Codes for Simultaneous Transmission of Quantum and Classical Information

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Abstract—**We consider the characterization as well as the construction of quantum codes that allow to transmit both quantum and classical information, which we refer to as 'hybrid codes'. We construct hybrid codes $[\![n, k{:}m, d]\!]_q$ with length n and distance d, that simultaneously transmit k qudits and m symbols from a classical alphabet of size q. Many good codes such as $[\![7, 1{:}1, 3]\!]_2$, $[\![9, 2{:}2, 3]\!]_2$, $[\![10, 3{:}2, 3]\!]_2$, $[\![11, 4{:}2, 3]\!]_2$, $[\![11, 1{:}2, 4]\!]_2$, $[\![13, 1{:}4, 4]\!]_2$, $[\![13, 1{:}1, 5]\!]_2$, $[\![14, 1{:}2, 5]\!]_2$, $[\![15, 1{:}3, 5]\!]_2$, $[\![19, 9{:}1, 4]\!]_2$, $[\![20, 9{:}2, 4]\!]_2$, $[\![21, 9{:}3, 4]\!]_2$, $[\![22, 9{:}4, 4]\!]_2$ have been found. All these codes have better parameters than hybrid codes obtained from the best known stabilizer quantum codes.**

I. Introduction

The simultaneous transmission of both quantum and classical information over a quantum channel was initially investigated in [6] from an information theoretic point of view, and followed up by many others (see, e. g. [9], [10], [14]). It was shown that there is an advantage to address the two tasks of transmitting both quantum and classical information simultaneously, compared to independent solutions.

For the finite length case, however, there are not many constructions of error-correcting codes for simultaneous transmission of quantum and classical information in the literature. In [12], the authors consider the problem in the context of so-called entanglement-assisted codes, i. e., when sender and receiver share perfect entanglement. The examples given in [12], however, fail to demonstrate an advantage in terms of the parameters of the resulting codes when compared to, e. g., stabilizer quantum codes.

Here we study codes for simultaneous transmission of quantum and classical information, which we refer to as 'hybrid quantum codes' or just 'hybrid codes'. Using the framework of stabilizer codes [2], [7] and its generalization, that is, codeword stabilized (CWS) codes [3] and union stabilizer codes [8], we obtain hybrid codes that have advantage over the best known quantum codes for transmitting quantum information only for up to eleven qubits by exhaustive or randomized search. A general construction yields codes for up to 22 qubits. We also formulate a linear program to bound the parameters of hybrid codes.

II. Background and Notation

Our discussion is based on the theory of stabilizer quantum codes and its connection to classical error-correcting codes (see, e. g., [2]). Although we consider only codes for qubit systems here, we state the theory for quantum systems composed of qudits of dimension $q = p^\ell$, where p is prime. A quantum error-correcting code, denoted by $\mathcal{C} = (\!(n, K, d)\!)_q$, is a K-dimensional subspace of the Hilbert space $\mathcal{H} = (\mathbb{C}^q)^{\otimes n}$, which is an n-fold tensor product of Hilbert spaces of dimension q. If the minimum distance of the code is d, then any error affecting no more than $d-1$ of the subsystems can be detected or acts as a multiple of identity on the code. For stabilizer codes, the dimension K is a power of p, and if $K = q^k$, we use the notation $\mathcal{C} = [\![n, k, d]\!]_q$. For classical block codes, the notation $C = (n, M, d)_q$ is used, and if the code is linear with cardinality $M = q^m$, we use the notation $C = [n, m, d]_q$. Following [12], we use the notation $\mathcal{C} = [\![n, k{:}m, d]\!]_q$ for a code that simultaneously transmits k qudits and m symbols from a classical alphabet of size q. Similarly, we use the notation $\mathcal{C} = (\!(n, K{:}M, d)\!)_q$ for such a code that encodes a quantum system of dimension K and one out of M classical messages.

Trivially, we have the following:

Lemma 1: Given a quantum code $\mathcal{C} = (\!(n, KM, d)\!)_q$ of composite dimension KM, there exists a hybrid code with parameters $(\!(n, K{:}M, d)\!)_q$.

Proof: First, factor the code space into two subsystems of dimension K and M, respectively. Then, one uses the first subsystem of dimension K to transmit quantum information, and the second subsystem of dimension M just to transmit classical information. ■

Similarly, we have the following conversion rule for hybrid stabilizer codes.

