International Journal of Approximate Reasoning 121 (2020) 125–134

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

L-partial metrics and their topologies

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ARTICLE INFO

Article history:
Received 17 September 2019
Received in revised form 16 March 2020
Accepted 17 March 2020
Available online 20 March 2020

Keywords:
Pointwise pseudo-quasi-metric
Weighted L-metric
L-partial metric
Remote-neighborhood mapping
L-topology

ABSTRACT

The aim of this paper is to study the theory of weighted (pseudo-quasi-)metrics based on L-fuzzy sets. We first prove that the category of weighted pointwise quasi-metric spaces is isomorphic to that of L-partial metric spaces. Then we introduce the remote-neighborhood mapping and use it to characterize the L-partial pseudo-quasi-metric. Finally, we investigate properties of the L-topology induced by the L-partial pseudo-quasi-metrics.

1. Introduction

Metrics were originally introduced by Fréchet [4] at the beginning of set-theoretic topology. A metric on a set is a function that defines a concept of distance between any two members of the set. As a part of the study of denotational semantics of dataflow networks, partial metrics [11] are also used to study the non-Hausdorff topology such as those required in the Tarskian approach to programming language semantics. To some extent, partial metrics can be considered as a generalization of metrics. The main difference with metrics is that the distance in partial metric spaces from a point to itself may not be zero. In 1994, it was proved that partial metrics are exactly the weighted quasi-metrics [11]. For years yet, the theory of partial metrics has received wide attention in both mathematics [6,9,12,13] and theoretical computer science [8,15,16].

In 1965, Zadeh introduced the theory of fuzzy set to the study of problems where a partial membership in a set was appropriate. Since then, some works have been proceeded to extend his ideas to other fields of mathematics including analysis, algebra, topology, convergence structure, convex structure, etc. The extension of topology to fuzzy setting (i.e., the well-known fuzzy topology) originates in the work of C.L. Chang [1] in the case that the truth value is considered in the unit interval. The notion of Chang's fuzzy topology was generalized to L-fuzzy setting by J.A. Goguen, which is now called L-topology.

There have been many spectacular and creative works about the theory of fuzzy metrics. The constructions are formed by those results in which a fuzzy metric on a set X is treated as a mapping $d : Y \times Y \longrightarrow [0, +\infty)$, where Y is the set of all fuzzy points of X. In this case, a fuzzy metric induces an L-topology [3,17,18]. Since fuzzy topologies and fuzzy convergence structures are closely linked, one important topic is to discuss the relationship between fuzzy metrics and fuzzy convergence

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https://doi.org/10.1016/j.ijar.2020.03.006
0888-613X/

structures [14,23]. In this paper, we shall present a theory of L-partial metrics based on Shi's pointwise metrics [17,18]. The advantages of using the pointwise metrics on L-fuzzy sets contain three aspects: it can nicely reflect the characteristics of pointwise fuzzy topology; its research methods are simpler and more direct; as essential examples, both the L-fuzzy unit interval and the L-fuzzy real line are pointwise pseudo-metrizable [17,18].

The theory of fuzzy partial metrics or probabilistic partial metrics has been considered by many other authors [7,21,24,25]. Recently, Xiu and Pang studied the theory of L-partial metrics [22]. This paper is a continuation of their work on L-partial metrics. The main results we proved include:

(1) The notion of weighted pointwise quasi-metrics is introduced, and it turns out that L-partial metrics are categorically isomorphic to weighted pointwise quasi-metrics;

(2) We introduce the notion of remote-neighborhood-mappings, and use it to characterize L-partial metrics;

(3) Two methods for generating L-topologies via L-partial pseudo-quasi-metrics are presented, and a sufficient condition is provided to ensure that the two types of L-topologies coincide.

2. Preliminaries

Let L be a complete lattice. A mapping $' : L \longrightarrow L$ is called an order-reversing involution provided that (i) $(a')' = a$ for all $a \in L$, and (ii) $a \leqslant b$ implies $b' \leqslant a'$ for all $a, b \in L$. Throughout this paper, L denotes a completely distributive lattice with an order-reversing involution $'$, i.e., L denotes a completely distributive De Morgan algebra. The smallest element and the largest element in L are denoted by $\perp$ and $\top$ respectively. For $a, b \in L$, we say that a is *well-below* b in L, denoted by $a \prec b$, if for any subset $A \subseteq L$, $b \leqslant \bigvee A$ implies $a \leqslant d$ for some $d \in A$ [2]. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L \mid a \prec b\}$ for each $b \in L$. The well-below relation in a completely distributive lattice has the interpolation property, that is, $a \prec b$ implies that $a \prec c \prec b$ for some $c \in L$. Moreover, one can easily obtain that $a \prec \bigvee_{i\in\Omega} b_i$ if and only if $a \prec b_i$ for some $i \in \Omega$. An element a in L is called *co-prime* if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$ [5].

For a nonempty set X, L^X denotes the set of all L-fuzzy sets on X. It is easy to verify that the family L^X is also a completely distributive lattice under the pointwise order. The set of all non-zero co-prime elements in L^X is denoted by $J\left(L^X\right)$. Each member in $J\left(L^X\right)$ is also called a *fuzzy point*. The smallest element and the largest element in L^X are denoted by $\perp_{L^X}$ and $\top_{L^X}$, respectively.

An *L-cotopological space* is a pair (X, η), where η is a subset of L^X which contains $\perp_{L^X}, \top_{L^X}$ and is closed under arbitrary infima and finite suprema. Each member of η is called a *closed L-set*. For any $A \in L^X$, the *closure* of A in an L-cotopological space (X, η) is the smallest closed L-sets containing A, denoted by A^-.

