CS3535: Algorithms Spring 2022
Assignment 02: Joy with Sort and Scan
Due Date: Monday, February 07; 23:00 hrs
Total Points: 25

Computational geometry is the field of informatics or computer science that deals with solving geometric problems using a computer. This field contains many areas of study, including graph drawing (given a description of a graph, how to draw it on a surface aesthetically?), algebraic modeling (solving a geometric problem as a vast set of equations, and finding fast solvers), computational topology (study of the structure of graphs on surfaces) and discrete geometry (studying the combinatorial properties of geometric problems). The core area of the field is to study algorithmic techniques and data structures in the context of geometric problems. Such techniques are widely used in computer graphics, geographical information systems, computer aided design, robot motion planning, databases and much more. In this assignment, we look at two very simple problems that are often used as black boxes in more complex algorithms.

1 Angular Sorting with Timsort

1.1 Sorting Angularly

In this problem, we will sort points in the two dimensional plane in anti-clockwise direction, based on the angle formed with the x axis. Each point is described as a tuple (x, y), where x and y are lengths of the point on the respective coordinate axes. Note that each point can be visualized as a vector, i.e., a ray from the origin in the direction of the point (see Figure (i) below).

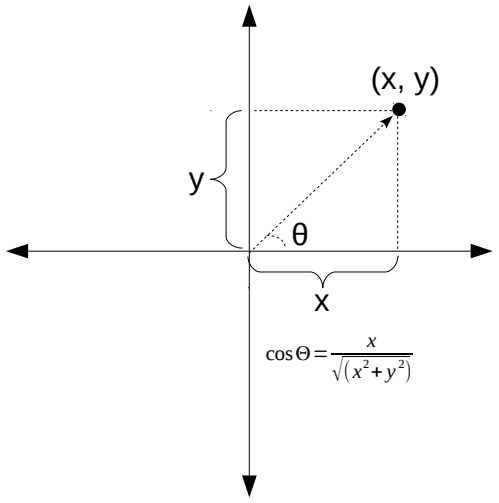
The angle θ formed with the x axis can be easily computed:

$$\begin{aligned} \cos\theta &= \frac{x}{\sqrt{x^2+y^2}} \\ \Rightarrow \quad \theta &= \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \end{aligned}$$

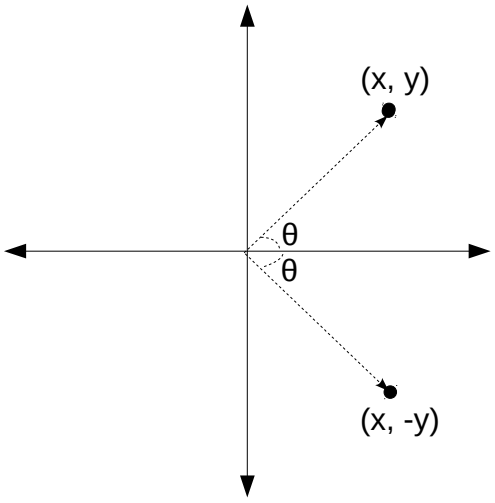
The use of the cosine function is nice because it is monotonically decreasing from 1 to -1 in $[0, \pi]$. The inverse of the cosine function, also called as arc cosine, is a common mathematical function provided in the math libraries of most programming languages. This however, gives the same value of θ for points (x, y) and $(x, -y)$ (see Figure (ii) below), since the range of the function is only $[0, \pi]$. This raises an important issue in sorting points angularly: the angle should be taken in the context of the quadrant[1]. in which it lies. For example, consider comparing two points p_1 and p_2 forming angles θ_1 and θ_2 respectively. Suppose θ_1 is larger than θ_2. It is possible for p_1 to appear earlier in the sorted ordering if p_2 lies in the fourth quadrant (see Figure (iii) below).