Lemma 2: Assume that a hybrid code $\mathcal{C} = [\![n, k{:}m, d]\!]_q$ with $k > 0$ exists. Then a code $\mathcal{C}' = [\![n, k-1{:}m+1, d]\!]_q$ exists as well.

Proof: One of the k qudits can be used to transmit classical information only, decreasing k and increasing m. ■

Note that the converse does not hold in general, as the transmission of quantum information over a quantum channel is more demanding than the transmission of classical information.

Another trivial construction is to independently use a quantum code of length n_1 and a classical code of length n_2.

Lemma 3: Assume that a quantum code $\mathcal{C}_1 = [\![n_1, k_1, d]\!]_q$ and a classical code $C_2 = [n_2, m_2, d]_q$ exist. Then there exists a hybrid code with parameters $\mathcal{C} = [\![n_1 + n_2, k_1{:}m_2, d]\!]_q$.

Our goal is to find codes that have better parameters than the codes that can be obtained by these trivial constructions.

III. Error correction conditions

A hybrid quantum code $\mathcal{C} = (\!(n, K{:}M)\!)_q$ can be described by a collection

$$\{\mathcal{C}^{(\nu)} \colon \nu = 1, \ldots, M\} \tag{1}$$

of M quantum codes $\mathcal{C}^{(\nu)} = (\!(n, K, d)\!)_q$. Each of the codes has length n, dimension K, and minimum distance d. The classical information ν determines which quantum code $\mathcal{C}^{(\nu)}$ is used to encode the quantum information. In the following, we will use Greek letters when referring to classical information. Assume that

$$\{|c_i^{(\nu)}\rangle \colon i = 1, \ldots, K\} \tag{2}$$

is an orthonormal basis for the code $\mathcal{C}^{(\nu)}$. In order to be able to correct the linear span of error operators $\{E_k \colon k = 1, 2, \ldots\}$, each of the codes $\mathcal{C}^{(\nu)}$ has to obey the Knill-Laflamme conditions [11], i. e.,

$$\langle c_i^{(\nu)}|E_k^\dagger E_\ell|c_j^{(\nu)}\rangle = \alpha_{k\ell}^{(\nu)}\delta_{ij}. \tag{3}$$

Note that the constants $\alpha_{k\ell}^{(\nu)} \in \mathbb{C}$ may depend on the classical information ν.

On the other hand, in order to be able to retrieve the classical information ν independently of the quantum information that is transmitted at the same time, one has to be able to perfectly distinguish the states $|c_i^{(\nu)}\rangle$ and $|c_j^{(\mu)}\rangle$ for $\nu \neq \mu$ and arbitrary i and j after an error. This is reflected by the condition

$$\langle c_i^{(\nu)}|E_k^\dagger E_\ell|c_j^{(\mu)}\rangle = 0, \qquad \text{for } \mu \neq \nu. \tag{4}$$

In particular, the states $|c_i^{(\nu)}\rangle$ and $|c_j^{(\mu)}\rangle$ have to be mutually orthogonal. Combining (3) and (4), we get the following necessary and sufficient condition for hybrid quantum codes.

Theorem 4: A hybrid quantum code $\mathcal{C} = (\!(n, K{:}M)\!)_q$ with orthonormal basis states $\{|c_i^{(\nu)}\rangle \colon i = 1, \ldots, K,\ \nu = 1, \ldots, M\}$ can correct all errors $\{E_k \colon k = 1, 2, \ldots\}$ if and only if

$$\langle c_i^{(\nu)}|E_k^\dagger E_\ell|c_j^{(\mu)}\rangle = \alpha_{k\ell}^{(\nu)}\delta_{ij}\delta_{\mu\nu}. \tag{5}$$