Definition 2.1 *([20]).* Let (X, η) be an L-cotopological space. A closed L-set F is called a *remote-neighborhood*, or simply *R-nbd* of $e \in J\left(L^X\right)$ if $e \nleqslant F$. The set of all R-nbds of $e \in J\left(L^X\right)$ is denoted by $\eta^-(e)$. A subfamily $\zeta \subseteq \eta^-(e)$ is called an *R-nbd base* of e if for all $F \in \eta^-(e)$, there exists $B \in \zeta$ such that $B \geqslant F$.

Definition 2.2 *([20]).* An L-cotopological space (X, η) is said to be C_I if every fuzzy point has a countable R-nbd base.

Theorem 2.3 *([20]).* Let (X, η) be an L-cotopological space. Then $e \leqslant A^-$ if and only if $A \nleqslant B$ for any $B \in \eta^-(e)$.

Definition 2.4 *([10,19]).* An *L-interior operator* on X is a mapping $\mathrm{Int} : L^X \longrightarrow L^X$ such that for any $A, B \in L^X$,

(LI1) $\mathrm{Int}(\top_{L^X}) = \top_{L^X}$;
(LI2) $\mathrm{Int}(A) \leqslant A$;
(LI3) $\mathrm{Int}(A \wedge B) = \mathrm{Int}(A) \wedge \mathrm{Int}(B)$;
(LI4) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

Definition 2.5 *([17,18]).* A *pointwise pseudo-quasi-metric*, or simply *pointwise pq-metric*, on X is a mapping $d : J\left(L^X\right) \times J\left(L^X\right) \longrightarrow [0, +\infty)$ satisfying that for any $a, b, c \in J\left(L^X\right)$,

(LD1) $d(a, a) = 0$;
(LD2) $d(a, c) \leqslant d(a, b) + d(b, c)$;
(LD3) $d(a, b) = \bigwedge\limits_{c \prec b} d(a, c)$;
(LD4) $a \leqslant b \Rightarrow d(a, c) \leqslant d(b, c)$.

A pointwise pq-metric d is called a pointwise *quasi-metric* if it satisfies

(LD5) $d(a, b) = 0$ if and only if $a \leqslant b$.

A pointwise quasi-metric d is said to be a *pointwise metric* if it satisfies

(LD6) $\forall u, v \in J\left(L^X\right), \bigwedge_{a \not\leqslant u'} d(a, v) = \bigwedge_{b \not\leqslant v'} d(b, u)$.

The pair (X, p) is called a *pointwise* (resp., *pq-metric*, *quasi-metric*) *metric space* if p is a pointwise (resp., pq-metric, quasi-metric) metric on X.

Definition 2.6 *([22]).* An *L-partial pseudo-quasi-metric*, or simply *L-partial pq-metric*, on X is a mapping $p : J\left(L^X\right) \times J\left(L^X\right) \longrightarrow [0, +\infty)$ satisfying that for any $a, b, c \in J\left(L^X\right)$,

(LPD1) $p(a, a) \leqslant p(a, b)$;
(LPD2) $p(a, b) \leqslant p(a, c) + p(c, b) - p(c, c)$;
(LPD3) $p(a, b) = \bigwedge_{c \prec b} p(a, c)$;
(LPD4) $a \leqslant b \Rightarrow p(a, c) - p(a, a) \leqslant p(b, c) - p(b, b)$.

The pair (X, p) is called an *L-partial pq-metric space* if p is an L-partial pq-metric on X.

Theorem 2.7 *([22]). Let p be an L-partial pq-metric on X and define $d^p : J\left(L^X\right) \times J\left(L^X\right) \longrightarrow [0, +\infty)$ as follows:*

$$\forall a, b \in J\left(L^X\right),\ d^p(a, b) = p(a, b) - p(a, a).$$

Then d^p is a pointwise pq-metric on X.

Definition 2.8 *([11]).* A *weighted quasi-metric* on X is a pair $(d, |\cdot|)$ consisting of a quasi-metric $d : X^2 \longrightarrow [0, +\infty)$ and a weight function $|\cdot| : X \longrightarrow [0, +\infty)$ satisfying

$$\text{(WQ)}\quad \forall x, y \in X,\ d(x, y) + |x| = d(y, x) + |y|.$$

A quasi-metric d is said to be *weightable* if there exists a function $|\cdot| : X \longrightarrow [0, +\infty)$ such that $(d, |\cdot|)$ is a weighted quasi-metric.

3. Weighted pointwise quasi-metrics

In this section, we shall introduce the notion of weighted pontwise quasi-metrics. As a generalization of Shi's [17] pointwise quasi-metrics, the category of weighted L-quasi-metrics is categorically isomorphic to that of L-partial metric.

To simplify some proofs on pointwise pq-metrics, the following lemma is necessary.

Lemma 3.1. *If a mapping $d : J\left(L^X\right) \times J\left(L^X\right) \longrightarrow [0, +\infty)$ satisfies* (LD3)*, then* (LD1) *is equivalent to*

(LD1*) $d(a, b) = 0$ *whenever* $a \leqslant b$.

