{\downarrow}$. Moreover, if P is a Lawson compact quasi-continuous dcpo, then $\omega(P) = \lambda(P)^{\downarrow}$ (see [2, Propositions III-3.17, III-3.18] or [6, Theorem 3.7]).
(2) For any continuous poset P, the family $\{\Uparrow x : x \in P\}$ forms a basis for $\sigma(P)$, and $\{\Uparrow x \cap (P \setminus \uparrow F) : x \in P, F \in \mathcal{P}_{fin}(P)\}$ forms a basis for $\lambda(P)$.

Lemma 2.2 *([3, Exercise 5.2.33]). Let P be a quasicontinuous dcpo and $A \subseteq P$. If A is a compact upper set in the Scott space $(P, \sigma(P))$, then there is a filtered family $\{F_i : i \in I\}$ of finite subsets of P such that $A = \bigcap_{i \in I} \uparrow F_i$.*

For a poset $(P, \leq)$ and a topology $\mathcal{T}$ on P, the triple $(P, \leq, \mathcal{T})$ is called a *partially ordered topological space*. We shall denote $(P, \leq, \mathcal{T})$ simply by $(P, \mathcal{T})$. The order $\leq$ on P is *semiclosed* if both $\downarrow x$ and $\uparrow x$ are closed in $(P, \mathcal{T})$ for each $x \in P$. The space $(P, \mathcal{T})$ is called *strongly order convex* if it has a basis of open sets each of which consists of the intersection of an open upper set and an open lower set.

Definition 2.3 *([5]).* Let $(P, \mathcal{T})$ be a partially ordered topological space. We say that $(P, \mathcal{T})$ is a *strictly completely regular ordered space* if

(S1) the order on P is semiclosed;
(S2) $(P, \mathcal{T})$ is strongly order convex;
(S3) for each nonempty closed lower set A and $x \notin A$, there is a monotone continuous map $f : P \longrightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(x) = 1$;
(S4) for each nonempty closed upper set A and $x \notin A$, there is a monotone continuous map $g : P \longrightarrow [0, 1]$ such that $g(A) = \{1\}$ and $g(x) = 0$,

where $[0, 1]$ is equipped with the usual topology.

Remark 2.4. Let $(P, \mathcal{T})$ be a partially ordered topological space. If $\mathcal{T}$ is a discrete topology, then $(P, \mathcal{T})$ is a strictly completely regular ordered space.

3. Strongly completely regular ordered spaces

For a poset P, we say that a map $f : P \longrightarrow [0, 1]$ is *lower* (resp., *Scott*) *continuous*, if it is continuous with respect to the lower (resp., the Scott) topologies on P and the interval $[0, 1]$ with the usual order.

Definition 3.1. Let P be a poset. We call the Lawson space $(P, \lambda(P))$ a *strongly completely regular ordered space* if it satisfies (S1) and (S2) of Definition 2.3, and the following conditions:

(SR3) for each nonempty closed lower set A and $x \notin A$, there is a lower and Scott continuous map $f : P \longrightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(x) = 1$;
(SR4) for each nonempty closed upper set A and $x \notin A$, there is a lower and Scott continuous map $g : P \longrightarrow [0, 1]$ such that $g(A) = \{1\}$ and $g(x) = 0$.

Remark 3.2. Since every lower and Scott continuous map $f : P \longrightarrow [0, 1]$ is Lawson continuous, we have the following implications:

$$\text{(SR3)} \Rightarrow \text{(S3) and (SR4)} \Rightarrow \text{(S4)}.$$

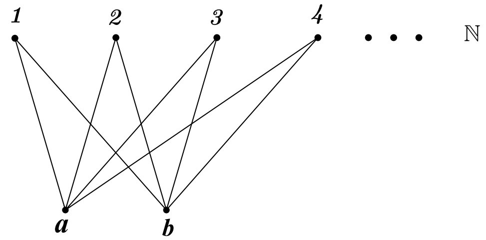

Fig. 1. Poset $\mathbb{B}$ in Remark 3.4.