Proof: As argued above, for $\mu = \nu$ condition (5) reduced to the Knill-Laflamme conditions. Now assume that $\mu \neq \nu$. When condition (4) is violated, i. e., $\langle c_i^{(\nu)}|E_k^\dagger E_\ell|c_j^{(\mu)}\rangle \neq 0$, the erroneous states $E_k|c_i^{(\nu)}\rangle$ and $E_\ell|c_j^{(\mu)}\rangle$ are non-orthogonal and can not be perfectly distinguished. On the other hand, when condition (4) holds, then the spaces $\mathcal{V}^{(\nu)}$ spanned by the images of the code $\mathcal{C}^{(\nu)}$ under all error operators, i. e.,

$$\mathcal{V}^{(\nu)} = \left\langle E_k|c_i^{(\nu)}\rangle \colon i = 1, \ldots K,\ k = 1, 2, \ldots \right\rangle \tag{6}$$

are mutually orthogonal. Therefore, there exists a measurement with associated orthogonal projections $P^{(\nu)}$ that can be used to retrieve the classical information ν. Then, knowing the index ν, one can apply the decoding algorithm for the code $\mathcal{C}^{(\nu)}$ to retrieve the quantum information. ■

Note that in the special case that the constants $\alpha_{k\ell}^{(\nu)}$ do not depend on ν, condition (5) reduces to the Knill-Laflamme conditions for a quantum code $\mathcal{C} = (\!(n, KM)\!)_q$ of dimension KM with basis states $\{|c_i^{(\nu)}\rangle \colon i = 1, \ldots, K,\ \nu = 1, \ldots, M\}$. Therefore, for hybrid codes to have better parameters than the codes given by Lemma 1, there should be at least a pair ν, μ and errors E_k, E_ℓ such that $\alpha_{k\ell}^{(\nu)} \neq \alpha_{k\ell}^{(\mu)}$. In particular, when the error operators E_k are unitary, $\alpha_{kk}^{(\nu)} = 1$. Then one should have $\alpha_{k\ell}^{(\nu)} \neq 0$ for some ν and $k \neq \ell$, which suggests that some of the codes $\mathcal{C}^{(\nu)}$ might be taken to be degenerate codes. In that case, the dimension of the spaces $\mathcal{V}^{(\nu)}$ in (6) is smaller, and hence one might be able to find a larger number of such spaces that are mutually orthogonal. In general, however, it is not excluded that all the subcodes $\mathcal{C}^{(\nu)}$ of a hybrid quantum code $\mathcal{C} = (\!(n, K{:}M, d)\!)_q$ are non-degenerate and at the same time the product KM is strictly larger than the maximal dimension K' of any quantum code $\mathcal{C}' = (\!(n, K', d)\!)_q$.

An alternative characterization of hybrid quantum codes in the Heisenberg picture of quantum mechanics was given as a special case in [1].

IV. Code construction

We outline the construction of hybrid quantum codes in the framework of CWS codes/union stabilizer codes. We start with a quantum code $\mathcal{C}^{(0)} = (\!(n, K, d)\!)_q$ which is a CWS code that might even be a stabilizer code $\mathcal{C}^{(0)} = [\![n, k, d]\!]_q$. The codes $\mathcal{C}^{(\nu)}$ are chosen as images of the seed code $\mathcal{C}^{(0)}$ under tensor products of generalized Pauli matrices, denoted by t_ν. Thus we have

$$\mathcal{C}^{(\nu)} = t_\nu \mathcal{C}^{(0)} \tag{7}$$

with $\{t_\nu \colon \nu = 1, \ldots M\}$ a set of M translation operators. When $\mathcal{C}^{(0)}$ is a non-degenerate quantum code, then all the codes $\mathcal{C}^{(\nu)}$ will also be non-degenerate. Furthermore, in this situation $\alpha_{k\ell}^{(\nu)} = \delta_{k\ell}$ for generalized Pauli errors E_k and E_ℓ. Then the resulting code will be a quantum code of dimension KM. Therefore, the seed code $\mathcal{C}^{(0)}$ is chosen to be degenerate.

Next we consider the classical codes associated with the quantum codes $\mathcal{C}^{(\nu)}$. For simplicity, we first consider the special case of stabilizer codes. The stabilizer group $\mathcal{S}$ of the code $\mathcal{C}^{(0)}$ corresponds to a self-orthogonal classical code C_0. The code C_0 is contained in its symplectic dual C_0^*, i. e., $C_0 \subseteq C_0^*$, which corresponds to the normalizer $\mathcal{N}$ of the stabilizer groups $\mathcal{S}$ in the generalized n-qudit Pauli group. For impure codes, we have

$$d = \min\{\operatorname{wgt} c \colon c \in C_0^* \setminus C_0\} > \min\{\operatorname{wgt} c \colon c \in C_0^* \setminus \{0\}\}. \tag{8}$$

