Level spacing distribution.

For given Hamiltonian H, we calculate the eigenenergy ν_i. the level ~~spa~~ gap between two consecutive state is

$$d_i = \nu_{i+1} - \nu_i .$$

the distribution $P(s)$ of the normalized level spacing s_i (after unfolding d_i) can be obtained. In the integrable limit $P_p(s) = e^{-s}$ is a Poisson distribution. In the chaotic limit the Wigner-Dyson statistics $P_w(s) = \frac{\pi}{2} s e^{-\frac{\pi s^2}{4}}$.

In general case, the level spacing could exhibit both statistics. and can be fitted with the Brody distribution [RMP, 53, 385(198

$$P_B(s) = b_\beta (\beta+1) s^\beta e^{-b_\beta s^{\beta+1}}.$$

where $b_\beta = [\Gamma(\frac{\beta+2}{\beta+1})]^{\beta+1}$ is a factor depending on β, and $\Gamma(x)$ is the gamma function. β measures the repulsion of levels, ~~de~~called the level repulsion exponent, and varys in $0 \le \beta \le 1$.

① when $\beta = 0$, $P_B(s) = P_p(s)$

② when $\beta = 1$. $P_B(s) = P_D(s)$.

Hence the larger the β is, the stronger the spectrum is

Mean ratio of the level spacing ②

[PRB, 75, 155111, 2007] & [PRL 110, 084101, 2013]

With the level spacing d_i, the mean ratio of level spacing is

$$\langle r\rangle = \frac{1}{N}\sum_{j=1}^{N} r_i, \qquad r_i = \min\left\{\frac{1}{\delta_i}, \delta_i\right\}$$

where N is the total number of r_i, and $\delta_i = \frac{d_{i+1}}{d_i}$.

the mean ratio of the level spacing $\langle r\rangle$ is an indicator of spectrum statistics. For ~~regu~~ regular, Poisson statistics,

$$\langle r\rangle_P \approx 0.386.$$

For chaotic system, $\langle r\rangle_W \approx 0.527$.

Multifractality of a state

[PRB 62, 7920, 2000] & [PRE 100, 032117, 2019]

denote $\{|M\rangle\}$ the orthonormal basis. ~~a~~ state is expanded with these state.

$$|\phi\rangle = \sum_{M=1}^{N} C_M |M\rangle.$$

with $C_M = \langle M|\phi\rangle$. and $\sum |C_M|^2 = 1$. ~~the~~

The mutifractality D_q is defined with

$$D_q = \frac{S_q}{\ln N},$$

with Rényi entropy $S_q = \frac{1}{1-q} \ln\left(\sum_{M=1}^{N} |C_M|^{2q}\right)$.

[if density matrix ρ. the $S_q = \frac{1}{1-q} \ln \mathrm{Tr}\, \rho^q$.]

For finite N, $D_q \in [0,1]$ and decreases with increasing q.

for localized states, $D_q^{\infty} = \lim_{N\to\infty} D_q = 0$, $(q>0, q\neq 1)$.

ergodic state $D_q^{\infty} = 1$.

* As we consider coherent state, ~~the~~ $q=1$ gives the Shannon entropy $S_1 = -\sum_m |C_m|^2 \ln |C_m|^2$. the dimension D_1 is called the information dimension, determining the scaling of Shannon entropy.

* $q=2$, $S_2 = -\ln \sum_m |C_m|^4 = \ln(\sum_m |C_m|^4)^{-1}$ gives the logarithm of the participation ratio [PRL, 123, 180601, 2019; PR-research 3, L032030, 2021,] determing the degree of delocalization of the state in Hilbert space

* when $q=\infty$, Reyni entropy $S_\infty = -\ln P_m$ with $P_m = \mathrm{Max}\{|C_m|^2\}$. and $D_\infty = -\frac{\ln P_m}{\ln N}$. determine the extreme value statistics of the q. state.

* In the fully chaotic regime

$$D_q = \begin{cases} 1 - \frac{g_q}{\ln N} & q=1,2, \\ 1 - g_\infty \frac{\ln(\ln N)}{\ln N}, & q=\infty \end{cases}$$

[PRE 100, 032117, 2019; PRL 126, 150601, 2021;]

provides nice scaling of the multi-fractality with $\ln N$.

entanglement and dynamics.