Proof. It is clear that (LD1*) implies (LD1). For the converse, let $a, b \in J\left(L^X\right)$ with $a \leqslant b$. By (LD1) and (LD3), we have

$$d(a, a) = \bigwedge_{c \prec a} d(a, c) = 0.$$

Hence for each $\varepsilon > 0$, there exists $c \prec a$ such that $d(a, c) < \varepsilon$. Since $c \prec b$, again using (LD3), we have

$$d(a, b) = \bigwedge_{e \prec b} d(a, e) \leqslant d(a, c) < \varepsilon,$$

which implies $d(a, b) = 0$. $\square$

Lemma 3.2. *In an L-partial pq-metric space (X, p),* (LPD1) *implies*

(LPD1*) $p(a, a) = p(a, b)$ *whenever* $a \leqslant b$.

Proof. Suppose that $p(a, a) < p(a, b)$ whenever $a \leqslant b$. By (LPD3), we have

$$p(a, a) = \bigwedge_{c \prec a} p(c, c) < p(a, b).$$

Hence there exists $c \prec a$ such that $p(a, c) < p(a, b)$. As $c \prec b$, by (LPD3), we have

$$p(c,b)=\bigwedge_{e\prec b} p(c,e)\leqslant p(c,c).$$

Thus, by (LPD2), we have $p(a,b)\leqslant p(a,c)+p(c,b)-p(c,c)\leqslant p(a,c)$, a contradiction. This shows that $p(a,a)=p(a,b)$. □

Lemma 3.3. *In an L-partial pq-metric space (X,p),* (LPD1*) *implies* (LPD4).

Proof. Suppose $a\leqslant b$. By (LPD2), we obtain $p(a,c)\leqslant p(a,b)+p(b,c)-p(b,b)=p(a,a)+p(b,c)-p(b,b)$, so (LPD4) holds. □

By Lemmas 3.2 and 3.3, the definition of L-partial pq-metrics can be simplified as follows.

Proposition 3.4. *A mapping $d: J\left(L^X\right)\times J\left(L^X\right)\longrightarrow[0,+\infty)$ is an L-partial pq-metric on X if and only if it satisfies* (LPD1)–(LPD3) *of Definition 2.6.*

Definition 3.5. An L-partial pq-metric p on X is called an *L-partial quasi-metric* if it satisfies

(LPD5) $p(a,a)=p(a,b)$ if and only if $a\leqslant b$.

An L-partial quasi-metric p is said to be an *L-partial metric* if it satisfies

(LPD6) $\forall u,v\in J\left(L^X\right),\ \bigwedge_{a\nleqslant u'} p(a,v)=\bigwedge_{b\nleqslant v'} p(b,u)$.

The pair (X,p) is called an *L-partial (quasi-)metric space* if p is an L-partial (*quasi*-)metric on X.

Remark 3.6. It is not difficult to obtain that an L-partial metric on X is a partial metric whenever $L=\{\bot,\top\}$.

A mapping $f:(X,p_X)\longrightarrow(Y,p_Y)$ between two L-partial metric spaces is called a *contraction* if $p_Y(f(a),f(b))\leqslant p_X(a,b)$ for all $a,b\in J\left(L^X\right)$. The category with objects being L-partial metric spaces and morphisms being contractions is denoted by **PMet**.

Definition 3.7. A *weighted pointwise quasi-metric* on X is a pair $(d,|\cdot|)$ consisting of a pointwise quasi-metric d on X and a weight function $|\cdot|: J\left(L^X\right)\longrightarrow[0,+\infty)$ satisfying

$$\text{(LWQ)}\quad \forall u,v\in J\left(L^X\right),\ \bigwedge_{a\nleqslant u'}(d(a,v)+|a|)=\bigwedge_{b\nleqslant v'}(d(b,u)+|b|).$$

In this case, the triple $(X,d,|\cdot|)$ is called a *weighted pointwise quasi-metric spaces*.

A pointwise quasi-metric d is said to be *weightable* if there exists a function $|\cdot|: J\left(L^X\right)\longrightarrow[0,+\infty)$ such that $(d,|\cdot|)$ is a weighted pointwise quasi-metric.

A mapping $f:(X,d_X,|\cdot|_X)\longrightarrow(Y,d_Y,|\cdot|_Y)$ between two weighted pointwise quasi-metric spaces is called a *partial contraction* if $d_Y(f(a),f(b))+|f(a)|_Y\leqslant d_X(a,b)+|a|_X$ for all $a,b\in J\left(L^X\right)$. The category with objects being weighted pointwise quasi-metric spaces and morphisms being partial contractions is denoted by **WQMet**.

Remark 3.8. (1) A weighted pointwise quasi-metric is a weighted quasi-metric [11] whenever $L=\{\bot,\top\}$.

(2) Every pointwise metric [17] is weightable by letting $|a|=0$ for all $a\in J\left(L^X\right)$.

The following theorem shows that every weighted pointwise quasi-metric can induce an L-partial metric.

Theorem 3.9. *Let $(X,d,|\cdot|)$ be a weighted pointwise quasi-metric space. Define a mapping $p^{(d,|\cdot|)}: J\left(L^X\right)\times J\left(L^X\right)\longrightarrow[0,+\infty)$ as follows:*

$$\forall a,b\in J\left(L^X\right),\ p^{(d,|\cdot|)}(a,b)=d(a,b)+|a|.$$

Then $p^{(d,|\cdot|)}$ is an L-partial metric on X.

Proof. It suffices to verify (LPD1)–(LPD3), (LPD5)–(LPD6).

(LPD1) For all $a, b \in J\left(L^X\right)$, since $|a| \leqslant d(a,b) + |a|$, by (LD1), we have $p^{(d,|\cdot|)}(a,a) \leqslant p^{(d,|\cdot|)}(a,b)$.