Thus, if $(P, \lambda(P))$ is a strongly completely regular ordered space, then it is a strictly completely regular ordered space. However, by Example 4.1, the converse conclusion is generally not true even for algebraic dcpos.

Proposition 3.3 *([8, Theorem 3.2]). Let P be a quasicontinuous poset. Then, $(P, \lambda(P))$ satisfies (S1), (S2) and (SR3). Furthermore, if $\omega(P) = \lambda(P)^{\downarrow}$, then $(P, \lambda(P))$ satisfies (SR4), and therefore $(P, \lambda(P))$ is a strongly completely regular ordered space (hence a strictly completely regular ordered space).*

Remark 3.4.

(1) The original Theorem 3.2 in [8] actually requires P to be a quasicontinuous dcpo; nevertheless, from the proof it can be reduced to be a quasicontinuous poset, as shown in Proposition 3.3 here.

(2) Condition $\omega(P) = \lambda(P)^{\downarrow}$ is not necessary for $(P, \lambda(P))$ being a strictly completely regular ordered space. In fact, every infinite antichain P is a continuous dcpo with $\omega(P) \neq \lambda(P)^{\downarrow}$, but $(P, \lambda(P))$ is a strictly completely regular ordered space, see Example 4.1 for details.
Here is another example.
Let $\mathbb{B} = \{a, b\} \cup \mathbb{N}$ with the order $\leq$ defined by $a \leq n$ and $b \leq n$ for each $n \in \mathbb{N}$, see Fig. 1. Then, we have the following facts:
(i) $\mathbb{B}$ is a continuous dcpo.
(ii) $(\mathbb{B}, \lambda(\mathbb{B}))$ is a strictly completely regular ordered space.
In fact, $\lambda(\mathbb{B})$ is discrete, since

$$\{a\} = \mathbb{B} \setminus \uparrow b, \ \{b\} = \mathbb{B} \setminus \uparrow a, \ \text{and} \ \{n\} = \uparrow n,$$

for each $n \in \mathbb{N}$. Then, by Remark 2.4 $(\mathbb{B}, \lambda(\mathbb{B}))$ is strictly completely regular ordered.
(iii) $\omega(\mathbb{B}) \neq \lambda(\mathbb{B})^{\downarrow}$.
Since $\lambda(\mathbb{B})$ is discrete, the lower set $U = \{a, b, 1\} \in \lambda(\mathbb{B})^{\downarrow}$, but $\mathbb{B} \setminus U = \mathbb{N} \setminus \{1\}$ is clearly not ω-closed, so $U \notin \omega(\mathbb{B})$. Hence, $\omega(\mathbb{B}) \neq \lambda(\mathbb{B})^{\downarrow}$.

The following theorem shows that the converse conclusion of Proposition 3.3 is also true.

Theorem 3.5. *For a quasicontinuous poset P, the following statements are equivalent:*

(1) $\omega(P) = \lambda(P)^{\downarrow}$;
(2) $(P, \lambda(P))$ *is a strongly completely regular ordered space;*
(3) *for each nonempty closed upper set A and $x \notin A$, there is a lower continuous map $f : P \longrightarrow [0, 1]$ such that $f(A) = \{1\}$ and $f(x) = 0$.*

Proof. The implication (1) ⇒ (2) follows from Proposition 3.3, and (2) ⇒ (3) is trivial.

(3) ⇒ (1): By Proposition 3.3, it suffices to prove that every Lawson closed upper subset A of P satisfies

$$A = \bigcap\{\uparrow F : F \in \mathcal{P}_{fin}(P) \text{ and } A \subseteq \uparrow F\}.$$

We denote by A^* the right-hand set. It is clear that $A \subseteq A^*$. To prove $A^* \subseteq A$, suppose $x \notin A$. Then by condition (3), there is a lower continuous map $f : P \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(A) = \{1\}$. Since $[\frac{1}{2}, 1]$ is an ω-closed subset of $[0,1]$, it follows that $f^{-1}([\frac{1}{2}, 1])$ is ω-closed on P. By the definition of lower topology,

$$f^{-1}([\frac{1}{2}, 1]) = \bigcap\{\uparrow F : F \in \mathcal{P}_{fin}(P) \text{ and } f^{-1}([\frac{1}{2}, 1]) \subseteq \uparrow F\}.$$