With a given initial state, we can calalute the time dependence of the entropy $S(t)$ and $D_q(t)$. The time average.

$$\hat{S} = \frac{1}{T}\int_0^T S(t)\,dt, \quad \hat{D}_q = \frac{1}{T}\int_0^T D_q(t)\,dt$$

will give ~~indicators of the~~ chaos as well. [PRE 70, 016217, 2004] partial trace.

The fock basis $|n_1, n_2, n_3\rangle$. Any eigen state can be expanded as $|j\rangle = \sum_M C_M^j |n_1, n_2, n_3\rangle$, with $H|j\rangle = E_j|j\rangle$ with $M = \{n_1, n_2, n_3\}$

Now for a given initial state. $|\psi_0\rangle = \sum_M a_M |n_1, n_2, n_3\rangle$.
The time-dependent state $|\psi(t)\rangle$ is

$$|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$$
$$= \sum_j e^{-iE_jt} \alpha_j |j\rangle$$

and $\alpha_j = \langle j|\psi_0\rangle = \sum_M \cdot \sum_{M'} (C_M^j)^* a_{M'} \langle n_1 n_2 n_3 | n_1' n_2' n_3'\rangle$

$$= \underline{\sum_M (C_M^j)^* a_M}.$$

At time t, the density matrix

$$\rho = |\psi(t)\rangle\langle\psi(t)|$$
$$= \sum_{j,j'} e^{-i(E_j - E_{j'})t} \alpha_j \alpha_{j'}^* |j\rangle\langle j'|.$$

We now trace one of the site, for example site 1.

$$\rho_r = \mathrm{Tr}_1 \rho = \sum_{n_1} \langle n_1|\rho|n_1\rangle.$$
$$= \sum_{n_1} \sum_{j,j'} e^{-i(E_j - E_{j'})t} \alpha_j \alpha_{j'}^* \langle n_1|j\rangle\langle j'|n_1\rangle.$$

$$= \sum_{j,j'} e^{-i(E_j - E_{j'})t} \alpha_j \alpha_{j'}^* \sum_{n_1} \langle n_1 | j \rangle\langle j' | n_1 \rangle$$

$$= \sum_{j,j'} e^{-i(E_j - E_{j'})t} \alpha_j \alpha_{j'}^* \underbrace{\sum_{M,M'} \sum_{n_1} \langle n_1 | C_M^j | \bar{n}_1 \bar{n}_2 \bar{n}_3 \rangle\langle n_1' n_2' n_3' | (C_{M'}^{j'})^* | n_1 \rangle}_{A}$$

Now we simplify A.

$$A = \sum_{M,M'} \sum_{n_1} C_M^j (C_{M'}^{j'})^* \langle n_1 | \bar{n}_1 \bar{n}_2, \bar{n}_3 \rangle\langle n_1' n_2' n_3' | n_1 \rangle.$$

$$= \sum_{M,M'} \sum_{n_1} C_M^j (C_{M'}^{j'})^* \delta_{n_1 \bar{n}_1} \cdot \delta_{n_1' n_1} | \bar{n}_2 \bar{n}_3 \rangle\langle n_2' n_3' |.$$

$$= \sum_{\bar{n}_2 \bar{n}_3} \sum_{n_2' n_3'} \left(\sum_{n_1} C_{\bar{n}_1 \bar{n}_2 \bar{n}_3}^{j} (C_{n_1 n_2' n_3'}^{j'})^* \right) | \bar{n}_2 \bar{n}_3 \rangle\langle n_2' n_3' |.$$

$$\rho_r = \sum_{\bar{n}_2 \bar{n}_3} \sum_{n_2' n_3'} \left[\sum_{j j'} e^{-i(E_j - E_{j'})t} \alpha_j \alpha_{j'}^* \times \sum_{n_1} C_{n_1 \bar{n}_2 \bar{n}_3}^{j} (C_{n_1 n_2' n_3'}^{j'})^* \right]$$

$$\times | \bar{n}_2 \bar{n}_3 \rangle\langle n_2' n_3' |.$$