(LPD2) For any $a, b, c \in J\left(L^X\right)$, we have $p^{(d,|\cdot|)}(a,a) = |a|$ by (LD1) and $d(a,b) \leqslant d(a,c) + d(c,b)$ by (LD2). Then we have

$$\begin{aligned} p^{(d,|\cdot|)}(a,b) &= d(a,b) + |a| \\ &\leqslant (d(a,c) + |a|) + (d(c,b) + |c|) - |c| \quad \text{(by (LD1))} \\ &= p^{(d,|\cdot|)}(a,c) + p^{(d,|\cdot|)}(c,b) - p^{(d,|\cdot|)}(c,c). \end{aligned}$$

(LPD3) For any $a, b, c \in J\left(L^X\right)$, we have

$$\begin{aligned} p^{(d,|\cdot|)}(a,b) &= d(a,b) + |a| \\ &= \bigwedge_{c \prec b} d(a,c) + |a| \ \ \text{(by (LD2))} \\ &= \bigwedge_{c \prec b} p^{(d,|\cdot|)}(a,c). \end{aligned}$$

(LPD5) It is straightforward.

(LPD6) For any $u, v \in J\left(L^X\right)$, we have

$$\begin{aligned} \bigwedge_{a \nleqslant u'} p^{(d,|\cdot|)}(a,v) &= \bigwedge_{a \nleqslant u'} (d(a,v) + |a|) \\ &= \bigwedge_{b \nleqslant v'} (d(b,u) + |b|) \quad \text{(by (LWQ))} \\ &= \bigwedge_{b \nleqslant v'} p^{(d,|\cdot|)}(b,u). \end{aligned}$$

The proof is completed. □

Theorem 3.10. *If (X, p) is an L-partial metric space, then the pair $(d^p, |\cdot|)$ defined by*

$$\forall a, b \in J\left(L^X\right),\ d^p(a,b) = p(a,b) - p(a,a) \text{ and } |a|^p = p(a,a)$$

is a weighted pointwise quasi-metric on X.

Proof. It has been proved that d^p is a pointwise pq-metric in [22, Theorem 3.5]. It remains to verify (LPD5) and (LWQ).

(LPD5). It is straightforward by (LD5) and Lemma 3.1.

(LWQ). For any $a \in J\left(L^X\right)$, we have

$$\begin{aligned} \bigwedge_{a \nleqslant u'} d^p(a,v) + |a|^p &= \bigwedge_{a \nleqslant u'} (p(a,v) - p(a,a)) + p(a,a) \\ &= \bigwedge_{a \nleqslant u'} p(a,v) \\ &= \bigwedge_{b \nleqslant v'} p(b,u) \\ &= \bigwedge_{b \nleqslant v'} (p(b,u) - p(b,b)) + p(b,b) \\ &= \bigwedge_{b \nleqslant v'} d^p(b,u) + |b|^p. \end{aligned}$$

The proof is completed. □

It is easy to check that $\mathbb{F}\colon \mathbf{PMet} \longrightarrow \mathbf{WQMet}$ defined by

$$\mathbb{F}(X,p) = (X, d^p, |\cdot|^p),\ \ \mathbb{F}(f) = f$$

is a functor. It turns out that $\mathbb{F}$ is an isomorphism in what follows.

Theorem 3.11. *For an L-partial metric p on X, we have $p^{(d^p,|\cdot|^p)} = p$.*

Proof. For any $a, b \in J\left(L^X\right)$, we have

$$p^{(d^p,|\cdot|^p)}(a,b) = d^p(a,b) + |a|^p = p(a,b) - p(a,a) + p(a,a) = p(a,b).$$

Hence $p^{(d^p,|\cdot|^p)} = p$. $\square$

Theorem 3.12. *For a weighted pointwise quasi-metric* $(d, |\cdot|)$ *on* X, *we have* $\left(d^{p^{(d,|\cdot|)}}, |\cdot|^{p^{(d,|\cdot|)}}\right) = (d, |\cdot|)$.

Proof. For any $a, b \in J\left(L^X\right)$, we have

$$d^{p^{(d,|\cdot|)}}(a,b) = p^{(d,|\cdot|)}(a,b) - p^{(d,|\cdot|)}(a,a) = d(a,b) + |a| - d(a,a) - |a| = d(a,b)$$

and

$$|a|^{p^{(d,|\cdot|)}}(a,b) = p^{(d,|\cdot|)}(a,a) = d(a,a) + |a| = |a|.$$

Hence $\left(d^{p^{(d,|\cdot|)}}, |\cdot|^{p^{(d,|\cdot|)}}\right) = (d, |\cdot|)$. $\square$

Corollary 3.13. *The category* **PMet** *is isomorphic to* **WQMet**.

Next, we end this section with an example, by which the relationship between L-partial metrics and weighted pointwise quasi-metrics is shown intuitively.

Example 3.14. Let $\mathbb{R}^+ = [0, +\infty)$ and let $\mathbb{I} = [0,1]$. For any $r \in \mathbb{R}^+$ and $\lambda \in \mathbb{I}$, define $r_\lambda : \mathbb{R}^+ \longrightarrow \mathbb{I}$ as follows

$$\forall s \in \mathbb{R}^+,\ r_\lambda(s) = \begin{cases} \lambda, & s = r; \\ 0, & s \neq r. \end{cases}$$

Note that $J(\mathbb{I}^{\mathbb{R}^+}) = \{r_\lambda \mid r \in \mathbb{R}^+, \lambda \in (0,1]\}$.