The codes $\mathcal{C}^{(\nu)} = t_\nu \mathcal{C}^{(0)}$ are associated with cosets $C_0^* + t_\nu$ of the normalizer code C_0^*, where we use the same symbol t_ν to denote the classical vector corresponding to the translation operator. When the cosets $C_0^* + t_\nu$ and $C_0^* + t_\mu$ are different, then the codes $\mathcal{C}^{(\nu)}$ and $\mathcal{C}^{(\mu)}$ will be orthogonal to each other. The hybrid code $\mathcal{C}$ is associated with the classical code

$$C^* = \bigcup_{\nu=1}^{M} C_0^* + t_\nu. \tag{9}$$

When the union of the codes in (9) is an additive code, the hybrid quantum code will be a stabilizer code. Note that, in general, we have the chain of classical codes

$$C \leq C_0 \leq C_0^* \leq C^*. \tag{10}$$

The minimum distance of the quantum code associated with C^* is computed as

$$d' = \min\{\operatorname{wgt} c \colon c \in C^* \setminus C\}. \tag{11}$$

It turns out that the minimum distance of a hybrid code associated with the codes $C_0 \leq C^*$ is given by

$$d = \min\{\operatorname{wgt} c \colon c \in C^* \setminus C_0\}. \tag{12}$$

Note that the minimum in (12) is taken over a smaller set compared to (11), as $C \leq C_0$, and hence $d \geq d'$.

In summary, we have the following construction.

Theorem 5: Let $C_0 = (n, q^{n-k}, d_0)_{q^2}$ be a classical additive code that is contained in its symplectic dual C_0^*. Further, let $C^* = (n, q^{n+k+m}, d')_{q^2}$ be an additive code containing C_0^*. Then there exists a hybrid stabilizer code $\mathcal{C} = [\![n, k{:}m, d]\!]_q$ encoding k qudits and m classical symbols. The minimum distance of $\mathcal{C}$ is given by

$$d = \min\{\operatorname{wgt} c \colon c \in C^* \setminus C_0\}. \tag{13}$$

Proof: There are q^m cosets of the code C_0^* in the code C^*. Using the representatives t_ν of the cosets C^*/C_0^*, we obtain the translated codes $\mathcal{C}^{(\nu)} = t_\nu \mathcal{C}^{(0)}$ which are mutually orthogonal. All these codes have the same minimum distance given by

$$\begin{aligned} d'' &= \min\{\operatorname{wgt} c \colon c \in C_0^* \setminus C_0\} && (14)\\ &\geq \min\{\operatorname{wgt} c \colon c \in C^* \setminus C_0\} = d. && (15)\end{aligned}$$

Hence, condition (5) holds for $\nu = \mu$. It remains to show that the distance between the quantum codes $C^{(\nu)}$ is at least d, i. e., that (4) holds for all operators $E_k^\dagger E_\ell$ of weight at most $d-1$. When we treat the linear span of all codes $\mathcal{C}^{(\nu)}$ as a larger stabilizer code, the minimum distance would be given by (11). When $E_k^\dagger E_\ell$ is an element of the stabilizer of $\mathcal{C}^{(0)}$, for $\nu \neq \mu$ we compute

$$\begin{aligned} \langle c_i^{(\nu)} | E_k^\dagger E_\ell | c_j^{(\mu)} \rangle &= \langle c_i^{(0)} | t_\nu^\dagger E_k^\dagger E_\ell t_\mu | c_j^{(0)} \rangle && (16)\\ &\propto \langle c_i^{(0)} | t_\nu^\dagger t_\mu E_k^\dagger E_\ell | c_j^{(0)} \rangle && (17)\\ &= \langle c_i^{(0)} | t_\nu^\dagger t_\mu | c_j^{(0)} \rangle = \langle c_i^{(\nu)} | c_j^{(\mu)} \rangle = 0. && (18)\end{aligned}$$

Hence we can not only exclude the elements of C, but also those of C_0 when computing the minimum distance in (13). ■

In terms of classical codes, the task of constructing a good hybrid stabilizer code can be carried out in two steps. First, one has to find a good additive code C_0^* that contains its symplectic dual C_0. This defines the seed code $\mathcal{C}^{(0)}$ used to encode the quantum information. Then, using m additional generators for encoding the classical information, one obtains the code C with $C_0^* \leq C^*$.