You may imagine angular sorting with a ray extending out of the origin in the direction of the

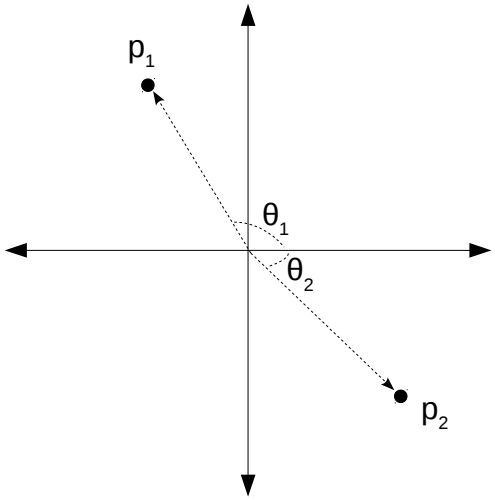
[1]In coordinate geometry, the first quadrant contains all points with $x \geq 0$ and $y \geq 0$, the second quadrant with points $x < 0$ and $y \geq 0$, third quadrant with points $x < 0$ and $y < 0$, and finally the fourth quadrant with points $x \geq 0$ and $y < 0$.


(x, y)
y
θ
x
$\cos\Theta = \frac{x}{\sqrt{(x^2+y^2)}}$

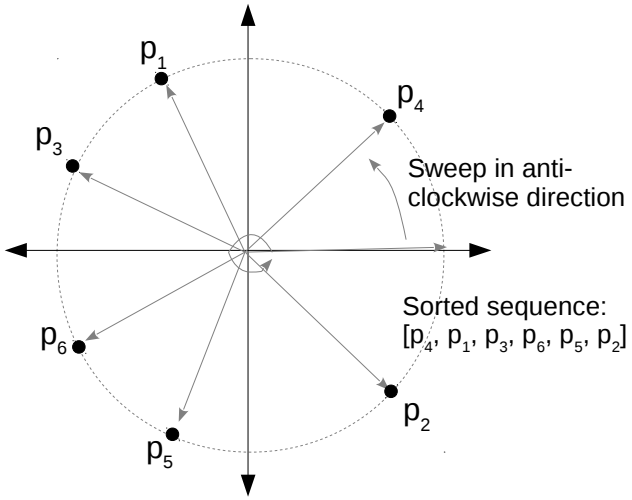
(i)


(x, y)
θ
θ
(x, -y)

(ii)


p_1
θ_1
θ_2
p_2

(iii)


p_1
p_4
p_3
Sweep in anti-clockwise direction
Sorted sequence:
[p_4, p_1, p_3, p_6, p_5, p_2]
p_6
p_2
p_5

(iv)

positive x axis, which sweeps in the anti-clockwise direction (see Figure (iv)). Each point is recorded in the output sorted sequence when it intersects this ray.

1.2 The Point Class

Since comparing two points is now non-trivial, it will be beneficial to implement a `Point` class that stores information about a point, and one that also knows to compare two points together. Please refer to the appropriate language documentation to see how to include a comparator or a comparison function to compare two `Point` objects. Depending on the language and implementation, this function may return an integer value that is < 0, 0 and > 0 if the first point is angularly smaller, equal and larger respectively than the second point. In some languages, it may be easier to override appropriate operators for the same task. Please make sure that you are returning the proper value (either `int` or `boolean`) depending on the implementation of the comparator function.

1.3 Timsort

Timsort is an industrial grade adaptive sorting algorithm whose running time adapts to the structure of data in the input. It is the sorting algorithm of choice in both Java and Python sorting library functions. It combines two sorting algorithms for its task – insertion sort to sort small $O(1)$ sized subproblems, and mergesort for larger subproblems. However, timsort does not divide the problem into equal sized chunks, rather it divides the problems into runs, which are consecutive elements that are non-decreasing in the input. Therefore it merges variable-sized sorted sequences into one. Timsort is also stable, and this is achieved by only merging consecutive runs. For this, it makes use of a run stack. Timsort has two distinct phases which are described below.

1.3.1 Scanning Phase

As a first step, timsort scans the input data from left to right and identifies runs. It performs a number of tasks during this process. Let the input array be a. We use the subscript notation a_i to refer to the i^{th} element in the array.

1. Timsort makes sure that each run is of sufficient length. Specifically, if timsort is scanning the i^{th} element a_i, it tries to decide if a_i can be included in the current run or whether a new run starts at a_i. If the number of elements in the current run is less than a pre-defined `minrun` length value, then a_i is insertion sorted into the current run. Note that a_i is not inserted into a sorted prefix of the entire input $a_1 \dots a_{i-1}$, but only the sorted prefix of the current run. If a_i is at least as large as the last element in the run, then the length of the run is increased. If a_i is smaller than the last element in the run and if the length of the current run is at least as large as `minrun`, then a new run is started.