Recall that $f(x) = 0$, which implies that $x \notin f^{-1}([\frac{1}{2}, 1])$. Thus, there is $F_0 \in \mathcal{P}_{fin}(P)$ such that $f^{-1}([\frac{1}{2}, 1]) \subseteq \uparrow F_0$ and $x \notin \uparrow F_0$. Note that $A \subseteq f^{-1}([\frac{1}{2}, 1]) \subseteq \uparrow F_0$, which implies that $x \notin \uparrow F_0 \supseteq A^*$, so $x \notin A^*$. This shows that $A^* \subseteq A$. Therefore, $A = A^*$, which gives (1). □

Remark 3.6. From the proof of Theorem 3.5, one easily observes that implications (2) ⇒ (3) and (3) ⇒ (1) hold for all posets (not just for quasicontinuous posets).

4. Some counterexamples

In the last section, we have seen that every strongly completely regular ordered Lawson space is strictly completely regular ordered. The following example shows that the converse conclusion is generally not valid.

Example 4.1. Let $P = \mathbb{N}$ with the discrete order (i.e., $\forall m, n \in P$, $m \le n$ iff $m = n$). Then, we have the following facts:

(1) P is an algebraic dcpo.
(2) The Lawson topology $\lambda(P)$ is discrete.
In fact, $\lambda(P)^{\downarrow} = \lambda(P)^{\uparrow} = \lambda(P) = \mathcal{P}(P)$, where $\mathcal{P}(P)$ is the power set of P. Note that each subset of P is lower and upper. Since $\{x\} = \Uparrow x$ is Scott open, hence is Lawson open, which follows that the Lawson topology $\lambda(P)$ is discrete.
(3) From (2) and Remark 2.4, it follows that $(P, \lambda(P))$ is a strictly completely regular ordered space.
(4) $(P, \lambda(P))$ is not a strongly completely regular ordered space.
Since $\lambda(P)$ is discrete, the set $P \setminus \{1\}$ is a Lawson closed upper set. However, it is not ω-closed, since $P \setminus \{1\} \nsubseteq \uparrow F = F$ for any finite subset F of P. Thus, $\lambda(P)^{\downarrow} \ne \omega(P)$. By Theorem 3.5, $(P, \lambda(P))$ is not a strongly completely regular ordered space.

Remark 4.2. The above example also demonstrates that the two topologies $\lambda(P)^{\downarrow}$ and $\omega(P)$ need not be identical even for an algebraic dcpo P. However, these two spaces do coincide for Lawson compact quasicontinuous dcpos or bounded complete dcpos (see Remark 2.1(3)).

We are now ready to answer Lawson's question:

- Is every continuous dcpo with the Lawson topology strictly completely regular ordered?

We will give a negative answer by constructing a counterexample.

Example 4.3. Let $\mathbb{N}^{\top} = \mathbb{N} \cup \{\infty\}$ with the usual order (i.e., $n \le n+1 \le \infty$, $\forall n \in \mathbb{N}$), $\mathbb{B} = \{a, b\} \cup \mathbb{N}$ endowed with the order of the example in Remark 3.4, and $\mathbb{D} = \mathbb{N} \times \mathbb{N} \times \{\infty\}$.