V. Linear programming bounds

In order to obtain bounds on the parameters of hybrid stabilizer codes $[\![n, k{:}m, d]\!]_q$, we consider the homogeneous weight enumerators of the associated code C_0 and its symplectic dual C_0^*, as well as the code C^* and its symplectic dual C:

$$\mathcal{W}_{C_0}(X,Y) = \sum_{w=0}^{n} A_w^\perp X^{n-w} Y^w, \tag{19}$$

$$\mathcal{W}_{C_0^*}(X,Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w, \tag{20}$$

$$\mathcal{W}_{C}(X,Y) = \sum_{w=0}^{n} B_w^\perp X^{n-w} Y^w, \tag{21}$$

$$\mathcal{W}_{C^*}(X,Y) = \sum_{w=0}^{n} B_w X^{n-w} Y^w. \tag{22}$$

The weight enumerators of C_0 and C_0^*, as well as those of C and C^*, are related by the MacWilliams transformation, i. e.,

$$\mathcal{W}_{C_0^*}(X,Y) = \frac{1}{|C_0|} \mathcal{W}_{C_0}\left(X + (q^2-1)Y, X - Y\right), \tag{23}$$

$$\mathcal{W}_{C^*}(X,Y) = \frac{1}{|C|} \mathcal{W}_{C}\left(X + (q^2-1)Y, X - Y\right). \tag{24}$$

Nestedness of the codes is reflected by the condition

$$0 \leq B_w^\perp \leq A_w^\perp \leq A_w \leq B_w, \qquad \text{for } w = 0, \ldots, n. \tag{25}$$

When the hybrid code has minimum distance d, we have

$$A_w^\perp = A_w = B_w, \qquad \text{for } w = 0, \ldots, d-1. \tag{26}$$

Additionally, we have:

$$A_0^\perp = A_0 = B_0 = 1, \tag{27}$$

$$\sum_{w=0}^{n} A_w^\perp = q^{n-k}, \qquad \sum_{w=0}^{n} A_w = q^{n+k}, \tag{28}$$

$$\sum_{w=0}^{n} B_w^\perp = q^{n-k-m}, \quad \sum_{w=0}^{n} B_w = q^{n+k+m}. \tag{29}$$

When a hybrid stabilizer code $[\![n, k{:}m, d]\!]_q$ exists, the linear program for the variables $B_w^\perp$, $A_w^\perp$, A_w, and B_w given by (23)–(29) has an integer solution. For qubit codes, we can strengthen the linear program by additionally considering the shadow enumerator [13]

$$\mathcal{S}_{C_0}(X,Y) = \frac{1}{|C_0|} \mathcal{W}_{C_0}\left(X + (q^2-1)Y, Y - X\right), \tag{30}$$

which has to have non-negative integer coefficients.

Using CPLEX V12.6.3.0, we checked whether the integer program is feasible. More precisely, we first fix the length n, number of qudits k, and number $M = 2^m$ of classical symbols. Then we look for the largest minimum distance d for which the integer program is found to be feasible. The resulting bounds on the parameters $[\![n, k{:}m, d]\!]_2$ are listed in Table I, i. e., for fixed parameters n, k, and d, the largest possible value for m is given.

VI. Results

Based on the construction discussed in Section IV, we perform a search for $\mathcal{C} = [\![n, k{:}m, d]\!]_2$ codes with distance $d \geq 3$. We start with the self-dual codes from the classification in [4], [5]. In a first step, we construct impure quantum codes $[\![n, 1, d]\!]_2$, and then look for additional vectors for the encoding

TABLE I: (LP bound) Upper bound on the number of classical bits m in any $[\![n,k{:}m,d]\!]_2$ hybrid stabilizer code with fixed length $n \le 14$ and dimension k for distance $d=3,4,5$. For $k=0$, we list the largest dimension of a classical linear binary code. Note that there is, e. g., no stabilizer code $[\![13,5,4]\!]_2$, excluding the corresponding entry in the table.

