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Question 1:

(a)

$$\begin{bmatrix} 27 & 44 & 51 \\ 35 & 39 & 62 \\ 33 & 50 & 47 \end{bmatrix} + \begin{bmatrix} 25 & 42 & 48 \\ 33 & 40 & 66 \\ 35 & 48 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 27+25 & 44+42 & 51+48 \\ 35+33 & 39+40 & 62+66 \\ 33+35 & 50+48 & 47+50 \end{bmatrix}$$

$$\begin{bmatrix} 52 & 86 & 99 \\ 68 & 79 & 128 \\ 68 & 98 & 97 \end{bmatrix}$$

(b) Solve AB and BA, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1*1+2*1 & 1*0+2*0 \\ 2*1+1*1 & 2*0+1*0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1*1+0*2 & 1*2+0*1 \\ 1*1+0*2 & 1*2+0*1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

In general, matrix multiplication is noncommutative as $BA \neq AB$.

(c) Compute $(A+B)^T$, for A and B below:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+3 & 2+1 \\ 3+-1 & 0+1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$(A + B)^T - B^T = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A^T + B^T - B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^T + B^T - B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Additive inverse on $M(\mathbb{R})_{2x2}$ yields:

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

and,

$$B^T = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B^T + A^T = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Additive commutativity in general linear systems (2x2) results in:

$$A^T + B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (A + B)^T$$

Thus: $A^T + B^T = (A + B)^T$ which was to be shown.

Question 2:

(a) Compute the following limits:

(a) $\lim_{x \to -2}(x^2 + 5x)$

$\lim_{x \to -2^+}(x^2 + 5x) = ((2^+)^2 + 5(2^+)) = (4 - 10)$

$$\lim_{x \to -2^+}(x^2 + 5x) = -6$$

$\lim_{x \to -2^-}(x^2 + 5x) = ((2^-)^2 + 5(-2^-)) = (4 - 10)$

$$\lim_{x \to -2^-}(x^2 + 5x) = -6$$

$$\lim_{x\to -2^-}(x^2+5x) = -6 = \lim_{x\to -2^+}(x^2+5x)$$

$$\lim_{x\to -2}(x^2+5x) = -6$$

(b) $\lim_{x\to 4} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15}$

$\lim_{x\to 4^+} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15} = \frac{2(2^+))^{\frac{3}{2}}-(4^+)^{\frac{1}{2}}}{(4^+)^2-15} = \frac{2(2^+)(2^+)(2^+)-(2^+)}{(16^+)-15} = \frac{16-2}{1}$

$$\lim_{x\to 4^+} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15} = 14$$

$\lim_{x\to 4^-} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15} = \frac{2(4^-))^{\frac{3}{2}}-(4^-)^{\frac{1}{2}}}{(4^-)^2-15} = \frac{2(2^-)(2^-)(2^-)-(2^-)}{(16^-)-15} = \frac{16-2}{1}$

$$\lim_{x\to 4^-} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15} = 14$$

$$\lim_{x\to 4^+} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15} = 14 = \lim_{x\to 4^-} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15}$$

$$\lim_{x\to 4} \frac{2x^{\frac{3}{2}}-x^{\frac{1}{2}}}{x^2-15} = 14$$

A limit of $\pm 15^{1/2} \in \mathbb{R}^1$ would be a different story.

(c) $\lim_{x\to a}(Ax^n)$

We need to be careful, as the following does not hold for $\forall x, a, n \in \mathbb{R}$, for example if $a = 0$ and $n < 0$ then we are dividing by zero which is undefined as the left sided and right sided limits diverge from each other.

Case 1:
Suppose n is a non-negative integer.
$\lim_{x\to a+}(Ax^n) = A(a^+)^n = Aa^n$
$\lim_{x\to a-}(Ax^n) = A(a^-)^n = Aa^n$

If n is a non-negative integer, then:

$$\lim_{x\to a^-}(Ax^n) = Aa^n = \lim_{x\to a+}(Ax^n)$$

$$\lim_{x\to a}(Ax^n) = Ax^n$$

Case 2:

Suppose $n \in \mathbb{Z}^-$ and x never crosses the 0 region on $x \to a^-$ and $a \neq 0$.
$\lim_{x \to a+}(Ax^n) = A(a^+)^n = Aa^n$
$\lim_{x \to a-}(Ax^n) = A(a^-)^n = Aa^n$

If $n \in \mathbb{Z}^-$ and x never crosses the 0 region on $x \to a^-$ and $a \neq 0$, then:

$$\lim_{x \to a^-}(Ax^n) = Aa^n = \lim_{x \to a+}(Ax^n)$$
$$\lim_{x \to a}(Ax^n) = Ax^n$$