(1) Define a mapping $p^{\mathbb{R}^+} : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow [0, +\infty)$ as follows:

$$p^{\mathbb{R}^+}(r_\lambda, s_\mu) = \max\{\lambda - \mu, 0\} + |r - s| + 1.$$

We prove that $p^{\mathbb{R}^+}$ is an $\mathbb{I}$-patrial metric on $\mathbb{I}$. The verification that $p^{\mathbb{R}^+}$ is an $\mathbb{I}$-partial pq-metric on $\mathbb{R}^+$ is straightforward. We only verify that $p^{\mathbb{R}^+}$ satisfies (LPD5) and (LPD6).

(LPD5) For any $r_\lambda, s_\mu \in J\left(\mathbb{I}^{\mathbb{R}^+}\right)$, we have

$$\begin{aligned} & p(r_\lambda, r_\lambda) = p(r_\lambda, s_\mu) \\ \Leftrightarrow\ & 1 = \max\{\lambda - \mu, 0\} + |r - s| + 1 \\ \Leftrightarrow\ & 0 = \max\{\lambda - \mu, 0\} + |r - s| \\ \Leftrightarrow\ & r = s, \lambda \leqslant \mu \\ \Leftrightarrow\ & r_\lambda \leqslant s_\mu. \end{aligned}$$

(LPD6) Suppose $r_\lambda \nleqslant (s_\mu)'$. This equals that $r = s$ and $\lambda > 1 - \mu$. So it suffices to show

$$\begin{aligned} & \textstyle\bigwedge_{u > 1-\lambda} p(r_u, s_\mu) = \bigwedge_{v > 1-\mu} p(s_v, r_\lambda) \\ \Leftrightarrow\ & \textstyle\bigwedge_{u > 1-\lambda} \max\{u - \mu, 0\} + |r - s| + 1 = \bigwedge_{v > 1-\mu} \max\{v - \lambda, 0\} + |r - s| + 1 \\ \Leftrightarrow\ & \textstyle\bigwedge_{u > 1-\lambda} \max\{u - \mu, 0\} = \bigwedge_{v > 1-\mu} \max\{v - \lambda, 0\}, \end{aligned}$$

which is trivial since both of them equal $\max\{1 - \lambda - \mu, 0\}$.

Therefore $p^{\mathbb{R}^+}$ is an $\mathbb{I}$-partial metric on $\mathbb{R}^+$.

(2) Define a mapping $d^{\mathbb{R}^+} : J\left(\mathbb{I}^{\mathbb{R}^+}\right) \times J\left(\mathbb{I}^{\mathbb{R}^+}\right) \longrightarrow [0, +\infty)$ as follows:

$$\forall r, s \in \mathbb{R}^+, \lambda, \mu \in \mathbb{I},\ d^{\mathbb{R}^+}(r_\lambda, s_\mu) = \max\{\lambda - \mu, 0\} + |r - s|,$$

and define a weight function $|\cdot|^{\mathbb{R}^+} : J\left(L^X\right) \longrightarrow \mathcal{R}$ by $|r_\lambda| = 1$ for all $r \in \mathbb{R}^+$ and $\lambda \in \mathbb{I}$. One can easily observe that

$$d^{\mathbb{R}^+}(r_\lambda, s_\mu) = p^{\mathbb{R}^+}(r_\lambda, s_\mu) - p^{\mathbb{R}^+}(r_\lambda, r_\lambda) \text{ and } |r_\lambda| = p^{\mathbb{R}^+}(r_\lambda, r_\lambda)$$

for all $r \in \mathbb{R}^+$ and $\lambda \in \mathbb{I}$. Thus, by Theorem 3.10, the pair $\left(\mathbb{R}^+, d^{\mathbb{R}^+}, |\cdot|^{\mathbb{R}^+}\right)$ is a weighted pointwise quasi-metric on $\mathbb{R}^+$. Further, by Theorems 3.11 and 3.12, we have $p^{\left(d^{\mathbb{R}^+}, |\cdot|^{\mathbb{R}^+}\right)} = p^{\mathbb{R}^+}$, $d^{p^{\mathbb{R}^+}} = p^{\mathbb{R}^+}$.

4. A characterization of L-partial pq-metrics

In the previous section, we have proved that the relationship between L-partial pq-metrics and weighted pointwise quasi-metrics is categorically isomorphic. As an application, we will show that every L-partial pq-metric can be characterized in terms of a family of mappings (called R-nbd mapping) with a weight function.

Definition 4.1. Let p be an L-partial pq-metric on X. For any $r \in (0, +\infty)$, a *remote-neighborhood mapping*, or simply *R-nbd mapping*, is a mapping $R^p_{r+|\cdot|} : J\left(L^X\right) \longrightarrow L^X$ defined by

$$\forall a \in J\left(L^X\right), R^p_{r+|a|}(a) = \bigvee \left\{b \in J\left(L^X\right) \mid p(a,b) \geqslant r + p(a,a)\right\}.$$

The family of R-nbd mappings induced by p is denoted by $\mathcal{R}^p$.

The notion of R-nbd mappings is not new, which is originally defined by Shi (2001) [17, Definition 3.4] to study L-pq-metrics.

Remark 4.2. For an L-partial pq-metric space (X, p), the R-nbd mapping $R^p_{r+|a|}(a)$ is equivalent to Shi's ([17, Definition 3.4]) R-nbd mapping

$$P_r(a) := \bigvee \{b \in J\left(L^X\right) \mid d^p(a,b) \geqslant r\}$$

for the pointwise pq-metric d^p, where $d^p(a,b) = p(a,b) - p(a,a)$ by Theorem 3.10.