$d=3$

$n \backslash k$	0	1	2	3	4	5	6	7	8
5	2	0	–	–	–	–	–	–	–
6	3	0	–	–	–	–	–	–	–
7	4	2	–	–	–	–	–	–	–
8	4	3	1	0	–	–	–	–	–
9	5	4	3	1	–	–	–	–	–
10	6	5	4	2	1	–	–	–	–
11	7	6	5	4	2	0	–	–	–
12	8	7	6	5	3	2	0	–	–
13	9	8	7	5	5	3	1	0	–
14	10	9	8	7	6	5	3	1	0

$d=4$

$n \backslash k$	0	1	2	3	4	5	6
5	1	–	–	–	–	–	–
6	2	–	–	–	–	–	–
7	3	–	–	–	–	–	–
8	4	–	–	–	–	–	–
9	4	–	–	–	–	–	–
10	5	3	1	–	–	–	–
11	6	4	2	–	–	–	–
12	7	5	4	2	0	–	–
13	8	6	5	4	2	0*	–
14	9	6	6	5	3	2	0

$d=5$

$n \backslash k$	0	1	2	3
5	1	–	–	–
6	1	–	–	–
7	1	–	–	–
8	2	–	–	–
9	2	–	–	–
10	3	–	–	–
11	4	0	–	–
12	4	2	–	–
13	5	4	–	–
14	6	5	3	1

of classical information, resulting in an $[\![n,1{:}m',d]\!]_2$ hybrid code. In some cases it turns out that we can in fact encode more than one qubit, i. e., the code $[\![n,1{:}m',d]\!]_2$ is in fact a hybrid code with parameters $[\![n,k{:}m'-k+1,d]\!]_2$.

For $d=4$ and $d=5$, we have exhaustively searched using all self-dual codes listed in [4], [5] up to $n=11$. For $d=3$, we have exhaustively searched all self-dual codes listed in [4], [5] up to $n=10$. We also have conducted randomized search for $n=11$. Finally, we appended some qubits in the state $|0\rangle$ to good quantum codes and found new hybrid codes. The results are summarized as follows.

Theorem 6: There exist hybrid codes with the following parameters:

$$[\![7,1{:}1,3]\!]_2,\ [\![9,2{:}2,3]\!]_2,\ [\![10,3{:}2,3]\!]_2,\ [\![11,4{:}2,3]\!]_2,$$
$$[\![11,1{:}2,4]\!]_2,\ [\![13,1{:}4,4]\!]_2,$$
$$[\![13,1{:}1,5]\!]_2,\ [\![14,1{:}2,5]\!]_2,\ [\![15,1{:}3,5]\!]_2,$$
$$[\![19,9{:}1,4]\!]_2,\ [\![20,9{:}2,4]\!]_2,\ [\![21,9{:}3,4]\!]_2,\ [\![22,9{:}4,4]\!]_2.$$

All these codes have better parameters than codes obtained from the best quantum codes using Lemma 2.

Below, we provide more details on these codes. In presenting each $[\![n,k{:}m,d]\!]_2$ code, we first list the generators of the stabilizer of the corresponding $[\![n,k,d]\!]_2$ impure quantum code $\mathcal{C}^{(0)}$, with its $2k$ logical operators between a single and a double horizontal line. The stabilizer of the code $\mathcal{C}^{(0)}$, corresponding to the code C_0, is generated by the rows above the single horizontal line, while the normalizer of the code $\mathcal{C}^{(0)}$, corresponding to the symplectic dual code C_0^*, is generated by the rows above the double horizontal line. Below the double horizontal line, we list the additional generators that are used to encode m classical bits.

A quantum code that encodes a single qubit and is able to correct a single error requires at least five qubits. For five and six qubits, linear programming shows that we can only transmit a single qubit and no additional classical bit when we want to correct a single errors, i. e., for distance $d \ge 3$.

Increasing the length to seven qubits, it is still only possible to encode a single qubit when a single error has to be corrected. The stabilizer of an impure code $[\![7,1,3]\!]_2$ is generated by the elements of the Pauli group given in first six lines above the single horizontal line in the matrix (31). Note that the element in the second line has only weight two. The next two elements between the single and the double horizontal line correspond to the logical operators on the encoded qubit. Starting with this impure code, we are able to transmit an extra classical bit, i. e., we obtain a hybrid code with parameters $[\![7,1{:}1,3]\!]_2$. The additional generator that is used to encode one classical bit is given below the double horizontal line.

$$\left(\begin{array}{ccccccc} X & I & I & Z & Y & Y & Z \\ Z & I & I & I & I & I & X \\ I & X & I & X & Z & I & I \\ I & Z & I & Z & I & X & X \\ I & I & X & X & I & Z & I \\ I & I & Z & Z & X & I & X \\ \hline I & I & I & X & Z & Z & X \\ I & I & I & Z & X & X & I \\ \hline\hline I & I & I & I & X & Y & Y \end{array}\right) \tag{31}$$

We have not found a hybrid code with parameters $[\![7,1{:}2,3]\!]_2$ which is not ruled out by linear programming.