   With lots of empirical testing, it is recommended to use a `minrun` value between 32 and 64. For this problem, we will set this value to 32.

2. Once an entire run is identified, timsort adds the current run onto a stack. Each run is identified using a tuple (i, l) where i is the index of the start of the run and l is its run length. It then checks to see if the following invariants hold for three consecutive runs z, y and x on top of the stack.

(a) $|z| > |x| + |y|$: If not, then z and y are merged.

(b) $|y| > |x|$: If not, then y with x are merged.

Note that $|x|$ is the length of run x. It only proceeds to the next run if the above invariants hold for the three topmost runs on the stack. Otherwise, it repeatedly merges two consecutive runs (either y with z or y with x) and checks the invariant again.

Figure (v) shows the above two steps working in tandem. Steps (a) and (e) indicate start of new runs. In steps (b), (f), (h) and (i), the current element is increasing. Therefore we move on. In steps (d), (j), (k) and (m), current runs are completed. In step (k), the first invariant is checked and the bottom two runs on the stack are merged. In step (m), the second invariant is violated and the top two runs on the stack are merged. Note that there could be multiple merges after the current run is determined. The algorithm only proceeds to the next run if the current invariant on the top of the stack is satisfied. Note that this does not mean that the invariants are satisfied for any three consecutive runs. A future run could merge things on the stack in myriad different ways, thereby potentially violating the invariant further along along in the stack. But we always guarantee that the invariants are satisfied on the top of the stack before finding the next run.