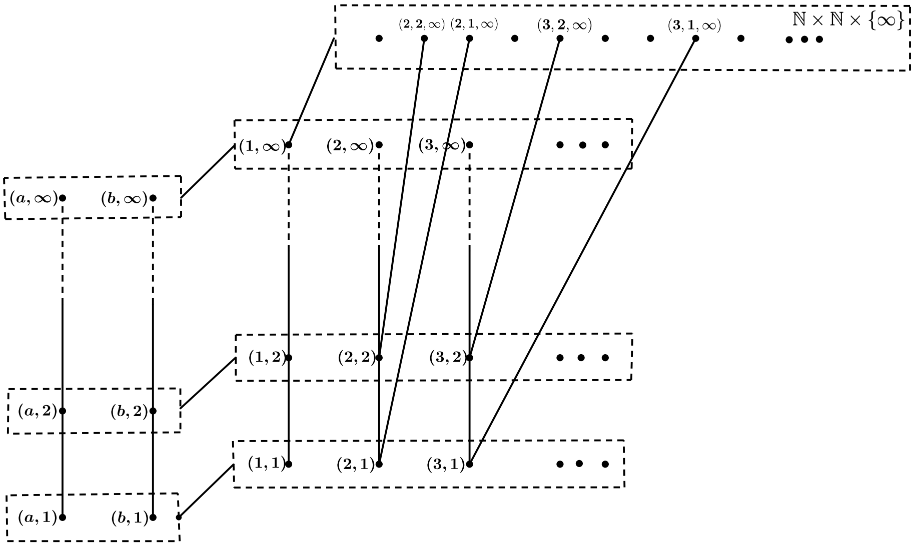

Fig. 2. Poset P in Example 4.3.

Now let $P = (\mathbb{B} \times \mathbb{N}^\top) \cup \mathbb{D}$ with an order $\leq$ defined as follows:

(i) $\leq$ is the coordinatewise order when restricting to $\mathbb{B} \times \mathbb{N}^\top$;
(ii) $\leq$ is the discrete order when restricting to $\mathbb{D}$;
(iii) $\forall (m,n,\infty) \in \mathbb{D}$, $(1,\infty) \leq (m,n,\infty)$ and $(m,n) \leq (m,n,\infty)$.

(1) P is a countable algebraic dcpo, refer to Fig. 2. In fact, we have the following facts:

(i) $\forall (x,n) \in \mathbb{B} \times \mathbb{N}$, $(x,n) \ll (x,n) \ll (x,\infty)$;
(ii) $\forall (m,n,\infty) \in \mathbb{D}$, $(m,n,\infty) \ll (m,n,\infty)$.

(2) $(P,\lambda(P))$ is not a strictly completely regular ordered space.

We confirm this in a few steps.

Step 1: $\forall m \in \mathbb{N}$, $\forall x \in P \setminus (\downarrow(1,\infty) \cup (\{m\} \times \mathbb{N}))$, $\forall n \in \mathbb{N}$,

$$x \leq (m,n,\infty) \quad \Leftrightarrow \quad x = (m,n,\infty).$$

This is trivial since $\downarrow(m,n,\infty) = \{(m,n,\infty)\} \cup \downarrow(1,\infty) \cup \{(m,k) : k \leq n\}$.

Step 2: $C \overset{def}{=} \downarrow\{(m,\infty) : m \geq 2\}$ is a Scott closed subset of P.

Suppose D is a directed subset of C. We claim that $D \subseteq \downarrow(m,\infty)$ for some integer $m \geq 2$. Assume on the contrary, $D \nsubseteq \downarrow(m,\infty)$ for any integer $m \geq 2$. Since $D \nsubseteq \downarrow(2,\infty)$, there is $(m_1,n_1) \in D$ such that $(m_1,n_1) \notin \downarrow(2,\infty)$. Since $\{a,b\} \times \mathbb{N}^\top \subseteq \downarrow(2,\infty)$ and $n_1 \in \mathbb{N}^\top$, it follows that $m_1 \notin \{a,b\}$; note that $(m_1,n_1) \in D \subseteq C = (\{a,b\} \cup (\mathbb{N} \setminus \{1\})) \times \mathbb{N}^\top$, so $m_1 \in \mathbb{N} \setminus \{1\}$. Then by our assumption, $D \nsubseteq \downarrow(m_1,\infty)$, so there exists $(m_2,n_2) \in D \setminus \downarrow(m_1,\infty)$; then $m_2 \neq m_1$ (otherwise, $(m_2,n_2) \in \downarrow(m_1,\infty)$). Note that $\{(m_1,n_1),(m_2,n_2)\} \subseteq D$ has no upper bound, which contradicts the directedness of D. Thus, there is $m_0 \in \mathbb{N} \setminus \{1\}$ such that $D \subseteq \downarrow(m_0,\infty)$, and hence $\bigvee D \in \downarrow(m_0,\infty) \subseteq C$. This shows that C is Scott closed.