For eight qubits, there is a quantum code with parameters $[\![8,3,3]\!]_2$. Using Lemma 2, we obtain an optimal hybrid code with parameters $[\![8,2{:}1,3]\!]_2$, as well as a code $[\![8,1{:}2,3]\!]_2$. We have not found a hybrid code with parameters $[\![8,1{:}3,3]\!]_2$ that might exist.

For nine qubits, we found a hybrid code $[\![9,2{:}2,3]\!]_2$ given in (32). The rows above the single horizontal line generate the stabilizer of an impure code $[\![9,2,3]\!]_2$. Taking all possible products of the two generators below the double horizontal line in (32) we obtain the four translation operators $t^{(1)}=id$, $t^{(2)}$, $t^{(3)}$, and $t^{(4)}=t^{(2)}t^{(3)}$ used to encode two extra classical bits.

A hybrid code $[\![10,3{:}2,3]\!]_2$ with ten qubits is given in (33). Via linear programming it is found that this code is optimal in the sense that it encodes the maximal possible number m of additional classical bits among all codes $[\![10,3{:}m,3]\!]_2$.

The first non-trivial hybrid code with distance $d=4$ has been found for eleven qubits. A hybrid code $[\![11,1{:}2,4]\!]_2$ is given in (34). We found a hybrid code $[\![11,4{:}2,3]\!]_2$ as well.

Appending two qubits in the state $|0\rangle$ to the impure quantum code $[\![11,1,4]\!]_2$ given above the double horizontal line in (34), one obtains an impure code $[\![13,1,4]\!]_2$. This code can

TABLE II: Generators of hybrid codes $[\![9,2{:}2,3]\!]_2$, $[\![10,3{:}2,3]\!]_2$, and $[\![11,1{:}2,4]\!]_2$.

$$\left(\begin{array}{ccccccccc}
X & I & I & Z & Y & Z & X & X & Y \\
Z & I & I & I & I & X & I & I & I \\
I & X & I & Z & Y & I & Y & I & Z \\
I & Z & I & I & I & I & X & I & I \\
I & I & X & Z & Z & I & I & I & X \\
I & I & Z & I & Y & X & I & Y & I \\
I & I & I & X & X & X & I & Z & I \\
\hline
I & I & I & Z & I & I & X & Y & X \\
I & I & I & I & X & I & I & Z & Y \\
I & I & I & I & Z & I & I & X & X \\
I & I & I & I & I & X & X & I & X \\
\hline\hline
I & I & I & I & I & Z & I & Z & X \\
I & I & I & I & I & I & Y & X & Z
\end{array}\right) \tag{32}$$

$$\left(\begin{array}{cccccccccc}
X & I & X & Y & I & X & Z & X & X & Y \\
Z & I & I & I & I & I & I & I & I & X \\
I & X & X & X & I & Y & X & Y & Z & X \\
I & Z & I & I & I & I & I & I & X & I \\
I & I & Z & Z & I & I & I & I & I & I \\
I & I & I & I & X & X & Y & Y & I & I \\
I & I & I & I & Z & Z & X & X & I & I \\
\hline
I & I & X & X & I & I & I & I & I & X \\
I & I & I & Z & I & I & I & I & X & X \\
I & I & I & I & I & X & I & Y & X & X \\
I & I & I & I & I & Z & I & X & I & X \\
I & I & I & I & I & I & X & X & X & X \\
I & I & I & I & I & I & Z & Z & I & X \\
\hline\hline
I & I & I & X & I & I & I & Z & X & Y \\
I & I & I & I & I & I & I & Y & Y & Z
\end{array}\right) \tag{33}$$