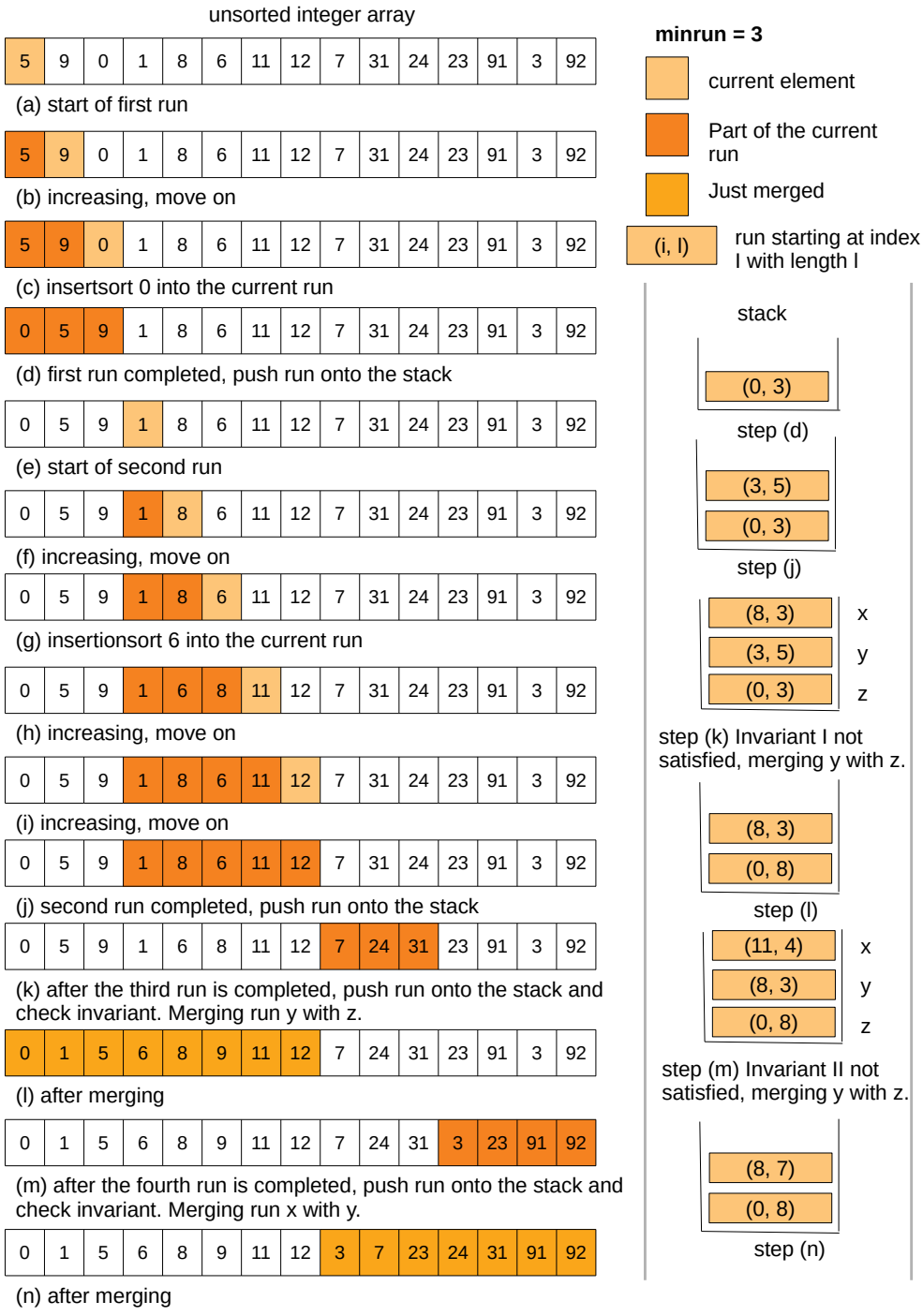
1.3.2 Bottom-up Merge

After the scanning phase is completed, there may be many runs on the stack. These runs are merged in a bottom-up fashion, two at a time. This can be implemented by popping two runs from the stack, merging them, and pushing the merged run onto another "helper" stack. Once the first stack is empty, we then copy all elements from the helper stack back onto the run stack and repeat.

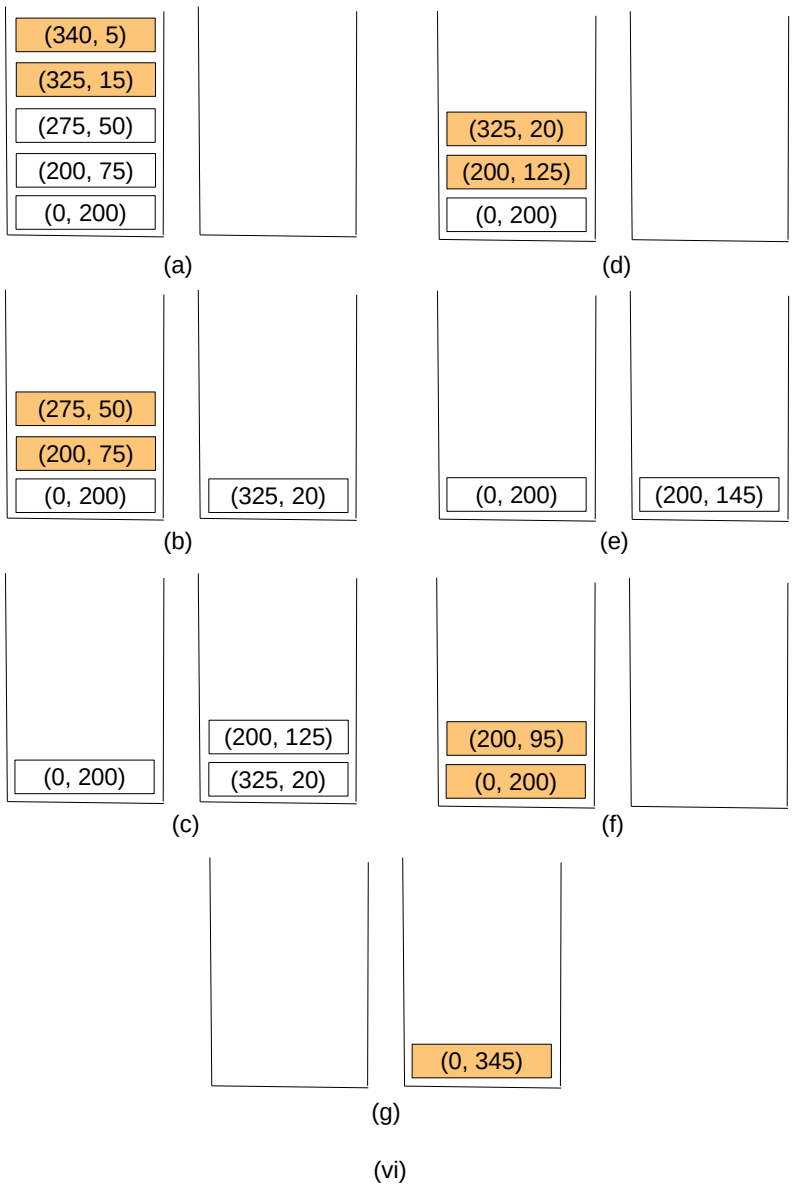
In Figure (vi), we see 5 runs still remaining on the stack after the scanning phase. The run stack is shown on the left, and the helper stack is shown on the right. In steps (a), (b), (d) and (f), we pop two runs from the run stack, merge them, and push the merged run onto the helper stack. In steps (c) and (e), there is only one run on the run stack. If the initial number of runs is even, then this stack will be empty. In steps (c) and (e), we pop all runs from the helper stack and push them onto the run stack. This merging process continues until we are left with a single run, which is the sorted array.