Step 3: $A \overset{def}{=} \{(m,\infty) : m \geq 2\}$ is a Lawson closed upper subset of P such that $(1,\infty) \notin A$.

It is clear that A is an upper set and $(1,\infty) \notin A$. By Step 2, we have that $\downarrow\{(m,\infty) : m \geq 2\}$ is Scott closed, hence is Lawson closed. Observe that $A = \uparrow(a,\infty) \cap \uparrow(b,\infty) \cap \downarrow\{(m,\infty) : m \geq 2\}$. Hence, A is a Lawson closed upper set.

Step 4: There is no monotone Lawson continuous map separating $(1,\infty)$ and A.

Assume, on the contrary, there is a monotone Lawson continuous map $f : P \longrightarrow [0,1]$ such that $f((1,\infty)) = 0$ and $f(x) = 1$ for all $x \in A$.

(i) $\forall m \in \mathbb{N} \setminus \{1\}$, there is $\widehat{m}$ such that $f((m,n)) \geq \frac{1}{2}$ whenever $n \geq \widehat{m}$.
In fact, as $(m,\infty) \in A$, it follows that $f((m,\infty)) = 1$. Note that $\{(m,n) : n \in \mathbb{N}\}$ is a directed set whose supremum is (m,∞), and since f is monotone Lawson continuous, it is Scott continuous (see [2, Exercise III-1.16]), i.e., it preserves directed-suprema, so we have that

$$\begin{aligned} & \bigvee\{f((m,n)) : n \in \mathbb{N}\} \\ = \ & f(\bigvee\{(m,n) : n \in \mathbb{N}\}) \\ = \ & f((m,\infty)) \\ = \ & 1 > \tfrac{1}{2}. \end{aligned}$$

Then, there is $n_0 \in \mathbb{N}$ such that $f((m,n_0)) \geq \frac{1}{2}$. Let

$$\widehat{m} \stackrel{def}{=} \min\{n \in \mathbb{N} : f((m,n)) \geq \frac{1}{2}\},$$

which satisfies the requirement. Obviously, $f((m,\widehat{m})) \geq \frac{1}{2}$ for all $m \in \mathbb{N} \setminus \{1\}$.

(ii) $f^{-1}([0,\frac{1}{2}))$ is not Lawson open.
We show this by proving that $(1,\infty)$ is not an interior point of $f^{-1}([0,\frac{1}{2}))$ with respect to the Lawson topology. Otherwise, there is a point $x_0 \in P$ and a finite subset F of P such that

$$(1,\infty) \in \Uparrow x_0 \cap (P \setminus \uparrow F) \subseteq f^{-1}([0,\frac{1}{2})).$$

Since F is finite, we may choose a large enough integer $k \in \mathbb{N} \setminus \{1\}$ satisfying the following conditions:
(a) $F \cap (\{k\} \times \mathbb{N}) = \emptyset$, and
(b) there is $l \geq \widehat{k}$ such that $(k,l,\infty) \notin F$.
Recall that $\widehat{k} = \min\{n \in \mathbb{N} : f((k,n)) \geq \frac{1}{2}\}$, as defined in (i).
Claim 1: $(k,l,\infty) \in \Uparrow x_0 \cap (P \setminus \uparrow F)$.
On the one hand, since $x_0 \ll (1,\infty) \leq (k,l,\infty)$, we have that $(k,l,\infty) \in \Uparrow x_0$. On the other hand, since $(1,\infty) \in P \setminus \uparrow F$, it follows that $\downarrow(1,\infty) \cap F = \emptyset$, that is, $F \subseteq P \setminus \downarrow(1,\infty)$, and since $F \cap (\{k\} \times \mathbb{N}) = \emptyset$, we have that $F \subseteq P \setminus (\downarrow(1,\infty) \cup (\{k\} \times \mathbb{N}))$. Then, for each $y \in F \subseteq P \setminus (\downarrow(1,\infty) \cup (\{k\} \times \mathbb{N}))$, since $(k,l,\infty) \notin F$, we have that $y \neq (k,l,\infty)$. By Step 1, $y \not\leq (k,l,\infty)$, so $(k,l,\infty) \in P \setminus \uparrow y$. By the arbitrariness of y, it follows that $(k,l,\infty) \in \bigcap_{y \in F} P \setminus \uparrow y = P \setminus \uparrow F$.
Claim 2: $(k,l,\infty) \notin f^{-1}([0,\frac{1}{2}))$.
By the definition of $\widehat{k}$, we have that $f((k,\widehat{k})) \geq \frac{1}{2}$. Recall that $l \geq \widehat{k}$, so $(k,l,\infty) \geq (k,l) \geq (k,\widehat{k})$. Since f is monotone, it follows that $f((k,l,\infty)) \geq f((k,\widehat{k})) \geq \frac{1}{2}$. Hence, $(k,l,\infty) \notin f^{-1}([0,\frac{1}{2}))$, as required.
Note that Claims 1 and 2 contradict the assumption $\Uparrow x_0 \cap (P \setminus \uparrow F) \subseteq f^{-1}([0,\frac{1}{2}))$. Thus, $f^{-1}([0,\frac{1}{2}))$ is not Lawson open. In addition, since $[0,\frac{1}{2})$ is a Lawson open set in $[0,1]$, it follows that f is not Lawson continuous.