Let f, g be two mappings from $J\left(L^X\right)$ to L^X. Define an operation $\odot$ (see [17]) as follows:

$$\forall a \in J\left(L^X\right), (f \odot g)(a) = \bigwedge \{f(b) \mid b \nleqslant g(a)\}.$$

Thus by Theorem 3.5 in [17] and Remark 4.2, we have the following result.

Theorem 4.3. *If p is an L-partial pq-metric on X, then the following statements are valid.*

(LR1) $\forall a \in J\left(L^X\right)$, $\bigwedge_{r>0} R^p_{r+|a|}(a) = \perp_{L^X}$.
(LR2) $\forall a \in J\left(L^X\right)$, $\forall r \in (0, +\infty)$, $a \nleqslant R^p_{r+|a|}(a)$.
(LR3) $\forall a \in J\left(L^X\right)$, $\forall r \in (0, +\infty)$, $R^p_{r+|a|}(a) = \bigwedge_{s<r} R^p_{s+|a|}(a)$.
(LR4) $\forall r, s > 0$, $R^p_{s+|\cdot|} \odot R^p_{r+|\cdot|} \geqslant R^p_{r+s+|\cdot|}$.

Remark 4.4. Suppose $\mathcal{R} = \left\{R_{r+|\cdot|} \mid r \in (0, +\infty)\right\}$ is a family of mappings from $J\left(L^X\right)$ to L^X with a weight function $|\cdot| : J\left(L^X\right) \longrightarrow [0, +\infty)$ satisfying (LR2)–(LR4). Then for any $r \in (0, +\infty)$, $R_{r+|\cdot|}$ is order-preserving. In detail, if $a \leqslant b$, then by (LR2), for any $\varepsilon > 0$ with $\varepsilon < r$, we have that $b \nleqslant R_{\varepsilon+|b|}(b)$, and by (LR4) $R_{r+|a|}(a) \leqslant R_{r-\varepsilon+|b|}(b)$. From (LR3), it follows that $R_{r+|a|}(a) \leqslant \bigwedge_{0<\varepsilon<r} R_{r-\varepsilon+|b|}(b) = R_{r+|b|}(b)$.

The following result is straightforward by Theorem 3.6 in [17] and Remark 4.2.

Theorem 4.5. *Let $\mathcal{R} = \{R_{r+|\cdot|} \mid r \in (0, +\infty)\}$ be a family of mappings from $J\left(L^X\right)$ to L^X with a weight function $|\cdot| : J\left(L^X\right) \longrightarrow [0, +\infty)$ satisfying* (LR1)–(LR4)*. Define a mapping $p^{\mathcal{R}} : J\left(L^X\right) \times J\left(L^X\right) \longrightarrow [0, +\infty)$ as follows:*

$$p^{\mathcal{R}}(a,b) = |a| + \bigwedge \{r \mid b \nleqslant R_{r+|a|}(a)\}.$$

Then the following statements are valid.
(1) *$p^{\mathcal{R}}$ is an L-partial pq-metric on X.*
(2) *The family of R-nbd mappings of $p^{\mathcal{R}}$ is exactly $\mathcal{R}$ with the weight function $|a| = p^{\mathcal{R}}(a,a)$ for all $a \in J\left(L^X\right)$.*

The following corollary is an immediate consequence of Theorem 4.5.

Corollary 4.6. *There is a bijective relationship between L-partial metrics and a family of mappings from $J\left(L^X\right)$ to L^X with a weight function satisfying* (LR1)–(LR4)*.*

5. L-partial pq-metrics topologies

This section concerns the connections between L-partial pq-metrics and topologies. First, we recall some denotations related to L-partial pq-metrics:

$$\forall a \in J\left(L^X\right),\ \mathcal{R}^p = \left\{R^p_{r+|a|}(a) \middle| a \in J\left(L^X\right), r \in (0,+\infty)\right\},$$

and for each $a \in J\left(L^X\right)$,

$$\mathcal{R}^p(a) = \left\{R^p_{r+|a|}(a) \middle| r \in (0,+\infty)\right\}.$$

Corresponding to Theorem 4.1 in [17], we have the following result.

Theorem 5.1. *If (X, p) is an L-partial pq-metric space, then $\mathcal{R}^p$ is a (closed) base for an L-cotopology on X. This L-cotopology is then called* the L-cotopology induced by p, *denoted by η^p.*

As an immediate consequence of Theorem 5.1, we have the following result.

Proposition 5.2. *The L-cotopology η^p induced by an L-partial pq-mertic p is C_I.*

For convenience, let Cl^p denote the L-closure operator of (X, η^p). Corresponding to Theorem 4.2 in [17], we can obtain the following lemma.

Lemma 5.3. *Let (X, p) be an L-partial pq-metric space. Then for any $a \in J\left(L^X\right)$ and $A \in L^X$, the following statements are equivalent.*

(1) $a \leqslant \mathrm{Cl}^p(A)$.
(2) *For any $r \in (0,+\infty)$, $A \nleqslant R^p_{r+|a|}(a)$.*
(3) $\bigwedge_{c \leqslant A} p(a,c) = p(a,a)$.

The following two results are straightforward by Theorem 5.1 and Lemma 5.3.