1.4 Sensible Testing

As there are many moving blocks, consider working with integer input as described in this handout. This allows you to test and debug your timsort routines. Also use good code design to modularize your code. For example, you can have separate functions for the scanning and the bottom-up parts. Moreover, you can have separate functions that perform insertion sort and mergesort. After you have thoroughly tested timsort for integers, you can work on the comparison function that compares two point objects. For this, you can use the small input instance provided in this handout and in the file `input-small.txt`. You may also use the library sorting function with your comparator function and test the sorted output, which is given in the file `output-small.txt`. Please also test for different input-output cases.


unsorted integer array
5 9 0 1 8 6 11 12 7 31 24 23 91 3 92
(a) start of first run
5 9 0 1 8 6 11 12 7 31 24 23 91 3 92
(b) increasing, move on
5 9 0 1 8 6 11 12 7 31 24 23 91 3 92
(c) insertsort 0 into the current run
0 5 9 1 8 6 11 12 7 31 24 23 91 3 92
(d) first run completed, push run onto the stack
0 5 9 1 8 6 11 12 7 31 24 23 91 3 92
(e) start of second run
0 5 9 1 8 6 11 12 7 31 24 23 91 3 92
(f) increasing, move on
0 5 9 1 8 6 11 12 7 31 24 23 91 3 92
(g) insertionsort 6 into the current run
0 5 9 1 6 8 11 12 7 31 24 23 91 3 92
(h) increasing, move on
0 5 9 1 8 6 11 12 7 31 24 23 91 3 92
(i) increasing, move on
0 5 9 1 8 6 11 12 7 31 24 23 91 3 92
(j) second run completed, push run onto the stack
0 5 9 1 6 8 11 12 7 24 31 23 91 3 92
(k) after the third run is completed, push run onto the stack and check invariant. Merging run y with z.
0 1 5 6 8 9 11 12 7 24 31 23 91 3 92
(l) after merging
0 1 5 6 8 9 11 12 7 24 31 3 23 91 92
(m) after the fourth run is completed, push run onto the stack and check invariant. Merging run x with y.
0 1 5 6 8 9 11 12 3 7 23 24 31 91 92
(n) after merging
minrun = 3
current element
Part of the current run
Just merged
(i, l)
run starting at index I with length l
stack
(0, 3)
step (d)
(3, 5)
(0, 3)
step (j)
(8, 3)
(3, 5)
(0, 3)
x
y
z
step (k) Invariant I not satisfied, merging y with z.
(8, 3)
(0, 8)
step (l)
(11, 4)
(8, 3)
(0, 8)
x
y
z
step (m) Invariant II not satisfied, merging y with z.
(8, 7)
(0, 8)
step (n)

(v)



1.5 Input and Output

For this program, we have one input file, and two output files. The file points `input-points.txt` contains 10000 points. The input file contains the number of points n on the first line followed by n lines that describe the x and y coordinates of a point. The sorted sequence is in `sorted-points.txt`. A sample input and output for 6 points is provided below.

```
Sample Input:

6
0.5 -0.8660254
-0.5 -0.8660254
-0.5 0.8660254
0.5 0.8660254
-1.0 0.0
1.0 0.0

Sample Sorted Output:

1.0 0.0
0.5 0.8660254
-0.5 0.8660254
-1.0 0.0
-0.5 -0.8660254
0.5 -0.8660254
```

In addition to the sorted sequence, we would also like to obtain information about the scanning and bottom-up merge phases. In particular, we would like to display the contents of the run stack after the entire scanning phase is completed. In the bottom-up merging phase, we would like to print the runs being merged. After the array is sorted, we print the total number of runs and merges the algorithm performs. Note that the number of merges is always one less than the total number of runs found in the scanning phase. The transcript output for the points in `input-points.txt` is shown below, and is also contained in `points-timsort-info.txt`.

```
after scanning phase, stack contents are
[9993, 7]
[9865, 128]
[7814, 2051]
[4066, 3748]
[0, 4066]
bottom-up merges
merging [9865, 128] [9993, 7]
merging [4066, 3748] [7814, 2051]
merging [4066, 5799] [9865, 135]
merging [0, 4066] [4066, 5934]
total number of runs found = 313
total number of merges performed = 312
```

Please make sure both output files match exactly. You may test the given input-output files using the `diff` Unix command.