All these show that $(P, \lambda(P))$ is not strictly completely regular ordered.

Definition 4.4 *([4]).* A space X equipped with two topologies $\mathcal{T}$ and $\mathcal{L}$ is called a *bitopological space*, and denoted by $(X, \mathcal{T}, \mathcal{L})$.

A subset of X is called $\mathcal{T}$*-closed* if it is closed in $(X, \mathcal{T})$. The notions of $\mathcal{T}$-open, $\mathcal{L}$-closed, and $\mathcal{L}$-open sets are defined analogously.

Definition 4.5 *([4,5]).* A bitopological space $(X, \mathcal{T}, \mathcal{L})$ is called *pairwise normal* if for each $\mathcal{T}$-closed set A and each $\mathcal{L}$-closed set B with $A \cap B = \emptyset$, there exist an $\mathcal{L}$-open set U and a $\mathcal{T}$-open set V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Theorem 4.6 *([4]). If $(X, \mathcal{T}, \mathcal{L})$ is a pairwise normal bitopological space, then for each $\mathcal{T}$-closed set A and each $\mathcal{L}$-closed set B with $A \cap B = \emptyset$, there exists a map $g : X \longrightarrow [0, 1]$ such that*

(i) *$g(A) = \{0\}$ and $g(B) = \{1\}$;*
(ii) *both $g : (X, \mathcal{T}) \longrightarrow \Sigma[0, 1]$ and $g : (X, \mathcal{L}) \longrightarrow \Omega[0, 1]$ are continuous.*

From Proposition 3.3 and Theorem 4.6, the following result is immediate.

Corollary 4.7. *For each quasicontinuous poset P, if $(P, \lambda(P)^{\uparrow}, \lambda(P)^{\downarrow})$ is a pairwise normal space, then $(P, \lambda(P))$ is a strongly completely regular ordered space.*

Through the above arguments, for each quasicontinuous poset P, we obtain the following relations:

$$\begin{tikzcd}
\text{strongly CRO} \arrow[r, Leftrightarrow] \arrow[d, shift left, "?"] & \lambda(P)^{\downarrow} = \omega(P) \arrow[d, shift left] \\
\text{pairwise normal} \arrow[u, shift left, "\times"] \arrow[r, shift left] & \text{strictly CRO,} \arrow[u, shift left, "\times"] \arrow[l, shift left, "\times"]
\end{tikzcd}$$

where "CRO" denotes "completely regular ordered", and "pairwise normal" means the bitopological space $(P, \lambda(P)^{\uparrow}, \lambda(P)^{\downarrow})$ is so.

We close the paper by posing the following open problem.

Problem 4.8. Is every continuous dcpo P equipped with the Lawson topology satisfying $\lambda(P)^{\downarrow} = \omega(P)$ pairwise normal?

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