Corollary 5.4. *In an L-partial pq-metric space (X, p), for any $a \in J\left(L^X\right)$, $\mathcal{R}^p(a)$ is a local R-nbd base of a with respect to η^p.*

Corollary 5.5. *Let (X, p) be an L-partial pq-metric space. Then for any $A \in L^X$, the L-closure of A in (X, η^p) is as follows:*

$$\begin{aligned}\mathrm{Cl}^p(A) &= \bigvee \left\{ a \in J\left(L^X\right) \middle| \bigwedge_{c \leqslant A} p(a,c) = p(a,a) \right\} \\ &= \bigvee \left\{ a \in J\left(L^X\right) \middle| \forall r \in (0,+\infty), A \nleqslant R^p_{r+|a|}(a) \right\}.\end{aligned}$$

Theorem 5.6. *Let (X, p) be an L-partial pq-metric space. For any $A \in L^X$, we define a mapping $\mathrm{Int}^p : L^X \longrightarrow L^X$ by*

$$\mathrm{Int}^p(A) = \bigvee_{r>0} \bigwedge_{e \nleqslant A} R^p_{r+|e|}(e).$$

Then Int^p is an L-interior operator on X. The L-topology induced by Int^p is denoted by τ^p.

Proof. It suffices to show that Int^p satisfies (LI1)–(LI4).

(LI1) Trivial.

(LI2) Take any $a \in J\left(L^X\right)$ with $a \prec \mathrm{Int}^p(A)$. Then there exists $r \in (0,+\infty)$ such that $a \leqslant R^p_{r+|e|}(e)$ for all $e \nleqslant A$. It follows from Lemma 5.3 that $p(e,a) - p(e,e) \geqslant r$ for all $e \nleqslant A$. If $a \nleqslant A$, then $p(a,a) - p(a,a) = 0 \geqslant r$, a contradiction. Thus $\mathrm{Int}^p(A) \leqslant A$ by the arbitrariness of $a \in J\left(L^X\right)$.

(LI3) First, one can easily check that Int^p is order-preserving, and thus $\mathrm{Int}^p(A \wedge B) \leqslant \mathrm{Int}^p(A) \wedge \mathrm{Int}^p(B)$. It remains to show that $\mathrm{Int}^p(A) \wedge \mathrm{Int}^p(B) \leqslant \mathrm{Int}^p(A \wedge B)$. Suppose $c \prec \mathrm{Int}^p(A) \wedge \mathrm{Int}^p(B)$. Then there exists $r_1, r_2 \in (0,+\infty)$ such that

$$\forall a \nleqslant A, c \leqslant R^p_{r_1+|a|}(a), \text{ and } \forall b \nleqslant B, c \leqslant R^p_{r_2+|b|}(b).$$

Now let $r_0 = \min\{r_1, r_2\}$. Then for any $e \nleqslant A \wedge B$, we have $e \nleqslant A$ or $e \nleqslant B$. It follows that

$$c \leqslant R^p_{r_1+|e|}(e) \leqslant R^p_{r_0+|e|}(e),\ \text{or}\ c \leqslant R^p_{r_2+|e|}(e) \leqslant R^p_{r_0+|e|}(e),$$

implying that $c \leqslant R^p_{r_0+|e|}(e)$. Thus $c \leqslant \bigwedge_{e \not\leqslant A\wedge B} R^p_{r_0+|e|}(e) \leqslant \mathrm{Int}^p(A \wedge B)$. Therefore $\mathrm{Int}^p(A) \wedge \mathrm{Int}^p(B) \leqslant \mathrm{Int}^p(A \wedge B)$.

(LI4) To prove $\mathrm{Int}^p(\mathrm{Int}^p(A)) = \mathrm{Int}^p(A)$, it only needs to verify $\mathrm{Int}^p(\mathrm{Int}^p(A)) \geqslant \mathrm{Int}^p(A)$. Let $a \in J\left(L^X\right)$ with $a \not\leqslant \mathrm{Int}^p(\mathrm{Int}^p(A))$. Then there exists $b \prec a$ such that $b \not\leqslant \mathrm{Int}^p(\mathrm{Int}^p(A))$. Further, for any $r \in (0, +\infty)$,

$$\begin{aligned}
&\exists e \not\leqslant \mathrm{Int}^p(A),\ b \not\leqslant R^p_{\frac{r}{2}+|e|}(e)\\
\Rightarrow\ &\exists c \not\leqslant A,\ e \not\leqslant R^p_{\frac{r}{2}+|c|}(c) \text{ and } b \not\leqslant R^p_{\frac{r}{2}+|e|}(e)\\
\Rightarrow\ &\exists c \not\leqslant A,\ b \not\leqslant \left(R^p_{\frac{r}{2}+|\cdot|} \odot R^p_{\frac{r}{2}+|\cdot|}\right)(c) \geqslant R^p_{r+|c|}(c)\\
\Rightarrow\ &\exists c \not\leqslant A,\ b \not\leqslant R^p_{r+|c|}(c)\\
\Rightarrow\ &b \not\leqslant \textstyle\bigwedge_{c \not\leqslant A} R^p_{r+|c|}(c),
\end{aligned}$$

which implies that $a \not\leqslant \mathrm{Int}^p(A)$. Therefore $\mathrm{Int}^p(A) \leqslant \mathrm{Int}^p(\mathrm{Int}^p(A))$. □

In general, for an L-partial pq-metric p, $(\eta^p)' \neq \tau^p$. The following theorem will show that $(\eta^p)' = \tau^p$ if p satisfies the following condition:

(LPD6*) $\bigwedge_{a \not\leqslant u'}(p(a,v) - p(a,a)) = \bigwedge_{b \not\leqslant v'}(p(b,u) - p(b,b))$.