1.6 Timsort Analysis

Let $c = O(1)$ be the `minrun` length. Then insertion sorting elements in each run takes only $O(1)$ time. We also spend $O(n)$ time to scan the input in the scanning phase. In the worst case, we end up with $\rho = n/c$ runs and need to merge them. Since each run has length at least c, every time we merge two runs together, we double the size of the run. The invariants also make sure that we always merge only similar sized chunks. So from the perspective of a given run, it participates in only $O(\log \rho)$ merges, and each of these take $O(n)$ time over all runs. So the total running time is $O(n \log \rho)$. Note that this running time does not violate the comparison based sorting lower bound, since if $\rho = O(n)$, it runs in $O(n \log n)$.

1.7 Further Optimizations

Apart from high-level ideas described above, timsort performs a sleuth of optimizations. We describe some of these below. Implementing these optimizations is beyond the scope of this assignment, but they do offer some food for thought. Feel encouraged to attempt these if you are in need of a challenge.

1. **Optimum Minrun Length:** Since the minrun length is recommended to be between 32 and 64, it is encouraged that the number of runs that can be formed to be close to a power of two. So the minrun length c is chosen in $[32, 64]$ such that n/c is closest to a power of two.

2. **Decreasing Runs:** Runs can also be formed using both increasing and decreasing sequences. However, in order to maintain stability, a decreasing run is strict, so it does not include equal elements. When scanning element a_i, we can perform a binary search within the run to see if a_i is already present. If it is not present, then we can go ahead and insert a_i within the run. Otherwise, we change the decreasing run into an increasing one by reversing it and then performing insertion sort on a_i. Taking into account both decreasing and increasing runs reduces the number of times we perform insertion sort.

3. **Merging Blocks:** Classical mergesort requires n units of extra space for merging. However, timsort can be implemented with at most $n/2$ units of memory. When merging two runs, we can copy the smaller run into the buffer. This therefore frees up space within the original array to perform the merge more efficiently. Also note that merging can be done in both directions. So when we consume the run stack, and move things from the helper stack back onto the run stack, we can perform merges by popping two runs at a time. This avoids unnecessary copying of runs from the helper stack to the run stack. This in effect, sweeps through the entire input data in one direction at a time until the data is sorted. This makes for better caching performance.

4. **Gallopping:** When timsort notices that certain blocks of runs always get merged together, it enters gallopping mode to move these together as a single unit in subsequent runs. This has the effect of speeding up future runs.

   For more information on timsort and its many optimizations, please see this Wikipedia article or this white paper from the original author Tim Peters.

2 Closest Pair of Points

In this problem, we will compute the closest pair of points among n points on the 2D plane. In Data Structures, we solved this problem using spatial hashing. This approach however, requires that the input point set come from a uniform distribution. We will explore a different approach that solves the problem in $O(n \log n)$ worst case time.

2.1 Sweep Line Algorithms

This is a very common and powerful algorithm design technique used to solve many problems in computational geometry. It involves sweeping one or more lines across a high dimensional scene, while maintaining and querying several data structures.

Given n points on the 2D plane, a line can be swept from left to right if we sort all the points based on the x coordinate. In fact, for this problem, we will scan two lines i and j across the point set from left to right (see Figure (vii)). Every time line j advances to a new point p_j , line i advances to a point p_i such that the difference between the x coordinates of p_j and p_i is at most the current minimum distance D. Only points between p_i and p_j in the x coordinate are "relevant" in comparing p_j, since all points before p_i are farther than length D with p_j. If we store all the relevant points in a binary search tree data structure keyed on the y coordinate, then we can query this data structure to find all points close to p_j. This binary search tree needs to be updated every time the lines move. Specifically, we insert point p_j every time line j moves, and remove point p_i when line i moves. It is enough if we only compare p_j against points in the shaded $D \times 2D$ box shown in Figure (vi) because only these points differ in both x and y coordinates with p_j by at most D. Since the binary search tree supports successor and predecessor queries, we can iterate through all the successors (and predecessors) until we reach a point beyond the shaded box. If a closer pair is found, the minimum distance D is updated.