Case 3:
Suppose $n \in \{m|m$ is even $\mathbb{R}^+ - \mathbb{Z}^+\}$ ie. 2.2, 51.2.
$\lim_{x \to a+}(Ax^n) = A(a^+)^n = Aa^n$
$\lim_{x \to a-}(Ax^n) = A(a^-)^n = Aa^n$

If $n \in \{m|m$ is even $\mathbb{R}^+ - \mathbb{Z}^+\}$, then:

$$\lim_{x \to a^-}(Ax^n) = Aa^n = \lim_{x \to a+}(Ax^n)$$
$$\lim_{x \to a}(Ax^n) = Ax^n$$

Case 4:
Suppose $n \in \{m|m$ is even $\mathbb{R}^- - \mathbb{Z}^-\}$ and x never crosses the 0 region on $x \to a^-$ and $a \neq 0$.
$\lim_{x \to a+}(Ax^n) = A(a^+)^n = Aa^n$
$\lim_{x \to a-}(Ax^n) = A(a^-)^n = Aa^n$

If $n \in \{m|m$ is even $\mathbb{R}^- - \mathbb{Z}^-\}$ and x never crosses the 0 region on $x \to a^-$ and $a \neq 0$, then:
$$\lim_{x \to a^-}(Ax^n) = Aa^n = \lim_{x \to a+}(Ax^n)$$
$$\lim_{x \to a}(Ax^n) = Ax^n$$

Case 5:
Suppose $n \in \{m|m$ is odd $\mathbb{R} - \mathbb{Z}\}$ ie. 1.3, -22.1 and $a > 0$.
$\lim_{x \to a+}(Ax^n) = A(a^+)^n = Aa^n$
$\lim_{x \to a-}(Ax^n) = A(a^-)^n = Aa^n$

If $n \in \{m|m$ is odd $\mathbb{R} - \mathbb{Z}\}$ and $a > 0$, then:

$$\lim_{x \to a^-}(Ax^n) = Aa^n = \lim_{x \to a+}(Ax^n)$$

$$\lim_{x\to a}(Ax^n) = Ax^n$$

Otherwise, the two sided limit does not exist in $\mathbb{R}$.

(b) Find an expression for dz in terms of dx and dy in the following:

(a) $z = Ax^a + By^b$

$$z = f(x,y) = Ax^a + By^b$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$dz = aAx^{a-1}dx + bBy^{b-1}dy$$

(b) $z = e^{xu}, where\ u = u(x,y).$

$$z = f(x,y) = e^{xu}, where\ u = u(x,y)$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}dx + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}dy$$

$$dz = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right)dx + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y}dy$$

$$dz = \left(ue^{xu} + xe^{xu}\frac{\partial u}{\partial x}\right)dx + xe^{xu}\frac{\partial u}{\partial y}dy$$

$$dz = e^{xu}\left(\left(u + x\frac{\partial u}{\partial x}\right)dx + x\frac{\partial u}{\partial y}dy\right)$$

$u = u(x,y)$ is unknown so its derivative with respect to x and y are unknown.

(c) $z = ln(x^2 + y)$

$$z = f(x,y) = ln(x^2 + y)$$

let $u = ln(w)$ and $w = x^2 + y$.

$$dz = \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}dx + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y}dy$$

$$dz = \frac{\partial u}{\partial w}\left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy\right)$$

$$dz = \frac{1}{w}(2xdx + dy)$$

$$dz = \frac{2xdx + 1dy}{x^2 + y}$$

Question 3:

Find A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$det(A) = (1*1*1) + (2*-1*1) + (3*0*2) - (2*0*1) - (1*-1*2) - (3*1*1)$$

$$det(A) = 1 + -2 + 0 - 0 - -2 - 3 = -2$$

Matrix is invertible.

$$A^{-1} = \frac{1}{det(A)} cof(A)^T = \frac{1}{det(A)} adj(A)$$

$$minors(A) = \begin{bmatrix} 3 & 1 & -1 \\ -4 & -2 & 0 \\ -5 & -1 & 1 \end{bmatrix}$$

$$cof(A) = \begin{bmatrix} 3 & -1 & -1 \\ 4 & -2 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

$$cof(A)^T = adjoint(A) = \begin{bmatrix} 3 & 4 & -5 \\ -1 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 3 & 4 & -5 \\ -1 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{-3}{2} & -2 & \frac{5}{2} \\ \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{1}{2} & 0 & \frac{-1}{2} \end{bmatrix}$$

Question 4: Consider the National -Income model with 3 endogenous variables, Y (national income), C (consumption), and t (taxes).

$$Q_d = a - bp \quad (a, b > 0)$$

$$Q_s = -c + dp \quad (c, d > 0)$$

Endogenous variables: $\{P, Q_s, Q_d\}$ which are functions of the exogenous variables :$\{a, b, c, d\}$.