Theorem 5.7. *If (X, p) is an L-partial pq-metric space satisfying* (LPD6*)*, then $(\eta^p)' = \tau^p$.*

Proof. For convenience, let Int^p be the interior operator of (X, τ^p) and let Cl^p be the closure operator of (X, η^p). We need to show that $U \in \tau^p$ if and only if $U' \in \eta^p$. It suffices to show $\mathrm{Int}^p(A) = A$ if and only if $\mathrm{Cl}^p(A') = A'$, which is equivalent to show that $\mathrm{Int}^p(A)' = \mathrm{Cl}^p(A')$ for all $A \in L^X$.

Step 1. $\mathrm{Cl}^p(A') \leqslant \mathrm{Int}^p(A)'$.

Let $a \in J\left(L^X\right)$ with $a \not\leqslant \mathrm{Int}^p(A)' = \bigwedge_{r>0}\bigvee_{e \not\leqslant A} R^p_{r+|e|}(e)'$. Then there exists $r \in (0, +\infty)$ such that $a \not\leqslant \bigvee_{e \not\leqslant A} R^p_{r+|e|}(e)'$. To prove $a \not\leqslant \mathrm{Cl}^p(A')$, by Lemma 5.3, it only needs to show $A' \leqslant R^p_{r+|a|}(a)$.

Now suppose $A' \not\leqslant R^p_{r+|a|}(a)$. Then there exists $b \leqslant A'$ such that $b \not\leqslant R^p_{r+|a|}(a)$, i.e., $p(a,b) - p(a,a) < r$. For any $d \not\leqslant a'$ (i.e., $a \not\leqslant d'$), we have

$$\begin{aligned}
&\textstyle\bigwedge_{e \not\leqslant b'} p(e,d) - p(e,e) \leqslant p(a,b) - p(a,a) < r \ \ \text{(by (LPD6*))}\\
\Rightarrow\ &\exists e_r \not\leqslant b',\ p(e_r,d) - p(e_r,e_r) < r\\
\Rightarrow\ &\exists e_r \not\leqslant b',\ d \not\leqslant R^p_{r+|e_r|}(e_r)\\
\Rightarrow\ &d \not\leqslant \textstyle\bigwedge_{e \not\leqslant b'} R^p_{r+|e|}(e).
\end{aligned}$$

Thus

$$a' \geqslant \bigwedge_{e \not\leqslant b'} R^p_{r+|e|}(e) \geqslant \bigwedge_{e \not\leqslant A} R^p_{r+|e|}(e),$$

meaning that $a \leqslant \bigvee_{e \not\leqslant b'} R^p_{r+|e|}(e)' \leqslant \bigvee_{e \not\leqslant A} R^p_{|e|+r}(e)'$. This contradicts $a \not\leqslant \bigvee_{e \not\leqslant A} R^p_{|e|+r}(e)'$. Therefore $A' \leqslant R^p_{r+|a|}(a)$.

Step 2. $\mathrm{Int}^p(A)' \leqslant \mathrm{Cl}^p(A')$.

Let $a \in J\left(L^X\right)$ with $a \prec \mathrm{Int}^p(A)' = \bigwedge_{r>0}\bigvee_{e \not\leqslant A} R^p_{r+|e|}(e)'$. Then there exists $l \in J\left(L^X\right)$ such that $a \prec l$ and $l \prec \mathrm{Int}^p(A)'$. Assume $r \in (0, +\infty)$. Then there exists $e \not\leqslant A$ such that $l \leqslant R^p_{|e|+r}(e)'$. Since $A' \not\leqslant e'$, there exists $b \leqslant A'$ such that $b \not\leqslant e'$. For any $k \not\leqslant l'$, since $R^p_{|e|+r}(e) \leqslant l'$, we have

$$\begin{aligned}
&k \not\leqslant R^p_{|e|+r}(e)\\
\Rightarrow\ &p(e,k) - p(e,e) < r\\
\Rightarrow\ &\textstyle\bigwedge_{m \not\leqslant k'} p(m,b) - p(m,m) \leqslant p(e,k) - p(e,e) < r \text{ (by (LPD6*))}\\
\Rightarrow\ &\exists m_k \not\leqslant k',\ p(m_k,b) - p(m_k,m_k) < r\\
\Rightarrow\ &\exists m_k \not\leqslant k',\ b \not\leqslant R^p_{r+|m_k|}(m_k)\\
\Rightarrow\ &\exists m_k \not\leqslant k',\ A' \not\leqslant R^p_{r+|m_k|}(m_k)\ \ (\text{by } b \leqslant A')
\end{aligned}$$

Further, as $k \not\leqslant m_k'$ for all $k \not\leqslant l'$, it follows that

$$\bigwedge\{m_k \mid k \in J\left(L^X\right), k \not\leqslant l'\} \leqslant l',$$

that is, $l \leqslant \bigvee\{m'_k \mid k \in J\left(L^X\right), k \nleqslant l'\}$. So from $a \prec l$ we know that there exists $m_{k_0} \in J\left(L^X\right)$ such that $k_0 \nleqslant l'$ and $k_0 \leqslant m_{k_0}$. Since $R^p_{r+|\cdot|}$ is order-preserving (see Remark 4.4), it follows that $A' \nleqslant R^p_{r+|m_{k_0}|}(m_{k_0}) \geqslant R^p_{r+|a|}(a)$, and hence $A' \nleqslant R^p_{r+|a|}(a)$. By Lemma 5.3 $a \leqslant \mathrm{Cl}^p(A')$. Therefore $\mathrm{Int}^p(A)' \leqslant \mathrm{Cl}^p(A')$. $\square$

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

Acknowledgements

The authors would like to express their sincere thanks to the anonymous reviewers and the area editor for their careful reading and constructive comments. The project is supported by the National Natural Science Foundation of China (11871097) and the China Scholarship Council (File No. 201806030073).

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