2.2 Note on Implementation

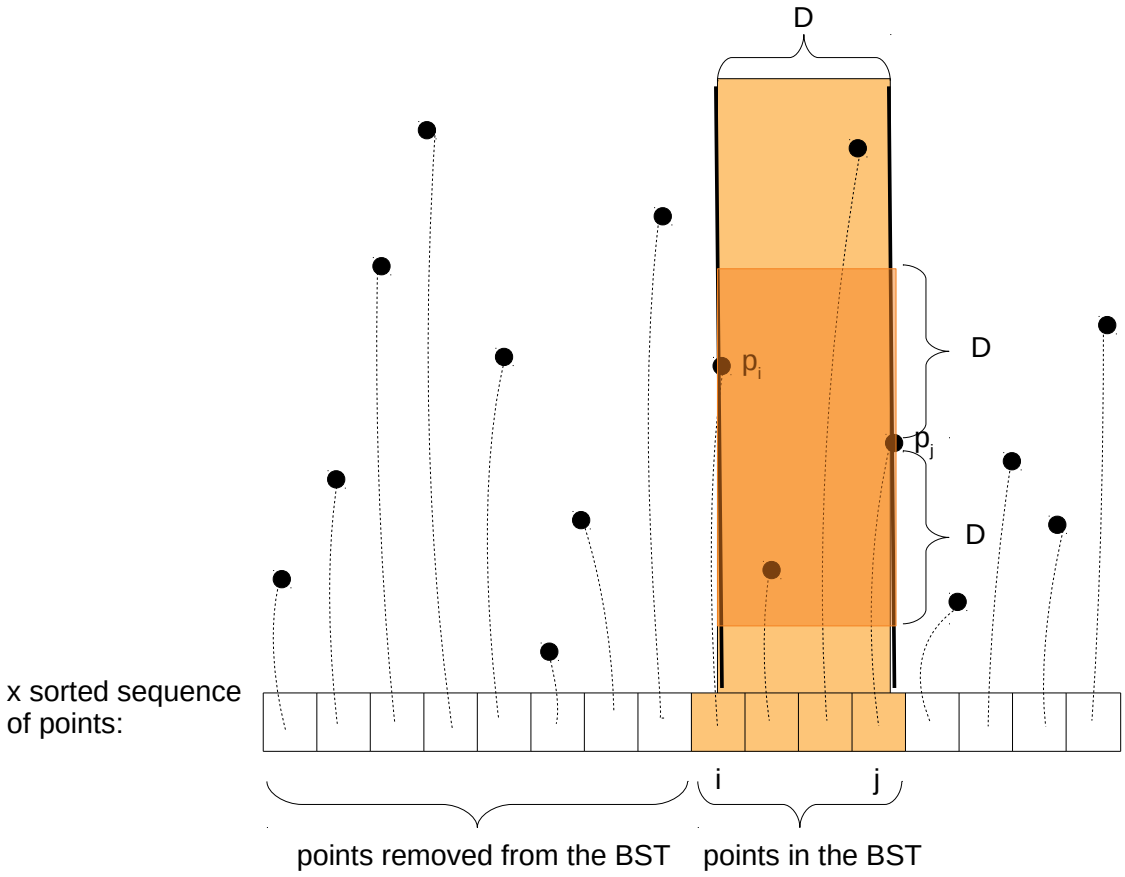
We can store the points in a class `Point2`[2]. Note that we need to sort this set of points two different ways – one along the x-axis and the other along the y-axis. This can be achieved several different ways, for instance, using two different comparator functions. In Java, this can be achieved by overriding the `compareTo` method for comparing on one axis, while having a comparator class to compare on the other. Please feel free to experiment with different code designs. As you write these functions, pay particular attention to the arguments passed, the return value of the functions, and how to call them.

In this problem, we will make use of the standard libraries for sorting and binary search trees. We will be using these libraries by supplying our own comparator functions.

One way to store these points is as a list or array. We can then iteratively scan this array using two indices i and j from left to right.