(a) Derive P^* and Q^* in equilibrium (when quantity supplied = to quantity demanded)

We want to find the intersection of the supply and demand curves. Since b and d are both positive, we know the demand and supply curves are opposite sloping. An intersection of the supply and demand curves must exist in $\mathbb{R}^2$.

$$D = \{(P, Q)|Q = a - bP\}$$
$$S = \{(P, Q)|Q = -c + dP\}$$
$$D \cap S = (P^*, Q^*)$$
$$Q_d = a - bP = -c + dP = Q_s \quad (c, d > 0), (a, b > 0)$$
$$a + c = bP + dP$$
$$a + c = P(b + d)$$
$$\frac{a + c}{b + d} = P^*$$
$$Q^* = a - b\left(\frac{a + c}{b + d}\right)$$
$$Q^* = \frac{a(b + d) - b(a + c)}{b + d}$$
$$Q^* = \frac{ab + ad - ba - bc}{b + d}$$
$$Q^* = \frac{ab - ab + ad - bc}{b + d}$$
$$Q^* = \frac{ad - bc}{b + d} \quad (a, b, c, d > 0), \quad (b + d > 0) \rightarrow (ad > bc)$$
$$P^* = \frac{a + c}{b + d} \quad (a, b, c, d > 0)$$

(b) Examine the comparative-static properties of the equilibrium quantity and provide the economic meaning of it? (Note compute partial derivatives of P^* with respect to parameters in the model. We discuss this in details in class during lecture)

This comparative-static model reflects the equilibrium point (Q^*, P^*) with respect to a single commodity. The exogenous variables a represents the quantity demanded if $P = 0$ and c reflect the quantity supplied if $P = 0$. Furthermore, b represents the decrease in quantity depanded per unit price, while d represents the additional quantity supplied per unit of price.

$$(a, b, c, d > 0), (ad > bc)$$
$$\frac{\partial Q^*}{\partial a} = \frac{d}{b + d}$$

$$\frac{\partial Q^*}{\partial c} = -\frac{b}{b+d}$$

$$\frac{\partial Q^*}{\partial d} = \frac{a(b+d)-(ad-bc)}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial d} = \frac{ab+ad-ad+bc}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial d} = \frac{ab+bc}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial d} = \frac{b(a-c)}{(b+d)^2}$$

lastly,

$$\frac{\partial Q^*}{\partial b} = \frac{-c(b+d)-(ad-bc)}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial b} = \frac{-cb+-cd-ad+bc}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial b} = \frac{-cd-ad}{(b+d)^2}$$

finally,

$$\frac{\partial Q^*}{\partial a} = \frac{d}{b+d}$$

$$\frac{\partial Q^*}{\partial b} = -\frac{d(c+a)}{(b+d)^2}$$

$$\frac{\partial Q^*}{\partial c} = -\frac{b}{b+d}$$

$$\frac{\partial Q^*}{\partial d} = \frac{b(a+c)}{(b+d)^2}$$

$$\frac{\partial P^*}{\partial a} = \frac{1}{b+d} = \frac{\partial P^*}{\partial c}$$

$$\frac{\partial P^*}{\partial b} = -\frac{a+c}{(b+d)^2} = \frac{\partial P^*}{\partial d}$$

$$Q_d = a - bP \quad (a, b > 0)$$

$$Q_s = -c + dP \quad (c, d > 0)$$

a represents the quantity demanded at $P = 0$.

If a increases, both the equilibrium price and quantity increase.

If a decreases, both the equilibrium price and quantity decrease.

c represents the quantity supplied at $P = 0$.

If c increases, the equilibrium price increases and the equilibrium quantity decrease.

If c decreases, the equilibrium price decreases and the equilibrium quantity increases.

b represents the per unit negation of the number of units demanded given a price P (I hope that makes sense). Negative slope.

If b increases, the demand slope becomes steeper, the equilibrium price decreases and the equilibrium quantity decrease.

If b decreases, the demand slope becomes more shallow, the equilibrium price increase and the equilibrium quantity increase.

d represents the per unit of additional units supplied given a price P. Positive slope.

If d increases, the supply slope becomes steeper, the equilibrium price decreases and the equilibrium quantity increases.

If d decreases, the supply slope becomes more shallow, the equilibrium price increases and the equilibrium quantity decreases.

For all cases, if all else is constant.

Hopefully I did not cross my wires.