$$\left(\begin{array}{ccccccccccc}
X & X & I & I & I & I & Z & Z & X & I & Z \\
Z & I & I & I & I & I & I & I & I & I & X \\
I & Z & I & I & I & I & I & I & I & I & X \\
I & I & X & I & I & Z & I & X & Z & I & I \\
I & I & Z & I & I & I & I & I & X & I & I \\
I & I & I & X & I & Z & Y & Z & X & Y & X \\
I & I & I & Z & I & I & I & I & I & X & I \\
I & I & I & I & X & Z & Z & I & I & X & I \\
I & I & I & I & Z & Z & X & X & I & I & I \\
I & I & I & I & I & Y & X & Y & I & X & I \\
\hline
I & I & I & I & I & Z & I & X & I & X & X \\
I & I & I & I & I & I & Z & Z & X & X & I \\
\hline\hline
I & X & I & I & I & I & I & X & Y & Z & I \\
I & I & I & I & I & I & X & I & X & Y & Z
\end{array}\right) \tag{34}$$

additionally transmit four classical bits, i. e., one obtains the hybrid code $[\![13,1{:}4,4]\!]_2$ given in (35).

$$\left(\begin{array}{ccccccccccc|cc}
\multicolumn{11}{c|}{\text{the stabilizer part of (34)}} & I & I \\
\hline
I & I & I & I & I & I & I & I & I & I & I & Z & I \\
I & I & I & I & I & I & I & I & I & I & I & I & Z \\
\hline
I & I & I & I & I & Z & I & X & I & X & X & I & I \\
I & I & I & I & I & I & Z & Z & X & X & I & I & I \\
\hline\hline
I & X & I & I & I & I & I & I & X & Y & X & X & X \\
I & I & I & I & I & I & X & I & X & I & I & X & X \\
I & I & I & I & I & I & I & X & Y & X & Y & X & X \\
I & I & I & I & I & I & I & X & I & Y & Y & X & I
\end{array}\right) \tag{35}$$

A related construction is the following:

Theorem 7: Let $\mathcal{C}_1 = [\![n,k_1,d_1]\!]_q \subset \mathcal{C}_2 = [\![n,k_2,d_2]\!]_q$ be nested quantum codes. Further, let $C_3 = [n_3, k_2-k_1, d_3]_q$ be a classical linear code. Then there is a hybrid quantum code $\mathcal{C} = [\![n+n_3, k_1{:}(k_2-k_1), d]\!]_q$ with $d \geq \min(d_1, d_2+d_3)$.

Proof: Let G_1 be a generator matrix for the normalizer of $\mathcal{C}_1$, and let G_{12} together with G_1 be a generator matrix for the normalizer of $\mathcal{C}_2$. Further, let G_3 be a generator matrix of C_3, and let $\omega \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. The hybrid code is given by the following matrix:

$$\left(\begin{array}{c|c}
0 & \omega I \\
\hline
G_1 & 0 \\
\hline\hline
G_{12} & G_3
\end{array}\right) \tag{36}$$

The matrix above the double horizontal line corresponds to the normalizer of the impure quantum code obtained by appending n_3 qudits in the state $|0\rangle$ to the code $\mathcal{C}_1$. The distance of this code is d_1. Any vector involving the matrix G_{12} will have weight at least $d_2 + d_3$. Hence, $d \geq \min(d_1, d_2+d_3)$. ∎

From the nested stabilizer codes $[\![11,1,5]\!]_2 \subset [\![11,4,3]\!]_2$ and classical codes $[n_3, n_3-1, 2]_2$, one obtains hybrid codes $[\![13,1{:}1,5]\!]_2$, $[\![14,1{:}2,5]\!]_2$, and $[\![15,1{:}3,5]\!]_2$. Similarly, from $[\![17,9,4]\!]_2 \subset [\![17,13,2]\!]_2$, one gets $[\![19,9{:}1,4]\!]_2$, $[\![20,9{:}2,4]\!]_2$, $[\![21,9{:}3,4]\!]_2$, and $[\![22,9{:}4,4]\!]_2$.

VII. Discussion

We have characterized hybrid quantum codes for the simultaneous transmission of quantum and classical information in terms of generalized Knill-Laflamme conditions. Using the framework of CWS codes/union stabilizer codes, we have formulated a linear program to obtain bounds on the parameters of codes. Moreover, we found several examples of hybrid codes that demonstrate the advantage of simultaneous transmission of quantum and classical information.

The code conditions derived in Section III suggest that one should start with good impure quantum codes. Theorem 7 uses trivial impure codes. In order to find a direct construction of hybrid codes with good parameters, a first step could be to develop methods to construct good non-trivial impure codes.

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