To create a binary search tree keyed on the y coordinates, we supply the appropriate comparator function as an argument while constructing the binary search tree. Note that unfortunately, Python does not have built-in libraries for binary search trees. Therefore, please implement this in Java or

[2]The choice of using `Point2` is to differ from the `Point` class written in Problem 1.



(vii)

C++, or if you feel a bit ambitious, implement your own version of a balanced binary search tree in Python. The *predecessor* and *successor* operations will allow you to iterate through the predecessor and successor points in the binary search tree respectively. This allows you to enumerate the relevant points for comparison within the $D \times 2D$ box as described above. Please see the appropriate language docs for information on using these functions.

2.3 Input and Output

You are given a point set with 1 million points in the file `points.txt`. Each point is described on a single line. Please produce a single line of output that indicates the distance of the closest pair. Please use the `diff` command to check if your output matches exactly with the one shown in file `closest.txt`.

3 Postamble

3.1 Points Breakup

The points breakup for each problem is shown below.

Angular Sort	15 points
Closest Pair	10 points

3.2 Makefile

As part of the submission, please submit a makefile that includes the following targets with their actions as described.

`copyfiles`	Copy all the test files from the directory specified by the argument `filepath`.
`compile-timsort`	Compiles all the files of your timsort solution.
`run-timsort`	Runs your timsort solution with arguments `output-info`, `input` and `output-sorted`.
`list-timsort`	Lists the files of your timsort solution.
`show-timsort`	Prints your timsort solution, one file at a time.
`compile-closest`	Compiles all the files of your closest solution.
`run-closest`	Runs your closest solution with arguments `input`, and `output`.
`list-closest`	Lists the files of your closest solution.
`show-closest`	Prints your closest solution, one file at a time.

A sample makefile for my solution is given below, and is also in `makefile`.

```
copyfiles:
    cp $(filepath) .

compile-timsort:
    @echo 'Python solution. Nothing to compile.'

run-timsort:
    python3 angular-sort.py $(output-info) < $(input) > $(output-sorted)

list-timsort:
    ls timsort.py

show-timsort:
    @echo 'Showing python file timsort.py'
    cat timsort.py

compile-closest:
    javac Point2.java
    javac Points2Compare.java
```

```
    javac Closest.java

run-closest:
    java Closest < $(input) > $(output)

list-closest:
    ls *.java

show-closest:
    cat Point2.java
    cat Points2Compare.java
    cat Closest.java
```

Note the descriptions of the arguments of the run commands below.

`run-timsort`	`input`	Input file for timsort
	`output-info`	Timsort output information file
	`output-sorted`	Sorted output from timsort
`run-closest`	`input`	Input file for closest
	`output`	Output file for closest

3.3 Submission

Please submit all the files as a single zip archive `h02.zip` through ASULearn. The zip archive should only contain the individual files of the assignment, and these files should not be inside folders. You can use the `zip` command in Unix for this.

```
$ zip h01.zip <space separated list of files>
```

Please only include files that represent your solution. Please do not include other extraneous files, including pre-compiled class or object files. We will compile your solutions before executing them so these files are not necessary.

3.4 Proper Use of Libraries

It is obvious that we cannot include sorting libraries for the timsort problem. You are welcome to store points in `ArrayList` or other collection objects. We will check not only the sorted output of your solution, but will also compare the timsort information file produced. Each test case will be only be graded as correct if both output files are correct.

For the closest problem, you are allowed to use the following data structure utility libraries. In general, you are allowed to use libraries for input and output, math and random functions.

C++	STL libraries `pair`, `vector`, `list`, `deque`, `queue`, `priority_queue`, `stack`, `set`, `multiset`, `map`, `multimap`, `unordered_set`, `unordered_multiset`, `unordered_map`, `unordered_multimap`
Java	`Arrays`, `Stack`, `ArrayList`, `LinkedList`, `TreeSet`, `TreeMap`, `HashSet`, `HashMap`, `Queue`, `Deque`, `PriorityQueue`
Python	tuples, lists, sets and dicts as defined in the Python language. Also allowed are `deque` and `heapq` from `collections`.

You may use these `Geeks for Geeks` resources for more information, or please check out the official language documentations.

C++ STL: `https://www.geeksforgeeks.org/the-c-standard-template-library-stl/`

Java Collection: `https://www.geeksforgeeks.org/collections-in-java-2/`

Python: `https://www.geeksforgeeks.org/python-programming-language/`